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Gauge-reversing maps on cones, and Hilbert and Thompson isometries

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We show that a cone admits a gauge-reversing map if and only if it is a symmetric cone. We use this to prove that every isometry of a Hilbert geometry is a projectivity unless the Hilbert geometry is the projective space of a non-Lorentzian symmetric cone, in which case the projectivity group is of index two in the isometry group. We also determine the isometry group of the Thompson geometry on a cone.

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1 Introduction

Consider a proper open convex cone C in a real finite-dimensional vector space V. Associated to C, there is a natural partial order on V defined by $x \leq_C y$ if $y - x \in \operatorname{cl} C$. The *gauge* on C is defined by

$$M_C(x, y) := \inf\{\lambda > 0 \mid x \leq_C \lambda y\} \quad \text{for all } x, y \in C.$$

Related to the gauge are the following two metrics on C. *Hilbert's projective metric* is defined to be

$$d_H(x, y) := \log M_C(x, y) M_C(y, x) \quad \text{for all } x, y \in C.$$

This is actually a pseudometric since $d_H(x, \lambda x) = 0$ for any $x \in C$ and $\lambda > 0$. On the projective space of the cone it is a genuine metric. *Thompson's metric* is defined to be

$$d_T(x, y) := \log \max(M_C(x, y), M_C(y, x)) \quad \text{for all } x, y \in C.$$

In this paper, we study the maps between cones that preserve or reverse the gauge, or are isometries of one of the two metrics. Recall that a map $\phi\colon C \to C'$ between two proper open convex cones is said to be gauge-preserving if $M_{C'}(\phi x, \phi y) = M_C(x, y)$ and gauge-reversing if $M_{C'}(\phi x, \phi y) = M_C(y, x)$ for all x and y in C. Obviously, both types of map are isometries of the Hilbert and Thompson metrics.

Any linear isomorphism between C and C' is clearly gauge-preserving. Noll and Schäffer [17] showed that the converse is also true: every gauge-preserving bijection between two finite-dimensional cones is a linear isomorphism.

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On a symmetric cone, that is, one that is homogeneous and self-dual, Vinberg's $*$–*map* is gauge-reversing; see Kai [11]. We show that gauge-reversing maps exist only on symmetric cones.

Theorem 1.1 *Let C be a proper open convex cone in a real finite-dimensional vector space. Then C admits a gauge-reversing map if and only if C is symmetric.*

Noll and Schäffer [17, page 377] raise the question of whether the existence of a gauge-reversing bijection between two cones requires that they be linearly isomorphic. We answer this question in the affirmative for finite-dimensional cones.

Corollary 1.2 *If there is a gauge-reversing bijection between two finite-dimensional cones, then the cones are linearly isomorphic.*

Let $D := P(C)$ be the projective space of the cone C, and consider the Hilbert metric on D. The isometry group of this metric was first studied by Busemann and Kelly; see (29.1) of [6]. They showed that, in the case where D is two-dimensional with strictly convex closure, every isometry is a projectivity, that is, arises as the projective action of a linear map on the cone.

De la Harpe [10] proved the same result in arbitrary finite dimension. He also noted that there exist isometries that are not projectivities in the case of the positive cone (where D is an open simplex) and in the case of the cone of positive-definite symmetric matrices. Both of these cones are symmetric. De la Harpe asked, in general, when do the isometry group $\mathrm{Isom}(D)$ and the projectivity group $\mathrm{Proj}(D)$ coincide?

Molnár [15] determined the isometry group of the Hilbert metric in the case of another symmetric cone, the cone $\mathrm{Pos}(\mathbb{C}, n)$ for $n \geq 3$ of positive definite Hermitian matrices with complex entries. One may interpret his results as saying that each isometry is the projective action of either a gauge-preserving or a gauge-reversing map on the cone. Molnár and Nagy [16] extended this result to the case of $n = 2$, where, of course, the Hilbert geometry is isometric to 3–dimensional hyperbolic space.

These results were generalised to all finite-dimensional symmetric cones by Bosché [4], using Jordan algebra techniques.

In [14], Matveev and Troyanov determine completely the isometry group in dimension two.

For polyhedral Hilbert geometries, it was shown by Lemmens and Walsh [13] that every isometry is a projectivity, unless the domain D is a simplex, in which case the projectivity group has index two in the isometry group.

It was proved by Speer [21] that, in general, the projectivity group is a subgroup of index at most two in the isometry group.

We show the following.

Theorem 1.3 *Let (D, d_H) be a finite-dimensional Hilbert geometry, and let C be a cone over D. Every isometry of (D, d_H) arises as the projective action of either a gauge-preserving or a gauge-reversing map of C.*

Combining this with Theorem 1.1 gives us the isometry group of any Hilbert metric.

Corollary 1.4 *If C is symmetric and not Lorentzian, then* $\mathrm{Proj}(D)$ *is a normal subgroup of index two in* $\mathrm{Isom}(D)$. *Otherwise,* $\mathrm{Isom}(D) = \mathrm{Proj}(D)$.

This result had been conjectured in [13]. It also resolves some conjectures of de la Harpe, namely that $\mathrm{Isom}(D)$ is a Lie group, and that $\mathrm{Isom}(D)$ acts transitively on D if and only if $\mathrm{Proj}(D)$ does.

We also determine the isometries of the Thompson metric.

Theorem 1.5 *Let C and C' be proper open convex cones, and let $\phi\colon C \to C'$ be a surjective isometry of the Thompson metric. Then there exist decompositions $C = C_1 \oplus C_2$ and $C' = C_1' \oplus C_2'$ such that ϕ takes the form $\phi(x_1 + x_2) = (\phi_1(x_1) + \phi_2(x_2))$, where ϕ_1 is a gauge-preserving map from C_1 to C_1', and ϕ_2 is a gauge-reversing map from C_2 to C_2'.*

The first to study the isometries of the Thompson metric were Noll and Schäffer [17]. They showed that, in the case where the cone order of either C or C' is *loose*, every such isometry is either gauge-preserving or gauge-reversing. Here loose means that for all x and y in the cone, the set $\{x, y\}$ has neither an infimum nor a supremum unless x and y are comparable. In particular, they showed that both the Lorentz cone and the cone of positive definite symmetric matrices are loose.

The isometry group of the Thompson metric has been worked out by Molnár [15] in the case of the cone of positive-definite complex Hermitian matrices, and by Bosché [4] for general symmetric cones.

The plan of the paper is as follows. We recall some background material in Section 2. We then prove the homogeneity of any cone admitting a gauge-reversing map in Section 3. An important tool we will use in much of the paper is the *horofunction* *boundary*; we recall its definition in Section 4 and describe known results about it in the case of the Hilbert geometry in Section 5. Using these results, we finish the proof of Theorem 1.1 and of Corollary 1.2 in Section 6. In Section 7, we study the isometries of the Hilbert geometry and prove Theorem 1.3 and Corollary 1.4. Sections 8 and 9 are devoted to the study of the horofunction boundaries of, respectively, product spaces and Thompson geometries. These results are then used in Section 10 to prove Theorem 1.5.

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2 Preliminaries

2.1 Gauge-preserving and gauge-reversing maps

Let C be an open convex cone in a real finite-dimensional vector space V. In other words, C is an open convex set that is invariant under multiplication by positive scalars. We use cl to denote the closure of a set. If $\operatorname{cl} C \cap (-\operatorname{cl} C) = \{0\}$, then C is called a *proper* open convex cone.

As described in the introduction, C induces a natural partial order $\leq_C$ on V, and this is used to define the gauge $M_C(\cdot,\cdot)$, which in turn is used to define Thompson's metric d_T and Hilbert's projective metric d_H on C.

Let $\phi\colon C \to C'$ be a map between two proper open convex cones in V. We say that ϕ is *isotone* if $x \leq_C y$ implies $\phi x \leq_{C'} \phi y$, and that it is *antitone* if $x \leq_C y$ implies $\phi y \leq_{C'} \phi x$. The map ϕ is called an *order embedding* if $x \leq_C y$ if and only if $\phi x \leq_{C'} \phi y$. An *order antiembedding* is defined in an analogous way. We say that ϕ is *homogeneous of degree* $\alpha \in \mathbb{R}$ if $\phi(\lambda x) = \lambda^\alpha \phi(x)$ for all $x \in C$ and $\lambda > 0$. Maps that are homogeneous of degree -1 we call *antihomogeneous*, and maps that are homogeneous of degree 1 we just call *homogeneous*.

For the proofs of the next two propositions, see [17].

Proposition 2.1 *A map $\phi\colon C \to C'$ is gauge-preserving if and only if it has any two of the following three properties: order embedding, homogeneous, Thompson-distance preserving.*

Proposition 2.2 *A map $\phi\colon C \to C'$ is gauge-reversing if and only if it has any two of the following three properties: order antiembedding, antihomogeneous, Thompson-distance preserving.*

We see from Proposition 2.1 that every linear isomorphism from C to C' is gauge-preserving. The following theorem shows that the converse is also true.

Theorem 2.3 [20; 17] *Let $\phi\colon C \to C'$ be a gauge-preserving bijection. Then, ϕ is the restriction to C of a linear isomorphism.*

2.2 Hilbert's metric

Hilbert originally defined his metric on bounded open convex sets. One can recover his definition by taking a cross-section of the cone, that is, by defining $D := \{x \in C \mid f(x) = 1\}$, where $f\colon V \to \mathbb{R}$ is some linear functional that is positive with respect to the partial order associated to C. Suppose we are given two distinct points x and y in D. Define w and z to be the points in the boundary ∂D of D such that w, x, y, and z are collinear and arranged in this order along the line in which they lie. The Hilbert distance between x and y is then defined to be the logarithm of the cross-ratio of these four points:

$$d_H(x, y) := \log \frac{|zx|\,|wy|}{|zy|\,|wx|}.$$

On D, this definition agrees with the previous one.

If D is an ellipsoid, then the Hilbert metric is Klein's model for hyperbolic space. At the opposite extreme, if D is an open simplex, then the Hilbert metric is isometric to a normed space with a polyhedral unit ball [10; 18].

One may of course identify the cross-section D with the projective space $P(C)$ of the cone.

Let (X, d) be a metric space and $I \subseteq \mathbb{R}$ an interval. A map $\gamma\colon I \to X$ is called a *geodesic* if

$$d(\gamma(s), \gamma(t)) = |s - t| \quad \text{for all } s, t \in I.$$

If I is a compact interval $[a, b]$, then the image of γ is called a *geodesic segment* connecting $\gamma(a)$ and $\gamma(b)$. In the Hilbert geometry, straight-line segments are geodesic segments. If $I = \mathbb{R}$, then we call the image of γ a *geodesic line*.

A subset of X of V is said to be *relatively open* if it is open in its affine hull. We denote by $\operatorname{rel\,int} X$ the *relative interior* of X, that is, its interior, considering it a subset of its affine hull.

A geodesic line is said to be *unique* if for each compact interval $[s, t] \subset \mathbb{R}$, the geodesic segment $\gamma([s, t])$ is the only one connecting $\gamma(s)$ and $\gamma(t)$. The following result characterises the unique geodesic lines [10].

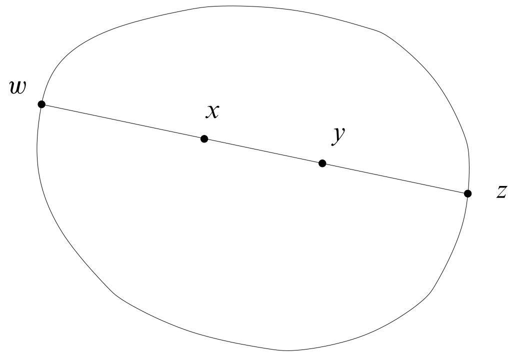

Figure 1: Definition of the Hilbert distance

Proposition 2.4 *Let (D, d_H) be a Hilbert geometry, and let $w, z \in \partial D$ be such that the relatively open line segment (w, z) lies in D. Then, (w, z) is a unique geodesic line if and only if there is no pair of relatively open line segments in* $\operatorname{cl} D$*, containing w and z, respectively, that span a two-dimensional affine space.*

Recall that an *exposed face* of a convex set is the intersection of the set with a supporting hyperplane. A convex subset E of a convex set D is an *extreme set* if the endpoints of any line segment in D are contained in E whenever any point of the relative interior of the line segment is. The relative interiors of the extreme sets of a convex set D partition D. If an extreme set consists of a single point, we call the point an *extreme point*. We call a point in the relative interior of a 1–dimensional extreme set of the closure of a cone an *extremal generator* of the cone. Alternatively, an extremal generator of a cone C is a point $x \in C$ such that $P(x)$ is an extreme point of $P(\operatorname{cl} C)$.

2.3 Symmetric cones

A proper open convex cone C in a real finite-dimensional vector space V is called *symmetric* if it is homogeneous and self-dual. Recall that C is *homogeneous* if its linear automorphism group $\operatorname{Aut}(C) := \{A \in \mathrm{GL}(V) \mid A(C) = C\}$ acts transitively on it, and it is self-dual if there exists an inner product $\langle \cdot, \cdot \rangle$ on V for which $C = C^\star$, where

$$C^\star := \{y \in V \mid \langle y, x \rangle > 0 \text{ for all } x \in \operatorname{cl} C \backslash \{0\}\}$$

is the *open dual* of C. The *characteristic function* $\phi\colon C \to \mathbb{R}$ is defined by

$$\phi(x) = \int_{C^\star} e^{-\langle y, x \rangle} \, \mathrm{d}y \quad \text{for all } x \in C.$$

This map is homogeneous of degree $-\dim V$, and so Vinberg's $*$–*map*,

$$C \to C^\star, \quad x \mapsto x^* := -\nabla \log \phi(x),$$

is antihomogeneous. On symmetric cones, the $*$–map coincides up to a scalar multiple with the inverse map in the associated Euclidean Jordan algebra [7]. It was shown in [11] that on symmetric cones the $*$–map is an order antiembedding. Hence, by Proposition 2.2, it is gauge-reversing for these cones. It was also shown in [11] that the symmetric cones are the only homogeneous cones for which this is true.

2.4 The Funk and reverse-Funk metrics

It will be convenient to consider the Hilbert and Thompson metrics as symmetrisations of the following function. Define

$$d_F(x,y) := \log M_C(x,y) \quad \text{for all } x \in V \text{ and } y \in C.$$

We call d_F the *Funk metric* after P Funk [8], and we call its reverse $d_R(x,y) := d_F(y,x)$ the *reverse-Funk metric*.

Like Hilbert's metric, the Funk metric was first defined on bounded open convex sets. On a cross-section D of the cone C, one can show that

$$d_F(x,y) = \log \frac{|xz|}{|yz|} \quad \text{and} \quad d_R(x,y) = \log \frac{|wy|}{|wx|}$$

for all $x, y \in D$. Here w and z are the points of the boundary ∂D shown in Figure 1.

On D, the Funk metric is a *quasimetric*; in other words, it satisfies the usual metric space axioms except that of symmetry. On C, it satisfies the triangle inequality but is not nonnegative. It has the following homogeneity property:

$$d_F(\alpha x, \beta y) = d_F(x,y) + \log\alpha - \log\beta \quad \text{for all } x, y \in C \text{ and } \alpha, \beta > 0.$$

Observe that both the Hilbert and Thompson metrics are symmetrisations of the Funk metric: for all $x, y \in C$,

$$d_H(x,y) = d_F(x,y) + d_R(x,y) \quad \text{and} \quad d_T(x,y) = \max(d_F(x,y), d_R(x,y)).$$

3 Homogeneity

In this section, we will prove that the existence of a gauge-reversing map on a cone implies that the cone is homogeneous.

Throughout the paper, we assume that C is a proper open convex cone in a real finite-dimensional vector space V.

Proposition 3.1 *Let $\phi\colon C \to C$ be gauge-reversing. Then ϕ is a bijection.*

Proof By Proposition 2.2, ϕ is an isometry of the Thompson metric. It is therefore injective and continuous. So, from invariance of domain, we get that $\phi(C)$ is an open set in C.

Let y_n be a sequence in $\phi(C)$ converging to $y \in C$. So there exists a sequence x_n in C such that $\phi(x_n) = y_n$, for all $n \in \mathbb{N}$. Moreover, y_n satisfies the Cauchy criterion, and so x_n does too. Therefore, since (C, d_T) is complete, x_n converges to some point $x \in C$. From continuity, we get that $\phi(x) = y$. We have proved that $\phi(C)$ is closed.

Since C is connected and $\phi(C)$ is nonempty and both open and closed, we conclude that $\phi(C) = C$. □

We use Id to denote the identity operator.

Lemma 3.2 *Let $\phi\colon C \to C$ be a gauge-reversing map that is differentiable at some point $x \in C$ with derivative $D_x\phi = -\operatorname{Id}$. Then x is a fixed point of ϕ.*

Proof Since ϕ is antihomogeneous, we have $\phi(x + \lambda x) = \phi(x)/(1+\lambda)$ for all $\lambda > 0$. This implies that $D_x\phi(x) = -\phi(x)$. But, by hypothesis, $D_x\phi(x) = -x$. Therefore, $\phi(x) = x$. □

Recall that an involution is a map ϕ satisfying $\phi \circ \phi = \operatorname{Id}$.

Lemma 3.3 *Assume there exists a gauge-reversing map $\phi\colon C \to C$. Then, for almost all x in C, there exists a gauge-reversing map $\phi_x\colon C \to C$ that fixes x, has derivative $D_x\phi_x = -\operatorname{Id}$ at x, and is an involution.*

Proof The map ϕ is 1–Lipschitz in the Thompson metric on C. However, this metric is Lipschitz equivalent to the Euclidean metric on any ball of finite radius in the Thompson metric. So, we may apply Rademacher's theorem to deduce that ϕ is differentiable almost everywhere within every ball of finite radius, and hence almost everywhere within all of C.

By Proposition 3.1, the map ϕ is bijective, and so has an inverse, which is also gauge-reversing, and hence, by the reasoning of the previous paragraph, differentiable almost everywhere.

We deduce that, for almost every point x in C, ϕ is differentiable at x and ϕ^{-1} is differentiable at $\phi(x)$. Necessarily, $D_{\phi(x)}\phi^{-1} = (D_x\phi)^{-1}$.

It follows from the antitonicity of ϕ that the linear map $D_x\phi$ is antitone, and hence that $-D_x\phi$ is isotone. Similarly, from the antitonicity of ϕ^{-1}, we deduce that $(-D_x\phi)^{-1} = -D_{\phi(x)}\phi^{-1}$ is isotone. Therefore, $(-D_x\phi)^{-1}$ is a linear isomorphism of C, and hence gauge-preserving.

So the map $\phi_x\colon C \to C$ defined by $\phi_x := (-D_x\phi)^{-1} \circ \phi$ is gauge-reversing. By the chain rule, $D_x\phi_x = -\mathrm{Id}$. So, from Lemma 3.2, x is a fixed point of ϕ_x. The map $\phi_x \circ \phi_x$ is gauge-preserving, and therefore linear, and its derivative at x is Id. We conclude that $\phi_x \circ \phi_x = \mathrm{Id}$. □

Lemma 3.4 *Assume there exists a gauge-reversing map $\phi\colon C \to C$. Let x and y be two points in $P(C)$ collinear with an extreme point of $P(\mathrm{cl}\, C)$. Then there exists an element of* $\mathrm{Proj}(P(C))$ *mapping x to y.*

Proof Let z be the midpoint, in the Hilbert metric on $P(C)$, between x and y on the straight line joining them. By Lemma 3.3, we may find a sequence z_n in $P(C)$ converging to z such that, for all $n \in \mathbb{N}$, there is a gauge-reversing map $\phi_{z_n}\colon C \to C$ that fixes some representative $\tilde{z}_n \in C$ of z_n, and has derivative $-\mathrm{Id}$ at $\tilde{z}_n$. Considering the action of ϕ_{z_n} on the projective space $P(C)$, we see that ϕ_{z_n} fixes $z_n = P(\tilde{z}_n)$ and its derivative there is $-\mathrm{Id}$.

By assumption, there is an extreme point a of $P(\mathrm{cl}\, C)$ such that a, x, and y are collinear. We may assume, by relabelling if necessary, that x lies between a and y. For each $n \in \mathbb{N}$, define the straight line segment $L_n := az_n \cap P(C)$, and let x_n and y_n be the two points on L_n satisfying

$$d_H(x_n, z_n) = d_H(z_n, y_n) = \tfrac{1}{2} d_H(x, y).$$

We label these two points in such a way that x_n lies between a and y_n; see Figure 2. Clearly, x_n and y_n converge, respectively, to x and y as n tends to infinity.

Since az_n passes through an extreme point of $P(\mathrm{cl}\, C)$, the line segment L_n is a unique geodesic line with respect to the Hilbert metric. So, $\phi_{z_n}(L_n)$ is also a unique geodesic line, and hence a straight line segment. Using now that ϕ_{z_n} fixes z_n and has derivative $-\mathrm{Id}$ there, we get that ϕ_{z_n} leaves L_n invariant and reverses its orientation. So we have $\phi_{z_n}(x_n) = y_n$ for all $n \in \mathbb{N}$.

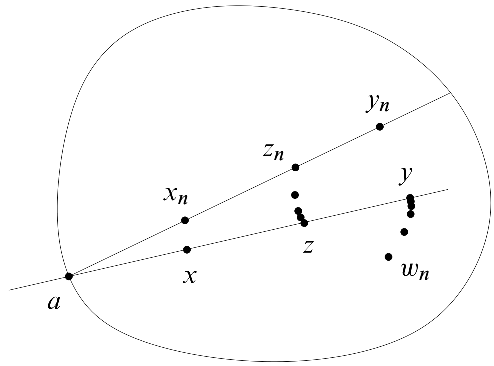

Figure 2: Illustration of the proof of Lemma 3.4

Again by Lemma 3.3, there exists a sequence w_n in $P(C)$ converging to y such that, for all $n \in \mathbb{N}$, there is a gauge-reversing map $\phi_{w_n}: C \to C$ that fixes a representative $\widetilde{w}_n \in C$ of w_n, and therefore fixes w_n. For each $n \in \mathbb{N}$, the map $f_n: C \to C$ defined by $f_n := \phi_{w_n} \circ \phi_{z_n}$ is gauge-preserving, and hence linear, by Theorem 2.3. Therefore, the action of f_n on $P(C)$ is in $\mathrm{Proj}(P(C))$ for each $n \in \mathbb{N}$.

Observe that the sequences y_n and w_n have the same limit, and that $\phi_{w_n}(w_n) = w_n$ converges to y. So, using that the $\{\phi_{w_n}\}$ are all 1–Lipschitz, we get that $\phi_{w_n}(y_n)$ converges to y. But $\phi_{w_n}(y_n) = f_n(x_n)$ for all $n \in \mathbb{N}$, and x_n converges to x. We conclude that $f_n(x)$ converges to y. This implies that the maps $\{f_n\}$ all lie in some bounded subset of $\mathrm{Proj}(P(C))$. It follows that there exists $f \in \mathrm{Proj}(P(C))$ such that some subsequence of f_n converges to f uniformly on compact sets of $P(C)$. Evidently, $f(x) = y$. □

Lemma 3.5 *Assume there exists a gauge-reversing map $\phi: C \to C$. Then C is a homogeneous cone.*

Proof Let x and y be points in C such that $y = x + z$, where z is an extremal generator of C. So $P(x)$, $P(y)$, and $P(z)$ are collinear in the projective space $P(C)$, and $P(z)$ is an extreme point of $P(\mathrm{cl}\, C)$. Therefore, by Lemma 3.4, some element of the linear automorphism group $\mathrm{Aut}(C)$ maps x to a positive multiple of y. Combining this automorphism with multiplication by a positive scalar, we can in fact find an element of $\mathrm{Aut}(C)$ that maps x to y. The result now follows since one can get from any element of C to any other by adding and subtracting a finite number of extremal generators of C. □

4 The horofunction boundary

We recall in this section the definition of the horofunction boundary, which will be used extensively in the rest of the paper. The setting will be that of quasimetric spaces, since some of the metrics with which we will be dealing, namely, the Funk and reverse-Funk metrics, are not symmetric.

Let (X,d) be a quasimetric space, that is, a space that satisfies the usual metric space axioms apart from that of symmetry. We endow X with the topology induced by the symmetrised metric $d_{\mathrm{sym}}(x,y) := d(x,y)+d(y,x)$, which for Funk and reverse-Funk metrics is the Hilbert metric.

To each point $z \in X$, associate the function $\psi_z\colon X \to \mathbb{R}$,

$$\psi_z(x) := d(x,z) - d(b,z),$$

where b is some fixed basepoint. Consider the map $\psi\colon X \to C(X)$, $z \mapsto \psi_z$, from X into $C(X)$, the space of continuous real-valued functions on X endowed with the topology of uniform convergence on bounded sets of d_{sym}. This map can be shown to be injective and continuous [2]. The *horofunction boundary* is defined to be

$$X(\infty) := \mathrm{cl}\{\psi_z \mid z \in X\} \backslash \{\psi_z \mid z \in X\},$$

and its elements are called horofunctions. This definition is due to Gromov [9], and is a development of an idea of Busemann [5], who considered the horofunctions obtained along geodesic rays.

It is easy to check that the horofunction boundaries obtained using different basepoints are homeomorphic to one another, and that indeed corresponding horofunctions differ only by an additive constant.

A geodesic in a quasimetric space (X,d) is a map γ from an interval of $\mathbb{R}$ to X such that $d(\gamma(s),\gamma(t)) = t-s$ for all s and t in the domain with $s<t$. The space (X,d) is said to be *geodesic* if, for any pair of points x and y in X, there is a geodesic $\gamma\colon [s,t] \to X$ with respect to d that starts at x and ends at y.

We make the following assumptions:

(I) The metric d_{sym} is proper, that is, its closed balls are compact.

(II) (X,d) is geodesic.

(III) For any point x and sequence x_n in X, we have $d(x_n,x) \to 0$ if and only if $d(x,x_n) \to 0$.

These assumptions are satisfied by the Funk and reverse-Funk metrics.

Under assumptions (I), (II) and (III), it can be shown that ψ is an embedding of X into $C(X)$; in other words, it is a homeomorphism from X to its image. From now on we identify X with its image.

We will need the next proposition in Section 8.

Proposition 4.1 *Let (X,d) be a proper geodesic metric space. Then $\inf \xi = -\infty$ for any horofunction ξ.*

Proof Let x_n be a sequence converging to ξ. Since X is proper, we have that $d(b,x_n)$ converges to infinity, where b is the basepoint. For each $n \in \mathbb{N}$, let $\gamma_n\colon [0, d(b,x_n)] \to X$ be a geodesic segment between b and x_n. Choose $t > 0$. For n large enough, $d(\gamma_n(t), x_n) = d(b,x_n) - t$. Since the sequence $(\gamma_n(t))_n$ lies in a compact set, we may, by taking a subsequence if necessary, assume that it has a limit y. Using that the functions $d(\cdot, x_n) - d(b,x_n)$ are 1–Lipschitz, we get that

$$\xi(y) = \lim_{n\to\infty} d(\gamma_n(t), x_n) - d(b,x_n) = -t.$$

The result follows since t is arbitrary. □

Isometries between quasimetric spaces extend continuously to homeomorphisms between their horofunction compactifications. Assume that f is an isometry from one quasimetric space (X,d) to another, (X',d'), with basepoints b and b', respectively. Then, for every horofunction ξ and point $x \in X$,

$$f \cdot \xi(x) = \xi(f^{-1}(x)) - \xi(f^{-1}(b')).$$

4.1 Almost-geodesics and Busemann points

Let (X,d) be a metric space. We call a path $\gamma\colon T \to X$, with T an unbounded subset of $\mathbb{R}_+$ containing 0, an *almost-geodesic* if, for each $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that

$$|d(\gamma(0),\gamma(s)) + d(\gamma(s),\gamma(t)) - t| < \epsilon \quad \text{for all } s \text{ and } t \text{ with } N \le s \le t.$$

Rieffel [19] proved that every almost-geodesic converges. We say that a horofunction is a *Busemann point* if there exists an almost-geodesic converging to it, and denote by $X_B(\infty)$ the set of all Busemann points in $X(\infty)$.

The following alternative characterisation of almost-geodesics will be useful.

Lemma 4.2 *Let (X,d) be a proper geodesic metric space. A map $\gamma\colon T \to X$, with T an unbounded subset of $\mathbb{R}_+$ containing 0, is an almost-geodesic if and only if, given any $\epsilon > 0$,*

$$|d(\gamma(0),\gamma(s)) - s| < \epsilon, \tag{1}$$

$$|d(\gamma(s),\gamma(t)) - t + s| < \epsilon, \tag{2}$$

for all $s,t \in T$ large enough with $s \le t$.

Proof That (1) and (2) hold for any almost-geodesic was proved by Rieffel [19]. The implication in the opposite direction is equally straightforward. □

When X is proper and geodesic, one may take T to be $\mathbb{R}_+$, as the following lemma demonstrates.

Lemma 4.3 *Let (X,d) be a proper geodesic metric space, and let ξ be a Busemann point. Then there exists an almost-geodesic defined on the whole of $\mathbb{R}_+$ that converges to ξ.*

Proof Since ξ is a Busemann point, there exists an almost-geodesic $\alpha\colon T \to X$ converging to it, where T is an unbounded subset of $\mathbb{R}_+$ containing 0. Choose a strictly increasing sequence (t_n) in T, starting at $t_0 := 0$ and converging to infinity. Since (X,d) is a geodesic space, we may find, for each $n \in \mathbb{N}$, a geodesic segment $\beta_n\colon [0, d(\alpha(t_n),\alpha(t_{n+1}))] \to X$ from $\alpha(t_n)$ to $\alpha(t_{n+1})$. We interpolate between the points $\alpha(t_n)$ for $n \in \mathbb{N}$ by reparametrising these geodesic segments and concatenating them. Define $\gamma\colon \mathbb{R}_+ \to X$ by

$$\gamma(t) := \beta_n\left(\frac{t-t_n}{t_{n+1}-t_n} d(\alpha(t_n),\alpha(t_{n+1}))\right) \quad \text{for all } t \in \mathbb{R}_+,$$

where n depends on t and is such that $t_n \le t < t_{n+1}$. Observe that $\gamma(t_n) = \alpha(t_n)$ for all $n \in \mathbb{N}$.

Suppose we are given $\epsilon > 0$. Let s and t both lie in $[t_n, t_{n+1}]$ for some $n \in \mathbb{N}$, with $s < t$. So

$$\frac{d(\gamma(s),\gamma(t))}{d(\alpha(t_n),\alpha(t_{n+1}))} = \frac{t-s}{t_{n+1}-t_n} \le 1.$$

From Lemma 4.2, if n is large enough, then $|d(\alpha(t_n),\alpha(t_{n+1})) - t_{n+1} + t_n| < \frac{\epsilon}{2}$. Therefore, in this case, $|d(\gamma(s),\gamma(t)) - t + s| < \frac{\epsilon}{2}$. This shows that (2) holds when s and t are large enough and lie in the same interval $[t_n, t_{n+1}]$.

Now let p lie in $[t_n, t_{n+1}]$ with $n \in \mathbb{N}$. The triangle inequality gives

$$\begin{aligned} d(\gamma(0), \gamma(t_{n+1})) - d(\gamma(p), \gamma(t_{n+1})) &\le d(\gamma(0), \gamma(p)) \\ &\le d(\gamma(0), \gamma(t_n)) + d(\gamma(t_n), \gamma(p)). \end{aligned}$$

From Lemma 4.2, both $|d(\gamma(0), \gamma(t_n)) - t_n|$ and $|d(\gamma(0), \gamma(t_{n+1})) - t_{n+1}|$ are less than $\frac{\epsilon}{2}$ if n is large enough. Using this and what we proved in the previous paragraph, we get that, when p is large enough, (1) holds, with p substituted for s.

The proof that (2) holds goes along similar lines. □

In the reverse-Funk metric, one may approach the boundary along a path of finite length. So, for such spaces, we must modify the definition of almost-geodesic. We drop the requirement that T be unbounded, and instead require that $\sup T$ be a limit point but not an element of T.

A path $\gamma\colon T \to X$ is now said to be an almost-geodesic if, for each $\epsilon > 0$, there exists $N < \sup T$ such that

$$|d(\gamma(0), \gamma(s)) + d(\gamma(s), \gamma(t)) - t| < \epsilon \quad \text{for all } s \text{ and } t \text{ with } N \le s \le t.$$

One may show again that every almost-geodesic converges, however the limit may now be a point in X. Again, a Busemann point is a horofunction that is the limit of an almost-geodesic. One may verify that most of the results concerning Busemann points carry over to this new definition.

The Busemann points provide enough information, in certain cases, to recover the metric. We will use the following result in Section 7.

Proposition 4.4 *Let (X, d) be a quasimetric space satisfying assumptions* (I), (II) *and* (III). *Also assume that for each pair of points x and y in X there exists a geodesic starting at x, passing through y, and converging to a Busemann point. Then*

$$d(x, y) = \sup_{\xi}(\xi(x) - \xi(y)) \quad \textit{for all } x, y \in X,$$

where the supremum is taken over all Busemann points ξ.

Proof Let $x, y \in X$. Horofunctions are 1–Lipschitz, that is, $\xi(x) \le d(x, y) + \xi(y)$ for each horofunction ξ. This implies that the left-hand side of the equation above is greater than or equal to the right-hand side.

To prove the opposite inequality, let γ be a geodesic starting at x, passing through y, and converging to a Busemann point ξ. For $t > d(x,y)$, we have $d(x,y)+d(y,\gamma(t)) = d(x,\gamma(t))$. It follows that $d(x,y)+\xi(y) = \xi(x)$. □

4.2 The detour metric

We define the *detour cost* for any two horofunctions ξ and η in $X(\infty)$ to be

$$H(\xi,\eta) := \sup_{W \ni \xi} \inf_{x \in W \cap X} (d(b,x)+\eta(x)),$$

where the supremum is taken over all neighbourhoods W of ξ in $X \cup X(\infty)$. An equivalent definition is

$$H(\xi,\eta) := \inf_{\gamma} \liminf_{t \to \sup T} \big(d(b,\gamma(t)) + \eta(\gamma(t))\big),$$

where the infimum is taken over all paths $\gamma\colon T \to X$ converging to ξ. This concept first appears in [1]. More detail about it can be found in [23].

The detour cost satisfies the triangle inequality and is nonnegative. The Busemann points can be characterised as those horofunctions ξ satisfying $H(\xi,\xi) = 0$.

By symmetrising the detour cost, we obtain a metric on the set of Busemann points,

$$\delta(\xi,\eta) := H(\xi,\eta) + H(\eta,\xi) \quad \text{for all Busemann points } \xi \text{ and } \eta.$$

We call δ the *detour metric*. It is possibly infinite-valued, so it is actually an *extended metric*. One may partition the set of Busemann points into disjoint subsets in such a way that $\delta(\xi,\eta)$ is finite if and only if ξ and η lie in the same subset. We call these subsets the *parts* of the horofunction boundary.

The following expression for the detour cost will prove useful in Sections 8 and 9.

Proposition 4.5 *Let ξ be a Busemann point, and η a horofunction of a metric space (X,d). Then*

$$H(\xi,\eta) = \sup_{x \in X} (\eta(x) - \xi(x)) = \inf\{\lambda \in \mathbb{R} \mid \eta(\cdot) \le \xi(\cdot) + \lambda\}. \tag{3}$$

Proof First observe that the second equality of (3) is easy to prove.

According to [23, Lemma 5.1], $\eta(\cdot) \le \xi(\cdot) + H(\xi,\eta)$. This implies that $H(\xi,\eta)$ is greater than or equal to the right-hand side of (3).

Let γ be an almost-geodesic converging to ξ. By [23, Lemma 5.2],

$$\lim_{t\to\infty}\big(d(b,\gamma(t))+\eta(\gamma(t))\big)=H(\xi,\eta) \quad\text{and}\quad \lim_{t\to\infty}\big(d(b,\gamma(t))+\xi(\gamma(t))\big)=0.$$

Therefore, $\lim_{t\to\infty}\big(\eta(\gamma(t))-\xi(\gamma(t))\big)=H(\xi,\eta)$. Thus, $H(\xi,\eta)\le\sup(\eta-\xi)$. □

5 The horofunction boundary of the Hilbert geometry

Let (D,d_H) be a Hilbert geometry with basepoint b. The horofunction boundary of this geometry is best understood by first considering the horofunction boundaries of the Funk and the reverse-Funk geometries.

5.1 The horofunction boundary of the reverse-Funk geometry

The following proposition says essentially that the horofunction boundary of the reverse-Funk geometry is just the usual boundary ∂D, and gives an explicit formula for the horofunctions. For each $x\in\operatorname{cl}D$, define the function $r_x(\cdot):=d_R(\cdot,x)-d_R(b,x)$.

Proposition 5.1 [22] *The set of horofunctions of the reverse-Funk geometry on D is $\{r_x \mid x\in\partial D\}$. Every horofunction is a Busemann point. A sequence in D converges to r_x, with $x\in\partial D$, in the horofunction boundary if and only if it converges to x in the usual topology.*

We also have a description of the detour metric for this geometry.

Proposition 5.2 [13] *Let x and y be in ∂D. If x and y are in the relative interior of the same extreme set E of $\operatorname{cl}D$, then the detour metric of the reverse-Funk geometry is $\delta_R(r_x,r_y)=d_H^E(x,y)$, where d_H^E is the Hilbert metric on $\operatorname{rel\,int}E$. Otherwise, $\delta_R(r_x,r_y)$ is infinite.*

Thus, the parts of the horofunction boundary are the relative interiors of the proper extreme sets of $\operatorname{cl}D$. We will be particularly interested in parts consisting of a single point. We call these parts *singletons*. In the reverse-Funk geometry, there is a singleton for each extreme point of $\operatorname{cl}D$.

5.2 The horofunction boundary of the Funk geometry

The horofunction boundary of the Funk geometry is more complicated than that of the reverse-Funk geometry. Its Busemann points were worked out explicitly in [22]. We will not need all the details here.

Proposition 5.3 [22] *There is a part associated to every proper extreme set of the polar D° of D. Endowed with its detour metric, each part is isometric to a Hilbert geometry of the same dimension as the associated extreme set. If one extreme set E_1 is contained in another, E_2, then the part associated to E_1 is contained in the closure of the part associated to E_2.*

Let C be the cone over D. Recall that the polar of D may be identified with a cross-section of

$$C^* := \{y \in V^* \mid \langle y, x\rangle \geq 0 \text{ for all } x \in C\},$$

the (closed) dual cone of C.

Proposition 5.4 [22] *Let y be an extremal generator of C^* normalised so that $\langle y, b\rangle = 1$. Then $\log\langle y, \cdot\rangle$ restricted to D is a singleton of the Funk geometry of D, and every singleton arises in this way.*

5.3 The horofunction boundary of the Hilbert geometry

From the expression of the Hilbert metric as the symmetrisation of the Funk metric, it is clear that every Hilbert horofunction is the sum of a Funk horofunction and a reverse-Funk horofunction. We have the following criterion for convergence.

Proposition 5.5 [22] *A sequence in D converges to a point in the Hilbert-geometry horofunction boundary if and only if it converges to a horofunction in both the Funk and reverse-Funk geometries.*

Combined with Proposition 5.1, this implies that every sequence converging to a Hilbert geometry horofunction also converges to a point in the usual boundary ∂D.

For each $x \in \partial D$, let $B(x)$ denote the set of Funk-geometry Busemann points that can be approached by a sequence converging to x in the usual topology.

Proposition 5.6 [22] *The set of Busemann points of the Hilbert geometry is*

$$\{r_x + f \mid x \in \partial D \text{ and } f \in B(x)\}.$$

The detour metric was calculated in [13].

Proposition 5.7 [13] *Every part of the horofunction boundary of the Hilbert geometry can be expressed as the Cartesian product of a part of the reverse-Funk geometry and* *a part of the Funk geometry. The detour metric distance between two Hilbert-geometry Busemann points* $h_1 := r_{x_1} + f_1$ *and* $h_2 := r_{x_2} + f_2$*, with* $f_1 \in B(x_1)$ *and* $f_2 \in B(x_2)$*, is*

$$\delta_H(h_1, h_2) = \delta_R(r_{x_1}, r_{x_2}) + \delta_F(f_1, f_2),$$

where δ_R *and* δ_F *are the detour metrics on the set of Busemann points of, respectively, the reverse-Funk and Funk geometries.*

So, each part is the ℓ_1–product of two lower-dimensional Hilbert geometries.

The following proposition tells us which reverse-Funk parts combine with which Funk parts to form a Hilbert part.

Proposition 5.8 [13] *Let* R *be the reverse-Funk part associated to a proper extreme set* E *of* $\operatorname{cl} D$*, and let* F *be the Funk part associated to a proper extreme set* E' *of* D°*. Then* $\{r + f \mid r \in R,\ f \in F\}$ *is a part of the Hilbert horofunction boundary if and only if* $\langle y, x\rangle = 0$ *for all* $y \in E'$ *and* $x \in E$*.*

Let $U = R \times F$ be a Hilbert part expressed as the product of a reverse-Funk part and a Funk part. If either of the component parts is a singleton, then U will itself be a Hilbert geometry when endowed with its detour metric. We call such parts of the Hilbert horofunction boundary *pure* parts. When R is a singleton, U will be called a *pure Funk part*, and when F is a singleton, U will be called a *pure reverse-Funk part*.

It was shown in [13] that the ℓ_1–product of two Hilbert geometries, each consisting of more than one point, cannot be isometric to a Hilbert geometry. It follows that the extension to the horofunction boundary of any isometry maps pure parts to pure parts.

Of particular interest will be the *maximal* pure parts. These are the pure parts that are not contained in the closure of any other pure part. This property is also preserved by isometries.

Let W be a maximal pure Funk part. From what we have seen above, there is some extreme point w of $\operatorname{cl} D$ such that each element of W can be written $r_w + f$, where f is in the part of the Funk horofunction boundary associated to the exposed face of D° defined by w.

Similarly, if U is a maximal pure reverse-Funk part, then there is a singleton Funk horofunction $f^{(u)}$ corresponding to some extreme point u of D°, such that any element of U can be written $r_x + f^{(u)}$, with $x \in \operatorname{rel\,int} F$, where F is the exposed face of $\operatorname{cl} D$ defined by u.

Note that

$$\operatorname{cl} W = \{r_w + f \mid f \in B(w)\} \quad \text{and} \quad \operatorname{cl} U = \{r_x + f^{(u)} \mid f^{(u)} \in B(x)\}.$$

Here the closures are taken in the set of Busemann points, not the set of horofunctions.

5.4 Horofunctions extended to the cone

Recall that Funk and reverse-Funk metrics can be extended to all of C. We may do likewise with the Funk and reverse-Funk horofunctions. In particular, for $x \in \partial D$, we have $r_x(y) = d_R(y,x) - d_R(b,x)$ for all $y \in C$.

Observe that the extension to C of a reverse-Funk horofunction is the logarithm of an antihomogeneous function, whereas that of a Funk horofunction is the logarithm of an homogeneous function. So the natural extension to C of any Hilbert horofunction is homogeneous of degree 0, that is, constant on the projective class of each point of C.

Recall that every isometry of a metric space extends continuously to a homeomorphism on the compactification. One may consider a gauge-reversing map on a cone C to be an isometry from C with the reverse-Funk metric to C with the Funk metric. This is formalised in the following proposition, which will be crucial in our study of gauge-reversing maps. It says that the extension of a gauge-reversing map to the horofunction boundary takes reverse-Funk horofunctions to Funk horofunctions and vice versa. Recall that we may consider D to be a cross-section of the cone C.

Proposition 5.9 *Let $\phi\colon C \to C$ be a gauge-reversing map. If $r_x\colon C \to \mathbb{R}$, with $x \in \partial D$, is an (extended) reverse-Funk horofunction, then*

$$\phi r_x(\cdot) := r_x \circ \phi^{-1}(\cdot) - r_x \circ \phi^{-1}(b)$$

is an (extended) Funk horofunction. Likewise, if $f\colon C \to \mathbb{R}$ is an (extended) Funk horofunction, then

$$\phi f(\cdot) := f \circ \phi^{-1}(\cdot) - f \circ \phi^{-1}(b)$$

is an (extended) reverse-Funk horofunction.

Proof Let x_n be a sequence in D converging to r_x in the reverse-Funk horofunction boundary. For each $n \in \mathbb{N}$, let $z_n \in D$ and $\lambda_n > 0$ be such that $z_n = \lambda_n \phi(x_n)$. For all $n \in \mathbb{N}$, we have

$$d_F(\cdot, z_n) - d_F(b, z_n) = d_F(\cdot, \phi x_n) - d_F(b, \phi x_n) = d_R(\phi^{-1}(\cdot), x_n) - d_R(\phi^{-1} b, x_n).$$

So, as n tends to infinity, the sequence z_n converges in the Funk geometry to the function $r_x \circ \phi^{-1}(\cdot) - r_x \circ \phi^{-1}(b)$, which must therefore be a horofunction.

The proof of the second part is similar. □

6 Self-duality

Max Koecher defined a *domain of positivity* for a symmetric nondegenerate bilinear form $\mathcal{B}$ on a real finite-dimensional vector space V to be a nonempty open set $\mathcal{C}$ such that $\mathcal{B}(x, y) > 0$ for all x and y in $\mathcal{C}$, and such that if $\mathcal{B}(x, y) > 0$ for all $y \in \operatorname{cl} \mathcal{C} \backslash \{0\}$, then $x \in \mathcal{C}$. See [12]

A domain of positivity is always a proper open convex cone. If the bilinear form is positive definite, then the cone is self-dual. A negative-definite bilinear form cannot have a domain of positivity.

Assume we have a proper open convex cone C in a finite-dimensional vector space that admits a gauge-reversing map. By Lemma 3.3, there exists a gauge-reversing map $\phi\colon C \to C$ that is an involution, has a fixed point b, and is differentiable at b with derivative $-\mathrm{Id}$. We take b to be the basepoint.

We wish to define a positive-definite bilinear form that makes the cone C a domain of positivity.

We use the fact that certain Funk geometry horofunctions are log-of-linear functions. For an illustration of our method, consider the positive cone $\operatorname{int} \mathbb{R}^n_+$ with the gauge-reversing map $\rho\colon \operatorname{int} \mathbb{R}^n_+ \to \operatorname{int} \mathbb{R}^n_+$ defined by $(\rho x)_i := 1/x_i$ for all $x \in \operatorname{int} \mathbb{R}^n_+$ and coordinates i. The singleton parts of the reverse-Funk horofunction boundary correspond to the extremal rays of the cone, in this case the (positive) coordinate axes of $\mathbb{R}^n$. More precisely, associated to the i^{th} coordinate axis e_i is the reverse-Funk horofunction $r_{e_i}(x) = -\log x_i$. Each of these singletons is mapped by ρ to a singleton of the Funk horofunction boundary. In the particular case under consideration, we have $\rho(r_{e_i})(x) = r_{e_i} \circ \rho^{-1}(x) = \log x_i$ for all $x \in \operatorname{int} \mathbb{R}^n_+$. In general, the singleton Funk horofunction is the logarithm of a linear functional. Thus, we have a correspondence between the extremal rays of the cone and linear functionals. This will allow us to define a bilinear form on the space.

For x an extremal generator of C, define $h_x(\cdot) := \exp(r_x \circ \phi(\cdot))$ on C, where $r_x(\cdot) := \log(M(x,\cdot)/M(x,b))$ is the reverse-Funk horofunction associated to the point $x \in \partial C \backslash \{0\}$. Observe that h_x is the exponential of the image of r_x under the map ϕ.

Lemma 6.1 *For each extremal generator x of C, the function h_x is linear on C. Moreover, h defines a bijection between the projective classes of extremal generators of C and those of its dual C^*.*

Proof Since ϕ reverses the gauge, it maps parts of the reverse-Funk horofunction boundary to parts of the Funk horofunction boundary. In particular, it maps singletons to singletons.

The parts of the reverse-Funk boundary correspond to the relative interiors of the extreme sets of $P(\mathrm{cl}\, C)$. So the singletons correspond to the projective classes of extremal generators of C.

We have seen in Proposition 5.4 that singletons of the Funk horofunction boundary correspond to projective classes of extremal generators of C^*, and that each of these horofunctions is the logarithm of a linear function. □

Definition 6.2 For y in C and x an extremal generator of C, let

$$B(y,x) := h_x(y)M(x,b) = M(x,\phi(y)).$$

Extend this definition to all $y \in V$ using Lemma 6.1. We obtain a function on V that is linear in y for every fixed extremal generator x of C.

Observe that $B(y,x)$ is homogeneous in x.

Lemma 6.3 *Let x and x' be extremal generators of C. Then $B(x,x') = B(x',x)$.*

Proof Let x_n and x'_n be sequences in C converging, respectively, to x and x'. Define

$$j_y(z) := \frac{M(z,y)}{M(b,y)} \quad \text{for all } y \in C \text{ and } z \in V.$$

Since x_n converges to x, the functions r_{x_n} converge pointwise to the reverse-Funk horofunction r_x. But ϕ is a gauge-reversing involution that fixes b, and so

$$r_{x_n} \circ \phi(\cdot) = \log \frac{M(x_n,\phi(\cdot))}{M(x_n,b)} = \log \frac{M(\cdot,\phi(x_n))}{M(b,\phi(x_n))} = \log j_{\phi(x_n)}(\cdot).$$

We deduce that the functions $\log j_{\phi(x_n)}$ converge pointwise to the Funk horofunction $\log h_x$. Therefore, $j_{\phi(x_n)}$ converges pointwise to h_x. By [22, Lemma 3.16], the functions $\{j_y\}$ for $y \in C$ are equi-Lipschitzian; in other words, there is some $\lambda > 0$ such that each of them is λ–Lipschitz. We deduce that $j_{\phi(x_n)}(x'_n)$ converges to $h_x(x')$ as n tends to infinity. Using in addition that $M(b, \phi(x_n)) = M(x_n, b)$ for all n, and that $M(\cdot\,, b)$ is continuous, we get

$$\lim_{n\to\infty} M(x'_n, \phi(x_n)) = \lim_{n\to\infty} j_{\phi(x_n)}(x'_n) M(b, \phi(x_n)) = h_x(x') M(x, b) = B(x', x).$$

But, for each $n \in \mathbb{N}$, we have that $M(x'_n, \phi(x_n))$ equals $M(x_n, \phi(x'_n))$, and similar reasoning to the above shows that the limit of this latter quantity is $B(x, x')$. □

Further extend the definition of B to $V \times V$ by taking $B(y, z) := \sum_j z_j B(y, x_j)$, where $z = \sum_j z_j x_j$ for some basis of extremal generators $\{x_j\}$ of C.

Proposition 6.4 *This definition is independent of the basis of extremal generators chosen.*

Proof Suppose $z = \sum_j z_j x_j = \sum_j z'_j x'_j$ for two bases of extremal generators $\{x_j\}$ and $\{x'_j\}$. Take any $y \in V$ and write $y = \sum_j y''_j x''_j$ for some basis of extremal generators $\{x''_j\}$ of C. Using Lemma 6.3, we get

$$\sum_j z_j B(y, x_j) = \sum_{j,k} z_j y''_k B(x''_k, x_j) = \sum_{j,k} z_j y''_k B(x_j, x''_k) = \sum_k y''_k B(z, x''_k).$$

Similarly, $\sum_j z'_j B(y, x'_j)$ can be shown to be equal to the same expression. □

Lemma 6.5 *The function B is a symmetric nondegenerate bilinear form, and C is a domain of positivity for B.*

Proof It is clear from its definition that B is bilinear, and the symmetry comes from the bilinearity and Lemma 6.3.

Let $z \in V$ be such that $B(z, x) = 0$ for all $x \in V$. So, in particular, for all extremal generators x of C we have $h_x(z) M(x, b) = B(z, x) = 0$, and hence $h_x(z) = 0$. But, by Lemma 6.1, h_x is an extremal generator of C^*, and all extremal generators of C^* arise in this way, up to scale. Also, C is proper, and so the extremal generators of C^* span V^*. It follows that $z = 0$. We deduce that B is nondegenerate.

If $y \in C$ and x is an extremal generator of C, then $B(y, x) = M(x, \phi(y)) > 0$. It follows that $B(y, z) > 0$ for all $y, z \in C$.

Let $y \in V$ be such that $B(y,z) > 0$ for all $z \in \operatorname{cl} C \backslash \{0\}$. So, for every extremal generator x of C, we have $h_x(y) = B(y,x)/M(x,b) > 0$. We use again that each h_x is an extremal generator of C^*, and that all extremal generators of C^* arise in this way, up to scale. We conclude that y is in C. □

We must now show that B is positive definite.

Given a symmetric nondegenerate bilinear form $\mathcal{B}$, diagonalise it to get a normal basis $\{e_j\}$ such that each $\mathcal{B}(e_j, e_j)$ is either -1 or $+1$. Let $S\colon V \to V$ be the map that changes the sign of each coordinate associated to a basis element satisfying $\mathcal{B}(e_j, e_j) = -1$. Also, let $\mathcal{E}(x,y) := \mathcal{B}(Sx, y)$ be the Euclidean bilinear form with the $\{e_j\}$ as an orthonormal basis. See [3] for a discussion of domains of positivity for bilinear forms that are not positive definite.

The next lemma uses the following result from [12]: let $\mathcal{C}$ be a domain of positivity with respect to $\mathcal{B}$; then $x \in \operatorname{cl}\mathcal{C}$ if and only if $\mathcal{B}(x,y) \geq 0$ for all $y \in \mathcal{C}$.

Lemma 6.6 *Let $\mathcal{C}$ be a domain of positivity with respect to a symmetric indefinite nondegenerate bilinear form $\mathcal{B}$. Then $\mathcal{B}(z,z) = 0$ for some $z \in \operatorname{cl}\mathcal{C}\backslash\{0\}$.*

Proof Define
$$f(y) := \frac{\mathcal{B}(y,y)}{\mathcal{E}(y,y)} \quad \text{for all } y \in V\backslash\{0\}.$$

One can calculate that $\nabla\mathcal{E}(y,y) = 2y$ and that $\nabla\mathcal{B}(y,y) = 2S(y)$. So the gradient of f is

$$\nabla f(y) = \frac{2\mathcal{E}(y,y)S(y) - 2\mathcal{B}(y,y)y}{\mathcal{E}(y,y)^2}. \tag{4}$$

We wish to minimise f over $\operatorname{cl}\mathcal{C}\backslash\{0\}$. Since f is homogeneous of degree zero and $\operatorname{cl}\mathcal{C}\backslash\{0\}$ is projectively compact, the minimum is attained at some point z of $\operatorname{cl}\mathcal{C}\backslash\{0\}$. The minimum is nonnegative since $\mathcal{C}$ is a domain of positivity. Let $v := \nabla f(z)$ be the gradient of f at z, and let $D_z f$ be the derivative of f at z. These quantities are related by the equation $D_z f(\cdot) = \mathcal{E}(v, \cdot)$. Near z, we have

$$f(z + \delta z) = f(z) + D_z f(\delta z) + o(\delta z).$$

Since the minimum of f over the convex set $\operatorname{cl}\mathcal{C}\backslash\{0\}$ is attained at z, we have $D_z f(x) \geq 0$ for all $x \in \mathcal{C}$, since all such tangent vectors x point into the cone. So $\mathcal{B}(Sv, x) = \mathcal{E}(v,x) \geq 0$ for all $x \in \mathcal{C}$. It follows that Sv is in $\operatorname{cl}\mathcal{C}$. We conclude that

$$\mathcal{B}(v,v) = \mathcal{B}(Sv, Sv) \geq 0. \tag{5}$$

Write $z = z_+ + z_-$, where z_+ is in the linear span of the basis vectors with $\mathcal{B}(e_j, e_j) = +1$ and z_- is in the linear span of the basis vectors with $\mathcal{B}(e_j, e_j) = -1$. So

$$\mathcal{E}(z,z) = \mathcal{E}(z_+, z_+) + \mathcal{E}(z_-, z_-) \quad \text{and} \quad \mathcal{B}(z,z) = \mathcal{E}(z_+, z_+) - \mathcal{E}(z_-, z_-).$$

Since $\mathcal{C}$ is open and $\mathcal{B}$ is not positive definite, $\mathcal{C}$ contains some element x such that $\mathcal{B}(x,x) < \mathcal{E}(x,x)$. So, from the minimising property of z, we get $\mathcal{B}(z,z) < \mathcal{E}(z,z)$, or, equivalently, $\mathcal{E}(z_-, z_-) > 0$.

We also have that $z \in \operatorname{cl} \mathcal{C}$, and so $\mathcal{B}(z,z) \geq 0$. Hence $\mathcal{E}(z_+, z_+) \geq \mathcal{E}(z_-, z_-) > 0$.

One can calculate from (4) and (5) that

$$0 \leq \mathcal{B}(v,v) = \frac{16}{\mathcal{E}(z,z)^4} \mathcal{E}(z_-, z_-)\mathcal{E}(z_+, z_+)(\mathcal{E}(z_-, z_-) - \mathcal{E}(z_+, z_+)).$$

So we see that $-\mathcal{B}(z,z) = \mathcal{E}(z_-, z_-) - \mathcal{E}(z_+, z_+) \geq 0$. In fact equality holds, since we proved the reverse inequality earlier. We have proved that $\mathcal{B}(z,z) = 0$. □

Lemma 6.7 *Let $\varphi\colon C \to C$ be a gauge-reversing map with fixed point b. Then b is an isolated fixed point of φ if and only if $P(b)$ is an isolated fixed point of the action of φ on $P(C)$.*

Proof Let x_n be a sequence of points in $P(C)$ distinct from $P(b)$ that converge to $P(b)$ and are fixed by the action of φ. Using the antihomogeneity of φ, we get that there exists a sequence y_n in C of fixed points of φ such that y_n is in the projective class x_n for each $n \in \mathbb{N}$. The set of elements greater than or equal to b and the set of elements less than or equal to b are exchanged by φ, and the only element they have in common is b. Therefore, each y_n is incomparable to b. It follows from this and the projective convergence of y_n to b that y_n converges to b in C. We have proved that b is not an isolated fixed point if $P(b)$ is not.

The converse is easy. □

Lemma 6.8 *Let $\varphi\colon C \to C$ be a gauge-reversing map, and consider its action on the projective space $P(C)$ of the cone. If $P(b)$ is an isolated fixed point of this action, then $P(b)$ is the unique fixed point.*

Proof The projective action of φ is an isometry of the Hilbert metric d_H. Let $P(b')$ be any fixed point of this projective action. For each $\alpha \in (0,1)$, the set

$$Z_\alpha := \{z \in P(C) \mid d_H(b,z) = \alpha d_H(b,b') \text{ and } d_H(z,b') = (1-\alpha) d_H(b,b')\}$$

is invariant under φ. It is also compact, convex and nonempty. Therefore, by the Brouwer fixed point theorem, Z_α contains a fixed point, which will be a distance $\alpha d_H(b,b')$ from b in the Hilbert metric. Since α can be made as small as we wish, and we have assumed that $P(b)$ is an isolated fixed point, we see that $P(b') = P(b)$. □

Lemma 6.9 *The projective action of ϕ has a unique fixed point in $P(C)$.*

Proof The derivative of ϕ at b satisfies $D_b\phi = -\mathrm{Id}$. It follows that b is an isolated fixed point of ϕ. So, by Lemma 6.7, $P(b)$ is an isolated fixed point of the projective action. Applying Lemma 6.8, we get the result. □

Lemma 6.10 *Let $z \in C$. Then $M(z,b)M(b,\phi(z)) = M(z,\phi(z))$.*

Proof Let

$$Y := \big\{y \in P(C) \mid d_H(z,y) = d_H(y,\phi(z)) = \tfrac{1}{2}d_H(z,\phi(z))\big\}.$$

This set is nonempty, closed, bounded, invariant under ϕ, and convex in the usual sense. So the Brouwer theorem implies that Y contains a fixed point, which by Lemma 6.9 must be $P(b)$. We deduce that $d_H(z,b) + d_H(b,\phi(z)) = d_H(z,\phi(z))$. This implies that $d_F(z,b) + d_F(b,\phi(z)) = d_F(z,\phi(z))$. The result now follows on taking exponentials. □

Lemma 6.11 *If x is an extremal generator of C, then $B(x,x) = M(x,b)^2$.*

Proof Let x_n be a sequence in C converging to x. Using the same reasoning as in the proof of Lemma 6.3, we get

$$B(x,x) = \lim_{n\to\infty} M(x_n,\phi(x_n)).$$

So, by Lemma 6.10,

$$B(x,x) = \lim_{n\to\infty} M(x_n,b)M(b,\phi(x_n)) = \lim_{n\to\infty} M(x_n,b)^2 = M(x,b)^2. \qquad \square$$

Lemma 6.12 *The bilinear form B is positive definite.*

Proof We have shown in Lemma 6.5 that C is a domain of positivity of B. Since C is nonempty, B cannot be negative definite. Let $y \in \operatorname{cl} C\backslash\{0\}$. So we can write $y = \sum_j y_j x_j$ as a positive combination of finitely many extremal generators $\{x_j\}$ of C. Therefore, $B(y,y) = \sum_{j,k} y_j y_k B(x_j,x_k)$. Since C is a domain of positivity, $B(x_j,x_k) \geq 0$ for all j and k. Also, $B(x_j,x_j) > 0$ for all j, by Lemma 6.11. We conclude that $B(y,y) > 0$. Therefore, by Lemma 6.6, B is positive definite. □

Lemma 6.13 *Assume there exists a gauge-reversing map $\phi\colon C \to C$. Then C is self-dual.*

Proof As we have seen, by Lemma 3.3, we may assume that ϕ is an involution, has b as a fixed point, and is differentiable at b with derivative $-\mathrm{Id}$. It was shown in Lemma 6.5 that the function $B(\cdot,\cdot)$ is a symmetric nondegenerate bilinear form, having C as a domain of positivity. But B is positive definite by Lemma 6.12, and so C is self-dual. □

Proof of Theorem 1.1 Assume there exists a gauge-reversing map on C. It was proved in Lemmas 3.5 and 6.13 that C is then, respectively, homogeneous and self-dual.

On the other hand, if C is symmetric, then Vinberg's $*$–map is gauge-reversing, as discussed in the introduction. □

Proof of Corollary 1.2 Suppose $\phi\colon C_1 \to C_2$ is a gauge-reversing bijection between the two cones. Let $C := C_1 \oplus C_2$ be the product cone, and define the map $\Phi\colon C \to C$ by

$$\Phi(x_1, x_2) := (\phi^{-1}(x_2), \phi(x_1)) \quad \text{for all } x_1 \in C_1 \text{ and } x_2 \in C_2.$$

Since C is a product cone,

$$M_C((x_1, x_2), (y_1, y_2)) = \max\{M_{C_1}(x_1, y_1), M_{C_2}(x_2, y_2)\}$$

for all (x_1, x_2) and (y_1, y_2) in C. Using this and the fact that both ϕ and ϕ^{-1} are gauge-reversing, it is easy to show that Φ is gauge-reversing. So, by Theorem 1.1, C is a symmetric cone. It follows that both C_1 and C_2 are symmetric. Vinberg's $*$–map $*_{C_2}\colon C_2 \to C_2$ on C_2 is gauge-reversing. So the map $*_{C_2} \circ \phi$ is a gauge-preserving map from C_1 to C_2, and hence, by Theorem 2.3, a linear isomorphism. □

7 Isometries of the Hilbert metric

In this section, we use Theorem 1.1 to determine the isometry group of the Hilbert geometry.

We begin with some lemmas.

Lemma 7.1 *Let (D, d_H) be a Hilbert geometry. Assume there exists a unique geodesic line connecting some point ξ in the horofunction boundary to a point η in a* *nonsingleton pure reverse-Funk part. Then there exists another unique geodesic line, connecting a point in the same part as ξ to a point, distinct from η, in the same part as η.*

Proof The unique geodesic line connecting ξ to η is a relatively open segment xy, with $x, y \in \partial D$. Let U be the part of the horofunction boundary containing η. The point y is contained in the relative interior E of some extreme set of $\operatorname{cl} D$. Since U is a pure reverse-Funk part, it may be written $U = \{r_z + f \mid z \in E\}$, where f is some Funk horofunction. By assumption, U contains a Hilbert horofunction η' distinct from η. So we have $\eta' = r_{y'} + f$ for some $y' \in E$ distinct from y.

Since xy is a unique geodesic line, there is, by Proposition 2.4, no pair of relatively open line segments in $\operatorname{cl} D$, containing x and y, respectively, that span a two-dimensional affine space. Observe that, given any relatively open line segment in $\operatorname{cl} D$ containing y', we may find a parallel one in $\operatorname{cl} D$ containing y. Therefore, there is no pair of relatively open line segments in $\operatorname{cl} D$, containing x and y', respectively, that span a two-dimensional affine space. We conclude, using Proposition 2.4 again, that xy' is a unique geodesic line.

Let xy and xy' be parametrised by unit-speed geodesics γ and γ', respectively, and denote by ξ' the point in the horofunction boundary connected to η' by xy'. It is not hard to show that $\lim_{t \to -\infty} d_H(\gamma(t), \gamma'(t))$ is finite. This implies that ξ' lies in the same part as ξ. We have already seen that η' lies in the same part as η. □

Lemma 7.2 *Let $\Phi\colon D \to D'$ be a surjective isometry from one Hilbert geometry (D, d_H) to another, (D', d'_H), and let U be a nonsingleton maximal pure Funk part associated to an extreme point u of $\operatorname{cl} D$. If U is mapped by Φ to a reverse-Funk part of D', then the line segment connecting u to any other extreme point of $\operatorname{cl} D$ lies in the boundary ∂D.*

Proof Suppose there is an extreme point v of $\operatorname{cl} D$ distinct from u such that the relatively open line segment vu is contained in D. Let V be the maximal pure Funk part associated to v. Since v is extreme, the geodesic line vu is unique. It connects some horofunction ξ in V to some horofunction η in U. Observe that any sequence in D converging to a horofunction in V must converge in the usual topology to v, and any sequence converging to a horofunction in U must converge in the usual topology to u. Since every unique geodesic line is a straight line segment, we conclude that vu is the only unique geodesic line connecting a horofunction in V to a horofunction in U. So $\Phi(vu)$ is the only unique geodesic line connecting a horofunction in ΦV to a horofunction in ΦU. Applying Lemma 7.1, we get that ΦU is not a pure reverse-Funk part. Since it is necessarily pure, it can not be a reverse-Funk part. □

Lemma 7.3 *Let W and Z be pure parts of the horofunction boundary of a Hilbert geometry. Assume that W is maximal, that $\operatorname{cl} W$ and $\operatorname{cl} Z$ have a point in common, and that $Z \not\subset \operatorname{cl} W$. Then W and Z are of opposite types.*

Proof We consider just the case where W is a pure Funk part; the other case is handled similarly. So $\operatorname{cl} W = \{r_x + f \mid f \in B(x)\}$ for some extreme point x of the Hilbert geometry. We deduce that $\operatorname{cl} Z$ contains a function of the form $r_x + f$, with $f \in B(x)$. Therefore, if Z was a pure Funk part, each of its elements would be of the form $r_x + f$ with $f \in B(x)$, and Z would be contained in $\operatorname{cl} W$, contrary to our assumption. We conclude that Z is a pure reverse-Funk part. □

The following is the key lemma of this section.

Lemma 7.4 *Let $\Phi\colon D \to D'$ be a surjective isometry between two Hilbert geometries that maps a nonsingleton maximal pure Funk part to a pure reverse-Funk part. Then Φ arises as the projective action of a gauge-reversing map from the cone over D to the cone over D'.*

Proof We may assume without loss of generality that $\Phi(b) = b'$, where b and b' are the basepoints of D and D', respectively.

Let U be the maximal pure Funk part in the statement of the lemma, and let $\Phi(U)$ be its image, which by assumption is a pure reverse-Funk part. Associated to U is an extreme point u of $\operatorname{cl} D$, and associated to $\Phi(U)$ is a Funk horofunction $f^{(u)}$. So we may write

(6) $$\operatorname{cl} U = \{r_u + f \mid f \in B(u)\} \quad \text{and} \quad \operatorname{cl} \Phi(U) = \{r_x + f^{(u)} \mid f^{(u)} \in B(x)\}.$$

Here, the closures are taken in the set of Busemann points.

Let C and C' be the cones over D and D', respectively, and make the identifications $P(C) = D$ and $P(C') = D'$. We extend the Funk, reverse-Funk and Hilbert horofunctions to these cones as described in Section 5.4.

We define a map $\phi\colon C \to C'$ as follows. For each $x \in C$, let $\phi(x)$ be such that $P(\phi(x)) = \Phi(P(x))$ and $f^{(u)}(\phi(x)) = r_u(x)$. Clearly, Φ is the projective action of ϕ, and, from the homogeneity properties of $f^{(u)}$ and r_u, we get that ϕ is antihomogeneous. Note also that $\phi(b) = b'$. Moreover, the push-forward of r_u is $f^{(u)}$, that is,

$$\phi r_u := r_u \circ \phi^{-1} = f^{(u)}. \tag{7}$$

Let v be any extreme point of $\mathrm{cl}\, D$ distinct from u, and denote by V the associated maximal pure Funk part. So $\mathrm{cl}\, V = \{r_v + f \mid f \in B(v)\}$. By Lemma 7.2, the straight line segment connecting u and v lies in the boundary of D. So there exists an element of $\partial(C^*)$ that supports C at both u and v. In fact, the set of elements of $\partial(C^*)$ that support C at both u and v forms a proper extreme set of C^*, and so contains an extremal generator w of C^*. Let $E := \mathrm{rel\,int}\{x \in \partial D \mid \langle w, x\rangle = 0\}$. So the maximal pure reverse-Funk part associated to w can be written as $W := \{r_x + f^{(w)} \mid x \in E\}$, where $f^{(w)}$ is the Funk Busemann point associated to w.

Observe that $\langle w, u\rangle = 0$. So $\{h_u := r_u + f^{(w)}\}$ is a singleton part of the Hilbert geometry horofunction boundary of D, and is contained in both $\mathrm{cl}\, U$ and $\mathrm{cl}\, W$. Therefore, W and U satisfy the assumptions of Lemma 7.3, and it follows that ΦW and ΦU do also. Applying the lemma, we get that ΦW is of opposite type to ΦU; in other words, ΦW is a pure Funk part.

Similarly, $\{h_v := r_v + f^{(w)}\}$ is a singleton part and is contained in both $\mathrm{cl}\, V$ and $\mathrm{cl}\, W$. Using the same reasoning as in the previous paragraph, we get that ΦW is also of opposite type to ΦV; in other words, ΦV is a pure reverse-Funk part.

Let z be the extreme point of the polar $(D')^\circ$ of D' associated to the maximal pure Funk part ΦW. Since Φh_u and Φh_v lie in $\mathrm{cl}\, \Phi W$, we may write them as

$$\Phi h_u = r_z + f^{(u)}, \tag{8}$$

$$\Phi h_v = r_z + f^{(v)}, \tag{9}$$

where r_z is the reverse-Funk horofunction on D' associated to z, and $f^{(u)}$ and $f^{(v)}$ are Funk horofunctions on D'. The $f^{(u)}$ here is the same as the one appearing in (6) because Φh_u also lies in $\mathrm{cl}\, \Phi U$.

Observe that if h is any function on C that just depends on the projective class, then ϕh is a function on C' that also just depends on the projective class, and ϕh agrees with Φh. Thus, (8) and (9) hold with Φ replaced by ϕ. Using this and (7), we get

$$\phi f^{(w)} = \phi(h_u - r_u) = \phi h_u - \phi r_u = r_z + f^{(u)} - f^{(u)} = r_z.$$

Therefore, $\phi r_v = \phi(h_v - f^{(w)}) = f^{(v)}$.

Let f be any Funk horofunction in $B(v)$. So $h := r_v + f$ is a Hilbert horofunction and is contained in the set $\mathrm{cl}\, V$. But we have seen that ΦV is a pure reverse-Funk part. It follows that ϕh is a horofunction of the form $r_p + f^{(v)}$ for some point p in $\partial D'$. Therefore, $\phi f = \phi(h - r_v) = r_p$.

But v was chosen to be an arbitrary extreme point of $\mathrm{cl}\, D$, and every Funk horofunction is contained in $B(v)$ for some choice of extreme point v of $\mathrm{cl}\, D$. So we have shown that every Funk horofunction is pushed forward by ϕ to a reverse-Funk horofunction.

By Proposition 4.4, we have the following two formulae:

$$d_F(x, y) = \sup_f (f(x) - f(y)) \quad \text{for all } x, y \in D, \tag{10}$$

$$d'_R(x, y) = \sup_r (r(x) - r(y)) \quad \text{for all } x, y \in D', \tag{11}$$

where the suprema are taken over the set of all Busemann points in, respectively, the Funk geometry on D and the reverse-Funk geometry on D'. In fact, since the quantities involved have the right homogeneity properties, the formulae extend to all x and y in C and C', respectively.

So, for every $x, y \in C$ and every Funk Busemann point f of D,

$$f(x) - f(y) = r(\phi x) - r(\phi y) \le d'_R(\phi x, \phi y),$$

where $r := \phi f := f \circ \phi^{-1}$ is the reverse-Funk horofunction of D' that is the push-forward of f. We deduce that $d_F(x, y) \le d'_R(\phi x, \phi y)$ for all $x, y \in C$. Using this inequality and the fact that ϕ preserves the Hilbert distance, we get the opposite inequality: for all $x, y \in C$,

$$d_F(x, y) = d_H(x, y) - d_F(y, x) \ge d'_H(\phi x, \phi y) - d'_R(\phi y, \phi x) = d'_R(\phi x, \phi y).$$

Therefore, $d_F(x, y) = d'_R(\phi x, \phi y)$ for all $x, y \in C$. It follows upon taking exponentials that ϕ is gauge-reversing. □

Let $\Phi\colon D \to D'$ be a surjective isometry between finite-dimensional Hilbert geometries D and D'. Consider the following property, which Φ may or may not have:

Property 7.5 *For every extreme point u of $\mathrm{cl}\, D$, there is an extreme point u' of $\mathrm{cl}\, D'$ such that, for all $v \in D$, we have $\Phi(uv) = u'\Phi(v)$ as an oriented line segment,*

In [13], it was shown that if D and D' are polyhedral and Φ and Φ^{-1} have Property 7.5, then Φ is a projectivity. The proof is in two parts. First, it was shown that Φ and Φ^{-1} extend continuously to the usual boundaries of D and D', respectively. Then it was shown that an isometry between Hilbert geometries that extends continuously to the boundary and has an inverse that does likewise is a projectivity. Inspecting the proof, one sees that the polyhedral assumption was not used in any essential way and that the same proof works in the general case provided one changes some terminology. In particular, one must consider the extreme points rather than the vertices, and the relative interiors of the extreme sets rather than the relatively open faces. Thus, we have the following theorem.

Theorem 7.6 [13] *Let Φ be a surjective isometry between two finite-dimensional Hilbert geometries such that both Φ and Φ^{-1} have Property 7.5. Then Φ is a projectivity.*

We now prove the main theorem of this section.

Proof of Theorem 1.3 Let Φ be an isometry of (D, d_H), and consider the action of Φ on the horofunction boundary. Either every maximal pure Funk part is mapped to a similar such part, or there is a nonsingleton maximal pure Funk part that is mapped to a pure reverse-Funk part.

In the latter case, Φ arises as the projective action of a gauge-reversing self-map on the cone C over D, by Lemma 7.4.

Consider now the former case. Let u be an extreme point of $\operatorname{cl} D$, and denote by U the associated maximal pure Funk part. By assumption, U is mapped by Φ to a maximal pure Funk part $U' := \Phi(U)$. This part is associated to some extreme point u' of $\operatorname{cl} D$.

Let $v \in D$. By Proposition 2.4, the open line segment uv is a unique geodesic half-line in the Hilbert geometry. So its image $\Phi(uv)$ is also a unique geodesic half-line, and is therefore a straight line segment.

Let x_n be sequence in uv converging to u in the usual topology. Since x_n is moving along a Hilbert geometry geodesic, it must converge to a Hilbert geometry horofunction, which will be in U. It follows that $\Phi(x_n)$ converges to a Hilbert horofunction in U', and this horofunction is necessarily of the form $r_{u'} + f$, with $f \in B(u')$. This implies that $\Phi(x_n)$ converges to u' in the usual topology. This establishes that Φ satisfies Property 7.5. That Φ^{-1} satisfies the same property can be shown in the same way. Applying Theorem 7.6 gives that Φ is a projectivity, and so arises as the projective action of a gauge-preserving self-map of C. □

Proof of Corollary 1.4 Denote by Λ^+ the set of self-maps of C that are gauge-preserving, and by Λ the set that are either gauge-preserving or gauge-reversing. By Theorem 1.3, every isometry of D arises as the projective action of a map in Λ.

If the cone C is not symmetric, then by Theorem 1.1, Λ consists of only gauge-preserving maps, and so every isometry is a projectivity.

So assume that C is symmetric.

Observe that the composition of a gauge-reversing map and a gauge-preserving map is gauge-reversing, and that the composition of two gauge-reversing maps is gauge-preserving. It follows easily that Λ^+ is a normal subgroup of index two in Λ. This implies that Λ is generated by Λ^+ and the Vinberg $*$–map associated to C, which is always gauge-reversing for symmetric cones. So $\mathrm{Isom}(D)$ is generated by the projectivities and the projective action of the $*$–map.

If C is Lorentzian, then the projective action of the $*$–map is a projectivity, so in this case $\mathrm{Isom}(D) = \mathrm{Proj}(D)$.

For all other symmetric cones, this projective action is not a projectivity, and therefore $\mathrm{Proj}(D)$ is a normal subgroup of index two in $\mathrm{Isom}(D)$. □

8 The horofunction boundary of product spaces

Let (X_1, d_1) and (X_2, d_2) be metric spaces. We define the ℓ_∞–product of these two spaces to be the space $X := X_1 \times X_2$ endowed with the metric d defined by

$$d((x_1, x_2), (y_1, y_2)) := \max\{d_1(x_1, y_1) d_2(x_2, y_2)\},$$

for all $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$, and we denote this space by

$$(X, d) =: (X_1, d_1) \oplus_\infty (X_2, d_2).$$

Our motivation for considering such spaces is that the Thompson metric on a product cone has such a structure, a fact we will use when studying the isometry group of the Thompson metric.

In this section, we will study the set of Busemann points and the detour cost for ℓ_∞–product spaces. We assume that X_1 and X_2 have basepoints b_1 and b_2, respectively, and we take (b_1, b_2) to be the basepoint of X.

Let $\vee$ and $\wedge$ denote, respectively, maximum and minimum. We use the convention that addition and subtraction take precedence over these operators. We write $x^+ := x \vee 0$ and $x^- := x \wedge 0$. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Given two real-valued functions f_1 and f_2, and $c \in \overline{\mathbb{R}}$, define

$$[f_1, f_2, c] := (f_1 + c^-) \vee (f_2 - c^+).$$

For the rest of this section, we will assume that (X_1, d_1) and (X_2, d_2) are proper geodesic metric spaces.

The following proposition shows that horofunctions of ℓ_∞–product spaces have a simple form.

Proposition 8.1 *Every horofunction of (X, d) is of the form*

$$[\xi_1, \xi_2, c]$$

with $\xi_1 \in X_1(\infty)$ and $\xi_2 \in X_2(\infty)$, and $c \in \overline{\mathbb{R}}$.

Proof Denote by $\overline{X}_1$ and $\overline{X}_2$ the horofunction compactifications of X_1 and X_2, respectively. Let $x^n = (x_1^n, x_2^n)$ be a sequence in X converging to a horofunction ξ. By passing to a subsequence if necessary, we may assume that x_1^n converges to $\xi_1 \in \overline{X}_1$, that x_2^n converges to $\xi_2 \in \overline{X}_2$, and that

$$d(b_1, x_1^n) - d(b_2, x_2^n) \to c,$$

with $c \in \overline{\mathbb{R}}$. At least one of ξ_1 and ξ_2 must be a horofunction of its respective space. Observe that, as n tends to infinity,

$$d(b, x^n) - d(b_1, x_1^n) \to -c^- \quad \text{and} \quad d(b, x^n) - d(b_2, x_2^n) \to c^+.$$

Therefore, for $y = (y_1, y_2)$ in X, we have the following limit as n tends to infinity:

$$\begin{aligned} d(y, x^n) - d(b, x^n) &= (d(y_1, x_1^n) \vee d(y_2, x_2^n)) - d(b, x^n) \\ &= (d(y_1, x_1^n) - d(b_1, x_1^n) + c^-) \vee (d(y_2, x_2^n) - d(b_2, x_2^n) - c^+) \\ &\to [\xi_1, \xi_2, c](y). \end{aligned}$$

If ξ_1 is not a horofunction, then $c = -\infty$, and ξ_1 is irrelevant in the expression $[\xi_1, \xi_2, c]$. Likewise, if ξ_2 is not a horofunction, then $c = +\infty$, and ξ_2 is irrelevant. □

Next, we will determine the Busemann points of product spaces. We will need the following lemma.

Lemma 8.2 *Let f_1 and g_1 be real-valued functions on a set Y_1, and let f_2 and g_2 be real-valued functions on a set Y_2. Assume that $f_1(x_1) \vee f_2(x_2) = g_1(x_1) \vee g_2(x_2)$ for all $(x_1, x_2) \in Y_1 \times Y_2$, and that $\inf f_2 = \inf g_2 = -\infty$. Then $f_1 = g_1$.*

Proof Let $x_1 \in Y_1$. Choose $x_2 \in Y_2$ such that $f_2(x_2) < f_1(x_1)$. So

$$f_1(x_1) = f_1(x_1) \vee f_2(x_2) = g_1(x_1) \vee g_2(x_2) \geq g_1(x_1).$$

The reverse inequality is proved similarly. □

We will also need the following characterisation of Busemann points from Theorem 6.2 of [1]: a horofunction is a Busemann point if and only if it cannot be written as the minimum of two 1–Lipschitz functions, both different from it.

Proposition 8.3 *For every pair of Busemann points $\xi_1 \in X_1(\infty)$ and $\xi_2 \in X_2(\infty)$, and every $c \in \overline{\mathbb{R}}$, the function $[\xi_1, \xi_2, c]$ is a Busemann point of X. Moreover, every Busemann point of X arises in this way.*

Proof Let ξ be a Busemann point of X. By Proposition 8.1, we may write $\xi = [\xi_1, \xi_2, c]$ with $\xi_1 \in X_1(\infty)$ and $\xi_2 \in X_2(\infty)$, and $c \in \overline{\mathbb{R}}$. Suppose $\xi_1 = f \wedge f'$, where f and f' are real-valued 1–Lipschitz functions on (X_1, d_1). We consider the case when $c \geq 0$; the other case is similar. So

$$\xi = (f \wedge f') \vee (\xi_2 - c) = (f \vee (\xi_2 - c)) \wedge (f' \vee (\xi_2 - c)).$$

Therefore, ξ is the minimum of two 1–Lipschitz functions on (X, d). This implies, since ξ is Busemann, that it is equal to one of them, say $\xi = f \vee (\xi_2 - c)$. But we also have that $\xi = \xi_1 \vee (\xi_2 - c)$. If $c = \infty$, then we have proved that $\xi_1 = f$; otherwise we apply Proposition 4.1 and Lemma 8.2 to get the same conclusion. We deduce that ξ_1 is a Busemann point of (X_1, d_1). The proof that ξ_2 is Busemann is similar.

Now assume that ξ_1 and ξ_2 are Busemann points of (X_1, d_1) and (X_2, d_2), respectively. So there exists an almost-geodesic γ_1 in X_1 converging to ξ_1, and an almost-geodesic γ_2 in X_2 converging to ξ_2. We may assume without loss of generality that γ_1 and γ_2 start, respectively, at b_1 and b_2, the basepoints of the spaces. By Lemma 4.3, we may also assume that the domain of definition of these almost-geodesics is $\mathbb{R}_+$. We furthermore assume that $c \geq 0$; the other case is handled similarly.

Define the path $\gamma\colon \mathbb{R}_+ \to X$ by

$$\gamma(t) := \big(\gamma_1(t), \gamma_2((t-c)^+)\big) \quad \text{for all } t \in \mathbb{R}_+.$$

By Lemma 4.2, we have, as t tends to infinity,

$$d_1(b_1,\gamma_1(t)) - t \to 0, \tag{12}$$

$$d_2(b_2,\gamma_2(t)) - t \to 0. \tag{13}$$

Observe that $(t-c)^+ - t$ converges to $-c$ as t tends to infinity. We deduce from this and (13) that

$$d_2\big(b_2,\gamma_2((t-c)^+)\big) - t \to -c \quad \text{as } t \to \infty.$$

From this and (12), we get

$$d(b,\gamma(t)) - t \to 0 \quad \text{as } t \to \infty. \tag{14}$$

This shows that condition (1) of Lemma 4.2 holds for γ. The proof of condition (2) of the same lemma is similar. So γ is an almost-geodesic.

Using (14), we get, for all $x := (x_1,x_2)$ in X,

$$\begin{aligned}\lim_{t\to\infty}\big(d(x,\gamma(t)) - d(b,\gamma(t))\big) &= \lim_{t\to\infty}\big(\big(d_1(x_1,\gamma_1(t)) \vee d_2(x_2,\gamma_2((t-c)^+))\big) - t\big) \\ &= \lim_{t\to\infty}\big(\big(d_1(x_1,\gamma_1(t)) - t\big) \vee \big(d_2(x_2,\gamma_2((t-c)^+)) - t\big)\big) \\ &= \xi_1(x_1) \vee (\xi_2(x_2) - c).\end{aligned}$$

In other words, $\gamma(t)$ converges to $\xi := [\xi_1,\xi_2,c]$. Therefore, ξ is a Busemann point of (X,d). □

We now calculate the detour cost in product spaces. We use the convention that $-\infty$ is absorbing for addition, that is, $(+\infty)+(-\infty) = -\infty$ and $(-\infty)-(-\infty) = -\infty$.

Proposition 8.4 *Let $\xi = [\xi_1,\xi_2,u]$ and $\eta = [\eta_1,\eta_2,v]$ be Busemann points of X. Then*

$$H(\xi,\eta) = \max\big(H(\xi_1,\eta_1) - u^- + v^-, H(\xi_2,\eta_2) + u^+ - v^+\big).$$

Proof We extend the definition of H somewhat by letting $H(\xi+u,\eta+v) := H(\xi,\eta) + v - u$ for all Busemann points ξ and η, and $u,v \in [-\infty,0]$.

Write

$$\overline{\xi}_1 := \xi_1 + u^-, \quad \overline{\xi}_2 := \xi_2 - u^+,$$
$$\overline{\eta}_1 := \eta_1 + v^-, \quad \overline{\eta}_2 := \eta_2 - v^+.$$

So $\xi = \overline{\xi}_1 \vee \overline{\xi}_2$ and $\eta = \overline{\eta}_1 \vee \overline{\eta}_2$.

By Proposition 4.5,

$$\eta_1(\cdot) \le H(\xi_1,\eta_1) + \xi_1(\cdot) \quad \text{and} \quad \eta_2(\cdot) \le H(\xi_2,\eta_2) + \xi_2(\cdot).$$

Let $M := \max(H(\overline{\xi}_1,\overline{\eta}_1), H(\overline{\xi}_2,\overline{\eta}_2))$. If $M = +\infty$, then clearly $H(\xi,\eta) \le M$, so assume that $M < +\infty$. This implies that it is not the case that $u = -\infty$ and $v > -\infty$, nor that $u = +\infty$ and $v < +\infty$. It follows that

$$\overline{\eta}_1(\cdot) \le H(\overline{\xi}_1,\overline{\eta}_1) + \overline{\xi}_1(\cdot) \quad \text{and} \quad \overline{\eta}_2(\cdot) \le H(\overline{\xi}_2,\overline{\eta}_2) + \overline{\xi}_2(\cdot).$$

Therefore, using Proposition 4.5,

$$\begin{aligned} H(\xi,\eta) &= \sup_{(x_1,x_2)\in X} \big((\overline{\eta}_1(x_1) \vee \overline{\eta}_2(x_2)) - (\overline{\xi}_1(x_1) \vee \overline{\xi}_2(x_2))\big) \\ &\le \sup_{(x_1,x_2)\in X} \big((H(\overline{\xi}_1,\overline{\eta}_1) + \overline{\xi}_1(x_1) \vee H(\overline{\xi}_2,\eta_2) + \overline{\xi}_2(x_2)) - (\overline{\xi}_1(x_1) \vee \overline{\xi}_2(x_2))\big) \\ &\le \sup_{(x_1,x_2)\in X} \big((M + \overline{\xi}_1(x_1) \vee M + \overline{\xi}_2(x_2)) - (\overline{\xi}_1(x_1) \vee \overline{\xi}_2(x_2))\big) \\ &= M. \end{aligned}$$

We now wish to prove the reverse inequality. We have

$$H(\xi,\eta) \ge \sup_{(x_1,x_2)\in X} \big(\overline{\eta}_1(x_1) - (\overline{\xi}_1(x_1) \vee \overline{\xi}_2(x_2))\big).$$

Fix $x_1 \in X_1$. From Proposition 4.1 and the fact that $u^+ \ge 0$, we get that $\inf \overline{\xi}_2 = -\infty$. Therefore, we can choose $x_2 \in X_2$ to make $\overline{\xi}_2(x_2)$ as negative as we wish. So we see that $H(\xi,\eta) \ge \overline{\eta}_1(x_1) - \overline{\xi}_1(x_1)$. We conclude that

$$H(\xi,\eta) \ge \sup_{x_1\in X_1} (\overline{\eta}_1(x_1) - \overline{\xi}_1(x_1)) = H(\overline{\xi}_1,\overline{\eta}_1).$$

Similar reasoning shows that $H(\xi,\eta) \ge H(\overline{\xi}_2,\overline{\eta}_2)$. □

Using this proposition, we can characterise the singletons of product spaces.

Corollary 8.5 *The following are equivalent:*

- ξ *is a singleton Busemann point in the horofunction boundary of* X.
- ξ *takes one of the following two forms:* $\xi(x_1,x_2) = \xi_1(x_1)$ *with* ξ_1 *a singleton Busemann point of* X_1, *or* $\xi(x_1,x_2) = \xi_2(x_2)$ *with* ξ_2 *a singleton Busemann point of* X_2.

Proof Let ξ be a Busemann point of X. By Proposition 8.3, we can write $\xi = [\xi_1, \xi_2, c]$, where ξ_1 and ξ_2 are Busemann points of X_1 and X_2, respectively, and $c \in \overline{\mathbb{R}}$.

Consider the case where c is finite. For each $\epsilon \in \mathbb{R}$, define the Busemann point $\xi^\epsilon := [\xi_1, \xi_2, c + \epsilon]$. We can calculate from Proposition 8.4 that $H(\xi, \xi^\epsilon) + H(\xi^\epsilon, \xi) = |\epsilon|$ for all $\epsilon \in \mathbb{R}$. This shows that ξ^ϵ is distinct from ξ but in the same part as it for all $\epsilon \in \mathbb{R} \backslash \{0\}$, and hence that ξ is not a singleton.

In the case where $c = \infty$, we have $\xi = \xi_1$. If ξ_1 is not a singleton of X_1, then let ξ'_1 be another Busemann point in the same part as ξ_1, and write $\xi' = \xi'_1$. From Proposition 8.4, we get $H(\xi, \xi') + H(\xi', \xi) = H(\xi_1, \xi'_1) + H(\xi'_1, \xi_1) < \infty$, and so ξ is not a singleton.

The case where $c = \infty$ and ξ_2 is not a singleton of X_2 is handled similarly.

Now let ξ_1 be a singleton Busemann point of X_1, and write $\xi(x_1, x_2) := \xi_1(x_1)$ for all $x_1 \in X_1$ and $x_2 \in X_2$. Observe that $\xi = [\xi_1, \xi_2, \infty]$ for any Busemann point ξ_2 of X_2. Let $\eta := [\eta_1, \eta_2, c]$ be a Busemann point of X in the same part of the horofunction boundary as ξ. Here, of course, η_1 and η_2 are Busemann points of X_1 and X_2, respectively, and $c \in \overline{\mathbb{R}}$. Since $H(\xi, \eta)$ is finite, looking at Proposition 8.4, we see that $H(\xi_2, \eta_2) + \infty - c^+ < \infty$, and hence $c = \infty$. Using this and the same proposition again, we get $H(\xi_1, \eta_1) = H(\xi, \eta) < \infty$ and $H(\eta_1, \xi_1) = H(\eta, \xi) < \infty$. Since ξ_1 was assumed to be a singleton, we conclude that $\eta_1 = \xi_1$, and therefore $\eta = \xi$. We have thus shown that ξ is a singleton.

That $\xi := \xi_2$ is a singleton point of X whenever ξ_2 is singleton point of X_2 may be proved in a similar manner. □

9 The horofunction boundary of the Thompson metric

In this section, we determine the horofunction boundary of the Thompson geometry and its set of Busemann points. We then calculate the detour metric on the boundary.

The results of this section will resemble somewhat those of the last. This is because the Thompson metric is the maximum of the Funk and reverse-Funk metrics, and, as a consequence, its boundary is related to those of these two metrics in a way similar to how the boundary of an ℓ_∞–product space is related to the boundaries of its components.

Recall that D is a cross-section of a proper open convex cone C. For each $x \in \partial D$, let $A(x)$ denote the set of horofunctions of the Funk geometry that may be approached by a sequence in D converging to x. Also, for each $x \in D$, define the following function on C:

$$f_{C,x}(\cdot) := \log \frac{M_C(\cdot, x)}{M_C(b, x)}.$$

We start off by describing the horofunctions of the Thompson metric.

Proposition 9.1 *Let (C, d_T) be a proper open convex cone with its Thompson metric. Its horofunction boundary is*

$$C(\infty) = \{r_x \mid x \in D\} \cup \{f_{C,x} \mid x \in D\} \cup \{[r_x, f, c] \mid x \in \partial D,\ f \in A(x), c \in \overline{\mathbb{R}}\},$$

where D is a cross-section of C.

Proof First, we show that each of the functions in the statement is a horofunction. Functions of the form r_x, with $x \in D$, may be approached by taking λx as $\lambda > 0$ tends to infinity. Similarly, $f_{C,x}$, with $x \in D$, is approached by x/λ as λ tends to infinity.

Let $x \in \partial D$, and $f \in A(x)$, and $c \in \overline{\mathbb{R}}$. So there exists a sequence $x_n \in D$ such that x_n converges to x and $d_F(\cdot, x_n) - d_F(b, x_n)$ converges pointwise to f. For each n, we may choose y_n in the same projective class as x_n such that $M(b, y_n) = M(y_n, b)$. It is not difficult to show that the limit of the sequence $y_n \exp\big(\frac{1}{2}c_n\big)$ in the horofunction compactification is $[r_x, f, c]$ for any sequence c_n in $\mathbb{R}$ converging to c.

Now we show that all horofunctions take one of the given forms. Let y_n be a sequence in C converging to a horofunction ξ. Using compactness, we may assume that y_n converges in both the Funk and reverse-Funk horofunction compactifications. If y_n converges projectively to the projective class of some point $y \in D$, then ξ must equal r_y if y_n heads away from the origin, or $f_{C,y}$ if y_n heads towards the origin. Otherwise, y_n converges to a horofunction f in the Funk geometry, and to a horofunction r_x in the reverse-Funk geometry, with $x \in \partial D$ and $f \in A(x)$. By taking a subsequence if necessary, we may assume that $d_R(b, y_n) - d_F(b, y_n)$ converges to a limit c in $\overline{\mathbb{R}}$. One may calculate then that $d_T(\cdot, y_n) - d_T(b, y_n)$ converges to $[r_x, f, c]$ as n tends to infinity. □

The following lemma parallels Lemma 8.2.

Lemma 9.2 *Let f_1, f_2, g_1, and g_2 be real-valued functions on a cone C satisfying*

$$f_1(\lambda x) = -\log\lambda + f_1(x), \quad f_2(\lambda x) = \log\lambda + f_2(x),$$
$$g_1(\lambda x) = -\log\lambda + g_1(x), \quad g_2(\lambda x) = \log\lambda + g_2(x),$$

for all $\lambda > 0$ and $x \in C$. Assume that $f_1 \vee f_2 = g_1 \vee g_2$ on C. Then $f_1 = g_1$ and $f_2 = g_2$.

Proof Let $x \in C$, and choose $\lambda > 0$ small enough that $f_2(\lambda x) < f_1(\lambda x)$. So

$$f_1(\lambda x) = f_1(\lambda x) \vee f_2(\lambda x) = g_1(\lambda x) \vee g_2(\lambda x) \geq g_1(\lambda x).$$

Therefore, $f_1(x) \geq g_1(x)$. The reverse inequality is proved similarly.

The proof that $f_2 = g_2$ goes along the same lines. □

In [1], the notion of almost-geodesic was defined slightly differently. A sequence (x_k) in a metric space (X, d) was said to be an *ϵ–almost-geodesic* if

$$d(x_0, x_1) + \cdots + d(x_m, x_{m+1}) \leq d(x_0, x_{m+1}) + \epsilon \quad \text{for all } m \in \mathbb{N}.$$

It was shown in [1] that every ϵ–almost-geodesic has a subsequence that may be parametrised to give an almost-geodesic in the sense of Rieffel. Conversely, given any almost-geodesic in the sense of Rieffel, it was shown that one may obtain an ϵ–almost-geodesic by taking a sequence of points along it.

Recall again that a horofunction is a Busemann point if and only if it can not be written as the minimum of two 1–Lipschitz functions, both different from it [1, Theorem 6.2]. In the context of a distance d that is not symmetric, a function f being 1–Lipschitz means that $f(x) \leq d(x, y) + f(y)$ for all points x and y.

Proposition 9.3 *The set of Busemann points of the Thompson geometry is*

$$\{r_x \mid x \in D\} \cup \{f_{C,x} \mid x \in D\} \cup \{[r_x, f, c] \mid x \in \partial D,\ f \in B(x),\ c \in \overline{\mathbb{R}}\}.$$

Proof Assume that ξ is a Busemann point. By Proposition 9.1, if ξ is not of the form r_x or $f_{C,x}$, with $x \in D$, then it is of the form $[r_x, f, c]$, with $x \in \partial D$ and $f \in A(x)$, and $c \in \overline{\mathbb{R}}$. Write $f = g \wedge g'$, where g and g' are real-valued functions on C that are 1–Lipschitz with respect to the Funk metric d_F. Observe that this property of g and of g' implies that each of them is the logarithm of an homogeneous function. We consider the case when $c \leq 0$; the other case is similar. We have

$$\xi = ((r_x + c) \vee (g \wedge g')) = ((r_x + c) \vee g) \wedge ((r_x + c) \vee g').$$

So ξ is the minimum of two functions on C that are 1–Lipschitz with respect to Thompson's metric d_T. Since ξ is a Busemann point, it must be equal to one of them, say $(r_x + c) \vee g$. If $c = -\infty$, then we have proved that $f = g$; otherwise we apply Lemma 9.2 to get the same conclusion. We have shown that if f is written as the minimum of two functions that are 1–Lipschitz with respect to the Funk metric, then it equals one of them. Since f is a Funk horofunction, it follows that f is a Busemann point of the Funk geometry.

Functions of the form r_x or $f_{C,x}$ with $x \in D$ are clearly Busemann points of the Thompson geometry since they are the limits, respectively, of the geodesics $t \mapsto e^t x$ and $t \mapsto e^{-t} x$.

Let $x \in \partial D$, and $f \in B(x)$, and $c \in \overline{\mathbb{R}}$. Choose $\epsilon > 0$. We must show that $[r_x, f, c]$ is a Busemann point. We consider only the case where c is finite; the case where it is infinite is similar and easier.

Since f is in $B(x)$, there exists, by the proof of [22, Lemma 4.3], a sequence x_n in D that converges to f and to r_x, respectively, in the Funk and reverse-Funk geometries, and furthermore is an ϵ–almost-geodesic with respect to both of these metrics. So, as discussed above, by passing to a subsequence if necessary and parametrising in the right way, we obtain an almost-geodesic converging to r_x in the reverse-Funk geometry. Applying [23, Lemma 5.2], we get that

$$d_R(b, x_n) + r_x(x_n) \to 0. \tag{15}$$

In a similar fashion, one may show that

$$d_F(b, x_n) + f(x_n) \to 0. \tag{16}$$

For each $n \in \mathbb{N}$, choose z_n in the same projective class as x_n such that $M(z_n, b) = M(b, z_n)e^c$. So, for all $n \in \mathbb{N}$,

$$d_T(b, z_n) = d_R(b, z_n) \vee d_F(b, z_n) = d_R(b, z_n) - c^- = d_F(b, z_n) + c^+.$$

Observe that both (15) and (16) also hold with z_n in place of x_n since, for each $n \in \mathbb{N}$, x_n and z_n are related by a positive scalar. Combining all this, we have

$$\begin{aligned} d_T(b, z_n) + [r_x, f, c](z_n) &= (d_T(b, z_n) + r_x(z_n) + c^-) \vee (d_T(b, z_n) + f(z_n) - c^+) \\ &= (d_R(b, z_n) + r_x(z_n)) \vee (d_F(b, z_n) + f(z_n)) \to 0 \end{aligned}$$

as n tends to infinity.

We have seen in the proof of Proposition 9.1 that z_n converges to $\xi := [r_x, f, c]$ in the Thompson geometry. We deduce that $H(\xi,\xi) = 0$, and so $[r_x, f, c]$ is a Busemann point. □

Recall that we have extended the definition of H by setting $H(\xi + u, \eta + v) := H(\xi,\eta) + v - u$ for all Busemann points ξ and η, and $u, v \in [-\infty, 0]$. We are also using the convention that $-\infty$ is absorbing for addition.

Proposition 9.4 *The detour distance between two Busemann points ξ and η in the horofunction boundary of the Thompson metric is $\delta(\xi,\eta) = d_H(x,y)$ if $\xi = r_x$ and $\eta = r_y$, with $x, y \in D$. The same formula holds when $\xi = f_{C,x}$ and $\eta = f_{C,y}$, with $x, y \in D$. If $\xi = [r_x, f, c]$ and $\eta = [r_{x'}, f', c']$, with $x, x' \in \partial D$, $f \in B(x)$, $f' \in B(x')$ and $c, c' \in \overline{\mathbb{R}}$, then*

$$\delta(\xi,\eta) = \max(H(\bar r_x, \bar r_{x'}), H(\bar f, \bar f')) + \max(H(\bar r_{x'}, \bar r_x), H(\bar f', \bar f)),$$

where

$$\bar r_x := r_x + c^-, \qquad \bar f := f - c^+,$$
$$\bar r_{x'} := r_{x'} + c'^-, \quad \bar f' := f' - c'^+.$$

In all other cases, $\delta(\xi,\eta) = \infty$.

Proof For x and y in D, we have, by Proposition 4.5,

$$\begin{aligned} H(r_x, r_y) &= \sup_{z\in C}(r_y(z) - r_x(z)) \\ &= \sup_{z\in C}(d_R(z,y) - d_R(b,y) - d_R(z,x) + d_R(b,x)) \\ &= d_R(x,y) - d_R(b,y) + d_R(b,x). \end{aligned}$$

Here we have used the triangle inequality to get an upper bound on the supremum, and taken $z = x$ to get a lower bound. Symmetrising, we get that $\delta(r_x, r_y) = d_H(x,y)$.

We use similar reasoning in the case where the two Busemann points are of the form $f_{C,x}$ and $f_{C,y}$, with $x, y \in D$.

Now let $\xi := [r_x, f, c]$ and $\eta := [r_{x'}, f', c']$, with $x, x' \in \partial D$, $f \in B(x)$, $f' \in B(x')$ and $c, c' \in \overline{\mathbb{R}}$. By Proposition 4.5,

$$r_{x'}(\cdot) \le r_x(\cdot) + H(r_x, r_{x'}) \quad \text{and} \quad f'(\cdot) \le f(\cdot) + H(f, f').$$

If either $H(\bar r_x, \bar r_{x'})$ or $H(\bar f, \bar f')$ equals $+\infty$, then $H(\xi,\eta)$ is trivially less than or equal to the maximum of the two. So assume that both quantities are less than $+\infty$.

This rules out the possibility that $c' > c = -\infty$, and the possibility that $c' < c = +\infty$. So

$$\overline{r}_{x'}(\cdot) \le \overline{r}_x(\cdot) + H(\overline{r}_x, \overline{r}_{x'}) \quad \text{and} \quad \bar{f}'(\cdot) \le \bar{f}(\cdot) + H(\bar{f}, \bar{f}').$$

Therefore,

$$\begin{aligned} (17) \qquad H(\xi,\eta) &= \sup_{z\in C}\big([r_{x'}, f', c'](z) - [r_x, f, c](z)\big) \\ &= \sup_{z\in C}\big((\overline{r}_{x'}(z) \vee \bar{f}'(z)) - (\overline{r}_x(z) \vee \bar{f}(z))\big) \\ &\le \sup_{z\in C}\big((\overline{r}_x(z) + H(\overline{r}_x, \overline{r}_{x'}) \vee \bar{f}(z) + H(\bar{f}, \bar{f}')) - (\overline{r}_x(z) \vee \bar{f}(z))\big) \\ &\le H(\overline{r}_x, \overline{r}_{x'}) \vee H(\bar{f}, \bar{f}'). \end{aligned}$$

We now wish to show that $H(\xi,\eta) \ge H(\overline{r}_x, \overline{r}_{x'})$. This is trivial if $H(\overline{r}_x, \overline{r}_{x'}) = -\infty$, so we assume the contrary, which is equivalent to assuming that $\overline{r}_{x'}$ is finite everywhere. Let $z \in C$. From (17), we have, for all $\lambda > 0$,

$$H(\xi,\eta) \ge \overline{r}_{x'}(\lambda z) - (\overline{r}_x(\lambda z) \vee \bar{f}(\lambda z)) = (\overline{r}_{x'} - \overline{r}_x)(\lambda z) \wedge (\overline{r}_{x'} - \bar{f})(\lambda z).$$

Observe that $\overline{r}_x(\lambda z) = \overline{r}_x(z) - \log\lambda$ and $\bar{f}(\lambda z) = \bar{f}(z) + \log\lambda$ for all $\lambda > 0$. So $(\overline{r}_{x'} - \overline{r}_x)(\lambda z)$ is independent of λ. Moreover, by choosing λ small enough, we may make $(\overline{r}_{x'} - \bar{f})(\lambda z)$ as large as we wish. We conclude that $H(\xi,\eta) \ge (\overline{r}_{x'} - \overline{r}_x)(z)$. Taking the supremum over $z \in C$ gives us what we wish.

The proof that $H(\xi,\eta) \ge H(\bar{f}, \bar{f}')$ is similar.

The result now follows on symmetrising. □

Corollary 9.5 *The singletons of the Thompson geometry are the Busemann points of the form r_x with x an extreme point of $\operatorname{cl} D$, or of the form $\log(\langle y,\cdot\rangle/\langle y,b\rangle)$ with y an extremal generator of C^*.*

Proof Since the Busemann points of the form r_x, with $x \in D$, all lie in the same part, none of them are singletons. Similarly, no Busemann point of the form $f_{C,x}$, with $x \in D$, is a singleton.

Consider now a Busemann point $\xi := [r_x, f, c]$, with $x \in \partial D$, and $f \in B(x)$, and $c \in \overline{\mathbb{R}}$.

If c is finite, we define the Busemann point $\xi^\epsilon := [r_x, f, c+\epsilon]$ for each $\epsilon \in \mathbb{R}$. A simple calculation using Proposition 9.4 then gives $\delta(\xi, \xi^\epsilon) = |\epsilon|$ for all $\epsilon \in \mathbb{R}$. So ξ^ϵ

is distinct from ξ but in the same part as it for all $\epsilon \in \mathbb{R}\backslash\{0\}$, and hence ξ is not a singleton.

In the case where $c = \infty$ and x is not an extreme point of $\operatorname{cl} D$, we have that x is in the relative interior of some extreme set of $\operatorname{cl} D$ that contains another point x' distinct from x. One can then show using Proposition 9.4 that $\xi' := [r_{x'}, f', \infty]$ is distinct from ξ but in the same part, where f' is any function.

The case where $c = -\infty$ and f is not a singleton of the Funk geometry is similar.

Now let x be an extreme point of $\operatorname{cl} D$, so that r_x is a singleton of the reverse-Funk geometry. Write $\xi := r_x = [r_x, f, \infty]$ for any Funk horofunction f in $B(x)$. Let $\xi' := [r_{x'}, f', c]$ be a Busemann point of the Thompson geometry lying in the same part of the horofunction boundary as ξ. Here, of course, $x' \in \partial D$, and $f' \in B(x')$, and $c \in \overline{\mathbb{R}}$. Since $\delta(\xi, \xi')$ is finite, looking at Proposition 9.4, we see that $H(f, f') + \infty - c^+ < \infty$, and hence $c = \infty$. Using this and the same proposition again, we get $\infty > \delta(\xi, \xi') = \delta(r_x, r_{x'})$. Since r_x was assumed to be a singleton, we conclude that $x' = x$, and therefore $\xi' = \xi$. We have thus shown that ξ is a singleton.

The proof is similar in the case of Busemann points of the form $\log(\langle y, \cdot\rangle / \langle y, b\rangle)$, with y an extremal generator of C^*. □

10 Isometries of the Thompson metric

Let C_1, C_2 and C be nonempty convex cones in the linear space V. We say that C is the direct product of C_1 and C_2 if $C = C_1 + C_2$ and $\operatorname{lin} C_1 \cap \operatorname{lin} C_2 = \{0\}$. Here lin denotes the linear span of a set. In this case we write $C = C_1 \oplus C_2$.

If $C = C_1 \oplus C_2$, then $\operatorname{lin} C$ is the (linear space) direct sum of $\operatorname{lin} C_1$ and $\operatorname{lin} C_2$. Denoting by P_1 and P_2 the corresponding projections, we have $P_1(C) = C_1$ and $P_2(C) = C_2$. We note that C is relatively open if and only if both C_1 and C_2 are.

Let $C = C_1 \oplus C_2$ be a product cone. We write the Thompson metrics on C_1 and C_2 as d_T^1 and d_T^2, respectively. Let $x_1, y_1 \in C_1$ and $x_2, y_2 \in C_2$. One may easily verify that $x_1 + x_2 \le_C y_1 + y_2$ if and only if both $x_1 \le_{C_1} y_1$ and $x_2 \le_{C_2} y_2$. It follows that

$$M_C(x_1 + x_2, y_1 + y_2) = M_{C_1}(x_1, y_1) \vee M_{C_2}(x_2, y_2),$$

and hence that the Thompson metric on C is

$$d_T(x_1 + x_2, y_1 + y_2) = d_T^1(x_1, y_1) \vee d_T^2(x_2, y_2). \tag{18}$$

Assume that C is proper, open and convex. Suppose that C_2 admits a gauge-reversing bijection ϕ_2. For example, one may think of $(0,\infty)$ with the map $x \mapsto 1/x$. Then, as pointed out in [17, Proposition 10.1], there exists a Thompson metric isometry of C that is neither gauge-preserving nor gauge-reversing, namely the map $\phi\colon C \to C$ defined by

$$\phi(x_1 + x_2) := x_1 + \phi_2(x_2) \quad \text{for all } x_1 \in C_1 \text{ and } x_2 \in C_2.$$

Indeed, we are applying here the identity map to the first component and ϕ_2 to the second. These maps are Thompson isometries on C_1 and C_2, respectively, and so (18) gives that ϕ is an isometry on C. However, ϕ is clearly neither homogeneous nor antihomogeneous, which implies by Propositions 2.1 and 2.2 that ϕ is neither gauge-preserving nor gauge-reversing.

We see from the following theorem that this is the only way in which such isometries arise.

Theorem 1.5 *Let C and C' be proper open convex cones, and let $\phi\colon C \to C'$ be a surjective isometry of the Thompson metric. Then there exist decompositions $C = C_1 \oplus C_2$ and $C' = C_1' \oplus C_2'$ such that ϕ takes the form $\phi(x_1+x_2) = \phi_1(x_1)+\phi_2(x_2)$, where ϕ_1 is a gauge-preserving map from C_1 to C_1' and ϕ_2 is a gauge-reversing map from C_2 to C_2'.*

We will prove this theorem by considering the action of ϕ on the horofunction boundary, or more specifically, its action on the singletons. Choose basepoints b and b' in C and C', respectively, such that $\phi(b) = b'$.

Recall that a singleton is a Busemann point that lies in a part consisting of a single point, or in other words, that is an infinite distance from every other Busemann point with respect to the detour metric. We have seen in Corollary 9.5 that each singleton of the Thompson geometry is either a singleton of the Funk geometry or a singleton of the reverse-Funk geometry on the same cone.

Let S be the set of functions of the form $\exp \circ g$, where g is a singleton of the Thompson geometry on C. So each element of S is the restriction to C of a function either of the form $\langle y, \cdot\rangle / \langle y, b\rangle$ with y an extremal generator of C^*, or of the form $M(x,\cdot)/M(x,b)$ with x an extremal generator of C. Denote by F those of the former kind, and by R those of the latter. On C', we define the sets of functions S', F' and R' in the same way.

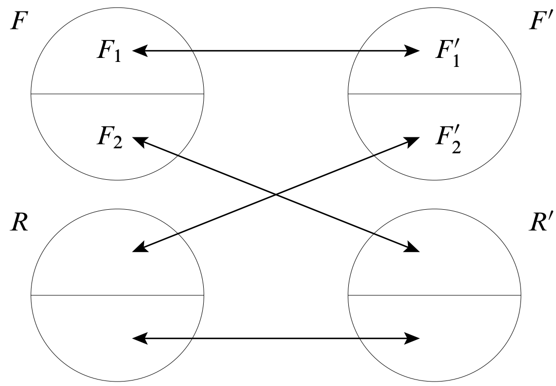

Figure 3: The action of an isometry on the singletons in the boundary

Since the action of ϕ on the horofunction boundary preserves the detour metric, it maps parts to parts, and hence maps singletons to singletons. Therefore, a function f is in S if and only if $f \circ \phi^{-1}$, which is its image under ϕ, is in S'.

Observe that each element of F is a linear functional on V, and together they span the dual space V^*. The idea of the proof is to examine which of these linear functionals get mapped by ϕ to a linear functional, and which get mapped to something nonlinear.

Let F_1 denote the elements of F that are mapped to elements of F', and F_2 denote those that are mapped to elements of R'. Similarly, define an element f' of F' to be in either F'_1 or F'_2 depending on whether its image $f' \circ \phi$ under ϕ^{-1} is in F or R. So we have the picture given in Figure 3.

Define

$$C_1 := \operatorname{rel\,int}\{z \in \operatorname{cl} C \mid f(z) = 0 \text{ for all } f \in F_2\},$$

$$C_2 := \operatorname{rel\,int}\{z \in \operatorname{cl} C \mid f(z) = 0 \text{ for all } f \in F_1\}.$$

Observe that $\operatorname{cl} C_1$ and $\operatorname{cl} C_2$ are exposed faces of $\operatorname{cl} C$. We define C'_1 and C'_2 in an analogous way.

Lemma 10.1 *If x and y are in C_1, and $f(x) = f(y)$ for all $f \in F_1$, then $x = y$. Similarly, if x and y are in C_2, and $f(x) = f(y)$ for all $f \in F_2$, then $x = y$.*

Proof Under the assumptions of the first statement, $f(x) = f(y)$ for all $f \in F_1 \cup F_2$. However, $F_1 \cup F_2$ is exactly the set of extreme points f of $\operatorname{cl} C^*$ satisfying $f(b) = 1$. Since this set spans V^*, we have $x = y$.

The proof of the second statement is similar. □

Lemma 10.2 *Suppose $z \in C$ can be written $z = x_1 + x_2 = y_1 + y_2$, with x_1 and y_1 in C_1, and x_2 and y_2 in C_2. Then $x_1 = y_1$ and $x_2 = y_2$.*

Proof Each $f \in F_2$ is linear and takes the value zero on C_1, and so $f(z) = f(x_2) = f(y_2)$. Therefore, by Lemma 10.1, $x_2 = y_2$.

The proof that $x_1 = y_1$ is similar. □

Define

$$P_1(z,\alpha) := \frac{1}{\alpha}\phi^{-1}(\alpha\phi(z)) \quad \text{and} \quad P_2(z,\alpha) := \frac{1}{\alpha}\phi^{-1}\Big(\frac{1}{\alpha}\phi(z)\Big)$$

for all $z \in C$ and $\alpha \in (0,\infty)$.

Lemma 10.3 *The cone C is the direct product of C_1 and C_2, and the maps $P_1(z) := \lim_{\alpha\to\infty} P_1(z,\alpha)$ and $P_2(z) := \lim_{\alpha\to\infty} P_2(z,\alpha)$ are the projection maps onto C_1 and C_2, respectively.*

Proof Let $z \in C$. For $f \in F_1$, we have that $f \circ \phi^{-1}$ is the exponential of a Funk horofunction, and is therefore homogeneous. In this case,

$$f(P_1(z,\alpha)) = \frac{1}{\alpha} f \circ \phi^{-1}(\alpha\phi(z)) = f(z).$$

On the other hand, for $f \in F_2$, we have that $f \circ \phi^{-1}$ is the exponential of a reverse-Funk horofunction, and is therefore antihomogeneous, which gives that $f(P_1(z,\alpha)) = f(z)/\alpha^2$.

So the limit of $f(P_1(z,\alpha))$ as α tends to infinity exists for all $f \in F$. This implies that the limit $P_1(z)$ defined in the statement of the lemma exists.

Moreover, we have

$$f(P_1(z)) = \begin{cases} f(z) & \text{for } f \in F_1, \\ 0 & \text{for } f \in F_2. \end{cases} \tag{19}$$

So $P_1(z)$ is in $\operatorname{cl} C_1$. Similarly, one can show that the limit $P_2(z)$ exists and lies in $\operatorname{cl} C_2$, and that

$$f(P_2(z)) = \begin{cases} 0 & \text{for } f \in F_1, \\ f(z) & \text{for } f \in F_2. \end{cases} \tag{20}$$

Observe that $f(P_1(z)) + f(P_2(z)) = f(z)$ for all $f \in F$. It follows that $z = P_1(z) + P_2(z)$. We have shown that C is a subset of $\operatorname{cl} C_1 + \operatorname{cl} C_2$. It follows that

$\operatorname{cl} C$ is also a subset of this set since the latter set is closed. That $\operatorname{cl} C_1 + \operatorname{cl} C_2$ is a subset of $\operatorname{cl} C$ follows from the fact that $\operatorname{cl} C_1$ and $\operatorname{cl} C_2$ are contained in $\operatorname{cl} C$.

Lemma 10.2 implies that

$$\operatorname{lin} \operatorname{cl} C_1 \cap \operatorname{lin} \operatorname{cl} C_2 = \operatorname{lin} C_1 \cap \operatorname{lin} C_2 = \{0\}.$$

So we see that the cone $\operatorname{cl} C$ is the direct product of $\operatorname{cl} C_1$ and $\operatorname{cl} C_2$. It follows immediately that C is the direct product of C_1 and C_2. □

We define maps P'_1 and P'_2 on C' analogously to how we defined P_1 and P_2.

Lemma 10.4 *Let x and y in C be such that $P_1(x) = P_1(y)$. Then $P'_1(\phi(x)) = P'_1(\phi(y))$.*

Proof From (19), we have $f(x) = f(y)$ for all $f \in F_1$. Equivalently,

$$f \circ \phi^{-1}(\phi(x)) = f \circ \phi^{-1}(\phi(y))$$

for all $f \in F_1$. But $f \circ \phi^{-1}$ is in F'_1 if and only if f is in F_1, and so $f'(\phi(x)) = f'(\phi(y))$ for all $f' \in F'_1$. It follows that $P'_1(\phi(x)) = P'_1(\phi(y))$. □

We say that a function f on a product cone $C_1 \oplus C_2$ is independent of the first component if $f(x) = f(y)$ whenever $P_2(x) = P_2(y)$. Similarly, we say that f is independent of the second component if $f(x) = f(y)$ whenever $P_1(x) = P_1(y)$.

Equations (19) and (20) imply, respectively, that each function in F_1 is independent of the second component, and each function in F_2 is independent of the first component.

Let d_T^1 and d_T^2 be the Thompson metrics on C_1 and C_2, respectively. Since C is the direct product of C_1 and C_2, we may write

$$d_T(x_1 + x_2, y_1 + y_2) = d_T^1(x_1, y_1) \vee d_T^2(x_2, y_2)$$

for all $x_1, y_1 \in C_1$ and $x_2, y_2 \in C_2$. So (C, d_T) is the ℓ_∞–product of the spaces (C_1, d_T^1) and (C_2, d_T^2) in the sense of Section 8. It was shown there, in Corollary 8.5, that each singleton of such a product depends only on one of the two components. We conclude that each element of S is either independent of the first component or independent of the second.

Lemma 10.5 *Let $f \in S$. Then f is independent of the first component if and only if $f \circ \phi^{-1}$ is. Likewise, f is independent of the second component if and only if $f \circ \phi^{-1}$ is.*

Proof Assume $f\circ\phi^{-1}$ is independent of the second component. Let x and y in C be such that $P_1(x)=P_1(y)$. So, by Lemma 10.4, $P_1'(\phi(x))=P_1'(\phi(y))$. Therefore, $f\circ\phi^{-1}(\phi(x))=f\circ\phi^{-1}(\phi(y))$, or equivalently, $f(x)=f(y)$. We conclude that f is independent of the second component.

The implication in the opposite direction is proved in a similar manner.

Now assume that $f\circ\phi^{-1}$ is independent of the first component, and so not independent of the second. Since f is in S, it must be independent of either the first or second component. However, the latter possibility is ruled out by what we have just proved. Again, the implication in the opposite direction is similar. □

Lemma 10.6 *Let x and y in C be such that $P_2(x)=P_2(y)$. Then $P_2'(\phi(x))=P_2'(\phi(y))$.*

Proof Let $f'\in F_2'$. So $f'\circ\phi$ is in R, and, by Lemma 10.5, it is independent of the first component since f' is. In particular, $f'(\phi(x))=f'(\phi(y))$. Using (20) we get that $f'(P_2'(\phi x))=f'(P_2'(\phi y))$ for all $f'\in F_2'$. But the same equation also holds for all $f'\in F_1'$, since, by (20), both sides are zero in this case. The conclusion follows, since the set of linear functions $F_1'\cup F_2'$ spans the dual space of $V':=\operatorname{lin}C'$. □

Proof of Theorem 1.5 It was shown in Lemma 10.3 that C and C' decompose in the way claimed, and in Lemmas 10.4 and 10.6 that ϕ is of the form $\phi(x_1+x_2)=\phi_1(x_1)+\phi_2(x_2)$ for all $x_1\in C_1$ and $x_2\in C_2$ for some maps $\phi_1\colon C_1\to C_1'$ and $\phi_2\colon C_2\to C_2'$.

Since C is a direct product of C_1 and C_2, its Thompson metric can be written as

$$d_T(x_1+x_2,y_1+y_2)=\max(d_T^1(x_1,y_1),d_T^2(x_2,y_2)),$$

in terms of the Thompson metrics on C_1 and C_2. A similar expression holds for d_T'. So, for $z\in C_1$ and $x_2,y_2\in C_2$, we have

$$d_T(z+x_2,z+y_2)=d_T^2(x_2,y_2)$$

and

$$\begin{aligned}d_T'(\phi(z+x_2),\phi(z+y_2))&=d_T'(\phi_1(z)+\phi_2(x_2),\phi_1(z)+\phi_2(y_2))\\&=d_T'^2(\phi_2(x_2),\phi_2(y_2)).\end{aligned}$$

We conclude that ϕ_2 is an isometry from (C_2,d_T^2) to $(C_2',d_T'^2)$.

Moreover, for all $x \in C$ and $\lambda > 0$,

$$P_2'(\phi(\lambda x)) = \lim_{\alpha\to\infty} \frac{1}{\lambda}\frac{\lambda}{\alpha}\phi\Big(\frac{\lambda}{\alpha}x\Big) = \frac{1}{\lambda} P_2'(\phi(x)).$$

Hence, ϕ_2 is antihomogeneous.

We now apply Proposition 2.2 to get that ϕ_2 is gauge-reversing.

A similar argument shows that ϕ_1 is also a Thompson-metric isometry, but this time homogeneous, and hence gauge-preserving by Proposition 2.1. □

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