

Central limit theorems for mapping class groups and $\text{Out}(F_N)$

CAMILLE HORBEZ

We prove central limit theorems for the random walks on either the mapping class group of a closed, connected, orientable, hyperbolic surface, or on $\text{Out}(F_N)$, each time under a finite second moment condition on the measure (either with respect to the Teichmüller metric, or with respect to the Lipschitz metric on outer space). In the mapping class group case, this describes the spread of the hyperbolic length of a simple closed curve on the surface after applying a random product of mapping classes. In the case of $\text{Out}(F_N)$, this describes the spread of the length of primitive conjugacy classes in F_N under random products of outer automorphisms. Both results are based on a general criterion for establishing a central limit theorem for the Busemann cocycle on the horoboundary of a metric space, applied to either the Teichmüller space of the surface or to the Culler–Vogtmann outer space.

20F65, 60B15

Introduction

Central limit theorems in noncommutative settings have a long history, and have already been established in various contexts. In the case of random products of matrices, a classical theorem of Furstenberg [19] asserts that if $(A_i)_{i \in \mathbb{N}}$ is a sequence of random matrices, all distributed with respect to some probability law μ on $\text{GL}(N, \mathbb{R})$ whose support generates a noncompact subgroup of $\text{GL}(N, \mathbb{R})$ that does not virtually preserve any proper linear subspace of \mathbb{R}^N , then under a first moment assumption on μ , there exists $\lambda > 0$ such that for all $v \in \mathbb{R}^N \setminus \{0\}$, almost surely, one has

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|A_n \cdots A_1 \cdot v\| = \lambda.$$

Central limit theorems in this context date back to the works of Furstenberg and Kesten [20], Le Page [37], Guivarc’h and Raugi [25], and Gol’dsheĭd and Margulis [23]. These assert, under some conditions on μ , that the variables

$$\frac{\log \|A_n \cdots A_1 \cdot v\| - n\lambda}{\sqrt{n}}$$

converge in law toward a centered Gaussian law on \mathbb{R} , which does not depend on the vector $v \in \mathbb{R}^N \setminus \{0\}$. These central limit theorems were classically established under an exponential moment assumption on the measure μ . Their proofs relied on establishing a spectral gap property for the transfer operator of the Markov chain corresponding to the random walk, in a well-chosen space of sufficiently regular functions. Recently, Benoist and Quint [3] gave a new approach to the central limit theorem on linear groups, which enabled them to relax the assumption made on the measure to a second moment condition.

Central limit theorems have also been established for free groups (Sawyer and Steger [49]) or, more generally, word-hyperbolic groups (Björklund [8]), describing the spread of the word length of the element obtained at time n of the random walk (again, the limiting law is Gaussian). Their proofs also required exponential moment assumptions on the measure. The new approach of Benoist and Quint also enabled them to similarly relax the moment assumption in this context [4].

The goal of the present paper is to prove central limit theorems on mapping class groups of closed, connected, orientable, hyperbolic surfaces, and on the group $\text{Out}(F_N)$ of outer automorphisms of a finitely generated free group. In the case of mapping class groups, we will establish a central limit theorem for the hyperbolic lengths of essential simple closed curves, under application of a random product of diffeomorphisms. In the case of $\text{Out}(F_N)$, we will establish a central limit theorem for the word lengths of primitive conjugacy classes of F_N , under application of a random product of outer automorphisms of F_N .

Central limit theorem on mapping class groups Let S be a closed, connected, oriented, hyperbolic surface, and let ρ be a hyperbolic metric on S . The mapping class group $\text{Mod}(S)$ is the group of isotopy classes of orientation-preserving diffeomorphisms of S . Karlsson [33] established a version of the law of large numbers for the random walk on $\text{Mod}(S)$, estimating the typical growth of curves under random products of diffeomorphisms of the surface. Given a probability measure μ on $\text{Mod}(S)$, the (*left*) random walk on $(\text{Mod}(S), \mu)$ is the Markov process whose position Φ_n at time n is obtained by successive multiplications on the left of n independent μ -distributed increments s_i , ie $\Phi_n = s_n \cdots s_1$. A probability measure on $\text{Mod}(S)$ is *nonelementary* if the subsemigroup of $\text{Mod}(S)$ generated by the support of μ is a subgroup of $\text{Mod}(S)$ that contains two independent pseudo-Anosov mapping classes.

Karlsson proved that if μ is a nonelementary probability measure on $\text{Mod}(S)$ with finite first moment with respect to the Teichmüller metric, then there exists a (deterministic) real number $\lambda > 0$ such that for all essential simple closed curves c on S , and almost

every sample path $(\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Mod}(S), \mu)$, one has

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log l_\rho(\Phi_n(c)) = \lambda.$$

Here $l_\rho(\Phi_n(c))$ is the smallest length of a curve in the isotopy class of $\Phi_n(c)$, measured by integration of the metric ρ . The growth rate λ is also equal to the drift of the random walk on $(\text{Mod}(S), \mu)$ with respect to the Teichmüller metric. We will establish the following central limit theorem, under a second moment condition on μ . In the statement, we denote by μ^{*n} the n^{th} convolution of μ .

Theorem 0.1 *Let S be a closed, connected, oriented, hyperbolic surface, and let ρ be a hyperbolic metric on S . Let μ be a nonelementary probability measure on $\text{Mod}(S)$ with finite second moment with respect to the Teichmüller metric. Let $\lambda > 0$ be the drift of the random walk on $(\text{Mod}(S), \mu)$ with respect to the Teichmüller metric.*

Then there exists a centered Gaussian law N_μ on \mathbb{R} such that for every compactly supported continuous function F on \mathbb{R} , and all essential simple closed curves c on S , one has

$$\lim_{n \rightarrow +\infty} \int_{\text{Mod}(S)} F\left(\frac{\log l_\rho(\Phi(c)) - n\lambda}{\sqrt{n}}\right) d\mu^{*n}(\Phi) = \int_{\mathbb{R}} F(t) dN_\mu(t)$$

uniformly in c .

Central limit theorem on $\text{Out}(F_N)$ Let $N \geq 2$, let F_N be a free group of rank N , and let $\text{Out}(F_N)$ denote its outer automorphism group. Let μ be a probability measure on $\text{Out}(F_N)$. We established in [28] the following analogue of Karlsson’s theorem for the random walk on $\text{Out}(F_N)$, estimating the growth of nontrivial conjugacy classes in F_N under application of a random product of outer automorphisms. Assume that the probability measure μ on $\text{Out}(F_N)$ is *nonelementary*, ie the subsemigroup of $\text{Out}(F_N)$ generated by its support is a subgroup which is not virtually cyclic and does not virtually preserve the conjugacy class of any proper free factor of F_N , and assume that μ has finite first moment with respect to the asymmetric Lipschitz metric d_{CV_N} on the outer space CV_N of Culler and Vogtmann. Then there exists a Lyapunov exponent $\lambda > 0$ such that for all primitive elements $g \in F_N$ and almost every sample path $(\Phi_n)_{n \in \mathbb{N}}$ of the left random walk on $(\text{Out}(F_N), \mu)$, one has

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|\Phi_n(g)\| = \lambda,$$

where $\|\Phi_n(g)\|$ denotes the smallest word length of a conjugate of $\Phi_n(g)$, written in some prescribed free basis of F_N . Here we recall that an element $g \in F_N$ is *primitive* if it belongs to some free basis of F_N , and we denote by \mathcal{P}_N the collection of all

primitive elements of F_N . Again, the Lyapunov exponent λ is also equal to the drift of the random walk on $(\text{Out}(F_N), \mu)$; ie

$$\lambda = \lim_{n \rightarrow +\infty} \frac{1}{n} d_{\text{CV}_N}(\Phi_n \cdot o, o)$$

for almost every sample path $(\Phi_n)_{n \in \mathbb{N}}$ of the random walk. We will establish a central limit theorem for the variables $\log \|\Phi_n(g)\|$ under a second moment assumption on μ .

Theorem 0.2 *Let μ be a nonelementary probability measure on $\text{Out}(F_N)$ with finite second moment with respect to d_{CV_N} . Let $\lambda > 0$ be the drift of the random walk on $(\text{Out}(F_N), \mu)$ with respect to d_{CV_N} . Then there exists a centered Gaussian law N_μ on \mathbb{R} such that for every compactly supported continuous function F on \mathbb{R} , and all primitive elements $g \in \mathcal{P}_N$, one has*

$$\lim_{n \rightarrow +\infty} \int_{\text{Out}(F_N)} F\left(\frac{\log \|\Phi(g)\| - n\lambda}{\sqrt{n}}\right) d\mu^{*n}(\Phi) = \int_{\mathbb{R}} F(t) dN_\mu(t)$$

uniformly in g .

Strategy of proofs The present paper was inspired by the new approach by Benoist and Quint to the central limit theorem for linear groups [3] and hyperbolic groups [4]. This relies on the study of various cocycles. Given a countable group G , a compact G -space X and a continuous cocycle $\sigma: G \times X \rightarrow \mathbb{R}$, Benoist and Quint developed a method for proving a central limit theorem for the cocycle σ . This follows the so-called Gordin’s method and requires proving that σ is *centerable*, ie can be written as

$$(1) \quad \sigma(g, x) = \sigma_0(g, x) + \psi(x) - \psi(gx)$$

for all $(g, x) \in G \times X$, where ψ is a bounded measurable function on X and where there exists $\lambda \in \mathbb{R}$ such that

$$\int_G \sigma_0(g, x) d\mu(g) = \lambda$$

for all $x \in X$. One is then left showing a central limit theorem for σ_0 , which can be done by using a classical central limit theorem for martingales due to Brown [9].

In order to prove a central limit theorem for the word length in a hyperbolic group G , Benoist and Quint [4] applied the above strategy to the Busemann cocycle on the horofunction boundary of G . We recall that the *horofunction compactification* of a proper geodesic metric G -space (X, d_X) is defined as the closure of the image of the embedding

$$\psi: X \rightarrow \mathcal{C}(X), \quad z \mapsto \{x \mapsto d_X(x, z) - d_X(o, z)\},$$

where $\mathcal{C}(X)$ is the space of continuous functions on X , equipped with the topology of uniform convergence on compact subsets of X , and $o \in X$ is a basepoint. The *Busemann cocycle* is the continuous cocycle β on $G \times \partial_h X$ (where $\partial_h X$ is the horoboundary of X) defined by letting $\beta(g, h) := h(g^{-1} \cdot o)$ for all $(g, h) \in G \times \partial_h X$.

In the mapping class group context, we will establish a central limit theorem for the Busemann cocycle on the horoboundary of the Teichmüller space $\mathcal{T}(S)$ of the surface, which we equip with the Teichmüller metric. It turns out that this is enough for proving [Theorem 0.1](#) since the Busemann cocycle on $\text{Mod}(S) \times \partial_h \mathcal{T}(S)$ is closely related to lengths of simple closed curves on S (the metric on $\mathcal{T}(S)$ can indeed be controlled using lengths of curves). Similarly, in the case of $\text{Out}(F_N)$, it will be enough to prove a central limit theorem for the Busemann cocycle on the horoboundary of the outer space of Culler and Vogtmann, which is closely related to lengths of conjugacy classes in F_N . A new difficulty arises however in the latter context: since the natural metric on outer space fails to be symmetric, outer space has in fact two horoboundaries (forward and backward), which appear to be rather different in nature (see [\[28\]](#), where the forward horoboundary of outer space is completely described, and the geometry of the backward horoboundary is also investigated). We will actually only prove a central limit theorem for the Busemann cocycle on the backward horoboundary, but our arguments will require working with both boundaries.

As a consequence of the work of Benoist and Quint, one can give a general condition under which the Busemann cocycle on the backward horoboundary $\partial_h^- X$ of a G -metric space X (where G is a countable group) satisfies a central limit theorem (a *dual* version holds for the Busemann cocycle on the forward horoboundary $\partial_h^+ X$ by reversing the roles of the forward and backward metrics). We denote by $\check{\mu}$ the *reflected measure* on G , defined by letting $\check{\mu}(g) := \mu(g^{-1})$ for all $g \in G$. We denote by $(\cdot | \cdot)$ the natural extension of the Gromov product on X to $\partial_h^- X \times \partial_h^+ X$, defined by letting

$$(x|y)_o := -\frac{1}{2} \inf_{z \in X} (h_x^-(z) + h_y^+(z))$$

for all $x, y \in \partial_h^- X \times \partial_h^+ X$, where h_x^- and h_y^+ denote the functions on X associated to x and y . We denote by d_X^{sym} the symmetrized metric on X , defined as the maximum of the forward and backward metrics.

Theorem 0.3 *Let (X, d_X) be a (possibly asymmetric) geodesic metric space, let $o \in X$, and let G be a countable group acting by isometries on X . Let μ be a probability measure on G with finite second moment with respect to d_X^{sym} . Assume that there exists a G -invariant measurable subset $Y^- \subseteq \partial_h^- X$, on which there exists a μ -ergodic μ -stationary probability measure ν , and a G -invariant subset $Y^+ \subseteq \partial_h^+ X$, on which there exists a $\check{\mu}$ -stationary probability measure ν^* . Further assume that*

(H1) there exists $\lambda \in \mathbb{R}$ such that

$$\int_{G \times Y^+} \beta^+(g, y) d\check{\mu}(g) dv^*(y) = \lambda,$$

(H2) there exists $\alpha > 0$ and a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $x \in Y^-$,

$$v^*(\{y \in Y^+ \mid (x|y)_o \geq \alpha n\}) \leq C_n.$$

Then $\beta_{|Y^-}^-$ is centerable. Let

$$V_v := \int_{G \times Y^-} (\beta_0^-(g, x) - \lambda)^2 d\mu(g) dv(x),$$

and let N_v be the centered Gaussian law on \mathbb{R} with variance V_v . Then

$$\lim_{n \rightarrow +\infty} \int_G F\left(\frac{\beta^-(g, x) - n\lambda}{\sqrt{n}}\right) d\mu^{*n}(g) = \int_{\mathbb{R}} F(t) dN_v(t)$$

for v -ae $x \in Y^-$ and every compactly supported continuous function F on \mathbb{R} . □

To derive [Theorem 0.3](#) from the work of Benoist and Quint, one is essentially left solving the cohomological equation (1) for the cocycle β^- : the solution is given as in [4] by

$$(2) \quad \psi(x) := -2 \int_{Y^+} (x|y)_o dv^*(y).$$

Hypothesis (H2) ensures that ψ is finite and bounded, and hypothesis (H1) ensures that the cocycle β_0^- defined as in (1) has constant average λ .

The intuition behind hypothesis (H1) is that if you pick an element $g \in G$ at random with respect to the probability measure μ , and a random point y in the forward horoboundary of X , then on average you will tend to move away from y (about distance λ) when going from o to go . This is a typical behavior if X is a hyperbolic space.

The intuition behind hypothesis (H2) is that if you pick two points x and y in the backward and forward horoboundaries of X , then with high probability, geodesic rays from o to x (for the backward metric) and from o to y (for the forward metric) will rapidly diverge. This is again a typical behavior if X is a hyperbolic space.

In order to prove [Theorem 0.1](#), we will establish hypotheses (H1) and (H2) for the Gromov product on the Teichmüller space $\mathcal{T}(S)$. The rough intuition is that, though not hyperbolic, Teichmüller space is *hyperbolic on average*, and typical rays in $\mathcal{T}(S)$ contain infinitely many subsegments with hyperbolic-like behavior; see Dowdall, Duchin and Masur [15].

To make sense of this intuition, we will take advantage of the mapping class group action on the curve graph $\mathcal{C}(S)$ of the surface, which is known to be Gromov hyperbolic thanks to work by Masur and Minsky [42]. Combining the arguments of Benoist and Quint with recent work by Maher and Tiozzo [40] extending results about random walks on hyperbolic spaces to nonproper settings, we will first establish hypotheses (H1) and (H2) for the Gromov product on $\mathcal{C}(S)$.

There is a well-behaved Lipschitz map from $\mathcal{T}(S)$ to $\mathcal{C}(S)$. In order to obtain the desired deviation estimates for the realization of the random walk on $\mathcal{T}(S)$, we will *lift* to $\mathcal{T}(S)$ our estimates for the realization of the random walk on $\mathcal{C}(S)$. This is done by appealing to a contraction property of typical geodesics in $\mathcal{T}(S)$, following a strategy that was already used in Dahmani and Horbez [14] for establishing spectral theorems for the random walks on $\text{Mod}(S)$ and $\text{Out}(F_N)$. Since the realizations of the random walk on $\mathcal{T}(S)$ and $\mathcal{C}(S)$ both escape the origin with positive speed, typical rays in $\mathcal{T}(S)$ must contain subsegments whose projections to $\mathcal{C}(S)$ make definite progress. Any subsegment I that makes progress also satisfies the following contraction property: if another Teichmüller segment has the same projection to $\mathcal{C}(S)$ as I , then it passes uniformly close to I in $\mathcal{T}(S)$. This will be the key observation for establishing hypotheses (H1) and (H2) for the Gromov product on $\mathcal{T}(S)$.

We use a similar strategy for establishing hypotheses (H1) and (H2) for the outer space of Culler and Vogtmann. This time, we will take advantage of the action of $\text{Out}(F_N)$ on the so-called *free factor graph*, which was proved to be Gromov hyperbolic by Bestvina and Feighn [6]. However, new technical difficulties arise, mainly coming from the asymmetry of the metric on outer space, and the contraction property we establish in this context is slightly weaker than the one we use in the context of mapping class groups.

Structure of the paper The paper is organized as follows. In [Section 1](#), we establish a central limit theorem for Busemann cocycles, in the general case of metric spaces that may fail to be symmetric. In [Section 2](#), we build on the works of Benoist and Quint [4] and Maher and Tiozzo [40] to establish quantitative deviation estimates for random walks on (possibly nonproper) hyperbolic spaces, under a second moment condition on the measure. The proof of the central limit theorem in the context of mapping class groups is carried in [Section 3](#), and we deal with the $\text{Out}(F_N)$ case in [Section 4](#).

Acknowledgements I would like to thank Yves Guivarc'h and Anders Karlsson for enlightening discussions and their interest in this project. I also thank the anonymous referee for valuable suggestions. I acknowledge support from ANR-11-BS01-013 and from the Lebesgue Center of Mathematics.

1 A central limit theorem for Busemann cocycles

1.1 Random walks on groups: general definitions and notation

General notation Let G be a group, and let μ be a probability measure on G . The (left) random walk on G with respect to the measure μ is the Markov chain on G with initial distribution the Dirac measure at the identity element, and with transition probabilities $p(x, y) := \mu(yx^{-1})$.

The product probability space $\Omega := (G^{\mathbb{N}^*}, \mu^{\otimes \mathbb{N}^*})$ is the space of increments of the random walk. The position of the random walk at time n is given from its position $g_0 = e$ at time 0 by successive multiplications on the left of independent μ -distributed increments s_i , ie $g_n = s_n \cdots s_1$. The path space $\mathcal{P} := G^{\mathbb{N}}$ is equipped with the σ -algebra generated by the cylinders $\{\mathbf{g} \in \mathcal{P} \mid g_i = g\}$ for all $i \in \mathbb{N}$ and all $g \in G$, and the probability measure \mathbb{P} induced by the map

$$\Omega \rightarrow \mathcal{P}, \quad (s_1, s_2, \dots) \mapsto (g_0, g_1, g_2, \dots).$$

Remark 1.1 We warn the reader that we will always be considering left random walks on groups in the present paper, because it is more natural when applying a random product of diffeomorphisms to a curve, or a random product of outer automorphisms to a conjugacy class. However, many results concerning random walks on either mapping class groups or outer automorphism groups of free groups are stated for right random walks in the literature: this is more natural when considering the random walk as an actual walk at random on the Cayley graph of the group, or on any space it acts on.

Moment and drift Assume now that G acts by isometries on a (possibly asymmetric) metric space (X, d_X) (ie d_X is assumed to satisfy the separation axiom and the triangle inequality, but it may fail to be symmetric). Let $o \in X$ be a basepoint. We say that μ has finite first moment with respect to d_X if

$$\int_G d_X(go, o) d\mu(g) < +\infty.$$

It has finite second moment with respect to d_X if

$$\int_G d_X(go, o)^2 d\mu(g) < +\infty.$$

Remark 1.2 In the case where the metric d_X is not symmetric, the reason why we use $d_X(go, o)$ rather than $d_X(o, go)$ in the above formula can be understood as follows. As explained in Remark 1.1, the natural realization of our random process $(\Phi_n)_{n \in \mathbb{N}}$ on G as a walk at random on the space X is by considering the sequence $(\Phi_n^{-1} \cdot o)_{n \in \mathbb{N}}$. Having this in mind, our definitions of the first and second moment really measure

the average step of the realization of the random walk on X . Notice however that in the case of the $\text{Out}(F_N)$ -action on the outer space CV_N (which is the application we have in mind in the present paper, where an asymmetric metric arises), the distances $d_X(g_o, o)$ and $d_X(o, g_o)$ are always within a bounded multiplicative constant from one another, so both possible definitions of moments will lead to the same condition in the $\text{Out}(F_N)$ -context.

By Kingman’s subadditive ergodic theorem [35], if μ has finite first moment with respect to d_X , then for \mathbb{P} -ae sample path $(g_n)_{n \in \mathbb{N}}$ of the random walk on (G, μ) , the limit

$$\lim_{n \rightarrow +\infty} \frac{1}{n} d_X(g_n o, o)$$

exists and is equal to

$$\inf_{n \in \mathbb{N}} \frac{1}{n} \int_G d_X(g_o, o) d\mu^{*n}(g),$$

where μ^{*n} denotes the n^{th} convolution of μ . This limit is called the *drift* of the random walk on (G, μ) with respect to d_X .

The *reflected measure* $\check{\mu}$ is the probability measure on G defined by letting $\check{\mu}(g) := \mu(g^{-1})$ for all $g \in G$. Notice that if d_X is symmetric, and if μ has finite first (or second) moment with respect to d_X , then the same also holds for $\check{\mu}$. The symmetry of d_X , together with the fact that the G -action on X is by isometries, also implies that the drifts of the random walks on (G, μ) and $(G, \check{\mu})$ with respect to d_X are equal in this situation.

Stationary measures A probability measure ν on X is said to be μ -stationary if $\nu = \mu * \nu$, where we recall that the convolution $\mu * \nu$ is the probability measure on X given by

$$\mu * \nu(S) := \int_G \nu(h^{-1}S) d\mu(h)$$

for all measurable subsets $S \subseteq X$. Any compact space admits a μ -stationary probability measure, obtained as a weak limit of the Cesàro averages of the measures $\mu^{*n} * \delta_o$, where δ_o is the Dirac measure at o .

1.2 Horoboundaries and Busemann cocycles

Horoboundaries Let (X, d_X) be a (possibly asymmetric) geodesic metric space. We let $d_X^+ := d_X$, and d_X^- be the (possibly asymmetric) metric on X defined by letting $d_X^-(x, y) := d_X(y, x)$ for all $x, y \in X$. We also let $d_X^{\text{sym}} := \max(d_X^+, d_X^-)$, which is a symmetric metric on X . Letting

$$h_z^+ : X \rightarrow \mathbb{R}, \quad x \mapsto d_X^+(x, z) - d_X^+(o, z),$$

for all $z \in X$, one obtains a continuous map

$$h^+ : X \rightarrow \mathcal{C}(X), \quad z \mapsto h_z^+,$$

where $\mathcal{C}(X)$ denotes the space of continuous real-valued functions on X , equipped with the topology of uniform convergence on compact sets of (X, d_X^{sym}) . The closure $\text{cl}_h^+(X) := \overline{h^+(X)}$ in $\mathcal{C}(X)$ is compact. In the case where (X, d_X^{sym}) is a proper metric space, and that d_X^+ and d_X^- determine the same topology on X , then the embedding of X in $\overline{h^+(X)}$ is a homeomorphism onto its image [50, Proposition 2.2]. In this situation, the horofunction boundary $\partial_h^+ X := \overline{h^+(X)} \setminus h^+(X)$ is compact. We will denote by $\partial_{\bar{h}}^- X$ the horofunction boundary of X for the metric d_X^- .

Extension of the Gromov product to the horoboundary For all $x, y \in X$, the Gromov product of x and y with respect to o is defined as

$$(x|y)_o := \frac{1}{2}(d_X(x, o) + d_X(o, y) - d_X(x, y)).$$

We extend it to $\text{cl}_{\bar{h}}^- X \times \text{cl}_h^+ X$ by letting

$$(x|y)_o := -\frac{1}{2} \inf_{z \in X} (h_x^-(z) + h_y^+(z))$$

for all $x \in \text{cl}_{\bar{h}}^- X$ and all $y \in \text{cl}_h^+ X$ (where we denote by h_x^- and h_y^+ the functions on X corresponding to x and y). When $x, y \in X$, this indeed coincides with the Gromov product defined above: in this case, the infimum is achieved at any point $z \in X$ lying on a geodesic segment from x to y . We note that $(x|y)_o$ may be infinite. We also note that this extension is not always continuous; see the example attributed to Walsh in [46, Appendix].

Busemann cocycles Let now G be a group acting by isometries on X . Then the G -action on X extends continuously to an action by homeomorphisms on $\partial_h^+ X$ by letting

$$g \cdot h_x^+(z) := h_x^+(g^{-1} \cdot z) - h_x^+(g^{-1} \cdot o)$$

for all $x \in \partial_h^+ X$ and all $z \in X$. The Busemann cocycle $\beta_X^+ : G \times \text{cl}_h^+ X \rightarrow \mathbb{R}$ is the continuous cocycle defined by

$$\beta_X^+(g, x) := h_x^+(g^{-1} \cdot o)$$

for all $(g, x) \in G \times \text{cl}_h^+ X$ (recall that if Y is a G -space, a map $\sigma : G \times Y \rightarrow \mathbb{R}$ is a cocycle if $\sigma(gh, y) = \sigma(g, hy) + \sigma(h, y)$ for all $g, h \in G$ and all $y \in Y$). We similarly define a cocycle β_X^- on $G \times \text{cl}_{\bar{h}}^- X$. Notice that, for all $x \in \text{cl}_h^+ X$ and all $g \in G$,

$$(3) \quad |\beta_X^+(g, x)| \leq d_X^{\text{sym}}(g o, o).$$

1.3 Deviation estimates for cocycles: review of the work of Benoist and Quint

Let G be a countable group acting continuously on a compact metrizable space X . Given a continuous cocycle $\sigma: G \times X \rightarrow \mathbb{R}$, we let $\sigma_{\text{sup}}: G \rightarrow \mathbb{R}$ be the function defined, for all $g \in G$, by

$$\sigma_{\text{sup}}(g) = \sup_{x \in X} |\sigma(g, x)|.$$

Proposition 1.3 (Benoist and Quint [3, Proposition 3.2]) *Let G be a discrete group, let X be a compact metrizable G -space, let μ be a probability measure on G . Let $\sigma: G \times X \rightarrow \mathbb{R}$ be a continuous cocycle such that $\sigma_{\text{sup}} \in L^2(G, \mu)$. Assume that there exists $\lambda \in \mathbb{R}$ such that for all μ -stationary probability measures ν on X , one has*

$$\int_{G \times X} \sigma(g, x) d\mu(g) d\nu(x) = \lambda.$$

Then for all $\epsilon > 0$, there exists a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $n \in \mathbb{N}$ and all $x \in X$, one has

$$\mu^{*n}(\{g \in G : |\sigma(g, x) - n\lambda| \geq \epsilon n\}) \leq C_n.$$

A way of checking the condition in the above statement is by using the following version of Birkhoff’s ergodic theorem in the context of cocycles.

Proposition 1.4 *Let G be a countable group acting by isometries on a space X , let μ be a probability measure on G , and let $\sigma: G \times X \rightarrow \mathbb{R}$ be a measurable cocycle. Assume that there exists $\lambda \in \mathbb{R}$, and a measurable subset $X' \subseteq X$ of full measure with respect to any μ -stationary probability measure on X , such that for all $x \in X'$ and \mathbb{P} -ae sample path $(g_n)_{n \in \mathbb{N}}$ of the random walk on (G, μ) , one has*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sigma(g_n, x) = \lambda.$$

Then for all μ -stationary probability measures ν on X ,

$$\int_{G \times X} \sigma(g, x) d\mu(g) d\nu(x) = \lambda.$$

Proof Recall that Ω denotes the step space of the random walk. Let ν be a μ -stationary probability measure on X . We equip the space $\Omega \times X$ with the probability measure $\mu^{\otimes \mathbb{N}^*} \otimes \nu$. Then the map

$$\tilde{\Omega}: \Omega \times X \rightarrow \Omega \times X, \quad ((s_n)_{n \geq 1}, x) \mapsto ((s_{n+1})_{n \geq 1}, s_1 \cdot x),$$

is measure preserving. We also let

$$\tilde{\sigma}: \Omega \times X \rightarrow \mathbb{R}, \quad ((s_n)_{n \geq 1}, x) \mapsto \sigma(s_1, x).$$

Then for all $\omega = (s_n)_{n \geq 1} \in \Omega$, all $n \in \mathbb{N}$, all $i \in \{1, \dots, n\}$ and all $x \in X$, the cocycle relation implies that

$$\sigma(g_n(\omega), x) = \sum_{i=1}^n \tilde{\sigma}(\tilde{U}^i(\omega, x)),$$

where $g_n(\omega) := s_n \cdots s_1$. Since

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sigma(g_n(\omega), x) = \lambda$$

for $\mu^{\otimes \mathbb{N}^*} \otimes \nu$ -ae $(\omega, x) \in \Omega \times X$, Birkhoff's ergodic theorem applied to each of the ergodic components of the transformation \tilde{U} , implies that

$$\int_{\Omega \times X} \tilde{\sigma}(\omega, x) d\mu^{\otimes \mathbb{N}^*}(\omega) d\nu(x) = \lambda.$$

This means that for all μ -stationary probability measures ν on X ,

$$\int_{G \times X} \sigma(g, x) d\mu(g) d\nu(x) = \lambda. \quad \square$$

1.4 Central limit theorem for Busemann cocycles

We will specify the central limit theorem for centerable cocycles of Benoist and Quint [3, Theorem 3.4] to the specific case of the Busemann cocycle on the horoboundary of a metric space (Theorem 1.6 below). In particular, we give a criterion, coming from [4], ensuring centerability of the Busemann cocycle (see below for a definition).

We start by recalling some terminology. Let G be a discrete group acting continuously on a metrizable G -space X , let μ be a probability measure on G , and let $\sigma: G \times X \rightarrow \mathbb{R}$ be a cocycle. Given $\lambda \in \mathbb{R}$, we say that σ has constant drift λ with respect to μ if for all $x \in X$, one has

$$\int_G \sigma(g, x) d\mu(g) = \lambda.$$

We say that σ is μ -centerable if there exist a measurable cocycle $\sigma_0: G \times X \rightarrow \mathbb{R}$ with constant drift with respect to μ , and a bounded measurable function $\psi: X \rightarrow \mathbb{R}$, such that for all $g \in G$ and all $x \in X$, one has

$$\sigma(g, x) = \sigma_0(g, x) + \psi(x) - \psi(gx).$$

The average of σ with respect to μ is defined as the drift of σ_0 with respect to μ . The following proposition gives a criterion for ensuring μ -centerability of the Busemann cocycle β^- on the backward horoboundary of X (by reversing the roles of d_X and d_X^- , a dual criterion can also be given for the cocycle β^+).

Proposition 1.5 *Let (X, d_X) be a (possibly asymmetric) geodesic metric space, let $o \in X$, and let G be a countable group acting by isometries on X . Let μ be a probability measure on G . Assume that there exist G -invariant measurable subsets $Y^- \subseteq \partial_h^- X$ and $Y^+ \subseteq \partial_h^+ X$, and a $\check{\mu}$ -stationary probability measure ν^* on Y^+ . Further assume that*

(H1) *there exists $\lambda \in \mathbb{R}$ such that*

$$\int_{G \times Y^+} \beta^+(g, y) d\check{\mu}(g) d\nu^*(y) = \lambda,$$

(H2) *there exists $\alpha > 0$ and a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $x \in Y^-$,*

$$\nu^*(\{y \in Y^+ \mid (x|y)_o \geq \alpha n\}) \leq C_n.$$

Then $\beta_{|Y^-}^-$ is μ -centerable with average λ .

Proof The argument follows the proofs of [4, Propositions 4.2 and 4.6]. For all $x \in Y^-$, we let

$$\psi(x) := -2 \int_{Y^+} (x|y)_o d\nu^*(y).$$

Hypothesis (H2) implies that ψ is finite and bounded: indeed, one bounds the integral by cutting Y^+ into subsets of the form $\{y \in Y^+ \mid \alpha n \leq (x|y)_o \leq \alpha(n+1)\}$ with n varying over \mathbb{N} , and then using summability of the sequence $(C_n)_{n \in \mathbb{N}}$. We then note that

$$(4) \quad \beta^-(g, x) = \beta^+(g^{-1}, y) + 2(gx|y)_o - 2(x|g^{-1}y)_o$$

for all $g \in G$, all $x \in Y^-$ and all $y \in Y^+$ (this relation is a consequence of the definitions; it was already noticed by Benoist and Quint [4, Lemma 1.2] in the case of a symmetric metric space). Since ν^* is $\check{\mu}$ -stationary, we have

$$-2 \int_{G \times Y^+} (x|g^{-1}y)_o d\mu(g) d\nu^*(y) = -2 \int_{Y^+} (x|y)_o d\nu^*(y) = \psi(x).$$

Using in addition hypothesis (H1), we obtain, by integrating (4) on $G \times Y^+$ with respect to $d\mu(g) d\nu^*(y)$, that

$$\int_G \beta^-(g, x) d\mu(g) = \lambda - \int_G \psi(g.x) d\mu(g) + \psi(x)$$

for all $x \in Y^-$. The cocycle β_0^- defined by letting

$$\beta_0^-(g, x) := \beta^-(g, x) + \psi(g.x) - \psi(x)$$

for all $g \in G$ and all $x \in Y^-$ has constant drift λ with respect to μ . Hence $\beta_{|Y^-}^-$ is μ -centerable with average λ . □

Once centerability of $\beta_{|Y^-}^-$ is established, the proof of the following central limit theorem is the same as the proofs of [3, Theorem 3.4] or [4, Theorem 4.7], which rely on a central limit theorem for martingales due to Brown [9]. The cocycle β_0^- appearing in the definition of V_ν is a cocycle provided by μ -centerability of $\beta_{|Y^-}^-$. As noticed in [3, Remark 3.3], the value of V_ν does not depend on the choice of β_0^- .

Theorem 1.6 *Let (X, d_X) be a (possibly asymmetric) geodesic metric space, let $o \in X$, and let G be a countable group acting by isometries on X . Let μ be a probability measure on G with finite second moment with respect to d_X^{sym} . Assume that there exists a G -invariant measurable subset $Y^- \subseteq \partial_h^- X$, on which there exists a μ -ergodic μ -stationary probability measure ν , and a G -invariant subset $Y^+ \subseteq \partial_h^+ X$, on which there exists a $\check{\mu}$ -stationary probability measure ν^* . Further assume that*

(H1) *there exists $\lambda \in \mathbb{R}$ such that*

$$\int_{G \times Y^+} \beta^+(g, y) d\check{\mu}(g) d\nu^*(y) = \lambda,$$

(H2) *there exists $\alpha > 0$ and a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $x \in Y^-$,*

$$\nu^*(\{y \in Y^+ \mid (x|y)_o \geq \alpha n\}) \leq C_n.$$

Let

$$V_\nu := \int_{G \times Y^-} (\beta_0^-(g, x) - \lambda)^2 d\mu(g) d\nu(x),$$

and let N_ν be the centered Gaussian law on \mathbb{R} with variance V_ν . Then

$$\lim_{n \rightarrow +\infty} \int_G F\left(\frac{\beta^-(g, x) - n\lambda}{\sqrt{n}}\right) d\mu^{*n}(g) = \int_{\mathbb{R}} F(t) dN_\nu(t)$$

for ν -ae $x \in Y^-$ and every compactly supported continuous function F on \mathbb{R} . □

Remark 1.7 If we assumed in addition that $Y^- = \partial_h^- X$, and that β^- has unique covariance in the sense of [3, Section 3.3] (which happens in particular if $\partial_h^- X$ carries a unique μ -stationary probability measure), then Theorem 1.6 would be a specification of the central limit theorem for centerable cocycles on compact spaces of Benoist and Quint [3, Theorem 3.4], in which case the convergence would be uniform in $x \in \partial_h^- X$ (notice that square-integrability of β^- follows from the second moment assumption on μ together with (3)). Nevertheless, the proof of Benoist and Quint, as it appears in [4, Theorem 4.7] applies as such (with Y^- in place of their space X) to prove Theorem 1.6.

2 Deviation results for random walks on hyperbolic spaces

2.1 Review on Gromov hyperbolic spaces

We briefly review basic facts about Gromov hyperbolic spaces, and refer the reader to [22] for details. A symmetric metric space X is *Gromov hyperbolic* if there exists $\delta \geq 0$ such that for all $x, y, z, o \in X$, one has

$$\langle x|y \rangle_o \geq \min(\langle x|z \rangle_o, \langle y|z \rangle_o) - \delta.$$

The smallest such δ is then called the *hyperbolicity constant* of X .

From now on, we let X be a geodesic Gromov hyperbolic metric space. A sequence $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ *converges to infinity* if $\langle x_n|x_m \rangle_o$ goes to $+\infty$ as n and m go to $+\infty$. Two sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ that converge to infinity are said to be *equivalent* if $\langle x_n|y_m \rangle_o$ goes to $+\infty$ as n and m go to $+\infty$. The *Gromov boundary* $\partial_\infty X$ of X is the set of equivalence classes of sequences that converge to infinity. The Gromov product on X extends to $X \cup \partial_\infty X$ by letting

$$\langle a|b \rangle_o := \inf \liminf_{i,j \rightarrow +\infty} \langle x_i|y_j \rangle_o$$

for all $a, b \in X \cup \partial_\infty X$, the infimum being taken over all sequences $(x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}$ converging to a and all sequences $(y_j)_{j \in \mathbb{N}} \in X^{\mathbb{N}}$ converging to b .

In the case of Gromov hyperbolic geodesic metric spaces, we have defined two extensions of the Gromov product: the extension to the Gromov boundary (denoted with brackets), and the extension to $\text{cl}_h X$ from the previous section (denoted with parentheses). Notice that since the metric on X is assumed to be symmetric in this section, we will just write $\text{cl}_h X$ without mentioning any superscript. The two extensions of the Gromov product are related in the following way. Following Maher and Tiozzo [40, Section 3.2], we define $\text{cl}_h^\infty X$ as the subspace of $\text{cl}_h X$ made of those horofunctions h such that $\inf_{x \in X} h(x) = -\infty$. Then there is a projection map $\pi: X \cup \text{cl}_h^\infty X \rightarrow X \cup \partial_\infty X$ which restricts to the identity on X , and such that for all $h \in \text{cl}_h^\infty X$, all sequences $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ converging to h for the topology on $\text{cl}_h X$, also converge to $\pi(h)$ for the topology on $X \cup \partial_\infty X$. All results below are adaptations of observations made by Benoist and Quint [4, Section 2] to the case of a possibly nonproper Gromov hyperbolic geodesic metric space X .

Lemma 2.1 *There exists $C > 0$, only depending on the hyperbolicity constant of X , such that for all $x, y \in X \cup \text{cl}_h^\infty X$ satisfying $\pi(x) \neq \pi(y)$,*

$$\langle \pi(x)|\pi(y) \rangle_o - C \leq \langle x|y \rangle_o \leq \langle \pi(x)|\pi(y) \rangle_o + C.$$

Given $C, K > 0$, a (C, K) -quasigeodesic in X is a map $\gamma: \mathbb{R} \rightarrow X$ such that

$$\frac{1}{C}|s - t| - K \leq d_X(\gamma(s), \gamma(t)) \leq C|s - t| + K$$

for all $s, t \in \mathbb{R}$.

Proof of Lemma 2.1 The key observation is that there exists a constant $K > 0$, only depending on the hyperbolicity constant of X , such that for all $x, y \in X \cup \text{cl}_h^\infty X$ satisfying $\pi(x) \neq \pi(y)$, there exists a $(1, K)$ -quasigeodesic $\gamma: I \rightarrow X$ (where $I \subseteq \mathbb{R}$ is either an interval, a half-line, or the entire real line) such that $\gamma(t)$ converges to $\pi(x)$ (resp. $\pi(y)$) as t goes to $-\infty$ (resp. $+\infty$). One then notices that up to a bounded additive error, the infimum in the formula defining $(x|y)_o$ can be taken over all points z lying on the image of γ . \square

Lemma 2.2 For all $x, y \in \text{cl}_h^\infty X$ such that $\pi(x) \neq \pi(y)$, there exists $C_{x,y} > 0$ such that, for all $m \in X$,

$$\max(h_x(m), h_y(m)) \geq d_X(o, m) - C_{x,y}.$$

Proof One has to choose $C_{x,y}$ to be sufficiently large compared to twice the distance from o to a quasigeodesic line joining x to y . Details of the proof are an exercise in hyperbolic metric spaces, and left to the reader. \square

Let now G be a group acting by isometries on X . We let

$$\kappa_X(g) := d_X(go, o)$$

for all $g \in G$. As a consequence of (3) from Section 1.2 and Lemma 2.2 applied to $m = g^{-1}o$, we obtain the following fact.

Corollary 2.3 For all $x, y \in \text{cl}_h^\infty X$ such that $\pi(x) \neq \pi(y)$, there exists $C_{x,y} > 0$ such that, for all $g \in G$,

$$\kappa_X(g) - C_{x,y} \leq \max(\beta_X(g, x), \beta_X(g, y)) \leq \kappa_X(g). \quad \square$$

Lemma 2.4 There exists $C > 0$, only depending on the hyperbolicity constant of X , such that for all $g \in G$ and all $x \in \text{cl}_h^\infty X$, one has

$$|(go|gx)_o - \frac{1}{2}(\kappa_X(g) + \beta_X(g, x))| \leq C$$

and

$$|(go|x)_o - \frac{1}{2}(\kappa_X(g) - \beta_X(g^{-1}, x))| \leq C.$$

Proof This follows from the definitions and the hyperbolicity of X by noticing that for all $m \in X$ and all $x \in \text{cl}_h^\infty X$, the infimum in the formula defining $(m|x)_o$ can be taken, up to a bounded additive error, over the points z lying on a $(1, K)$ -quasigeodesic ray from m to $\pi(x)$. \square

2.2 A large deviation principle for the Busemann cocycle

Building on work by Benoist and Quint [4] and Maher and Tiozzo [40], we will establish quantitative deviation results for random walks on groups acting on (possibly nonproper) hyperbolic spaces, under a second moment assumption on the measure. Deviation estimates were also obtained by Mathieu and Sisto [43] under an exponential moment assumption on the measure. Throughout the section, we let X be a separable Gromov hyperbolic geodesic metric space, and G be a countable group acting by isometries on X . We fix a basepoint $o \in X$. A subgroup $H \subseteq G$ is *nonelementary* if it contains two loxodromic isometries of X with disjoint fixed point sets in $\partial_\infty X$. A probability measure μ on G is *nonelementary* if the subsemigroup of G generated by the support of μ is a nonelementary subgroup of G . In this case, the reflected measure $\check{\mu}$ is also nonelementary. We start by recalling the following result of Maher and Tiozzo.

Proposition 2.5 [40, Theorem 1.1] *Let G be a countable group acting by isometries on a separable Gromov hyperbolic geodesic metric space (X, d_X) , let $o \in X$, and let μ be a nonelementary probability measure on G . Then for \mathbb{P} -ae every sample path $\mathbf{g} := (g_n)_{n \in \mathbb{N}}$ of the random walk on (G, μ) , the sequence $(g_n^{-1} \cdot o)_{n \in \mathbb{N}}$ converges to a point $\text{bnd}(\mathbf{g}) \in \partial_\infty X$. The hitting measure ν^* on $\partial_\infty X$ defined by letting*

$$\nu^*(S) = \mathbb{P}[\text{bnd}(\mathbf{g}) \in S]$$

for all measurable subsets $S \subseteq \partial_\infty X$, is nonatomic, and it is the unique $\check{\mu}$ -stationary probability measure on $\partial_\infty X$.

We will first prove a deviation principle for the Busemann cocycle $\beta_X: G \times \text{cl}_h X \rightarrow \mathbb{R}$, under a second moment assumption on μ (Proposition 2.8 below). We recall that $\kappa_X(g) := d_X(g \cdot o, o)$ for all $g \in G$. The following lemma is an extension of [4, Proposition 3.2] to the case where X is no longer assumed to be proper.

Lemma 2.6 *Let G be a countable group acting by isometries on a separable Gromov hyperbolic metric space X , and let μ be a nonelementary probability measure on G . Then for all $\epsilon > 0$, there exists $T > 0$ such that for all $x \in \text{cl}_h^\infty X$, one has*

$$\mathbb{P} \left[\sup_{n \in \mathbb{N}} |\kappa_X(g_n) - \beta_X(g_n, x)| \leq T \right] \leq 1 - \epsilon.$$

Proof The proof is similar to the proof of [4, Proposition 3.2], by using the convergence statement recalled in Proposition 2.5 and the adaptation of the Benoist–Quint estimates given in Section 2.1. The rough idea is to show that if $x \in \text{cl}_h^\infty X$ then, \mathbb{P} -almost surely, one has

$$\sup_{n \in \mathbb{N}} |\kappa_X(g_n) - \beta_X(g_n, x)| < +\infty,$$

from which the lemma follows. This fact is shown by noticing that, \mathbb{P} -almost surely, the sequence $(g_n^{-1}.o)_{n \in \mathbb{N}}$ converges to a boundary point $\text{bnd}(\mathbf{g})$ distinct from x (because the hitting measure is nonatomic), and denoting by z a coarse center for the triangle made of o, x and $\text{bnd}(\mathbf{g})$. Then $|\kappa_X(g_n) - \beta_X(g_n, x)|$ is equal, up to a bounded additive error, to $2d_X(o, z)$, for all sufficiently large $n \in \mathbb{N}$. □

Corollary 2.7 *Let G be a countable group acting by isometries on a separable Gromov hyperbolic metric space X , and let μ be a nonelementary probability measure on G with finite first moment with respect to d_X . Let λ_X be the drift of the random walk on (G, μ) with respect to d_X . Then for all μ -stationary probability measures θ on $\text{cl}_h X$,*

$$\int_{G \times \text{cl}_h X} \beta_X(g, x) d\mu(g) d\theta(x) = \lambda_X.$$

Proof Lemma 2.6 implies that for all $x \in \text{cl}_h^\infty X$ and \mathbb{P} -ae sample path $(g_n)_{n \in \mathbb{N}}$ of the random walk on (G, μ) , one has

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \beta_X(g_n, x) = \lambda_X.$$

By [40, Proposition 4.4], all μ -stationary probability measures on $\text{cl}_h X$ are supported on $\text{cl}_h^\infty X$. As β_X is a cocycle, Corollary 2.7 follows by applying Birkhoff’s ergodic theorem. □

The following proposition can be viewed both as an extension of [4, Proposition 4.1] to the case of a random walk on a group acting by isometries on a (not necessarily proper) hyperbolic space, and as an extension of [40, Theorem 1.2] to the case of a measure with finite second moment with respect to d_X .

Proposition 2.8 (Benoist and Quint [4, Proposition 4.1]) *Let G be a countable group acting by isometries on a separable Gromov hyperbolic geodesic metric space X , and let μ be a nonelementary probability measure on G with finite second moment with respect to d_X . Let λ_X be the drift of the random walk on (G, μ) with respect to d_X . Then for all $\epsilon > 0$, there exists a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $x \in \text{cl}_h X$*

and all $n \in \mathbb{N}$, one has

$$\mu^{*n}(\{g \in G : |\beta_X(g, x) - n\lambda_X| \geq \epsilon n\}) \leq C_n$$

and

$$\mu^{*n}(\{g \in G : |\kappa_X(g) - n\lambda_X| \geq \epsilon n\}) \leq C_n.$$

Proof The deviation principle for the Busemann cocycle β_X follows from [Corollary 2.7](#) and the large deviation principle for cocycles of Benoist and Quint ([Proposition 1.3](#)). Notice that metrizable of $\text{cl}_h X$ was established in [\[40, Proposition 3.1\]](#). The deviation principle for the function κ_X then follows from the deviation principle for the Busemann cocycle by using [Corollary 2.3](#). \square

2.3 Sublinear tracking

Using the fact that $\lambda_X > 0$ by [\[40, Theorem 1.2\]](#), and arguing as in [\[4, Lemma 4.5\]](#), one can deduce the following estimate from [Proposition 2.8](#).

Lemma 2.9 (Benoist and Quint [\[4, Lemma 4.5\]](#)) *Let G be a countable group acting by isometries on a separable Gromov hyperbolic geodesic metric space X , and let μ be a nonelementary probability measure on G with finite second moment with respect to d_X . Let ν be the unique μ -stationary probability measure on $\partial_\infty X$. Then for all $\alpha > 0$, there exists a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $n \in \mathbb{N}$ and all $x, y \in X \cup \partial_\infty X$, one has*

$$\mu^{*n}(\{g \in G \mid \langle go|y \rangle_o \geq \alpha n\}) \leq C_n$$

and

$$\mu^{*n}(\{g \in G \mid \langle gx|y \rangle_o \geq \alpha n\}) \leq C_n,$$

and hence

$$\nu(\{x \in \partial_\infty X \mid \langle x|y \rangle_o \geq \alpha n\}) \leq C_n.$$

Remark 2.10 [Corollary 2.7](#) guarantees that hypothesis (H1) from [Theorem 1.6](#) is satisfied, and [Lemma 2.9](#) implies that hypothesis (H2) is also satisfied. One can deduce that the Busemann cocycle β_X and the function κ_X both satisfy a central limit theorem, extending the central limit theorem for hyperbolic groups of Benoist and Quint [\[4\]](#) to nonproper settings.

We recall from [Proposition 2.5](#) that for \mathbb{P} -ae sample path $\mathbf{g} = (g_n)_{n \in \mathbb{N}}$ of the random walk on (G, μ) , the sequence $(g_n^{-1}.o)_{n \in \mathbb{N}}$ converges to a boundary point $\text{bnd}(\mathbf{g}) \in \partial_\infty X$. We will denote by $\tau_{\mathbf{g}, X}$ a $(1, K)$ -quasigeodesic ray from o to $\text{bnd}(\mathbf{g})$ (where K only depends on the hyperbolicity constant of X). We now obtain the following quantitative version of sublinear tracking under a second moment assumption on the measure μ .

Proposition 2.11 *Let G be a countable group acting by isometries on a separable Gromov hyperbolic geodesic metric space X , and let μ be a nonelementary probability measure on G with finite second moment with respect to d_X .*

Then for all $\epsilon > 0$, there exists a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that

$$\mathbb{P}[d_X(g_n^{-1}.o, \tau_{\mathbf{g}, X}(\mathbb{R}_+)) \geq \epsilon n] \leq C_n.$$

Proof The proof follows the argument from the proof of [40, Proposition 5.7], using the quantitative estimate from Lemma 2.9; it goes as follows. There exists $K > 0$, only depending on the hyperbolicity constant of X , such that

$$|d_X(g_n^{-1}.o, \tau_{\mathbf{g}, X}(\mathbb{R}_+)) - \langle o | \text{bnd}(\mathbf{g}) \rangle_{g_n^{-1}.o}| \leq K,$$

and hence

$$|d_X(g_n^{-1}.o, \tau_{\mathbf{g}, X}(\mathbb{R}_+)) - \langle g_n.o | g_n.\text{bnd}(\mathbf{g}) \rangle_o| \leq K.$$

Therefore, the condition

$$d_X(g_n^{-1}.o, \tau_{\mathbf{g}, X}(\mathbb{R}_+)) \geq \epsilon n$$

implies that

$$\langle g_n.o | g_n.\text{bnd}(\mathbf{g}) \rangle_o \geq \epsilon n - K.$$

In addition, the boundary point $g_n.\text{bnd}(\mathbf{g})$ is independent from g_n , and its distribution is given by the hitting measure ν^* . Conditioning over the value of $g_n.\text{bnd}(\mathbf{g})$, we get

$$\mathbb{P}[d_X(g_n^{-1}.o, \tau_{\mathbf{g}, X}(\mathbb{R})) \geq \epsilon n] \leq \int_{\partial_\infty X} \mu^{*n}(\{g \in G \mid \langle go|y \rangle_o \geq \epsilon n - K\}) d\nu^*(y).$$

Proposition 2.11 therefore follows from Lemma 2.9. □

Given $\kappa > 0$, two quasigeodesic segments $\gamma: [a, b] \rightarrow X$ and $\gamma': [a', b'] \rightarrow X$ are said to *fellow travel up to distance κ* if there exists an increasing homeomorphism $\theta: [a, b] \rightarrow [a', b']$ such that $d_X(\gamma(t), \gamma' \circ \theta(t)) \leq \kappa$ for all $t \in [a, b]$. As a consequence of Lemma 2.9, one also gets the following fact.

Proposition 2.12 *Let G be a countable group acting by isometries on a separable Gromov hyperbolic geodesic metric space X , and let μ be a nonelementary probability measure on G with finite second moment with respect to d_X . Let λ_X be the drift of the random walk on (G, μ) with respect to d_X .*

Then for all $\beta > 0$, there exists a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$, such that for all $x \in X \cup \partial_\infty X$, one has

$$\mathbb{P}[\langle x | \text{bnd}(\mathbf{g}) \rangle_o \leq \frac{1}{2}\beta n] \geq 1 - C_n.$$

In particular, if $K > 0$, then with probability at least $1 - C_n$, any (K, K) -quasigeodesic from x to $\text{bnd}(\mathbf{g})$ contains a subsegment that fellow travels $\tau_{\mathbf{g}, X}([\beta n, +\infty))$ up to a distance κ that only depends on K and on the hyperbolicity constant of X . \square

3 Central limit theorem on mapping class groups

The goal of this section is to establish a central limit theorem in the context of mapping class groups of surfaces (Theorem 0.1).

3.1 Background on mapping class groups

Let S be a closed, connected, oriented, hyperbolic surface. The *mapping class group* $\text{Mod}(S)$ is defined as the group of all isotopy classes of orientation-preserving diffeomorphisms of S . We start by reviewing classical material on mapping class groups.

Teichmüller space and two of its metrics The *Teichmüller space* $\mathcal{T}(S)$ is the space of isotopy classes of conformal structures on S . Up to isotopy, there is a unique hyperbolic metric on S in a given conformal class, so $\mathcal{T}(S)$ can alternatively be defined as the space of isotopy classes of hyperbolic metrics on S . We review the definition of two metrics on $\mathcal{T}(S)$.

The *Teichmüller metric* is defined by letting

$$d_{\mathcal{T}}(x, y) := \frac{1}{2} \inf_f \log K(f)$$

for all $x, y \in \mathcal{T}(S)$, where the infimum is taken over the collection of all quasiconformal maps f from (S, x) to (S, y) , and $K(f)$ denotes the quasiconformal dilatation of f . The Teichmüller metric is uniquely geodesic: any two points in $\mathcal{T}(S)$ are joined by a unique geodesic segment. This metric can alternatively be described in terms of ratios of extremal lengths of curves, as follows. A simple closed curve c on S is *essential* if it does not bound a disk on S . Given an essential simple closed curve c on S and $x \in \mathcal{T}(S)$, the *extremal length* of c in the conformal structure x is

$$\text{Ext}_x(c) = \sup_{\rho} \frac{l_{\rho}(c)^2}{\text{Area}(\rho)},$$

where the supremum is taken over all metrics ρ in the conformal class x , where $l_{\rho}(c)$ denotes the infimal ρ -length of a curve isotopic to c , and $\text{Area}(\rho)$ is the area of S equipped with the metric ρ . We denote by \mathcal{S} the collection of all isotopy classes of

essential simple closed curves on S . Kerckhoff proved in [34, Theorem 4] that for all $x, y \in \mathcal{T}(S)$,

$$d_{\mathcal{T}}(x, y) = \frac{1}{2} \log \sup_{c \in \mathcal{S}} \frac{\text{Ext}_y(c)}{\text{Ext}_x(c)}.$$

Thurston’s asymmetric metric is defined by letting

$$d_{\text{Th}}(x, y) := \inf_f \log \text{Lip}(f)$$

for all $x, y \in \mathcal{T}(S)$, where the infimum is taken over the collection of all Lipschitz maps f from (S, x) to (S, y) , and $\text{Lip}(f)$ denotes the Lipschitz constant of f . This metric can also be described using hyperbolic lengths of curves: Thurston established that

$$d_{\text{Th}}(x, y) = \log \sup_{c \in \mathcal{S}} \frac{l_y(c)}{l_x(c)}$$

for all $x, y \in \mathcal{T}(S)$, where l_x denotes the length measured in the unique hyperbolic metric in the conformal class x . The next proposition, due to Lenzhen, Rafi and Tao, states that, up to a bounded additive error, one can actually take the supremum over a finite collection of curves in the above formula. Let $x \in \mathcal{T}(S)$. Following [38, Section 2], we define a *marking* μ_x on (S, x) to be a collection of curves of S , made of both a finite set \mathcal{P} of pairwise disjoint curves on S that cut S into a finite collection of pairs of pants, and a set of transverse curves \mathcal{Q} that satisfy the following property: each curve $\alpha \in \mathcal{P}$ intersects exactly one curve $\beta \in \mathcal{Q}$, and β intersects α minimally, and does not intersect any other curve in \mathcal{P} . The marking μ_x is a *short marking* if \mathcal{P} is constructed by first picking a curve c on S that minimizes $l_x(c)$, then a second shortest curve, and so on, and curves in \mathcal{Q} are then chosen to be as short as possible. Notice that a short marking on (S, x) is not unique (but the set of all curves that can appear in a short marking on (S, x) defines a bounded set of the curve graph of S , whose definition is recalled a bit later).

Proposition 3.1 (Lenzhen, Rafi and Tao [38, Theorem E]) *There exists $K > 0$ (only depending on the topological type of S) such that for all $x, y \in \mathcal{T}(S)$, one has*

$$\left| d_{\text{Th}}(x, y) - \log \max_{c \in \mu_x} \frac{l_y(c)}{l_x(c)} \right| \leq K,$$

where the maximum is taken over the collection of all curves in a short marking μ_x on (S, x) .

Given $\epsilon > 0$, the ϵ -thick part $\mathcal{T}(S)^\epsilon$ is the subspace of $\mathcal{T}(S)$ made of those hyperbolic metrics for which no essential simple closed curve on S has length smaller than ϵ . The two metrics defined above are comparable in restriction to the thick part of $\mathcal{T}(S)$.

Proposition 3.2 (Choi and Rafi [10, Theorem B]) *For all $\epsilon > 0$, there exists $C = C(\epsilon) > 0$ such that for all $x, y \in \mathcal{T}(S)^\epsilon$, one has $|d_{\mathcal{T}}(x, y) - d_{\text{Th}}(x, y)| \leq C$.*

The Thurston boundary and the Gardiner–Masur boundary Thurston defined a compactification of $\mathcal{T}(S)$, as the closure of the image of the embedding

$$\mathcal{T}(S) \rightarrow \mathbb{P}\mathbb{R}^S, \quad x \mapsto \mathbb{R}^*(l_x(c))_{c \in \mathcal{S}},$$

and he identified the boundary $\overline{\mathcal{T}(S)} \setminus \mathcal{T}(S)$ with the space \mathcal{PMF} of projective Whitehead equivalence classes of measured foliations on S ; see [17]. We denote by \mathcal{MF} the space of (unprojectivized) Whitehead equivalence classes of measured foliations on S . The length pairing between curves in S and points in $\mathcal{T}(S)$ extends to continuous intersection pairings (denoted by i) from $\mathcal{MF} \times \mathcal{T}(S)$ to \mathbb{R} and from $\mathcal{MF} \times \mathcal{MF}$ to \mathbb{R} .

A measured foliation F on S is *arational* if no leaf of F contains a simple closed curve on S . It is *uniquely ergodic* if in addition, every measured foliation $F' \in \mathcal{MF}$ with the same topological support as F is homothetic to F . We will denote by $\mathcal{UE} \subseteq \mathcal{PMF}$ the space of uniquely ergodic arational foliations, and let $\mathcal{PMF}_0 := \mathcal{UE} \cup \mathcal{S}$. Given any two transverse measured foliations $x, y \in \mathcal{PMF}_0$, there exists a Teichmüller geodesic $\gamma: \mathbb{R} \rightarrow \mathcal{T}(S)$ such that $\gamma(t)$ converges to x (resp. to y) as t goes to $-\infty$ (resp. $+\infty$).

Gardiner and Masur have constructed [21] another compactification $\text{cl}_{\text{GM}} \mathcal{T}(S)$ of $\mathcal{T}(S)$, using extremal lengths instead of hyperbolic lengths, by taking the closure of the image of the embedding

$$\mathcal{T}(S) \rightarrow \mathbb{P}\mathbb{R}^S, \quad x \mapsto \mathbb{R}^*(\text{Ext}_x(c))_{c \in \mathcal{S}}$$

in the projective space $\mathbb{P}\mathbb{R}^S$. We will denote by $\partial_{\text{GM}} \mathcal{T}(S) := \text{cl}_{\text{GM}} \mathcal{T}(S) \setminus \mathcal{T}(S)$ the *Gardiner–Masur boundary*. Liu and Su have identified the horoboundary of $(\mathcal{T}(S), d_{\mathcal{T}})$ with the Gardiner–Masur boundary [39]. There exists an injective map from \mathcal{PMF} to $\partial_{\text{GM}} \mathcal{T}(S)$, whose restriction to \mathcal{PMF}_0 is a homeomorphism onto its image [45, Theorem 2]. In particular, the Busemann cocycle β on $\partial_{\text{GM}} \mathcal{T}(S)$ restricts to a continuous cocycle (again denoted by β) on \mathcal{PMF}_0 . Miyachi also proved [45, Corollary 1] that for all $F \in \mathcal{PMF}_0$, all Teichmüller rays with vertical foliation equal to F converge to F in $\partial_{\text{GM}} \mathcal{T}(S)$. For all $F \in \mathcal{PMF}_0$, the horofunction h_F associated to F is given, for all $z \in \mathcal{T}(S)$, by

$$h_F(z) = \log \sup_{\alpha \in \mathcal{S}} \frac{i(F, \alpha)}{\text{Ext}_z(\alpha)^{1/2}} - \log \sup_{\alpha \in \mathcal{S}} \frac{i(F, \alpha)}{\text{Ext}_o(\alpha)^{1/2}}.$$

The curve graph The *curve graph* $\mathcal{C}(S)$ is the simplicial graph whose vertices are the isotopy classes of essential simple closed curves on S , in which two vertices are joined by an edge whenever there are disjoint representatives in the isotopy classes

of the corresponding curves. We denote by d_C the simplicial metric on $\mathcal{C}(S)$. Masur and Minsky proved in [42] that $(\mathcal{C}(S), d_C)$ is Gromov hyperbolic, and that an element $\Phi \in \text{Mod}(S)$ acts loxodromically on $\mathcal{C}(S)$ if and only if Φ is a pseudo-Anosov mapping class. There is a coarsely $\text{Mod}(S)$ -equivariant, coarsely Lipschitz map $\pi: \mathcal{T}(S) \rightarrow \mathcal{C}(S)$, which sends every point $x \in \mathcal{T}(S)$ to the isotopy class of one of the essential simple closed curves with minimal hyperbolic length in (S, x) . Masur and Minsky also proved in [42] that π -images of Teichmüller geodesics are uniform unparametrized quasigeodesics in $\mathcal{C}(S)$.

The Gromov boundary of the curve graph was identified by Klarreich [36] with the space of equivalence classes of arational foliations, two arational foliations being equivalent if they have the same topological support, and only differ by their transverse measure. Klarreich also proved that there is a well-defined $\text{Mod}(S)$ -equivariant map

$$\psi: \mathcal{PMF}_0 \rightarrow \mathcal{C}(S) \cup \partial_\infty \mathcal{C}(S)$$

sending any element in \mathcal{S} to the corresponding vertex of $\mathcal{C}(S)$, and such that for all $x \in \mathcal{UE}$ and all sequences $(x_n)_{n \in \mathbb{N}} \in \mathcal{T}(S)^\mathbb{N}$ converging to x , the sequence $(\pi(x_n))_{n \in \mathbb{N}}$ converges to $\psi(x) \in \partial_\infty \mathcal{C}(S)$.

Random walks on mapping class groups We finish this section by reviewing a result of Kaimanovich and Masur [30] about random walks on $\text{Mod}(S)$. A subgroup $H \subseteq \text{Mod}(S)$ is *nonelementary* if it contains two pseudo-Anosov mapping classes that generate a free subgroup of H . A probability measure on $\text{Mod}(S)$ is *nonelementary* if the subsemigroup of $\text{Mod}(S)$ generated by the support of μ is a nonelementary subgroup of $\text{Mod}(S)$. In view of the definition of nonelementarity in Section 2, this is equivalent to nonelementarity with respect to the action on the curve graph $\mathcal{C}(S)$.

Theorem 3.3 (Kaimanovich and Masur [30, Theorem 2.2.4]) *Let μ be a nonelementary probability measure on $\text{Mod}(S)$. Then for \mathbb{P} -ae sample path $\Phi := (\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Mod}(S), \mu)$, the sequence $(\Phi_n^{-1} \cdot o)_{n \in \mathbb{N}}$ converges to a point $\text{bnd}(\Phi) \in \mathcal{UE}$. The hitting measure ν^* on \mathcal{PMF} , defined by letting*

$$\nu^*(S) = \mathbb{P}[\text{bnd}(\Phi) \in S]$$

for all measurable subsets $S \subseteq \mathcal{PMF}$, is nonatomic, and it is the unique $\check{\mu}$ -stationary measure on \mathcal{PMF} .

3.2 Relating the length cocycle to the Busemann cocycle

Let $\sigma: \text{Mod}(S) \times \mathcal{S} \rightarrow \mathbb{R}$ be the *length cocycle*, defined by letting

$$\sigma(\Phi, c) := \log \frac{l_o(\Phi(c))}{l_o(c)}$$

for all $\Phi \in \text{Mod}(S)$ and all $c \in \mathcal{S}$, where o is a fixed basepoint in $\mathcal{C}(S)$. The following proposition will enable us to get control over the length cocycle in terms of the Busemann cocycle. We recall that h_c denotes the horofunction in $\partial_h \mathcal{T}(S)$ associated to c .

Proposition 3.4 *For all $\epsilon > 0$, there exists $C > 0$ such that for all $c \in \mathcal{S}$ and all $z \in \mathcal{T}(S)^\epsilon$, one has*

$$\left| h_c(z) - \log \frac{l_z(c)}{l_o(c)} \right| \leq C.$$

Proof It follows from work by Minsky [44, Lemma 4.3] that there exists $C_1 > 0$ such that for all $z \in \mathcal{T}(S)^\epsilon$ and all $c \in \mathcal{S}$, one has

$$\left| \log(l_z(c)) - \frac{1}{2} \log(\text{Ext}_z(c)) \right| \leq C_1.$$

Hence for all $z \in \mathcal{T}(S)^\epsilon$ and all $c \in \mathcal{S}$, one has

$$\left| h_c(z) - \left(\log \sup_{\alpha \in \mathcal{S}} \frac{i(c, \alpha)}{l_z(\alpha)} - \log \sup_{\alpha \in \mathcal{S}} \frac{i(c, \alpha)}{l_o(\alpha)} \right) \right| \leq 2C_1.$$

By Proposition 3.1, together with the fact that $i(\alpha, y) = l_y(\alpha)$ for all $y \in \mathcal{T}(S)$ and all $\alpha \in \mathcal{S}$, we know that there exists $C_2 > 0$ such that

$$\left| \log \sup_{\alpha \in \mathcal{S}} \frac{i(\alpha, y)}{l_x(\alpha)} - \log \sup_{\alpha \in \mu_x} \frac{i(\alpha, y)}{l_x(\alpha)} \right| \leq C_2$$

for all $y \in \mathcal{T}(S)$. Using the continuity of the extension of the intersection form to the boundary, we will show that this is also true if $y \in \mathcal{T}(S)$ is replaced by $\xi \in \mathcal{MF}$: indeed, if $\alpha_0 \in \mathcal{S}$, then there exists $y \in \mathcal{T}(S)$ such that

$$\left| \log \frac{i(\xi, \alpha_0)}{l_x(\alpha_0)} - \log \frac{i(y, \alpha_0)}{l_x(\alpha_0)} \right| \leq C_2$$

and

$$\left| \log \sup_{\alpha \in \mu_x} \frac{i(\xi, \alpha)}{l_x(\alpha)} - \log \sup_{\alpha \in \mu_x} \frac{i(y, \alpha)}{l_x(\alpha)} \right| \leq C_2.$$

Therefore,

$$\log \frac{i(\xi, \alpha_0)}{l_x(\alpha_0)} - \log \sup_{\alpha \in \mu_x} \frac{i(\xi, \alpha)}{l_x(\alpha)} \leq 3C_2,$$

and since this is true for all $\alpha_0 \in \mathcal{S}$ we get

$$0 \leq \log \sup_{\alpha \in \mathcal{S}} \frac{i(\alpha, \xi)}{l_x(\alpha)} - \log \sup_{\alpha \in \mu_x} \frac{i(\alpha, \xi)}{l_x(\alpha)} \leq 3C_2.$$

By applying this observation to both $x = z$ and $x = o$, we get the existence of $C_3 > 0$ such that for all $z \in \mathcal{T}(S)^\epsilon$ and all $c \in \mathcal{S}$, one has

$$\left| h_c(z) - \left(\log \sup_{\alpha \in \mu_z} \frac{i(c, \alpha)}{l_z(\alpha)} - \log \sup_{\alpha \in \mu_o} \frac{i(c, \alpha)}{l_o(\alpha)} \right) \right| \leq C_3.$$

In addition, when $z \in \mathcal{T}(S)^\epsilon$, all curves in a short marking μ_z (including the transverse curves) coarsely have hyperbolic length 1, up to a bounded multiplicative error; see eg [48, Section 3.1]. Therefore, there exists $C_4 > 0$ such that for all $z \in \mathcal{T}(S)^\epsilon$ and all $c \in \mathcal{S}$, one has

$$(5) \quad \left| h_c(z) - \log \frac{\sup_{\alpha \in \mu_z} i(c, \alpha)}{\sup_{\alpha' \in \mu_o} i(c, \alpha')} \right| \leq C_4.$$

Finally, in view of [38, Proposition 3.1], there exists $C_5 > 0$ such that for all $z \in \mathcal{T}(S)$ and all $c \in \mathcal{S}$, one has

$$\frac{1}{C_5} \sum_{\alpha \in \mu_z} i(c, \alpha) l_z(\bar{\alpha}) \leq l_z(c) \leq C_5 \sum_{\alpha \in \mu_z} i(c, \alpha) l_z(\bar{\alpha}),$$

where the curve $\bar{\alpha}$ is the transverse curve to the curve α to the short marking μ_z . One derives that

$$\frac{1}{C_5} \sup_{\alpha \in \mu_z} (i(c, \alpha) l_z(\bar{\alpha})) \leq l_z(c) \leq C_5 \left(\sum_{\alpha \in \mu_z} l_z(\bar{\alpha}) \right) \sup_{\alpha \in \mu_z} i(c, \alpha).$$

Using again the fact that all curves in μ_z coarsely have length 1, one can therefore deduce that there exists $C_6 > 0$ such that for all $z \in \mathcal{T}(S)^\epsilon$ and all $c \in \mathcal{S}$, we have

$$(6) \quad \left| \log(l_z(c)) - \log \left(\sup_{\alpha \in \mu_z} i(c, \alpha) \right) \right| \leq C_6.$$

The claim follows from the estimates (5) and (6). □

As a consequence of Proposition 3.4 applied to $z = \Phi^{-1}o$, we obtain the following:

Corollary 3.5 *There exists $C > 0$ such that for all $\Phi \in \text{Mod}(S)$ and all $c \in \mathcal{S}$,*

$$|\beta(\Phi, c) - \sigma(\Phi, c)| \leq C. \quad \square$$

3.3 A deviation principle in Teichmüller space

In [33], Karlsson established a law of large numbers for $\text{Mod}(S)$. This is stated in the case where $x \in \mathcal{S}$ is a simple closed curve on S , however Karlsson’s proof extends to the case where $x \in \mathcal{MF}$. We fix once and for all a basepoint $o \in \mathcal{T}(S)$. We recall that i denotes the intersection form on $\mathcal{MF} \times \mathcal{T}(S)$.

Theorem 3.6 (Karlsson [33, Corollary 4]) *Let μ be a nonelementary probability measure on $\text{Mod}(S)$ with finite first moment with respect to $d_{\mathcal{T}}$, and let λ be the drift of the random walk on $(\text{Mod}(S), \mu)$ with respect to $d_{\mathcal{T}}$. Then for all $x \in \mathcal{MF}$ and \mathbb{P} -ae sample path of the random walk on $(\text{Mod}(S), \mu)$, one has*

$$\lambda = \lim_{n \rightarrow +\infty} \frac{1}{n} \log i(\Phi_n \cdot x, o).$$

We deduce the following deviation estimate for the realization in $\mathcal{T}(S)$ of the random walk on $(\text{Mod}(S), \mu)$.

Proposition 3.7 *Let μ be a nonelementary probability measure on $\text{Mod}(S)$ with finite second moment with respect to $d_{\mathcal{T}}$, and let λ be the drift of the random walk on $(\text{Mod}(S), \mu)$ with respect to $d_{\mathcal{T}}$. Then for every $\epsilon > 0$, there exists a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $n \in \mathbb{N}$ and all $c \in S$, one has*

$$\mu^{*n} \left(\left\{ \Phi \in \text{Mod}(S) : \left| \log \frac{l_o(\Phi(c))}{l_o(c)} - n\lambda \right| \geq \epsilon n \right\} \right) \leq C_n.$$

Proof Let $\sigma: \text{Mod}(S) \times \mathcal{PMF} \rightarrow \mathbb{R}$ be the continuous cocycle defined by

$$\sigma(\Phi, x) := \log \frac{i(\Phi, x, o)}{i(x, o)}$$

for all $\Phi \in \text{Mod}(S)$ and all $x \in \mathcal{PMF}$. [Theorem 3.6](#) states that for all $x \in \mathcal{PMF}$ and \mathbb{P} -ae sample path of the random walk on $(\text{Mod}(S), \mu)$, one has

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sigma(\Phi_n, x) = \lambda.$$

Birkhoff’s ergodic theorem, in the form provided by [Proposition 1.4](#), then implies that

$$\int_{\text{Mod}(S) \times \mathcal{PMF}} \sigma(\Phi, x) d\mu(\Phi) d\nu(x) = \lambda,$$

where ν denotes the unique μ -stationary probability measure on \mathcal{PMF} . [Proposition 3.7](#) follows by applying the deviation estimate of Benoist and Quint ([Proposition 1.3](#)) to the cocycle σ (square-integrability of σ_{sup} follows from the fact that $\sigma_{\text{sup}}(\Phi, x) = d_{\text{Th}}(\Phi, o, o)$ for all $\Phi \in \text{Mod}(S)$). \square

We now establish an analogue of [Proposition 2.8](#) for the realization in $\mathcal{T}(S)$ of the random walk on $(\text{Mod}(S), \mu)$. In the following statement, for all $\Phi \in \text{Mod}(S)$, we let

$$\kappa_{\mathcal{T}}(\Phi) := d_{\mathcal{T}}(\Phi, o, o).$$

Proposition 3.8 *Let μ be a nonelementary probability measure on $\text{Mod}(S)$ with finite second moment with respect to $d_{\mathcal{T}}$. Let λ be the drift of the random walk on $(\text{Mod}(S), \mu)$ with respect to $d_{\mathcal{T}}$. Then for every $\epsilon > 0$, there exists a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $n \in \mathbb{N}$,*

$$\mu^{*n} (\{ \Phi \in \text{Mod}(S) : |\kappa_{\mathcal{T}}(\Phi) - n\lambda| \geq \epsilon n \}) \leq C_n.$$

Proof In view of [Proposition 3.2](#), it is enough to prove the analogous statement where $d_{\mathcal{T}}$ is replaced by d_{Th} in the definition of $\kappa_{\mathcal{T}}$. [Proposition 3.8](#) therefore follows from [Proposition 3.7](#) applied to each of the finitely many curves in μ_o given by [Proposition 3.1](#). \square

3.4 Lifting estimates from $\mathcal{C}(S)$ to $\mathcal{T}(S)$

3.4.1 Deviation estimates for the Gromov product: hypothesis (H2) We will now check hypothesis (H2) from [Theorem 1.6](#) for the Gromov product on the horoboundary of $\mathcal{T}(S)$.

Proposition 3.9 *Let μ be a nonelementary probability measure on $\text{Mod}(S)$, and let ν^* be the unique $\check{\mu}$ -stationary probability measure on \mathcal{PMF}_0 . Then there exist $\alpha > 0$ and a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $x \in \mathcal{PMF}_0$, one has*

$$\nu^*(\{y \in \mathcal{PMF}_0 \mid (x|y)_o \geq \alpha n\}) \leq C_n.$$

The strategy of our proof of [Proposition 3.9](#) will consist in *lifting* to $\mathcal{T}(S)$ the analogous estimate for the Gromov product on $\mathcal{C}(S)$. In order to make the lifting argument possible, we will appeal to a contraction property for typical Teichmüller geodesics. A similar strategy was already used in [\[14\]](#) for proving a spectral theorem for the random walk on $\text{Mod}(S)$.

Let $K > 0$ be a constant such that all π -images of Teichmüller geodesics are (K, K) -unparametrized quasigeodesics in $\mathcal{C}(S)$. We fix once and for all a large enough constant $\kappa > 0$ such that all triangles in $\mathcal{C}(S)$ whose sides are (K, K) -quasigeodesics, are κ -thin (in particular, κ is assumed to satisfy the conclusion of [Proposition 2.12](#)). We equip $\mathcal{T}(S)$ with the Teichmüller metric $d_{\mathcal{T}}$. Let I be a Teichmüller geodesic segment. Given $B, C > 0$, we say that I is (B, C) -*progressing* if $\text{diam}_{\mathcal{T}(S)}(I) \leq B$ and $\text{diam}_{\mathcal{C}(S)}(\pi(I)) \geq C$. Given $D, \tau > 0$, we say that I is (D, τ) -*contracting* if for all geodesic segments J in $\mathcal{T}(S)$, if $\pi(I)$ and $\pi(J)$ fellow travel up to distance κ in $\mathcal{C}(S)$ (with a slight abuse of terminology, as we are identifying $\pi(I)$ and $\pi(J)$ with their parametrizations), then there exists $J_1 \subseteq J$ at $d_{\mathcal{T}}$ -Hausdorff distance at most D from I , such that $\pi(J_1)$ has diameter at least $\text{diam}(\pi(I)) - \tau$ in $\mathcal{C}(S)$. The following proposition, established in [\[14\]](#), essentially follows from work by Dowdall, Duchin and Masur.

Proposition 3.10 (Dowdall, Duchin and Masur [\[15, Theorem A\]](#); Dahmani and Horbez [\[14, Proposition 3.6\]](#)) *There exist $C_0, \tau > 0$ such that for all $B > 0$, there exists $D > 0$ such that for all $C > C_0$, all (B, C) -progressing Teichmüller geodesic segments are (D, τ) -contracting.*

We fix once and for all the constants $C_0, \tau > 0$ given by [Proposition 3.10](#). We say that I is D -*supercontracting* if for all geodesic segments J in $\mathcal{T}(S)$, if $\pi(I)$ and $\pi(J)$ fellow travel up to distance κ in $\mathcal{C}(S)$, then J contains a (D, τ) -contracting subsegment.

Corollary 3.11 *There exists $C > 0$ such that for all $B > 0$, there exists $D > 0$ such that all (B, C) -progressing Teichmüller geodesic segments are D -supercontracting.*

Proof Let $C > C_0 + \tau$. If I is (B, C) -progressing and J is such that $\pi(I)$ and $\pi(J)$ fellow travel up to distance κ , then in view of Proposition 3.10, there exists a subsegment $J_1 \subseteq J$ at $d_{\mathcal{T}}$ -Hausdorff distance at most D from I , whose π -image has diameter at least $C - \tau$. In particular, the segment J_1 is $(B + 2D, C - \tau)$ -progressing. Since $C - \tau > C_0$, Proposition 3.10 applies to J_1 , showing that I is D' -supercontracting for some $D' > 0$ only depending on B and C . \square

From now on, we fix $C > 0$ provided by Corollary 3.11. We now assume that the basepoint in $\mathcal{C}(S)$ is the π -image of the basepoint in $\mathcal{T}(S)$ (we will denote both of them by o). We recall from Section 2.3 that $\tau_{\Phi, \mathcal{C}(S)}$ denotes a $(1, K)$ -quasigeodesic ray (where K is a universal constant) from o to the limit point in $\partial_{\infty} \mathcal{C}(S)$ of the sequence $(\Phi_n^{-1} \cdot o)_{n \in \mathbb{N}}$.

Proposition 3.12 *Let μ be a nonelementary probability measure on $\text{Mod}(S)$ with finite second moment with respect to $d_{\mathcal{T}}$.*

Then there exist constants $B, \beta > 0$ and a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that the probability that the Teichmüller segment $[o, \Phi_n^{-1} \cdot o]$ contains a (B, C) -progressing subsegment whose π -image in $\mathcal{C}(S)$ fellow travels a subsegment of $\tau_{\Phi, \mathcal{C}(S)}([\beta n, +\infty))$ up to distance κ , is at least $1 - C_n$.

Proof We denote by $\lambda_{\mathcal{C}}$ (resp. $\lambda_{\mathcal{T}}$) the drift of the random walk on $(\text{Mod}(S), \mu)$ with respect to $d_{\mathcal{C}}$ (resp. $d_{\mathcal{T}}$). We let B be a positive real number greater than $5C\lambda_{\mathcal{T}}/\lambda_{\mathcal{C}}$. We denote by $\gamma_{n, \Phi}: [0, \kappa_{\mathcal{T}}(\Phi_n)] \rightarrow \mathcal{T}(S)$ the parametrization of the Teichmüller segment from o to $\Phi_n^{-1} \cdot o$.

We first claim that there exists a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $n \in \mathbb{N}$, with probability at least $1 - C_n$, the following hold:

- $\kappa_{\mathcal{T}}(\Phi_n) \leq \lambda_{\mathcal{T}}n + \frac{\lambda_{\mathcal{C}}Bn}{5C} - B$.
- $\kappa_{\mathcal{C}}(\Phi_n) > \frac{4}{5}\lambda_{\mathcal{C}}n$.
- $\pi \circ \gamma_{n, \Phi}([0, t_n^2(\Phi)])$ fellow travels a subsegment of $\tau_{\Phi, \mathcal{C}(S)}$ up to distance κ , where $t_n^2(\Phi) > 0$ is the infimum of all real numbers such that $\pi \circ \gamma_{n, \Phi}([0, t_n^2(\Phi)])$ has $d_{\mathcal{C}}$ -diameter at least $\frac{4}{5}\lambda_{\mathcal{C}}n$.

The first conclusion can be ensured by Proposition 3.8, the second by Proposition 2.8, and the third by Proposition 2.11.

Now, we claim that in this situation, denoting by $t_n^1(\Phi) > 0$ the infimum of all real numbers such that $\pi \circ \gamma_{n, \Phi}([0, t_n^1(\Phi)])$ has $d_{\mathcal{C}}$ -diameter at least $\frac{1}{5}\lambda_{\mathcal{C}}n$, the segment $\gamma_{n, \Phi}([t_n^1(\Phi), t_n^2(\Phi)])$ contains a (B, C) -progressing subsegment whose π -image

fellow travels a subsegment of $\tau_{\Phi, \mathcal{C}(S)}$ up to distance κ . Proposition 3.12 will follow from this claim (with $\beta := \frac{1}{5}\lambda_C$).

To prove the claim, we subdivide the Teichmüller segment $\gamma_{n, \Phi}([t_n^1(\Phi), t_n^2(\Phi)])$ into $\lceil \kappa_{\mathcal{T}}(\Phi_n)/B \rceil$ subsegments of $d_{\mathcal{T}}$ -length at most B , whose π -images all fellow travel some subsegment of $\tau_{\Phi, \mathcal{C}(S)}$ up to distance κ . If none of these segments had a π -image of diameter at least C , then the image $\pi \circ \gamma_{n, \Phi}([t_n^1(\Phi), t_n^2(\Phi)])$ would have d_C -diameter at most

$$\left(\frac{\kappa_{\mathcal{T}}(\Phi_n)}{B} + 1\right)C \leq \frac{\lambda_{\mathcal{T}}Cn}{B} + \frac{\lambda_C n}{5} \leq \frac{2\lambda_C n}{5},$$

a contradiction. The claim follows. □

We recall from Theorem 3.3 that for \mathbb{P} -ae sample path $\Phi = (\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Mod}(S), \mu)$, the sequence $(\Phi_n^{-1} \cdot o)_{n \in \mathbb{N}}$ converges to a point $\text{bnd}_{\mathcal{T}}(\Phi) \in \mathcal{UE}$. We denote by $\tau_{\Phi, \mathcal{T}}$ the Teichmüller ray from o to $\text{bnd}_{\mathcal{T}}(\Phi)$. Notice that the π -image of $\tau_{\Phi, \mathcal{T}}$ fellow travels $\tau_{\Phi, \mathcal{C}(S)}$ up to distance κ .

Proposition 3.13 *Let μ be a nonelementary probability measure on $\text{Mod}(S)$ with finite second moment with respect to $d_{\mathcal{T}}$.*

Then there exist constants $D, \alpha, \beta > 0$, and a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$, such that the probability that $\tau_{\Phi, \mathcal{T}}([0, \alpha n])$ contains a (D, τ) -contracting subsegment whose π -image lies at d_C -distance at least βn from o is at least $1 - C_n$.

Proof Let $B, \beta > 0$ be the constants given by Proposition 3.12. Let $D > 0$ be the constant corresponding to B provided by Corollary 3.11. Let λ be the drift of the random walk on $(\text{Mod}(S), \mu)$ with respect to $d_{\mathcal{T}}$. Propositions 3.8 and 3.12 imply that there exists a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that with probability at least $1 - C_n$, the Teichmüller segment $[o, \Phi_n^{-1} \cdot o]$ has length at most $2\lambda n$, and contains a subsegment I whose π -image lies at d_C -distance at least βn from o , which is (B, C) -progressing, and such that $\pi(I)$ fellow travels the π -image of a subsegment J of $\tau_{\Phi, \mathcal{T}}$ up to distance κ . In view of Corollary 3.11, the segment I is D -supercontracting. This implies that J contains a (D, τ) -contracting subsegment. □

Proof of Proposition 3.9 Let $x \in \mathcal{PMF}_0$, and let $D, \alpha, \beta > 0$ and $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ be as in Proposition 3.13. We will show that, for all $n \in \mathbb{N}$,

$$\mathbb{P}[(x | \text{bnd}_{\mathcal{T}}(\Phi))_o \leq 2\alpha n] \geq 1 - C_n.$$

Using nonatomicity of the hitting measure ν^* and Proposition 3.13, we get the existence a measurable subset X of the path space, of measure at least $1 - C_n$, such that for all $\Phi \in X$, we have $\text{bnd}_{\mathcal{T}}(\Phi) \in \mathcal{PMF}_0 \setminus \{x\}$, and the segment $\tau_{\Phi, \mathcal{T}}([0, \alpha n])$ contains a (D, τ) -contracting subsegment J such that $\pi(J)$ lies at d_C -distance at least βn from o .

In view of [Proposition 2.12](#), we can also assume that for all $\Phi \in X$, the Teichmüller geodesic line (or ray) from x to $\text{bnd}_{\mathcal{T}}(\Phi)$ contains a subsegment whose π -image fellow travels $\pi(J)$ up to distance κ . One can thus find a point $y \in [x, \text{bnd}_{\mathcal{T}}(\Phi)]$ and a point $y' \in \tau_{\Phi, \mathcal{T}}([0, \alpha n])$, such that $d_{\mathcal{T}}(y, y') \leq D$. From now on, we let $\Phi \in X$.

Let $\gamma: \mathbb{R} \rightarrow \mathcal{T}(S)$ be a parametrization of the Teichmüller line from x to $\text{bnd}_{\mathcal{T}}(\Phi)$. Then

$$(x | \text{bnd}_{\mathcal{T}}(\Phi))_o = -\frac{1}{2} \inf_{z \in \mathcal{T}(S)} \lim_{n \rightarrow +\infty} (h_{\gamma(-n)}(z) + h_{\gamma(n)}(z)),$$

with the notation from [Section 1.2](#). It then follows from the triangle inequality that the infimum in the above formula is achieved at any point lying on the image of γ . In particular, one has

$$(x | \text{bnd}_{\mathcal{T}}(\Phi))_o = -\frac{1}{2}(h_x(y) + h_{\text{bnd}_{\mathcal{T}}(\Phi)}(y)).$$

Since $\tau_{\Phi}(k)$ also converges to $\text{bnd}_{\mathcal{T}}(\Phi)$, using Miyachi's result [[45](#), Corollary 1] and the identification between the horofunction boundary and the Gardiner–Masur boundary, we have (all limits are taken as k goes to $+\infty$, and we write d instead of $d_{\mathcal{T}}$ for ease of notation) that

$$\begin{aligned} & -2(x | \text{bnd}_{\mathcal{T}}(\Phi))_o \\ &= \lim [d(y, \gamma(-k)) + d(y, \tau_{\Phi, \mathcal{T}}(k)) - d(o, \gamma(-k)) - d(o, \tau_{\Phi, \mathcal{T}}(k))] \\ &\geq \lim [d(y, \gamma(-k)) + (d(y', \tau_{\Phi, \mathcal{T}}(k)) - D) - d(o, \gamma(-k)) - d(o, \tau_{\Phi, \mathcal{T}}(k))] \\ &= \lim [d(y, \gamma(-k)) - d(o, \gamma(-k))] - d(o, y') - D \\ &\geq -d(o, y) - d(o, y') - D \\ &\geq -2d(o, y') - 2D \\ &\geq -2\alpha n - 2D. \end{aligned}$$

This implies that

$$(x | \text{bnd}_{\mathcal{T}}(\Phi))_o \leq \alpha n + D,$$

and concludes the proof of [Proposition 3.9](#). □

3.4.2 Mean value of the Busemann cocycle: hypothesis (H1) Using similar arguments as in the proof of [Proposition 3.9](#), we will now establish hypothesis (H1) from [Theorem 1.6](#) for the Gromov product on the horoboundary of $\mathcal{T}(S)$: this is the content of [Corollary 3.15](#) below.

Proposition 3.14 *Let μ be a nonelementary probability measure on $\text{Mod}(S)$. For all $\epsilon > 0$, there exists $M > 0$ such that for all $x \in \mathcal{PMF}_0$, one has*

$$\mathbb{P} \left[\sup_{n \in \mathbb{N}} |\beta(\Phi_n, x) - \kappa(\Phi_n)| \leq M \right] \geq 1 - \epsilon.$$

Proof It suffices to show that for all $x \in \mathcal{PMF}_0$ and \mathbb{P} -ae sample path $(\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Mod}(S), \mu)$, one has

$$\sup_{n \in \mathbb{N}} |\beta(\Phi_n, x) - \kappa(\Phi_n)| < +\infty.$$

We observe that there exists $D > 0$ such that for all $x \in \mathcal{PMF}_0$ and \mathbb{P} -ae sample path $\Phi := (\Phi_n)_{n \in \mathbb{N}}$ of the random walk, the π -image of the Teichmüller geodesic from x to $\text{bnd}_{\mathcal{T}}(\Phi)$ crosses the π -image of a (D, τ) -contracting subsegment I of $\tau_{\Phi, \mathcal{T}}$ up to distance κ . This observation relies on the fact that \mathbb{P} -almost surely, the Teichmüller ray from o to $\text{bnd}_{\mathcal{T}}(\Phi)$ contains infinitely many (D, τ) -contracting subsegments: this fact was established in [14, Proposition 3.11]. In particular, there exists a point $z \in I$ such that for all $n \in \mathbb{N}$ sufficiently large, both the Teichmüller segment $[o, \Phi_n^{-1}.o]$ and the Teichmüller ray from $\Phi_n^{-1}.o$ to x pass at bounded distance from z . So for all $n \in \mathbb{N}$ sufficiently large, the difference $|\beta(\Phi_n, x) - \kappa(\Phi_n)|$ is equal to $|h_x(z) - d_{\mathcal{T}}(o, z)|$ up to a bounded error (by a similar computation as in the proof of Proposition 3.9), from which the claim follows. \square

Corollary 3.15 *Let μ be a nonelementary probability measure on $\text{Mod}(S)$ with finite first moment with respect to $d_{\mathcal{T}}$. Let λ be the drift of the random walk on $(\text{Mod}(S), \mu)$ with respect to $d_{\mathcal{T}}$. Let ν be the μ -stationary probability measure on \mathcal{PMF}_0 . Then*

$$\int_{\text{Mod}(S) \times \mathcal{PMF}_0} \beta(\Phi, y) d\mu(\Phi) d\nu(y) = \lambda.$$

Proof Proposition 3.14 implies that for all $y \in \mathcal{PMF}_0$ and \mathbb{P} -ae sample path $(\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Mod}(S), \mu)$, one has

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \beta(\Phi_n, y) = \lambda.$$

Corollary 3.15 then follows by applying Birkhoff’s ergodic theorem, in the form provided by Proposition 1.4. \square

3.5 Central limit theorem

We now complete the proof of the central limit theorem for mapping class groups.

Theorem 3.16 *Let S be a closed, connected, oriented, hyperbolic surface, and let ρ be a hyperbolic metric on S . Let μ be a nonelementary probability measure on $\text{Mod}(S)$ with finite second moment with respect to the Teichmüller metric. Let λ be the drift of the random walk on $(\text{Mod}(S), \mu)$ with respect to $d_{\mathcal{T}}$.*

Then there exists a centered Gaussian law N_μ on \mathbb{R} such that for every compactly supported continuous function F on \mathbb{R} , and all essential simple closed curves c on S , one has, uniformly in c ,

$$\lim_{n \rightarrow +\infty} \int_{\text{Mod}(S)} F\left(\frac{\log l_\rho(\Phi(c)) - n\lambda}{\sqrt{n}}\right) d\mu^{*n}(\Phi) = \int_{\mathbb{R}} F(t) dN_\mu(t).$$

Proof In view of [Corollary 3.5](#), it is enough to prove a central limit theorem for the Busemann cocycle β , ie show that there exists a gaussian law N_μ on \mathbb{R} such that for every compactly supported continuous function F on \mathbb{R} , and all $x \in \mathcal{PMF}_0$, one has

$$\lim_{n \rightarrow +\infty} \int_{\text{Mod}(S)} F\left(\frac{\beta(\Phi, x) - n\lambda}{\sqrt{n}}\right) d\mu^{*n}(\Phi) = \int_{\mathbb{R}} F(t) dN_\mu(t),$$

uniformly in $x \in \mathcal{PMF}_0$. In view of [Proposition 3.14](#), it is enough to prove that there exists $x \in \mathcal{PMF}_0$ for which the limit holds. Since the Busemann cocycle β satisfies hypotheses (H1) ([Corollary 3.15](#), applied to the nonelementary probability measure $\check{\mu}$) and (H2) ([Proposition 3.9](#)) from [Theorem 1.6](#) (with $Y^- = Y^+ = \mathcal{PMF}_0$), the result follows from [Theorem 1.6](#). \square

4 Central limit theorem on $\text{Out}(F_N)$

Let $N \geq 2$. The goal of this section is to establish a central limit theorem on $\text{Out}(F_N)$ ([Theorem 0.2](#)). The proof will follow the same outline as in the mapping class group case; the main novelties come from the need to take care of asymmetry of the metric on CV_N , which implies in particular that we will only have a *one-sided* version of [Corollary 3.11](#) in the context of $\text{Out}(F_N)$.

4.1 Background on $\text{Out}(F_N)$

Outer space and its metric *Outer space* CV_N was introduced by Culler and Vogtmann in [\[13\]](#), and can be defined as the space of equivalence classes of free, minimal, simplicial, isometric F_N -actions on simplicial metric trees, two trees being equivalent whenever there exists an F_N -equivariant homothety between them. *Unprojectivized outer space* cv_N is defined in a similar way, by considering trees up to F_N -equivariant isometry, instead of homothety. The group $\text{Out}(F_N)$ acts on both CV_N and cv_N on the right by precomposing the F_N -actions. These $\text{Out}(F_N)$ -actions can be turned into left actions by letting $\Phi.T := T.\Phi^{-1}$ for all $\Phi \in \text{Out}(F_N)$ and all $T \in \text{CV}_N$. Given $\epsilon > 0$, the ϵ -*thick part* CV_N^ϵ is the subspace of CV_N made of those trees T such that the volume-one representative of the quotient graph T/F_N does not contain any embedded loop of length smaller than ϵ .

Outer space comes equipped with a natural asymmetric metric d_{CV_N} [18], the distance between two trees $T, T' \in \text{CV}_N$ being equal to the logarithm of the infimal Lipschitz constant of an F_N -equivariant map from the covolume-one representative of T , to the covolume-one representative of T' . The $\text{Out}(F_N)$ -action on CV_N is by isometries for this metric. White has proved (see [18, Proposition 3.15] or [1, Proposition 2.3]) that

$$d_{\text{CV}_N}(T, T') = \log \sup_{g \in F_N \setminus \{e\}} \frac{\|g\|_{T'}}{\|g\|_T}$$

for all $T, T' \in \text{CV}_N$, identified with their covolume-one representatives in the above formula. In addition, the supremum in the above formula can be taken over a finite set $\text{Cand}(T)$ that only depends on T . Elements in $\text{Cand}(T)$ are called *candidates* for T , they are primitive elements of F_N (recall that an element of F_N is *primitive* if it belongs to some free basis of F_N).

Currents on free groups Let $\partial^2 F_N := \partial F_N \times \partial F_N \setminus \Delta$, where ∂F_N is identified with the Gromov boundary of a Cayley tree R_0 of F_N , and Δ denotes the diagonal subset. A *current* on F_N is an F_N -invariant Borel measure on $\partial^2 F_N$ that is finite on compact subsets of $\partial^2 F_N$. We denote by Curr_N the space of currents on F_N , which is topologized as in [31]. Every $g \in F_N$ which is not of the form h^k for any $h \in F_N$ and $k > 1$ determines a *rational current* η_g , where for all closed-open subsets $S \subseteq \partial^2 F_N$, the number $\eta_g(S)$ is the number of F_N -translates of the axis of g in R_0 , whose pairs of endpoints belong to S . The group $\text{Out}(F_N)$ acts on the set of currents on the left in the following way: given $\Phi \in \text{Out}(F_N)$, a current η , and a compact set $K \subseteq \partial^2 F_N$, we let $\Phi(\eta)(K) := \eta(\phi^{-1}(K))$, where $\phi \in \text{Aut}(F_N)$ is any representative of Φ . The length pairing between trees in cv_N and elements of F_N extends continuously [32] to an intersection pairing $\langle \cdot, \cdot \rangle: \overline{\text{cv}_N} \times \text{Curr}_N \rightarrow \mathbb{R}_+$.

Forward and backward horoboundaries of outer space We denote by \mathcal{P}_N the collection of all primitive elements of F_N . The *primitive compactification* $\overline{\text{CV}_N}^{\text{prim}}$ was introduced in [29, Section 2.4] by taking the closure of the image of the embedding

$$i: \text{CV}_N \rightarrow \mathbb{P}\mathbb{R}^{\mathcal{P}_N}, \quad g \mapsto \mathbb{R}^*(\|g\|_T)_{g \in \mathcal{P}_N},$$

in the projective space $\mathbb{P}\mathbb{R}^{\mathcal{P}_N}$. The forward horofunction compactification of CV_N was identified in [28, Theorem 2.2] with the primitive compactification $\overline{\text{CV}_N}^{\text{prim}}$. This is a quotient of the Culler–Morgan compactification $\overline{\text{CV}_N}$, which was introduced in [12], and identified by Cohen and Lustig [11] and Bestvina and Feighn [5] (see also [27]) with the space of equivariant homothety classes of *very small* minimal F_N -trees, ie trees whose arc stabilizers are either trivial, or maximally cyclic, and whose tripod stabilizers are trivial. The fibers of the quotient map from $\overline{\text{CV}_N}$ to $\overline{\text{CV}_N}^{\text{prim}}$ were described in [29].

Among trees in $\partial\text{CV}_N := \overline{\text{CV}_N} \setminus \text{CV}_N$, *arational trees* will be of particular interest to us. These are defined in the following way. A subgroup $A \subseteq F_N$ is a *free factor* if there exists $B \subseteq F_N$ such that $F_N = A * B$. A tree $T \in \partial\text{CV}_N$ is *arational* if no proper free factor of F_N has a global fixed point in T , and the action of every proper free factor of F_N on its minimal subtree in T is free and simplicial. We denote by \mathcal{UE} the subspace of ∂CV_N made of those arational trees T which are both *uniquely ergometric* and *dually uniquely ergodic*, ie those that admit, up to homothety, a unique length measure (see the definition in [24, Section 5.1], attributed to Paulin) and a unique geodesic current $\eta \in M_N$ satisfying $\langle T, \eta \rangle = 0$ (we say that η is *dual* to T). We note that the quotient map from $\overline{\text{CV}_N}$ to $\overline{\text{CV}_N}^{\text{prim}}$ is one-to-one in restriction to the set of trees with dense orbits [29], and in particular, in restriction to \mathcal{UE} .

Properties of the backward horoboundary $\partial_{\bar{h}}\text{CV}_N$ were also investigated in [28, Section 4]. Backward horofunctions are described in terms of geodesic currents on F_N . We denote by M_N the minimal set of currents, defined in [41] as the closure of the set of rational currents associated to primitive conjugacy classes. Given a finite set $S \subseteq M_N$, we define a function f_S on CV_N by setting

$$f_S(T) := \log \frac{\sup_S \langle T, \eta \rangle}{\sup_S \langle o, \eta \rangle}$$

for all $T \in \text{CV}_N$ (here again trees are identified with their covolume-one representatives). By [28, Proposition 4.5], for all $\xi \in \partial_{\bar{h}}\text{CV}_N$, there exists a finite set $S \subseteq M_N$ such that $\xi = f_S$. For all trees $T \in \mathcal{UE}$ with dual current η , and all geodesic lines $\gamma: \mathbb{R} \rightarrow \text{CV}_N$ such that $\lim_{t \rightarrow -\infty} \gamma(t) = T$ in the Culler–Morgan compactification $\overline{\text{CV}_N}$, one has $\lim_{t \rightarrow -\infty} \gamma(t) = f_{[\eta]}$ in $\text{CV}_N \cup \partial_{\bar{h}}\text{CV}_N$; see [28, Remark 4.6].

Folding lines in outer space A nice collection of paths in outer space is the collection of so-called *folding lines*, whose definition we now review. A *morphism* between two \mathbb{R} -trees T and T' is a map $f: T \rightarrow T'$, such that every segment in T can be subdivided into finitely many subsegments, in restriction to which f is an isometry. Every morphism defines a partition of the set of connected components of $T \setminus \{x\}$ (called *directions*), at each point $x \in T$: two directions belong to the same class of the partition if and only if their f -images overlap in T' . The data of all these partitions is called a *train-track structure* on T . A morphism $f: T \rightarrow T'$ is *optimal* if there are at least two distinct equivalence classes of directions at every point in T , and f realizes the infimal Lipschitz constant of an F_N -equivariant map from T to T' .

An *optimal folding path* in cv_N is a continuous map $\gamma: I \rightarrow \text{cv}_N$, with $I \subseteq \mathbb{R}$ an interval, together with a collection of F_N -equivariant optimal morphisms $f_{t,t'}: \gamma(t) \rightarrow \gamma(t')$ for all $t < t'$, such that $f_{t,t''} = f_{t',t''} \circ f_{t,t'}$ for all $t < t' < t''$. It is a *greedy folding path*

if for all $t_0 \in \mathbb{R}$, there exists $\epsilon > 0$ such that for all $t \in [t_0, t_0 + \epsilon]$, the tree $\gamma(t)$ is obtained from $\gamma(t_0)$ by identifying two segments of length ϵ in $\gamma(t_0)$ whenever they have a common endpoint, and have the same $f_{t_0,t}$ -image. The projection to CV_N of a (greedy) folding path in cv_N will again be called a (greedy) folding path.

Any two trees $T, T' \in \text{CV}_N$ are joined by a (nonunique) geodesic segment, which is the concatenation of a segment contained in a *simplex* of CV_N (ie the subspace of CV_N made of all trees obtained by only varying the edge lengths of T), and an optimal greedy folding path. A geodesic segment obtained in this way will be called a *standard geodesic segment*. If $T' \in \mathcal{UE}$, then one can similarly find a (nonunique) *standard geodesic ray* in CV_N starting at o and limiting at T' , consisting of the concatenation of an initial segment contained in a simplex, and an optimal greedy folding ray; see [7, Lemma 6.11]. Given any two distinct trees $T, T' \in \mathcal{UE}$, there exists a (nonunique) optimal greedy folding line $\gamma: \mathbb{R} \rightarrow \text{CV}_N$ such that $\gamma(t)$ converges to T (resp. to T') as t goes to $-\infty$ (resp. $+\infty$), as follows from the work of Bestvina and Reynolds [7, Theorem 6.6 and Lemma 6.11].

The free factor graph The *free factor graph* FF_N is the simplicial graph whose vertices are the conjugacy classes of proper free factors of F_N , in which two vertices $[A]$ and $[B]$ are joined by an edge whenever there exist representatives A, B in the corresponding conjugacy classes, such that either $A \not\subset B$ or $B \not\subset A$. The graph FF_N is Gromov hyperbolic; see Bestvina and Feighn [6]. The group $\text{Out}(F_N)$ has a natural left action on FF_N . There is a natural coarsely Lipschitz, coarsely $\text{Out}(F_N)$ -equivariant map $\pi: \text{CV}_N \rightarrow \text{FF}_N$, which sends any tree $T \in \text{CV}_N$ to a proper free factor A such that T collapses to a tree T' in which A fixes a point. Bestvina and Feighn also established in [6] that π -images of standard geodesic lines in CV_N are uniform unparametrized quasigeodesics in FF_N .

Bestvina and Reynolds [7] and Hamenstädt [26] independently described the Gromov boundary $\partial_\infty \text{FF}_N$ as the space of equivalence classes of arational trees in ∂CV_N , two trees being equivalent whenever they have the same underlying topological tree (and only differ by the metric). In particular, there is a continuous $\text{Out}(F_N)$ -equivariant map $\psi: \mathcal{UE} \rightarrow \partial_\infty \text{FF}_N$, such that for all $T \in \mathcal{UE}$, and all sequences $(S_n)_{n \in \mathbb{N}} \in \text{CV}_N^{\mathbb{N}}$ converging to T (for the topology of $\overline{\text{CV}_N}$), the sequence $(\pi(S_n))_{n \in \mathbb{N}}$ converges to $\psi(T)$ (for the topology of $\text{FF}_N \cup \partial_\infty \text{FF}_N$).

Review on projections to folding paths Let $\gamma: I \rightarrow \text{CV}_N$ (where $I \subseteq \mathbb{R}$ is an interval) be an optimal greedy folding path determined by a morphism f . The morphism f determines train-track structures on all trees $\gamma(t)$ with $t \in I$. A segment $[a, b] \subseteq \gamma(t)$ is *legal* if for every $x \in J$, the intervals $[a, x]$ and $(x, b]$ belong to components

of $\gamma(t) \sim \{x\}$ in distinct equivalence classes of the train-track structure. Following Bestvina and Feighn [6], for all $g \in \mathcal{P}_N$, we define $\text{right}_\gamma(g)$ as the infimal $t \in I$ such that every segment of length M_{BF} in the axis of g in $\gamma(t)$, contains a legal subsegment of length 3 (here M_{BF} is the constant defined in [6, Section 6], which only depends on N). We then let

$$\text{Pr}_\gamma(S) := \gamma\left(\sup_{g \in \text{Cand}(S)} \text{right}_\gamma(g)\right)$$

for all $S \in \text{CV}_N$. The following lemma relies on the work of Bestvina and Feighn [6, Section 4]; it was established as such in [14, Lemma 4.7].

Lemma 4.1 (Bestvina and Feighn [6]) *There exists $K_0 > 0$ (only depending on N) such that for all greedy folding paths $\gamma: I \rightarrow \text{CV}_N$, all $g \in \mathcal{P}_N$, and all $t, t' \geq \text{right}_\gamma(g)$ satisfying $t \leq t'$, one has*

$$\left| d_{\text{CV}_N}(\gamma(t), \gamma(t')) - \log \frac{\|g\|_{\gamma(t')}}{\|g\|_{\gamma(t)}} \right| \leq K_0.$$

The Bestvina–Feighn projection satisfies the following contraction property.

Lemma 4.2 (Bestvina and Feighn [6, Proposition 7.2]) *There exists $D_1 > 0$ such that for every standard geodesic line $\gamma: I \rightarrow \text{CV}_N$, and all $H, H' \in \text{CV}_N$, if $d_{\text{CV}_N}(H, H') \leq d_{\text{CV}_N}(H, \mathfrak{S}(\gamma))$, then $d_{\text{FF}_N}(\pi(\text{Pr}_\gamma(H)), \pi(\text{Pr}_\gamma(H'))) \leq D_1$.*

The following lemma of Dowdall and Taylor, relates the Bestvina–Feighn projection and the closest-point projection in FF_N . Given an optimal greedy folding path $\gamma: I \rightarrow \text{CV}_N$, we denote by $\mathbf{n}_{\pi \circ \gamma}$ a closest-point projection map to the image of $\pi \circ \gamma$ in FF_N .

Lemma 4.3 (Dowdall and Taylor [16, Lemma 4.2]) *There exists $D_2 > 0$ such that*

$$d_{\text{FF}_N}(\pi(\text{Pr}_\gamma(H)), \mathbf{n}_{\pi \circ \gamma}(\pi(H))) \leq D_2$$

for all optimal greedy folding paths γ and all $H \in \text{CV}_N$.

Random walks on $\text{Out}(F_N)$ A subgroup $H \subseteq \text{Out}(F_N)$ is *nonelementary* if H is not virtually cyclic, and does not virtually fix the conjugacy class of any proper free factor of F_N . A probability measure on $\text{Out}(F_N)$ is *nonelementary* if the subsemigroup generated by its support is a nonelementary subgroup of $\text{Out}(F_N)$. With the terminology from Section 2, this is equivalent to nonelementarity with respect to the action on the free factor graph FF_N .

Proposition 4.4 (Namazi, Pettet and Reynolds [47, Theorem 7.21]) *Let μ be a nonelementary probability measure on $\text{Out}(F_N)$ with finite first moment with respect to d_{CV_N} . Then for \mathbb{P} -ae sample path $\Phi := (\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Out}(F_N), \mu)$, and any $o \in \text{CV}_N$, the sequence $(\Phi_n^{-1} \cdot o)_{n \in \mathbb{N}}$ converges to a point $\text{bnd}(\Phi) \in \mathcal{UE}$. The hitting measure ν^* defined by setting*

$$\nu^*(S) := \mathbb{P}(\text{bnd}(\Phi) \in S)$$

for all measurable subsets $S \subseteq \partial \text{CV}_N$ is nonatomic, and it is the unique $\check{\mu}$ -stationary probability measure on ∂CV_N .

It follows from the description of $\partial_h^+ \text{CV}_N$ as a quotient of the Culler–Morgan boundary that ν^* can also be viewed as the unique $\check{\mu}$ -stationary probability measure on $\partial_h^+ \text{CV}_N$.

4.2 Progress and contraction for folding paths

In this section, we will establish a contraction property for folding lines in outer space (Proposition 4.6 below, which is a variation on [14, Proposition 4.17]). This will play the same role in our proof of the central limit theorem as Corollary 3.11 in the mapping class group case, though we only get a *one-sided* version in the $\text{Out}(F_N)$ context.

Let $\kappa > 0$ be a sufficiently large constant, such that all quasigeodesic triangles in FF_N whose sides are π -images of folding lines in outer space, are κ -thin (in particular, κ is assumed to satisfy the conclusion of Proposition 2.12, if K is a constant such that π -images of folding lines are (K, K) -unparametrized quasigeodesics). Given $D > 0$, a geodesic line $\gamma: \mathbb{R} \rightarrow \text{CV}_N$ and a subsegment $I = [a, b] \subseteq \mathbb{R}$, we say that γ is *D-bicontracting along I* if for all geodesic segments $\gamma': [a', b'] \rightarrow \text{CV}_N$, if $\pi \circ \gamma'$ contains a subsegment which fellow travels $\pi \circ \gamma_I$ up to distance κ , then there exists $t \in [a', b']$ such that $d_{\text{CV}_N}^{\text{sym}}(\gamma(a), \gamma'(t)) \leq D$. We say that γ is *D-right-contracting along I* if the above holds for all geodesic segments γ' which are further assumed to be such that $\gamma'(b')$ belongs to the image of γ and lies to the right of γ_I .

Given $B, C > 0$, we say that a geodesic segment $\gamma: I = [a, b] \rightarrow \text{CV}_N$ is *(B, C)-progressing* if $d_{\text{CV}_N}(\gamma(a), \gamma(b)) \leq B$ and $\pi \circ \gamma(I)$ has d_{FF_N} -diameter at least C . The following lemma is based on an observation due to Dowdall and Taylor [16, Lemma 4.3]. We include a proof for completeness.

Lemma 4.5 (Dowdall and Taylor [16, Lemma 4.3]) *There exists $C_0 > 0$ such that for all $B > 0$, there exists $\epsilon > 0$ such that for all $C > C_0$, if $\gamma: [a, b] \rightarrow \text{CV}_N$ is a (B, C) -progressing geodesic segment, then $\gamma(a) \in \text{CV}_N^\epsilon$.*

Proof Let $\epsilon < \exp(-B)$, and assume by contradiction that $\gamma(a) \notin \text{CV}_N^\epsilon$: there exists $g \in F_N$ represented by a loop of length smaller than ϵ in the volume-one representative of $\gamma(a)/F_N$. Let $t \in [a, b]$ be such that $d_{\text{FFN}}(\pi \circ \gamma(a), \pi \circ \gamma(t)) \geq 11$ (this exists as soon as C_0 is sufficiently large). Then $\|g\|_{\gamma(t)} \geq 1$ (see the argument in [16, Lemma 4.2] for the precise constant 11). Hence

$$B \geq d_{\text{CV}_N}(\gamma(a), \gamma(t)) \geq \log \frac{1}{\epsilon}$$

(the first inequality follows from the fact that γ is (B, C) -progressing, and the second follows from White’s formula for the distance on CV_N). This is a contradiction. \square

Proposition 4.6 *There exists $C > 0$ such that for all $B > 0$, there exists $D > 0$ such that the following holds.*

Let $\gamma: \mathbb{R}_+ \rightarrow \text{CV}_N$ be a geodesic ray. Let $I \subseteq \mathbb{R}$ be an interval such that $\gamma|_I$ is a (B, C) -progressing optimal greedy folding path. Then γ is D -right-contracting along I .

The proof of Proposition 4.6 is a variation on the argument from [14, Proposition 4.17]: in [14], the folding path γ was supposed to satisfy a stronger condition, but the conclusion was that γ is D -bicontracting along I .

Proof The proof is illustrated in Figure 1. Let $I = [a, b]$ and $C \geq 2\kappa + D_1 + 2D_2 + C_0$, where $D_1, D_2, C_0 > 0$ are the constants from Lemmas 4.2, 4.3 and 4.5, respectively. Let $B > 0$. Assume that $\gamma|_I$ is (B, C) -progressing. Let $\gamma': [a', b'] \rightarrow \text{CV}_N$ be a geodesic segment such that $\gamma'(b') \in \gamma([b, +\infty))$, and there exists $I' \subseteq [a', b']$ such that $\pi \circ \gamma'_{I'}$ fellow travels $\pi \circ \gamma|_I$ up to distance κ . We aim at showing that there exists $t \in I'$ such that $d_{\text{CV}_N}^{\text{sym}}(\gamma(a), \gamma'(t)) \leq D$, where D is a constant that only depends on B .

Let $S := \gamma(a)$. As $C \geq C_0$ and $\gamma|_I$ is (B, C) -progressing, we have $S \in \text{CV}_N^\epsilon$, where $\epsilon := \epsilon(B)$ is the constant provided by Lemma 4.5. Hence by [2], there exists $M > 0$ (only depending on B) such that, for all $y \in \text{CV}_N$,

$$(7) \quad d_{\text{CV}_N}^{\text{sym}}(S, y) \leq M d_{\text{CV}_N}(y, S).$$

As $\pi \circ \gamma'_{I'}$ fellow travels $\pi \circ \gamma|_I$ up to distance κ , there exists $U \in \gamma'(I')$ such that $d_{\text{FFN}}(\pi(U), \pi(S)) \leq \kappa$. Therefore, $d_{\text{FFN}}(\mathbf{n}_{\pi \circ \gamma|_I}(\pi(U)), \pi(S)) \leq 2\kappa$, and by Lemma 4.3 we have (writing Pr_I for $\text{Pr}_{\gamma|_I}$ for ease of notation) that

$$(8) \quad d_{\text{FFN}}(\pi(\text{Pr}_I(U)), \pi(S)) \leq 2\kappa + D_2.$$

Let $S_0 \in \gamma(I)$ be a point closest to U in $\gamma(I)$, and let $U' \in \gamma'(I)$ be a point lying to the right of U on the image of γ' , and such that

$$d_{\text{CV}_N}(U, U') = d_{\text{CV}_N}(U, \gamma(I)) = d_{\text{CV}_N}(U, S_0).$$

Lemma 4.2 shows that $d_{\text{FF}_N}(\pi(\text{Pr}_I(U)), \pi(\text{Pr}_I(S_0))) \leq D_1$. By **Lemma 4.3**, we also have $d_{\text{FF}_N}(\pi(S_0), \pi(\text{Pr}_I(S_0))) \leq D_2$. Together with (8), the triangle inequality then yields $d_{\text{FF}_N}(\pi(S_0), \pi(S)) \leq 2\kappa + D_1 + 2D_2$. Recall that $S, S_0 \in \gamma(I)$, and S_0 lies to the right of S . As $C \geq 2\kappa + D_1 + 2D_2$ and $\gamma|_I$ is (B, C) -progressing, this implies that $d_{\text{CV}_N}(S, S_0) \leq B$, and in view of (7) we obtain

$$(9) \quad d_{\text{CV}_N}^{\text{sym}}(S, S_0) \leq MB.$$

Lemma 4.2 also shows that $d_{\text{FF}_N}(\pi(\text{Pr}_I(U')), \pi(\text{Pr}_I(U))) \leq D_1$. Together with (8), the triangle inequality then yields $d_{\text{FF}_N}(\pi(\text{Pr}_I(U')), \pi(S)) \leq 2\kappa + D_1 + D_2$. This implies as above that $d_{\text{CV}_N}(S, \text{Pr}_I(U')) \leq B$, and hence, in view of (7),

$$(10) \quad d_{\text{CV}_N}^{\text{sym}}(S, \text{Pr}_I(U')) \leq MB.$$

Let now $g \in \mathcal{P}_N$ be such that

$$(11) \quad d_{\text{CV}_N}(U', \text{Pr}_I(U')) = \log \frac{\|g\|_{\text{Pr}_I(U')}}{\|g\|_{U'}}.$$

Let $S' := \gamma'(b') \in \gamma([b, +\infty))$. Notice that $\text{Pr}_I(U')$ is necessarily to the right of $\text{Pr}_\gamma(U')$ on the image of γ (they may coincide), and S' lies to the right of $\text{Pr}_I(U')$. **Lemma 4.1**, applied to the optimal greedy folding path γ , shows that there exists $K_0 > 0$ (which only depends on N) such that

$$(12) \quad d_{\text{CV}_N}(\text{Pr}_I(U'), S') \leq \log \frac{\|g\|_{S'}}{\|g\|_{\text{Pr}_I(U')}} + K_0.$$

By adding (11) and (12), we get

$$(13) \quad d_{\text{CV}_N}(U', \text{Pr}_I(U')) + d_{\text{CV}_N}(\text{Pr}_I(U'), S') - K_0 \leq \log \frac{\|g\|_{S'}}{\|g\|_{U'}} \leq d_{\text{CV}_N}(U', S').$$

Then (10) shows that there exists $K > 0$ (which only depends on B and on the rank N of the free group) such that

$$(14) \quad d_{\text{CV}_N}(U', S) + d_{\text{CV}_N}(S, S') - K \leq d_{\text{CV}_N}(U', S').$$

Therefore, the total green length (the sum of the lengths of the segments S_0U , S_0U' and S_0S') on **Figure 1** is equal, up to a bounded error, to $d_{\text{CV}_N}(U, S')$. This implies

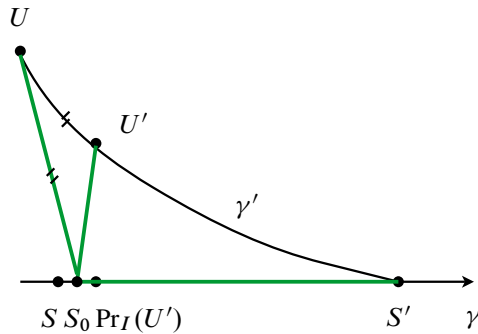


Figure 1: The situation (represented in CV_N) in the proof of Proposition 4.6

that $d_{\text{CV}_N}(U', S)$ is uniformly bounded. Precisely, we have

$$\begin{aligned}
 d_{\text{CV}_N}(U', S) &\leq d_{\text{CV}_N}(U', S') - d_{\text{CV}_N}(S, S') + K \\
 &\leq d_{\text{CV}_N}(U', S') - d_{\text{CV}_N}(U, S') + d_{\text{CV}_N}(U, S) + K \\
 &= -d_{\text{CV}_N}(U, U') + d_{\text{CV}_N}(U, S) + K \\
 &\leq -d_{\text{CV}_N}(U, U') + d_{\text{CV}_N}(U, S_0) + d_{\text{CV}_N}(S_0, S) + K \\
 &= d_{\text{CV}_N}(S_0, S) + K \\
 &\leq MB + K,
 \end{aligned}$$

and therefore,

$$d_{\text{CV}_N}^{\text{sym}}(U', S) \leq M(MB + K)$$

as required, in view of (7). □

4.3 A deviation principle in outer space

Let μ be a probability measure on $\text{Out}(F_N)$ with finite first moment with respect to d_{CV_N} . We recall that the drift of the random walk on $(\text{Out}(F_N), \mu)$ with respect to d_{CV_N} is defined with the conventions of the present paper as being equal to the limit

$$\lim_{n \rightarrow +\infty} \frac{1}{n} d_{\text{CV}_N}(\Phi_n \cdot o, o)$$

for \mathbb{P} -ae sample path $(\Phi_n)_{n \in \mathbb{N}}$ of the left random walk on $(\text{Out}(F_N), \mu)$.

Theorem 4.7 (Horbez [28, Corollary 5.4]) *Let μ be a nonelementary probability measure on $\text{Out}(F_N)$ with finite first moment with respect to d_{CV_N} , and let λ be the drift of the random walk on $(\text{Out}(F_N), \mu)$ with respect to d_{CV_N} .*

Then for all $x \in \partial_h^- \text{CV}_N$ and \mathbb{P} -ae sample path $(\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Out}(F_N), \mu)$, one has

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \beta^-(\Phi_n, x) = \lambda.$$

Proof With the notation from Section 4.1, we have $x = f_S$ for some finite subset $S \subseteq M_N$, and

$$\beta^-(\Phi_n, x) = f_S(\Phi_n^{-1}.o) = \log \frac{\sup_S \langle o, \Phi_n \eta \rangle}{\sup_S \langle o, \eta \rangle}.$$

The conclusion of Theorem 4.7 was established in [28, Corollary 5.4] when S consists of a single rational current, corresponding to a primitive conjugacy class in F_N . However, the proof from [28] extends in the same way to any singleton S , by noticing that the supremum in the formula giving the distance between two trees in CV_N can be taken over all currents in M_N , if one uses the continuous extension of the length pairing between conjugacy classes and trees to $M_N \times \overline{CV_N}$. The conclusion then also holds for any finite subset $S \subseteq M_N$. \square

We derive the following large deviation principle for the Busemann cocycle on the backward horoboundary of outer space, and for the function κ_{CV_N} defined for all $\Phi \in \text{Out}(F_N)$ by

$$\kappa_{CV_N}(\Phi) := d_{CV_N}(\Phi.o, o).$$

Proposition 4.8 *Let μ be a nonelementary probability measure on $\text{Out}(F_N)$ with finite second moment with respect to d_{CV_N} , and let λ be the drift of the random walk on $(\text{Out}(F_N), \mu)$ with respect to d_{CV_N} . Then for every $\epsilon > 0$, there exists a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $n \in \mathbb{N}$ and all $x \in \partial_{\bar{h}} CV_N$, one has*

$$\mu^{*n}(\{\Phi \in \text{Out}(F_N) : |\beta^-(\Phi, x) - n\lambda| \geq \epsilon n\}) \leq C_n.$$

In particular, one has

$$\mu^{*n}(\{\Phi \in \text{Out}(F_N) : |\log \|\Phi(g)\| - n\lambda| \geq \epsilon n\}) \leq C_n$$

for all $n \in \mathbb{N}$ and all $g \in \mathcal{P}_N$, and

$$\mu^{*n}(\{\Phi \in \text{Out}(F_N) : |\kappa_{CV_N}(\Phi) - n\lambda| \geq \epsilon n\}) \leq C_n.$$

Proof Theorem 4.7, together with Birkhoff’s ergodic theorem (in the form provided by Proposition 1.4), implies that

$$\int_{\text{Out}(F_N) \times \partial_{\bar{h}} CV_N} \beta^-(\Phi, x) d\mu(\Phi) d\nu(x) = \lambda$$

for every μ -stationary probability measure ν on $\partial_{\bar{h}} CV_N$. The first assertion then follows from Proposition 1.3 (the fact that $\beta_{\text{sup}}^- \in L^2(G, \mu)$ follows from (3) in Section 1). The second assertion is a specification of the first to the case where $x = f_S$, with S consisting of a single rational current corresponding to the primitive element g . The last assertion follows from the second, applied to each of the finitely many candidates in $\text{Cand}(o)$. \square

4.4 Lifting estimates from FF_N to CV_N

4.4.1 Deviation estimate for the Gromov product: hypothesis (H2) We will now check hypothesis (H2) from [Theorem 1.6](#) for the Gromov product on $\partial_h^- \text{CV}_N \times \partial_h^+ \text{CV}_N$. The proof follows the same outline as in the mapping class group case ([Section 3.4](#)).

Proposition 4.9 *Let μ be a nonelementary probability measure on $\text{Out}(F_N)$ with finite second moment with respect to d_{CV_N} , and let ν^* be the unique $\check{\mu}$ -stationary probability measure on $\partial_h^+ \text{CV}_N$. Then there exist $\alpha > 0$ and a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $x \in \mathcal{UE} \subseteq \partial_h^- \text{CV}_N$, one has*

$$\nu^*(\{y \in \partial_h^+ \text{CV}_N \mid (x|y)_o \geq \alpha n\}) \leq C_n.$$

We assume that the basepoint in FF_N is the π -image of the basepoint in CV_N (both are denoted by o). Recall from [Proposition 4.4](#) that for \mathbb{P} -ae sample path $\Phi := (\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Out}(F_N), \mu)$, the sequence $(\Phi_n^{-1}.o)_{n \in \mathbb{N}}$ converges to a point $\text{bnd}(\Phi) \in \mathcal{UE}$. We then let τ_Φ be a standard geodesic ray from o to $\text{bnd}(\Phi)$, and for all $n \in \mathbb{N}$, we let $t_n(\Phi) \in \mathbb{R}_+$ be the infimum of all $t \in \mathbb{R}_+$ such that

$$d_{\text{CV}_N}(\Phi_n^{-1}.o, \tau_\Phi(t)) = \inf_{t' \in \mathbb{R}_+} d_{\text{CV}_N}(\Phi_n^{-1}.o, \tau_\Phi(t')).$$

Notice the image of $\pi \circ \tau_\Phi$ in FF_N lies at bounded Hausdorff distance from any $(1, K)$ -quasigeodesic ray from o to $\text{bnd}_{\text{FF}_N}(\Phi)$. We start by establishing the following lemma.

Lemma 4.10 *Let μ be a nonelementary probability measure on $\text{Out}(F_N)$ with finite second moment with respect to d_{CV_N} . Then there exist $K_1, K_2 > 0$ and a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that*

$$\mathbb{P}[d_{\text{CV}_N}(o, \tau_\Phi(t_n(\Phi))) \leq K_1 n \text{ and } d_{\text{FF}_N}(o, \pi \circ \tau_\Phi(t_n(\Phi))) \geq K_2 n] \geq 1 - C_n.$$

Proof Let λ be the drift of the random walk on $(\text{Out}(F_N), \mu)$ with respect to d_{CV_N} . In view of [Proposition 4.8](#), there exists a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $n \in \mathbb{N}$, there exists a measurable subspace X_n of the path space with $\mathbb{P}(X_n) \geq 1 - C_n$, and such that $d_{\text{CV}_N}(o, \Phi_n^{-1}.o) \leq 2\lambda n$ for all $\Phi := (\Phi_k)_{k \in \mathbb{N}} \in X_n$. For all $\Phi \in X_n$, since o and $\Phi_n^{-1}.o$ both belong to the thick part of CV_N , we have

$$d_{\text{CV}_N}(\Phi_n^{-1}.o, o) \leq 2M\lambda n$$

for some constant $M > 0$ only depending on the rank N of the free group [\[2\]](#). Since $\tau_\Phi(t_n(\Phi))$ is a closest-point projection of $\Phi_n^{-1}.o$ to the image of τ_Φ , we then have

$$d_{\text{CV}_N}(\Phi_n^{-1}.o, \tau_\Phi(t_n(\Phi))) \leq 2M\lambda n,$$

and the triangle inequality implies that

$$d_{\text{CV}_N}(o, \tau_{\Phi}(t_n(\Phi))) \leq 2(M + 1)\lambda n.$$

In view of Lemmas 4.2 and 4.3, there also exists $K > 0$, only depending on the rank N of the free group, such that

$$d_{\text{FF}_N}(\pi \circ \tau_{\Phi}(t_n(\Phi)), \mathbf{n}_{\pi \circ \tau_{\Phi}}(\pi(\Phi_n^{-1} \cdot o))) \leq K.$$

On the other hand, Propositions 2.8 and 2.11 imply that we can find a constant $K_2 > 0$, a sequence $(C'_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$, and for all $n \in \mathbb{N}$, a measurable subset X'_n of the path space with $\mathbb{P}(X'_n) \geq 1 - C'_n$, and such that for all $\Phi \in X'_n$,

$$d_{\text{FF}_N}(o, \mathbf{n}_{\pi \circ \tau_{\Phi}}(\pi(\Phi_n^{-1} \cdot o))) \geq K_2 n.$$

Then all sample paths $\Phi \in X_n \cap X'_n$ satisfy the required estimates, hence the proof. \square

Proposition 4.11 *Let μ be a nonelementary probability measure on $\text{Out}(F_N)$ with finite second moment with respect to d_{CV_N} . Then there exist $K_1, D, \epsilon, \beta > 0$ and a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that with probability at least $1 - C_n$, there exists a subsegment $I := [a, b] \subseteq [0, K_1 n]$ such that $\tau_{\Phi}(a) \in \text{CV}_N^{\epsilon}$, and $\pi \circ \tau_{\Phi}(I)$ lies at distance at least βn from o in FF_N , and τ_{Φ} is D -right-contracting along I .*

Proof The proof is similar to the proof of Proposition 3.12. Let C be a constant that satisfies the conclusions of Lemma 4.5 and Proposition 4.6. Let $K_1, K_2 > 0$ be the constants provided by Lemma 4.10, and let B be a positive real number greater than $4CK_1/K_2$. By Lemma 4.10, there exists a sequence $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ such that for all $n \in \mathbb{N}$, with probability at least $1 - C_n$, one has

$$(15) \quad d_{\text{CV}_N}(o, \tau_{\Phi}(t_n(\Phi))) \leq K_1 n \quad \text{and} \quad d_{\text{FF}_N}(o, \pi \circ \tau_{\Phi}(t_n(\Phi))) \geq K_2 n.$$

We claim that when (15) holds, assuming in addition that $n \geq 4C/K_2$, if we denote by $t_n^0(\Phi) > 0$ the smallest real number such that $\pi \circ \tau_{\Phi}([0, t_n^0(\Phi)])$ has d_{FF_N} -diameter at least $\frac{1}{4}K_2 n$, then the segment $\tau_{\Phi}([t_n^0(\Phi), t_n(\Phi)])$ contains a (B, C) -progressing subsegment. Proposition 4.11 will follow from this claim (with $\beta := \frac{1}{4}K_2$), together with Lemma 4.5, which states that progressing subsegments are thick, and Proposition 4.6, which states that progressing subsegments are right-contracting.

To prove the claim, we subdivide $\tau_{\Phi}([t_n^0(\Phi), t_n(\Phi)])$ into $\lceil K_1 n/B \rceil$ subsegments of d_{CV_N} -length at most B . If none of these segments had a π -image of diameter at least C , then the image $\pi \circ \tau_{\Phi}([t_n^0(\Phi), t_n(\Phi)])$ would have d_{FF_N} -diameter at most $(K_1 n/B + 1)C$, which is smaller than $\frac{1}{2}K_2 n$ since $n \geq 4C/K_2$. The image $\pi \circ \tau_{\Phi}([0, t_n(\Phi)])$ would then have diameter at most $\frac{3}{4}K_2 n$, which is a contradiction. The claim follows. \square

Proof of Proposition 4.9 The proof is similar to the proof of Proposition 3.9. Let $K_1, D, \epsilon, \beta > 0$ and $(C_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N})$ be as in Proposition 4.11. Let $x \in \mathcal{UE}$. We will show that there exists $\alpha' > 0$ such that for all $n \in \mathbb{N}$,

$$\mathbb{P}[(x | \text{bnd}(\Phi))_o \leq \alpha' n] \geq 1 - C_n.$$

Let $n \in \mathbb{N}$. By Proposition 4.11, and since the hitting measure ν^* on $\partial_h^+ \text{CV}_N$ is nonatomic, there exists a measurable subset X_n of the path space, of measure at least $1 - C_n$, such that for all $\Phi \in X_n$, we have $\text{bnd}(\Phi) \in \mathcal{UE} \setminus \{x\}$, and there exists a subsegment $I := [a, b] \subseteq [0, K_1 n]$, such that $\tau_\Phi(a) \in \text{CV}_N^\epsilon$, and $\pi \circ \tau_\Phi(I)$ lies at d_{FF_N} -distance at least βn from o , and τ_Φ is D -right-contracting along I . In view of Proposition 2.12 (and the existence of the map $\psi: \mathcal{UE} \rightarrow \partial_\infty \text{FF}_N$), we can also assume that for all $\Phi \in X$, and all sequences $(x_k)_{k \in \mathbb{N}} \in \text{CV}_N^\mathbb{N}$ converging to x , any geodesic segment from x_k to $\tau_\Phi(k)$ with $k \in \mathbb{N}$ sufficiently large contains a subsegment whose π -image fellow travels $\pi \circ \tau_\Phi(I)$ up to distance κ .

From now on, we let $\Phi \in X_n$. Let $(x_k)_{k \in \mathbb{N}} \in \text{CV}_N^\mathbb{N}$ be a sequence that converges to x . For all sufficiently large $k \in \mathbb{N}$, there exists a point $y_k \in \text{CV}_N$ on a standard geodesic segment γ_k from x_k to $\tau_\Phi(k)$ satisfying $d_{\text{CV}_N}^{\text{sym}}(y_k, \tau_\Phi(a)) \leq D$. The segments γ_k then accumulate [7, Theorem 6.6] to an optimal greedy folding line $\gamma: \mathbb{R} \rightarrow \text{CV}_N$ from x to $\text{bnd}(\Phi)$. There exists a point y lying on the image of γ (obtained as an accumulation point of the points y_k) such that $d_{\text{CV}_N}^{\text{sym}}(y, \tau_\Phi(a)) \leq D$. We have

$$(x | \text{bnd}(\Phi))_o = -\frac{1}{2} \inf_{z \in \text{CV}_N} \lim_{n \rightarrow +\infty} (h_{\gamma(-n)}(z) + h_{\gamma(n)}(z)),$$

with the notation from Section 1.2. It then follows from the triangle inequality that the infimum in the above formula is achieved at any point z lying on the image of γ . In particular, one has

$$(x | \text{bnd}(\Phi))_o = -\frac{1}{2} (h_x(y) + h_{\text{bnd}(\Phi)}(y)).$$

Since $\tau_\Phi(k)$ also converges to $\text{bnd}(\Phi)$ as k goes to $+\infty$, we have (all limits are taken as k goes to $+\infty$, and we use d to denote d_{CV_N} , and d^{sym} to denote $d_{\text{CV}_N}^{\text{sym}}$, for ease of notation) that

$$\begin{aligned} & -2(x | \text{bnd}(\Phi))_o \\ &= \lim [d(\gamma(-k), y) + d(y, \tau_\Phi(k)) - d(\gamma(-k), o) - d(o, \tau_\Phi(k))] \\ &\geq \lim [d(\gamma(-k), y) + (d(\tau_\Phi(a), \tau_\Phi(k)) - D) - d(\gamma(-k), o) - d(o, \tau_\Phi(k))] \\ &= \lim [d(\gamma(-k), y) - d(\gamma(-k), o)] - d(o, \tau_\Phi(a)) - D \\ &\geq -d(y, o) - d(o, \tau_\Phi(a)) - D \\ &\geq -2d^{\text{sym}}(o, \tau_\Phi(a)) - 2D. \end{aligned}$$

Since $\tau_{\Phi}(a) \in \text{CV}_N^\epsilon$, there exists [2] a constant $M > 0$ such that $d_{\text{CV}_N}^{\text{sym}}(o, \tau_{\Phi}(a)) \leq Md_{\text{CV}_N}(o, \tau_{\Phi}(a))$. This implies that

$$(x | \text{bnd}(\Phi))_o \leq MK_1 n + D$$

and concludes the proof of Proposition 4.9. □

4.4.2 Mean value of β^+ : hypothesis (H1) We now establish hypothesis (H1) from Theorem 1.6 for the cocycle β^+ : this is the content of Corollary 4.13 below.

Proposition 4.12 *Let μ be a nonelementary probability measure on $\text{Out}(F_N)$ with finite first moment with respect to d_{CV_N} , and let ν be the μ -stationary probability measure on $\partial_h^+ \text{CV}_N$. Then there exists a measurable subset $Y \subseteq \partial_h^+ \text{CV}_N$ with $\nu(Y) = 1$, such that for all $y \in Y$ and all $\epsilon > 0$, there exists $C > 0$ such that*

$$\mathbb{P} \left[\sup_{n \in \mathbb{N}} |\beta^+(\Phi_n, y) - d_{\text{CV}_N}(\Phi_n^{-1}.o, o)| \leq C \right] \geq 1 - \epsilon.$$

Proof Let $Y \subseteq \mathcal{UE}$ be a subset of $\partial_h^+ \text{CV}_N$ of ν -measure 1 such that for all $y \in Y$, any standard geodesic ray from o to y contains infinitely many D -right-contracting subsegments (the existence of Y was established in [14, Proposition 4.25]). It suffices to show that for all $y \in Y$ and \mathbb{P} -ae sample path $(\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Out}(F_N), \mu)$, one has

$$\sup_{n \in \mathbb{N}} |\beta^+(\Phi_n, y) - d_{\text{CV}_N}(\Phi_n^{-1}.o, o)| < +\infty.$$

Let $y \in Y$. For \mathbb{P} -ae sample path $(\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Out}(F_N), \mu)$, there exists a D -right-contracting subsegment I on a standard geodesic ray τ from o to y , and $z \in I$, such that for all sufficiently large $n, k \in \mathbb{N}$, any geodesic ray from $\Phi_n^{-1}.o$ to $\tau(k)$ passes at $d_{\text{CV}_N}^{\text{sym}}$ -distance at most D from z . For all $n \in \mathbb{N}$, we have (we write d instead of d_{CV_N} for ease of notation) that

$$|\beta^+(\Phi_n, y) - d(\Phi_n^{-1}.o, o)| = \lim_{k \rightarrow +\infty} |d(\Phi_n^{-1}.o, \tau(k)) - d(o, \tau(k)) - d(\Phi_n^{-1}.o, o)|.$$

This is equal up to a bounded error to $|d_{\text{CV}_N}(\Phi_n^{-1}.o, z) - d_{\text{CV}_N}(o, z) - d_{\text{CV}_N}(\Phi_n^{-1}.o, o)|$, which is bounded above by $2d_{\text{CV}_N}^{\text{sym}}(o, z)$ by the triangle inequality. □

Corollary 4.13 *Let μ be a nonelementary probability measure on $\text{Out}(F_N)$ with finite first moment with respect to d_{CV_N} . Let $\check{\lambda}$ be the drift of the random walk on $(\text{Out}(F_N), \check{\mu})$ with respect to d_{CV_N} , and let ν be the unique μ -stationary probability measure on $\partial_h^+ \text{CV}_N$. Then*

$$\int_{\text{Out}(F_N)} \int_{\partial_h^+ \text{CV}_N} \beta^+(g, y) d\nu(y) d\mu(g) = \check{\lambda}.$$

Proof We first notice that by Kingman’s subadditive ergodic theorem, for \mathbb{P} -ae sample path $(\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Out}(F_N), \mu)$, the limit

$$\lim_{n \rightarrow +\infty} \frac{1}{n} d_{\text{CV}_N}(\Phi_n^{-1}.o, o)$$

exists, and is equal to

$$\inf_{n \in \mathbb{N}} \int_{\text{Out}(F_N)} d(\Phi^{-1}.o, o) d\mu^{*n}(\Phi),$$

which is nothing but the drift $\check{\lambda}$. **Proposition 4.12** therefore implies that for ν -ae $y \in \partial_h^+ \text{CV}_N$ and \mathbb{P} -ae sample path $(\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Out}(F_N), \mu)$, one has

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \beta^+(\Phi_n, y) = \check{\lambda}.$$

Corollary 4.13 then follows from Birkhoff’s ergodic theorem, in the form provided by **Proposition 1.4**. □

4.5 Relating β^- , κ_{CV_N} and the length cocycle

We will relate the Busemann cocycle β^- , the function κ_{CV_N} and the length cocycle, in order to reduce the proof of the central limit theorem (**Theorem 0.2**) to proving a central limit theorem for β^- . The following proposition is a *dual* version of **Proposition 4.12** for the cocycle β^- .

Proposition 4.14 *Let μ be a nonelementary probability measure on $\text{Out}(F_N)$ with finite first moment with respect to d_{CV_N} . Then for all $x \in \mathcal{UE}$ and all $\epsilon > 0$, there exists $C > 0$ such that*

$$\mathbb{P} \left[\sup_{n \in \mathbb{N}} |\beta^-(\Phi_n, x) - \kappa(\Phi_n)| \leq C \right] \geq 1 - \epsilon.$$

Proof It suffices to prove that for all $x \in \mathcal{UE}$ and \mathbb{P} -ae sample path $\Phi := (\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Out}(F_N), \mu)$, one has

$$\sup_{n \in \mathbb{N}} |\beta^-(\Phi_n, x) - \kappa(\Phi_n)| < +\infty.$$

For \mathbb{P} -ae sample path Φ of the random walk, the standard geodesic ray τ_Φ contains infinitely many D -bicontracting subsegments [14, Proposition 4.25]. Let $(x_k)_{k \in \mathbb{N}} \in \text{CV}_N^{\mathbb{N}}$ be a sequence that converges to x . We have

$$|\beta^-(\Phi_n, x) - \kappa(\Phi_n)| = \lim_{k \rightarrow +\infty} |d_{\text{CV}_N}(x_k, \Phi_n^{-1}.o) - d_{\text{CV}_N}(x_k, o) - d_{\text{CV}_N}(o, \Phi_n^{-1}.o)|.$$

By the bicontraction property, there exists $z \in \tau_\Phi(\mathbb{R}_+)$ such that for all sufficiently large $k, n \in \mathbb{N}$, any standard geodesic segment from x_k to $\Phi_n^{-1}.o$ passes at bounded $d_{\text{CV}_N}^{\text{sym}}$ -distance from z , and similarly any standard geodesic segment from o to $\Phi_n^{-1}.o$

passes at bounded $d_{\text{CV}_N}^{\text{sym}}$ -distance from z . Therefore, $|\beta^-(\Phi_n, x) - \kappa(\Phi_n)|$ is equal up to a bounded error to

$$\lim_{k \rightarrow +\infty} |d_{\text{CV}_N}(x_k, z) - d_{\text{CV}_N}(x_k, o) - d_{\text{CV}_N}(o, z)| = |h_x^-(z) - d_{\text{CV}_N}(o, z)|,$$

which concludes the proof of Proposition 4.14. □

The following proposition is a version of Proposition 4.14 above in the case where x is a rational current associated to a primitive conjugacy class, instead of $x \in \mathcal{UE}$.

Proposition 4.15 *Let μ be a nonelementary probability measure on $\text{Out}(F_N)$ with finite first moment with respect to d_{CV_N} . Then for all $g \in \mathcal{P}_N$ and all $\epsilon > 0$, there exists $C > 0$ such that*

$$\mathbb{P} \left[\sup_{n \in \mathbb{N}} \left| \kappa(\Phi_n) - \log \frac{\|\Phi_n(g)\|_o}{\|g\|_o} \right| \leq C \right] \geq 1 - \epsilon.$$

Proof It is enough to prove that for \mathbb{P} -ae sample path $\Phi := (\Phi_n)_{n \in \mathbb{N}}$ of the random walk on $(\text{Out}(F_N), \mu)$, we have

$$\sup_{n \in \mathbb{N}} \left| \kappa(\Phi_n) - \log \frac{\|\Phi_n(g)\|_o}{\|g\|_o} \right| < +\infty.$$

By [14, Proposition 4.25], for \mathbb{P} -ae sample path $\Phi := (\Phi_n)_{n \in \mathbb{N}}$ of the random walk, the standard geodesic ray τ_Φ contains infinitely many D -bicontracting subsegments. Let $g \in \mathcal{P}_N$, and let I be a D -bicontracting subsegment of τ_Φ lying to the right of $\tau_\Phi(\text{Pr}_{\tau_\Phi}(g))$.

For all $n \in \mathbb{N}$, let β_n be a standard geodesic segment from o to $\Phi_n^{-1}.o$. Then there exists $z' \in I$ such that for all sufficiently large $n \in \mathbb{N}$, the segment β_n contains a point z_n at bounded $d_{\text{CV}_N}^{\text{sym}}$ -distance of z' . In addition, Lemma 4.1 applied to β_n implies that for all $n \in \mathbb{N}$, we have

$$\left| d_{\text{CV}_N}(z_n, \Phi_n^{-1}.o) - \log \frac{\|\Phi_n(g)\|_o}{\|g\|_{z_n}} \right| \leq K.$$

Since all points z_n lie at the same $d_{\text{CV}_N}^{\text{sym}}$ -distance from o (up to a bounded additive error), there exists $K' > 0$ such that

$$\left| d_{\text{CV}_N}(o, \Phi_n^{-1}.o) - \log \frac{\|\Phi_n(g)\|_o}{\|g\|_o} \right| \leq K',$$

which is the desired inequality. □

4.6 Central limit theorem

We now complete the proof of the central limit theorem for the variables $\log \|\Phi(g)\|$ with $g \in \mathcal{P}_N$.

Theorem 4.16 *Let μ be a nonelementary probability measure on $\text{Out}(F_N)$ with finite second moment with respect to d_{CV_N} . Let $\lambda > 0$ be the drift of the random walk on $(\text{Out}(F_N), \mu)$ with respect to d_{CV_N} . Then there exists a centered Gaussian law N_μ on \mathbb{R} such that for every compactly supported continuous function F on \mathbb{R} , and all primitive elements $g \in \mathcal{P}_N$, one has*

$$\lim_{n \rightarrow +\infty} \int_{\text{Out}(F_N)} F\left(\frac{\log \|\Phi(g)\| - n\lambda}{\sqrt{n}}\right) d\mu^{*n}(\Phi) = \int_{\mathbb{R}} F(t) dN_\mu(t),$$

uniformly in g .

Proof In view of Propositions 4.14 and 4.15, it is enough to show that there exist $x \in \mathcal{UE}$ and a centered Gaussian law N_μ on \mathbb{R} such that for every compactly supported continuous function F on \mathbb{R} , one has

$$\lim_{n \rightarrow +\infty} \int_{\text{Mod}(S)} F\left(\frac{\beta^-(\Phi, x) - n\lambda}{\sqrt{n}}\right) d\mu^{*n}(\Phi) = \int_{\mathbb{R}} F(t) dN_\mu(t).$$

This follows from Theorem 1.6 applied to $Y^- := \mathcal{UE}$ and to $Y^+ := \partial_h^+ \text{CV}_N$ (hypothesis (H1) is checked in Corollary 4.13, applied to $\check{\mu}$, and hypothesis (H2) is checked in Proposition 4.9). \square

References

- [1] **Y Algom-Kfir**, *Strongly contracting geodesics in outer space*, *Geom. Topol.* 15 (2011) 2181–2233 [MR](#)
- [2] **Y Algom-Kfir**, **M Bestvina**, *Asymmetry of outer space*, *Geom. Dedicata* 156 (2012) 81–92 [MR](#)
- [3] **Y Benoist**, **J-F Quint**, *Central limit theorem for linear groups*, *Ann. Probab.* 44 (2016) 1308–1340 [MR](#)
- [4] **Y Benoist**, **J-F Quint**, *Central limit theorem on hyperbolic groups*, *Izv. Ross. Akad. Nauk Ser. Mat.* 80 (2016) 5–26 [MR](#) In Russian; translated in *Izvestiya: Mathematics* 80 (2016) 3–23
- [5] **M Bestvina**, **M Feighn**, *Outer limits*, preprint (1994) Available at <http://andromeda.rutgers.edu/~feighn/papers/outer.pdf>

- [6] **M Bestvina, M Feighn**, *Hyperbolicity of the complex of free factors*, Adv. Math. 256 (2014) 104–155 [MR](#)
- [7] **M Bestvina, P Reynolds**, *The boundary of the complex of free factors*, Duke Math. J. 164 (2015) 2213–2251 [MR](#)
- [8] **M Björklund**, *Central limit theorems for Gromov hyperbolic groups*, J. Theoret. Probab. 23 (2010) 871–887 [MR](#)
- [9] **B M Brown**, *Martingale central limit theorems*, Ann. Math. Statist. 42 (1971) 59–66 [MR](#)
- [10] **Y-E Choi, K Rafi**, *Comparison between Teichmüller and Lipschitz metrics*, J. Lond. Math. Soc. 76 (2007) 739–756 [MR](#)
- [11] **MM Cohen, M Lustig**, *Very small group actions on \mathbf{R} -trees and Dehn twist automorphisms*, Topology 34 (1995) 575–617 [MR](#)
- [12] **M Culler, JW Morgan**, *Group actions on \mathbf{R} -trees*, Proc. London Math. Soc. 55 (1987) 571–604 [MR](#)
- [13] **M Culler, K Vogtmann**, *Moduli of graphs and automorphisms of free groups*, Invent. Math. 84 (1986) 91–119 [MR](#)
- [14] **F Dahmani, C Horbez**, *Spectral theorems for random walks on mapping class groups and $\text{Out}(F_N)$* , preprint (2015) [arXiv](#)
- [15] **S Dowdall, M Duchin, H Masur**, *Statistical hyperbolicity in Teichmüller space*, Geom. Funct. Anal. 24 (2014) 748–795 [MR](#)
- [16] **S Dowdall, S J Taylor**, *Hyperbolic extensions of free groups*, preprint (2014) [arXiv](#)
- [17] **A Fathi, F Laudenbach, V Poenaru** (editors), *Travaux de Thurston sur les surfaces*, Astérisque 66, Soc. Math. France, Paris (1979) [MR](#)
- [18] **S Francaviglia, A Martino**, *Metric properties of outer space*, Publ. Mat. 55 (2011) 433–473 [MR](#)
- [19] **H Furstenberg**, *Noncommuting random products*, Trans. Amer. Math. Soc. 108 (1963) 377–428 [MR](#)
- [20] **H Furstenberg, H Kesten**, *Products of random matrices*, Ann. Math. Statist. 31 (1960) 457–469 [MR](#)
- [21] **FP Gardiner, H Masur**, *Extremal length geometry of Teichmüller space*, Complex Variables Theory Appl. 16 (1991) 209–237 [MR](#)
- [22] **E Ghys, P de la Harpe** (editors), *Sur les groupes hyperboliques d’après Mikhael Gromov*, Progress in Mathematics 83, Birkhäuser, Boston (1990) [MR](#)
- [23] **IY Gol’dsheĭd, G A Margulis**, *Lyapunov exponents of a product of random matrices*, Uspekhi Mat. Nauk 44 (1989) 13–60 [MR](#) In Russian; translated in [Russian Math. Surveys](#) 44 (1989), 11–71

- [24] **V Guirardel**, *Dynamics of $\text{Out}(F_N)$ on the boundary of outer space*, Ann. Sci. École Norm. Sup. 33 (2000) 433–465 [MR](#)
- [25] **Y Guivarc’h, A Raugi**, *Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence*, Z. Wahrsch. Verw. Gebiete 69 (1985) 187–242 [MR](#)
- [26] **U Hamenstädt**, *The boundary of the free splitting graph and the free factor graph*, preprint (2012) [arXiv](#)
- [27] **C Horbez**, *The boundary of the outer space of a free product*, preprint (2014) [arXiv](#)
- [28] **C Horbez**, *The horoboundary of outer space, and growth under random automorphisms*, Ann. Sci. Éc. Norm. Supér. 49 (2016) 1075–1123 [MR](#)
- [29] **C Horbez**, *Spectral rigidity for primitive elements of F_N* , J. Group Theory 19 (2016) 55–123 [MR](#)
- [30] **V A Kaimanovich, H Masur**, *The Poisson boundary of the mapping class group*, Invent. Math. 125 (1996) 221–264 [MR](#)
- [31] **I Kapovich**, *Currents on free groups*, from “Topological and asymptotic aspects of group theory” (R Grigorchuk, M Mihalik, M Sapir, Z Šuník, editors), Contemp. Math. 394, Amer. Math. Soc., Providence, RI (2006) 149–176 [MR](#)
- [32] **I Kapovich, M Lustig**, *Geometric intersection number and analogues of the curve complex for free groups*, Geom. Topol. 13 (2009) 1805–1833 [MR](#)
- [33] **A Karlsson**, *Two extensions of Thurston’s spectral theorem for surface diffeomorphisms*, Bull. Lond. Math. Soc. 46 (2014) 217–226 [MR](#)
- [34] **SP Kerckhoff**, *The asymptotic geometry of Teichmüller space*, Topology 19 (1980) 23–41 [MR](#)
- [35] **J F C Kingman**, *The ergodic theory of subadditive stochastic processes*, J. Roy. Statist. Soc. Ser. B 30 (1968) 499–510 [MR](#)
- [36] **E Klarreich**, *The boundary at infinity of the curve complex and the relative Teichmüller space*, preprint (1999) Available at <http://tinyurl.com/Klarreich-cc>
- [37] **E Le Page**, *Théorèmes limites pour les produits de matrices aléatoires*, from “Probability measures on groups” (H Heyer, editor), Lecture Notes in Math. 928, Springer (1982) 258–303 [MR](#)
- [38] **A Lenzen, K Rafi, J Tao**, *Bounded combinatorics and the Lipschitz metric on Teichmüller space*, Geom. Dedicata 159 (2012) 353–371 [MR](#)
- [39] **L Liu, W Su**, *The horofunction compactification of the Teichmüller metric*, from “Handbook of Teichmüller theory, IV” (A Papadopoulos, editor), IRMA Lect. Math. Theor. Phys. 19, Eur. Math. Soc., Zürich (2014) 355–374 [MR](#)
- [40] **J Maher, G Tiozzo**, *Random walks on weakly hyperbolic groups*, preprint (2015) [arXiv](#)

- [41] **R Martin**, *Non-uniquely ergodic foliations of thin type*, Ergodic Theory Dynam. Systems 17 (1997) 667–674 [MR](#)
- [42] **H A Masur, Y N Minsky**, *Geometry of the complex of curves, I: Hyperbolicity*, Invent. Math. 138 (1999) 103–149 [MR](#)
- [43] **P Mathieu, A Sisto**, *Deviation inequalities for random walks*, preprint (2014) [arXiv](#)
- [44] **Y N Minsky**, *Extremal length estimates and product regions in Teichmüller space*, Duke Math. J. 83 (1996) 249–286 [MR](#)
- [45] **H Miyachi**, *Teichmüller rays and the Gardiner–Masur boundary of Teichmüller space, II*, Geom. Dedicata 162 (2013) 283–304 [MR](#)
- [46] **H Miyachi**, *Unification of extremal length geometry on Teichmüller space via intersection number*, Math. Z. 278 (2014) 1065–1095 [MR](#)
- [47] **H Namazi, A Pettet, P Reynolds**, *Ergodic decompositions for folding and unfolding paths in outer space*, preprint (2014) [arXiv](#)
- [48] **K Rafi**, *Thick-thin decomposition for quadratic differentials*, Math. Res. Lett. 14 (2007) 333–341 [MR](#)
- [49] **S Sawyer, T Steger**, *The rate of escape for anisotropic random walks in a tree*, Probab. Theory Related Fields 76 (1987) 207–230 [MR](#)
- [50] **C Walsh**, *The horoboundary and isometry group of Thurston’s Lipschitz metric*, from “Handbook of Teichmüller theory, IV” (A Papadopoulos, editor), IRMA Lect. Math. Theor. Phys. 19, Eur. Math. Soc., Zürich (2014) 327–353 [MR](#)

Laboratoire de Mathématiques d’Orsay, CNRS - Université Paris Sud
Orsay, France

camille.horbez@math.u-psud.fr

Proposed: Martin Bridson

Received: 27 September 2015

Seconded: Bruce Kleiner, Jean-Pierre Otal

Revised: 7 July 2016