Geometry & Topology 22 (2018) 235–303

A mathematical theory of the gauged linear sigma model

HUIJUN FAN
TYLER J JARVIS
YONGBIN RUAN

We construct a mathematical theory of Witten's Gauged Linear Sigma Model (GLSM). Our theory applies to a wide range of examples, including many cases with nonabelian gauge group.

Both the Gromov–Witten theory of a Calabi–Yau complete intersection X and the Landau–Ginzburg dual (FJRW theory) of X can be expressed as gauged linear sigma models. Furthermore, the Landau–Ginzburg/Calabi–Yau correspondence can be interpreted as a variation of the moment map or a deformation of GIT in the GLSM. This paper focuses primarily on the algebraic theory, while a companion article will treat the analytic theory.

14D23, 14L24, 14N35, 53D45, 81T60; 14J32, 14L30, 32G81, 81T40

1 Introduction

In 1991 a celebrated conjecture of Witten [50] asserted that the intersection theory of Deligne–Mumford moduli space is governed by the KdV hierarchy. His conjecture was soon proved by Kontsevich [34]. The KdV hierarchy is the first of a family of integrable hierarchies (Drinfeld–Sokolov/Kac–Wakimoto hierarchies) associated to integrable representations of affine Kac–Moody algebras. Immediately after Kontsevich's solution

Published: 6 November 2017 DOI: 10.2140/gt.2018.22.235

of Witten's conjecture, a great deal of effort was spent in investigating other integrable hierarchies in Gromov–Witten theory. In fact, this question was very much in Witten's mind when he proposed his famous conjecture in the first place. Around the same time, he also proposed a sweeping generalization of his conjecture [51; 52]. The core of his generalization is a remarkable first-order, nonlinear, elliptic PDE associated to an arbitrary quasihomogeneous singularity. It has the simple form

$$\bar{\partial} u_i + \overline{\partial W / \partial u_i} = 0, \tag{1}$$

where W is a quasihomogeneous polynomial, and u_i is interpreted as a section of an appropriate orbifold line bundle on an orbifold Riemann surface $\mathscr{C}$.

During the last decade, a comprehensive treatment of the Witten equation has been carried out, and a new theory like Gromov–Witten has been constructed by Fan, Jarvis and Ruan [23; 24; 25]. In particular, Witten's conjecture for ADE-integrable hierarchies has been verified (for the A series by Faber, Shadrin and Zvonkine [22] and Lee [36], and for the D and E series by Fan, Jarvis and Ruan [25]).

The so-called *FJRW theory* has applications beyond the ADE-integrable hierarchy conjecture. For example, it can be viewed as the Landau–Ginzburg dual of a Calabi–Yau hypersurface

$$X_W = \{W = 0\} \subset W\mathbb{P}^{N-1}$$

in weighted projective space. The relation between the Gromov–Witten theory of X_W and the FJRW theory of W is the subject of the Landau–Ginzburg/Calabi–Yau correspondence, a famous duality from physics. More recently, the LG/CY correspondence has been reformulated as a precise mathematical conjecture, and a great deal of progress has been made on this conjecture; see Chiodo and Ruan [13; 14], Chiodo, Iritani and Ruan [12], Lee, Priddis and Shoemaker [37] and Priddis and Shoemaker [44].

A natural question is whether the LG/CY correspondence can be generalized to complete intersections in projective space, or more generally to toric varieties. The physicists' answer is "yes". In fact, Witten considered this question in the early 1990s [51] in his effort to give a physical derivation of the LG/CY correspondence. In the process, he invented an important model in physics called the *gauged linear sigma model* (*GLSM*). From the point of view of partial differential equations, the gauged linear sigma model generalizes the Witten equation (1) to the *gauged Witten equation*

$$\bar{\partial}_A u_i + \overline{\partial W / \partial u_i} = 0, \tag{2}$$

$$*F_A = \mu, \tag{3}$$

where A is a connection of a certain principal bundle, and μ is the moment map of the GIT quotient, viewed as a symplectic quotient. In general, both the Gromov–

Witten theory of a Calabi–Yau complete intersection X and the LG dual of X can be expressed as gauged linear sigma models. Furthermore, the LG/CY correspondence can be interpreted as a variation of the moment map μ (or a deformation of GIT) in the GLSM.

The main purpose of this article and its companion [26] is to construct a rigorous mathematical theory for the gauged linear sigma model. This new model has many applications and some of them are already under way (see for example Ross and Ruan [47] and work in progress of Ross, Ruan and Shoemaker, and Clader, Janda and Ruan [19]).

An important phenomenon in FJRW theory is that the state space is a direct sum of *narrow* and *broad* sectors. The theory for the narrow sectors admits a purely algebraic construction in terms of cosection localization. A similar situation holds for the GLSM: we have both broad and narrow sectors, but the narrow sectors are a subset of a larger class called *compact type*. We show in this paper how to use cosection localization to describe the GLSM algebraically for sectors of compact type. The analytic theory for more general broad sectors, and the relation to other approaches like that of Tian and Xu [49], will appear in a companion article [26].

1.1 Brief description of the theory

The *input data* of our new theory is

(I) A finite-dimensional vector space V over $\mathbb{C}$.

(II) A reductive algebraic group $G \subseteq \mathrm{GL}(V)$.

(III) A G–character θ with the property $V^s_G(\theta) = V^{ss}_G(\theta)$. We say that it defines a *strongly regular phase* $\mathscr{X}_\theta = [V /\!\!/_\theta G]$.

(IV) A choice of $\mathbb{C}^*$–action (R–charge) on V (denoted by $\mathbb{C}^*_R$) that is *compatible* with G, ie commuting with G–action, and such that $G \cap \mathbb{C}^*_R = \langle J \rangle$ has finite order d. Denote the subgroup of $\mathrm{GL}(V)$ generated by G and $\mathbb{C}^*_R$ by Γ.

(V) A G–invariant *superpotential* $W\colon V \to \mathbb{C}$ of degree d with respect to the $\mathbb{C}^*_R$–action with the property that the GIT quotient $\mathscr{CR}_\theta$ of the critical locus $\mathrm{Crit}(W)$ is compact.

(VI) A stability parameter $\varepsilon > 0$ in $\mathbb{Q}$. We also often write $\varepsilon = 0+$ to indicate the limit as $\varepsilon \downarrow 0$ or $\varepsilon = \infty$ to indicate the limit as $\varepsilon \to \infty$.

(VII) If $\varepsilon > 0$, a Γ character ϑ that defines a *good lift* of θ, meaning that $\vartheta|_G = \theta$ and $V^{ss}_\Gamma(\vartheta) = V^{ss}_G(\theta)$. The good lift provides some stability conditions for the moduli space. But in the case of $\varepsilon = 0+$ the good lift is unnecessary.

With the above input data we construct a theory with following main ingredients:

(I) **A state space** This is the relative Chen–Ruan cohomology of the quotient $\mathscr{X}_\theta = [V/\!\!/_\theta G]$ with an additional shift by $2q$:

$$\mathscr{H}_{W,G} = \bigoplus_{\alpha\in\mathbb{Q}} \mathscr{H}^{\alpha}_{W,G} = \bigoplus_{\Psi} \mathscr{H}_\Psi,$$

where the sum runs over those conjugacy classes Ψ of G for which $\mathscr{X}_{\theta,\Psi}$ is nonempty, and where

$$\mathscr{H}^{\alpha}_{W,G} = H^{\alpha+2q}_{\mathrm{CR}}(\mathscr{X}_\theta, W^\infty, \mathbb{Q}) = \bigoplus_{\Psi} H^{\alpha-2\operatorname{age}(\gamma)+2q}(\mathscr{X}_\Psi, W^\infty_\Psi, \mathbb{Q}),$$

and

$$\mathscr{H}_\Psi = H^{\bullet+2q}_{\mathrm{CR}}(\mathscr{X}_{\theta,\Psi}, W^\infty, \mathbb{Q}) = \bigoplus_{\alpha\in\mathbb{Q}} H^{\alpha-2\operatorname{age}(\gamma)+2q}(\mathscr{X}_{\theta,\Psi}, W^\infty_\Psi, \mathbb{Q}).$$

Here $W^\infty = \mathfrak{Re}(W)^{-1}(M,\infty) \subset [V/\!\!/_\theta G]$ for some large, real M (see Section 4.1 for details).

(II) **The moduli space of LG quasimaps** We denote by $\mathscr{CR}_\theta = [\mathrm{Crit}^{\mathrm{ss}}_G(\theta)/G] \subset [V/\!\!/_\theta G] = [V^{\mathrm{ss}}_G(\theta)/G]$ the GIT quotient (with polarization θ) of the critical locus of W. Our main object of study is the stack

$$\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta)$$

of *Landau–Ginzburg quasimaps to* $\mathscr{CR}_\theta$ (see the precise definition in Section 4.2). The definition of the stack works equally well if the vector space V is replaced by a closed subvariety, but we have focused on the case of $V = \mathbb{C}^n$ for simplicity.

The main technical theorem of the article is:

Theorem 1.1.1 $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta)$ *is a proper Deligne–Mumford stack whenever* $\mathscr{CR}_\theta$ *is proper.*

(III) **A virtual cycle** The stack $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta)$ is naturally embedded into the stack $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}([V/\!\!/_\theta G], \beta)$. The latter is not compact, but it admits a two-term perfect obstruction theory with a cosection whose degeneracy locus is precisely $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta)$.

Applying Kiem and Li's theory [32] of cosection localized virtual cycles, and adapting the cosection introduced to the LG model by Chang, Li and Li [6; 7] to the stack $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}([V/\!\!/_\theta G], \beta)$, we can construct a virtual cycle

$$[\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta)]^{\mathrm{vir}} \in H_*(\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta), \mathbb{Q})$$

with virtual dimension

$$\dim_{\mathrm{vir}} = \int_\beta c_1(V/\!\!/_\theta G) + (\hat{c}_{W,G} - 3)(1-g) + k - \sum_i (\mathrm{age}(\gamma_i) - q),$$

where $\hat{c}_{W,G}$ is the *central charge* (see Definition 3.2.3).

(IV) **Numerical invariants** Once we construct the virtual cycle, we can define correlators

$$\langle \tau_{l_1}(\alpha_1), \dots, \tau_{l_k}(\alpha_k) \rangle = \int_{[\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta,\beta)]^{\mathrm{vir}}} \prod_i \mathrm{ev}_i^*(\alpha_i)\psi_i^{l_i},$$

where $\alpha_i \in \mathscr{H}_{W,G}$ is of compact type (see Definition 4.1.4). One can define a generating function in the standard fashion. These invariants satisfy the usual gluing axioms whenever all insertions are narrow.

Almost all known examples in physics satisfy the conditions of our input data, and hence our theory applies. We list several examples in the paper. To keep this article to a reasonable length, we will not spend much time on the many applications, but rather we focus on the algebraic construction of the theory in this paper and on the analytic construction in its companion article [26].

We should mention that the equation for the case $W = 0$ has been studied already in mathematics under the name *symplectic vortex equation*. There is a large amount of work on this in both the algebraic and symplectic setting. A particularly important piece of work for us is the theory of stable quotients — see Marian, Oprea and Pandharipande [38] — and stable quasimaps; see Cheong, Ciocan-Fontanine, Kim and Maulik [10; 15; 16; 33]. In fact, our new theory can be treated as a unification of FJRW theory with stable quasimaps.

There are two important special cases which we use to check the consistency of our theory. The first one is the theory of stable maps with p–fields by Chang and Li [6], which corresponds to the geometric phase of our theory with an $\varepsilon = \infty$ stability condition. The other one is the *hybrid model* of Clader [17; 18]. Unfortunately, that hybrid model only works for a very restrictive situation. The theory we describe here corresponds to a much more general situation, including complete intersections of toric varieties and even quotients by nonabelian groups. But an understanding of the failure of the hybrid model for general complete intersections motivated much of our construction. In this article, we will focus on the sectors of compact type and our construction will be completely algebraic. Finally, the virtual cycle construction relies on Kiem and Li's theory of cosection localized virtual cycles — see Kiem and Li [32] — and using the cosection introduced to the LG model by Chang, Li and Li [7].

The GLSM can be viewed as a generalization of FJRW theory from a hypersurface with a finite abelian gauge group G to more general spaces with an arbitrary reductive gauge groups.

The results of this article were first announced by the second author at the workshop *Geometry and Physics of the Gauged Linear Sigma Model* in March, 2013 in Michigan. In the lecture, the second author gave a complete construction of the moduli space. The only thing missing was the full detail of the proof of various properties of the moduli space. We apologize for the long delay in producing those details.

Acknowledgments Jarvis thanks Emily Clader, Dan Edidin and Bumsig Kim for helpful conversations. Ruan would like to thank Kentaro Hori for many helpful conversations on GLSM, Huai-Liang Chang, Jun Li, Wei-Ping Li for many helpful discussions on the cosection technique, and Emily Clader for helpful discussions on the hybrid model. A special thank goes to E Witten for introducing us to the gauged Witten equation and for many insightful conversation over the years. Finally, we thank the referee for many helpful suggestions, which greatly improved the paper. Ruan's work was partially supported by NSF grants DMS 1159265 and DMS 1405245. Jarvis's work was partially supported by NSF grant DMS 1564502.

2 Brief review of FJRW theory

In this section, we review the basic elements of FJRW theory. Our new generalization will follow the blueprint of this older case closely.

2.1 The basic construction

The basic starting point is a $\mathbb{C}^*$–action on $\mathbb{C}^N$ with positive weights $(c_1, \dots, c_N)$ and a nondegenerate polynomial $W \in \mathbb{C}[x_1, \dots, x_N]$ of degree $d > 1$ with respect to the $\mathbb{C}^*$–action. We also choose a *gauge group* G of diagonal symmetries of W. We think of both $\mathbb{C}^*$ and G as subgroups of $\mathrm{GL}(N, \mathbb{C})$. Let $J = (\exp(2\pi i c_1/d), \dots, \exp(2\pi i c_N/d)) \in \mathbb{C}^* \subset \mathrm{GL}(N, \mathbb{C})$. We require $\mathbb{C}^* \cap G = \langle J \rangle$.

In order for the Witten equation (1) to make sense, we work with roots of the log-canonical bundle

$$\omega_{\log,\mathscr{C}} = \omega_{\mathscr{C}}\left(\sum_{i=1}^{k} y_i\right).$$

Specifically, we work on the space of W–curves, which are tuples

$$(\mathscr{C}, \varphi_j \colon \mathscr{L}_j^{\otimes d} \to \omega_{\log,\mathscr{C}}^{\otimes c_j}) \quad \text{for } j \in \{1, \dots, N\},$$

where $\mathscr{C}$ is a stable orbifold curve, each $\mathscr{L}_j$ is an orbifold line bundle on $\mathscr{C}$, and each $\varphi_j\colon \mathscr{L}_j^{\otimes d} \to \omega_{\log,\mathscr{C}}^{\otimes c_j}$ makes $\mathscr{L}_j$ in to a d^{th} root of the c_j^{th} power of $\omega_{\log,\mathscr{C}}$. Some additional conditions are also required of the W–structure, namely:

(I) At each point y the induced representation $\rho_p\colon G_y \to (\mathbb{C}^*)^N$ of the local group G_y of $\mathscr{C}$ at y on the sum $\bigoplus_{i=1}^N \mathscr{L}_i$ is faithful.

(II) If s is the number of monomials $W_1, \dots, W_s$ of W, then for each $i = 1, \dots, s$ the isomorphisms $\{\varphi_j\}_{i=1}^N$ induce isomorphisms

$$W_i(\mathscr{L}_1, \dots, \mathscr{L}_N) = \bigotimes_{j=1}^N \mathscr{L}_j^{\otimes e_{ij}} \xrightarrow{\sim} \omega_{\log,\mathscr{C}},$$

where the e_{ij} are the exponents of W_i.

Let $\mathscr{W}_{g,k}^{W,G}$ be the stack of stable W–curves with the property that at each marked point y the image of the local group G_y under the representation $\rho_y\colon G_y \to (CC^*)^N$ lies in G. In the formulation we have given here, FJRW theory naturally corresponds to the orbifolded Landau–Ginzburg A–model for the superpotential W on the orbifold $[\mathbb{C}^N/G_{\max}]$. As we will describe below, it is possible to generalize it to $[\mathbb{C}^N/G]$ for any subgroup G containing the element J. But the current theory does not work for any group smaller than $\langle J\rangle$ in any generality.

A marked point y_j of a W–curve is called *narrow* if the fixed point locus $\mathrm{Fix}(\rho_y(G_y)) \subseteq \mathbb{C}^N$ is just $\{0\}$. The point y_j is called *broad* otherwise, and any coordinates z_i for $\mathbb{C}^N$ fixed by G_y are called broad variables.

There are several natural morphisms of $\mathscr{W}_{g,k}^{W,G}$ analogous to the morphisms of the stack $\overline{\mathscr{M}}_{g,k}(X,\beta)$ of stable maps, including a *stabilization* map. Forgetting the W–structure and the orbifold structure gives a morphism

$$\mathrm{st}\colon \mathscr{W}_{g,k}^{W,G} \to \overline{\mathscr{M}}_{g,k}.$$

A key result in the theory states that $\mathscr{W}_{g,k}^{W,G}$ is a compact, smooth complex orbifold with projective coarse moduli space, and st is a finite morphism (but not representable).

2.2 The Polishchuk–Vaintrob construction

Polishchuk and Vaintrob [42] have given an alternative formulation for the W–structures in terms of principal bundles. Although it is maybe not quite as easy to see how to define the Witten equation in this construction, it simplifies the description of the stack of W–curves and has the advantage of making clear that the resulting stacks depend only on the (finite, abelian) group G and not on the superpotential W. This construction also inspires part of our generalization to the more general theory for arbitrary (infinite and possibly nonabelian) groups.

Let $G \subset \mathrm{Aut}(W)$ be a finite subgroup containing J, and let Γ be the subgroup of $(\mathbb{C}^*)^N$ generated by G and $\mathbb{C}^*_R = \{(\lambda^{c_1}, \dots, \lambda^{c_N}) \mid \lambda \in \mathbb{C}^*\}$, where this $\mathbb{C}^*_R$ corresponds the quasihomogeneity of W. It is easy to see that

$$G \cap \mathbb{C}^*_R = \langle J \rangle. \tag{4}$$

We can define a surjective homomorphism

$$\zeta\colon \Gamma \to \mathbb{C}^*$$

by sending G to 1 and $(\lambda^{c_1}, \dots, \lambda^{c_N})$ to λ^d. Equation (4) shows that the map ζ is well-defined and that $\ker(\zeta) = G$. Let $\mathring{\omega}_{\log,\mathscr{C}}$ denote the principal $\mathbb{C}^*$–bundle associated to $\omega_{\log,\mathscr{C}}$.

Definition 2.2.1 A Γ–structure on an orbicurve $\mathscr{C}$ is:

(I) A principal Γ–bundle $\mathscr{P}$ on $\mathscr{C}$ such that the corresponding map $\mathscr{C} \to \mathrm{B}\Gamma$ to the classifying stack $\mathrm{B}\Gamma$ is representable.

(II) A choice of isomorphism $\varkappa\colon \zeta_*\mathscr{P} \cong \mathring{\omega}_{\log,\mathscr{C}}$. Here $\zeta_*\mathscr{P}$ denotes the principal $\mathbb{C}^*$–bundle on $\mathscr{C}$ induced from $\mathscr{P}$ by the homomorphism ζ.

An equivalent way to state (II) is to recognize that the homomorphism ζ induces a morphism of stacks $\mathrm{B}\zeta\colon \mathrm{B}\Gamma \to \mathrm{B}\mathbb{C}^*$ and (II) is equivalent to the requirement that the composition $\mathrm{B}\zeta \circ \mathscr{P}\colon \mathscr{C} \to \mathrm{B}\mathbb{C}^*$ be equal to the morphism of stacks $\mathscr{C} \to \mathrm{B}\mathbb{C}^*$ induced by the line bundle $\omega_{\log,\mathscr{C}}$.

Let's match this new definition with the definition of a W–structure. The projections $\pi_i\colon \Gamma \subseteq (\mathbb{C}^*)^N \to \mathbb{C}^*$ to the i^{th} factor for each $i \in \{1, \dots N\}$ define a collection of line bundles $(\mathscr{L}_1, \dots, \mathscr{L}_N)$. It is easy to check that $\pi_i^d = \zeta^{c_i}$. And thus we have

$$\mathscr{L}_i^d = \omega_{\mathscr{C},\log}^{c_i}.$$

Let $W = \sum_j W_j$. We want to show $W_j(\mathscr{L}_1, \dots, \mathscr{L}_N) = \omega_{\mathscr{C},\log}$ for each $j \in \{1, \dots, N\}$. The monomial W_j induces a homomorphism $(\mathbb{C}^*)^N \to \mathbb{C}^*$. By our initial assumptions, we have $W_j|_G = 1$. Therefore, $W_j\colon \Gamma/G \to \mathbb{C}^*$. By checking W_j on the subgroup $\mathbb{C}^*_R = \{(\lambda^{c_1}, \dots, \lambda^{c_N})\}$ we can easily show that the above homomorphism is an isomorphism. Hence, $W_j(\pi_1, \dots, \pi_N) = \zeta$. This implies $W_j(\mathscr{L}_1, \dots, \mathscr{L}_N) = \omega_{C,\log}$.

Let G_{y_i} be the local group of $\mathscr{C}$ at the marked point y_i the morphism $\mathscr{C} \to \mathrm{B}\Gamma$ implies that each G_{y_i} has a homomorphism to Γ. Let γ_{y_i} be the canonical generator of G_{y_i}. Its image $(\gamma_1, \dots, \gamma_N)$ in $(\mathbb{C}^*)^N$ gives us the familiar presentation of the local group. The fact that $\omega_{C,\log}$ has no orbifold structure implies that G_{y_i} actually maps to $\ker(\zeta) = G \subset \Gamma$. And representability of the morphism $\mathscr{C} \to \mathrm{B}\Gamma$ implies that the map $G_{y_i} \to \ker(\zeta) = G$ is injective, so we have $(\gamma_1, \dots, \gamma_N) \in G \subset (\mathbb{C}^*)^N$.

A complete proof of the equivalence of this definition with our original definition is given in [42, Proposition 3.2.2].

2.3 The virtual cycle

A choice of W–structure does not solve the problem completely. Suppose $u_i \in \Omega^0(\mathscr{L}_i)$ and $\mathscr{L}_1, \dots, \mathscr{L}_N$ is a W–structure. Then

$$\bar{\partial} u_i \in \Omega^{0,1}(\mathscr{L}_i), \quad \overline{\partial W/\partial u_i} \in \Omega^{0,1}_{\log}(\bar{\mathscr{L}}_i^{-1}),$$

where $\Omega^{0,1}_{\log}$ means a $(0,1)$–form with possible singularities of order 1. So the Witten equation (1) has singular coefficients! This is a fundamental phenomenon for the application of the Witten equation. One of the most difficult conceptual advances in the entire theory was to generate the A–model state space from the study of the Witten equation. Now it is understood that the singularity of the Witten equation is the key. Unfortunately, the appearance of singularities makes the Witten equation very difficult to study analytically. The general construction of the FJRW virtual cycle is analytic. However, there is a subsector (the *narrow sector*) which admits a purely algebraic treatment in terms of cosection localization.

In any case, our treatment of the moduli space of solutions of the Witten equation allows us to construct a virtual cycle

$$[\mathscr{W}_{g,k}(\gamma_1, \dots, \gamma_k)]^{\mathrm{vir}} \in H_*(\mathscr{W}_{g,k}(\gamma_1, \dots, \gamma_k), \mathbb{Q}) \otimes \prod_i H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}, W^{\infty}_{\gamma_i}, \mathbb{Q})^G.$$

This naturally leads us to the state space

$$\mathscr{H}_{W,G} = \prod_i H^{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}, W^{\infty}_{\gamma_i}, \mathbb{Q})^G.$$

The space $\mathscr{H}_{W,G}$ in FJRW theory is analogous to the cohomology of the target in Gromov–Witten theory.

Pushing down $[\mathscr{W}_{g,k}(\boldsymbol{\gamma})]^{\mathrm{vir}}$ to the stack of stable curves $\overline{\mathscr{M}}_{g,k}$ and Poincaré dualizing

$$\Lambda^W_{g,k}(\alpha_1, \dots, \alpha_k) := \frac{|G|^g}{\deg(\mathrm{st})} \mathrm{PD}\, \mathrm{st}_* \biggl([\mathscr{W}_{g,k}(\boldsymbol{\gamma})]^{\mathrm{vir}} \cap \prod_{i=1}^k \alpha_i \biggr).$$

gives a cohomological field theory, in the sense of Kontsevich and Manin.

The general construction of [25] is analytic. However, for narrow sectors the Witten equation has only the zero solution. This leads to an algebraic treatment in this subsector.

In this case, the Witten equation breaks into two separate equations,

$$\bar{\partial} u_i = 0, \quad \overline{\partial W/\partial u_i} = 0.$$

The first equation says that u_i is a holomorphic section. The second equation implies all the u_i vanish, by nondegeneracy of W. In this case, the virtual cycle can be formulated in terms of the topological Euler class. Our original construction of this cycle was not quite algebraic because we used the complex conjugate at one point. The effort to remove it leads to several algebraic treatments, including those of Polishchuk and Vaintrob [41; 42], Chiodo [11] and Chang, Kiem, Li and Li [32; 6; 7]. We will use many of their ideas in this paper to construct the virtual cycle for the compact type sector of the gauged linear sigma model.

3 Gauged linear sigma model (GLSM)

We will describe a broad generalization of FJRW theory and use it to provide a mathematical theory of gauged linear sigma models.

3.1 Quotients

Geometric invariant theory (GIT) is a fundamental tool in our constructions. It is also often useful to describe quotients in terms of symplectic reductions. Here we briefly fix notation and conventions and also describe the connection between the GIT and symplectic pictures.

3.1.1 Geometric invariant theory Unless otherwise indicated, we will always work with a reductive algebraic group G acting on a finite-dimensional vector space V. For a given character $\theta\colon G \to \mathbb{C}^*$, we write $\boldsymbol{L}_\theta$ for the line bundle $V \times \mathbb{C}$ with the induced linearization.

We call a point of $v \in V$ *stable* with respect to the linearization θ (or θ–stable) if

(I) the stabilizer $\mathrm{Stab}_G(v) = \{g \in G \mid gv = v\}$ is finite, and

(II) there exists a $k > 0$ and an $f \in H^0(V, \boldsymbol{L}_\theta^k)$ such that $f(v) \neq 0$ and every G–orbit in $D_f = \{f \neq 0\}$ is closed.[1]

Mumford, Fogarty and Kirwan [39] use the name *properly stable* to describe what we call stable.

For a closed G–invariant subvariety $Z \subset V$, we are interested in several different quotients:

[1] This always holds if all the points in D_f have finite stabilizer.

- $[Z/G]$ the stack quotient of Z by G.
- $Z/_{\text{aff}}G$, the *affine quotient* given by $Z/_{\text{aff}}G = \operatorname{Spec}(\mathbb{C}[Z^*]^G)$, where $\mathbb{C}[Z^*]$ is the ring of regular functions on Z.
- $[Z/\!\!/_\theta G] = [Z^{\mathrm{ss}}_G(\theta)/G]$, the GIT quotient stack.
- $Z/\!\!/_\theta G = \operatorname{Proj}_{Z/_{\text{aff}}G}\big(\bigoplus_{k\geq 0} H^0(Z, \boldsymbol{L}^k_\theta)^G\big)$, the underlying coarse moduli space of $[Z/\!\!/_\theta G]$.

In this paper we are primarily concerned with characters $\theta \in \widehat{G}_{\mathbb{Q}} = \operatorname{Hom}(G, \mathbb{C}^*) \otimes \mathbb{Q}$ such that every semistable point of Z is stable: $Z^s_G(\theta) = Z^{\mathrm{ss}}_G(\theta)$. This implies that the GIT quotient is a Deligne–Mumford stack.

Definition 3.1.1 We say that $\theta \in \widehat{G}_{\mathbb{Q}}$ (or the corresponding linearization $\boldsymbol{L}_\theta$) is *strongly regular* on Z if $Z^{\mathrm{ss}}_G(\theta)$ is not empty and $Z^s_G(\theta) = Z^{\mathrm{ss}}_G(\theta)$.

The linearization $\boldsymbol{L}_\theta$ induces a line bundle on $[Z/\!\!/_\theta G]$, which we denote by L_θ. GIT guarantees that there is a line bundle M on $Z/\!\!/_\theta G$ that is relatively ample over the affine quotient and that pulls back to L^k_θ for some positive integer k.

For a fixed Z, changing the linearization gives a different quotient. The space of (fractional) linearizations is divided into chambers, and any two linearizations lying in the same chamber have isomorphic GIT quotients. We will call the isomorphism classes of these quotients *phases*. If the linearizations lie in distinct chambers, the quotients are birational to each other, and are related by flips [48; 20]. This variation of GIT and the way the quotients change when crossing a wall of a chamber is important in the theory of the gauged linear sigma model.

3.1.2 Symplectic reductions It is often useful to think of GIT quotients as symplectic reductions. Take $Z \subseteq \mathbb{C}^n$ with the standard Kähler form $\omega = \sum_i dz_i \wedge d\bar{z}_i$. Since G is reductive, it is the complexification of a maximal compact Lie subgroup H, acting on Z via a faithful unitary representation $H \subseteq U(n)$. Denote the Lie algebra of H by $\mathfrak{h}$.

We have a Hamiltonian action of H on Z with moment map $\mu_Z\colon Z \to \mathfrak{h}^*$ for the action of H on Z, given by

$$\mu_Z(v)(Y) = \tfrac{1}{2}\bar{v}^{\mathsf{T}} Y v = \tfrac{1}{2}\sum_{i,j\leq n} \bar{v}_i Y_{i,j} v_j$$

for $v \in Z$ and $Y \in \mathfrak{h}$. If $\tau \in \mathfrak{h}^*$ is a value of the moment map, then the locus $\mu^{-1}(H\tau)$ is an H–invariant set, and the symplectic orbifold quotient of Z at τ is defined as

$$[Z/\!\!/^{\mathrm{spl}}_\tau H] = [\mu_Z^{-1}(H\tau)/H] = [\mu_Z^{-1}(\tau)/H_\tau],$$

where H_τ is the stabilizer in H of τ. The symplectic quotients $[Z/\!\!/_\tau^{\mathrm{spl}} H]$ depend on a choice of $\tau \in \mathfrak{h}^*$. As in the GIT case, there is a chamber structure for the image of μ such that:

(I) For any two regular values τ and τ' in the same chamber, the quotients $[Z/\!\!/_\tau^{\mathrm{spl}} H]$ and $[Z/\!\!/_{\tau'}^{\mathrm{spl}} H]$ are isomorphic.

(II) The quotients associated to regular values in different chambers are birational to each other.

(See [39, Section 8] for details.)

3.1.3 Relation between GIT and symplectic quotients Although we are primarily interested in GIT quotients, identifying the phases is sometimes easier in the symplectic setting, so it is useful to understand the relation between the two formulations.

To do this, we first observe we can G–equivariantly compactify the vector space V by embedding it into $\overline{V} = \mathbb{P}(V \oplus \mathbb{C})$ in the obvious way, with the trivial G–action on the factor $\mathbb{C}$. For any integer $n > 0$, define a G–linearization on $\overline{V}$ by letting G act on the fiber of $\mathscr{O}(n) = \mathscr{O}_{\overline{V}}(n)$ by multiplication by θ.

Proposition 3.1.2 *For each $n > 0$, let $\overline{V}^{\mathrm{ss}}_{\theta,n}$ denote the semistable locus in $\overline{V}$ with respect to the previously defined linearization on $\mathscr{O}(n)$. There exists a finite $M > 0$ such that $V \cap \overline{V}^{\mathrm{ss}}_{\theta,n}$ is equal to the affine semistable locus $V^{\mathrm{ss}}_G(\theta)$ for all $n \geq M$.*

Proof We have $V^{\mathrm{ss}}_G(\theta) = \bigcup_t D_t$, where the union runs over all G–invariant global sections t of $\boldsymbol{L}^k$ for all $k > 0$, and D_t is the distinguished open set $\{x \mid t(x) \neq 0\}$. Any such t corresponds to a polynomial $g \in \mathbb{C}[V^*]$ such that G acts on g as θ^{-k}.

Similarly, we have $\overline{V}^{\mathrm{ss}}_{\theta,n} = \bigcup_s D_s$, where the union runs over all G–invariant sections s of $\mathscr{O}(kn)$ for all $k > 0$. Any such s corresponds to a polynomial $f \in \mathbb{C}[V^*]$ of degree at most kn such that G acts on f as θ^{-k}. Clearly, every such section s defines a section of $\boldsymbol{L}^k_\theta$ on V, and hence $(\overline{V}^{\mathrm{ss}}_{\theta,n} \cap V) \subset V^{\mathrm{ss}}_G(\theta)$ for every $n > 0$.

Conversely, since V is quasicompact in the Zariski topology, we may choose a finite number of G–invariant sections $t_1, \dots, t_m$ such that $V^{\mathrm{ss}}_G(\theta) = \bigcup_{i=1}^m D_{t_i}$. For each i, let $g_i \in \mathbb{C}[V^*]$ be the polynomial corresponding to the section t_i of $\boldsymbol{L}^{k_i}_\theta$ and let d_i be the degree of g_i. Letting $M = \max(d_1/k_1, \dots, d_m/k_m)$ implies that each g_i has degree no more than Mk_i and thus defines a G–invariant section of $\mathscr{O}(nk_i)$ for every $n \geq M$. Therefore, $V^{\mathrm{ss}}_G(\theta) \subset (\overline{V}^{\mathrm{ss}}_{\theta,n} \cap V)$ for all $n \geq M$. □

We can also extend the action of H to a Hamiltonian action of H on $\overline{V}$ with an extended moment map $\widetilde{\mu}\colon \overline{V} \to \mathfrak{h}^*$ such that $V \cap \mu^{-1}(\tau) = V \cap \widetilde{\tau}$.

To relate the GIT quotient $[\overline{V}/\!\!/ G]$ to the symplectic quotient $[\overline{V}/\!\!/^{\mathrm{spl}}_{\tau} H]$ we use the Kempf–Ness theorem and the so-called *shifting trick*. For our purposes, these can be combined into the following theorem, which is essentially [20, Theorem 2.2.4].

Theorem 3.1.3 *Taking derivations of the character θ defines a weight $\tau_\theta \in \mathfrak{h}^*$ and a very ample line bundle $\mathfrak{L}_\theta$ on G/B for some Borel subgroup B of G. The manifold G/B inherits the Fubini–Study symplectic structure via the projective embedding of G/B defined by $\mathfrak{L}_\theta$. Let $\mu_{\mathfrak{L}_\theta}\colon G/B \to \mathfrak{h}^*$ be the corresponding moment map. This also defines a line bundle $\mathrm{pr}_2^*(\mathfrak{L}_\theta)$ on $\overline{V}\times G/B$ and a moment map $\mu_\theta\colon \overline{V}\times G/B \to \mathfrak{h}^*$ by $\mu_\theta(v, gB) = \mu_V(v) + \mu_{\mathfrak{L}_\theta}(gB)$. We have*

$$[(\overline{V}\times G/B)/\!\!/_{\mathrm{pr}_2^*(\mathfrak{L}_\theta)} G] = [\mu_\theta^{-1}(0)/H] = [\mu^{-1}(-\tau_\theta)/H_{-\tau_\theta}] = [\overline{V}/\!\!/^{\mathrm{spl}}_{-\tau_\theta} H],$$

where $H_{-\tau_\theta}$ is the stabilizer of $-\tau_\theta$ in H. This can be extended to rational characters $\theta \in \widehat{G}_{\mathbb{Q}}$ by taking appropriate powers of the corresponding line bundles.

Corollary 3.1.4 *Whenever the coadjoint orbit of τ_θ in $\mathfrak{h}^*$ is trivial, so that G/B is a single point* (eg in the case that G is abelian), *then we have $\mathrm{pr}_2^*\,\mathfrak{L}_\theta = \boldsymbol{L}_\theta$ and*

$$[Z/\!\!/^{\mathrm{spl}}_{-\tau_\theta} H] = [Z/\!\!/_\theta G]$$

for any G–invariant quasiprojective subvariety $Z \subseteq \overline{V}$.

For us the main use of this corollary is that it allows us to identify the phases of the GIT quotient by examining the critical points of the moment map.

3.2 GLSM

The gauged linear sigma model (GLSM) requires an additional $\mathbb{C}^*$–action on V called the *R–charge* and a superpotential on the quotient. We will be especially interested in the critical locus of the superpotential.

Our basic setup is the following: Let V be an n–dimensional vector space over $\mathbb{C}$, and let $G \subset \mathrm{GL}(V)$ be a reductive algebraic group over $\mathbb{C}$ with identity component G_0 such that G/G_0 is finite. We call G the *gauge group*. If the gauge group action on V factors through $\mathrm{SL}(V)$ then we say that it satisfies the *Calabi–Yau condition.*

Assume that V also admits a $\mathbb{C}^*$–action $(z_1,\dots,z_n) \to (\lambda^{c_1} z_1,\dots,\lambda^{c_n} z_n)$, which we denote by $\mathbb{C}^*_R$. We think of $\mathbb{C}^*_R$ as a subgroup of $\mathrm{GL}(V,\mathbb{C})$. This means we require $\gcd(c_1,\dots,c_n)=1$. Unlike the case of FJRW theory, we allow the weights c_i of $\mathbb{C}^*_R$ to be zero or negative.

Definition 3.2.1 Fix a polynomial $W\colon V \to \mathbb{C}$ of degree $d \neq 0$ with respect to the $\mathbb{C}^*_R$–action (ie quasihomogeneous) and invariant under the action of G. The polynomial W will be called the *superpotential* for our theory.

Remark 3.2.2 For any strongly regular phase θ, the complex dimension of $\mathscr{X}_\theta = [V/\!\!/_\theta G]$ is $n - \dim(G)$.

Definition 3.2.3 Let $N = n - \dim(G)$. We define the *central charge* of the theory for the pair (W, G) to be

$$\hat{c}_{W,G} = N - 2\sum_{j=1}^{n} \frac{c_j}{d}. \tag{5}$$

And we define

$$J = \left(\exp\left(2\pi i \frac{c_1}{d}\right), \dots, \exp\left(2\pi i \frac{c_n}{d}\right)\right), \tag{6}$$

which is an automorphism of W of order d.

It will sometimes be convenient to write $q_i = c_i/d$ and $q = \sum_{i=1}^n q_i$ so that

$$\hat{c}_{W,G} = N - 2q \quad \text{and} \quad J = (\exp(2\pi i q_1), \dots, \exp(2\pi i q_n)).$$

Note that the $\mathbb{C}^*_R$–action is closely related to what the physics literature calls *R–charge*. More precisely, R–charge is the $\mathbb{C}^*$–action given by the weights $(2c_1/d, \dots, 2c_n/d)$; but for our purposes, $\mathbb{C}^*_R$ is more useful, and we will sometimes abuse language and call it the R–charge.

Definition 3.2.4 We say that the actions of G and $\mathbb{C}^*_R$ are *compatible* if:

(I) They commute: $gr = rg$ for any $g \in G$ and any $r \in \mathbb{C}^*_R$.

(II) We have $G \cap \mathbb{C}^*_R = \langle J \rangle$.

Definition 3.2.5 We define Γ to be the subgroup of $\mathrm{GL}(V, \mathbb{C})$ generated by G and $\mathbb{C}^*_R$. If G and $\mathbb{C}^*_R$ are compatible, then every element γ of Γ can be written as $\gamma = gr$ for $g \in G$ and $r \in \mathbb{C}^*_R$; that is,

$$\Gamma = G\mathbb{C}^*_R.$$

The representation $\gamma = gr$ is unique up to an element of $\langle J \rangle$. Moreover, there is a well-defined homomorphism

$$\zeta\colon \Gamma = G\mathbb{C}^*_R \to \mathbb{C}^*, \quad g(\lambda^{c_1}, \dots, \lambda^{c_n}) \mapsto \lambda^d. \tag{7}$$

We denote the target of ζ by $H = \zeta(\mathbb{C}^*_R) = \mathbb{C}^*$, to distinguish it from $\mathbb{C}^*_R$. This gives the exact sequence

(8) $$1 \to G \to \Gamma \xrightarrow{\zeta} H \to 1.$$

Moreover, there is another homomorphism

(9) $$\xi\colon \Gamma \to G/\langle J\rangle, \quad gr \mapsto g\langle J\rangle.$$

This is also well-defined, and gives another exact sequence,

$$1 \to \mathbb{C}^*_R \to \Gamma \xrightarrow{\xi} G/\langle J\rangle \to 1.$$

Definition 3.2.6 Let $\theta\colon G \to \mathbb{C}^*$ define a strongly regular phase $\mathscr{X}_\theta = [V/\!\!/_\theta G]$. The superpotential W descends to a holomorphic function $W\colon \mathscr{X}_\theta \to \mathbb{C}$. Let $\mathrm{Crit}^{\mathrm{ss}}_G(\theta) = \{v \in V^{\mathrm{ss}}_G(\theta) \mid \partial W/\partial x_i = 0 \text{ for all } i = 1, \dots, n\} \subset V^{\mathrm{ss}}$ denote the semistable points of the critical locus. The group G acts on $\mathrm{Crit}^{\mathrm{ss}}_G(\theta)$ and the stack quotient is

$$\mathscr{CR}_\theta = [\mathrm{Crit}^{\mathrm{ss}}_G(\theta)/G] = \{x \in \mathscr{X}_\theta \mid dW = 0\} \subset \mathscr{X}_\theta,$$

where $dW\colon T\mathscr{X}_\theta \to T\mathbb{C}^*$ is the differential of W on $\mathscr{X}_\theta$. We say that the pair (W, G) is *nondegenerate for* $\mathscr{X}_\theta$ if the critical locus $\mathscr{CR}_\theta \subset \mathscr{X}_\theta$ is compact.

3.2.1 Characters, lifts and GIT stability

Definition 3.2.7 Given any G–character $\theta \in \widehat{G}$, we say that a character $\vartheta \in \widehat{\Gamma} = \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ is a *lift* of θ if its restriction to G is equal to θ:

$$\vartheta|_G = \theta.$$

Proposition 3.2.8 *Given any character $\theta \in \widehat{G}$, there is a lift of θ to some $\vartheta \in \widehat{\Gamma}$. Composition of this lift with the inclusion $\mathbb{C}^*_R \subset \Gamma$ induces a character $\mathbb{C}^*_R \to \mathbb{C}^*$.*

Given any two lifts $\vartheta, \vartheta' \in \widehat{\Gamma}$, the ratio $\vartheta^{-1}\vartheta'$ induces a character on $\mathbb{C}^*_R$ of weight divisible by d, which factors through the composition $\mathbb{C}^*_R \subset \Gamma \xrightarrow{\zeta} \mathbb{C}^*$.

Conversely, given any lift ϑ of θ and given any $l \in \mathbb{Z}$, there is a unique lift ϑ' of θ such that $\vartheta^{-1}\vartheta'$ induces a character on $\mathbb{C}^*_R$ of weight ld.

Finally, the d^{th} power θ^d of any character θ factors through $G/\langle J\rangle$, inducing a character $\overline{\theta}\colon G/\langle J\rangle \to \mathbb{C}^*$. This gives a lift of θ^d via $\Gamma \xrightarrow{\xi} G/\langle J\rangle \xrightarrow{\overline{\theta}} \mathbb{C}^*$, which we denote by ϑ^d_0. The induced character on $\mathbb{C}^*_R$ has weight 0.

Proof Given a character $\theta \in \widehat{G}$, the element $J \in G \cap \mathbb{C}_R^*$ must satisfy $\theta(J) = \exp(2\pi i a/d)$ for some $a \in \mathbb{Z}$. For any $\gamma \in \Gamma$, write $\gamma = gr$ with $g \in G$ and $r = (\lambda^{c_1}, \dots, \lambda^{c_n}) \in \mathbb{C}_R^*$. Define $\vartheta(gr) = \theta(g)\lambda^a$. This is a well-defined group homomorphism because the only possible ambiguity in the representation of γ is due to elements in $G \cap \mathbb{C}_R^* = \langle J \rangle$. That is to say, the only ambiguity is whether an element is written as $g \cdot rJ^k$ or $gJ^k \cdot r$. We calculate

$$\vartheta(g\,rJ^k) = \theta(g)\lambda^a J^{ka} = \theta(gJ^k)\lambda^a = \vartheta(gJ^k\,r).$$

So the lift ϑ is well-defined. This proves the existence of a lift of θ.

The ratio of any two lifts of θ is a lift of the trivial G–character. The induced character on $\mathbb{C}_R^*$ must therefore be trivial on J, and hence must have weight divisible by d. Moreover, given any lift ϑ with $\mathbb{C}_R^*$–weight a, we can define a new character by $\vartheta'(gr) = \theta(g)\lambda^{a+ld}$. It is immediate that $\vartheta^{-1}\vartheta'$ induces a character on $\mathbb{C}_R^*$ of weight ld

The final statement about θ^d is immediate from the definition. □

It will also be useful to consider fractional characters rather than just integral characters.

Definition 3.2.9 We write $\widehat{G}_{\mathbb{Q}}$ as a shorthand for $\widehat{G} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Hom}(G, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\widehat{\Gamma}_{\mathbb{Q}}$ as a shorthand for $\widehat{\Gamma} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Hom}(\Gamma, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$. A *lift of* $\theta \in \widehat{G}_{\mathbb{Q}}$ is a fractional character $\vartheta \in \widehat{\Gamma}_{\mathbb{Q}}$ such that $\vartheta|_G = \theta$.

Corollary 3.2.10 *Given any $\theta \in \widehat{G}_{\mathbb{Q}}$ and any $\xi \in \widehat{\mathbb{C}_R^*} \otimes \mathbb{Q} \cong \mathbb{Q}$, there exists a unique lift $\vartheta \in \widehat{\Gamma}_{\mathbb{Q}}$ of θ that induces ξ.*

In the next proposition we list many of the properties of Γ– and G–actions and characters that are relevant for our use of geometric invariant theory. Many of these are simple, but we find it useful to state them explicitly.

Proposition 3.2.11 (I) *Given any character $\vartheta \in \widehat{\Gamma}$, the ϑ–semistable locus $V_\Gamma^{\mathrm{ss}}(\vartheta)$ for the Γ–action on V is a subset of the $\vartheta|_G$–semistable locus $V_G^{\mathrm{ss}}(\vartheta)$ for the G–action on V*

$$V_\Gamma^{\mathrm{ss}}(\vartheta) \subseteq V_G^{\mathrm{ss}}(\vartheta).$$

(II) *For any two characters $\vartheta, \vartheta' \in \widehat{\Gamma}$ such that $\vartheta|_G = \vartheta'|_G$, we have*

$$H^0(V, \boldsymbol{L}_\vartheta^k)^G = H^0(V, \boldsymbol{L}_{\vartheta'}^k)^G$$

for every nonnegative integer k. Furthermore, the G–semistable loci agree:

$$V_G^{\mathrm{ss}}(\vartheta) = V_G^{\mathrm{ss}}(\vartheta').$$

(III) *For any character $\vartheta \in \widehat{\Gamma}$ and any nonnegative integer k, the group Γ acts on the G–invariant space of sections $H^0(V, \boldsymbol{L}_\vartheta^k)^G$, hence this space gives a representation of $\mathbb{C}_R^*$ and can be decomposed into eigenspaces:*

$$H^0(V, \boldsymbol{L}_\vartheta^k)^G = \bigoplus_{l \in \mathbb{Z}} E_{l,\vartheta},$$

where $\mathbb{C}_R^*$ acts on $E_{l,\vartheta}$ with weight $l \in \mathbb{Z}$.
Moreover, d must divide l for any nontrivial (nonzero) *component $E_{l,\vartheta}$.*

(IV) *For any two characters $\vartheta, \vartheta' \in \widehat{\Gamma}$ that agree when restricted to G, and for any integer l, the eigenspace $E_{l,\vartheta}$ is equal to an eigenspace $E_{l',\vartheta'}$ for some integer l'. That is, the decomposition into components is the same for ϑ and ϑ', but the weight of the $\mathbb{C}_R^*$–action on each component depends on the choice of character.*

(V) *For any character $\theta \in \widehat{G}$ and any positive integer k, let ϑ be any lift of θ^k* (ϑ is not necessarily equal to the k^{th} power of a lift of θ). *We have*

$$H^0(V, \boldsymbol{L}_\vartheta)^\Gamma \subseteq H^0(V, \boldsymbol{L}_\theta^k)^G$$

and

$$V_\Gamma^{\mathrm{ss}}(\vartheta) \subseteq V_G^{\mathrm{ss}}(\theta^k) = V_G^{\mathrm{ss}}(\theta).$$

(VI) *Given any character $\theta \in \widehat{G}$ and any nonnegative integer k, the set of G–invariant sections $H^0(V, \boldsymbol{L}_\theta^k)^G$ is the direct sum, over all ϑ lifting θ^k, of the Γ–invariant sections:*

$$H^0(V, \boldsymbol{L}_\theta^k)^G = \bigoplus_{\vartheta \text{ lifting } \theta^k} H^0(V, \boldsymbol{L}_\vartheta)^\Gamma.$$

Moreover, the θ–semistable locus $V_G^{\mathrm{ss}}(\theta)$ is the union of all the semistable loci for the Γ–action with characters ϑ, where ϑ ranges over all lifts of θ^k and k ranges over all positive integers:

$$V_G^{\mathrm{ss}}(\theta) = \bigcup_{k \in \mathbb{Z}^{>0}} \bigcup_{\vartheta \text{ lifting } \theta^k} V_\Gamma^{\mathrm{ss}}(\vartheta).$$

(VII) *For any character $\theta \in \widehat{G}$ the G–semistable locus $V_G^{\mathrm{ss}}(\theta)$ and its complement, the G–unstable locus $V_G^{\mathrm{un}}(\theta)$, are both preserved by Γ.*

Proof (I) This is immediate from the definition of semistable.

(II) Again, this is immediate from the definitions.

(III) The fact that Γ acts on $H^0(V, \boldsymbol{L}_\vartheta^k)^G$ is a straightforward computation, which follows from the fact that the action of $\mathbb{C}_R^*$ and G commute.

This implies, in particular, that $H^0(V, \boldsymbol{L}_\vartheta^k)^G$ is a finite-dimensional representation of $\mathbb{C}_R^*$ and can be decomposed into eigenspaces:

$$H^0(V, \boldsymbol{L}_\vartheta^k)^G = \bigoplus_{l \in \mathbb{Z}} E_{l,\vartheta},$$

where $\mathbb{C}_R^*$ acts on $E_{l,\vartheta}$ with weight $l \in \mathbb{Z}$.

Finally, we note that if $f \in E_{l,\vartheta}$ is nontrivial, then since f is G–invariant it must also be fixed by $J \in G$, but since $J \in \mathbb{C}_R^*$ we must have $J \cdot f = J^l f = f$, and hence d divides l.

(IV) The character $\vartheta\colon \Gamma \to \mathbb{C}^*$ induces a character of $\mathbb{C}_R^*$ with some weight $w \in \mathbb{Z}$. The action of $r = (\lambda^{c_1}, \dots, \lambda^{c_n}) \in \mathbb{C}_R^*$ on any section $f \in H^0(V, L_\vartheta^k)^G$ is given by $(r \cdot f)(v') = \vartheta(r^{-1}) f(rv')$ for every $v' \in V$, so for $f \in E_l$ we have $\lambda^{-w} f(rv') = \lambda^l f(v')$ and thus $f(rv') = \lambda^{l+w} f(v')$. That is, there exists an integer m such that $f(rv') = \lambda^m f(v')$ for every $v' \in V$. This last result is independent of ϑ. Applying this in the case of ϑ', the action of r on f is $(r \cdot f)(v') = \vartheta'(r^{-1}) f(rv') = \lambda^{m-w'} f(v')$, where w' is the weight of the character of $\mathbb{C}_R^*$ induced by ϑ'.

Thus any eigenspace $E_{l,\vartheta}$ of $H^0(V, \boldsymbol{L}_\vartheta)^G$ is also an eigenspace of $H^0(V, \boldsymbol{L}_{\vartheta'})^G$ but with possibly a different weight l'.

(V) As a G–linearization, the line bundle $\boldsymbol{L}_\vartheta$ is identical to $\boldsymbol{L}_{\theta^k} = \boldsymbol{L}_\theta^k$, so any Γ–invariant section $\sigma' \in H^0(V, \boldsymbol{L}_\vartheta)^\Gamma$ is also a G–invariant section of $H^0(V, \boldsymbol{L}_\theta^k)$, and hence $V_\Gamma^{\mathrm{ss}}(\vartheta) \subseteq V_G^{\mathrm{ss}}(\theta^k) = V_G^{\mathrm{ss}}(\theta)$.

(VI) Given any lift ϑ of θ^d we have $H^0(V, \boldsymbol{L}_\theta^k)^G = \bigoplus_l E_{dl,\vartheta}$. For each l let ϑ' be the character $\vartheta'(gr) = \vartheta(gr)\lambda^{-dl}$, where $g \in G$ and $r = (\lambda^{c_1}, \dots, \lambda^{c_n}) \in \mathbb{C}_R^*$. This shows that $E_{dl,\vartheta} = E_{0,\vartheta'} = H^0(V, \boldsymbol{L}_{\vartheta'})^\Gamma$. By Proposition 3.2.8, there is precisely one such lift for each l. Thus we have

$$H^0(V, \boldsymbol{L}_\theta^k)^G = \bigoplus_{\vartheta' \text{ lifting } \theta^k} H^0(V, \boldsymbol{L}_{\vartheta'})^\Gamma.$$

Now we obviously have

$$\bigcup_{k \in \mathbb{Z}^{>0}} \bigcup_{\vartheta \text{ lifting } \theta^k} V_\Gamma^{\mathrm{ss}}(\vartheta) \subseteq V_G^{\mathrm{ss}}(\theta).$$

Conversely, given any $v \in V_G^{\mathrm{ss}}(\theta)$ and any $f \in H^0(V, \boldsymbol{L}_\theta^k)^G$ for some positive integer k with $f(v) \neq 0$, fix a choice of lift ϑ of θ^k. We can decompose f as a sum $f_i + \dots + f_n$ with each f_j in the eigenspace $E_{j,\vartheta}$. Since f does not vanish at $v \in V$, then $f_l(v) \neq 0$ for at least one integer l, so we may assume that, with respect to the character ϑ, the group $\mathbb{C}_R^*$ acts on f by multiplication by λ^l for some integer l. By (III) the integer l is divisible by d. Choosing $\vartheta'(gr) = \vartheta(gr)\lambda^{-ell}$ shows that $f \in E_{0,\vartheta'} = H^0(V, \boldsymbol{L}_{\vartheta'})^\Gamma$, so $v \in V_\Gamma^{\mathrm{ss}}(\vartheta')$, as desired.

(VII) By (VI), given any $v \in V_G^{\mathrm{ss}}(\theta)$ there is a ϑ lifting some θ^k such that $v \in V_\Gamma^{\mathrm{ss}}(\vartheta)$. But $V_\Gamma^{\mathrm{ss}}(\vartheta)$ is preserved by Γ, and hence the Γ–orbit of V must lie in $V_\Gamma^{\mathrm{ss}}(\vartheta) \subset V_G^{\mathrm{ss}}(\theta)$. □

Lemma 3.2.12 *For any $\theta \in \widehat{G}_{\mathbb{Q}}$, let $\vartheta_-, \vartheta_0, \vartheta_+ \in \widehat{\Gamma}_{\mathbb{Q}}$ be the unique lifts such that the induced characters of $\mathbb{C}_R^*$ have weight -1, 0 and 1, respectively. We have*

$$V_G^{\mathrm{ss}}(\theta) = V_\Gamma^{\mathrm{ss}}(\vartheta_-) \cup V_\Gamma^{\mathrm{ss}}(\vartheta_0) \cup V_\Gamma^{\mathrm{ss}}(\vartheta_+).$$

Proof Taking powers as necessary, we may assume that $\theta \in \widehat{G}$. The algebra $\bigoplus_{k>0} H^0(V, \boldsymbol{L}_\theta^k)^G = \bigoplus_{k>0} H^0(V, \boldsymbol{L}_{\theta^k})^G$ is finitely generated, so there exists a finite set $f_1, \dots, f_K \in \bigoplus_{k>0} H^0(V, \boldsymbol{L}_{\theta^k})^G$ such that for every $v \in V_G^{\mathrm{ss}}(\theta)$ at least one of the f_i does not vanish on v. We may further assume that each f_i is an element of some $H^0(V, \boldsymbol{L}_\theta^{k_i})^G$. Taking appropriate powers, we may assume that k_i is the same for all i and is divisible by d. Let $k = k_i$ be that common choice of k_i.

Decompose $H^0(V, \boldsymbol{L}_{\theta^k})^G = H^0(V, \boldsymbol{L}_{\vartheta_0^k})^G = \bigoplus_l E_{l,\vartheta_0^k}$ into isotypical pieces. By Proposition 3.2.11, each l is divisible by d. When $l = 0$, we have

$$E_{0,\vartheta_0^k} \subset H^0(V, \boldsymbol{L}_{\vartheta_0}^k)^\Gamma.$$

When $l > 0$ it is straightforward to see that

$$E_{l,\vartheta_0^k} \subset H^0(V, \boldsymbol{L}_{\vartheta_+}^{kl/d})^\Gamma,$$

and when $l < 0$, we have

$$E_{l,\vartheta_0^k} \subset H^0(V, \boldsymbol{L}_{\vartheta_-}^{-kl/d})^\Gamma.$$

Therefore, we have

$$V_G^{\mathrm{ss}}(\theta) = V_\Gamma^{\mathrm{ss}}(\vartheta_-) \cup V_\Gamma^{\mathrm{ss}}(\vartheta_0) \cup V_\Gamma^{\mathrm{ss}}(\vartheta_+).$$ □

In many cases, however, we can do much better than the previous lemma.

Definition 3.2.13 We say that a lift $\vartheta \in \widehat{\Gamma}_{\mathbb{Q}}$ of $\theta \in \widehat{G}_{\mathbb{Q}}$ is a *good lift* if $V_\Gamma^{\mathrm{ss}}(\vartheta) = V_G^{\mathrm{ss}}(\theta)$.

Although not every $\theta \in \widehat{G}$ has a good lift for every choice of (G–compatible) $\mathbb{C}_R^*$–action, most of the examples we discuss in this paper have this property.

Remark 3.2.14 Even when a point is both θ–stable and ϑ–semistable for some lift ϑ of θ, the stabilizer in Γ of the point will often be infinite. Hence ϑ–stability and θ–stability are not easily related, even if ϑ is a good lift of θ.

3.2.2 Input data From now on we will assume that we have the following input data:

(I) A finite-dimensional vector space V over $\mathbb{C}$.

(II) A reductive algebraic group $G \subseteq \mathrm{GL}(V)$.

(III) A choice of $\mathbb{C}^*_R$–action on V which is compatible with G, and such that $G \cap \mathbb{C}^*_R = \langle J \rangle$ has order d.

(IV) A G–character θ defining a strongly regular phase $\mathscr{X}_\theta = [V /\!\!/_\theta G]$

(V) A good lift ϑ of θ, except when the stability parameter ε is $0+$ (otherwise any lift will work and all give the same results).

(VI) A nondegenerate, G–invariant superpotential $W \colon V \to \mathbb{C}$ of degree d with respect to the $\mathbb{C}^*_R$–action.

Here we provide one simple example to illustrate the ideas. In Section 7 we consider many more of the important examples studied by Witten [53]. The reader who wants to get right to the main results may skip this example on first reading; whereas, others may wish to look at the additional examples in Section 7 before proceeding.

Example 3.2.15 (hypersurfaces) Suppose that $G = \mathbb{C}^*$ and $F \in \mathbb{C}[x_1, \dots, x_K]$ is a nondegenerate quasihomogeneous polynomial of G–weights $(b_1, \dots, b_K)$ and total degree b, as in FJRW theory. Let

$$W = pF \colon \mathbb{C}^{K+1} \to \mathbb{C}.$$

Here, we assign G–weight $-b$ to the variable p, so that W is G–invariant.

The critical set of W is given by the equations

$$\partial_p W = F = 0, \quad \partial_{x_i} W = p \partial_{x_i} F = 0.$$

This implies that either $p \neq 0$ and $(x_1, \dots, x_K) = (0, \dots, 0)$ or that $p = 0$ and $F(x_1, \dots, x_K) = 0$. Suppose that $b_i > 0$ for $i = 1, \dots, K$ and $b > 0$. Consider the quotient of $\mathbb{C}^{K+1}$ by $G = \mathbb{C}^*$ with weights $(b_1, \dots, b_K, -b)$. If $b = \sum_{i=1}^K b_i$, then we have a Calabi–Yau weight system, but we do not assume that here. The affine moment map

$$\mu = \frac{1}{2}\left(\sum_{i=1}^{K} b_i |x_i|^2 - b|p|^2\right)$$

is a quadratic function whose only critical point is at zero. Therefore, the only critical value is $\tau = 0$ and there are two phases, $\tau > 0$ or $\tau < 0$.

$\boldsymbol{\tau > 0}$ We have

$$\sum_i^K b_i |x_i|^2 = b|p|^2 + 2\tau.$$

For each choice of p, the set of $(x_1, \dots, x_K) \in \mathbb{C}^K$ such that $(x_1, \dots, x_K, p) \in \mu^{-1}(\tau)$ is a nontrivial ellipsoid E, isomorphic to S^{2K-1}; and we obtain a map from the quotient $\mathscr{X}_\tau^{\mathrm{sympl}}$ to $\mathscr{X}_{\mathrm{base}} = [E/U(1)] = W\mathbb{P}(b_1, \dots, b_K)$, corresponding to the maximal collection of column vectors $(b_1, \dots, b_K)$ of B. The space $\mathscr{X}_\tau^{\mathrm{sympl}}$ can be expressed as the total space of the line bundle $\mathscr{O}(-b)$ over $\mathscr{X}_\tau^{\mathrm{sympl}}$. If $\sum_i b_i = b$, this is the canonical bundle $\omega_{W\mathbb{P}(b_1,\dots,b_K)}$.

Alternatively, we can consider the GIT quotient $[\mathbb{C}^{K+1} /\!\!/_\theta G]$, where θ has weight $-e$, with $e > 0$. One can easily see that the $\boldsymbol{L}_\theta$–semistable points are $((\mathbb{C}^K - \{\mathbf{0}\}) \times \mathbb{C}) \subset \mathbb{C}^K \times \mathbb{C} = \mathbb{C}^{K+1}$, and the first projection $\mathrm{pr}_1\colon (\mathbb{C}^K - \{\mathbf{0}\}) \times \mathbb{C} \to (\mathbb{C}^K - \{\mathbf{0}\})$ induces the map $[V /\!\!/_\theta G] \to W\mathbb{P}(b_1, \dots, b_K)$.

Now we choose $\mathbb{C}_R^*$–weights $c_{x_i} = 0$ and $c_p = 1$, so that W has $\mathbb{C}_R^*$–weight $d = 1$. The element J is trivial, and the group Γ is a direct product $\Gamma \cong G \times \mathbb{C}_R^*$, with ξ and ζ just the first and second projections, respectively.

The critical locus $\mathscr{CR}_\theta = \{p = 0 = F(x_1, \dots, x_K)\}$ is a degree-b hypersurface in the image of the zero section of $[V /\!\!/_\theta G] \cong \mathscr{O}(-d) \to W\mathbb{P}(b_1, \dots, b_K)$. We call this phase the *Calabi–Yau phase* or *geometric phase*.

We wish to find a good lift of θ. To do this, consider any $v \in V_G^{\mathrm{ss}}(\theta) = ((\mathbb{C}^K - \{\mathbf{0}\}) \times \mathbb{C})$. If ℓ is a generator of $\boldsymbol{L}_\theta^*$ over $\mathbb{C}[V^*]$ with G acting on ℓ with weight $-e$, if we choose the trivial lift ϑ_0 of θ, which corresponds to $\mathbb{C}_R^*$ acting trivially on ℓ, then a monomial of the form $x_i^{ke} \ell^k$ is Γ–invariant and does not vanish on points with $x_i \neq 0$, so every point of $\mathbb{C}^K \times \mathbb{C}$ with $x_i \neq 0$ is in $V_\Gamma^{\mathrm{ss}}(\vartheta_0)$. Letting i range from 1 to K shows that $V_\Gamma^{\mathrm{ss}}(\vartheta_0) = V_G^{\mathrm{ss}}(\theta)$. Thus ϑ_0 is a good lift of the character θ. It is easy to see that ϑ_0 is the only good lift of θ.

$\boldsymbol{\tau < 0}$ We have

$$\mu^{-1}(\tau) = \left\{ (x_1, \dots, x_K, p) \;\middle|\; \sum_{i=1}^K b_i |x_i|^2 - \tau = b|p|^2 \right\}$$

For each choice of $x_1, \dots, x_K \in \mathbb{C}^K$ the set of $p \in \mathbb{C}$ such that $(x_1, \dots, x_K, p) \in \mu^{-1}(\tau)$ is the circle $S^1 \subset \mathbb{C}$, corresponding to the maximal collection $(-b)$, and we obtain a map $\mathscr{X}_\tau^{\mathrm{sympl}} \to [S^1/U(1)]$. If we choose the basis of $U(1)$ to be λ^{-1}, then p can be considered to have positive weight b. Moreover, every p has isotropy equal to the b^{th} roots of unity (isomorphic to $\mathbb{Z}_b$). The quotient $[S^1/U(1)]$ is $W\mathbb{P}(b) = \mathrm{B}\mathbb{Z}_b = [\mathrm{pt}/\mathbb{Z}_b]$.

In the GIT formulation of this quotient, with θ of weight $-e$, and $e < 0$, the L_θ–semistable points are equal to $(\mathbb{C}^K \times \mathbb{C}^*) \subset \mathbb{C}^{K+1}$. The second projection $\mathrm{pr}_2\colon (\mathbb{C}^K \times \mathbb{C}^*) \to \mathbb{C}^*$ induces the map $[V/\!\!/_\theta G] \to \mathrm{B}\mathbb{Z}_b$.

The toric variety $\mathscr{X}_\theta = [V/\!\!/_\theta G]$ can be viewed as the total space of a rank-K orbifold vector bundle over $\mathrm{B}\mathbb{Z}_b$. This bundle is actually just a $\mathbb{Z}_b$–bundle, where $\mathbb{Z}_b$ acts by

$$(x_1, \dots, x_K) \mapsto (\xi_b^{b_1} x_1, \dots, \xi_b^{b_K} x_K), \quad \xi_b = \exp\Bigl(\frac{2\pi i}{b}\Bigr).$$

If W has $\mathbb{C}^*_R$–weight b, then this is exactly the action of the element J in FJRW theory. So the bundle $\mathscr{X}_\theta$ is isomorphic to $[\mathbb{C}^K/\langle J\rangle]$. This is a special phase which is sort of like a toric variety of a finite group instead of $\mathbb{C}^*$.

We can choose $\mathbb{C}^*_R$ to have weights $c_{x_i} = b_i$ and $c_p = 0$. Now W has $\mathbb{C}^*_R$–weight $d = b$, and $J = (\xi^{b_1}, \dots, \xi^{b_K}, 1)$, where $\xi = \exp(2\pi i/d)$. We have $\Gamma = \{((st)^{b_1}, \dots, (st)^{b_K}, t^{-d}) \mid s, t \in \mathbb{C}^*\} = \{(\alpha^{b_1}, \dots, \alpha^{b_K}, \beta) \mid \alpha, \beta \in \mathbb{C}^*\}$, with $\zeta\colon \Gamma \to \mathbb{C}^*$ given by $(\alpha^{b_1}, \dots, \alpha^{b_K}, \beta) \mapsto \alpha^d \beta$. Also the map $\xi\colon \Gamma \to G/\langle J\rangle$ is given by $(\alpha^{b_1}, \dots, \alpha^{b_K}, \beta) \mapsto \beta$.

A similar argument to the one we gave above (for the geometric phase) shows that the trivial lift ϑ_0 is again a good lift of θ.

The critical subset is the single point $\{(0, \dots, 0)\}$ in the quotient $\mathscr{X}_\tau = [\mathbb{C}^K/\mathbb{Z}_d]$. It is clearly compact, so the polynomial W is nondegenerate. We call $\mathscr{X}_\tau$ a *Landau–Ginzburg phase* or a *pure Landau–Ginzburg phase* [53]. This example underlies Witten's physical argument of the Landau–Ginzburg/Calabi–Yau correspondence for Calabi–Yau hypersurfaces of weighted projective spaces.

3.2.3 Choice of $\mathbb{C}^*_R$

Our theory does not really depend on $\mathbb{C}^*_R$, but rather only on the embedding of the groups $G \subseteq \Gamma \subseteq \mathrm{GL}(V)$, on the sum $q = \sum_{i=1}^n q_i = \sum_{i=1}^n c_i/d$ of the $\mathbb{C}^*_R$ weights, and on the choice of lift ϑ. Of course the choice of q and the embedding of Γ in $\mathrm{GL}(V)$ put many constraints on $\mathbb{C}^*_R$; but they still allow some flexibility.

For an example of this, consider the case when the gauge group $G = (\mathbb{C}^*)^m$ is an algebraic torus. Let the action of the i^{th} copy of $\mathbb{C}^*$ on $V = \mathbb{C}^n$ be given by

$$\lambda_i(x_1, \dots, x_n) = (\lambda_i^{b_{i1}} x_1, \dots, \lambda_i^{b_{in}} x_n).$$

We call the integral matrix $B = (b_{ij})$ the *gauge weight matrix*. If the weight matrix $B = (b_{ij})$ satisfies the *Calabi–Yau condition* $\sum_j b_{ij} = 0$ for each i, then we have a lot of flexibility in our choice of $\mathbb{C}^*_R$, as shown by the following lemma:

Lemma 3.2.16 *If the gauge group G is a torus with weight matrix $B = (b_{ij})$ and if we have a compatible $\mathbb{C}^*_R$–action with weights $(c_1, \dots, c_n)$ such that W has $\mathbb{C}^*_R$–weight d, then for any $\mathbb{Q}$–linear combination $(b'_1, \dots, b'_n)$ of rows of the gauge weight matrix B, we define a new choice of R–weights $(c'_1, \dots, c'_n) = (c_1 + b'_1, \dots, c_n + b'_n)$. Denote the corresponding $\mathbb{C}^*$–action by $\mathbb{C}^*_{R'}$.*

Since the group Γ' generated by G and $\mathbb{C}^*_{R'}$ lies inside the maximal torus of $\mathrm{GL}(n, \mathbb{C})$, it is abelian; and so we automatically have that G and $\mathbb{C}^*_{R'}$ commute. We also have the following:

- (I) *The group Γ' generated by G and $\mathbb{C}^*_R$ is the same as the group Γ generated by G and $\mathbb{C}^*_R$.*
- (II) *The $\mathbb{C}^*_{R'}$–weight of W is equal to d.*
- (III) *$G \cap \mathbb{C}^*_{R'} = G \cap \mathbb{C}^*_R = \langle J \rangle$, where J is the element defined by (6) for the original $\mathbb{C}^*_R$–action.*
- (IV) *If B is a Calabi–Yau weight system, then for both $\mathbb{C}^*_R$ and $\mathbb{C}^*_{R'}$ the sum of the weights $q = \sum q_i = \sum c_i/d$ is the same and the central charge $\widehat{c}_W$ is the same.*

Proof For any element $h' \in \mathbb{C}^*_{R'}$ we have $h' = (t^{c'_1}, \dots, t^{c'_n}) \in \mathbb{C}^*_{R'}$ for some $t \in \mathbb{C}^*$. Letting $h = (t^{c_1}, \dots, t^{c_n}) \in \mathbb{C}^*_R$ and $g = (t^{b'_1}, \dots, t^{b'_n}) \in G$, we have $h' = gh$.

- (I) From the equation $h' = gh$, it is now immediate that $G\mathbb{C}^*_R = G\mathbb{C}^*_{R'}$.
- (II) Since the G–weight of W is zero we also have that $\mathbb{C}^*_{R'}$–weight of W is the same as the $\mathbb{C}^*_R$–weight of W.
- (III) If $h' \in G \cap \mathbb{C}^*_{R'}$ then $\gamma = gh$ for some $g \in G$ and $h \in \mathbb{C}^*_R$, but $\gamma \in G$ implies that $h \in G$, so $G \cap \mathbb{C}^*_{R'} \subseteq G \cap \mathbb{C}^*_R$, and a similar argument shows that $G \cap \mathbb{C}^*_R \subseteq G \cap \mathbb{C}^*_{R'}$.
- (IV) For a Calabi–Yau weight system we have $\sum_j b_{ij} = 0$ for each i, hence $\sum_j b'_j = 0$, and the invariance of q and $\widehat{c}_W$ follows. □

Remark 3.2.17 Since Γ is preserved in the preceding lemma and lifts depend only on Γ, any good lift ϑ of $\theta \in \widehat{G}$ for the original $\mathbb{C}^*_R$–action is also a good lift for the new $\mathbb{C}^*_{R'}$–action.

4 Moduli space and evaluation maps

Throughout this section we assume that we have a reductive $G \subseteq \mathrm{GL}(V)$ and that $\mathbb{C}^*_R \subset \mathrm{GL}(V)$ is a diagonal embedding of $\mathbb{C}^*$ into $\mathrm{GL}(V)$ such that G and $\mathbb{C}^*_R$ are compatible. Let $\Gamma \subset \mathrm{GL}(V)$ be the subgroup generated by G and $\mathbb{C}^*_R$.

We further assume that we have chosen a superpotential $W\colon V \to \mathbb{C}$ which is G–invariant and has degree d with respect to the $\mathbb{C}^*_R$–action.

Assume that $\theta \in \widehat{G}$ defines a polarization $\boldsymbol{L}_\theta$ such that $V^{\mathrm{ss}}_G(\theta)$ is nonempty and is equal to $V^s_G(\theta)$. Denote by $\mathscr{X}_\theta = [V/\!\!/_\theta G]$ the corresponding phase of the quotient of V by the action of G and by $\mathscr{CR}_\theta = [\mathrm{Crit}(W)/\!\!/_\theta G]$ the phase of the critical locus of W. Furthermore assume that ϑ is a good lift of θ if the stability parameter ε is not $0+$.

Finally, assume that W defines a nondegenerate holomorphic map $W\colon [V/\!\!/_\theta G] \to \mathbb{C}$.

4.1 State space

The GLSM has a state space similar to that of FJRW theory. For complete intersections, it has already been studied by Chiodo and Nagel (work in progress).

Definition 4.1.1 Let

$$\mathbb{I}\mathscr{X} = \big[\{(v,g) \in V^{\mathrm{ss}}_\theta \times G \mid gv = v\}/G\big]$$

denote the inertia stack of $\mathscr{X}$ (the group G acts on the second factor in the quotient by conjugation).

For each conjugacy class $\Psi \subset G$, let

$$I(\Psi) = \{(v,g) \in V^{\mathrm{ss}}_\theta \times G \mid gv = v,\ g \in \Psi\}$$

and

$$\mathscr{X}_{\theta,\Psi} = [I(\Psi)/G].$$

We have

$$\mathbb{I}\mathscr{X} = \bigsqcup_{\Psi} \mathscr{X}_{\theta,\Psi}, \tag{10}$$

where Ψ runs over all conjugacy classes of G. However, since the action of G on $V^{\mathrm{ss}}_\theta = V^s_\theta$ is proper (see [21, Section 2.1] for more on proper group actions), the set $I(\Psi)$ is empty unless all the elements of Ψ are of finite order. Moreover, by [21, Lemma 2.10] all but finitely many of the $I(\Psi)$ are empty, so the union in (10) has only a finite number of nonempty terms.

Definition 4.1.2 We will abuse notation and denote the map induced by W on $\mathscr{X}_\theta$ as $W\colon \mathscr{X} \to \mathbb{C}$. Let W^∞ be the set $W^\infty = (\mathfrak{Re}\, W)^{-1}(M,\infty) \subseteq [V/\!\!/_\theta G]$ for some large, real M. Similarly, for each conjugacy class Ψ in G, denote by $W^\infty_\Psi = (\mathfrak{Re}\, W)^{-1}(M,\infty) \subseteq \mathscr{X}_\Psi$.

We define the *state space* to be the vector space

$$\mathscr{H}_{W,G} = \bigoplus_{\alpha \in \mathbb{Q}} \mathscr{H}^{\alpha}_{W,G} = \bigoplus_{\Psi} \mathscr{H}_{\Psi},$$

where the sum runs over those conjugacy classes Ψ of G for which $\mathscr{X}_{\theta,\Psi}$ is nonempty, and where

$$\mathscr{H}^{\alpha}_{W,G} = H^{\alpha+2q}_{\mathrm{CR}}(\mathscr{X}_\theta, W^\infty, \mathbb{Q}) = \bigoplus_{\Psi} H^{\alpha-2\,\mathrm{age}(\gamma)+2q}(\mathscr{X}_\Psi, W^\infty_\Psi, \mathbb{Q})$$

and

$$\mathscr{H}_\Psi = H^{\bullet+2q}_{\mathrm{CR}}(\mathscr{X}_{\theta,\Psi}, W^\infty, \mathbb{Q}) = \bigoplus_{\alpha \in \mathbb{Q}} H^{\alpha-2\,\mathrm{age}(\gamma)+2q}(\mathscr{X}_{\theta,\Psi}, W^\infty_\Psi, \mathbb{Q}),$$

That is, the state space is the relative Chen–Ruan cohomology with an additional shift by $2q$.

For each element $g \in G$ we write $[\![g]\!] \subset G$ for the conjugacy class of g in G. We often call the factor $\mathscr{H}_{[\![g]\!]}$ the $[\![g]\!]$*–sector*, and we call the factor $\mathscr{H}_{[\![1]\!]}$ the *untwisted sector*.

Recall (see Definition 3.2.3) that N is the complex dimension of the GIT quotient $\mathscr{X}_\theta = [V/\!\!/_\theta G]$

$$N = \dim([V/\!\!/_\theta G]) = n - \dim(G).$$

And similarly, for each $[\![\gamma]\!]$ we let N_γ denote the complex dimension of the sector $\mathscr{X}_{[\![\gamma]\!]}$:

$$N_\gamma = \dim(\mathscr{X}_{\theta,[\![\gamma]\!]}) = \dim(\mathrm{Fix}(\gamma)) - \dim(Z_G(\gamma)),$$

where $Z_G(\gamma)$ is the centralizer of γ in G.

Similar to the classical case, for every $i \in \mathbb{Q}$, there is a perfect pairing

$$H^i(\mathscr{X}_{[\![\gamma]\!]}, W^\infty_{[\![\gamma]\!]}) \otimes H^{2N_\gamma - i}(\mathscr{X}_{[\![\gamma]\!]}, W^\infty_{[\![\gamma]\!]}) \to \mathbb{C},$$

dual to the intersection pairing of relative homology (see [25, Section 3] for more details). Recall that the age satisfies

$$\mathrm{age}(\gamma) + \mathrm{age}(\gamma^{-1}) = \mathrm{codim}(\mathscr{X}_{\theta,[\![\gamma]\!]}) = N - N_\gamma,$$

so applying the previous pairing to each sector, we obtain a nondegenerate pairing

$$\langle\, , \rangle\colon \mathscr{H}^p_{W,G} \otimes \mathscr{H}^{2\hat{c}-p}_{W,G} \to \mathbb{C},$$

where $\hat{c} = \hat{c}_{W,G} = N - 2q$ (see Definition 3.2.3).

Definition 4.1.3 An element $\gamma \in G$ is called *narrow* if the corresponding component $\mathscr{X}_{[\![\gamma]\!]} \subset \mathbb{I}\mathscr{X}_\theta$ is compact (or, equivalently, if its underlying coarse moduli space is compact). In this case we also say that the corresponding sector $\mathscr{H}_{[\![\gamma]\!]}$ is narrow. If γ is not narrow, we call it (and the corresponding sector) *broad*.

The theory for narrow sectors is generally much easier to understand than for the broad sectors, but some elements of the broad sectors also behave well, namely those which are supported on a compact substack of $\mathbb{I}\mathscr{X}_\theta$.

Definition 4.1.4 If W and G are nondegenerate for $\mathscr{X}_\theta$ (that is, if $\mathscr{CR}_\theta$ is compact) then we say an element of $\mathscr{H}_{W,G}$ is *of compact type* if its Poincaré dual is supported on a compact substack of the inertial stack of $\mathbb{I}\mathscr{X}_\theta$. Any narrow element is obviously of compact type. Define $\mathscr{H}_{W,G,\mathrm{comp}} \subset \mathscr{H}_{W,G}$ to be the span of all the compact-type elements.

If G is finite and W is nondegenerate, then narrow insertions are the only nonzero elements of compact type.

4.2 Moduli space

Our moduli space will be a sort of unification of the quasimaps of [15; 16; 33; 10] with an extension of the Polishchuk–Vaintrob description of the FJRW moduli space [42] to reductive algebraic groups.

As before, we denote by $\mathscr{CR}_\theta = \mathscr{X}_\theta = [\mathrm{Crit}^{\mathrm{ss}}_G(\theta)/G] \subset [V/\!\!/_\theta G] = [V^{\mathrm{ss}}_G(\theta)/G]$ the GIT quotient (with polarization θ) of the critical locus of W. It will be useful also to consider other affine varieties, so we let $Z \subseteq V$ be a closed subvariety of V such that $Z^s_G(\theta) = Z^{\mathrm{ss}}_G(\theta) \neq \varnothing$, and we denote by $\mathscr{Z}_\theta$ the quotient $\mathscr{Z}_\theta = [Z/\!\!/_\theta G] = [Z^{\mathrm{ss}}_G(\theta)/G]$.

Our main object of study is the stack of *Landau–Ginzburg quasimaps to* $\mathscr{Z}_\theta$,

$$\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{Z}_\theta, \beta),$$

with a special interest in the case of $\mathscr{Z}_\theta = \mathscr{CR}_\theta$. We will embed $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta)$ into $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{X}_\theta, \beta)$, which plays a role analogous to the stack of stable maps with p–fields [6; 7].

Before we define our moduli problem, we recall the definition of a prestable orbicurve.

Definition 4.2.1 A *prestable orbicurve* is a balanced twisted curve $\mathscr{C}$ (see Section 4 of [1]).

A prestable orbicurve has a prestable underlying coarse curve (ie the only singularities are nodes) and there is a contraction $\mathscr{C} \to \mathscr{C}'$ to a stable orbicurve $\mathscr{C}'$ (see [1, Section 9]).

Definition 4.2.2 Assume that the actions of G and $\mathbb{C}^*_R$ are compatible and that $\theta \in \widehat{G}$ defines a polarization $\boldsymbol{L}_\theta$ such that the stable and semistable loci of $Z \subset V$ are nonempty and coincide. A *prestable, k–pointed, genus-g, LG quasimap to $\mathscr{X}_\theta$* is a tuple $(\mathscr{C}, y_1, \dots, y_k, \mathscr{P}, u, \varkappa)$ consisting of

(A) a prestable, k–pointed orbicurve $(\mathscr{C}, y_1, \dots, y_k)$ of genus g,

(B) a principal (orbifold) Γ–bundle $\mathscr{P}\colon \mathscr{C} \to \mathrm{B}\Gamma$ over $\mathscr{C}$,

(C) a global section $\sigma\colon \mathscr{C} \to \mathscr{E} = \mathscr{P} \times_\Gamma V$,

(D) an isomorphism $\varkappa\colon \zeta_*\mathscr{P} \to \mathring{\omega}_{\log,\mathscr{C}}$ of principal $\mathbb{C}^*$–bundles ($\mathring{\omega}_{\log,\mathscr{C}}$ indicates the principal $\mathbb{C}^*$–bundle associated to the line bundle $\omega_{\log,\mathscr{C}}$),

such that:

(I) The morphism of stacks $\mathscr{P}\colon \mathscr{C} \to \mathrm{B}\Gamma$ is representable (ie for each point y of $\mathscr{C}$, the induced map from the local group G_y to Γ is injective).

(II) The set of points $b \in \mathscr{C}$ such that any point p of the fiber $\mathscr{P}_b$ over b is mapped by σ into an $\boldsymbol{L}_\theta$–unstable G–orbit of V is finite, and this set is disjoint from the nodes and marked points of $\mathscr{C}$.

(III) The image of the induced map $[\sigma]\colon \mathscr{P} \to V$ lies in Z.

Definition 4.2.3 The points b occurring in condition (II) above are called *basepoints* of the quasimap. That is, $b \in \mathscr{C}$ is a basepoint if there is at least one point of the fiber $\mathscr{P}_b$ over b that is mapped by σ into an $\boldsymbol{L}_\theta$–unstable G–orbit of V.

Definition 4.2.4 Any G–character χ defines a G–linearized line bundle $\boldsymbol{L}_\chi$ on V, and hence a line bundle on $[V/\!\!/_\theta G]$. We denote this line bundle by L_χ.

Alternatively, we may construct L_χ as follows. Note that the stable locus $V^{\mathrm{ss}}_G(\theta)$ is a principal G–bundle over $[V/\!\!/_\theta G]$ and thus defines a morphism $[V/\!\!/_\theta G] \to \mathrm{B}G$ to the classifying stack of G. The character χ induces a map of classifying stacks $\mathrm{B}\chi\colon \mathrm{B}G \to \mathrm{B}\mathbb{C}^*$. Composing these maps gives a morphism $[V/\!\!/_\theta G] \to \mathrm{B}\mathbb{C}^*$ and hence a line bundle on $[V/\!\!/_\theta G]$. This is L_χ.

Definition 4.2.5 For any prestable LG quasimap $\mathscr{Q} = (\mathscr{C}, y_1, \dots, y_k, \mathscr{P}, \sigma, \varkappa)$, a Γ–equivariant line bundle $\boldsymbol{L} \in \mathrm{Pic}^\Gamma(V)$ determines a line bundle $\mathscr{L} = \mathscr{P} \times_\Gamma \boldsymbol{L}$ over $\mathscr{E} = \mathscr{P} \times_\Gamma V$, and pulling back along σ gives a line bundle $\sigma^*(\mathscr{L})$ on $\mathscr{C}$.

In particular, any character $\alpha \in \widehat{\Gamma}$ determines a Γ–equivariant line bundle $\boldsymbol{L}_\alpha$ on V and hence a line bundle $\sigma^*(\mathscr{L}_\alpha)$ on $\mathscr{C}$. Alternatively, we may construct $\sigma^*(\mathscr{L}_\alpha)$ by composing the map $\mathscr{P}\colon \mathscr{C} \to \mathrm{B}\Gamma$ with the map $\mathrm{B}\alpha\colon \mathrm{B}\Gamma \to \mathrm{B}\mathbb{C}^*$ to get $\sigma^*\mathscr{L}_\alpha\colon \mathscr{C} \xrightarrow{\mathrm{B}\alpha\circ\mathscr{P}} \mathrm{B}\mathbb{C}^*$.

Definition 4.2.6 For any $\alpha \in \widehat{\Gamma}$, we define the *degree* of α on $\mathcal{Q}$ to be

$$\deg_{\mathcal{Q}}(\alpha) = \deg_{\mathscr{C}}(\sigma^*(\mathscr{L}_\alpha)) \in \mathbb{Q}.$$

This defines a homomorphism $\deg_{\mathcal{Q}}\colon \widehat{\Gamma} \to \mathbb{Q}$.

For any $\beta \in \mathrm{Hom}(\widehat{\Gamma}, \mathbb{Q})$ we say that an LG quasimap $\mathcal{Q} = (\mathscr{C}, x_1, \dots, x_k, \mathscr{P}, \sigma, \varkappa)$ has *degree* β if $\deg_{\mathcal{Q}} = \beta$.

Remark 4.2.7 If $\vartheta \in \widehat{\Gamma}_{\mathbb{Q}}$ is any character of Γ, then geometric invariant theory guarantees the existence of a line bundle M on $Z/\!\!/_{\vartheta}\Gamma$ such that M is relatively ample over $Z/_{\mathrm{aff}}\Gamma$ and such that for some $n > 0$ we have $\overline{\phi}^* M = L_{\vartheta}^{\otimes n}$ on $[Z/\!\!/_{\vartheta}\Gamma]$, or equivalently,

$$p^*\pi^*\overline{\phi}^* M = \boldsymbol{L}_{\vartheta}^{\otimes n} \tag{11}$$

as a Γ–equivariant bundle on Z^{ss} (see for example [3, Theorem 11.5]).

If ϑ is also a good lift of $\theta \in \widehat{G}_{\mathbb{Q}}$ and $Z \subseteq V$ is a closed subvariety of V, we have the following diagram of quotients:

$$\begin{tikzcd} Z^{\mathrm{ss}} \arrow[r, "p"] & {[Z/\!\!/_{\theta}G]} \arrow[r, "\phi"] \arrow[d, "\pi"] & Z/\!\!/_{\theta}G \arrow[d, "\pi'"] \\ & {[Z/\!\!/_{\vartheta}\Gamma]} \arrow[r, "\overline{\phi}"] & Z/\!\!/_{\vartheta}\Gamma \end{tikzcd} \tag{12}$$

The bundle $\pi'^*(M)$ is ample over $Z/_{\mathrm{aff}}G$, and we have

$$\pi^*\overline{\phi}^* M = \phi^*\pi'^*(M) = L_{\theta}^{\otimes n}. \tag{13}$$

Definition 4.2.8 A *family of prestable, k–pointed, genus-g, LG quasimaps to $\mathscr{X}_\theta$ over a scheme T* is a tuple $(\varpi\colon \mathscr{C} \to T, y_1, \dots, y_k, \mathscr{P}, \sigma, \varkappa)$ consisting of

(A) a flat family of prestable, genus-g, k–pointed orbicurves $(\varpi\colon \mathscr{C} \to T,\ y_1, \dots, y_k)$ over T with (gerbe) markings $\mathscr{S}_i \subset \mathscr{C}$, and sections $y_i\colon T \to \mathscr{S}_i$ which induce isomorphisms between T and the coarse moduli of $\mathscr{S}_i$ for each $i \in \{1, \dots, k\}$,

(B) a principal Γ–bundle $\mathscr{P}\colon \mathscr{C} \to \mathrm{B}\Gamma$ over $\mathscr{C}$,

(C) a section $\sigma\colon \mathscr{C} \to \mathscr{E} = \mathscr{P} \times_{\Gamma} V$,

(D) an isomorphism $\varkappa\colon \zeta_*\mathscr{P} \to \mathring{\omega}_{\log,\mathscr{C}}$ of principal $\mathbb{C}^*$–bundles,

such that the restriction to every geometric fiber of $\varpi\colon \mathscr{C} \to T$ induces a prestable, k–pointed, genus-g, LG quasimap to $\mathscr{X}_\theta$.

Definition 4.2.9 A morphism between LG quasimaps $(\varpi\colon \mathscr{C}\to T, y_1,\dots,y_k,\mathscr{P},\sigma,\varkappa)$ and $(\varpi'\colon \mathscr{C}'\to T', y_1',\dots,y_k',\mathscr{P}',\sigma',\varkappa')$ is a tuple of morphisms (τ,ξ,ρ), where (τ,ξ) form a morphism of prestable orbicurves

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\xi"] \arrow[d, "\varpi"'] & \mathscr{C}' \arrow[d, "\varpi'"] \\ T \arrow[r, "\tau"] & T' \end{tikzcd}$$

and $\rho\colon \mathscr{P}\to \xi^*(\mathscr{P}')$ is a morphism of principal Γ–bundles such that the obvious diagrams commute:

$$\begin{tikzcd} \zeta_*(\mathscr{P}) \arrow[r, "\varkappa"] \arrow[d, "\zeta_*(\rho)"'] & \mathring{\omega}_{\log,\mathscr{C}} \arrow[d] \\ \zeta_*(\xi^*(\mathscr{P}')) \arrow[r, "\xi^*(\varkappa')"] & \xi^*(\mathring{\omega}_{\log,\mathscr{C}'}) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{C} \arrow[r, "\sigma"] \arrow[dr, "\xi^*(\sigma)"'] & \mathscr{P}\times_\Gamma Z \arrow[d, "\rho\times\mathbf{1}"] \\ & \xi^*(\mathscr{P}')\times_\Gamma Z \end{tikzcd}$$

We now wish to define a stability condition for LG quasimaps. To do this we must first define the *length* of an LG quasimap at a point.

Definition 4.2.10 Choose a polarization $\theta\in\widehat{G}$ and a lift ϑ of θ. Given a prestable LG quasimap $\mathcal{Q}=(\mathscr{C},y_1,\dots,y_k,\mathscr{P},\sigma,\varkappa)$ to $[Z/\!\!/_\theta G]$ and any point $y\in\mathscr{C}$ such that the generic point of the component of $\mathscr{C}$ containing y maps to a ϑ–semistable point, we define the *length* of y with respect to $\mathcal{Q}$ and ϑ to be

$$\ell(y)=\min\left\{\frac{(\sigma^*(s))_y}{m}\;\middle|\; s\in H^0(Z,\boldsymbol{L}_\vartheta^m)^\Gamma,\ m>0\right\},$$

where $(\sigma^*(s))_y$ is the order of the vanishing of the section $\sigma^*(s)\in H^0(\mathscr{C},\sigma^*\mathscr{L}_\vartheta^{\otimes m})$ at y.

This definition differs from that in [16, Definition 7.1.1], in that it depends on the choice of the lift $\vartheta\in\widehat{\Gamma}$ rather than on the polarization $\theta\in\widehat{G}$, but the following properties listed in [16, Section 7.1] still hold:

(I) For every $y\in\mathscr{C}$, if the generic point of the component of $\mathscr{C}$ containing y maps to a ϑ–semistable point, then we have

$$\deg_{\mathscr{C}}(\sigma^*(\mathscr{L}_\vartheta))\ge \ell(y)\ge 0$$

with $\ell(y)>0$ if and only if y is a ϑ–basepoint of $\mathcal{Q}$.

(II) If ϑ is a good lift and B is the set of basepoints of $\mathcal{Q}$, then the map σ, when restricted to $\mathscr{C} \smallsetminus B$, defines a map

$$\sigma\colon \mathscr{C} \smallsetminus B \to [Z/\!\!/_\vartheta \Gamma] \xrightarrow{\bar{\phi}} Z/\!\!/_\vartheta \Gamma.$$

Since B is disjoint from nodes and marks and $Z/\!\!/_\vartheta \Gamma$ is projective over $Z/_{\mathrm{aff}}\Gamma$, this extends to a morphism $\sigma_{\mathrm{reg}}\colon \mathscr{C} \to Z/\!\!/_\vartheta \Gamma$. Choose $M \in \operatorname{Pic}(Z/\!\!/_\vartheta \Gamma)$, as in Remark 4.2.7, with $p^*\pi^*\bar{\phi}^*M = \boldsymbol{L}^{\otimes n}$ for some $n > 0$. We have

$$\deg_{\mathscr{C}}(\sigma^*(\mathscr{L}_\vartheta)) - \frac{1}{n}\deg_{\mathscr{C}}(\sigma^*_{\mathrm{reg}}(M)) = \sum_{y \in \mathscr{C}} \ell(y).$$

(III) For any family of prestable LG quasimaps $(\mathscr{C}/T, y_1, \dots, y_k, \mathscr{P}, \varkappa)$ over T, the function $\ell\colon \mathscr{C} \to \mathbb{Q}$ is upper semicontinuous.

Definition 4.2.11 Choose a polarization $\theta \in \widehat{G}$ and a good lift ϑ of θ (see Definition 3.2.13).

Given a prestable LG quasimap $\mathcal{Q} = (\mathscr{C}, x_1, \dots, x_k, \mathscr{P}, \sigma, \varkappa)$ and any positive rational ε, we say that $\mathcal{Q}$ is *ε–stable* (for the lift ϑ) if

(I) $\omega_{\log,\mathscr{C}} \otimes \sigma^*(\mathscr{L}_\vartheta)^\varepsilon$ is ample, and

(II) $\varepsilon\ell(y) \le 1$ for every $y \in \mathscr{C}$.

We say that $\mathcal{Q}$ is ∞–stable if there exists an $n > 0$ such that $\mathcal{Q}$ is ε–stable for all $\varepsilon > n$.

Remark 4.2.12 The ∞–stability condition is equivalent to saying that there are no basepoints (by condition (II) when ε is large) and that on each component of $\mathscr{C}$ the line bundle $\sigma^*(\mathscr{L}_\vartheta)$ has nonnegative degree (by condition (I) when ε is large), with the degree only being able to vanish on components where $\omega_{\log}$ is ample.

We also wish to define another stability condition we call $0+$ stability. This is the limiting stability condition as $\varepsilon \downarrow 0$; but where ε–stability requires a good lift, $0+$ stability does not.

Definition 4.2.13 Given a polarization $\theta \in \widehat{G}$ and a prestable LG quasimap $\mathcal{Q} = (\mathscr{C}, x_1, \dots, x_k, \mathscr{P}, \sigma, \varkappa)$, we say that $\mathcal{Q}$ is *$(0+)$–stable* if there exists a lift ϑ (not necessarily a good lift), such that

(I) every rational component has at least two special points (a mark y_i or a node), and

(II) on every component $\mathscr{C}'$ with trivial $\omega_{\log,\mathscr{C}'}$, the line bundle $\sigma^*(\mathscr{L}_\vartheta)$ has positive degree.

It turns out that condition (II) holds for some lift if and only if it holds for all lifts. This follows from the next proposition and its corollary.

Proposition 4.2.14 *For any two lifts ϑ and ϑ' of θ, the bundles $\sigma^*(\mathscr{L}_\vartheta)$ and $\sigma^*(\mathscr{L}_{\vartheta'})$ differ by a power of $\omega_{\log,\mathscr{C}}$:*

$$\sigma^*(\mathscr{L}_\vartheta)^{-1} \otimes \sigma^*(\mathscr{L}_{\vartheta'}) = \omega_{\log,\mathscr{C}}^a$$

for some $a \in \mathbb{Q}$.

Proof We have (after clearing denominators, if necessary) that

$$(\vartheta^{-1}\vartheta')(g) = \theta^{-1}(g)\theta(g) = 1$$

for any $g \in G$. Hence $\vartheta^{-1}\vartheta'$ factors through ζ, and in fact we have $\vartheta^{-1}\vartheta' = \zeta^l$ for some l. This gives $\sigma^*(\mathscr{L}_{\vartheta^{-1}\vartheta'}) = \mathscr{L}_\zeta^l = \omega_{\log,\mathscr{C}}^{l/d}$. □

Corollary 4.2.15 *A prestable LG quasimap $\mathcal{Q} = (\mathscr{C}, y_1, \dots, y_k, \mathscr{P})$ satisfies condition (II) for $(0+)$–stability for one lift of θ if and only if it satisfies that condition for every lift of θ.*

Proof By the previous proposition, the difference between the various lifts is a power of $\omega_{\log,\mathscr{C}}$ and hence is trivial on these components. □

Definition 4.2.16 A *family of ε–stable, k–pointed, genus-g, LG quasimaps to $\mathscr{X}_\theta$ over a scheme T* is a family of prestable k–pointed, genus-g, LG quasimaps to $\mathscr{X}_\theta$ over T (see Definition 4.2.8) such that the induced LG quasimap on each geometric fiber is ε–stable.

Proposition 4.2.17 *The automorphism group of any ε–stable LG quasimap $\mathcal{Q} = (\mathscr{C}, x_1, \dots, x_k, \mathscr{P}, \sigma, \varkappa)$ is finite and reduced.*

Proof Observe that we have an exact sequence

$$1 \to \mathrm{Aut}_{\mathscr{C}}(\mathcal{Q}) \to \mathrm{Aut}(\mathcal{Q}) \to \mathrm{Aut}_{\mathscr{C}},$$

where $\mathrm{Aut}_{\mathscr{C}}(\mathcal{Q})$ is the group of automorphisms of $\mathcal{Q}$ fixing $\mathscr{C}$. Thus we may break the proof into two parts. First, the same argument as given in [16, Proposition 7.1.5] shows that if $\mathscr{C}$ is irreducible but unstable (ie $\mathrm{Aut}(\mathscr{C})$ is infinite), then $\mathrm{Aut}(\mathcal{Q})$ is finite. Second, we prove that $\mathrm{Aut}_{\mathscr{C}}(\mathcal{Q})$ is finite.

The quasimap $\mathcal{Q}$ induces a morphism $\overline{\sigma}\colon \mathring{\omega}_{\log,\mathscr{C}} \smallsetminus F \to [V/\!\!/_\theta G]$, where F is the fiber in $\mathring{\omega}_{\log,\mathscr{C}}$ over the set of basepoints B of σ. Any element of $\mathrm{Aut}_{\mathscr{C}}(\mathcal{Q})$ must fix $\mathring{\omega}_{\log,\mathscr{C}}$ and the morphism $\overline{\sigma}\colon \mathring{\omega}_{\log,\mathscr{C}} \smallsetminus F \to [V/\!\!/_\theta G]$. Since $[V/\!\!/_\theta G]$ is a DM stack, the set of automorphisms of $\overline{\sigma}$ restricted to the generic point must be finite. But any automorphism of the Γ–bundle $\mathscr{P}$ over a curve is completely determined by its value on the generic point. Hence $\mathrm{Aut}_{\mathscr{C}}\,\mathcal{Q}$ is finite.

Finally, the automorphism group is reduced because we have restricted ourselves to characteristic 0. $\square$

Definition 4.2.18 For a given choice of compatible G– and $\mathbb{C}^*_R$–actions on a closed affine variety $Z \subseteq V$, a strongly regular character $\theta \in \widehat{G}$, a good lift ϑ of θ and a nondegenerate W, we denote the corresponding stack of k–pointed, genus-g, ε–stable (for ϑ) LG quasimaps into $\mathscr{X}_\theta$ of degree β by

$$\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{X}_\theta, \beta).$$

If $\varepsilon = 0+$, we can dispense with the good lift and instead define

$$\mathrm{LGQ}^{0+}_{g,k}(\mathscr{X}_\theta, \beta)$$

to be the stack of k–pointed, genus-g, LG quasimaps into $\mathscr{X}_\theta$ of degree β that are $(0+)$–stable for any (and hence every) lift of θ.

4.3 Example: hypersurfaces

We illustrate these ideas with the hypersurface described in Example 3.2.15. The reader who wishes to move directly to the main results of this paper may skip this example on first reading. For many more examples, see Section 7.

Example 4.3.1 (hypersurfaces, geometric phase) Consider again the situation of a hypersurface in weighted projective space, as in Example 3.2.15, where $G = \mathbb{C}^*$ and $V = \mathbb{C}^K \times \mathbb{C}$ with coordinates $(x_1, \dots, x_K, p)$. Let $W = Fp\colon \mathbb{C}^{K+1} \to \mathbb{C}$ have G–weights $(b_1, \dots, b_K, -b)$.

In the geometric phase we have semistable locus $(x_1, \dots, x_K) \neq (0, \dots, 0)$, and critical locus $\{p = 0, F(x_1, \dots, x_K) = 0\}$. So the quotient $\mathscr{CR}_\theta = \{p = 0 = F(x_1, \dots, x_K)\}$ of the critical locus is a degree-b hypersurface in $W\mathbb{P}(b_1, \dots, b_K)$.

Choosing the $\mathbb{C}^*_R$–weights $(0, \dots, 0, 1)$ gives a hybrid model in which W has $\mathbb{C}^*_R$–weight $d = 1$ and Γ is a direct product $\Gamma \cong G \times \mathbb{C}^*_R$, with ξ and ζ just the first and second projections, respectively. We use the trivial lift ϑ_0 as our good lift.

A principal Γ–bundle $\mathscr{P}$ on $\mathscr{C}$ with $\zeta_*(\mathscr{P}) \cong \mathring{\omega}_{\log,\mathscr{C}}$ is equivalent to a line bundle $\mathscr{L}$ on $\mathscr{C}$ with $\mathscr{P} = \mathring{\mathscr{L}} \times \mathring{\omega}_{\log,\mathscr{C}}$, where $\mathring{\mathscr{L}}$ is the principal $\mathbb{C}^*$–bundle associated to the line bundle $\mathscr{L}$.

The vector bundle $\mathscr{P}\times_\Gamma V$ is $\mathscr{L}^{\oplus K}\oplus(\mathscr{L}^{\otimes(-b)}\otimes\omega_{\log,\mathscr{C}})$, so the stack is

$$\{(\mathscr{C},\mathscr{L},s_1,\dots,s_K,p)\mid s_i\in H^0(\mathscr{C},\mathscr{L}),\ p\in H^0(\mathscr{L}^{-b}\otimes\omega_{\log,\mathscr{C}})\}$$

satisfying the stability conditions. Here $\mathscr{C}$ is a marked orbicurve and $\mathscr{L}$ is a line bundle over $\mathscr{C}$.

A particularly simple case is the ∞–stable LG quasimaps to the critical locus $\mathscr{CR}_\theta$. Since there are no basepoints in this case, $(s_1,\dots,s_K)\neq 0$. The critical locus requires $p=0$ and $F=0$; the quasimap $\sigma=(s_1,\dots,s_K,p)$ corresponds to a map $\mathscr{C}\to W\mathbb{P}(b_1,\dots,b_K)$. Moreover, the image of the map must lie in $X_F=\{F=0\}\subset W\mathbb{P}(b_1,\dots,b_K)$ and we have $\mathscr{L}=\sigma^*\mathscr{O}(1)=\sigma^*\mathscr{L}_{\vartheta_0}$. So the ∞–stability condition for the trivial lift exactly corresponds to this map's being a stable map to X_F. Therefore, $\mathrm{LGQ}^{\infty,\vartheta_0}_{g,k}(\mathscr{CR}_\theta,\beta)$ is the stack of stable maps to the critical locus $\mathscr{CR}_\theta=\{F=0\}\subseteq W\mathbb{P}(b_1,\dots,b_K)$ of degree β. Moreover, the stack $\mathrm{LGQ}^{\infty,\vartheta_0}_{g,k}([V/\!\!/_\theta G],\beta)$ is the space of stable maps with p–fields, studied in [6; 7].

There is a parallel theory of quasimaps into X_F that has the same moduli space as our construction in this example (the geometric phase of the hypersurface), but the virtual cycle constructions are different. For $\varepsilon=\infty$, Chang, Li, Li and Liu [8] proved equivalence of the two theories using a sophisticated degeneration argument. A similar argument probably works for other choices of ε.

Example 4.3.2 (hypersurfaces, LG phase) Let's now consider the LG phase of the hypersurface in weighted projective space. The unstable locus is $\{p=0\}$. We first consider the same R–charge as before, ie $c_{x_i}=0$ and $c_p=1$. We have a similar moduli space

$$\{(\mathscr{C},\mathscr{L},s_1,\dots,s_K,p)\mid s_i\in H^0(\mathscr{C},\mathscr{L}),\ p\in H^0(\mathscr{L}^{-b}\otimes\omega_{\log,\mathscr{C}})\},$$

satisfying the stability condition that $p\neq 0$. For the LG quasimaps to lie in the critical locus requires $s_i=0$. The basepoints are precisely the zeros of p, and the base locus forms an effective divisor D with $\mathscr{L}^{-b}\otimes\omega_{\log,\mathscr{C}}\cong O(D)$. So we can reformulate the moduli problem as

$$\{(\mathscr{C},\mathscr{L},s_1,\dots,s_K)\mid s_i\in H^0(\mathscr{C},\mathscr{L}),\ \mathscr{L}^b\cong\omega_{\log,\mathscr{C}}(-D)\},$$

which can be viewed as a *weighted b–spin* condition (see [47]). When $\varepsilon=\infty$, there is no basepoint, ie $D=0$, and we obtain the usual b–spin moduli space corresponding to

$$\mathscr{L}^d\cong\omega_{\log,\mathscr{C}}.$$

There are other choices of R–charge. For example, we can choose the $\mathbb{C}^*_R$–action to have weights $c_{x_i}=b_i$ and $c_p=0$. We have $\Gamma=\{(\alpha^{b_1},\dots,\alpha^{b_K},\beta)\mid\alpha,\beta\in\mathbb{C}^*\}$, with $\zeta\colon \Gamma \to \mathbb{C}^*$ given by $(\alpha^{b_1}, \dots, \alpha^{b_K}, \beta) \mapsto \alpha^d \beta$. Also the map $\xi\colon \Gamma \to G/\langle J \rangle$ is given by $(\alpha^{b_1}, \dots, \alpha^{b_K}, \beta) \mapsto \beta$.

The stack $\mathrm{LGQ}^{\varepsilon,\vartheta_0}_{g,k}(\mathscr{X}_\theta, \beta)$ consists of pointed orbicurves $\mathscr{C}$ with line bundles $\mathscr{L}$ and $\mathscr{B}$ such that $\mathscr{B} \cong \omega_{\log,\mathscr{C}} \otimes \mathscr{L}^{-d}$ and sections $s_1, \dots, s_N$ of $\mathscr{L}$ and p of $\mathscr{B}$ satisfying the stability conditions. Again let's consider $\mathrm{LGQ}^{\infty,\vartheta_0}_{g,k}(\mathscr{C}\mathscr{R}_\theta, \beta)$. In this case, since the semistable locus $\mathrm{Crit}^{\mathrm{ss}}_G(\theta)$ consists of points of the form $(0, \dots, 0, p)$ with $p \neq 0$, the sections $s_1, \dots, s_N$ must all vanish. Again, we can identify $\mathscr{B} = O(D)$ for an effective divisor. This implies

$$\mathscr{L}^d \cong \omega_{\log,\mathscr{C}}(-D).$$

Moreover, since θ has weight $-e$ for some $e < 0$, the trivial lift ϑ_0 corresponds to the map $\Gamma \to \mathbb{C}^*$ given by $(\alpha^{b_1}, \dots, \alpha^{b_K}, \beta) \mapsto \beta^{-e/d}$, and the pullback line bundle $\sigma^*(\mathscr{L}_{\vartheta_0})$ is precisely $\mathscr{B}^{-e/d}$, which is a d^{th} root of $\mathscr{O}_{\mathscr{C}}$. So the stability condition just reduces to the requirement that $\omega_{\log,\mathscr{C}}$ be ample — that is, that the orbicurve $\mathscr{C}$ be stable.

4.4 Evaluation maps

LG quasimaps to $\mathscr{Z}_\theta = [Z /\!\!/_\theta G]$ are not quasimaps into $\mathscr{Z}_\theta$. Their target is the Artin stack $[Z^{\mathrm{ss}}_G(\theta)/\Gamma]$, so one might expect that evaluation maps would only land in the inertia stack $\mathbb{I}[Z^{\mathrm{ss}}_G(\theta)/\Gamma]$ of the stack $[Z^{\mathrm{ss}}_G(\theta)/\Gamma]$. But we can define evaluation maps

$$\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{Z}_\theta, \beta) \xrightarrow{\mathrm{ev}_i} \mathbb{I}\mathscr{Z}_\theta = \bigsqcup_{\Psi} \mathscr{Z}_{\theta,\Psi}$$

to the inertia stack of the GIT quotient stack $\mathscr{Z}_\theta$, as follows.

Observe first that the log-canonical bundle $\omega_{\log,\mathscr{C}}$ and its corresponding principal $\mathbb{C}^*$–bundle $\mathring{\omega}_{\log,\mathscr{C}}$ have a canonical section at each marked point y_i (call this section dz/z). Since G is the kernel of ζ, the preimage $\zeta_*^{-1}\varkappa^{-1}(dz/z) \subset \mathscr{P}|_{y_i}$ is a principal G–orbit in $\mathscr{P}$, and hence defines a principal G–bundle $\mathscr{Q}$ over the (orbifold) marked point y_i. The section $\sigma\colon \mathscr{C} \to \mathscr{P} \times_\Gamma V$ induces a section $\mathscr{C} \to \mathscr{Q} \times_G V$, which gives a map $\{y_i\} \to [Z/G]$. Since the section σ is never G–unstable at nodes and marked points, this actually gives a map to $\mathscr{Z}_\theta$ and not just to $[Z/G]$. Moreover, since y_i is an orbifold point of the form $y_i = [\tilde{y}_i / G_{y_i}] \cong \mathrm{B}G_{y_i}$, the generator of the local group G_{y_i} must map to an element of the stabilizer of the image of $\tilde{y}_i$. That is, the evaluation map takes values in the inertia stack $\mathbb{I}\mathscr{Z}_\theta$.

Applying this construction to all LG quasimaps gives the desired evaluation morphisms

$$\mathrm{ev}_i\colon \mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{Z}_\theta, \beta) \to \mathbb{I}\mathscr{Z}_\theta.$$

The existence of the evaluation maps shows that we can decompose

$$\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{Z}_\theta,\beta)=\bigsqcup_{\Psi_1,\dots,\Psi_k}\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{Z}_\theta,\beta)(\Psi_1,\dots,\Psi_k),\tag{14}$$

where the Ψ_i are conjugacy classes in G indexing the twisted sectors of $\mathbb{I}\mathscr{Z}_\theta$, and the factors $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{Z}_\theta,\beta)(\Psi_1,\dots,\Psi_k)$ are the open and closed substacks where the i^{th} evaluation morphism maps to the component (sector) $\mathscr{Z}_{\theta,\Psi_i}$ of $\mathbb{I}\mathscr{Z}_\theta$.

Proposition 4.4.1 *There is an integer $\boldsymbol{e}$, depending only on W, G and the action of Γ on V, such that for any prestable LG quasimap $\mathcal{Q}=(\mathscr{C},y_1,\dots,y_k,\mathscr{P},\sigma,\varkappa)$, the degree of every line bundle on $\mathscr{C}$ lies in $\frac{1}{\boldsymbol{e}}\mathbb{Z}$, and for any marked point or node y of $\mathcal{Q}$, the order of the local group G_y at y is bounded by $\boldsymbol{e}$.*

Proof Recall that $\mathbb{I}\mathscr{X}$ is indexed by a finite number of conjugacy classes, each of finite order (see the discussion after Definition 4.1.1). Let $\boldsymbol{e}$ be the least common multiple of these orders.

Let y be a marked point or node of $\mathscr{C}$ and let G_y be the local group of the orbifold $\mathscr{C}$ at y. Since $\mathscr{P}\colon\mathscr{C}\to B\Gamma$ is representable, the corresponding homomorphism $G_y\to G\subset\Gamma$ must be injective, and hence $G_y\cong\langle\gamma\rangle$ for some $\gamma\in G$ fixing $\sigma(y)\in V$. Therefore γ must lie in one of the finite number of conjugacy classes corresponding to nonempty components of $\mathbb{I}\mathscr{X}$, and hence the order of G_y must divide $\boldsymbol{e}$.

This also shows that for any line bundle $\mathscr{N}$ on $\mathscr{C}$ the tensor power $\mathscr{N}^{\otimes\boldsymbol{e}}$ is the pullback of a line bundle on the coarse curve underlying $\mathscr{C}$, and hence $\boldsymbol{e}$ times the degree of $\mathscr{N}$ is an integer. □

Example 4.4.2 Consider again the geometric phase of a hypersurface X_F in weighted projective space of Examples 3.2.15 and 4.3.1. The untwisted sector $\mathscr{X}_{\theta,[\![1]\!]}$ is broad and is the line bundle $\mathscr{O}(-d)$ over weighted projective space. Any subvariety of weighted projective space defines an element of the state space of compact type, and $\mathscr{H}_{W,G,\mathrm{comp}}$ can be identified with the ambient classes of $H^*_{\mathrm{CR}}(X_F,\mathbb{Q})$.

The elements of the state space which are not of compact type correspond to the so-called *primitive cohomology* of $H^*_{\mathrm{CR}}(X_F,\mathbb{Q})$. These correspond to broad insertions in FJRW theory.

5 Properties of the moduli space

5.1 Boundedness

In this section we develop some boundedness results that will be used in the proof of Theorem 5.2.3 (specifically, to show that the stack of LG quasimaps is of finite type).

Proposition 5.1.1 *Given a lift ϑ of θ and any prestable LG quasimap*

$$\mathcal{Q} = (\mathscr{C}, y_1, \dots, y_k, \mathscr{P}, \sigma, \varkappa)$$

such that σ maps the generic point of a component $\mathscr{C}'$ of $\mathscr{C}$ to a ϑ–semistable point of V, the degree of the pullback bundle $\sigma^*(\mathscr{L}_\vartheta)$ on $\mathscr{C}'$ is nonnegative:

$$\deg_{\mathscr{C}'} \sigma^*(\mathscr{L}_\vartheta) \geq 0.$$

Moreover, $\deg_{\mathscr{C}'} \sigma^*(\mathscr{L}_\vartheta) = 0$ if and only if there are no ϑ–basepoints on $\mathscr{C}'$ and composing σ with the natural map $[V /\!/_\vartheta \Gamma] \to V /\!/_\vartheta \Gamma$ induces a constant map $\mathscr{C}' \to V /\!/_\vartheta \Gamma$.

Proof We may assume that $\mathscr{C}$ is irreducible. Since the generic point of $\mathscr{C}$ maps to a Γ–semistable point of V with respect to ϑ, we must have some $n > 0$ for which there exists a nonzero $f \in H^0(V, \boldsymbol{L}_\vartheta^n)^\Gamma$ such that $f(\sigma(y)) \neq 0$ for some $y \in \mathscr{C}$. Thus $\sigma^*(f)$ is a nonzero element of $H^0(\mathscr{C}, \sigma^*\mathscr{L}_\vartheta^{\otimes n})$, and hence the degree of $\sigma^*\mathscr{L}_\vartheta^{\otimes n}$ must be nonnegative.

Moreover, if σ has no basepoints, but $\sigma^*\mathscr{L}_\vartheta$ has degree 0, then the only global sections of $\sigma^*\mathscr{L}_\vartheta^n$ are constant on $\mathscr{C}$ for every $n > 0$, hence the induced map $\mathscr{C} \to V /\!/_\vartheta \Gamma$ is constant. The converse follows from Remark 4.2.7 — if there are no basepoints and the induced map $\bar{\sigma}\colon \mathscr{C} \to V /\!/_\vartheta \Gamma$ is constant, then there is an ample line bundle M on $V /\!/_\vartheta \Gamma$ such that $\sigma^*\mathscr{L}_\vartheta^n = \bar{\sigma}^* M = \mathscr{O}_{\mathscr{C}}$.

Finally, if b is a ϑ–basepoint of σ, then every section in $H^0(V, \boldsymbol{L}_\vartheta^n)^\Gamma$ must vanish at $\sigma(b)$ and hence $\sigma^*(f)$ is a nonzero section of $\sigma^*\mathscr{L}_\vartheta^{\otimes n}$ on $\mathscr{C}$ that has at least one zero, and hence $\sigma^*\mathscr{L}_\vartheta^{\otimes n}$ must have positive degree. □

Corollary 5.1.2 (compare to [15, Corollary 3.1.5]) *The number of irreducible components of the underlying curve of a k–pointed, genus-g, ε–stable LG quasimap $\mathcal{Q} = (\mathscr{C}, y_1, \dots, y_k, \mathscr{P}, \sigma, \varkappa)$ of degree β is bounded in terms of g, k and $\beta(\vartheta)$.*

Proof Because the genus is bounded, the number of irreducible components of genus greater than zero is bounded. Because the number of marked points is bounded, the number of genus-0 components with at least three points is also bounded. It remains only to consider the components of genus 0 with two or fewer marked points or those of genus 1 with no marked points.

The existence of any unstable component (genus 0 and two or fewer marked points, or genus 1 and no marks) for which $\deg_{\mathscr{C}} \sigma^*\mathscr{L}_\vartheta$ vanishes would contradict the conditions of stability. This implies $\deg_{\mathscr{C}} \sigma^*\mathscr{L}_\vartheta > 0$ on each such component. By Proposition 4.4.1, there is a uniform bound $\boldsymbol{e}$ such that $\deg_{\mathscr{C}} \sigma^*\mathscr{L}_\vartheta \geq \frac{1}{\boldsymbol{e}}$ on each such component, and hence the number of such components is bounded. □

Remark 5.1.3 The previous corollary also holds for $(0+)$–stable curves. The only adjustment that must be made to the proof is that one may use any lift — not just a good lift — in the argument that $\deg_{\mathscr{C}} \sigma^* \mathscr{L}_\vartheta \geq \frac{1}{e}$.

Theorem 5.1.4 *Fixing a prestable orbicurve $\mathscr{C}$, a polarization $\theta \in \widehat{G}$, any character $\xi \in \widehat{\Gamma}_{\mathbb{Q}}$ and a rational number b, the family of prestable LG quasimaps $\mathcal{Q}$ from $\mathscr{C}$ to $\mathscr{X}_\theta$ such that $\deg_{\mathscr{C}} \sigma^* \mathscr{L}_\xi = b$ is bounded.*

The proof is similar to that of [16, Theorem 3.2.4], with additional complications arising from the difference between Γ and G and from the fact that for any lift ϑ, the set $V^s_\Gamma(\vartheta)$ may be empty, even if ϑ is a good lift of θ.

It suffices to prove boundedness of the set S of principal Γ–bundles $\mathscr{P}$ over a fixed, irreducible, orbicurve $\mathscr{C}$ with an isomorphism $\varkappa\colon \zeta(\mathscr{P}) \to \mathring{\omega}_{\log,\mathscr{C}}$ which *admit* an ε–stable LG quasimap $\sigma\colon \mathscr{P} \to V$ of class β to $[V/\!\!/_\theta G]$ (but the particular choice of quasimap σ is not fixed). We can also reduce to the case where $\mathscr{C}$ is nonsingular because a principal Γ–bundle on a nodal orbicurve $\mathscr{C}$ is given by a principal bundle $\widetilde{\mathscr{P}}$ on the normalization $\widetilde{C}$ and a choice of an identification of the fibers $\widetilde{\mathscr{P}}_{p_+} \cong \widetilde{\mathscr{P}}_{p_-}$ over each node p, and for each node, these identifications are parametrized by the group Γ.

We first consider the case that G is connected. In this case Γ is also connected, because there is a surjective map from $G \times \mathbb{C}^*_R$ to Γ.

Lemma 5.1.5 *Let $G \leq \mathrm{GL}(V)$ be a connected reductive algebraic group. Let $T' \leq G$ be a maximal torus containing $\langle J \rangle$ and let $B' \leq G$ be a Borel subgroup containing T'. Let $B \leq \Gamma = G\mathbb{C}^*_R$ be the subgroup of Γ generated by B' and $\mathbb{C}^*_R$, and let $T \leq \Gamma$ be the subgroup generated by T' and $\mathbb{C}^*_R$.*

The group Γ is reductive, and the subgroup B is a Borel subgroup of Γ containing T, which is a maximal torus of Γ. Moreover, we have $B \cap G = B'$, and the unipotent radical B'_u of B' is the same as the unipotent radical B_u of B.

Proof First, Γ is reductive because it is the quotient of the reductive group $G \times \mathbb{C}^*_R$ by the finite subgroup $\langle J \rangle$.

To see that B is Borel in Γ, observe first that since $B' \triangleleft B$ is normal in B, and the quotient B/B' is abelian, then B is solvable in Γ. If $C \subset \Gamma$ is any solvable subgroup in Γ such that $B' \leq C \cap G$, we claim that $B' = C \cap G$. To see this, note that given any subnormal series $\{1\} = C_0 \triangleleft C_1 \triangleleft \cdots \triangleleft C_n = C$ whose quotients C_k/C_{k-1} are all abelian, the corresponding series $C_0 \cap G \triangleleft \cdots \triangleleft C_n \cap G$ shows that $C \cap G$ is solvable in G. But B' is Borel, hence is a maximal solvable subgroup in G (since G is connected, Borel subgroups are maximal among all solvable subgroups — not just among those that are Zariski-closed and connected — see [5, 11.17]). Thus, since $B' \leq C \cap G$, we must have $B' = C \cap G$.

To see that B is Borel in Γ, it remains to show that B is maximal among the solvable subgroups of Γ. Assume that $S \leq \Gamma$ is a solvable subgroup of Γ with $B \leq S$. Any element $s \in S \leq \Gamma$ can be written as $s = gr$, where $g \in G$ and $r \in \mathbb{C}_R^*$, and $g = sr^{-1} \in SB \leq S$, so $g \in S \cap G$. By the previous paragraph, we have $S \cap G = B'$, so $g \in B'$ and $gr \in B'\mathbb{C}_R^* = B$. Therefore $S = B$, and B is maximal among solvable subgroups of Γ, hence B is Borel in Γ.

The group T is abelian and contains T', and the quotient T/T' is isomorphic to $H \cong \mathbb{C}^*$, by the map $\zeta\colon T \to H$; see (7). By [5, Corollary, page 149] we have that T is also a torus. Since $\langle J \rangle \leq T'$, we have $T' \cap \mathbb{C}_R^* = \langle J \rangle = B' \cap \mathbb{C}_R^*$. So the sequence (8) gives us $B/B' = T/T' = H$, and we have the diagram of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & T' \arrow[r] \arrow[d, "\cong"'] & T \arrow[r, "\zeta"] \arrow[d] & H \arrow[r] \arrow[d, "\cong"'] & 1 \\
1 \arrow[r] & B'/B'_u \arrow[r] & B/B'_u \arrow[r, "\zeta"] & B/B' \arrow[r] & 1
\end{tikzcd}$$

where the leftmost vertical arrow is an isomorphism because T' is the maximal torus of B'. Thus we have $B/B'_u \cong T$. The maximal torus $\widetilde{T}$ of B must contain T and is isomorphic to B/B_u. Also B_u contains B'_u, so we have

$$\begin{tikzcd}
T \arrow[r, "\cong"] \arrow[d, hook] & B/B'_u \arrow[d, two heads] \\
\widetilde{T} \arrow[r, "\cong"] & B/B_u
\end{tikzcd}$$

Thus $T = \widetilde{T}$ must be the maximal torus and $B_u = B'_u$. □

We can now finish the proof of the theorem. By [45, Section 2.11] (see also [16, Theorem 3.2.4]) we may choose a reduction to a principal B–bundle $\mathscr{P}'$ for each principal Γ–bundle $\mathscr{P}$ in the set S. Let

$$R = \{\overline{\mathscr{P}}' = \mathscr{P}'/B_u \mid \mathscr{P} \in S\}.$$

For each $\overline{\mathscr{P}}'$, let $d_{\overline{\mathscr{P}}'}\colon \hat{T} \to \mathbb{Q}$ be given by

$$d_{\overline{\mathscr{P}}'}(\xi) = \deg_{\mathscr{C}}(\overline{\mathscr{P}}' \times_T \mathbb{C}_\xi).$$

By [16, Lemma 3.2.7] and [29, Proposition 3.1 and Lemma 3.3], the set S is bounded if the set R is bounded, and R is bounded if the set

$$D = \{d_{\overline{\mathscr{P}}'} \colon \widehat{T} \to \mathbb{Q} \mid \overline{\mathscr{P}}' \in R\}$$

is bounded.

The argument in the proof of [16, Lemma 3.2.8] shows there is a $\mathbb{Q}$–basis $\{\theta_1, \ldots, \theta_m\}$ of $\widehat{T}' \otimes \mathbb{Q}$ such that for each θ_i we have $V^s_{T'}(\theta) \subseteq V^{\mathrm{ss}}_{T'}(\theta_i)$, and $\theta = \sum_i a_i \theta_i$, with $a_i > 0$ for every i. For each choice of $\mathscr{C}$, $\mathscr{P}$ and σ, the generic point of $\mathscr{C}$ maps by σ into

$$V^s_G(\theta) \subseteq V^s_{T'}(\theta) \subseteq V^{\mathrm{ss}}_{T'}(\theta_i).$$

By Lemma 3.2.12, for each i there are three standard lifts ϑ_-, ϑ_0 and ϑ_+ of θ_i such that

$$V^{\mathrm{ss}}_{T'}(\theta_i) = V^{\mathrm{ss}}_T(\vartheta_-) \cup V^{\mathrm{ss}}_T(\vartheta_0) \cup V^{\mathrm{ss}}_T(\vartheta_+),$$

and such that the $\mathbb{C}^*_R$–weight of ϑ_-, ϑ_0 and ϑ_+ is -1, 0 and 1, respectively.

Therefore, for at least one of these three lifts (denote it simply by ϑ_i), the generic point of $\mathscr{C}$ must map to $V^{\mathrm{ss}}_{\vartheta_i}(T)$. That means there must exist some section $s \in H^0(V, \boldsymbol{L}^n_{\vartheta_i})^T$ for some $n > 0$. This induces a section of $\overline{\mathscr{P}}' \times_T \boldsymbol{L}^n_{\vartheta_i}$ that does not vanish on the generic point of $\mathscr{C}$, and thus $\deg_{\mathscr{C}} \overline{\mathscr{P}}' \times_T \boldsymbol{L}_{\vartheta_i} \geq 0$.

If $2g-2+k \geq 0$, let φ be the unique lift of the trivial T'–character with $\mathbb{C}^*_R$–weight 1. The LG quasimap structure $\mathscr{P} \to \mathring{\omega}_{\log,\mathscr{C}}$ means that $\deg_{\mathscr{C}}(\overline{\mathscr{P}}' \times_T \mathbb{C}_\varphi) = \deg_{\mathscr{C}} \omega_{\log,\mathscr{C}} = 2g-2+k \geq 0$. Similarly, if $2g-2+k < 0$, let φ be the unique lift of the trivial T'–character with $\mathbb{C}^*_R$–weight -1. Again we have $\deg_{\mathscr{C}}(\overline{\mathscr{P}}' \times_T \boldsymbol{L}_\varphi) = \deg_{\mathscr{C}} \omega^{-1}_{\log,\mathscr{C}} = 2-2g-k > 0$.

The characters $\{\vartheta_1, \ldots, \vartheta_n, \varphi\}$ form a basis for $\widehat{T} \otimes \mathbb{Q}$. Since the degree

$$\deg_{\mathscr{C}} \overline{\mathscr{P}}' \times_T \boldsymbol{L}_\varphi = |2g-2+k|$$

is fixed by g and k, it suffices to prove there are a finite number of possible values for each $\deg_{\mathscr{C}} \overline{\mathscr{P}}' \times_T \boldsymbol{L}_{\vartheta_i}$. To do this, note that there is a unique $r \in \mathbb{Z}$ such that the character ξ can be written as

$$\xi = \sum_{i=1}^n a_i \vartheta_i + r\varphi.$$

This gives

$$\deg_{\mathscr{C}} \sigma^*(\mathscr{L}_\xi) - r|2g-2+k| = \sum_{i=1}^n a_i \deg_{\mathscr{C}} \overline{\mathscr{P}}' \times_T \boldsymbol{L}_{\vartheta_i}.$$

All the coefficients a_i and r are independent of $\mathscr{C}$ and $\overline{\mathscr{P}}'$ and depend only on the action of T on V and on characters $\xi, \varphi, \vartheta_1, \ldots, \vartheta_n \in \widehat{T}$. Since the a_i are all positive, every $\deg_{\mathscr{C}} \overline{\mathscr{P}}' \times_T \boldsymbol{L}_{\vartheta_i}$ is nonnegative and the left-hand side of this equation is determined by g, k and b, it follows that the possible values for $\deg_{\mathscr{C}} \overline{\mathscr{P}}' \times_T \boldsymbol{L}_{\vartheta_i}$ are all bounded. Since these degrees must all lie in $\frac{1}{e}\mathbb{Z}$, there can only be a finite number of them. Hence the set D is bounded, and we have shown the theorem in the case that G is connected.

In the general case we assume that G is reductive with identity component G_0 such that G/G_0 is finite, but G is not necessarily connected. Let Γ_0 be the component of Γ containing the identity element. Clearly $G_0\mathbb{C}_R^* \subseteq \Gamma_0$, so the group Γ/Γ_0 is a quotient of G/G_0, and hence it is finite.

Given an LG quasimap $\mathcal{Q} = (\mathscr{P} \to \mathring{\omega}_{\log,\mathscr{C}} \to \mathscr{C}, \mathscr{P} \xrightarrow{\sigma} V)$, the quotient $\mathscr{P}/\Gamma_0$ is a principal Γ/Γ_0–bundle over $\mathscr{C}$, hence it is a prestable orbicurve, which we denote by $\widetilde{\mathscr{C}}$. The morphism $\mathscr{P} \to \mathring{\omega}_{\log,\mathscr{C}}$ induces a morphism $\mathscr{P} \to \mathring{\omega}_{\log,\mathscr{C}} \times_{\mathscr{C}} \widetilde{\mathscr{C}} \cong \mathring{\omega}_{\log,\widetilde{\mathscr{C}}}$, which, when combined with $\mathscr{P} \xrightarrow{\sigma} V$, defines an LG quasimap $\widetilde{\mathcal{Q}}$ with gauge group G_0 over $\widetilde{\mathscr{C}}$ and with polarization $\theta|_{G_0}$ induced from θ by restriction to G_0. The degree $\deg_{\widetilde{\mathcal{Q}}}^{G_0}(\theta|_{G_0})$ is just $\frac{1}{m}\deg_{\mathcal{Q}}\theta = b/m$, so by the proof of the theorem in the connected case, the subfamily of these LG quasimaps $\widetilde{\mathcal{Q}}$ with gauge group G_0 over a fixed $\widetilde{\mathscr{C}}$ is bounded. But the number of étale maps $\widetilde{\mathscr{C}} \to \mathscr{C}$ of fixed degree $|\Gamma/\Gamma_0|$ is finite, so the family of all such prestable LG quasimaps over $\mathscr{C}$ is bounded.

Corollary 5.1.6 *For any $\beta \in \operatorname{Hom}(\widehat{\Gamma}_{\mathbb{Q}}, \mathbb{Q})$ the family of prestable LG quasimaps $\mathcal{Q}$ from $\mathscr{C}$ to $\mathscr{X}_\theta$ of degree β is bounded.*

5.2 Finite-type Deligne–Mumford stack

To prove that $\mathrm{LGQ}_{g,k}^{\varepsilon,\vartheta}(\mathscr{X}_\theta, \beta)$ is a Deligne–Mumford stack we first define an intermediate stack.

Definition 5.2.1 Given $\Gamma \xrightarrow{\zeta} \mathbb{C}^*$, let $\mathfrak{A}_{g,k} \to \mathfrak{Bun}_{\Gamma,g,k}$ denote the stack of tuples $(\mathscr{C}, y_1, \dots, y_k, \mathscr{P}, \varkappa)$ consisting of a k–pointed, genus-g prestable orbicurve, a principal Γ–bundle $\mathscr{P}$ on $\mathscr{C}$, and an isomorphism $\varkappa\colon \zeta_*(\mathscr{P}) \to \mathring{\omega}_{\log,\mathscr{C}}$ with the property that the induced morphism $\mathscr{C} \to \mathrm{B}\Gamma$ is representable.

Lemma 5.2.2 *The stack $\mathfrak{A}_{g,k}$ is a smooth Artin stack, locally of finite type over $\mathbb{C}$.*

Proof By [16, Proposition 2.1.1] the stack $\mathfrak{Bun}_\Gamma$ is a smooth Artin stack, locally of finite type over $\mathbb{C}$. Let $\mathfrak{C}$ denote the universal curve over $\mathfrak{Bun}_\Gamma$; let $\mathring{\mathfrak{w}}_{\log,\mathfrak{C}}$ denote the principal $\mathbb{C}^*$–bundle associated to the log-canonical bundle of $\mathfrak{C}$; and let $\mathfrak{P}$ denote the universal Γ–bundle over $\mathfrak{C}$. As a stack $\mathfrak{A}_{g,k}$ is isomorphic to $\operatorname{Isom}_{\mathfrak{C}/\mathfrak{Bun}_\Gamma}(\zeta_*(\mathfrak{P}), \mathring{\mathfrak{w}}_{\log,\mathfrak{C}})$, hence it is representable over $\mathfrak{Bun}_\Gamma$. This proves that $\mathfrak{A}_{g,k}$ is an Artin stack of finite type over $\mathfrak{Bun}_\Gamma$, hence locally of finite type over $\mathbb{C}$.

To see that it is smooth, we use an argument similar to the proof that $\mathfrak{Bun}_\Gamma$ is smooth (see [16, Proposition 2.1.1]). First note that since $\mathbb{C}^*_R$ commutes with G, it lies in the center of Γ, therefore the subgroup $G \subset \Gamma$ inherits a Γ–action from the adjoint action of Γ on itself. Since $\ker(\zeta) = G$, the infinitesimal automorphisms of the data $(\mathscr{P}, \varkappa)$ are precisely those automorphisms of $\mathscr{P}$ over $\mathscr{C}$ which are also automorphisms of $\mathscr{P}$ as a G–bundle over $\mathring{\omega}_{\log,\mathscr{C}}$. Over $\mathring{\omega}_{\log,\mathscr{C}}$ these are given by $\operatorname{Hom}_\Gamma(\mathscr{P}, G) = H^0(\mathring{\omega}_{\log,\mathscr{C}}, \mathscr{P} \times_\Gamma G)$.

Assume we are given a family $\mathscr{C}_0$ of prestable orbicurves over $\operatorname{Spec}(A_0)$, where A_0 is a finitely generated $\mathbb{C}$–algebra A_0, and that we are given the data $(\mathscr{P}_0, \varkappa_0)$ over $\mathscr{C}_0$. Given a square-zero extension A of A_0 with kernel I and an extension $\mathscr{C}$ of $\mathscr{C}_0$ over $\operatorname{Spec}(A)$, extensions of $(\mathscr{P}_0, \varkappa_0)$ to $\mathscr{C}$ are parametrized by $H^1(\mathring{\omega}_{\log,\mathscr{C}_0}, \mathscr{P} \times_\Gamma \mathfrak{g}) \otimes_{A_0} I$, where $\mathfrak{g}$ is the Lie algebra of G. The obstruction to extending $(\mathscr{P}_0, \varkappa_0)$ to $\mathscr{C}$ lies in $H^2(\mathring{\omega}_{\log,\mathscr{C}_0}, \mathscr{P} \times_\Gamma \mathfrak{g}) \otimes_{A_0} I$. Since the fibers of the projection $q\colon \mathring{\omega}_{\log,\mathscr{C}_0} \to \mathscr{C}$ are affine, the higher derived push forwards $R^i q_* \mathscr{P} \times_\Gamma \mathfrak{g}$ vanish for $i > 0$ and the Leray spectral sequence degenerates. So $H^2(\mathring{\omega}_{\log,\mathscr{C}_0}, \mathscr{P} \times_\Gamma \mathfrak{g}) = H^2(\mathscr{C}_0, q_*(\mathscr{P} \times_\Gamma \mathfrak{g}))$. Since $\mathscr{C}_0$ is a family of curves over an affine scheme, $H^2(\mathscr{C}_0, \mathscr{P} \times_\Gamma \mathfrak{g})$ vanishes, and the deformations are unobstructed. Hence $\mathfrak{A}_{g,k}$ is smooth over the stack $\mathfrak{M}_{g,k}$ of prestable orbicurves, which is also smooth. □

Theorem 5.2.3 *Let $\beta \in \operatorname{Hom}(\widehat{G}, \mathbb{Q})$. Fix either $\varepsilon = 0+$ or $\varepsilon > 0$ and a good lift ϑ. Let $\mathscr{Q}$ be*

$$\mathscr{Q} = \mathrm{LGQ}^{0+}_{g,k}([V /\!\!/_\theta G], \beta) \quad \text{or} \quad \mathscr{Q} = \mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}([V /\!\!/_\theta G], \beta).$$

Let $\mathfrak{M} = \mathfrak{M}_{g,k}$ denote the stack of prestable orbicurves $\mathfrak{M}_{g,k}$ and let $\mathfrak{A} = \mathfrak{A}_{g,k}$. The stack $\mathscr{Q}$ is a Deligne–Mumford stack of finite type over $\mathfrak{M}$. And if $Z \subset V$ is a closed subvariety with GIT quotient $\mathscr{Z}_\theta = [Z /\!\!/_\theta G]$, then $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{Z}_\theta, \beta)$ (or $\mathrm{LGQ}^{0+}_{g,k}(\mathscr{Z}_\theta, \beta)$) is a closed substack of $\mathscr{Q}$.

If $\mathscr{P} \to \mathscr{C}$ denotes the universal principal Γ–bundle $\mathscr{P}$ on the universal curve $\pi\colon \mathscr{C} \to \mathscr{Q}$ and $\mathscr{E} = \mathscr{P} \times_\Gamma V$, then $\mathscr{Q} \to \mathfrak{A}$ is representable and has a relative perfect obstruction theory

$$\phi_{\mathscr{Q}/\mathfrak{A}}\colon \mathbb{T}_{\mathscr{Q}/\mathfrak{A}} \to \mathbb{E}_{\mathscr{Q}/\mathfrak{A}} = R^\bullet \pi_* \mathscr{E}, \tag{15}$$

where $\mathbb{T}_{\mathscr{Q}/\mathfrak{A}}$ is the relative tangent complex (dual to the relative cotangent complex $\mathbb{L}_{\mathscr{Q}/\mathfrak{A}}$).

Proof Let $\pi\colon \mathfrak{C} \to \mathfrak{A}$ be the universal curve and let $\mathfrak{P}$ be the universal Γ–bundle on $\mathfrak{C}$. Let $\mathfrak{E} = \mathfrak{P} \times_\Gamma V$ and let $q\colon \mathfrak{E} \to \mathfrak{C}$ be the projection. Chang and Li [6, Section 2.1] show that the direct image cone $\mathfrak{Q} = C(q_* \mathfrak{E})$, consisting of sections σ of $\mathfrak{E}$ over $\mathfrak{C}$ is an Artin stack, and the projection $\mu\colon \mathfrak{Q} \to \mathfrak{A}$ is representable and quasiprojective with relative perfect obstruction theory

$$\phi_{\mathfrak{Q}/\mathfrak{A}}\colon \mathbb{T}_{\mathfrak{Q}/\mathfrak{A}} \to \mathbb{E}_{\mathfrak{Q}/\mathfrak{A}} = R^{\bullet}\pi_*\sigma^*\Omega_{\mathfrak{E}/\mathfrak{C}}. \tag{16}$$

We can realize $\mathscr{Q}$ as the open substack of $\mathfrak{Q}$ where the following conditions hold:

(I) The degree of σ is β.

(II) The section σ maps the generic points of components of $\mathscr{C}$ to $V_\Gamma^{\mathrm{ss}}(\vartheta)$.

(III) The section σ maps the nodes and the marked points to $V_\Gamma^{\mathrm{ss}}(\vartheta)$.

(IV) $\omega_{\log,\mathscr{C}} \otimes \sigma^*\mathscr{L}_\vartheta^\varepsilon$ is ample.

(V) $\varepsilon\ell(y) \le 1$ for all $y \in \mathscr{C}$.

Therefore $\mathscr{Q}$ is an Artin stack with relative perfect obstruction theory (16) over $\mathfrak{A}$. Since $\mathfrak{E}$ is a vector bundle and σ is a section, we have $\sigma^*\Omega^\vee_{\mathfrak{E}/\mathfrak{C}} = \mathfrak{E}$, giving the desired relative perfect obstruction theory (15) for $\mathscr{Q}$. Note that the obstruction theory for a more general target $[Z/\!\!/_\theta G]$ is not necessarily perfect.

The fact that $\mathscr{Q}$ is of finite type over $\mathbb{C}$ follows from the boundedness results of the previous sections, as follows. Consider the obvious projection morphisms $\nu\colon \mathscr{Q} \to \mathfrak{M}$ and $\mu\colon \mathscr{Q} \to \mathfrak{A}$. By Corollary 5.1.2, the image of ν is contained in an open and closed substack $\mathfrak{S} \subset \mathfrak{M}$ of finite type. By Theorem 5.1.4, μ factors thorough an open substack of finite type $\mathfrak{A}_\beta \subset \mathfrak{A}$ lying over $\mathfrak{S}$. Since μ is quasiprojective, this implies that $\mathscr{Q}$ is of finite type.

The fact that $\mathscr{Q}$ is Deligne–Mumford follows from the finiteness of the automorphism group. Finally, the condition that the image of σ lie in Z is a closed condition, so $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}([Z/\!\!/_\theta G],\beta)$ is a closed substack of $\mathscr{Q}$. □

5.3 Separatedness

Theorem 5.3.1 *For any ε and for any closed subvariety $Z \subset V$ with GIT quotient $\mathscr{X}_\theta = [Z/\!\!/_\theta G]$, the Deligne–Mumford stack $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{X}_\theta,\beta)$ (or $\mathrm{LGQ}^{0+}_{g,k}(\mathscr{X}_\theta,\beta)$) is a separated stack.*

The proof of the theorem follows, by the valuative criterion, from the following lemma.

Lemma 5.3.2 *Let R be a discrete valuation ring over $\mathbb{C}$. Let η be the generic point of $\mathrm{Spec}(R)$, and let 0 be the closed point. Consider two prestable LG quasimaps*

$$\mathcal{Q}_1 = (\mathscr{C}_1, y_{1,1},\dots,y_{1,k},\mathscr{P}_1,\sigma_1,\varkappa_1) \quad \text{and} \quad \mathcal{Q}_2 = (\mathscr{C}_2, y_{2,1},\dots,y_{2,k},\mathscr{P}_2,\sigma_2,\varkappa_2)$$

over Spec(R) *that are isomorphic over* η. *Given a lift* ϑ (not necessarily good), *if for each* $i \in \{1,2\}$ *the quasimap* $\mathcal{Q}_i$ *satisfies the stability condition that* $\omega_{\log,\mathscr{C}_i} \otimes \sigma_i^*(\mathscr{L}_\vartheta)^\varepsilon$ *is ample on every fiber of* $\mathscr{C}_i$, *then after possibly replacing* R *with a cover ramified at* 0, *the isomorphism of* $\mathcal{Q}_1$ *with* $\mathcal{Q}_2$ *over* η *extends to an isomorphism over all of* R.

Proof The proof is similar to that in [38; 16], but with additional complications arising from the difference between Γ and G.

If C_1 and C_2 are the coarse underlying curves of $\mathscr{C}_1$ and $\mathscr{C}_2$, respectively, then semistable reduction (see [28, Proposition 3.48]) guarantees that, after possibly replacing R with a cover ramified over 0, there is a prestable, k–pointed curve $C, y_1, \dots, y_k$ over $\Delta = \mathrm{Spec}(R)$ and dominant morphisms $\pi_1\colon C \to C_1$ and $\pi_2\colon C \to C_2$ compatible with the sections and such that each π_i is an isomorphism away from the nodes of the central fibers $(C_i)_0$.

The description of the universal deformation of twisted nodal curves in Remark 1.11 of [40] shows that one can define an orbicurve $\mathscr{C}$ with coarse underlying space C, and $\mathscr{C}$ is compatible with the maps π_i; that is, we have dominant maps $\widetilde{\pi}_i\colon \mathscr{C} \to \mathscr{C}_i$ such that the diagrams

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\widetilde{\pi}_i"] \arrow[d] & \mathscr{C}_i \arrow[d] \\ C \arrow[r, "\pi_i"] & C_i \end{tikzcd}$$

commute for $i \in \{1,2\}$, and the maps $\widetilde{\pi}_i$ are isomorphisms except possibly at the nodes of the central fibers $(\mathscr{C}_i)_0$.

Pulling back $\mathcal{Q}_1$ and $\mathcal{Q}_2$ to $\mathscr{C}$ gives prestable LG quasimaps on $\mathscr{C}$ that are isomorphic over the generic fiber. For each $i \in \{1,2\}$ let B_i denote the base locus in $\mathscr{C}$ of $\mathcal{Q}_i$. Let $U = \mathscr{C} \smallsetminus B_1 \cup B_2$. The maps σ_i induce maps $\overline{\sigma}_i\colon \mathring{\omega}_{\log,\mathscr{C}}|_U \to [V/\!\!/_\theta G]$. These maps agree on the generic fiber, and the target $[V/\!\!/_\theta G]$ is separated, so $\overline{\sigma}_1 = \overline{\sigma}_2$ on U. The isomorphism $f_\eta\colon \mathscr{P}_1 \to \mathscr{P}_2$ (which is Γ–equivariant over $\mathscr{C}_\eta$) must, therefore, extend to an isomorphism f over $\mathscr{C}_\eta \cup U$ in such a way that it is G–equivariant over $\mathring{\omega}_{\log,\mathscr{C}}$.

The question now is whether the G–equivariant morphism $f\colon \mathscr{P}_1 \to \mathscr{P}_2$ over $\mathring{\omega}_{\log,\mathscr{C}}$ defines a Γ–equivariant morphism over $\mathscr{C}_\eta \cup U$. For any $p \in \mathscr{P}_1$ over the special fiber and for any $r \in \mathbb{C}_R^*$, there is a Δ–valued point $\widetilde{p}$ of $\mathscr{P}_1$ which specializes to p. Since f is Γ–equivariant over the generic fibers, we have

$$f(r\,\widetilde{p}_\eta) = rf(\widetilde{p}_\eta).$$

And since the space $\mathscr{P}_2$ is separated, we must have

$$f(r\,\widetilde{p}) = rf(\widetilde{p}),$$

and hence $f(rp) = rf(p)$. Therefore, f is Γ–equivariant and defines an isomorphism $\mathscr{P}_1 \to \mathscr{P}_2$ of principal Γ–bundles over $\mathscr{C}_\eta \cup U$.

Since the base loci B_i are disjoint from all nodes, the isomorphism is defined everywhere but a finite collection of (unorbifolded) points $(B_1 \cup B_2) \cap \mathscr{C}_0$ in the central fiber $\mathscr{C}_0$, and by Hartogs' theorem it must extend to all of $\mathscr{C}$. Therefore, we may assume that on $\mathscr{C}$ the bundles $\mathscr{P}_i$ are isomorphic and the maps σ_i are identified by that isomorphism.

The morphisms $\tilde{\pi}_i$ must contract precisely those components of the special fiber $\mathscr{C}_0$ for which $\omega_{\log,\mathscr{C}} \otimes \sigma^*(\mathscr{L}_\vartheta)^\varepsilon$ is not ample. But this condition depends only on $\mathscr{P}$ and σ, so the same components are contracted for each i, and the isomorphisms $(\mathscr{C}_1)_\eta \to (\mathscr{C}_2)\eta$ and $(\mathscr{P}_1)_\eta \to (\mathscr{P}_2)_\eta$ extend to isomorphisms $\mathscr{C}_1 \to \mathscr{C}_2$ and $\mathscr{P}_1 \to \mathscr{P}_2$, which gives the isomorphism $\mathcal{Q}_1 \cong \mathcal{Q}_2$. □

5.4 Properness

Theorem 5.4.1 *If $Z/\!\!/_\theta G \subset V/\!\!/_\theta G$ is projective, then for every good lift ϑ of θ, every pair g and k, every $\beta \in \mathrm{Hom}(\widehat{\Gamma}, \mathbb{Q})$ and every ε, the stack $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{X}_\theta, \beta)$ (or $\mathrm{LGQ}^{0+}_{g,k}(\mathscr{X}_\theta, \beta)$, with any lift) is proper (over $\mathrm{Spec}(\mathbb{C})$).*

Proof To begin, note that $Z/\!\!/_\theta G$ is always projective over $Z/_{\mathrm{aff}}G$ and it is projective (over $\mathrm{Spec}(\mathbb{C})$) if and only if $Z/_{\mathrm{aff}}G = \mathrm{Spec}(\mathbb{C})$. And thus $Z/\!\!/_\theta G$ is projective implies that $Z/\!\!/_\vartheta \Gamma$ is projective as well (but the Artin stack $[Z/\!\!/_\vartheta \Gamma]$ need not be separated).

To prove properness of the stack $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{X}_\theta, \beta)$, we use the valuative criterion. If Δ is the spectrum of a complete discrete valuation ring with generic point η and special point 0, assume we have a k–pointed, genus-g, ε–stable LG quasimap $\mathcal{Q}_\eta$

$$\mathscr{C}_\eta \leftarrow \mathring{\omega}_{\log,\mathscr{C}_\eta} \leftarrow \mathscr{P}_\eta \xrightarrow{\sigma_\eta} Z \tag{17}$$

over the generic fiber. After possibly shrinking Δ and making a base change ramified only over 0, we may assume that the basepoints of $\mathcal{Q}_\eta$ are sections $b_i \colon \eta \to \mathscr{C}_\eta$, with $i \in \{1, \dots, m\}$, and that the lengths $\ell(b_i)$ are constant. We may also assume that the generic fiber $\mathscr{C}_\eta$ is smooth and irreducible. Let $C_\eta \to \eta$ be the coarse underlying curve of $\mathscr{C}_\eta$ with corresponding sections $\bar{y}_1, \dots, \bar{y}_k$, and $\bar{b}_1, \dots, \bar{b}_m$.

The first step of the proof is to choose a suitable open set $U' \subset \mathscr{C}_\eta$, where the LG quasimap is sufficiently well behaved that we can extend it to most of the central fiber. Gluing this to the original LG quasimap and using Hartogs' theorem will allow us to extend this to an LG quasimap on the entire curve. For any semistable curve extending $\mathscr{C}_\eta$ to all of Δ, we also choose a section τ of the log-canonical bundle on a coarse, stable model (or on a special semistable model if $2g - 2 + k \le 0$).

We choose

$$U' = \mathscr{C}_\eta \smallsetminus \{c_1, \dots, c_n, b_1, \dots, b_m\},$$

where the $c_1, \dots c_n$ are sections of $\mathscr{C}_\eta$ chosen as follows.

If $\mathscr{C}_\eta$ with sections $y_1, \dots, y_k, b_1, \dots, b_m$ is not stable as a pointed curve, then the genus g of $\mathscr{C}_\eta$ satisfies $2g-2+k+m \le 0$. If $2g-2+k+m = 0$, then either $k=2$ and $\omega_{\log,\mathscr{C}_\eta}$ is trivial, or $m > 0$, and $\omega_{\log,\mathscr{C}_\eta}$ is trivial on $\mathscr{C}_\eta \smallsetminus \{b_1, \dots, b_m\}$. Letting $\widetilde{\mathscr{C}} \to \Delta$ be any semistable curve over Δ whose coarse generic fiber agrees with $\mathscr{C}_\eta$, then repeatedly contracting all the -1–curves in the special fiber will give a new (unorbifolded) curve C with no -1–curves in the special fiber. Every component of this new curve C will either have genus 1 with no marked points, or have genus 0 with two marked points or nodes, in either case, every component has a trivial log-canonical bundle. In this case we take no additional sections (that is, $n=0$), and we fix a section $\tau\colon C \to \mathring{\omega}_{\log,C}$.

Similarly, if $2g-2+k+m < 0$ and $m=1$, then $g=k=0$. Taking any semistable curve over Δ whose coarse generic fiber agrees with $\mathscr{C}_\eta$ and repeatedly contracting -1–curves not containing b_1 gives a curve C with only one component in the special fiber, and it must contain the point b_1. The log-canonical bundle is trivial over $C \smallsetminus \{b_1\}$. Again, take no additional sections (that is, $n=0$), and fix a section $\tau\colon C \smallsetminus \{b_1\} \to \mathring{\omega}_{\log,C}$.

If $2g-2+k+m < 0$ and $m=0$, then $g=0$, and $0 \le k \le 1$. Choose c_1 to be any section that is disjoint from the section y_1 (or let c_1 be any section if $k=0$). If $\widetilde{\mathscr{C}} \to \Delta$ is any semistable curve (pointed with c_1 and with y_1 if $k=1$) over Δ whose coarse generic fiber agrees with $\mathscr{C}_\eta$, then repeatedly contracting all the -1–curves in the special fiber (relative to both the y_1 and c_1) will give a new (unorbifolded) curve C with no -1–curves in the special fiber. This new curve C will have trivial log-canonical bundle. Fix a section $\tau\colon C \to \mathring{\omega}_{\log,C}$.

Finally, consider the case where $\mathscr{C}_\eta$ with sections $y_1, \dots, y_k, b_1, \dots, b_m$ is a stable curve. Since the stack of stable curves is proper, there is a unique family of genus-g, $k+m$–pointed stable curves $\overline{C} \to \Delta$ extending $C_\eta \to \eta$. We also denote by $\bar{y}_1, \dots, \bar{y}_k$ and $\bar{b}_1, \dots, \bar{b}_m$ the extensions to C of the corresponding (coarse) sections of C_η.

If the log-canonical bundle $\omega_{\log,\overline{C}}$ is trivial over $\overline{C} \smallsetminus \{\bar{b}_1, \dots, \bar{b}_m\}$, then we need no additional sections, so $n=0$.

If $\omega_{\log,\overline{C}}$ is not trivial over $\overline{C} \smallsetminus \{\bar{b}_1, \dots, \bar{b}_m\}$, we may choose a finite set of sections $\bar{c}_1, \dots \bar{c}_n$ of $\overline{C} \to \Delta$, disjoint from the marks $\bar{y}_i$ and the basepoints $\bar{b}_i$, such that the corresponding $\mathbb{C}^*$–bundle $\mathring{\omega}_{\log,\overline{C}}$ is trivial on the complement

$$U = \overline{C} \smallsetminus \{\bar{c}_1, \dots, \bar{c}_n, \bar{b}_1, \dots, \bar{b}_m\}.$$

Let $c_1, \dots, c_n$ be the corresponding sections of $\mathscr{C}_\eta$ and let $\tau\colon U \to \mathring{\omega}_{\log,\overline{C}}$ be a section of the log-canonical bundle.

Now that we have chosen the $c_1, \dots, c_n$ in every case, we set

$$U' = \mathscr{C}_\eta \smallsetminus \{c_1, \dots, c_n, b_1, \dots, b_m\}.$$

Over U' the morphism $\sigma_\eta\colon \mathscr{P}_\eta \to Z$ has no basepoints, so its image lies entirely in $Z_G^{\mathrm{ss}}(\theta)$, and it corresponds to a morphism $\overline{\sigma}'\colon \mathring{\omega}_{\log,\mathscr{C}}|_{U'} \to \mathscr{Z}_\theta$. Composing with $\tau\colon U' \to \mathring{\omega}_{\log,\mathscr{C}}|_{U'}$ gives a morphism $\alpha' = \overline{\sigma}' \circ \tau\colon U' \to \mathscr{Z}_\theta$.

The quotient $\mathscr{Z}_\theta$ is Deligne–Mumford with a projective coarse moduli space, so by [10, Lemma 2.5] there is a unique orbicurve $\widetilde{\mathscr{C}_\eta}$ constructed from $\mathscr{C}_\eta$ by possibly adding additional orbifold structure at the points of $\mathscr{C}_\eta \smallsetminus U'_\eta = \{b_1, \dots, b_m, c_1, \dots c_n\}$, and a unique representable morphism $\alpha_\eta\colon \widetilde{\mathscr{C}_\eta} \to \mathscr{Z}_\theta$ such that $\alpha_\eta|_{U'_\eta} = \alpha'$. The stability conditions on the generic fiber imply that the morphism α_η is a balanced twisted stable $(k+m+n)$–pointed map to $\mathscr{Z}_\theta$. By [1, Theorem 1.4.1] or [9, Theorem A], the stack $\mathscr{K}^{\mathrm{bal}}_{g,k+m+n}(\mathscr{Z}_\theta)$ of such maps is proper, so α_η extends uniquely to a balanced twisted $(k+m+n)$–pointed stable map $\alpha\colon \widetilde{\mathscr{C}} \to \mathscr{Z}_\theta$.

If $\varepsilon \neq 0+$, we have a good lift ϑ, and there is an obvious morphism $p\colon \mathscr{Z}_\theta \to [Z/\!\!/_\vartheta \Gamma]$, given by sending any $T \leftarrow Q \xrightarrow{f} Z$ to

$$T \leftarrow Q \times_G \Gamma \xrightarrow{\tilde{f}} Z,$$

where $\tilde{f}(q, \gamma) = \gamma f(q)$. Composing with α, we have $p \circ \alpha\colon \widetilde{\mathscr{C}} \to [Z/\!\!/_\vartheta \Gamma]$. It is straightforward to see that

$$\deg_{\mathscr{C}_\eta} \sigma^* \mathscr{L}_\vartheta - \deg_{\mathscr{C}_\eta} (p \circ \alpha)^* \mathscr{L}_\vartheta = \sum_{b \in \mathscr{C}} \ell(b). \tag{18}$$

As in [16, 7.1.6], for each subcurve D of the special fiber of $\widetilde{\mathscr{C}}$ we define

$$\deg(D, \mathscr{L}_\vartheta) = \deg_D((p \circ \alpha)^* \mathscr{L}_\vartheta) + \sum_{b_i \cap D \neq \varnothing} \ell(b_i).$$

For each -1–curve D of the special fiber $\widetilde{\mathscr{C}_0}$ (ie an irreducible, rational component that does not contain any of the marked points y_i and only intersects the rest of the special fiber in one point z), we contract this -1–curve if and only if

$$\deg(D, \mathscr{L}_\vartheta) \leq \frac{1}{\varepsilon}. \tag{19}$$

Repeat this process until there are no -1–curves satisfying (19). If $\varepsilon = 0+$ then we just contract all -1–curves. We call the contracted curve $\widehat{\mathscr{C}}$. Denote the set of all the resulting points on $\widehat{\mathscr{C}}$ by $\{z_1,\dots,z_s\}$. For each z_i, denote by Ψ_i the tree of rational curves in $\widetilde{\mathscr{C}}$ that was contracted to z_i. Let

$$\widehat{U}=\widehat{\mathscr{C}}\smallsetminus\{b_1,\dots,b_m,c_1,\dots,c_n,z_1,\dots,z_s\},$$

Note that the generic fiber of $\widehat{\mathscr{C}}$ is not the same as the original generic fiber $\mathscr{C}_\eta$ because $\widehat{\mathscr{C}}$ may have additional orbifold structure at the points $\{b_1,\dots,b_m,c_1,\dots c_n\}$. Nevertheless, these are equal on the open set $\widehat{U}\cap\mathscr{C}_\eta$.

Let $\varrho\colon\widehat{\mathscr{C}}\to C$ be the natural map to the coarse underlying curve C of $\widehat{\mathscr{C}}$. Forgetting the orbifold structure of $\widehat{\mathscr{C}}$ at all the sections $b_1,\dots,b_m,c_1,\dots,c_n$ gives a unique (balanced, prestable) orbicurve $\mathscr{C}$ over Δ with coarse underlying curve C, with orbifold structure matching $\mathscr{C}_\eta$ on the generic fiber and with orbifold structure at the nodes of the central fiber matching $\widehat{\mathscr{C}}$. From now on we will think of $\widehat{U}$ as an open subset of $\mathscr{C}$ rather than of $\widehat{\mathscr{C}}$.

If the generic fiber $\mathscr{C}_\eta$ with its sections $y_1,\dots,y_k,b_1,\dots,b_m$ is stable, then let $f\colon C\to\overline{C}$ be the obvious contraction to the (coarse) stable model $\overline{C}$ of C. Otherwise, let $f\colon C\to\overline{C}$ be the curve obtained by repeatedly contracting all rational curves that do not contain any marked points y_i, basepoints b_i, or additional sections c_i.

We now use the pullback of the trivialization τ along $f\circ\varrho$ to construct a trivialization of $\mathring{\omega}_{\log,\mathscr{C}}$ over $\widehat{U}$.

First note that the log-canonical bundles $\omega_{\log}$ satisfy the following two properties:

(I) For any prestable curve $f\colon C\to\overline{C}$ with sections $y_1,\dots,y_k$ lying over $\overline{y}_1,\dots,\overline{y}_k$, with generic fiber C_η and *no rational tails* (with respect to the marks $y_1,\dots,y_k$) in the central fiber, we have

$$\omega_{\log,C}=f^*(\omega_{\log,\overline{C}}).$$

(II) For any pointed prestable orbicurve $\mathscr{C},y_1,\dots,y_k$ with the map to its coarse underlying curve $C,y_1,\dots,y_k$ denoted by $\varrho\colon\mathscr{C}\to C$, we have

$$\varrho^*(\omega_{\log,C})=\omega_{\log,\mathscr{C}}.$$

Except on uncontracted -1–trees of the central fiber, we have $\varrho^*f^*(\mathring{\omega}_{\log,\overline{C}})=\mathring{\omega}_{\log,\mathscr{C}}$, so in this case pulling back the trivialization τ along $f\circ\varrho$ immediately gives a trivialization of $\mathring{\omega}_{\log,\mathscr{C}}$.

For each -1–tree Ψ of the central fiber, there is a neighborhood N of Ψ in $\mathscr{C}$ which is the result of a sequence of successive blowups of points of Ψ. For each blowup, let x be a local coordinate of the curve before blowing up (so the curve is locally of the form $\operatorname{Spec} R[\![x]\!]$). Removing the section $x=0$, we can trivialize the log-canonical bundle (before blowing up) by $dx/x \mapsto 1 \in \mathscr{O}$. Removing the strict transform of $x=0$ from the blowup, we have that $\omega_{\log,\mathscr{C}}$ is again trivialized by $dx/x \mapsto 1$. Repeat this process for each blowup and for each -1–tree, and denote the sections removed (the strict transform of each $x=0$) by $\{d_1,\dots,d_t\}$. We abuse notation and redefine $\widehat{U}$ to be

$$\widehat{U} = \widehat{\mathscr{C}} \smallsetminus \{b_1,\dots,b_m,c_1,\dots,c_n,z_1,\dots,z_s,d_1,\dots,d_t\}.$$

Combining the local trivialization with the pullback of τ, we have now constructed a trivialization of $\mathring{\omega}_{\log,\mathscr{C}}$ on all of $\widehat{U}$.

Let $\widehat{U} \leftarrow \mathscr{Q} \rightarrow Z^{\mathrm{ss}}_G(\theta)$ be the principal G–bundle on $\widehat{U}$ corresponding to the morphism $\alpha\colon \widehat{U} \rightarrow \mathscr{X}_\theta$. The space $\widetilde{\mathscr{P}} = \mathscr{Q} \times_G \Gamma$ is a principal Γ–bundle over $\widehat{U}$ with a natural morphism $i\colon \mathscr{Q} \rightarrow \widetilde{\mathscr{P}}$ (given by sending a point with local coordinate $(u,g) \in \widehat{U} \times G$ to the point with local coordinate $((u,g),1) \in (\widehat{U} \times G) \times_G \Gamma$), and we may construct a corresponding Γ–equivariant morphism $\widetilde{\sigma}\colon \widetilde{\mathscr{P}} \rightarrow Z^{\mathrm{ss}}_G(\theta)$ (by sending points of the form $i(q) \in \widetilde{\mathscr{P}}$ to $\alpha(q)$ and extending Γ–equivariantly to the rest of $\widetilde{\mathscr{P}}$).

Denote by $b_{i,0}$, $c_{i,0}$ and $d_{i,0}$ be the intersection of the sections b_i, c_i and d_i, respectively, with the central fiber $\mathscr{C}_0$ of $\mathscr{C}$. The bundles and morphisms $\mathscr{P}_\eta$ and σ_η and $\widetilde{\mathscr{P}}$ and $\widetilde{\sigma}$ agree on the intersection $\mathscr{C}_\eta \cap \widehat{U}$, so they glue together to give a principal Γ–bundle $\mathscr{P}$ and a morphism $\sigma\colon \mathscr{P} \rightarrow Z$ defined on the open set $\mathscr{C}_\eta \cup \widetilde{U} = \mathscr{C} \smallsetminus \{b_{1,0},\dots,b_{m,0},c_{1,0},\dots,c_{n,0},z_1,\dots,z_s,d_{1,0},\dots,d_{t,0}\}$.

By [16, Lemma 4.3.2], the principal Γ–bundle $\mathscr{P}$ extends from $\mathscr{C}_\eta \cup \widehat{U}$ to all of $\mathscr{C}$. We also denote this extension by $\mathscr{P}$. By Hartogs' theorem, the morphism $\sigma\colon \mathscr{P} \rightarrow Z$ extends uniquely over all of $\mathscr{C}$.

Over $\widetilde{U}$ we have $\zeta_*\colon \mathscr{P}' \rightarrow \mathbb{C}^* \times \widetilde{U}$, and we may combine this with the trivialization

$$\varrho^* f^* \tau\colon \mathbb{C}^* \times \widetilde{U} \rightarrow \mathring{\omega}_{\log,\mathscr{C}}|_{\widetilde{U}}.$$

By construction this composition agrees with the LG quasimap structure $\mathscr{P}_\eta \rightarrow \mathring{\omega}_{\log,\mathscr{C}}$ on $\mathscr{C}_\eta \cap \widetilde{U}$, so these glue together to give a morphism $\mathscr{P} \rightarrow \mathring{\omega}_{\log,\mathscr{C}}$ on all of $\mathscr{C}_\eta \cup \widehat{U}$. Again, by Hartogs' theorem this morphism extends uniquely to a morphism over all of $\mathscr{C}$. Thus we have constructed a family $\mathcal{Q}$ of prestable LG quasimaps

$$\Delta \leftarrow \mathscr{C} \leftarrow \mathring{\omega}_{\log,\mathscr{C}} \leftarrow \mathscr{P} \xrightarrow{\sigma} Z$$

whose generic fiber is the stable LG quasimap (17). It remains to show that the central fiber of $\mathcal{Q}$ is ε–stable.

The rest of the proof is very similar to the corresponding part of the proof of [16, Theorem 7.1.6], but we include a sketch here for completeness. Let $\Psi_1,\dots,\Psi_s$ be the

-1–trees in $\widetilde{\mathscr{C}}_0$ that were contracted (and the resulting points in the special fiber of $\mathscr{C}_0$ are $z_1, \dots, z_s$). The analog of (18) for the special fiber gives

$$\deg_{\mathscr{C}_0} \sigma^* \mathscr{L}_\vartheta = \deg_{\widetilde{\mathscr{C}}_0} (p \circ \alpha)^* \mathscr{L}_\vartheta + \sum_{i=1}^{s} \ell(z_i) + \sum_{b \in \mathscr{C}_0, b \notin \{z_1, \dots, z_s\}} \ell(b).$$

The degree of $\sigma^* \mathscr{L}_\vartheta$ is constant in the fibers, and combining this with semicontinuity of ℓ and using the previous equation we obtain, for each basepoint b_i,

$$\ell(b_{i,0}) = \ell(b_i) \le \frac{1}{\varepsilon}$$

and similarly, for each contracted -1–tree Ψ_j we have

$$\ell(z_j) = \deg((p \circ \alpha)^* \mathscr{L}_\vartheta |_{\Psi_j}) + \sum_{b \in \Psi_j} \ell(b) = \deg(\Psi_j, \mathscr{L}_\vartheta) \le \frac{1}{\varepsilon}.$$

Finally we verify that the ampleness criterion holds. First, any uncontracted -1–curve D must have $\deg_D(\sigma^* \mathscr{L}_\vartheta) > \frac{1}{\varepsilon}$ by construction, hence $\omega_{\log,\mathscr{C}} \otimes \sigma^* \mathscr{L}_\vartheta^\varepsilon$ is ample on D.

Second, for any component D with $\deg_D \omega_{\log,\mathscr{C}} = 0$, if D contains a ϑ–basepoint, we must have $\deg_D \sigma^* \mathscr{L}_\vartheta > 0$ by Proposition 5.1.1. If there are no ϑ–basepoints of σ on D, then since ϑ is a lift of θ there must be no θ basepoints on D, and D lies entirely in $\widehat{U}$, and so on D the bundle $\sigma^*(\mathscr{L}_\vartheta)$ is equal to $\alpha^*(L_\theta)$ (see Remark 4.2.7). In that case, $\alpha\colon \widetilde{\mathscr{C}} \to \mathscr{X}_\theta$ is a stable map that does not contract the component D, so $\sigma^*(\mathscr{L}_\vartheta) = \alpha^*(L_\theta)$ is ample on the component D. $\square$

6 The virtual cycle

In this section, we construct the virtual cycle for the case where all insertions are of compact type (see Definition 4.1.4). To do this, we use the cosection localization techniques of Kiem and Li [32] as applied in [6; 7]. To use the cosection technique we need a relative perfect obstruction theory for $\mathscr{Q} = \mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}([\mathscr{X}_\theta], \beta)$ over $\mathfrak{M}$ and a cosection

$$\mathrm{Obs}_{\mathscr{Q}} \to \mathscr{O}$$

whose degeneracy locus is $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta) \subset \mathscr{Q}$.

6.1 Cosection and virtual cycle

As shown in Theorem 5.2.3, if $\mathscr{P} \to \mathscr{C}$ is the universal principal Γ–bundle on the universal curve $\pi\colon \mathscr{C} \to \mathscr{Q}$ and $\mathscr{E} = \mathscr{P} \times_\Gamma V$, then the map $\mathscr{Q} = \mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}([V /\!\!/_\theta G], \beta) \to \mathfrak{A}$

is representable and has a relative perfect obstruction theory

$$\phi_{\mathscr{Q}/\mathfrak{A}}\colon \mathbb{L}_{\mathscr{Q}/\mathfrak{A}} \to \mathbb{E}_{\mathscr{Q}/\mathfrak{A}} = R^{\bullet}\pi_*\mathscr{E},$$

where $\mathfrak{A} = \mathfrak{A}_{g,k}$ is the smooth Artin stack of principal Γ–bundles P on twisted, k–pointed, genus-g prestable curves $\mathscr{C}$ with an isomorphism $\zeta_* P \to \mathring{\omega}_{\log,\mathscr{C}}$ to the (punctured) log-canonical bundle such that the corresponding morphism $\mathscr{C} \to \mathrm{B}\Gamma$ is representable.

We wish to define a cosection, that is, a homomorphism $\mathrm{Obs}_{\mathscr{Q}} \to \mathscr{O}_{\mathscr{Q}}$ from the obstructions of $\mathscr{Q}$ over the stack of prestable curves. To do this, we will proceed in several steps. First we define a relative cosection $\delta\colon \mathrm{Obs}_{\mathscr{Q}/\mathfrak{A}} = R^1\pi_*\mathscr{E} \to \mathscr{O}_{\mathscr{Q}}$ from the relative obstruction space over $\mathfrak{A}$. We then show this also induces a relative cosection $\mathrm{Obs}_{\mathscr{Q}/\mathfrak{M}_{g,k}} \to \mathscr{O}_{\mathscr{Q}}$. Finally we show that this induces a cosection $\mathrm{Obs}_{\mathscr{Q}} \to \mathscr{O}_{\mathscr{Q}}$.

To begin, note that the superpotential $W\colon V \to \mathbb{C}$ is equivariant with respect to the homomorphism $\zeta\colon \Gamma \to \mathbb{C}^*$, so for any LG quasimap $(\mathscr{C}, y_1, \dots, y, \mathscr{P}, \sigma, \varkappa)$ the map W defines a morphism of vector bundles $W\colon \mathscr{E} = \mathscr{P} \times_\Gamma V \to \mathring{\omega}_{\log,\mathscr{C}} \times_{\mathbb{C}^*} \mathbb{C} = \omega_{\log,\mathscr{C}}$. Differentiating along the section σ gives another morphism of vector bundles

$$dW_\sigma\colon T\mathscr{E}|_\sigma \to T\omega_{\log,\mathscr{C}}|_\sigma.$$

But we have canonical isomorphisms $T\mathscr{E}|_\sigma \cong \mathscr{E}$ and $T\omega_{\log,\mathscr{C}}|_\sigma \cong \omega_{\log,\mathscr{C}}$, so this gives a map

$$dW_\sigma\colon \mathscr{E} \to \omega_{\log,\mathscr{C}}. \tag{20}$$

Lemma 6.1.1 *For any LG quasimap $Q = (\mathscr{C}, y_1, \dots, y_k, \mathscr{P}, \sigma, \varkappa)$ into $[V/\!\!/_\theta G]$, if $\mathrm{ev}_i(Q)$ lies in a compact substack of $\mathscr{X}_{\theta,g}$, then the map $dW_\sigma\colon \mathscr{E} \to \omega_{\log,\mathscr{C}}$ factors through the obvious inclusion $\omega_{\log,\mathscr{C}}(-y_i) \subset \omega_{\log,\mathscr{C}}$.*

In particular, if all the marked points are narrow, then dW_σ factors through the canonical inclusion $\iota\colon \omega_{\mathscr{C}} \to \omega_{\log,\mathscr{C}}$.

Proof To prove that dW_σ factors through $\omega_{\log,\mathscr{C}}(-y_i) \subset \omega_{\log,\mathscr{C}}$ is a local problem, so it suffices to show that the map $dW|_{\sigma(y_i)}\colon V \to \mathbb{C}$ vanishes.

Assume that $\mathrm{ev}_i(Q)$ lies in a compact substack of $\mathscr{X}_{\theta,g} = [V^{\mathrm{ss},g}/\!\!/_\theta Z_G(g)]$, where $V^{\mathrm{ss},g}$ is the fixed point locus of g in V^{ss}. In particular, $\sigma(y_i) \in V^{\mathrm{ss},g}$.

As observed in Section 4.1, g must have finite order, and since we are in characteristic zero, g must be semisimple. Choose coordinates $x_1, \dots, x_N$ on V to diagonalize g.

We may assume that for some k the coordinates $x_1, \dots, x_k$ are fixed by g, while $x_{k+1}, \dots, x_N$ are not fixed by g. Therefore

$$V^{\mathrm{ss},g} = \{x_{k+1} = x_{k+2} = \cdots = x_N = 0\} \cap V^{\mathrm{ss}}.$$

If $W|_{V^{\mathrm{ss},g}}$ is not zero, then it defines a polynomial on $V^{\mathrm{ss},g}$. But the G–invariance of W implies that W defines a function on $\mathscr{X}_{\theta,[\![g]\!]} = [V^{\mathrm{ss},g}/Z_G(g)]$. And since $\mathscr{X}_{\theta,[\![g]\!]}$ is compact, this function must be constant, hence $W|_{V^{\mathrm{ss},g}}$ is constant.

Since $W|_{V^{\mathrm{ss},g}}$ is constant, every monomial of W is either constant or contains at least one x_j for some $j > k$. Since W is G–invariant, every monomial that contains such an x_j must also contain another x_l for $l > k$ (otherwise the monomial is not fixed by γ). Therefore every monomial in each partial derivative $\partial W/\partial x_i$ must also contain at least one x_j for $j > k$, and hence each $\partial W/\partial x_i$ and also dW must vanish on $V^{\mathrm{ss},g}$. $\square$

We can now define the homomorphism $\delta\colon R^1\pi_*\mathscr{E} = \mathrm{Obs}_{\mathscr{Q}/\mathfrak{A}} \to \mathscr{O}_{\mathscr{Q}}$ in any situation where dW_σ factors through the canonical inclusion $\iota\colon \omega_{\mathscr{C}} \to \omega_{\log,\mathscr{C}}$.

Definition 6.1.2 If dW_σ factors through the canonical inclusion $\iota\colon \omega_{\mathscr{C}} \to \omega_{\log,\mathscr{C}}$, let $\delta\colon \mathscr{E} \to \omega_{\mathscr{C}}$ be the homomorphism corresponding to that factorization:

$$\begin{tikzcd} \mathscr{E} \arrow[r, "dW_\sigma"] \arrow[dr, "\delta"'] & \omega_{\log,\mathscr{C}} \\ & \omega_{\mathscr{C}} \arrow[u, hook, "\iota"] \end{tikzcd}$$

By Serre duality, we have $\delta \in \mathrm{Hom}(\mathscr{E}, \omega_{\mathscr{C}}) \cong H^0(\mathscr{C}, \mathscr{E}^\vee \otimes \omega_{\mathscr{C}}) \cong H^1(\mathscr{C}, \mathscr{E})^\vee$, hence δ defines a homomorphism $H^1(\mathscr{C}, \mathscr{E}) \to \mathscr{O}_{\mathscr{C}}$, and on the stack $\mathscr{Q}$ we have $\delta\colon R^1\pi_*\mathscr{E} \to \mathscr{O}_{\mathscr{Q}}$, as desired.

Proposition 6.1.3 *If* $\mathrm{ev}_i(Q)$ *lies in either a narrow sector* $\mathscr{X}_{\theta,g}$ *or a compact substack of* $\mathscr{X}_{\theta,g}$ *in the broad case, then the degeneracy locus of* δ (the locus on $\mathscr{Q}$ where δ vanishes) *is precisely the closed substack* $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta) \subset \mathscr{Q}$.

Proof The hypothesis guarantees that dW_σ factors through the canonical inclusion $\iota\colon \omega_{\mathscr{C}} \to \omega_{\log,\mathscr{C}}$, and hence that δ is defined. The stack $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta)$ embeds in $\mathscr{Q}$ as the locus where the image of σ lies in $\mathrm{Crit}(W)$, and this is, by definition, the locus where dW_σ vanishes. Since $dW_\sigma = \iota \circ \delta$ and ι is injective, this is precisely the locus where δ vanishes. $\square$

Next we show that δ induces a relative cosection $\mathrm{Obs}_{\mathscr{Q}/\mathfrak{M}_{g,k}} \to \mathscr{O}_{\mathscr{Q}}$ by generalizing the arguments of [17, Section 3.3]. To reduce clutter in our notation, we write $\mathfrak{M} = \mathfrak{M}_{g,k}$ and continue to use $\mathfrak{A}$ to denote $\mathfrak{A}_{g,k}$ and $\mathscr{Q}$ to denote $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}([V/\!\!/_\theta G], \beta)$.

Lemma 6.1.4 *The homomorphism* $\delta\colon \mathrm{Obs}_{\mathscr{Q}/\mathfrak{A}} \to \mathscr{O}_{\mathscr{Q}}$ *induces a homomorphism* $\mathrm{Obs}_{\mathscr{Q}/\mathfrak{M}} \to \mathscr{O}_{\mathscr{Q}}$ (which we also denote by δ).

Proof If $p\colon \mathscr{Q} \to \mathfrak{A}$ is the obvious forgetful morphism, then we have the deformation exact sequence

$$T_{\mathfrak{A}/\mathfrak{M}} \xrightarrow{\tau} \mathrm{Obs}_{\mathscr{Q}/\mathfrak{A}} \to \mathrm{Obs}_{\mathscr{Q}/\mathfrak{M}} \to 0.$$

So to verify that the cosection δ induces a cosection $\mathrm{Obs}_{\mathscr{Q}/\mathfrak{M}} \to \mathscr{O}_{\mathscr{Q}}$, we must verify that $\delta \circ \tau = 0$.

As we saw in the proof of Lemma 5.2.2, at any point $\mathcal{A} = (\mathscr{C}, y_1, \dots, y_k, \mathscr{P} \to \mathring{\omega}_{\log,\mathscr{C}})$ of $\mathfrak{A}$ the deformation space $T_{\mathfrak{A}/\mathfrak{M}}$ at $\mathcal{A}$ is $H^1(\mathscr{C}, \mathscr{P} \times_\Gamma \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G with the adjoint action of Γ. Let $e\colon \mathscr{P} \times_\Gamma \mathfrak{g} \to \mathscr{P} \times_\Gamma V = \mathscr{E} \cong T\mathscr{E}|_\sigma$ be given by sending $(z, \alpha) \in \mathscr{P} \times_\Gamma \mathfrak{g}$ to $(z, \alpha\sigma(z))$. Since W is G–invariant, we have that $dW|_\sigma \circ e = 0$, hence $\delta \circ e = 0$. Fiberwise, over any point $(\mathscr{C}, y_1, \dots, y_k, \mathscr{P} \to \mathring{\omega}_{\log,\mathscr{C}}, \sigma\colon \mathscr{P} \to V) \in \mathscr{Q}$, the map τ is just $h^1(e)$, and hence $\delta \circ \tau = 0$. Thus $\delta\colon \mathrm{Obs}_{\mathscr{Q}/\mathfrak{A}} \to \mathscr{O}_{\mathscr{Q}}$ induces a homomorphism from the cokernel $\mathrm{Obs}_{\mathscr{Q}/\mathfrak{M}}$ of τ to $\mathscr{O}_{\mathscr{Q}}$. □

Finally, to apply the general theory of Kiem and Li [32], we must show that the relative cosection $\delta\colon \mathrm{Obs}_{\mathscr{Q}/\mathfrak{A}} \to \mathscr{O}_{\mathscr{Q}}$ induces an absolute cosection $\mathrm{Obs}_{\mathscr{Q}} \to \mathscr{O}_{\mathscr{Q}}$, where $\mathrm{Obs}_{\mathscr{Q}}$ is the absolute obstruction bundle, defined as the cokernel of a homomorphism η as described below.

We have a distinguished triangle

$$p^*\mathbb{L}_{\mathfrak{A}} \to \mathbb{L}_{\mathscr{Q}} \to \mathbb{L}_{\mathscr{Q}/\mathfrak{A}} \xrightarrow{\partial} p^*\mathbb{L}_{\mathfrak{A}/\mathfrak{M}}[1].$$

Composing the dual $\partial^\vee$ of the connecting homomorphism and the map $\phi_{\mathscr{Q}/\mathfrak{A}}$ gives

$$\phi_{\mathscr{Q}/\mathfrak{A}} \circ \partial^\vee\colon\ p^*\mathbb{T}_{\mathfrak{A}} \to \mathbb{E}_{\mathscr{Q}/\mathfrak{A}}[1]$$

and hence a map

$$\eta = h^0(\phi_{\mathscr{Q}/\mathfrak{A}} \circ \partial^\vee)\colon\ H^0(p^*\mathbb{T}_{\mathfrak{A}}) \to \mathrm{Obs}_{\mathscr{Q}/\mathfrak{A}}.$$

We define $\mathrm{Obs}_{\mathscr{Q}}$ to be the cokernel of η.

To extend $\delta\colon \mathrm{Obs}_{\mathscr{Q}/\mathfrak{A}} \to \mathscr{O}_{\mathscr{Q}}$ to $\mathrm{Obs}_{\mathscr{Q}/\mathfrak{M}_{g,k}}$, we must verify that $\delta \circ \eta = 0$. Because η factors through $H^1(\mathbb{T}_{\mathscr{Q}/\mathfrak{A}}) \to \mathrm{Obs}_{\mathscr{Q}/\mathfrak{A}}$, the vanishing of $\delta \circ \eta$ follows from the following lemma.

Lemma 6.1.5 *If* $(\mathscr{C}_{\mathscr{Q}}, \mathscr{P} \to \mathring{\omega}_{\log,\mathscr{C}}, \sigma)$ *denotes the universal LG quasimap structure on* $\mathscr{Q}$ *and* $\mathscr{E}$ *is the sheaf of sections of the vector bundle* $\mathscr{P} \times_\Gamma V$*, then the composition*

$$H^1(\mathbb{T}_{\mathscr{Q}/\mathfrak{A}}) \to \mathrm{Obs}_{\mathscr{Q}/\mathfrak{A}} = R^1\pi_*\mathscr{E} \xrightarrow{\delta} R^1\pi_*\omega_{\mathscr{C}_{\mathscr{Q}}} = \mathscr{O}_{\mathscr{Q}} \tag{21}$$

is zero.

Proof The proof is very similar to the proofs of [6, Lemma 3.6] and [17, Lemma 3.4.4]. We sketch the proof here and refer the reader to [6; 17] for more details. Let $\omega_{\mathscr{C}_{\mathfrak{A}}/\mathfrak{A}}$ be the relative dualizing sheaf of the universal curve $\mathscr{C}_{\mathfrak{A}}$ over $\mathfrak{A}$, and let $\mathrm{Vb}(\omega_{\mathscr{C}_{\mathfrak{A}}/\mathfrak{A}})$ denote its corresponding vector bundle. Let $Q = C(\pi_*\omega_{\mathscr{C}_{\mathfrak{A}}/\mathfrak{A}})$ be the direct image cone of $\omega_{\mathscr{C}_{\mathfrak{A}}/\mathfrak{A}}$ parametrizing global sections of $\omega_{\mathscr{C}_{\mathfrak{A}}/\mathfrak{A}}$ on curves in $\mathfrak{A}$ (see [6, Definition 2.1]), and let $\mathscr{C}_Q$ be the universal curve over Q. Composing the function W with the section $\sigma\colon \mathscr{C}_{\mathscr{Q}} \to \mathscr{E}$ defines a section $\varepsilon = W \circ \sigma \in H^0(\mathscr{C}_{\mathscr{Q}}, \omega_{\mathscr{C}_{\mathscr{Q}}})$, and hence a morphism $\mathscr{Q} \to Q$. Denote by $\widetilde{\Phi}_\varepsilon\colon \mathscr{C}_{\mathscr{Q}} \to \mathscr{C}_Q$ the morphism of curves induced by (lifting) Φ_ε. This gives a commutative diagram

$$\begin{tikzcd}
\mathscr{C}_{\mathscr{Q}} \arrow[r, "\sigma"] \arrow[d, "\Phi_\varepsilon"'] & \mathrm{Vb}(\mathscr{E}) \arrow[d, "W"'] \\
\mathscr{C}_Q \arrow[r, "\mathfrak{e}'"] & \mathrm{Vb}(\omega_{\mathscr{C}_{\mathfrak{A}}/\mathfrak{A}})
\end{tikzcd}$$

where $\mathfrak{e}'$ is the tautological morphism. From this we see that the following diagram is commutative:

$$\begin{tikzcd}
\pi^*\mathbb{T}_{\mathscr{Q}/\mathfrak{A}} \arrow[r, equal] \arrow[d] & \mathbb{T}_{\mathscr{C}/\mathscr{C}_{\mathfrak{A}}} \arrow[r] \arrow[d] & \sigma^*\Omega^\vee_{\mathrm{Vb}(\mathscr{E})/\mathscr{C}_{\mathfrak{A}}} \arrow[d, "dW"'] \\
\pi^*\widetilde{\Phi}_\varepsilon^*\mathbb{T}_{Q/\mathfrak{A}} \arrow[r, equal] & \widetilde{\Phi}_\varepsilon^*\mathbb{T}_{\mathscr{C}_Q/\mathscr{C}_{\mathfrak{A}}} \arrow[r] & \widetilde{\Phi}_\varepsilon^*\mathfrak{e}'^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathscr{C}_{\mathfrak{A}}/\mathfrak{A}})/\mathscr{C}_{\mathfrak{A}}}
\end{tikzcd}$$

Applying $R^1\pi_*$ to the bottom right-hand arrow gives a homomorphism

$$H^1(\Phi_\varepsilon^*\mathbb{T}_{Q/\mathfrak{A}}) \to \Phi_\varepsilon^* R^1\pi_{Q*}\omega_{\mathscr{C}_Q/Q},$$

which vanishes because Q is a vector bundle over $\mathfrak{A}$ and $\mathscr{C}_Q$ is smooth over $\mathscr{C}_{\mathfrak{A}}$. As described in [6, Equation (3.13)], this implies that the composition

$$H^1(\mathbb{T}_{\mathscr{Q}/\mathfrak{A}}) \to R^1\pi_*\sigma^*\Omega^\vee_{\mathrm{Vb}(\mathscr{E})/\mathscr{C}_{\mathfrak{A}}} \to R^1\pi_*\sigma^*W^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathscr{C}_{\mathfrak{A}}/\mathfrak{A}})/\mathscr{C}_{\mathfrak{A}}}$$

is equal to the composition

$$H^1(\mathbb{T}_{\mathscr{Q}/\mathfrak{A}}) \to H^1(\Phi_\varepsilon^*\mathbb{T}_{Q/\mathfrak{A}}) \xrightarrow{0} \Phi_\varepsilon^* R^1\pi_{Q*}\omega_{\mathscr{C}_Q/Q},$$

and hence it vanishes. Using $\sigma^*W^*\Omega^\vee_{\mathrm{Vb}(\omega_{\mathscr{C}_{\mathfrak{A}}/\mathfrak{A}})/\mathscr{C}_{\mathfrak{A}}} = \omega_{\mathscr{C}_{\mathscr{Q}}}$ we see that the composition (21) vanishes. □

Now we can apply the general cosection localization theory of Kiem and Li [32] to construct our virtual cycle.

Definition 6.1.6 Suppose that all the marked points have a narrow insertion or $\mathrm{ev}_i(\mathscr{Q})$ lies in a compact substack of $\mathscr{X}_{\theta,g}$ in the broad case, and that the cosection δ is defined as in Definition 6.1.2. The virtual cycle of the stack $\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{C}\mathscr{R}_\theta,\beta)$ is defined as

$$[\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{C}\mathscr{R}_\theta,\beta)]^{\mathrm{vir}}=[\mathscr{Q}]^{\mathrm{vir}}_{\mathrm{loc}},$$

taken with respect to the cosection δ.

Lemma 6.1.7 $[\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{C}\mathscr{R}_\theta,\beta)]^{\mathrm{vir}}$ *has virtual dimension*

$$\dim_{\mathrm{vir}}=\int_\beta c_1(V/\!\!/_\theta G)+(\hat{c}_{W,G}-3)(1-g)+k-\sum_i \mathrm{age}(\gamma_i-q).$$

Proof Cosection localization preserves the virtual dimension. Therefore, the virtual dimension is the sum of the dimension of stack of Γ–bundle and the index of vector bundle $\mathscr{P}\times_\Gamma V$. Namely,

$$\begin{aligned}\dim_{\mathrm{vir}}&=3g-3+k+\dim(G)g+c_1(\mathscr{P}\times_\Gamma V)-\sum_i \mathrm{age}(\gamma_i)+n(1-g)-\dim G\\&=(n-\dim(G)-3)(1-g)+k+c_1(\mathscr{P}\times_\Gamma \det(V))-\sum_i \mathrm{age}(\gamma_i).\end{aligned}$$

Note that $\mathscr{P}\times_\Gamma \det(V)$ is defined by a Γ–character. We choose its zero lift $\widehat{\det(V)}$ and define

$$\int_\beta c_1(V/\!\!/_\theta G)=c_1(\widehat{\det(V)}).$$

By taking a higher multiple if necessary, we can assume $\mathscr{P}\times_\Gamma \det(V)$ is a $G\times\mathbb{C}^*_R$–bundle, ie the tensor product of $\widehat{\det(V)}$ and a $\mathbb{C}^*_R$–bundle $\det(V)_R$. The $\mathbb{C}^*_R$–bundle has the property

$$\det(V)^d_R=\omega_{\log}^{\sum_i c_i}.$$

Hence,

$$c_1(\det(V)_R)=q(2(g-1)-k).$$

We can put everything together to obtain

$$\dim_{\mathrm{vir}}=\int_\beta c_1(V/\!\!/_\theta G)+(\hat{c}_{W,G}-3)(1-g)+k-\sum_i \mathrm{age}(\gamma_i-q). \qquad \square$$

6.2 Correlators

Definition 6.2.1 Suppose that $\alpha_i\in\mathscr{H}_{W,G,\mathrm{comp}}$. We define correlator

$$\langle\tau_{l_1}(\alpha_1),\dots,\tau_{l_k}(\alpha_k)\rangle=\int_{[\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{C}\mathscr{R}_\theta,\beta)]^{\mathrm{vir}}}\prod_i \mathrm{ev}_i^*(\alpha_i)\psi_i^{l_i}.$$

One can define the generating function in a standard fashion.

These invariants satisfy the gluing axioms for nodes that are narrow. The argument is standard and we leave it to the reader (see for example the proof of [7, Theorem 4.8]). For insertions that are not narrow, but where the evaluation map factors through a compact substack of $\mathscr{X}_\theta$, a form of the gluing axioms should also hold. We will treat this in a future paper. We do not expect a forgetful morphism or string/dilaton equations to hold, except in the chamber where $\varepsilon = \infty$.

7 Examples

In this section we consider several more examples of the GLSM, including some important examples studied by Witten [53]. We begin with some general considerations about toric quotients.

7.1 Toric quotients

The hypersurface in Examples 3.2.15 and 4.3.1 is a special case of a toric quotient, that is, where the group $G = (\mathbb{C}^*)^m$ is an algebraic torus. The geometric and combinatorial properties of the polarization are encoded in the weights of the $(\mathbb{C}^*)^m$–action. Let $B = (b_{ij})$ be the gauge weight matrix, as described in Section 3.2.3. Note that some b_{ij} could be negative, and hence the resulting quotient could be fail to be compact, but we always assume that B is of maximal rank (ie rank m).

An important case is that of *Calabi–Yau weights*, where $\sum_j b_{ij} = 0$ for all i. In this case, the quotient $[V /\!\!/_\theta G]$ is Calabi–Yau and cannot be compact. In fact, $[V /\!\!/_\theta G]$ or $\mathscr{X}_\tau^{\mathrm{sympl}}$ is compact if and only if $B^{-1}(0) \cap \mathbb{R}^n_{\geq 0} = \{0\}$, meaning that if $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ are the column vectors of B, then the only nonnegative solution $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (\mathbb{R}^{\geq 0})^n$ to the linear equation

$$\alpha_1 \boldsymbol{b}_1 + \dots + \alpha_n \boldsymbol{b}_n = 0$$

is the zero solution (see [43, Section 2]). Note that this condition is entirely independent of the phase (θ or τ).

If the above condition fails, the quotient is not compact. However, one can choose a maximal collection of column vectors of B with the property above. After possibly reindexing, we may write $\mathbb{C}^n = \mathbb{C}^K \times \mathbb{C}^M$ with variables $x_1, \dots, x_K, p_1, \dots, p_M$ such that $\mathbb{C}^K$ corresponds to the maximal collection of column vectors. In this case the subset $[(\mathbb{C}^K \times \{\mathbf{0}\}) /\!\!/_\theta (\mathbb{C}^*)^m] \subset [\mathbb{C}^n /\!\!/_\theta (\mathbb{C}^*)^m]$ is compact and depends on a choice of phase (τ or θ). This compact piece may be empty, but if it is not empty, we call it a *maximal compact piece*. In general, there may be several maximal compact pieces.

A particularly interesting case is when $[\mathbb{C}^n /\!\!/_\theta (\mathbb{C}^*)^m] = [(\mathbb{C}^K \times \mathbb{C}^M) /\!\!/_\theta (\mathbb{C}^*)^m]$ is a toric vector bundle over the maximal compact piece $\mathscr{X}_{\text{base}} = [\mathbb{C}^K /\!\!/_\theta (\mathbb{C}^*)^m]$. Each remaining variable p_j defines a line bundle $\mathscr{F}_j \to \mathscr{X}_{\text{base}}$. Each corresponding column vector $\boldsymbol{b}_{p_j}$ of B can be written as

$$\boldsymbol{b}_{p_j} = \alpha_{1,j}\boldsymbol{b}_{x_1} + \cdots + \alpha_{K,j}\boldsymbol{b}_{x_n}$$

for some choice of $\alpha_{i,j} \le 0$. Letting D_i be the toric divisor corresponding to $\boldsymbol{b}_{x_i}$, we have

$$c_1(\mathscr{F}_j) = \sum_{i=1}^{K} \alpha_{ij} D_i, \quad \text{or} \quad \mathscr{F}_j = \bigotimes_{i=1}^{K} \mathscr{O}(\alpha_{ij} D_i).$$

A very important subclass of the toric examples consists of the so-called *hybrid models*.

Definition 7.1.1 For a torus $G = (\mathbb{C}^*)^m$, a phase θ of (W, G) is called a *hybrid model* if

(I) the quotient $\mathscr{X}_\theta \to \mathscr{X}_{\text{base}}$ has the structure of a toric bundle over a compact base $\mathscr{X}_{\text{base}}$, and

(II) the $\mathbb{C}_R^*$–weights of the base variables are all zero.

Both the geometric and the LG phases of the hypersurface in Example 3.2.15 were hybrid models. Several examples of hybrid models have been worked out in detail by E Clader [18].

7.2 Complete intersections

Suppose that $G = \mathbb{C}^*$ and that we have several quasihomogeneous polynomials $F_1, F_2, \dots, F_M \in \mathbb{C}[x_1, \dots, x_K]$ of G–degree $(d_1, \dots, d_M)$, where each variable x_i has G–weight $b_i > 0$. We assume that the F_j intersect transversely in $W\mathbb{P}(b_1, \dots, b_K)$ and define a complete intersection. Let

$$W = \sum_i p_i F_i \colon \mathbb{C}^{K+M} \to \mathbb{C},$$

where we assign G–weight $-d_i$ to p_i. In the special case that $\sum_i b_i = \sum_j d_j$, the complete intersection defined by $F_1 = \cdots = F_M = 0$ is a Calabi–Yau orbifold in $W\mathbb{P}(b_1, \dots, b_K)$. One can view this as a *toric LG model for the complete intersection*. We do not assume the Calabi–Yau condition here.

The critical set of W is defined by the equations

$$\partial_{p_j} W = F_j = 0, \quad \partial_{x_i} W = \sum_j p_j \partial_{x_i} F_j = 0. \tag{22}$$

The moment map is

$$\mu = \sum_i \tfrac{1}{2} b_i |x_i|^2 - \tfrac{1}{2} \sum_j d_j |p_j|^2.$$

7.2.1 Phases As with the hypersurface, there are two phases, $\tau > 0$ and $\tau < 0$. When $\tau > 0$, any choice of $p_1, \dots p_M$ determines a nontrivial ellipsoid $E \subset \mathbb{C}^K$ of points $(x_1, \dots, x_K)$ such that $(x_1, \dots, x_K, p_1, \dots, p_M)$ lies in $\mu^{-1}(\tau)$. Quotienting by $U(1)$, the first projection $\mathrm{pr}_1 \colon E \times \mathbb{C}^M \to E$ induces a map $\mathscr{X}_\tau^{\text{sympl}} \to \mathscr{X}_{\text{base}} = W\mathbb{P}(b_1, \dots, b_K)$, corresponding to the maximal collection of column vectors $(b_1, \dots, b_K)$. The full quotient is $\mathscr{X}_\tau^{\text{sympl}} = \bigoplus_j \mathscr{O}(-d_j)$ over $\mathscr{X}_{\text{base}}$. Similarly, for $\tau < 0$, the toric variety is $\bigoplus_i \mathscr{O}(-b_i)$ over $W\mathbb{P}(d_1, \dots, d_M)$.

$\boldsymbol{\tau > 0}$ The chamber $\tau > 0$ is called the *geometric phase*. Here we have $(x_1, \dots, x_K) \neq (0, \dots, 0)$. In this case, we can choose our $\mathbb{C}^*_R$–action to have weights $c_{x_i} = 0$ and $c_{p_j} = 1$, which gives a hybrid model, and the trivial lift ϑ_0 is a good lift of θ. The polynomial W has $\mathbb{C}^*_R$–degree $d = 1$, and the element J is trivial, so $\Gamma \cong G \times \mathbb{C}^*_R$. The critical locus is defined by (22). Since the F_i intersect transversely, the dF_i are linearly independent for $(x_1, \dots, x_K) \neq (0, \dots, 0)$. Therefore, all the p_i vanish, and the critical set is the complete intersection

$$\{F_1 = \cdots = F_M = 0\}$$

in the zero section of $\mathscr{X}_\tau \to W\mathbb{P}(b_1, \dots, b_K)$.

$\boldsymbol{\tau < 0}$ The chamber $\tau < 0$ is called the *LG phase*. If we happen to have $d_1 = \cdots = d_r = d$, we may take $c_{x_i} = b_i, c_{p_j} = 0$, and we again have a hybrid model with good lift ϑ_0. In this hybrid model case, we have

$$J = \left(\exp\left(2\pi i \frac{c_1}{d}\right), \dots, \exp\left(2\pi i \frac{c_n}{d}\right), 1, \dots, 1\right),$$

and we have

$$\begin{aligned}\Gamma &= \{((st)^{b_1}, \dots, (st)^{b_n}, s^{-d}, \dots, s^{-d}) \mid s, t \in \mathbb{C}^*\} \\ &= \{(\alpha^{b_1}, \dots, \alpha^{b_n}, \beta^d, \dots, \beta^d) \mid \alpha, \beta \in \mathbb{C}^*\}\end{aligned}$$

with

$$\zeta(\alpha^{b_1}, \dots, \alpha^{b_n}, \beta, \dots, \beta) = \alpha^d \beta.$$

Again, the critical locus is defined by (22), but now we have $(p_1, \dots, p_r) \neq (0, \dots, 0)$. This implies that $(x_1, \dots, x_K) = (0, \dots, 0)$. So the critical set is the zero section of the corresponding quotient $\mathscr{X}_\tau = \bigoplus_i \mathscr{O}(-b_i) \to W\mathbb{P}(d, \dots, d)$. Thus, for each choice of $(p_1, \dots, p_n) \in W\mathbb{P}(d, \dots, d)$, we have a pure LG model of superpotential $\sum_i p_i F_i$. One can view this as a family of pure LG theories.

7.2.2 LG quasimaps into complete intersections Now assume that $d_1 = \cdots = d_r = d$. In the geometric phase, with the trivial lift ϑ_0, the stack of LG quasimaps is

$$\{(\mathscr{C}, \mathscr{L}, s_1, \dots, s_K, p_1, \dots, p_r) : s_i \in H^0(\mathscr{C}, \mathscr{L}),\ p_j \in H^0(\mathscr{C}, \mathscr{L}^{-d} \otimes \omega_{\log,\mathscr{C}})\}$$

satisfying the stability condition. We obtain a theory similar to that of the geometric phase of the hypersurface in Example 4.3.1 and the corresponding p–field theory.

At the LG phase with $\varepsilon = \infty$, with the trivial lift ϑ_0, the moduli space consists of

$$\sigma = (p_1, \dots, p_r)\colon \mathscr{C} \to W\mathbb{P}^{r-1}(d, d, \dots, d),$$

where $W\mathbb{P}^{r-1}(d, d, \dots, d)$ is weighted projective space, corresponding to usual (unweighted) projective space with an order-d gerbe, and $\mathscr{L}^{-d} \otimes \omega_{\log,\mathscr{C}} \cong \sigma^*\mathscr{O}(1)$. Similar to FJRW theory, we have the condition

$$\mathscr{L}^d \cong \omega_{\log,\mathscr{C}} \otimes \sigma^*\mathscr{O}(-1).$$

This is the hybrid theory constructed by Clader [18].

Remark 7.2.1 When the F_j have different degrees d_j, there is generally no good lift. Moreover, the sections $p_j \in H^0(\mathscr{C}, \mathscr{L}^{-d_j} \otimes \omega_{\log,\mathscr{C}})$ are sections of different bundles, so we do not have a simple stable map description as before. Physicists have referred to this case as a *pseudohybrid model* [4]. We will come back to this on a different occasion.

7.3 Hypersurface in a product

The previous examples all have a one-dimensional parameter space for τ. We now give an example of multiparameter model, namely a hypersurface of bidegree (b, b') in a product of weighted projective spaces

$$W\mathbb{P}(b_1, \dots, b_K) \times W\mathbb{P}(b'_1, \dots, b'_M).$$

Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^K$ with positive weights $(b_1, \dots, b_K)$ and let $z_1, \dots, z_K$ be the coordinates on $\mathbb{C}^K$. Its quotient is weighted projective space $W\mathbb{P}(b_1, \dots, b_K)$. Consider another weighted projective space given by a different $\mathbb{C}^*$ acting on $\mathbb{C}^M$ with weights $(b'_1, \dots, b'_M)$, and let $w_1, \dots, w_M$ be the coordinates on $\mathbb{C}^M$. We combine these by setting $G = \mathbb{C}^* \times \mathbb{C}^*$ and letting G act on $\mathbb{C}^{K+M} \times \mathbb{C}$ with weights

$$\begin{pmatrix} b_1 & \cdots & b_K & 0 & \cdots & 0 & -b \\ 0 & \cdots & 0 & b'_1 & \cdots & b'_M & -b' \end{pmatrix}.$$

That is, if the last factor $\mathbb{C}$ has coordinate p, then p has bidegree $(b, -b')$.

Let F be any bihomogeneous polynomial in $\mathbb{C}[z_1, \dots, z_K] \otimes \mathbb{C}[w_1, \dots, w_M]$ of bidegree (b, b') that is *nondegenerate*, in the sense that if

$$\frac{\partial F}{\partial z_i} = 0 = \frac{\partial F}{\partial w_j} \quad \text{for all } i \in \{1, \dots, K\} \text{ and } j \in \{1, \dots, M\},$$

then either $z_1 = \dots = z_K = 0$ or $w_1 = \dots = w_M = 0$. As in Example 3.2.15 let

$$W = pF,$$

so that W is G–invariant. The critical locus of W is

$$\left\{ p\frac{\partial F}{\partial z_i} = 0,\ p\frac{\partial F}{\partial w_j} = 0,\ F = 0 \right\}.$$

The moment map $\mu\colon \mathbb{C}^{K+M} \times \mathbb{C} \to \mathfrak{u}(1) \oplus \mathfrak{u}(1) = \mathbb{R}^2$ is

$$\mu_1 = \frac{1}{2}\left(\sum_i b_i |z_i|^2 - b|p|^2\right), \quad \mu_2 = \frac{1}{2}\left(\sum_j b'_j |w_j|^2 - b'|p|^2\right).$$

The critical loci are

(i) $\{z_1 = \dots = z_K = 0,\ p = 0\}$;

(ii) $\{w_1 = \dots = w_M = 0,\ p = 0\}$;

(iii) $\{z_1 = \dots = z_K = w_1 = \dots = w_M = 0\}$.

The corresponding critical values are

(i) $\tau_1 = 0,\ \tau_2 \geq 0$;

(ii) $\tau_2 = 0,\ \tau_1 \geq 0$;

(iii) $\tau_1, \tau_2 < 0,\ \tau_1/b = \tau_2/b'$.

These divide $\mathbb{R}^2$ into three phases.

$\boldsymbol{\tau_1, \tau_2 > 0}$ In this phase we have $(z_1, \dots, z_K) \neq (0, \dots, 0)$ and $(w_1, \dots, w_M) \neq (0, \dots, 0)$. The maximal collection is

$$\begin{pmatrix} b_1 & \cdots & b_K & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b'_1 & \cdots & b'_M \end{pmatrix}$$

The quotient can be expressed as the total space of the line bundle

$$\mathscr{O}_1(-b) \otimes \mathscr{O}_2(-b') = K_{W\mathbb{P}(b_1, \dots, b_K)} \otimes K_{W\mathbb{P}(b'_1, \dots, b'_M)}$$

of bidegree $(-b, -b')$ over $W\mathbb{P}(b_1, \dots, b_K) \times W\mathbb{P}(b'_1, \dots, b'_M)$.

In the GIT formulation, let $\boldsymbol{L}_\theta$ have a generating section ℓ, and let θ have G–weights $(-e, -e')$ with $e, e' > 0$. Any G–invariant section of $\boldsymbol{L}_\theta$ is given by a polynomial in the z_i and ℓ, and can be written as a sum of G–invariant monomials in the z_i and ℓ, so to find the unstable and semistable points it suffices to consider only the G–invariant monomials of the form

$$\prod_{i=1}^{K} z_i^{a_i} \prod_{j=1}^{M} z_{K+j}^{a'_j} \ell^k.$$

Since both e and e' are positive, any G–invariant monomial must have at least one a_i and at least one a'_j not vanishing. This implies that the locus $\{z_1 = z_2 = \cdots = z_{K+M}\}$ is unstable. But any monomial of the form $z_i^e w_j^{e'} \ell$ will be G–invariant and will vanish only on the locus $z_i = w_j = 0$. Letting i and j range over all possible values shows that every point that is not in $\{z_1 = z_2 = \cdots = z_{K+M}\}$ is semistable.

Choose $\mathbb{C}^*_R$ to have weights $(0, \dots, 0, 1)$, so that W has $\mathbb{C}^*_R$–weight 1. Let ϑ_0 be the lift of θ with $\mathbb{C}^*_R$–weight 0. Every monomial of the form $z_i^e w_j^{e'} \ell$ is also $\mathbb{C}^*_R$–invariant, so ϑ_0 is a good lift of θ.

The semistable points of the critical locus of W are given by the equations

$$p = 0, \quad F = 0,$$

which is the hypersurface defined by the vanishing of F in the image of the zero section of $\mathscr{O}_1(-b) \otimes \mathscr{O}_2(-b') \to W\mathbb{P}(b_1, \dots, b_K) \times W\mathbb{P}(b'_1, \dots, b'_M)$. This is the geometric phase.

$\boldsymbol{\tau_2 < 0,\ \tau_1/b > \tau_2/b'}$ In this phase a similar analysis implies $(z_1, \dots, z_K) \neq (0, \dots, 0)$ and $p \neq 0$.

The maximal collection is

$$\begin{pmatrix} b_1 & \cdots & b_K & -b \\ 0 & \cdots & 0 & -b' \end{pmatrix}$$

We can quotient by $(\mathbb{C}^*)^2$, but since the two actions on z_{K+M+1} intertwine, we do not obtain $W\mathbb{P}(b_1, \dots, b_K) \times \mathrm{B}\mathbb{Z}_b$. Instead, we obtain a nontrivial gerbe over $W\mathbb{P}(b_1, \dots, b_K)$.

To be more specific, dividing by the first copy of $\mathbb{C}^*$, we obtain

$$\mathscr{O}_1(-b) \to W\mathbb{P}(b_1, \dots, b_K),$$

where $\mathscr{O}_1(1)$ is the standard $\mathbb{C}^*$–bundle associated with the first $\mathbb{C}^*$. Then, we quotient out the second $\mathbb{C}^*$. We obtain a nontrivial $\mathrm{B}\mathbb{Z}_{b'}$–bundle over $W\mathbb{P}(b_1, \dots, b_K)$, called a *gerbe*. We denote it by $W\mathbb{P}(b_1, \dots, b_K)^{-b/b'}$. Our quotient is the total space of $\bigoplus_i \mathscr{O}_2(-b'_i)$.

We choose the $\mathbb{C}^*_R$–action in this phase to have weights $(0, \dots, 0, b'_1, \dots, b'_M, 0)$. Again, the lift ϑ_0 is a good lift of θ.

The semistable points of the critical locus are those with $w_1 = \cdots = w_M = 0 = F$, so this phase gives us a mixture of LG and geometric phases with the w–directions corresponding to an LG model and the z–directions corresponding to a geometric model.

$\tau_1 < 0,\ \tau_1/b < \tau_2/b'$ The analysis for this phase is similar to the previous one and yields a different mixture of LG and geometric phase with the z–directions now corresponding to an LG model and the w–directions corresponding to a geometric model.

7.4 Nonabelian examples

The subject of gauged linear sigma models for nonabelian groups is a very active area of research in physics and is far from complete. Here, we discuss the example of complete intersection of Grassmannian varieties. One should be able to discuss everything in the setting of complete intersections of quiver varieties, although the details have not been worked out. It would be very interesting to explore mirror symmetry among Calabi–Yau complete intersections in quiver varieties.

7.4.1 Complete intersection in a Grassmannian Consider a complete intersection in the Grassmannian $\mathrm{Gr}(k,n)$. The space $\mathrm{Gr}(k,n)$ can be constructed as the GIT quotient $M_{k,n}/\!\!/\mathrm{GL}(k,\mathbb{C})$, where $M_{k,n}$ is the space of $k \times n$ matrices and $\mathrm{GL}(k,\mathbb{C})$ acts as matrix multiplication on the left.

The Grassmannian $\mathrm{Gr}(k,n)$ can also be embedded into $\mathbb{P}^K$ for $K = n!/k!(n-k)! - 1$ by the Plücker embedding

$$A \mapsto (\ldots, \det(A_{i_1,\ldots,i_k}), \ldots),$$

where $A_{i_1,\ldots,i_k}$ is the $k \times k$ submatrix of A consisting of the columns $i_1, \ldots, i_k$.

The group $G = \mathrm{GL}(k,\mathbb{C})$ acts on the Plücker coordinates $B_{i_1,\ldots,i_k}(A) = \det(A_{i_1,\ldots,i_k})$ by the determinant, that is, for any $U \in G$, and $A \in M_{k,n}$ we have

$$B_{i_1,\ldots,i_k}(UA) = \det(U) B_{i_1,\ldots,i_k}(A),$$

Let $F_1, \ldots, F_s \in \mathbb{C}[B_{1,\ldots,k}, \ldots, B_{n-k+1,\ldots,n}]$ be degree-d_j homogeneous polynomials such that the zero loci $Z_{F_j} = \{F_j = 0\}$ and the Plücker embedding of $\mathrm{Gr}(k,n)$ all intersect transversely in $\mathbb{P}^K$. We let

$$Z_{d_1,\ldots,d_s} = \mathrm{Gr}(k,n) \cap \bigcap_j Z_{F_j}$$

denote the corresponding complete intersection.

The analysis of $Z_{d_1,\dots,d_s}$ is similar to the abelian case. Namely, let

$$W = \sum_j p_j F_j \colon M_{k,n} \times \mathbb{C}^s \to \mathbb{C}$$

be the superpotential. We assign an action of $G = \mathrm{GL}(k,\mathbb{C})$ on p_j by $p_j \to \det(U)^{-d_j}$.

The phase structure is similar to that of a complete intersection in projective space. The moment map is given by $\mu(A, p_1, \dots, p_s) = \frac{1}{2}\bigl(A\bar{A}^T - \sum_{i=1}^s d_i |p_i|^2\bigr)$. Alternatively, to construct a linearization for GIT, the only characters of $\mathrm{GL}(k,\mathbb{C})$ are powers of the determinant, so $\theta(U) = \det(U)^{-e}$ for some e, and τ will be positive precisely when e is positive.

Let ℓ be a generator of $\mathbb{C}[\boldsymbol{L}^*]$ over $\mathbb{C}[V^*]$. Any element of $H^0(V, \boldsymbol{L}_\theta)$ can be written as a sum of monomials in the Plücker coordinates $B_{i_1,\dots,i_k}$ and the p_j times ℓ. Any $U \in G$ will act on a monomial of the form $\prod B_{i_1,\dots,i_k}^{b_{i_1,\dots,i_k}} \prod p_j^{a_j} \ell^m$ by multiplication by $\det(U)^{\sum b_{i_1,\dots,i_k} - \sum d_j a_j - me}$.

$e > 0$ In order to be G–invariant, a monomial must have $\sum b_{i_1,\dots,i_k} > 0$, which implies that any point with every $B_{i_1,\dots,i_k} = 0$ must be unstable, but for each $m > 0$ and each k–tuple $(i_1, \dots, i_k)$ the monomial $B_{i_1,\dots,i_k}^{me} \ell^m$ is G–invariant, so every point with at least one nonzero $B_{i_1,\dots,i_k}$ must be θ–semistable. Thus $[V /\!\!/_\theta G]$ is isomorphic to the bundle $\bigoplus_j \mathscr{O}(-d_j)$ over $\mathrm{Gr}(k,n)$.

Furthermore, W is quasihomogeneous of degree 1 with respect to the compatible $\mathbb{C}_R^*$–action

$$\lambda(A, p_1, \dots, p_s) = (A, \lambda p_1, \dots, \lambda p_s).$$

The trivial lift ϑ_0 is a good lift because each monomial of the form $B_{i_1,\dots,i_k}^{me} \ell^m$ is Γ–invariant for the action induced by ϑ_0.

As in Section 7.2, the critical locus in this phase is given by $p_1 = \dots = p_s = 0 = F_1 = \dots = F_s$, so we recover the complete intersection $F_1 = \dots = F_s$ in $\mathrm{Gr}(k,n)$.

As in the toric case, we call this phase the *geometric phase*.

$e < 0$ We call the case where $e < 0$ the *LG phase*. In this case, in order to be G–invariant a monomial $\prod B_{i_1,\dots,i_k}^{b_{i_1,\dots,i_k}} \prod p_j^{a_j} \ell^m$ must have $\sum a_j > 0$, which implies that any point with every $p_j = 0$ must be unstable, but for each $m > 0$ and each j the monomial $p_j^{me} \ell^{md_j}$ is G–invariant, so every point with at least one nonzero p_j is θ–semistable. Therefore $V_G^{\mathrm{ss}}(\theta) = M_{k,n} \times (\mathbb{C}^s \smallsetminus \{\mathbf{0}\})$. Again, since the F_j and the image of the Plücker embedding are transverse, the equations $\partial_{B_{i_1,\dots,i_k}} W = \sum_j p_j \partial_{B_{i_1,\dots,i_k}} F_j = 0$ imply that the critical locus is $\bigl[\bigl(\{\mathbf{0}\} \times (\mathbb{C}^s \smallsetminus \{\mathbf{0}\})\bigr)/\mathrm{GL}(k,\mathbb{C})\bigr]$ inside $[V /\!\!/_\theta G] = \bigl[\bigl(M_{k,n} \times (\mathbb{C}^s \smallsetminus \{\mathbf{0}\})\bigr)/\mathrm{GL}(k,\mathbb{C})\bigr]$. This phase does not immediately fit into our theory because we have an infinite stabilizer $\mathrm{SL}(k,\mathbb{C})$ for any points of the form $(\mathbf{0}, p_1,\dots,p_s)$. This means that the quotient $[V/\!\!/_\theta G]$ is an Artin stack.

Hori and Tong [30] have analyzed the gauged linear sigma model of the Calabi–Yau complete intersection $Z_{1,\dots,1}\subset \mathrm{Gr}(2,7)$ which is defined by seven linear equations in the Plücker coordinates. They gave a physical derivation that its LG phase is equivalent to the Gromov–Witten theory of the so-called *Pfaffian variety*

$$\mathrm{Pf}\big(\textstyle\bigwedge^2\mathbb{C}^7\big)=\big\{A\in \textstyle\bigwedge^2\mathbb{C}^7 : A\wedge A\wedge A=0\big\}.$$

It is interesting to note that the Pfaffian $\mathrm{Pf}\big(\bigwedge^2\mathbb{C}^7\big)$ is not a complete intersection. For additional work on this example, see [46; 35; 31; 2]

7.4.2 Complete intersections in a flag variety Another class of interesting examples is that of complete intersections in partial flag varieties. The partial flag variety $\mathrm{Flag}(d_1,\dots,d_k)$ parametrizes the space of partial flags

$$0\subset V_1\subset\cdots V_i\subset\cdots V_k=\mathbb{C}^n$$

such that $\dim V_i=d_i$. The combinatorial structure of the equivariant cohomology of $\mathrm{Flag}(d_1,\dots,d_k)$ is a very interesting subject in its own right.

For our purposes, $\mathrm{Flag}(d_1,\dots,d_k)$ can be constructed as a GIT or symplectic quotient of the vector space

$$\prod_{i=1}^{k-1} M_{d_i,d_{i+1}}$$

by the group

$$G=\prod_{i=1}^{k-1}\mathrm{GL}(d_i,\mathbb{C}).$$

The moment map sends the element $(A_1,\dots,A_{k-1})\in\prod_{i=1}^{k-1}M_{i,i+1}$ to the element $\frac{1}{2}(A_1\bar{A}_1^T,\dots,A_{k-1}\bar{A}_{k-1}^T)\in\prod_{i=1}^{k-1}\mathfrak{u}(d_i)$.

Let χ_i be the character of $\prod_j \mathrm{GL}(d_j)$ given by the determinant of the i^{th} factor. Each character χ_i defines a line bundle on the vector space $M_{d_1,d_2}\times\cdots\times M_{d_{k-1},d_k}$, which descends to a line bundle L_i on $\mathrm{Flag}(d_1,\dots,n_k)$. A hypersurface of multidegree $(l_1,\dots,l_k)$ is a section of $\bigotimes_j L_j^{l_j}$. To consider the gauged linear sigma model for the complete intersection $F_1=\cdots=F_s=0$ of such sections, we again consider the vector space

$$V=\prod_{i=1}^{k-1}M_{d_i,d_i+1}\times\mathbb{C}^s,$$

with coordinates $(p_1, \dots, p_s)$ on $\mathbb{C}^s$ and superpotential

$$W = \sum_{j=1}^{s} p_j F_j .$$

We define an action of G on p_i by $(g_1, \dots, g_{k-1}) \in G$ acts on p_i as $\prod_{j=1}^{k-1} \det(g_j)^{-l_{ij}}$, where l_{ij} is the j^{th} component of the multidegree degree of F_i.

We may describe the polarization as

$$\theta = \prod_{i=1}^{k-1} \det(g_i)^{-e_i},$$

or the moment map as

$$\mu(A_1, \dots, A_{k-1}, p_1, \dots, p_s) \\ = \frac{1}{2}\Big(A_1 \bar{A}_1^T - \sum_{i=1}^{s} l_{1j} |p_j|^2, \dots, A_{k-1} \bar{A}_{k-1}^T - \sum_{i=1}^{s} l_{k-1,j} |p_j|^2\Big).$$

This gives a phase structure similar to the complete intersection in a product of projective spaces.

For example, when $e_i > 0$ for all $i \in \{1, \dots, k-1\}$ we can choose a compatible $\mathbb{C}^*_R$–action with weight 1 on p_j and weight 0 on each A_i, and the trivial lift ϑ_0 is a good lift of θ in this phase.

This example should be easy to generalize to complete intersections in quiver varieties. It would be very interesting to calculate the details of our theory for these examples.

7.5 Graph spaces and generalizations

7.5.1 Graph spaces The graph moduli space is very important in Gromov–Witten theory. It is used to define the I–function and prove genus zero mirror symmetry (see for example [27]). We can construct it in the GLSM setting as follows. Suppose that we have a phase θ of a GLSM $W\colon \mathbb{C}^n/G \to \mathbb{C}$ with a certain R–charge $\mathbb{C}^*_R$, defining Γ and a good lift ϑ of θ.

We construct a new GLSM as follows. Let $V' = V \times \mathbb{C}^2$, and let $\mathbb{C}^*$ act on $\mathbb{C}^2$ with weights $(1,1)$. Let $G' = G \times \mathbb{C}^*$ act on V' with the product action, so G acts trivially on the last two coordinates and $\mathbb{C}^*$ acts trivially on the first n coordinates. Let $\theta'\colon G' \to \mathbb{C}^*$ be given by sending any $(g,h) \in G \times \mathbb{C}^*$ to $\theta'(g,h) = \theta(g)h^{-e}$ for some $e > 0$. The GIT quotient is the product $[V' /\!\!/_{\theta'} G'] = [V /\!\!/_{\theta} G] \times \mathbb{P}^1$. Let W' be defined on V' by the same polynomial as W, so that the critical locus of W' is $\mathbb{C}^2$ times the critical locus of W, and the GIT quotient of the critical locus is $\mathbb{P}^1$ times the corresponding quotient in the original GLSM.

Keeping the same R–charge (that is, letting $\mathbb{C}^*_R$ acts trivially on the last two coordinates of V'), we have $\Gamma' = \Gamma \times \mathbb{C}^*$, and we construct a lift ϑ' of θ' by sending $(\gamma, h) \in \Gamma \times \mathbb{C}^*$ to $\vartheta'(\gamma, h) = \vartheta(\gamma) h^{-e}$. It is easy to see that ϑ' is a good lift of θ' if ϑ is a good lift of θ.

In the $\varepsilon = \infty$ case, the last two coordinates (z_1, z_2) induce a stable map $\mathscr{C} \to \mathbb{P}^1$. For the other ε–stable case, we choose $e \gg 0$ such that stability condition for the second $\mathbb{C}^*$ is always in the ∞–chamber. There is no basepoint for (z_1, z_2) which induces a stable map $\mathscr{C} \to \mathbb{P}^1$. Therefore, it can be reformulated as a usual GLSM moduli space of $[V /\!\!/_\theta G]$ with the additional data of a stable map $f \colon \mathscr{C} \to \mathbb{P}^1$.

7.5.2 Generalization of the graph space We can generalize slightly the graph moduli space to obtain a new moduli space with a remarkable property. Let's take the quintic GLSM as an example. Now, we consider a new GLSM on $\mathbb{C}^{6+2}/(\mathbb{C}^*)^2$ with charge matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & -5 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

for an integer $d > 0$.

Let's look at its chamber structure. The moment maps are

$$\mu_1 = \tfrac{1}{2}(|x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 + |x_5|^2 - 5|p|^2 + d|z_1|^2), \quad \mu_2 = \tfrac{1}{2}(|z_1|^2 + |z_2|^2).$$

It has three chambers. We are interested in the chamber $0 < \mu_1 < d\mu_2$. This corresponds to a character θ of $G = \mathbb{C}^* \times \mathbb{C}^*$ with weights $(-e_1, -e_2)$ and $0 < e_1 < de_2$. The unstable locus for this character is

$$\{(x_1, x_2, x_3, x_4, x_5, z_1) = (0,0,0,0,0,0)\} \cup \{(p, z_2) = (0,0)\} \cup \{(z_1, z_2) = (0,0)\}.$$

Taking the superpotential

$$W = \sum_{i=1}^{5} x_i^5$$

and the R–charge of weight $(0,0,0,0,0,1,0,0)$, we have

$$\Gamma = G \times \mathbb{C}^*_R = \{(a,a,a,a,a,w,ba^d,b) \mid a,b,w \in \mathbb{C}^*\}$$

and the map ζ takes (a,a,a,a,a,w,ba^d,b) to wa^5. There is no good lift of θ, so we restrict to the case of $\varepsilon = 0+$. We must choose some lift for the stability condition, so we take $\vartheta(a,b,w) = a^{-e_1} b^{-e_2}$. Any other lift will give the same stability conditions.

The resulting moduli problem consists of

$$\Bigl\{(\mathscr{C}, y_1, \dots, y_k, \mathscr{A}, \mathscr{B}, x_1, \dots, x_5, p, z_1, z_2) \Bigm| x_i \in H^0(\mathscr{C}, \mathscr{A}),\ p \in H^0(\mathscr{C}, \mathscr{A}^{-5} \otimes \omega_{\log,\mathscr{C}}), \\ z_1 \in H^0(\mathscr{C}, \mathscr{A}^d \otimes \mathscr{B}),\ z_2 \in H^0(\mathscr{C}, \mathscr{B})\Bigr\}$$

satisfying the stability condition that $\sigma^*\mathscr{L}_\vartheta = \mathscr{A}^{-e_1}\mathscr{B}^{-e_2}$ is ample on all components where $\omega_{\log,\mathscr{C}}$ has degree 0.

This GLSM admits a $\mathbb{C}^*$–action on z_2. The induced action on the moduli space has three types of fixed point loci: the Gromov–Witten locus, FJRW locus and the theory of a point. This remarkable property gives us the hope that we can extract a relation between Gromov–Witten theory and FJRW theory geometrically by using localization techniques on this moduli space. A program is being carried out right now for the $\varepsilon = 0+$ theory by Ross, Ruan and Shoemaker, and Clader, Janda and Ruan [19]. A theory based on the same GIT quotient, but with a different stability condition, was discovered and the localization argument was carried out independently by Chang, Li, Li and Liu [8].

References

[1] **D Abramovich**, **A Vistoli**, *Compactifying the space of stable maps*, J. Amer. Math. Soc. 15 (2002) 27–75 MR

[2] **N Addington**, **W Donovan**, **E Segal**, *The Pfaffian–Grassmannian equivalence revisited*, Algebr. Geom. 2 (2015) 332–364 MR

[3] **J Alper**, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) 63 (2013) 2349–2402 MR

[4] **P S Aspinwall**, **M R Plesser**, *Decompactifications and massless D-branes in hybrid models*, J. High Energy Phys. (2010) art. id. 078 MR

[5] **A Borel**, *Linear algebraic groups*, 2nd edition, Graduate Texts in Mathematics 126, Springer (1991) MR

[6] **H-L Chang**, **J Li**, *Gromov–Witten invariants of stable maps with fields*, Int. Math. Res. Not. 2012 (2012) 4163–4217 MR

[7] **H-L Chang**, **J Li**, **W-P Li**, *Witten's top Chern class via cosection localization*, Invent. Math. 200 (2015) 1015–1063 MR

[8] **H-L Chang**, **J Li**, **W-P Li**, **C-C M Liu**, *Mixed-spin-p fields of Fermat quintic polynomials*, preprint (2015) arXiv

[9] **W Chen**, **Y Ruan**, *Orbifold Gromov–Witten theory*, from "Orbifolds in mathematics and physics" (A Adem, J Morava, Y Ruan, editors), Contemp. Math. 310, Amer. Math. Soc., Providence, RI (2002) 25–85 MR

[10] **D Cheong**, **I Ciocan-Fontanine**, **B Kim**, *Orbifold quasimap theory*, Math. Ann. 363 (2015) 777–816 MR

[11] **A Chiodo**, *The Witten top Chern class via K–theory*, J. Algebraic Geom. 15 (2006) 681–707 MR

[12] **A Chiodo**, **H Iritani**, **Y Ruan**, *Landau–Ginzburg/Calabi–Yau correspondence, global mirror symmetry and Orlov equivalence*, Publ. Math. Inst. Hautes Études Sci. 119 (2014) 127–216 MR

[13] **A Chiodo**, **Y Ruan**, *Landau–Ginzburg/Calabi–Yau correspondence for quintic three-folds via symplectic transformations*, Invent. Math. 182 (2010) 117–165 MR

[14] **A Chiodo**, **Y Ruan**, *LG/CY correspondence: the state space isomorphism*, Adv. Math. 227 (2011) 2157–2188 MR

[15] **I Ciocan-Fontanine**, **B Kim**, *Moduli stacks of stable toric quasimaps*, Adv. Math. 225 (2010) 3022–3051 MR

[16] **I Ciocan-Fontanine**, **B Kim**, **D Maulik**, *Stable quasimaps to GIT quotients*, J. Geom. Phys. 75 (2014) 17–47 MR

[17] **E Clader**, *The Landau–Ginzburg/Calabi–Yau correspondence for certain complete intersections*, PhD thesis, University of Michigan (2014) Available at `http://www-personal.umich.edu/~eclader/CladerDissertation.pdf`

[18] **E Clader**, *Landau–Ginzburg/Calabi–Yau correspondence for the complete intersections $X_{3,3}$ and $X_{2,2,2,2}$*, Adv. Math. 307 (2017) 1–52 MR

[19] **E Clader**, **F Janda**, **Y Ruan**, *Higher-genus wall-crossing in the gauged linear sigma model*, preprint (2017) arXiv

[20] **I V Dolgachev**, **Y Hu**, *Variation of geometric invariant theory quotients*, Inst. Hautes Études Sci. Publ. Math. 87 (1998) 5–56 MR

[21] **D Edidin**, **T J Jarvis**, **T Kimura**, *Logarithmic trace and orbifold products*, Duke Math. J. 153 (2010) 427–473 MR

[22] **C Faber**, **S Shadrin**, **D Zvonkine**, *Tautological relations and the r–spin Witten conjecture*, Ann. Sci. Éc. Norm. Supér. 43 (2010) 621–658 MR

[23] **H Fan**, **T J Jarvis**, **Y Ruan**, *The Witten equation and its virtual fundamental cycle*, preprint (2007) arXiv

[24] **H Fan**, **T J Jarvis**, **Y Ruan**, *Geometry and analysis of spin equations*, Comm. Pure Appl. Math. 61 (2008) 745–788 MR

[25] **H Fan**, **T Jarvis**, **Y Ruan**, *The Witten equation, mirror symmetry, and quantum singularity theory*, Ann. of Math. 178 (2013) 1–106 MR

[26] **H Fan**, **T J Jarvis**, **Y Ruan**, *Analytic theory of the gauged linear sigma model*, preprint (2017)

[27] **A Givental**, *A mirror theorem for toric complete intersections*, from "Topological field theory, primitive forms and related topics" (M Kashiwara, A Matsuo, K Saito, I Satake, editors), Progr. Math. 160, Birkhäuser, Boston (1998) 141–175 MR

[28] **J Harris**, **I Morrison**, *Moduli of curves*, Graduate Texts in Mathematics 187, Springer (1998) MR

[29] **Y I Holla**, **M S Narasimhan**, *A generalisation of Nagata's theorem on ruled surfaces*, Compositio Math. 127 (2001) 321–332 MR

[30] **K Hori**, **D Tong**, *Aspects of non-abelian gauge dynamics in two-dimensional $\mathcal{N} = (2,2)$ theories*, J. High Energy Phys. (2007) art. id. 079 MR

[31] **S Hosono**, **Y Konishi**, *Higher genus Gromov–Witten invariants of the Grassmannian, and the Pfaffian Calabi–Yau 3–folds*, Adv. Theor. Math. Phys. 13 (2009) 463–495 MR

[32] **Y-H Kiem**, **J Li**, *Localizing virtual cycles by cosections*, J. Amer. Math. Soc. 26 (2013) 1025–1050 MR

[33] **B Kim**, *Stable quasimaps*, Commun. Korean Math. Soc. 27 (2012) 571–581 MR

[34] **M Kontsevich**, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. 147 (1992) 1–23 MR

[35] **A Kuznetsov**, *Lefschetz decompositions and categorical resolutions of singularities*, Selecta Math. 13 (2008) 661–696 MR

[36] **Y-P Lee**, *Witten's conjecture and the Virasoro conjecture for genus up to two*, from "Gromov–Witten theory of spin curves and orbifolds" (T J Jarvis, T Kimura, A Vaintrob, editors), Contemp. Math. 403, Amer. Math. Soc., Providence, RI (2006) 31–42 MR

[37] **Y-P Lee**, **N Priddis**, **M Shoemaker**, *A proof of the Landau–Ginzburg/Calabi–Yau correspondence via the crepant transformation conjecture*, Ann. Sci. Éc. Norm. Supér. 49 (2016) 1403–1443 MR

[38] **A Marian**, **D Oprea**, **R Pandharipande**, *The moduli space of stable quotients*, Geom. Topol. 15 (2011) 1651–1706 MR

[39] **D Mumford**, **J Fogarty**, **F Kirwan**, *Geometric invariant theory*, 3rd edition, Ergeb. Math. Grenzgeb. 34, Springer (1994) MR

[40] **M C Olsson**, *(Log) twisted curves*, Compos. Math. 143 (2007) 476–494 MR

[41] **A Polishchuk**, **A Vaintrob**, *Algebraic construction of Witten's top Chern class*, from "Advances in algebraic geometry motivated by physics" (E Previato, editor), Contemp. Math. 276, Amer. Math. Soc., Providence, RI (2001) 229–249 MR

[42] **A Polishchuk**, **A Vaintrob**, *Matrix factorizations and cohomological field theories*, J. Reine Angew. Math. 714 (2016) 1–122 MR

[43] **A M Popa**, *Two-point Gromov–Witten formulas for symplectic toric manifolds*, PhD thesis, State University of New York at Stony Brook (2012) MR Available at `https://search.proquest.com/docview/1095122515`

[44] **N Priddis**, **M Shoemaker**, *A Landau–Ginzburg/Calabi–Yau correspondence for the mirror quintic*, Ann. Inst. Fourier (Grenoble) 66 (2016) 1045–1091 MR

[45] **A Ramanathan**, *Moduli for principal bundles over algebraic curves, I*, Proc. Indian Acad. Sci. Math. Sci. 106 (1996) 301–328 MR

[46] **E A Rødland**, *The Pfaffian Calabi–Yau, its mirror, and their link to the Grassmannian $G(2,7)$*, Compositio Math. 122 (2000) 135–149 MR

[47] **D Ross**, **Y Ruan**, *Wall-crossing in genus zero Landau–Ginzburg theory*, preprint (2014) arXiv

[48] **M Thaddeus**, *Geometric invariant theory and flips*, J. Amer. Math. Soc. 9 (1996) 691–723 MR

[49] **G Tian**, **G Xu**, *Analysis of gauged Witten equation*, preprint (2014) arXiv

[50] **E Witten**, *Two-dimensional gravity and intersection theory on moduli space*, from "Surveys in differential geometry", Lehigh Univ., Bethlehem, PA (1991) 243–310 MR

[51] **E Witten**, *The N matrix model and gauged WZW models*, Nuclear Phys. B 371 (1992) 191–245 MR

[52] **E Witten**, *Algebraic geometry associated with matrix models of two-dimensional gravity*, from "Topological methods in modern mathematics" (L R Goldberg, A V Phillips, editors), Publish or Perish, Houston (1993) 235–269 MR

[53] **E Witten**, *Phases of $N = 2$ theories in two dimensions*, from "Mirror symmetry, II" (B Greene, S-T Yau, editors), AMS/IP Stud. Adv. Math. 1, Amer. Math. Soc., Providence, RI (1997) 143–211 MR

HF: *School of Mathematical Science, Beijing (Peking) University*
Beijing, China

TJ: *Department of Mathematics, Brigham Young University*
Provo, UT, United States

YR: *Mathematics Department, University of Michigan*
Ann Arbor, MI, United States

YR: *Beijing International Center for Mathematical Science, Peking University*
Beijing, China

fanhj@math.pku.edu.cn, jarvis@math.byu.edu, ruan@umich.edu

Proposed: Jim Bryan
Seconded: Richard Thomas, Simon Donaldson

Received: 2 November 2015
Revised: 23 December 2016