Chord arc properties for constant mean curvature disks

WILLIAM H MEEKS, III

GIUSEPPE TINAGLIA

We prove a chord arc type bound for disks embedded in $\mathbb{R}^3$ with constant mean curvature that does not depend on the value of the mean curvature. This bound is inspired by and generalizes the weak chord arc bound of Colding and Minicozzi in Proposition 2.1 of Ann. of Math. 167 (2008) 211–243 for embedded minimal disks. Like in the minimal case, this chord arc bound is a fundamental tool for studying complete constant mean curvature surfaces embedded in $\mathbb{R}^3$ with finite topology or with positive injectivity radius.

1 Introduction

We apply results of Colding and Minicozzi [2], Meeks and Rosenberg [5] and Meeks and Tinaglia [9; 10; 8] to derive a chord arc bound for compact disks embedded in $\mathbb{R}^3$ with constant mean curvature. For clarity of exposition, we will call an oriented surface Σ immersed in $\mathbb{R}^3$ an *H–surface* if it is *embedded*, *connected* and it has *nonnegative constant mean curvature* H. We will call an H–surface an *H–disk* if the H–surface is homeomorphic to a closed unit disk in the Euclidean plane; in general we will allow an H–disk Σ to be nonsmooth along its boundary. We remark that this definition of H–surface agrees with the one given in [10; 8], but differs from the one given in [9], where we restrict to the case when $H > 0$.

It will be important to distinguish between intrinsic and extrinsic balls centered at points of Σ; given $p \in \Sigma$ and $R > 0$, we will denote by $B_\Sigma(p, R)$ (resp. $\mathbb{B}(p, R)$) the *open* intrinsic (resp. open extrinsic) ball of center p and radius R and let $\mathbb{B}(R) = \mathbb{B}(\vec{0}, R)$, where $\vec{0}$ is the origin in $\mathbb{R}^3$. We will denote by $\bar{B}_\Sigma(p, R)$ (resp. $\bar{\mathbb{B}}(p, R)$) the *closed* intrinsic (resp. closed extrinsic) ball of center p and radius R and let $\bar{\mathbb{B}}(R)$ be the closure of $\mathbb{B}(\vec{0}, R)$.

Definition 1.1 Given a point p on a surface $\Sigma \subset \mathbb{R}^3$, $\Sigma(p, R)$ denotes the closure of the component of $\Sigma \cap \mathbb{B}(p, R)$ passing through p.

We note that if the surface Σ in the above definition is transverse to $\partial\mathbb{B}(p, R)$, then $\Sigma(p, R)$ is the component of $\Sigma \cap \overline{\mathbb{B}}(p, R)$ passing through p. The main result of this paper is the following theorem.

Theorem 1.2 (weak chord arc property for H–disks)   *There exists a $\delta_1 \in \left(0, \frac{1}{2}\right)$ such that the following holds:*

Let Σ be an H–disk in $\mathbb{R}^3$. Then for all intrinsic closed balls $\overline{B}_\Sigma(x, R)$ in $\Sigma - \partial\Sigma$:

   (i) $\Sigma(x, \delta_1 R)$ *is a disk with piecewise smooth boundary* $\partial\Sigma(x, \delta_1 R) \subset \partial\mathbb{B}(x, \delta_1 R)$.

   (ii) $\Sigma(x, \delta_1 R) \subset B_\Sigma\left(x, \frac{R}{2}\right)$.

Theorem 1.2 gives rise to a more standard chord arc type result that closely resembles the chord arc type result for 0–disks given by Colding and Minicozzi [2, Theorem 0.5]; see [10, Theorem 1.2] for this application.

We clarify that Theorem 1.2 in this manuscript depends on the extrinsic one-sided curvature estimate for H–disks [8, Theorem 1.1]. On the other hand, the intrinsic one-sided curvature estimate for H–disks [8, Theorem 4.5] relies on Theorem 1.2 in this manuscript.

Other applications of the results in this manuscript can be found in Meeks and Tinaglia [11].

2   Proof of Theorem 1.2

The proof of Theorem 1.2 relies on three results that appear in [10; 8], and for the sake of completeness, we include their statements here. Theorems 2.1 and 2.2 are generalizations of results that were proved by Colding and Minicozzi in the minimal case in [1].

Theorem 2.1 (one-sided curvature estimate for H–disks [8, Theorem 1.1]) *There exist $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $C > 0$ such that for any $R > 0$, the following holds: Let $\mathcal{D}$ be an H–disk such that*

$$\mathcal{D} \cap \mathbb{B}(R) \cap \{x_3 = 0\} = \varnothing \quad \text{and} \quad \partial\mathcal{D} \cap \mathbb{B}(R) \cap \{x_3 > 0\} = \varnothing.$$

Then

$$(1) \qquad\qquad \sup_{x \in \mathcal{D} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\}} |A_{\mathcal{D}}|(x) \leq \frac{C}{R},$$

where $|A_{\mathcal{D}}|$ denotes the norm of the second fundamental form of $\mathcal{D}$. In particular, if $\mathcal{D} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\} \neq \varnothing$, then $H < \frac{C}{R}$.

Theorem 2.2 (limit lamination theorem for H–disks [10, Theorem 1.1]) *Fix $\varepsilon > 0$ and let $\{M_n\}_n$ be a sequence of H_n–disks in $\mathbb{R}^3$ containing the origin and such that $\partial M_n \subset [\mathbb{R}^3 - \mathbb{B}(n)]$ and $|A_{M_n}|(\vec{0}) \geq \varepsilon$. Then, after replacing by some subsequence, exactly one of the following two statements hold:*

(A) *The surfaces M_n converge smoothly with multiplicity one or two on compact subsets of $\mathbb{R}^3$ to a helicoid M_∞ containing the origin. Furthermore, every component Δ of $M_n \cap \mathbb{B}(1)$ is an open disk whose closure $\overline{\Delta}$ in M_n is a compact disk with piecewise smooth boundary, and where the intrinsic distance in M_n between any two points in $\overline{\Delta}$ is less than 10.*

(B) *There are points $p_n \in M_n$ such that*

$$\lim_{n \to \infty} p_n = \vec{0} \quad \text{and} \quad \lim_{n \to \infty} |A_{M_n}|(p_n) = \infty,$$

and the following hold:

   (a) *The surfaces M_n converge to a foliation of $\mathbb{R}^3$ by planes and the convergence is C^α for any $\alpha \in (0, 1)$ away from the line containing the origin and orthogonal to the planes in the foliation.*

   (b) *There exists compact subdomains $\mathcal{C}_n$ of M_n with $[M_n \cap \overline{\mathbb{B}}(1)] \subset \mathcal{C}_n \subset \mathbb{B}(2)$ and $\partial\mathcal{C}_n \subset \mathbb{B}(2) - \overline{\mathbb{B}}(1)$, each $\mathcal{C}_n$ consisting of one or two pairwise disjoint disks, where each disk component has intrinsic diameter less than 3 and intersects $\mathbb{B}\left(\frac{1}{n}\right)$. Moreover, each connected component of $M_n \cap \mathbb{B}(1)$ is an open disk whose closure in M_n is a compact disk with piecewise smooth boundary.*

Corollary 2.3 [10, Corollary 4.6] *There exist constants $\varepsilon \in (0, 1)$ and $C > 1$ such that the following holds: Let Σ_1, Σ_2 and Σ_3 be three pairwise disjoint H_i–disks with $\partial\Sigma_i \subset [\mathbb{R}^3 - \mathbb{B}(1)]$ for $i = 1, 2, 3$. If $\mathbb{B}(\varepsilon) \cap \Sigma_i \neq \varnothing$ for $i = 1, 2, 3$, then*

$$\sup_{\mathbb{B}(\varepsilon) \cap \Sigma_i, i=1,2,3} |A_{\Sigma_i}| \leq C.$$

2.1 A weak chord arc property for certain H–disks.

Throughout this section and the next one, Σ will denote a compact H–disk in $\mathbb{R}^3$ with piecewise smooth boundary.

The following main result in this section generalizes the similar [2, Proposition 2.1] for minimal disks to certain H–disks. The reader should keep in mind that the convex hull property of minimal surfaces fails in the case of H–surfaces with $H > 0$, and this failure contributes to making the proof of the next proposition and some other results in this paper more difficult than in the $H = 0$ case.

Proposition 2.4 *There exists $\delta_2 \in \left(0, \frac{1}{2}\right)$ such that the following holds:*

If Σ satisfies $\partial\Sigma \subset \partial\mathbb{B}(p, R)$ and $p \in \Sigma$, then for all $s \in (0, R]$:

  (i) *$\Sigma(p, \delta_2 s)$ is a disk with piecewise smooth boundary $\partial\Sigma(p, \delta_2 s) \subset \partial\mathbb{B}(p, \delta_2 s)$.*

  (ii) *$\Sigma(p, \delta_2 s) \subset \bar{B}_\Sigma\left(p, \frac{s}{2}\right)$.*

Proof Suppose that Σ satisfies $\partial\Sigma \subset \partial\mathbb{B}(p, R)$ and $p \in \Sigma$. We first prove that items (i) and (ii) of the proposition hold for some $\delta_2 \in \left(0, \frac{1}{2}\right)$ in the special case that $s = R$. Arguing by contradiction, suppose there is no such universal δ_2. Then there exists a sequence $\Sigma(n)$ of H_n–disks and a sequence R_n of positive numbers such that

  (i) $\vec{0} \in \Sigma(n)$.

  (ii) $\partial\Sigma(n) \subset \partial\mathbb{B}(R_n)$.

  (iii) Either $\Sigma(n)\left(\vec{0}, \frac{1}{n} R_n\right)$ is not a disk or it is not contained in $\bar{B}_{\Sigma(n)}\left(\vec{0}, \frac{1}{2} R_n\right)$.

Let $\widetilde{\Sigma}(n)$ be the sequence of rescaled disks $(n/R_n)\Sigma(n)$; see Figure 1. Note that, for all n, $\partial\widetilde{\Sigma}(n) \subset \partial\mathbb{B}(n)$ and either $\widetilde{\Sigma}(n)(\vec{0}, 1)$ is not a disk or it is not contained in $\bar{B}_{\widetilde{\Sigma}(n)}\left(\vec{0}, \frac{n}{2}\right)$.

After replacing by a subsequence, one of the following three cases holds:

  (i) $\lim_{n\to\infty} \max_{x\in\widetilde{\Sigma}(n)\cap\bar{\mathbb{B}}(1)} |A_{\widetilde{\Sigma}(n)}(x)| = 0$;

  (ii) $\lim_{n\to\infty} \max_{x\in\widetilde{\Sigma}(n)\cap\bar{\mathbb{B}}(1)} |A_{\widetilde{\Sigma}(n)}(x)| = \infty$;

  (iii) $\lim_{n\to\infty} \max_{x\in\widetilde{\Sigma}(n)\cap\bar{\mathbb{B}}(1)} |A_{\widetilde{\Sigma}(n)}(x)| = L \in (0, \infty)$.

First consider the case that $\lim_{n\to\infty} \max_{x\in\widetilde{\Sigma}(n)\cap\bar{\mathbb{B}}(1)} |A_{\widetilde{\Sigma}(n)}(x)| = 0$. Then, for n large, $\widetilde{\Sigma}(n)(\vec{0}, 1)$ is an almost totally geodesic disk whose diameter is bounded by 3 and that is a small graph over its projection to the unit disk in the (x_1, x_3)–plane, which gives a contradiction.
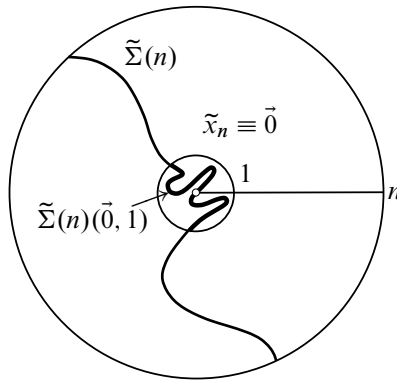
Figure 1: The component $\widetilde{\Sigma}(n)(\vec{0}, 1)$ inside the rescaled surface $\widetilde{\Sigma}(n)$

Next, suppose that $\lim_{n \to \infty} \max_{x \in \widetilde{\Sigma}(n) \cap \overline{\mathbb{B}}(1)} |A_{\widetilde{\Sigma}(n)}(x)| = \infty$ and, after going to a subsequence, let $p_n \in \widetilde{\Sigma}(n) \cap \overline{\mathbb{B}}(1)$ be a sequence of points such that

$$\lim_{n \to \infty} p_n = p \in \overline{\overline{\mathbb{B}}}(1) \quad \text{and} \quad \lim_{n \to \infty} |A_{\widetilde{\Sigma}(n)}(p_n)| = \infty.$$

Then we can apply Theorem 2.2 to the sequence of translated surfaces $\Sigma'(n) = \frac{1}{3}[\widetilde{\Sigma}(n) - p_n]$. In particular, since $\lim_{n \to \infty} |A_{\Sigma'(n)}(\vec{0})| = \infty$, case (B) of Theorem 2.2 applies. Note that the origin, as a point contained in $\widetilde{\Sigma}(n)$, has become the point $-\frac{1}{3} p_n$ in $\Sigma'(n)$ and, by our hypothesis, $\Sigma'(n)\left(-\frac{1}{3} p_n, \frac{1}{3}\right)$ is not a disk or it is not contained in $\overline{B}_{\Sigma'(n)}\left(-\frac{1}{3} p_n, \frac{n}{6}\right)$. By case (B) of Theorem 2.2, since $\left|\frac{1}{3} p_n\right| \leq \frac{1}{3}$, we have that

$$\Sigma'(n)\left(-\tfrac{1}{3} p_n, \tfrac{1}{3}\right) \subset [\Sigma'(n) \cap \overline{\mathbb{B}}(1)] \subset \mathcal{D}_n \subset \mathbb{B}(2),$$

where $\mathcal{D}_n$ is a disk with intrinsic diameter bounded by 3 and $\partial \mathcal{D}_n \subset \mathbb{B}(2) - \overline{\mathbb{B}}(1)$. Let Δ_n denote $\Sigma'(n)\left(-\frac{1}{3} p_n, \frac{1}{3}\right)$. Since $\Delta_n \subset \mathcal{D}_n$ and the intrinsic diameter of $\mathcal{D}_n$ is bounded by 3, in order to obtain a contradiction it suffices to prove that Δ_n is a disk. In the case where $\mathcal{D}_n$ intersects $\partial \mathbb{B}\left(-\frac{1}{3} p_n, \frac{1}{3}\right)$ transversely, Δ_n is a smooth compact surface and the following arguments can be simplified; therefore, on a first reading of the next paragraph the reader might want to consider this special generic case first.

Since Δ_n is a two-dimensional semianalytic set in $\mathbb{R}^3$ and $\Delta_n \cap \partial \mathbb{B}\left(-\frac{1}{3} p_n, \frac{1}{3}\right)$ is an analytic subset of the sphere $\partial \mathbb{B}\left(-\frac{1}{3} p_n, \frac{1}{3}\right)$, by [3] it follows that Δ_n admits a triangulation by analytic simplices, and the interiors of the 2–dimensional simplices are contained in $\mathbb{B}\left(-\frac{1}{3} p_n, \frac{1}{3}\right)$ because otherwise, by analyticity, $\Delta_n \subset \partial \mathbb{B}\left(-\frac{1}{3} p_n, \frac{1}{3}\right)$, which is false. Since the inclusion map of $\mathcal{D}_n$ is an injective immersion, then it follows that Δ_n is a semianalytic subset of $\mathcal{D}_n$ that can be triangulated with a finite number of closed 2–dimensional analytic simplices whose interiors are contained in $\Delta_n \cap \mathbb{B}\left(-\frac{1}{3} p_n, \frac{1}{3}\right)$ and $\Delta_n \cap \partial \mathbb{B}\left(-\frac{1}{3} p_n, \frac{1}{3}\right)$ is a connected 1–dimensional analytic

subset of $\mathcal{D}_n$, where we identify $\mathcal{D}_n$ with its image in $\mathbb{R}^3$; note that $\Delta_n \cap \partial\mathbb{B}\left(-\frac{1}{3}p_n, \frac{1}{3}\right)$ does not contain any isolated points by the mean curvature comparison principle. By the elementary topology of the disk $\mathcal{D}_n$ and using arguments as in [12], one can check that Δ_n fails to be a disk with piecewise smooth analytic boundary if and only if there exists a simple closed piecewise analytic curve $\Gamma(n)$ contained in the 1–dimensional simplicial subcomplex of $\Delta_n \cap \mathbb{B}\left(-\frac{1}{3}p_n, \frac{1}{3}\right)$ such that $\Gamma(n)$ does not bound a disk in $\Delta_n \cap \mathbb{B}\left(-\frac{1}{3}p_n, \frac{1}{3}\right)$. In the case that $\mathcal{D}_n$ is transverse to $\partial\mathbb{B}\left(-\frac{1}{3}p_n, \frac{1}{3}\right)$, we can choose $\Gamma(n)$ to be the boundary curve of a component of $\mathcal{D}_n \cap \left(\mathbb{R}^3 - \mathbb{B}\left(-\frac{1}{3}p_n, \frac{1}{3}\right)\right)$ that has its entire boundary in $\partial\mathbb{B}\left(-\frac{1}{3}p_n, \frac{1}{3}\right)$.

Arguing by contradiction, suppose Δ_n is not a compact disk. Let D_n denote the compact subdisk of $\mathcal{D}_n$ with boundary $\Gamma(n) \subset \mathcal{D}_n \cap \partial\mathbb{B}\left(-\frac{1}{3}p_n, \frac{1}{3}\right)$ and notice that $D_n \not\subset \overline{\mathbb{B}}\left(-\frac{1}{3}p_n, \frac{1}{3}\right)$. Hence, there is a point $q_n \in D_n$ that has maximal distance $T_n > \frac{1}{3}$ from the $-\frac{1}{3}p_n$. Since the boundary of D_n lies in $\partial B\left(-\frac{1}{3}p_n, \frac{1}{3}\right)$ and D_n lies in $\mathbb{R}^3 - \mathbb{B}\left(-\frac{1}{3}p_n, \frac{1}{3}\right)$ near ∂D_n, then q_n is an interior point of D_n not contained in $\mathbb{B}\left(-\frac{1}{3}p_n, \frac{1}{3}\right)$ and D_n lies inside the closed ball $\overline{\mathbb{B}}\left(-\frac{1}{3}p_n, T_n\right)$ and intersects $\partial\mathbb{B}\left(-\frac{1}{3}p_n, T_n\right)$ at the point q_n. By the mean curvature comparison principle applied at the point q_n, the constant mean curvature of $\mathcal{D}_n$ is at least $1/T_n$. By Theorem 2.2 the constant mean curvature values of the surfaces $\mathcal{D}_n$ are tending to zero as n goes to infinity (portions of the surfaces $\Sigma'(n)$ are converging to planes), so the interior points $q_n \in D_n \subset \mathcal{D}_n \subset \mathbb{B}(2)$ are diverging to infinity in $\mathbb{R}^3$ as n goes to infinity, which is a contradiction and proves that Δ_n must be a disk.

Finally, suppose that

$$\lim_{n\to\infty} \max_{x\in\widetilde{\Sigma}(n)\cap\overline{\mathbb{B}}(1)} |A_{\widetilde{\Sigma}(n)}(x)| = L \in (0, \infty)$$

and, after going to a subsequence, let $p_n \in \widetilde{\Sigma}(n) \cap \overline{\mathbb{B}}(1)$ be a sequence of points such that

$$\lim_{n\to\infty} p_n = p \in \overline{\mathbb{B}}(1) \quad \text{and} \quad \lim_{n\to\infty} |A_{\widetilde{\Sigma}(n)}(p_n)| = L.$$

In this case, after going to a subsequence, one of the following subcases holds:

(3A) $\lim_{n\to\infty} \max_{x\in\widetilde{\Sigma}(n)\cap\overline{\mathbb{B}}(2)} |A_{\widetilde{\Sigma}(n)}(x)| < \infty$;

(3B) $\lim_{n\to\infty} \max_{x\in\widetilde{\Sigma}(n)\cap\overline{\mathbb{B}}(2)} |A_{\widetilde{\Sigma}(n)}(x)| = \infty$.

If case (3A) holds, then by applying Theorem 2.2 to the sequence of translated surfaces $\widetilde{\Sigma}(n) - p_n$ we obtain that, after going to a subsequence, $\widetilde{\Sigma}(n) - p_n$ converges smoothly with multiplicity one or two on compact subsets of $\mathbb{R}^3$ to a helicoid containing the origin, which is a surface of negative curvature. This being the case then, for n sufficiently large,

$$|A_{\widetilde{\Sigma}(n)}(\vec{0})| > \varepsilon \in (0, 1)$$

and case (A) of Theorem 2.2 applies to the sequence of surfaces $\widetilde{\Sigma}(n)$. In particular, every component Δ of $\widetilde{\Sigma}(n) \cap \mathbb{B}(1)$ is an open disk whose closure $\overline{\Delta}$ in $\widetilde{\Sigma}(n) \cap \mathbb{B}(1)$ is a compact disk with piecewise smooth boundary, and where the intrinsic distance in $\widetilde{\Sigma}(n)$ between any two points in $\overline{\Delta}$ is less than 10. This contradicts our assumption.

If case (3B) holds then one can obtain a contradiction by arguing similarly to case (ii). This completes the proof that for some $\delta_2 \in \left(0, \frac{1}{2}\right)$ in the case $s = R$. This fixes the value of δ_2.

Fix $s \in (0, R)$. By the arguments in the previous case where $s = R$, $\Sigma(p, s)$ admits an analytic triangulation and it is the closure in Σ of a connected open surface. Hence by the elementary topology of a disk, the set $\widehat{\Sigma}[p, s] \subset \Sigma$ that is the closure of the complement of the annular component of $\Sigma - \Sigma(p, s)$ that contains $\partial \Sigma$ is a piecewise-smooth subdisk of Σ; also note that $p \in \widehat{\Sigma}[p, s]$ and $\partial \widehat{\Sigma}[p, s] \subset \partial \mathbb{B}(s)$. Applying the previously proved case where $s = R$ to the H–disk $\widehat{\Sigma}[p, s]$ and the subdomain $\widehat{\Sigma}[p, s](p, s)$ (which is equal to the domain $\Sigma(p, s)$), one has that $\Sigma(p, \delta_2 s)$ is a disk with $\partial \Sigma(p, \delta_2 s) \subset \partial \mathbb{B}(p, \delta_2 s)$ and $\Sigma(p, \delta_2 s) \subset \overline{B}_\Sigma\left(p, \frac{s}{2}\right)$.

This finishes the proof of the proposition. $\qquad\square$

2.2 Expanding the scale of being δ_2 weakly chord arc

We begin by giving a definition characterizing certain intrinsic geodesic balls of Σ.

Definition 2.5 (weakly chord arc) Given $\delta \in \left(0, \frac{1}{2}\right)$, an intrinsic ball $\overline{B}_\Sigma(x, R) \subset \Sigma$ is said to be δ *weakly chord arc* if:

  (i) For all $s \in (0, R)$, $\overline{B}_\Sigma(x, s) \subset \mathrm{Int}(\Sigma)$.

  (ii) For all $s \in (0, R]$,

     (a) $\Sigma(x, \delta s)$ is a disk;
     (b) $\Sigma(x, \delta s) \subset \overline{B}_\Sigma\left(x, \frac{s}{2}\right)$.

Remark 2.6 Suppose that $x \in \Sigma$. If an intrinsic ball $\overline{B}_\Sigma(x, R) \subset \Sigma - \partial \Sigma$, then for any $s \in (0, R)$, $\overline{B}_\Sigma(x, s)$ is contained in the interior of Σ. Also, if $\partial \Sigma \subset \partial \mathbb{B}(x, R)$, then Proposition 2.4 implies that $\overline{B}_\Sigma(x, R)$ is δ_2 weakly chord arc.

Definition 2.7 Given $\delta \in \left(0, \frac{1}{2}\right)$ and $x \in \Sigma - \partial \Sigma$,

$$R(x, \delta) = \sup\{R < \mathrm{dist}(x, \partial \Sigma) \mid \text{the ball } \overline{B}_\Sigma(x, R) \text{ is } \delta \text{ weakly chord arc}\}.$$

Our definition of the $R(x, \delta)$ function is the same as the one given in [5] and differs somewhat from the related $R_\delta(x)$ function defined in [2].

We now state and prove a proposition that in certain cases allows us to prove that if a given ball $\overline{B}_\Sigma(x, R)$ in Σ is δ_2 weakly chord arc, then $\overline{B}_\Sigma(x, 5R)$ is also δ_2 weakly chord arc; here, δ_2 is the constant defined in Proposition 2.4. The next result corresponds to the closely related [2, Proposition 3.4] and [5, Proposition 8].

Proposition 2.8 *There exists a constant $C_b > 5$ such that if $\overline{B}_\Sigma(y, C_b R) \subset \Sigma - \partial\Sigma$ satisfies*

"*every intrinsic subball $\overline{B}_\Sigma(z, R) \subset \overline{B}_\Sigma(y, C_b R)$ is δ_2 weakly chord arc*",

then $\overline{B}_\Sigma(y, 5R)$ is δ_2 weakly chord arc. In particular, $R(y, \delta_2) \geq 5R$.

Proof Arguing by contradiction, suppose that Proposition 2.8 fails. Then there exist a sequence of H_n–disks $\Sigma(n)$ and constants $C_n > 5n$ and $R_n > 0$ satisfying:

(i) $\overline{B}_{\Sigma(n)}(y_n, C_n R_n) \subset \Sigma(n) - \partial\Sigma(n)$.

(ii) Every intrinsic subball $\overline{B}_{\Sigma(n)}(z, R_n) \subset \overline{B}_{\Sigma(n)}(y_n, C_n R_n)$ is δ_2 weakly chord arc.

(iii) $\overline{B}_{\Sigma(n)}(y_n, 5R_n)$ is not δ_2 weakly chord arc.

Let us first assume that, after passing to a subsequence,

$$\Sigma(n)(y_n, 5R_n) \cap \partial B_{\Sigma(n)}(y_n, C_n R_n) = \varnothing \quad \text{for all } n.$$

Since $\overline{B}_{\Sigma(n)}(y_n, C_n R_n) \subset \Sigma(n) - \partial\Sigma(n)$, the above intersection equation implies $\Sigma(n)(y_n, 5R_n) \subset \Sigma(n) - \partial\Sigma(n)$. By the arguments in the last paragraph of the proof of Proposition 2.4, $\Sigma(n)$ contains a compact subdisk $\widehat{\Sigma}(n) \subset \Sigma - \partial\Sigma$ with $\partial\widehat{\Sigma}(n) \subset \partial\mathbb{B}(y_n, 5R_n)$ and $\Sigma(n)(y_n, 5R_n) = \widehat{\Sigma}(n)(y_n, 5R_n)$. Since

$$\overline{B}_{\Sigma(n)}(y_n, 5R_n) = \overline{B}_{\widehat{\Sigma}(n)}(y_n, 5R_n),$$

Remark 2.6 implies $\overline{B}_{\Sigma(n)}(y_n, 5R_n)$ is δ_2 weakly chord arc, which is a contradiction to item (iii) above. Hence, for the remainder of the proof we shall assume that

$$\Sigma(n)(y_n, 5R_n) \cap \partial B_{\Sigma(n)}(y_n, C_n R_n) \neq \varnothing \quad \text{for all } n.$$

Since $\Sigma(n)(y_n, 5R_n)$ is path connected, we can find a path $\gamma_n \subset \Sigma(n)(y_n, 5R_n)$ starting at y_n and ending at a point of $\partial B_{\Sigma(n)}(y_n, C_n R_n)$. Homothetically scale the surfaces $\Sigma(n)$ by $1/R_n$ from the points y_n to obtain new surfaces $\widetilde{\Sigma}(n)$ passing through y_n; we will use tilde to denote other related scaled objects as well. Balls of radius $C_n R_n$ then become balls of radius $C_n > 5n$, and balls of radius R_n become balls of radius 1.
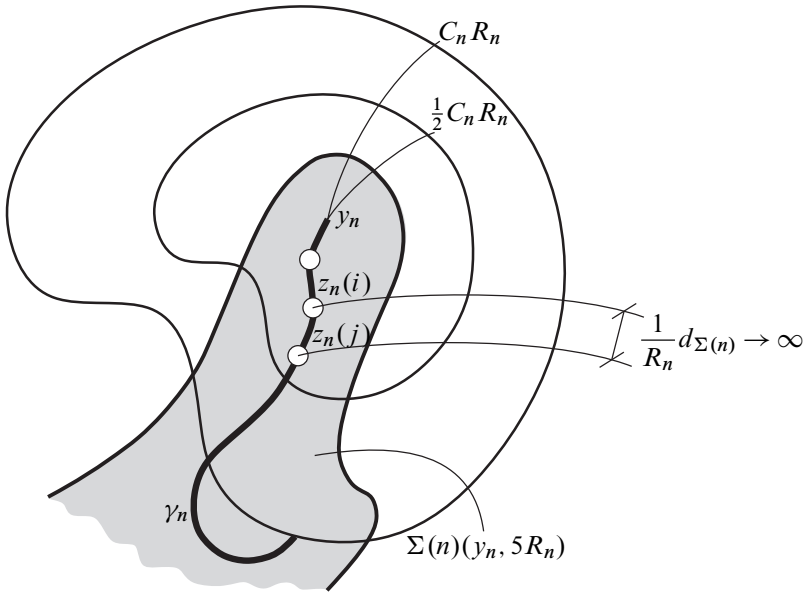
Figure 2: $d_{\Sigma(n)}$ denotes the intrinsic distance in $\Sigma(n)$.

The corresponding expanded path $\widetilde{\gamma}_n \subset \widetilde{\Sigma}(n)(\widetilde{y}_n, 5) \subset \overline{\mathbb{B}}(\widetilde{y}_n, 5)$ joins $\widetilde{y}_n = y_n$ with a point of $\partial \mathbb{B}(\widetilde{y}_n, 5)$ at an intrinsic distance C_n from $\widetilde{y}_n$. Since $\lim_{n\to\infty} C_n = \infty$, there exists a subset $\widetilde{S}_n = \{\widetilde{z}_n(1), \widetilde{z}_n(2), \ldots, \widetilde{z}_n(k(n))\} \subset \widetilde{\gamma}_n \cap \overline{B}_{\widetilde{\Sigma}(n)}(\widetilde{y}_n, \frac{1}{2}C_n)$ satisfying:

(P1) $\lim_{n\to\infty} k(n) = \infty$.

(P2) The intrinsic distance in $\widetilde{\Sigma}(n)$ between any two of the points of $\widetilde{S}_n$ tends to infinity as n goes to infinity.

In the original scale, we have corresponding finite sets

$$S_n = \{z_n(1), z_n(2), \ldots, z_n(k(n))\} \subset \gamma_n \cap \overline{B}_{\Sigma(n)}(y_n, \tfrac{1}{2}C_n R_n);$$

see Figure 2.

We claim that $\overline{B}_{\widetilde{\Sigma}(n)}(\widetilde{z}, 1)$ is δ_2 weakly chord arc for any $\widetilde{z} \in \widetilde{S}_n$, which after scaling, is equivalent to proving that $\overline{B}_{\Sigma(n)}(z, R_n)$ is δ_2 weakly chord arc. Since $\widetilde{z} \in \overline{B}_{\widetilde{\Sigma}(n)}(\widetilde{y}_n, \frac{1}{2}C_n)$, $\overline{B}_{\widetilde{\Sigma}(n)}(\widetilde{y}_n, C_n) \subset \widetilde{\Sigma}(n) - \partial\widetilde{\Sigma}(n)$ and $C_n > 5$, the intrinsic triangle inequality implies that $\overline{B}_{\widetilde{\Sigma}(n)}(\widetilde{z}, 1) \subset \overline{B}_{\widetilde{\Sigma}(n)}(\widetilde{y}_n, C_n)$ and so $\overline{B}_{\Sigma(n)}(z, R_n) \subset \overline{B}_{\Sigma(n)}(y_n, C_n R_n)$. The main hypothesis in the statement of the proposition implies under these conditions that $\overline{B}_{\Sigma(n)}(z, R_n)$ is δ_2 weakly chord arc. This proves the desired claim and so, in particular, $\widetilde{\Sigma}(n)(\widetilde{z}, \delta_2)$ is a disk contained in $\overline{B}_{\widetilde{\Sigma}(n)}(\widetilde{z}, \frac{1}{2})$.

By the last claim and condition (P2) above, the disks $\widetilde{\Sigma}(\widetilde{z}, \delta_2)$ are pairwise disjoint for distinct points $\widetilde{z}$ of $\widetilde{S}_n$ for n large. At this point it is useful to make another normalization of the surfaces $\Sigma(n)$ by translating them by the vector $-\vec{y}_n$ and so the points y_n are now equal to the origin: with this normalization, $\widetilde{y}_n = y_n = \vec{0}$. Note that for any $\widetilde{z} \in \widetilde{S}_n$, the disk $\widetilde{\Sigma}(n)(\widetilde{z}, \delta_2)$ is contained in $\mathbb{B}(6)$. This follows since $\widetilde{z} \in \overline{\mathbb{B}}(5)$ and $\delta_2 \in (0, \frac{1}{2})$. The number of disks $\widetilde{\Sigma}(n)(\widetilde{z}, \delta_2)$ centered at points $\widetilde{z} \in \widetilde{S}_n$ goes to infinity as n goes to infinity, by condition (P1) in our choice of the points. Hence, after replacing by a subsequence, there is a point $q \in \overline{\mathbb{B}}(5)$, where the number of points in $\mathbb{B}(q, \frac{1}{n}) \cap \widetilde{S}_n$ goes to infinity as n goes to infinity. In particular, we may assume that for each n, there exist three distinct points $\widetilde{z}_1(n)$, $\widetilde{z}_2(n)$ and $\widetilde{z}_3(n)$ in $\mathbb{B}(q, \frac{1}{n}) \cap \widetilde{S}_n$. Since the intrinsic distances between any two of these points is diverging to infinity and $\widetilde{\Sigma}(n)(\widetilde{z}_i(n), \delta_2) \subset \overline{B}_{\widetilde{\Sigma}(n)}(\widetilde{z}_i(n), \frac{1}{2})$ for each i, the disks $\{\widetilde{\Sigma}(n)(\widetilde{z}_i(n), \delta_2) \mid i = 1, 2, 3\}$ form a pairwise disjoint collection.

It follows that for n large, the boundaries of the disks $\{\widetilde{\Sigma}(n)(\widetilde{z}_i(n), \delta_2) \mid i = 1, 2, 3\}$ are contained in $\mathbb{R}^3 - \mathbb{B}(q, \frac{1}{2}\delta_2)$. Hence, by Corollary 2.3, there exists a small $\delta' > 0$ such that the components $\{\widehat{\Sigma}(n, i) \mid i = 1, 2, 3\}$ of $\widetilde{\Sigma}(n)(\widetilde{z}_i(n), \delta_2) \cap \overline{\mathbb{B}}(q, \delta')$ containing the respective points $\widetilde{z}_i(n)$ have second fundamental forms bounded by a universal constant and so, after possibly replacing δ' by a smaller positive number, these components are disks which are graphical over their projections to a plane passing through q. After replacing by another subsequence and after reindexing, a sequence of pairs $\widehat{\Sigma}(n, 1)$, $\widehat{\Sigma}(n, 2)$ of these graphs converges to a stable compact H–disk $\mathcal{D}$ passing through q, for some value of H. Moreover, we will assume that the inner products of the unit normal vectors of $\widehat{\Sigma}(n, 1)$ and $\widehat{\Sigma}(n, 2)$ are positive, even when $H = 0$. Repeated applications of Corollary 2.3 together with a prolongation argument, as carried out in the proof of [2, Proposition 3.4] and in the proof of [5, Proposition 8], demonstrates that $\mathcal{D}$ is contained in the image of a complete immersion $f \colon F \looparrowright \mathbb{R}^3$ of constant mean curvature $H \geq 0$ of bounded norm of the second fundamental form for some complete surface F. Indeed, after possibly replacing by a subsequence and reindexing, the sequence of surfaces

$$\overline{B}_{\widetilde{\Sigma}(n)}(\widetilde{z}_1(n), n) \cup \overline{B}_{\widetilde{\Sigma}(n)}(\widetilde{z}_2(n), n)$$

converges smoothly to $f(F)$ with multiplicity at least two.

Since $f(F)$ is a limit of embedded surfaces, f satisfies the additional property that it is almost-embedded in the sense that if $p_1 \neq p_2$ are points in F with the same image p in $f(F)$, then images of small intrinsic neighborhoods of p_1 and p_2 locally lie on one side of each other at p; note that by the maximum principle for H–surfaces, this nonembeddedness property of F can only occur if $H > 0$ and the mean curvature vectors at p of these two respective neighborhoods are negatives of each other. Since

complete almost-embedded H–surfaces of bounded norm of the second fundamental form are properly immersed in $\mathbb{R}^3$ by [7, Corollary 2.5] (also see [6, Theorem 6.1]), f is a proper immersion of F into $\mathbb{R}^3$.

If the universal cover of the limit surface $f(F)$ is a stable H–surface, then $f(F)$ is a plane [4; 13] that intersects $\bar{\bar{\mathbb{B}}}(5)$. But if the image surface $f(F)$ is a plane P, then for n sufficiently large, $\bar{B}_{\widetilde{\Sigma}(n)}(\tilde{z}_1(n), 11)$ would become arbitrarily close to a planar disk of radius 11 contained in P and which intersects $\bar{\bar{\mathbb{B}}}(5)$, and so $\partial B_{\widetilde{\Sigma}(n)}(\tilde{z}_1(n), 11) \subset [\mathbb{R}^3 - \bar{\bar{\mathbb{B}}}(5)]$. Since the intrinsic distance between $\tilde{z}_1(n)$ and $\tilde{z}_2(n)$ is going to infinity as n goes to infinity, this implies that $\tilde{z}_1(n)$ and $\tilde{z}_2(n)$ cannot be connected by a curve in $\widetilde{\Sigma}(n) \cap \bar{\mathbb{B}}(5)$. This contradiction would prove the proposition. Thus, it suffices to prove the following claim:

Claim 2.9 *The universal cover of F is stable.*

Proof Let $\Pi\colon \langle \widetilde{F}, \tilde{p} \rangle \to \langle F, p \rangle$ denote the universal cover of the pointed surface $\langle F, p \rangle$, where $f(\Pi(\tilde{p})) = f(p) = q \in \mathcal{D}$. The proof of this claim uses standard arguments to construct a nonzero Jacobi field on an arbitrary open connected subset $\Omega \subset \widetilde{F}$ with compact closure. Since the norm of the second fundamental form of $\widetilde{F}$ is bounded, there is a normal disk bundle $\mathcal{N}$ of fixed radius that submerses in $\mathbb{R}^3$. Let $i\colon \mathcal{N} \to \mathbb{R}^3$ denote the submersion; then we give $\mathcal{N}$ the flat metric induced by i. We consider the surface $\widetilde{F}$ to be the zero section of $\mathcal{N}$ and the map $\Gamma\colon \mathcal{N} \to \widetilde{F}$ given by the nearest point projection is smooth. Let ξ denote the unit normal vector field to $\widetilde{F}$.

Let Δ_n be the lift (preimage) in $\mathcal{N}$ of

$$[\bar{B}_{\widetilde{\Sigma}(n)}(\tilde{z}_1(n), n) \cup \bar{B}_{\widetilde{\Sigma}(n)}(\tilde{z}_2(n), n)] \cap i(\mathcal{N})$$

via the submersion i, where we can make choices of preimages $z_1'(n) \in i^{-1}(\tilde{z}_1(n))$ and $z_2'(n) \in i^{-1}(\tilde{z}_2(n))$ that converge to the same point $q' \in i^{-1}(q)$. By the nature of the convergence, Δ_n converges smoothly to $\widetilde{F}$ in $\mathcal{N}$ with multiplicity at least two. Namely, each point $p \in \widetilde{F}$ has a neighborhood U_p which is the uniform limit of at least two disjoint domains $U_{p,1}(n)$ and $U_{p,2}(n)$ in Δ_n. Each of these domains is a graph over U_p via the nearest point projection Γ and the normal vectors of such graphs at related points have inner products converging to 1 as n goes to infinity.

Let $\Omega \subset \widetilde{F}$ be an arbitrary open connected subset with compact closure and let $\widetilde{\Omega} \subset \widetilde{F}$ be a precompact, simply connected domain containing Ω and the point q'. The usual holonomy construction and the convergence with multiplicity at least two gives that there exist two disjoint domains $\Omega_1(n)$ and $\Omega_2(n)$ in Δ_n such that the following holds:

each $\Omega_i(n)$ is a graph over $\widetilde{\Omega}$ via the nearest point projection, and $z_1'(n) \in \Omega_1(n)$ and $z_2'(n) \in \Omega_2(n)$. Namely, there exists $u_i^n \colon \widetilde{\Omega} \to \mathbb{R}$ such that

$$\Omega_i(n) = \{ p + u_i^n(p)\xi(p) \mid p \in \Omega \}.$$

Because $\Omega_1(n)$ and $\Omega_2(n)$ are disjoint, we can assume that $u_2^n > u_1^n$. Moreover, by our previous choice of the points $\widetilde{z}_1(n)$ and $\widetilde{z}_2(n)$, we may assume that the unit normal vectors of $\Omega_1(n)$ and of $\Omega_2(n)$ at corresponding points over points of Ω have positive inner products converging to 1 as n goes to infinity. A standard compactness argument using the Harnack inequality shows that the positive function

$$\frac{u_2^n - u_1^n}{u_2^n(q') - u_1^n(q')}$$

converges to a positive Jacobi function w over $\widetilde{\Omega}$. Thus $w|_\Omega$ is a positive Jacobi function over Ω, which implies that Ω is stable. Since Ω was an arbitrary precompact domain in F, this finishes the proof of the claim. $\qquad\square$

As mentioned previously, Claim 2.9 completes the proof of the proposition. $\qquad\square$

2.3 The function a_δ

For the remainder of Section 2, Σ will be the H–disk in the statement of Theorem 1.2. We claim that if the theorem holds whenever Σ is a smooth H–disk, then it holds in general. To see that this claim holds, assume $\delta_1 \in \left(0, \frac{1}{2}\right)$ is such that Theorem 1.2 holds for smooth H–disks and let Σ be an H–disk which is nonsmooth along its boundary. Suppose $\overline{B}_\Sigma(x, R) \subset \Sigma - \partial\Sigma$ and consider the conditions below:

(C1) $\Sigma(x, \delta_1 R)$ is a disk with piecewise smooth boundary $\partial\Sigma(\vec{0}, \delta_1 R) \subset \partial\mathbb{B}(\delta_1 R)$.

(C2) $\Sigma(x, \delta_1 R) \subset \overline{B}_\Sigma\left(x, \frac{R}{2}\right)$.

The compact intrinsic ball $\overline{B}_\Sigma(x, R)$ is contained in the interior of a smooth sub-H–disk $\Sigma' \subset (\Sigma - \partial\Sigma)$, so $\overline{B}_\Sigma(x, R) = \overline{B}_{\Sigma'}(x, R)$ and $\Sigma(x, \delta_1 R) = \Sigma'(x, \delta_1 R)$. Using that Theorem 1.2 holds for Σ' then implies that the two conditions (C1) and (C2) hold. So henceforth we will assume that Σ is a *smooth H–disk*.

We claim that for $\delta \in \left(0, \frac{1}{2}\right)$, the function

$$G_\delta(z) = \frac{d_\Sigma(z, \partial\Sigma)}{R(z, \delta)} \colon \Sigma - \partial\Sigma \to (0, \infty)$$

is bounded on $\Sigma - \partial\Sigma$ and is equal to 1 in some small neighborhood of $\partial\Sigma$. To see this, first note that if $p \in \Sigma - \partial\Sigma$ has distance at least ε from $\partial\Sigma$ for some $\varepsilon > 0$, then $R(p, \delta)$ is greater than some positive constant that only depends on ε and Σ. This is because the

norm of the second fundamental form of Σ is bounded and so for any $\varepsilon' < \varepsilon$ sufficiently small, there exists $\varepsilon'' < \varepsilon'$ such that $\bar{B}_\Sigma(p, \varepsilon')$ is a graph over its projection to its tangent plane at p with small norm of its gradient and $\Sigma\left(p, \frac{1}{2}\varepsilon'\right) \subset \bar{B}_\Sigma\left(p, \frac{1}{2}\varepsilon' + \varepsilon''\right)$, with $\lim_{\varepsilon' \to 0} \varepsilon''/\varepsilon' = 0$. Since $\delta < \frac{1}{2}$, $R(p, \delta)$ is bounded from below outside of any small ε–regular neighborhood of $\partial\Sigma$. On the other hand, since the geodesic curvature of $\partial\Sigma$ and the norm of the second fundamental form are both bounded, the same argument shows that for some sufficiently small $\varepsilon > 0$, $R(p, \delta)$ is equal to $d_\Sigma(p, \partial\Sigma)$, when $p \in \Sigma - \partial\Sigma$ and $d_\Sigma(p, \partial\Sigma) < \varepsilon$. This proves that the function G is bounded and is equal to 1 in some neighborhood of $\partial\Sigma$.

Definition 2.10 Let $\delta \in \left(0, \frac{1}{2}\right)$. Then we define

$$ a_\delta = \sup_{z \in (\Sigma - \partial\Sigma)} \frac{d_\Sigma(z, \partial\Sigma)}{R(z, \delta)} = \sup(G_\delta). $$

The next lemma and its proof correspond to [5, Lemma 11].

Lemma 2.11 Let $\delta' \in \left(0, \frac{1}{2}\right)$. If $a_{\delta'} < c$ with $c \in [2, \infty)$, then Theorem 1.2 holds for Σ with $\delta_1 = \delta'/c$.

Proof Suppose that $\bar{B}_\Sigma(x, R) \subset \Sigma - \partial\Sigma$ for some $R > 0$ and $x \in \Sigma$. By definition of $a_{\delta'}$,

$$ a_{\delta'} \geq \frac{d_\Sigma(x, \partial\Sigma)}{R(x, \delta')} > \frac{R}{R(x, \delta')}, $$

which implies that $R(x, \delta') > \frac{R}{c}$. Since $R(x, \delta') > \frac{R}{c}$, the definition of $R(x, \delta')$ implies that $\Sigma\left(x, \delta'\frac{R}{c}\right)$ is a disk and

$$ \Sigma\left(x, \delta'\frac{R}{c}\right) \subset \bar{B}_\Sigma\left(x, \frac{1}{2} \cdot \frac{R}{c}\right). $$

Thus, since $\Sigma\left(x, \frac{1}{c}\delta'R\right) = \Sigma\left(x, \delta'\frac{R}{c}\right)$ and $\frac{R}{c} < R$, we conclude that $\Sigma\left(x, \frac{1}{c}\delta'R\right)$ is a disk and

$$ \Sigma\left(x, \frac{\delta'}{c}R\right) \subset \bar{B}_\Sigma\left(x, \frac{1}{2}R\right), $$

and so Theorem 1.2 holds for Σ with $\delta_1 = \frac{1}{c}\delta'$. $\qquad \square$

2.4 Locating the smallest scale which is not δ weakly chord arc

The proof of the next lemma uses a standard technique for finding a smallest scale for which some property holds on a surface. The property we are considering here is that of being δ weakly chord arc. In this case we take the proof directly from the proof of the similar [2, Lemma 3.39].

Lemma 2.12 Let $\delta \in \left(0, \frac{1}{2}\right)$. Then there exists a point $y \in \Sigma$ and a number $R_1 > 0$ such that:

(i) $a_\delta R_1 < \frac{1}{2} d_\Sigma(y, \partial\Sigma)$.

(ii) $R(x, \delta) > R_1$ for every $x \in \bar{B}_\Sigma(y, a_\delta R_1)$.

(iii) $\bar{B}_\Sigma(y, 5R_1)$ is **not** δ weakly chord arc.

Proof Recall the function G_δ on $\Sigma - \partial\Sigma$ is defined by $G_\delta(x) = d_\Sigma(x, \partial\Sigma)/R(x, \delta)$ and extends to a bounded function on Σ which has a constant value 1 near $\partial\Sigma$. Thus, $a_\delta = \sup(G_\delta)$ is a finite number that is greater than or equal to 1. Choose y to be a point in $\Sigma - \partial\Sigma$ such that $G_\delta(y)$ is greater than $\frac{1}{2} a_\delta$. Hence, if we define $d_\partial = d_\Sigma(y, \partial\Sigma)$, then $\frac{1}{2} a_\delta < d_\partial/R(y, \delta) = G_\delta(y)$, or equivalently,

$$(2) \qquad\qquad\qquad a_\delta R(y, \delta) < 2d_\partial.$$

Now choose $R_1 = \frac{1}{4} R(y, \delta)$ and we will show this definition of R_1 satisfies the statements in the lemma. This value of R_1 and (2) give the inequality $a_\delta R_1 < \frac{1}{2} d_\partial$, which is statement (i) in the lemma. By definition of R_1, $R(y, \delta) = 4R_1$ and by the definition of $R(y, \delta)$ as a supremum, the ball $\bar{B}_\Sigma(y, 5R_1)$ is not δ weakly chord arc, which proves (iii).

By (i), $a_\delta R_1 < \frac{1}{2} d_\partial$, and so, $\bar{B}_\Sigma(y, a_\delta R_1) \subset \bar{B}_\Sigma\left(y, \frac{1}{2} d_\partial\right)$. So if we check that (ii) holds for points in $\bar{B}_\Sigma\left(y, \frac{1}{2} d_\partial\right)$, then (ii) holds. If $x \in \bar{B}_\Sigma\left(y, \frac{1}{2} d_\partial\right)$, then by the triangle inequality, $\frac{1}{2} d_\partial \leq d_\Sigma(x, \partial\Sigma)$. This inequality, the definition of G_δ and the choice of y give the inequalities

$$\frac{d_\partial}{2R(x, \delta)} \leq \frac{d_\Sigma(x, \partial\Sigma)}{R(x, \delta)} = G_\delta(x) \leq a_\delta < 2G_\delta(y) = \frac{2d_\partial}{R(y, \delta)}.$$

Therefore, $R(x, \delta) > \frac{1}{4} R(y, \delta) = R_1$. This completes the proof of statement (ii) and the lemma now follows.   □

2.5 The proof of Theorem 1.2

We now prove Theorem 1.2. By Lemma 2.11, we just need to prove that a_δ is bounded independently of Σ for some fixed constant $\delta \in \left(0, \frac{1}{2}\right)$.

Let $\delta = \delta_2$, where δ_2 is as given in Proposition 2.4. We now prove a_δ is bounded from above by C_b, where C_b is as given in Proposition 2.8. Suppose there exists a Σ with $a_\delta > C_b$.

By Lemma 2.12, there exists a point $y \in \Sigma$ and an R_1 such that:

(i) $\bar{B}_\Sigma(y, a_\delta R_1) \subset \bar{B}_\Sigma\left(y, \frac{1}{2} d_\Sigma(y, \partial\Sigma)\right)$.

(ii) $R(x, \delta) > R_1$ for every $x \in \bar{B}_\Sigma(y, a_\delta R_1)$.

(iii) $\bar{B}_\Sigma(y, 5R_1)$ is *not* δ weakly chord arc.

By the definition of $R(x, \delta)$ and statement (ii), we have that $\bar{B}_\Sigma(x, R_1)$ is δ weakly chord arc for every $x \in \bar{B}_\Sigma(y, C_b R_1) \subset \bar{B}_\Sigma(y, a_\delta R_1)$. But Proposition 2.8 implies that $\bar{B}_\Sigma(y, 5R_1)$ is δ weakly chord arc, contradicting statement (iii) above. This contradiction completes the proof of Theorem 1.2.

3   Applications of Theorem 1.2

The next result gives a useful intrinsic one-sided curvature estimate; its proof uses Theorem 1.2 and the extrinsic one-sided curvature estimate given in Theorem 2.1. This next result is also stated as [8, Theorem 4.5]; in the case that $H = 0$, the next theorem follows from [2, Corollary 0.8].

Theorem 3.1 (intrinsic one-sided curvature estimate for H–disks)   *There exist $\varepsilon_I \in \left(0, \frac{1}{2}\right)$ and $C_I \geq 2\sqrt{2}$ such that for any $R > 0$, the following holds: Let $\mathcal{D}$ be an H–disk such that*

$$\mathcal{D} \cap \mathbb{B}(R) \cap \{x_3 = 0\} = \varnothing$$

and $x \in \mathcal{D} \cap \mathbb{B}(\varepsilon_I R)$, where $d_\mathcal{D}(x, \partial \mathcal{D}) \geq R$. Then

$$|A_\mathcal{D}|(x) \leq \frac{C_I}{R}. \tag{3}$$

In particular, $H < C_I / R$.

Proof   Let ε and C be the constants given in Theorem 2.1 and let δ_1 be the constant given in Theorem 1.2. We next check that the constants $\varepsilon_I = \delta_1 \varepsilon$ and $C_I = 2C/\delta_1$ satisfy the conditions in the theorem.

Without loss of generality, we may assume that $x \in \mathcal{D} \cap \mathbb{B}(\varepsilon_I R) \cap \{x_3 > 0\}$, where $d_\mathcal{D}(x, \partial \mathcal{D}) \geq R$. By Theorem 1.2, the surface $\mathcal{D}' := \Sigma(x, \delta_1 R) \subset \mathcal{D}$ is an H–disk with its boundary in $\partial \mathbb{B}(x, \delta_1 R)$. Since $\varepsilon \in \left(0, \frac{1}{2}\right)$, we have $d_{\mathbb{R}^3}(x, \vec{0}) < \varepsilon \delta_1 R < \frac{1}{2}\delta_1 R$, and so the triangle inequality implies $\mathbb{B}\left(\frac{1}{2}\delta_1 R\right) \cap \partial \mathbb{B}(x, \delta_1 R) = \varnothing$. Hence, $\Sigma(x, \delta_1 R)$ must have its boundary in $\mathbb{R}^3 - \mathbb{B}\left(\frac{1}{2}\delta_1 R\right)$. Therefore, the scaled surface $(2/\delta_1)\mathcal{D}'$ satisfies the conditions of the disk described in Theorem 2.1; in other words,

$$\left(\frac{2}{\delta_1}\mathcal{D}'\right) \cap \mathbb{B}(R) \cap \{x_3 = 0\} = \varnothing \quad \text{and} \quad \partial\left(\frac{2}{\delta_1}\mathcal{D}'\right) \cap \mathbb{B}(R) \cap \{x_3 > 0\} = \varnothing.$$

Since the scaled point $(2/\delta_1)x \in ((2/\delta_1)\mathcal{D}') \cap \mathbb{B}(\varepsilon) \cap \{x_3 > 0\}$, Theorem 2.1 gives $|A_{(2/\delta_1)\mathcal{D}'}|((2/\delta_1)x) \leq \frac{C}{R}$, and so

$$|A_{\mathcal{D}'}|(x) \leq \frac{2}{\delta_1}\frac{C}{R} = \frac{C_I}{R},$$

which completes the proof of the theorem. □

The following result is a direct consequence of some of the arguments in the proof of Theorem 1.2.

Theorem 3.2 *Given $\varepsilon > 0$ and $m \in (0, \infty)$, there exists $R(m, \varepsilon) > m$ such that the following holds: Let Σ be a complete H–surface with boundary such that for any $x \in \Sigma$,*

$$\mathrm{Inj}_\Sigma(x) \geq \min\{\varepsilon, d_\Sigma(x, \partial\Sigma)\}.$$

If $\bar{B}_\Sigma(y, R) \subset \Sigma - \partial\Sigma$ with $R \geq R(m, \varepsilon)$, then

$$\Sigma(y, m) \subset \bar{B}_\Sigma\left(y, \frac{R}{2}\right).$$

Proof Arguing by contradiction, suppose there exist $\varepsilon > 0$ and $m \in (0, \infty)$ such that for any $n > 1$, there exists a compact H_n–surface $\Sigma(n)$ with

$$\mathrm{Inj}_{\Sigma(n)}(x) \geq \min\{\varepsilon, d_{\Sigma(n)}(x, \partial\Sigma(n))\},$$

and $y_n \in \Sigma(n)$ such that $\bar{B}_\Sigma(y_n, n) \subset \Sigma - \partial\Sigma$ but

$$\Sigma(n)(y_n, m) \not\subset \bar{B}_{\Sigma(n)}\left(y_n, \frac{n}{2}\right).$$

We now follow the arguments in the proof of Proposition 2.8 to give a sketch of the proof, leaving the details to the reader. Without loss of generality, after rescaling by $\frac{1}{m}$ and normalizing by translations, we can assume that $m = 1$ and that $y_n = \vec{0}$. Also we may assume that $\varepsilon \in (0, 1)$, since if the theorem holds for a smaller positive choice of ε, then it holds for the original choice.

Since $\Sigma(n)(\vec{0}, 1)$ is path connected, we can find an embedded path $\gamma_n \subset \Sigma(n)(\vec{0}, 1)$ starting at $\vec{0}$ and ending at some point of $\partial B_{\Sigma(n)}(\vec{0}, \frac{n}{2})$. As n goes to infinity, there exists a subset $S_n = \{z_n(1), \ldots, z_n(k(n))\} \subset \gamma_n$ with

(i) $\lim_{n\to\infty} k(n) = \infty$.

(ii) The intrinsic distance in $\Sigma(n)$ between any two of the points of S_n tends to infinity as n goes to infinity.

The hypothesis $\mathrm{Inj}_\Sigma(x) \geq \min\{\varepsilon, d_\Sigma(x, \partial\Sigma)\}$ implies that if $d_{\Sigma(n)}(x, \partial\Sigma(n)) > \varepsilon$ for some $x \in \Sigma(n)$, then the geodesic ball $\bar{B}_{\Sigma(n)}(x, \varepsilon)$ is a disk. Thus, Theorem 1.2 gives that $\Sigma(n)(x, \delta_1\varepsilon)$ is a compact disk with piecewise smooth boundary in $\partial\mathbb{B}(x, \delta_1\varepsilon)$ and $\Sigma(n)(x, \delta_1\varepsilon) \subset \bar{B}_{\Sigma(n)}\left(x, \frac{\varepsilon}{2}\right)$. In particular this is true for any $z \in S_n$. Recall that the number of disks $\Sigma(n)(z, \delta_1\varepsilon)$ centered at points z in S_n diverges as $n \to \infty$. Hence, after reindexing and replacing by a subsequence, there is a point $q \in \bar{\bar{\mathbb{B}}}(1)$, where the number of points in $\mathbb{B}\left(q, \frac{1}{n}\right) \cap S_n$ goes to infinity as n goes to infinity. In particular, we may assume that for each n, there exist three distinct points $z_1(n), z_2(n), z_3(n)$ in $\mathbb{B}\left(q, \frac{1}{n}\right) \cap S_n$. Since the intrinsic distances between any two of these points is diverging to infinity and $\Sigma(n)(z_i(n), \delta_1\varepsilon) \subset \bar{B}_{\Sigma(n)}\left(z_i(n), \frac{\varepsilon}{2}\right)$ for $i \in \{1, 2, 3\}$, the disks $\{\Sigma(n)(z_i(n), \delta_1\varepsilon) \mid i = 1, 2, 3\}$ form a pairwise disjoint collection. Arguing similarly as in the prolongation argument in the proof of Proposition 2.8, in the limit we obtain a complete stable minimal surface F (which is a plane), which can be used to prove that the points $z_1(n), z_2(n), z_3(n)$ can not be contained in the embedded arc $\gamma_n \subset \bar{\bar{\mathbb{B}}}(1)$. This gives a contradiction and completes sketch of the proof of the theorem. $\qquad\square$

Remark 3.3 Theorem 3.2 can be improved in various ways. For example, Theorem 1.2 implies that the hypothesis

$$\mathrm{Inj}_\Sigma(x) \geq \min\{\varepsilon, d_\Sigma(x, \partial\Sigma)\}$$

can be replaced by the weaker condition that there exists an $\varepsilon > 0$ such that for all $x \in \Sigma$ such that $d_\Sigma(x, \partial\Sigma) > \varepsilon$, the intrinsic ball $\bar{B}_\Sigma(x, \varepsilon)$ is contained in a simply connected subdomain of Σ.

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Department of Mathematics, University of Massachusetts
Amherst, MA, United States

Department of Mathematics, King's College London
London, United Kingdom

profmeeks@gmail.com,   giuseppe.tinaglia@kcl.ac.uk