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Gromov–Witten invariants of the Hilbert schemes of points of a K3 surface

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We study the enumerative geometry of rational curves on the Hilbert schemes of points of a K3 surface.

Let S be a K3 surface and let $\mathsf{Hilb}^d(S)$ be the Hilbert scheme of d points of S. In the case of elliptically fibered K3 surfaces $S \to \mathbb{P}^1$, we calculate genus-0 Gromov–Witten invariants of $\mathsf{Hilb}^d(S)$, which count rational curves incident to two generic fibers of the induced Lagrangian fibration $\mathsf{Hilb}^d(S) \to \mathbb{P}^d$. The generating series of these invariants is the Fourier expansion of a power of the Jacobi theta function times a modular form, hence of a Jacobi form.

We also prove results for genus-0 Gromov–Witten invariants of $\mathsf{Hilb}^d(S)$ for several other natural incidence conditions. In each case, the generating series is again a Jacobi form. For the proof we evaluate Gromov–Witten invariants of the Hilbert scheme of two points of $\mathbb{P}^1 \times E$, where E is an elliptic curve.

Inspired by our results, we conjecture a formula for the quantum multiplication with divisor classes on $\mathsf{Hilb}^d(S)$ with respect to primitive curve classes. The conjecture is presented in terms of natural operators acting on the Fock space of S. We prove the conjecture in the first nontrivial case $\mathsf{Hilb}^2(S)$. As a corollary, we find that the full genus-0 Gromov–Witten theory of $\mathsf{Hilb}^2(S)$ in primitive classes is governed by Jacobi forms.

We present two applications. A conjecture relating genus-1 invariants of $\mathsf{Hilb}^d(S)$ to the Igusa cusp form was proposed in joint work with R Pandharipande. Our results prove the conjecture when $d = 2$. Finally, we present a conjectural formula for the number of hyperelliptic curves on a K3 surface passing through two general points.

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0 Introduction

0.1 The Yau–Zaslow formula

Let S be a smooth projective K3 surface, and let $\beta_h \in H_2(S, \mathbb{Z})$ be a primitive effective curve class of self-intersection $\beta_h^2 = 2h-2$. The Yau–Zaslow formula [42] predicts the number N_h of rational curves in class β_h via the generating series

$$\sum_{h \geq 0} \mathsf{N}_h q^{h-1} = \frac{1}{q} \prod_{m \geq 1} \frac{1}{(1-q^m)^{24}}. \tag{1}$$

The right-hand side is the reciprocal of the Fourier expansion of a classical modular form of weight 12, the modular discriminant

$$\Delta(\tau) = q \prod_{m \geq 1} (1-q^m)^{24}, \tag{2}$$

where $q = \exp(2\pi i \tau)$ and $\tau \in \mathbb{H}$. The prediction (1) was proven by Beauville [1] and Chen [6] using the compactified Jacobian, and by Bryan and Leung [3] using Gromov–Witten theory. It is the starting point of further research into the enumerative geometry of algebraic curves on K3 surfaces; see for example Katz, Klemm and Pandharipande [18], Maulik, Pandharipande and Thomas [30] and Pandharipande and Thomas [37].

The Hilbert scheme of d points on S, denoted

$$\mathsf{Hilb}^d(S),$$

is the moduli space of zero-dimensional subschemes of S of length d; see Lehn [22] and Nakajima [32] for an introduction. It is a nonsingular projective variety of dimension $2d$, which is simply connected and carries a holomorphic symplectic form. For $d = 1$ we recover the original surface,

$$\mathsf{Hilb}^1(S) = S,$$

while for $d \geq 2$ the Hilbert schemes $\mathsf{Hilb}^d(S)$ may be thought of as analogues of K3 surfaces in higher dimensions.

In this paper we study the enumerative geometry of rational curves on the Hilbert scheme of points $\mathsf{Hilb}^d(S)$ for all $d \geq 1$. In particular, we obtain a generalization of the Yau–Zaslow formula (1).

0.2 Gromov–Witten invariants

For all $\alpha \in H^*(S;\mathbb{Q})$ and $i > 0$ let

$$\mathfrak{p}_{-i}(\alpha)\colon H^*(\mathsf{Hilb}^d(S);\mathbb{Q}) \to H^*(\mathsf{Hilb}^{d+i}(S);\mathbb{Q}), \quad \gamma \mapsto \mathfrak{p}_{-i}(\alpha)\gamma$$

be the Nakajima creation operator obtained by adding length-i punctual subschemes incident to a cycle Poincaré dual to α. By Grojnowski [15] and Nakajima [32, Corollary 8.16], the cohomology of $\mathsf{Hilb}^d(S)$ is completely described by the cohomology of S via the action of the operators $\mathfrak{p}_{-i}(\alpha)$ on the vacuum vector

$$1_S \in H^*(\mathsf{Hilb}^0(S);\mathbb{Q}) = \mathbb{Q}.$$

Let ω be the class of a point on S. For $\beta \in H_2(S;\mathbb{Z})$, define the class

$$C(\beta) = \mathfrak{p}_{-1}(\beta)\mathfrak{p}_{-1}(\omega_S)^{d-1}1_S \in H_2(\mathsf{Hilb}^d(S);\mathbb{Z}).$$

If $\beta = [C]$ for a curve $C \subset S$, then $C(\beta)$ is the class of the curve obtained by fixing $d-1$ distinct points away from C and letting a single point move on C. For brevity, we often write β for $C(\beta)$. For $d \geq 2$ let

$$A = \mathfrak{p}_{-2}(\omega_S)\mathfrak{p}_{-1}(\omega_S)^{d-2}1_S$$

be the class of an exceptional curve, ie the locus of 2–fat points centered at a point $P \in S$ plus $d-2$ distinct points away from P. For $d \geq 2$ we have

$$H_2(\mathsf{Hilb}^d(S);\mathbb{Z}) = \{\beta + kA \mid \beta \in H_2(S;\mathbb{Z}), k \in \mathbb{Z}\}.$$

Let $\beta + kA \in H_2(\mathsf{Hilb}^d(S))$ be a nonzero effective curve class and consider the moduli space

$$\overline{M}_{g,m}(\mathsf{Hilb}^d(S), \beta + kA) \tag{3}$$

of m–marked stable maps[1] $f\colon C \to \mathsf{Hilb}^d(S)$ of genus g and class $\beta + kA$. Since $\mathsf{Hilb}^d(S)$ carries a holomorphic symplectic 2–form, the virtual class on (3) defined by ordinary Gromov–Witten theory vanishes; see Kiem and Li [20]. A modified reduced theory was defined in Maulik and Pandharipande [29] and gives rise to a nonzero *reduced* virtual class

$$[\overline{M}_{g,m}(\mathsf{Hilb}^d(S), \beta + kA)]^{\mathrm{red}}$$

of dimension $(1-g)(2d-3) + m + 1$; see also Pridham [39] and Schürg, Toën and Vezzosi [41]. Let

$$\mathrm{ev}_i\colon \overline{M}_{g,m}(\mathsf{Hilb}^d(S), \beta + kA) \to \mathsf{Hilb}^d(S) \quad \text{for } i = 1, \dots, n$$

be the evaluation maps. The *reduced Gromov–Witten invariant* of $\mathsf{Hilb}^d(S)$ in genus g and class $\beta + kA$ with primary insertions

$$\gamma_1, \dots, \gamma_m \in H^*(\mathsf{Hilb}^d(S))$$

is defined by

$$\langle \gamma_1, \dots, \gamma_m \rangle^{\mathsf{Hilb}^d(S)}_{g,\beta+kA} = \int_{[\overline{M}_{g,m}(\mathsf{Hilb}^d(S),\beta+kA)]^{\mathrm{red}}} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_m^*(\gamma_m). \tag{4}$$

For $d = 1$ and $k \neq 0$, the moduli space $\overline{M}_{g,m}(\mathsf{Hilb}^d(S), \beta + kA)$ is empty by convention and the invariant (4) is defined to vanish.

0.3 The Yau–Zaslow formula in higher dimensions

Let $\pi\colon S \to \mathbb{P}^1$ be an elliptically fibered K3 surface and let

$$\pi^{[d]}\colon \mathsf{Hilb}^d(S) \to \mathsf{Hilb}^d(\mathbb{P}^1) = \mathbb{P}^d$$

be the induced *Lagrangian* fibration with generic fiber a smooth Lagrangian torus. Let

$$L_z \subset \mathsf{Hilb}^d(S)$$

denote the fiber of $\pi^{[d]}$ over a point $z \in \mathbb{P}^d$.

[1] The domain of a stable map is always taken here to be connected.

Let $F \in H_2(S;\mathbb{Z})$ be the class of a fiber of π, and let β_h be a primitive effective curve class on S with

$$F \cdot \beta_h = 1 \quad \text{and} \quad \beta_h^2 = 2h - 2.$$

For points $z_1, z_2 \in \mathbb{P}^d$ and for all $d \geq 1$ and $k \in \mathbb{Z}$, define the Gromov–Witten invariant

$$\mathsf{N}_{d,h,k} = \langle L_{z_1}, L_{z_2} \rangle^{\mathsf{Hilb}^d(S)}_{\beta_h + kA} = \int_{[\overline{M}_{0,2}(\mathsf{Hilb}^d(S), \beta_h + kA)]^{\mathrm{red}}} \mathrm{ev}_1^*(L_{z_1}) \cup \mathrm{ev}_2^*(L_{z_2}),$$

which (virtually) counts the number of rational curves in class $\beta_h + kA$ incident to the Lagrangians L_{z_1} and L_{z_2}. The first result of this paper is a complete evaluation of the invariants $\mathsf{N}_{d,h,k}$.

Define the Jacobi theta function

$$F(z,\tau) = \frac{\vartheta_1(z,\tau)}{\eta^3(\tau)} = (y^{1/2} + y^{-1/2}) \prod_{m \geq 1} \frac{(1 + yq^m)(1 + y^{-1}q^m)}{(1 - q^m)^2}, \tag{5}$$

considered as a power series in the variables

$$y = -e^{2\pi i z} \quad \text{and} \quad q = e^{2\pi i \tau},$$

where $|q| < 1$.

Theorem 1 *For all $d \geq 1$, we have*

$$\sum_{h \geq 0} \sum_{k \in \mathbb{Z}} \mathsf{N}_{d,h,k} y^k q^{h-1} = F(z,\tau)^{2d-2} \cdot \frac{1}{\Delta(\tau)} \tag{6}$$

under the variable change $y = -e^{2\pi i z}$ and $q = e^{2\pi i \tau}$.

The right-hand side of (6) is the Fourier expansion of a Jacobi form[2] of index $d - 1$. For $d = 1$ the class A vanishes on S and by convention only the term $k = 0$ is taken in the sum on the left side of (6). Then (6) specializes to the Yau–Zaslow formula (1).

0.4 Further Gromov–Witten invariants

Let S be a smooth projective K3 surface, let $\beta_h \in H_2(S,\mathbb{Z})$ be a primitive curve class of square

$$\beta_h^2 = 2h - 2,$$

[2] See Eichler and Zagier [8] for an introduction.

and let $\gamma \in H^2(S,\mathbb{Z})$ be a cohomology class with $\gamma \cdot \beta_h = 1$ and $\gamma^2 = 0$. We define three sets of invariants.

For $d \geq 2$, consider the classes

$$C(\gamma) = \mathfrak{p}_{-1}(\gamma)\mathfrak{p}_{-1}(\omega)^{d-1} 1_S,$$
$$A = \mathfrak{p}_{-2}(\omega)\mathfrak{p}_{-1}(\omega)^{d-2} 1_S,$$

which were defined in Section 0.2. Define the first two invariants

$$\mathsf{N}^{(1)}_{d,h,k} = \langle C(\gamma) \rangle^{\mathsf{Hilb}^d(S)}_{\beta_h + kA}, \quad \mathsf{N}^{(2)}_{d,h,k} = \langle A \rangle^{\mathsf{Hilb}^d(S)}_{\beta_h + kA},$$

counting rational curves incident to a cycle of class $C(\gamma)$ and A, respectively.

For a point $P \in S$ consider the incidence scheme of P,

$$I(P) = \{\xi \in \mathsf{Hilb}^d(S) \mid P \in \xi\}.$$

For generic points $P_1, \dots, P_{2d-2} \in S$ define the third invariant

$$\mathsf{N}^{(3)}_{d,h,k} = \langle I(P_1), \dots, I(P_{2d-2}) \rangle^{\mathsf{Hilb}^d(S)}_{\beta_h + kA}. \tag{7}$$

By geometric recursion, the invariants $\mathsf{N}^{(i)}_{d,h,k}, i = 1,2,3$ and $\mathsf{N}_{d,h,k}$ determine the full Gromov–Witten theory of $\mathsf{Hilb}^2(S)$ in genus 0; see Section 5.6.3. For $d = 2$ the invariants (7) are also related to counting hyperelliptic curves on a K3 surface passing through two generic points; see Section 0.7.

The following theorem provides a full evaluation of the invariants $\mathsf{N}^{(i)}_{d,h,k}$. Consider the formal variables

$$y = -e^{2\pi i z} \quad \text{and} \quad q = e^{2\pi i \tau}$$

expanded in the region $|y| < 1$ and $|q| < 1$, and the function

$$\begin{aligned} G(z,\tau) &= F(z,\tau)^2 \Big(y \frac{d}{dy}\Big)^2 \log F(z,\tau) \\ &= F(z,\tau)^2 \cdot \Big\{ \frac{y}{(1+y)^2} - \sum_{d \geq 1} \sum_{m|d} m\big((-y)^{-m} + (-y)^m\big) q^d \Big\} \\ &= 1 + (y^{-2} + 4y^{-1} + 6 + 4y^1 + y^2)q \\ &\qquad + (6y^{-2} + 24y^{-1} + 36 + 24y + 6y^2)q^2 + \cdots . \end{aligned} \tag{8}$$

Theorem 2 *For all $d \geq 2$, we have*

$$\sum_{h\geq 0}\sum_{k\in\mathbb{Z}} \mathsf{N}^{(1)}_{d,h,k} y^k q^{h-1} = G(z,\tau)^{d-1}\frac{1}{\Delta(\tau)},$$

$$\sum_{h\geq 0}\sum_{k\in\mathbb{Z}} \mathsf{N}^{(2)}_{d,h,k} y^k q^{h-1} = \frac{1}{2-2d}\Big(y\frac{d}{dy}(G(z,\tau)^{d-1})\Big)\frac{1}{\Delta(\tau)},$$

$$\sum_{h\geq 0}\sum_{k\in\mathbb{Z}} \mathsf{N}^{(3)}_{d,h,k} y^k q^{h-1} = \frac{1}{d}\binom{2d-2}{d-1}\Big(q\frac{d}{dq}F(z,\tau)\Big)^{2d-2}\frac{1}{\Delta(\tau)}$$

under the variable change $y=-e^{2\pi i z}$ and $q=e^{2\pi i\tau}$.

0.5 Quantum cohomology

0.5.1 Reduced quantum cohomology Let $\hbar$ be a formal parameter with $\hbar^2=0$. The reduced quantum cohomology of $\mathsf{Hilb}^d(S)$ is a formal deformation of the ordinary cup-product multiplication in $H^*(\mathsf{Hilb}^d(S))$ defined by

(9) $$\langle a*b,c\rangle = \langle a\cup b,c\rangle + \hbar\sum_{\beta>0}\langle a,b,c\rangle_{0,\beta}^{\mathsf{Hilb}^d(S)} q^\beta,$$

where $\langle a,b\rangle = \int_{\mathsf{Hilb}^d(S)} a\cup b$ is the standard intersection form, β runs over all nonzero elements of the cone $\mathrm{Eff}_{\mathsf{Hilb}^d(S)}$ of effective curve classes in $\mathsf{Hilb}^d(S)$, the symbol q^β denotes the corresponding element in the semigroup algebra, and $\langle a,b,c\rangle_{0,\beta}^{\mathsf{Hilb}^d(S)}$ denotes the *reduced* genus-0 Gromov–Witten invariants of $\mathsf{Hilb}^d(S)$ in class β; see Maulik and Okounkov [28] for the related case of equivariant quantum cohomology.

By the WDVV equation for reduced virtual classes (see Appendix A), the equality (9) defines a commutative and associative product on

$$H^*(\mathsf{Hilb}^d(S),\mathbb{Q})\otimes\mathbb{Q}[\![\mathrm{Eff}_{\mathsf{Hilb}^d(S)}]\!]\otimes\mathbb{Q}[\hbar]/\hbar^2,$$

which we call the reduced quantum cohomology ring

(10) $$QH^*(\mathsf{Hilb}^d(S)).$$

The ordinary cohomology ring structure on $H^*(\mathsf{Hilb}^d(S),\mathbb{Q})$ has been explicitly determined by Lehn and Sorger in [23]. In this paper, we put forth several conjectures and results about its quantum deformation (10). Our results will concern only the quantum multiplication with a divisor class on $\mathsf{Hilb}^d(S)$. In other cases (see Lehn [21], Li, Qin and Wang [24], Maulik and Oblomkov [27], Maulik and Okounkov [28] and

Okounkov and Pandharipande [36]) this has been the first step towards a more complete understanding. We will also restrict to *primitive* classes β below.

0.5.2 Elliptic K3 surfaces Let $\pi\colon S \to \mathbb{P}^1$ be an elliptic K3 surface with a section, and let B and F denote the class of a section and fiber, respectively. For every $h \geq 0$, we set

$$\beta_h = B + hF.$$

For $d \geq 1$ and cohomology classes $\gamma_1, \dots, \gamma_m \in H^*(\mathsf{Hilb}^d(S); \mathbb{Q})$, define the quantum bracket

$$\langle \gamma_1, \dots, \gamma_m \rangle_q^{\mathsf{Hilb}^d(S)} = \sum_{h \geq 0} \sum_{k \in \mathbb{Z}} y^k q^{h-1} \langle \gamma_1, \dots, \gamma_m \rangle_{\beta_h + kA}^{\mathsf{Hilb}^d(S)}.$$

Define the *primitive* quantum multiplication $*_{\mathrm{prim}}$ by

$$\langle a, b *_{\mathrm{prim}} c \rangle = \langle a, b \cup c \rangle + \hbar \cdot \langle a, b, c \rangle_q. \tag{11}$$

for all a, b, c. Since $\hbar^2 = 0$, different curve classes β don't interact, and $*_{\mathrm{prim}}$ defines a commutative and associative product on

$$H^*(\mathsf{Hilb}^d(S), \mathbb{Q}) \otimes \mathbb{Q}((y))((q)) \otimes \mathbb{Q}[\hbar]/\hbar^2. \tag{12}$$

The main content of Section 5 is a conjecture for the primitive quantum multiplication with divisor classes. By the divisor axiom and by deformation invariance the conjecture explicitly predicts the full 2–point genus-0 Gromov–Witten theory of all Hilbert schemes of points of a K3 surface in primitive classes. By direct calculations using the WDVV equation and the evaluations of Section 3, we prove the conjecture for $\mathsf{Hilb}^2(S)$.

0.5.3 Quasi-Jacobi forms Let $(z, \tau) \in \mathbb{C} \times \mathbb{H}$. The ring QJac of quasi-Jacobi forms is defined as the linear subspace

$$\mathrm{QJac} \subset \mathbb{Q}\big[F(z,\tau), E_2(\tau), E_4(\tau), \wp(z,\tau), \wp^\bullet(z,\tau), J_1(z,\tau)\big]$$

of functions which are holomorphic at $z = 0$ for generic τ; here $F(z,\tau)$ is the Jacobi theta function (5), E_{2k} are the classical Eisenstein series, $\wp$ is the Weierstrass elliptic function, $\wp^\bullet$ is its derivative with respect to z, and J_1 is the logarithmic derivative of F with respect to z; see Appendix B.

We will identify a quasi-Jacobi form $\psi \in \mathrm{QJac}$ with its power series expansion in the variables

$$q = e^{2\pi i \tau} \quad \text{and} \quad y = -e^{2\pi i z}.$$

The space QJac is naturally graded by index m and weight k:

$$\mathrm{QJac} = \bigoplus_{m\geq 0}\bigoplus_{k\geq -2m} \mathrm{QJac}_{k,m},$$

with finite-dimensional summands $\mathrm{QJac}_{k,m}$.

Based on the proven case of $\mathsf{Hilb}^2(S)$ and effective calculations for $\mathsf{Hilb}^d(S)$ for any d, we have the following results linking curve counting on $\mathsf{Hilb}^d(S)$ to quasi-Jacobi forms.

Theorem 3 *For all $\mu,\nu \in H^*(\mathsf{Hilb}^2(S))$, we have*

$$\langle \mu,\nu\rangle_q^{\mathsf{Hilb}^2(S)} = \frac{\psi(z,\tau)}{\Delta(\tau)}$$

for a quasi-Jacobi form $\psi(z,\tau)$ of index 1 and weight ≤ 6.

Since $\overline{M}_0(\mathsf{Hilb}^2(S),\gamma)$ has virtual dimension 2 for all γ, Theorem 3 implies that the full genus-0 Gromov–Witten theory of $\mathsf{Hilb}^2(S)$ in primitive classes is governed by quasi-Jacobi forms.

Conjecture J *For $d\geq 1$ and for all $\mu,\nu\in H^*(\mathsf{Hilb}^d(S))$, we have*

$$\langle \mu,\nu\rangle_q^{\mathsf{Hilb}^d(S)} = \frac{\psi(z,\tau)}{\Delta(\tau)}$$

for a quasi-Jacobi form $\psi(z,\tau)$ of index $d-1$ and weight $\leq 2+2d$.

A sharper formulation of Theorem 3 and Conjecture J specifying the weight appears in Lemma* 42.

0.6 Application 1: Genus 1 invariants of $\mathsf{Hilb}^d(S)$

Let S be a K3 surface and let $\beta_h \in H^2(S,\mathbb{Z})$ be a primitive curve class of square $\beta_h^2 = 2h-2$. Let $(E,0)$ be a nonsingular elliptic curve with origin $0\in E$, and let

$$\overline{M}_{(E,0)}(\mathsf{Hilb}^d(S),\beta_h+kA) \tag{13}$$

be the fiber of the forgetful map

$$\overline{M}_{1,1}(\mathsf{Hilb}^d(S),\beta_h+kA)\to \overline{M}_{1,1}$$

over the moduli point $(E,0)\in\overline{M}_{1,1}$. Hence, (13) is the moduli space parametrizing stable maps to $\mathsf{Hilb}^d(S)$ with 1–pointed domain with complex structure *fixed* after stabilization to be $(E,0)$. The moduli space (13) carries a reduced virtual class of dimension 1.

For $d>0$ consider the reduced Gromov–Witten potential

$$\mathsf{H}_d(y,q) = \sum_{k\in\mathbb{Z}} \sum_{h\geq 0} y^k q^{h-1} \int_{[\overline{M}_{(E,0)}(\mathsf{Hilb}^d(S),\beta_h+kA)]^{\mathrm{red}}} \mathrm{ev}_0^*(\beta_{h,k}^\vee), \tag{14}$$

where the divisor class $\beta_{h,k}^\vee \in H^2(\mathsf{Hilb}^d(S),\mathbb{Q})$ is chosen to satisfy

$$\int_{\beta_h+kA} \beta_{h,k}^\vee = 1. \tag{15}$$

The invariants (14) virtually count the number of maps from the elliptic curve E to the Hilbert scheme $\mathsf{Hilb}^d(S)$ in the classes β_h+kA. By degenerating E to a nodal curve, resolving and using the divisor equation, the series $\mathsf{H}_d(y,q)$ is seen to not depend on the choice of $\beta_{h,k}^\vee$.

The case $d=1$ of the series $\mathsf{H}_d(y,q)$ is determined by the Katz–Klemm–Vafa formula; see Maulik, Pandharipande and Thomas [30]. For $d=2$ we have the following result.

Proposition 4 *Under the variable change $y=-e^{2\pi i z}$ and $q=e^{2\pi i\tau}$,*

$$\mathsf{H}_2(y,q) = F(z,\tau)^2 \cdot \big(54\cdot \wp(z,\tau)\cdot E_2(\tau) - \tfrac{9}{4}E_2(\tau)^2 + \tfrac{3}{4}E_4(\tau)\big)\frac{1}{\Delta(\tau)}.$$

In joint work with Rahul Pandharipande [35] a correspondence between curve counting on $\mathsf{Hilb}^d(S)$ and the enumerative geometry of the product Calabi–Yau $S\times E$ was proposed. This in turn led to an explicit conjecture for $\mathsf{H}_d(y,q)$ for all d in terms of the reciprocal of the Igusa cusp form χ_{10}. Proposition 4 verifies this conjecture for $d=2$.

0.7 Application 2: Hyperelliptic curves

A projective nonsingular curve C of genus $g\geq 2$ is *hyperelliptic* if C admits a degree-2 map to $\mathbb{P}^1$,

$$C\to\mathbb{P}^1.$$

The locus of hyperelliptic curves in the moduli space M_g of nonsingular curves of genus g is a closed substack of codimension $g-2$. Let

$$\mathcal{H}_g \in H^{2(g-2)}(\overline{M}_g,\mathbb{Q})$$

be the stack fundamental class of the closure of nonsingular hyperelliptic curves inside $\overline{M}_g$. By results of Faber and Pandharipande [9], $\mathcal{H}_g$ is a tautological class (see Faber and Pandharipande [10]) of codimension $g-2$.

Let S be a K3 surface, let $\beta_h \in H^2(S)$ be a primitive curve class of square $\beta_h^2 = 2h-2$, and let

$$\pi\colon \overline{M}_{g,2}(S,\beta_h) \to \overline{M}_g$$

be the forgetful map from the moduli space of genus-g stable maps to S in class β_h. A virtual count of genus $g \geq 2$ hyperelliptic curves on S in class β_h passing through two general points is defined by the integral

$$\mathsf{H}_{g,h} = \int_{[\overline{M}_{g,2}(S,\beta_h)]^{\mathrm{red}}} \pi^*(\mathcal{H}_g)\,\mathrm{ev}_1^*(\omega)\,\mathrm{ev}_2^*(\omega),$$

where $\omega \in H^4(S,\mathbb{Z})$ is the class of a point.

T Graber [13] used the genus-0 Gromov–Witten theory of $\mathsf{Hilb}^2(\mathbb{P}^2)$ to obtain enumerative results on hyperelliptic curves in $\mathbb{P}^2$. A similar strategy has been applied for $\mathbb{P}^1 \times \mathbb{P}^1$ (see Pontoni [38]) and for abelian surfaces, modulo a transversality result; see Bryan, Oberdieck, Pandharipande and Yin [4] and Rose [40]. By arguments parallel to the abelian case (see [4]), Theorem 2 leads to the following prediction for $\mathsf{H}_{g,h}$.

Let $\Delta(\tau) = q\prod_{m\geq 1}(1-q^m)^{24}$ be the modular discriminant and let

$$F(z,\tau) = u\exp\biggl(\sum_{k\geq 1}\frac{(-1)^k B_{2k}}{2k(2k)!}E_{2k}(\tau)u^{2k}\biggr)$$

be the Jacobi theta function, which appeared already in (5), expanded in the variables

(16) $$q = e^{2\pi i\tau} \quad \text{and} \quad u = 2\pi z.$$

Conjecture H *Under the variable change (16),*

$$\sum_{h\geq 0}\sum_{g\geq 2} u^{2g+2}q^{h-1}\mathsf{H}_{g,h} = \Bigl(q\frac{d}{dq}F(z,\tau)\Bigr)^2 \cdot \frac{1}{\Delta(\tau)}.$$

By a direct verification using results of Bryan and Leung [3] and Maulik, Pandharipande and Thomas [30] and an explicit expression (see Harris and Mumford [16]) for $\mathcal{H}_3 \in H^2(\overline{M}_3,\mathbb{Q})$, Conjecture H holds in the first nontrivial case $g=3$. Similar conjectures relating the Gromov–Witten count of r–gonal curves on the K3 surface S to the genus-0 Gromov–Witten invariants of $\mathsf{Hilb}^r(S)$ can be made.

The virtual counts $\mathsf{H}_{g,h}$ have contributions from the boundary of the moduli space, and do *not* correspond to the actual, enumerative count of hyperelliptic curves on S. For example, $\mathsf{H}_{3,1} = -\frac{1}{4}$ is both rational and negative. For $h \geq 0$ BPS numbers $\mathsf{h}_{g,h}$ of genus-g hyperelliptic curves on S in class β_h are defined by the expansion

$$\sum_{g \geq 2} \mathsf{h}_{g,h} \big(2 \sin \tfrac{u}{2}\big)^{2g+2} = \sum_{g \geq 2} \mathsf{H}_{g,h} u^{2g+2}. \tag{17}$$

The invariants $\mathsf{h}_{g,h}$ are expected to yield the enumerative count of genus-g hyperelliptic curves in class β_h on a generic K3 surface S carrying a curve class β_h; compare [4, Section 0.2.4].

The invariants $\mathsf{h}_{g,h}$ vanish for $h = 0, 1$. The first nonvanishing values of $\mathsf{h}_{g,h}$ are presented in Table 1. The distribution of the nonzero values in Table 1 matches precisely the results of Ciliberto and Knutsen [7, Theorem 0.1]: there exist curves on a generic K3 surface in class β_h with normalization a hyperelliptic curve of genus g if and only if

$$h \geq g + \lfloor \tfrac{1}{2} g \rfloor \big(g - 1 - \lfloor \tfrac{1}{2} g \rfloor\big).$$

$h \downarrow \quad g \rightarrow$	2	3	4	5	6
2	1	0	0	0	0
3	36	0	0	0	0
4	672	6	0	0	0
5	8728	204	0	0	0
6	88830	3690	9	0	0
7	754992	47160	300	0	0
8	5573456	476700	5460	0	0
9	36693360	4048200	70848	36	0
10	219548277	29979846	730107	1134	0
11	1210781880	198559080	6333204	19640	0
12	6221679552	1197526770	47948472	244656	36
13	30045827616	6666313920	324736392	2438736	1176
14	137312404502	34612452966	2002600623	20589506	20895
15	597261371616	169017136848	11396062440	152487720	265860

Table 1: The first values for the counts $\mathsf{h}_{g,h}$ of hyperelliptic curves of genus g and class β_h on a generic K3 surface S passing through two general points, as predicted by Conjecture H and the BPS expansion (17).

0.8 Plan of the paper

In Section 1, we introduce the bare notational necessities and prove a few general lemmas. In Section 2 we prove Theorem 1 by reducing to an elliptic K3 surface with

24 rational nodal fibers and by comparison with rational curve counts on a Kummer K3. In Sections 3 and 4 we prove Theorem 2 by reducing the statement to a calculation of Gromov–Witten invariants of $\mathsf{Hilb}^2(\mathbb{P}^1 \times E)$. This approach is mainly independent from the Kummer K3 geometry used in Section 2, and yields a second proof of Theorem 1. In Section 5, we present the conjectures and results on the quantum cohomology ring of $\mathsf{Hilb}^d(K3)$. Here we also prove Theorem 3 and Proposition 4. In Appendix A, we present the precise form of the WDVV equations for reduced invariants. In Appendix B, we introduce the notion of a quasi-Jacobi form, and list numerical results related to the conjectures of Section 5.

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1 Preliminaries

Let S be a smooth projective surface and let $\mathsf{Hilb}^d(S)$ be the Hilbert scheme of d points of S. By definition, $\mathsf{Hilb}^0(S)$ is a point parametrizing the empty subscheme.

1.1 Notation

We always work over $\mathbb{C}$. All cohomology coefficients are in $\mathbb{Q}$ unless specified otherwise. We let $[V]$ denote the homology class of an algebraic cycle V.

On a connected smooth projective variety X, we will freely identify cohomology and homology classes by Poincaré duality. We write

$$\omega = \omega_X \in H^{2\dim(X)}(X;\mathbb{Z}),$$
$$e = e_X \in H^0(X;\mathbb{Z})$$

for the class of a point and the fundamental class of X, respectively. Using the degree map we identify the top cohomology group with the underlying ring:

$$H^{2\dim(X)}(X,\mathbb{Q}) \equiv \mathbb{Q}.$$

The tangent bundle of X is denoted by T_X.

A homology class $\beta \in H_2(X,\mathbb{Z})$ is an *effective curve class* if X admits an algebraic curve C of class $[C]=\beta$. The class β is *primitive* if it is indivisible in $H_2(X,\mathbb{Z})$.

1.2 Cohomology of $\mathsf{Hilb}^d(S)$

1.2.1 The Nakajima basis Let $(\mu_1,\dots,\mu_l)$ with $\mu_1 \geq \dots \geq \mu_l \geq 1$ be a partition and let

$$\alpha_1,\dots,\alpha_l \in H^*(S;\mathbb{Q})$$

be cohomology classes on S. We call the tuple

$$\mu = \big((\mu_1,\alpha_1),\dots,(\mu_l,\alpha_l)\big) \tag{18}$$

a cohomology-weighted partition of size $|\mu| = \sum \mu_i$.

If the set $\{\alpha_1,\dots,\alpha_l\}$ is ordered, we call (18) ordered if for all $i \leq j$ we have either

$$\mu_i \geq \mu_j \quad \text{or} \quad (\mu_i = \mu_j \text{ and } \alpha_i \geq \alpha_j).$$

For $i>0$ and $\alpha \in H^*(S;\mathbb{Q})$, let

$$\mathfrak{p}_{-i}(\alpha)\colon H^*(\mathsf{Hilb}^d(S),\mathbb{Q}) \to H^*(\mathsf{Hilb}^{d+i}(S),\mathbb{Q})$$

be the Nakajima creation operator [31], and let

$$1_S \in H^*(\mathsf{Hilb}^0(S),\mathbb{Q}) = \mathbb{Q}$$

be the vacuum vector. A cohomology-weighted partition (18) defines the cohomology class

$$\mathfrak{p}_{-\mu_1}(\alpha_1)\cdots\mathfrak{p}_{-\mu_l}(\alpha_l)1_S \in H^*(\mathsf{Hilb}^{|\mu|}(S)).$$

Let $\alpha_1,\dots,\alpha_p$ be a homogeneous ordered basis of $H^*(S;\mathbb{Q})$. By a theorem of Grojnowski [15] and Nakajima [31], the cohomology classes associated to all ordered cohomology-weighted partitions of size d with cohomology weighting by the α_i, not repeating factors (μ_j,α_j) with α_j odd, form a basis of the cohomology $H^*(\mathsf{Hilb}^d(S);\mathbb{Q})$.

1.2.2 Special cycles We will require several natural cycles and their cohomology classes. In the definitions below, we set $\mathfrak{p}_{-m}(\alpha)^k = 0$ whenever $k < 0$.

(i) The diagonal The diagonal divisor

$$\Delta_{\mathsf{Hilb}^d(S)} \subset \mathsf{Hilb}^d(S)$$

is the reduced locus of subschemes $\xi \in \mathsf{Hilb}^d(S)$ such that $\mathrm{len}(\mathcal{O}_{\xi,x}) \geq 2$ for some $x \in S$. It has cohomology class

$$[\Delta_{\mathsf{Hilb}^d(S)}] = \frac{1}{(d-2)!}\mathfrak{p}_{-2}(e)\mathfrak{p}_{-1}(e)^{d-2}1_S = -2\cdot c_1(\mathcal{O}_S^{[d]}),$$

where we let $E^{[d]}$ denote the tautological bundle on $\mathsf{Hilb}^d(S)$ associated to a vector bundle E on S; see [21; 22].

(ii) The exceptional curve Let $\mathrm{Sym}^d(S)$ be the d^{th} symmetric product of S and let

$$\rho\colon \mathsf{Hilb}^d(S) \to \mathrm{Sym}^d(S), \quad \xi \mapsto \sum_{x\in S} \mathrm{len}(\mathcal{O}_{\xi,x})x$$

be the Hilbert–Chow morphism.

For distinct points $x, y_1, \dots, y_{d-2} \in S$ where $d \geq 2$, the fiber of ρ over

$$2x + \sum_i y_i \in \mathrm{Sym}^d(S)$$

is isomorphic to $\mathbb{P}^1$ and called an *exceptional curve*. For all d define the cohomology class

$$A = \mathfrak{p}_{-2}(\omega)\mathfrak{p}_{-1}(\omega)^{d-2}1_S,$$

where $\omega \in H^4(S,\mathbb{Z})$ is the class of a point on S. If $d \geq 2$ every exceptional curve has class A.

(iii) The incidence subschemes Let $z \subset S$ be a zero-dimensional subscheme. The incidence scheme of z is the locus

$$I(z) = \{\xi \in \mathsf{Hilb}^d(S) \mid z \subset \xi\}$$

endowed with the natural subscheme structure.

(iv) Curve classes For $\beta \in H_2(S)$ and $a, b \in H_1(S)$, define

$$(19) \qquad \begin{aligned} C(\beta) &= \mathfrak{p}_{-1}(\beta)\mathfrak{p}_{-1}(\omega)^{d-1}1_S &&\in H_2(\mathsf{Hilb}^d(S)),\\ C(a,b) &= \mathfrak{p}_{-1}(a)\mathfrak{p}_{-1}(b)\mathfrak{p}_{-1}(\omega)^{d-2}1_S &&\in H_2(\mathsf{Hilb}^d(S)).\end{aligned}$$

In unambiguous cases, we write β for $C(\beta)$. By Nakajima's theorem, the assignment (19) induces for $d \geq 2$ the isomorphism

$$\begin{aligned} H_2(S,\mathbb{Q}) \oplus \textstyle\bigwedge^2 H_1(S,\mathbb{Q}) \oplus \mathbb{Q} &\to H_2(\mathsf{Hilb}^d(S);\mathbb{Q}), \\ (\beta, a\wedge b, k) &\mapsto \beta + C(a,b) + kA. \end{aligned}$$

If $d \leq 1$, we write

$$\beta + \sum_i C(a_i, b_i) + kA \in H_2(\mathsf{Hilb}^d(S),\mathbb{Q})$$

for some β, a_i, b_i, k; we *always assume* $a_i = b_i = 0$ and $k = 0$. If $d = 0$, we also assume $\beta = 0$. This will allow us to treat $\mathsf{Hilb}^d(S)$ simultaneously for all d at once; see for example Section 1.3.

(v) Partition cycles Let $V \subset S$ be a subscheme, let $k \geq 1$ and consider the diagonal embedding

$$\iota_k\colon S \to \mathrm{Sym}^k(S)$$

and the Hilbert–Chow morphism

$$\rho\colon \mathsf{Hilb}^k(S) \to \mathrm{Sym}^k(S).$$

The k*-fattening* of V is the subscheme

$$V[k] = \rho^{-1}(i_k(V)) \subset \mathsf{Hilb}^k(S).$$

Let $d = d_1 + \cdots + d_r$ be a partition of d into integers $d_i \geq 1$, and let

$$V_1, \ldots, V_r \subset S$$

be pairwise disjoint subschemes on S. Consider the open subscheme

$$U = \bigl\{(\xi_1, \ldots, \xi_r) \in \mathsf{Hilb}^{d_1}(S) \times \cdots \times \mathsf{Hilb}^{d_r}(S) \mid \xi_i \cap \xi_j = \varnothing \text{ for all } i \neq j\bigr\} \tag{20}$$

and the natural map $\sigma\colon U \to \mathsf{Hilb}^d(S)$, which sends a tuple of subschemes $(\xi_1, \ldots, \xi_r)$ defined by ideal shaves I_{ξ_i} to the subscheme $\xi \in \mathsf{Hilb}^d(S)$ defined by the ideal sheaf $I_{\xi_1} \cap \cdots \cap I_{\xi_r}$. We often use the shorthand notation[3]

$$\sigma(\xi_1, \ldots, \xi_r) = \xi_1 + \cdots + \xi_r. \tag{21}$$

[3] For functions $f_i\colon X \to \mathsf{Hilb}^{d_i}(S)$, $i = 1, \ldots, r$ with $(f_1, \ldots, f_r)\colon X \to U$, we also write $f_1 + \cdots + f_r = \sigma \circ (f_1, \ldots, f_r)\colon X \to \mathsf{Hilb}^d(S)$.

We define the *partition cycle* as

$$V_1[d_1]\cdots V_r[d_r] = \sigma(V_1[d_1]\times\cdots\times V_r[d_r]) \subset \mathsf{Hilb}^d(S). \tag{22}$$

By [32, Theorem 9.10], the subscheme (22) has cohomology class

$$\mathfrak{p}_{-d_1}(\alpha_1)\cdots\mathfrak{p}_{-d_r}(\alpha_r)1_S \in H^*(\mathsf{Hilb}^d(S)),$$

where $\alpha_i = [V_i]$ for all i.

1.3 Curves in $\mathsf{Hilb}^d(S)$

1.3.1 Cohomology classes Let C be a projective curve and let $f\colon C\to \mathsf{Hilb}^d(S)$ be a map. Let $p\colon \mathcal{Z}_d \to \mathsf{Hilb}^d(S)$ be the universal subscheme and let $q\colon \mathcal{Z}_d\to S$ be the universal inclusion. Consider the fiber diagram

$$\begin{array}{ccccc} \widetilde{C} & \xrightarrow{\ \tilde f\ } & \mathcal{Z}_d & \xrightarrow{\ q\ } & S \\ \big\downarrow{\scriptstyle \tilde p} & & \big\downarrow{\scriptstyle p} & & \\ C & \xrightarrow{\ f\ } & \mathsf{Hilb}^d(S) & & \end{array} \tag{23}$$

and let $f' = q\circ\tilde f$. The embedded curve $\widetilde{C}\subset C\times S$ is flat of degree d over C. By the universal property of $\mathsf{Hilb}^d(S)$, we can recover f from $\widetilde{C}$. Here, even when C is a smooth connected curve, $\widetilde{C}$ could be disconnected, singular and possibly nonreduced.

Lemma 5 *Let C be a reduced projective curve and let $f\colon C\to\mathsf{Hilb}^d(S)$ be a map with*

$$f_*[C] = \beta + \sum_j C(\gamma_j,\gamma_j') + kA \tag{24}$$

for some $\beta\in H_2(S)$, $\gamma_j,\gamma_j'\in H_1(S)$ and $k\in\mathbb{Z}$. Then

$$(q\circ\tilde f)_*[\widetilde{C}] = \beta.$$

Proof We may assume that $d\geq 2$ and that C is irreducible. Since $\tilde p$ is flat,

$$f'_*[\widetilde{C}] = f'_*\tilde p^*[C] = q_*p^*f_*[C].$$

Therefore, the claim of Lemma 5 follows from (24) and

$$q_*p^*A = 0,\quad q_*p^*C(\beta)=\beta,\quad q_*p^*C(a,b)=0$$

for all $\beta \in H_2(S)$ and $a, b \in H_1(S)$. By considering an exceptional curve of class A, one finds $q_* p^* A = 0$. We will verify that $q_* p^* C(\beta) = \beta$; the equation $q_* p^* C(a,b) = 0$ is similar.

Let $U \subset S^d$ be the open set defined in (20) with $d_i = 1$ for all i, and let $\sigma\colon U \to \mathrm{Hilb}^d(S)$ be the sum map. We have $C(\beta) = \sigma_*(\omega^{d-1} \times \beta)$. Consider the fiber square

$$\begin{array}{ccccc} \widetilde{U} & \longrightarrow & \mathcal{Z}_d & \xrightarrow{q} & S \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \\ U & \longrightarrow & \mathrm{Hilb}^d(S) & & \end{array}$$

Let $\Delta_{i,d+1} \subset S^d \times S$ be the $(i, d+1)$ diagonal. Then $\widetilde{U} \subset S^d \times S$ is the disjoint union $\bigcup_{i=1,\dots,d} \Delta_{i,d+1} \cap (U \times S)$. Therefore

$$\begin{aligned} q_* p^* C(\beta) &= q_* p^* \sigma_*(\omega^{d-1} \times \beta) \\ &= \mathrm{pr}_{d+1*}\, p'^*(\omega^{d-1} \times \beta) \\ &= \sum_{i=1}^{d} \mathrm{pr}_{d+1*}\big([\Delta_{i,d+1}] \cdot (\omega^{d-1} \times \beta \times e_S)\big) \\ &= \beta. \end{aligned}$$

□

Lemma 6 *Let C be a smooth, projective, connected curve of genus g and let $f\colon C \to \mathrm{Hilb}^d(S)$ be a map of class (24). Then*

$$k = \chi(\mathcal{O}_{\widetilde{C}}) - d(1-g).$$

Proof The intersection of $f_*[C]$ with the diagonal class $\Delta = -2c_1(\mathcal{O}_S^{[d]})$ is $-2k$. Therefore

$$k = \deg(c_1(\mathcal{O}_S^{[d]}) \cap f_*[C]) = \deg(f^* \mathcal{O}_S^{[d]}) = \chi(f^* \mathcal{O}_S^{[d]}) - d(1-g),$$

where we used Riemann–Roch in the last step. Since we have

$$f^* \mathcal{O}_S^{[d]} = f^* p_* q^* \mathcal{O}_S = \tilde{p}_* \tilde{f}^* q^* \mathcal{O}_S = \tilde{p}_* \mathcal{O}_{\widetilde{C}}$$

and $\tilde{p}$ is finite, we obtain $\chi(f^* \mathcal{O}_S^{[d]}) = \chi(\tilde{p}_* \mathcal{O}_{\widetilde{C}}) = \chi(\mathcal{O}_{\widetilde{C}})$. □

Corollary 7 *Let $\gamma \in H_2(\mathrm{Hilb}^d(S), \mathbb{Z})$ and let $\overline{M}_0(\mathrm{Hilb}^d(S), \gamma)$ be the moduli space of stable maps of genus 0 in class γ. Then for $m \ll 0$,*

$$\overline{M}_0(\mathrm{Hilb}^d(S), \gamma + mA) = \varnothing.$$

Proof Let $f\colon \mathbb{P}^1 \to \mathsf{Hilb}^d(S)$ be a map in class $\gamma + mA$. The cohomology class of the corresponding curve $\widetilde{C} = f^*\mathcal{Z}_d \subset \mathbb{P}^1 \times S$ is independent of m. Hence the holomorphic Euler characteristic $\chi(\mathcal{O}_{\widetilde{C}})$ is bounded from below by a constant independent of m. Therefore, by Lemma 6, we find m to be bounded from below when the domain curve is $\mathbb{P}^1$. Since an effective class $\gamma + mA$ decomposes in at most finitely many ways in a sum of effective classes, the claim is proven. □

1.3.2 Irreducible components Let $f\colon C \to \mathsf{Hilb}^d(S)$ be a map and consider the fiber diagram

$$\begin{array}{ccc} \widetilde{C} = f^*\mathcal{Z}_d & \longrightarrow & \mathcal{Z}_d \\ \big\downarrow{\scriptstyle \tilde{p}} & & \big\downarrow{\scriptstyle p} \\ C & \xrightarrow{\ f\ } & \mathsf{Hilb}^d(S) \end{array}$$

where $p\colon \mathcal{Z}_d \to \mathsf{Hilb}^d(S)$ is the universal family.

Definition 8 The map f is *irreducible* if $f^*\mathcal{Z}_d$ is irreducible.

Let $d \geq 1$ and let $f\colon C \to \mathsf{Hilb}^d(S)$ be a map from a connected nonsingular projective curve C. Consider the (reduced) irreducible components

$$G_1, \dots, G_r$$

of the curve $\widetilde{C} = f^*\mathcal{Z}_d$, and let

$$\xi = \bigcup_{i \neq j} \tilde{p}(G_i \cap G_j) \subset C$$

be the image of their intersection points under $\tilde{p}$. Every *connected* component D of $\widetilde{C} \setminus \tilde{p}^{-1}(\xi)$ is an irreducible curve and flat over $C \setminus \xi$. Since C is a nonsingular curve, also the closure $\overline{D}$ is flat over C, and by the universal property of $\mathsf{Hilb}^{d'}(S)$ yields an associated irreducible map

$$C \to \mathsf{Hilb}^{d'}(S)$$

for some $d' \leq d$. Let $\phi_1, \dots, \phi_r$ be the irreducible maps associated to all connected components of $\widetilde{C} \setminus \tilde{p}^{-1}(\xi)$. We say f *decomposes into the irreducible components* $\phi_1, \dots, \phi_r$.

Conversely, let $\phi_i\colon C \to \mathsf{Hilb}^{d_i}(S), i = 1, \dots, n$ be irreducible maps such that

- $\sum_i d_i = d$, and
- $\phi_i^*\mathcal{Z}_{d_i} \cap \phi_j^*\mathcal{Z}_{d_j}$ is of dimension 0 for all $i \neq j$.

Let U be the open subset defined in (20). The map

$$(\phi_1, \dots, \phi_n)\colon C \to \mathsf{Hilb}^{d_1}(S) \times \cdots \times \mathsf{Hilb}^{d_n}(S)$$

meets the complement of U in a finite number of points $x_1, \dots, x_m \in C$. By smoothness of C, the composition

$$\sigma \circ (\phi_1, \dots, \phi_n)\colon C \setminus \{x_1, \dots, x_m\} \to \mathsf{Hilb}^d(S)$$

extends uniquely to a map $f\colon C \to \mathsf{Hilb}^d(S)$.

A direct verification shows that the two operations above are inverse to each other. We write

$$f = \phi_1 + \cdots + \phi_r$$

for the decomposition of f into the irreducible components $\phi_1, \dots, \phi_r$.

Let $\beta, \beta_i \in H_2(S)$, $\gamma_j, \gamma'_j, \gamma_{i,j}, \gamma'_{i,j} \in H_1(S)$ and $k, k_i \in \mathbb{Z}$ be such that

$$f_*[C] = C(\beta) + \sum_j C(\gamma_j, \gamma'_j) + kA \in H_2(\mathsf{Hilb}^d(S)),$$

$$\phi_{i*}[C] = C(\beta_i) + \sum_j C(\gamma_{i,j}, \gamma'_{i,j}) + k_i A \in H_2(\mathsf{Hilb}^{d_i}(S)).$$

Lemma 9 *We have*

$$\sum_i \beta_i = \beta \in H_2(S;\mathbb{Z}),$$

$$\sum_{i,j} \gamma_{i,j} \wedge \gamma'_{i,j} = \sum_j \gamma_j \wedge \gamma'_j \in \textstyle\bigwedge^2 H_1(S).$$

Proof This follows directly from [32, Theorem 9.10]. □

2 The Yau–Zaslow formula in higher dimensions

2.1 Overview

In the remainder of Section 2 we give a proof of Theorem 1. The proof proceeds in the following steps.

In Section 2.2 we use the deformation theory of K3 surfaces to reduce Theorem 1 to an evaluation on a specific elliptic K3 surface S. Here, we also analyze rational curves on $\mathsf{Hilb}^d(S)$ and prove a few lemmas. This discussion will be used also later on.

In Section 2.3, we study the structure of the moduli space of stable maps which are incident to the Lagrangians L_{z_1} and L_{z_2}. The main result is a splitting statement (Proposition 16), which reduces the computation of Gromov–Witten invariants to integrals associated to fixed elliptic fibers.

In Section 2.4, we evaluate these remaining integrals using the geometry of the Kummer K3 surfaces, the Yau–Zaslow formula and a theta function associated to the D_4 lattice.

2.2 The Bryan–Leung K3

2.2.1 Definition Let $\pi\colon S \to \mathbb{P}^1$ be an elliptic K3 surface with a unique section $s\colon \mathbb{P}^1 \to S$ and 24 rational nodal fibers. We call S a *Bryan–Leung K3 surface*.

Let $x_1, \dots, x_{24} \in \mathbb{P}^1$ be the basepoints of the nodal fibers of π, let B_0 be the image of the section s, and let

$$F_x \subset S$$

denote the fiber of π over a point $x \in \mathbb{P}^1$.

The Picard group

$$\mathrm{Pic}(S) = H^{1,1}(S;\mathbb{Z}) = H^2(S;\mathbb{Z}) \cap H^{1,1}(S;\mathbb{C})$$

is of rank 2 and generated by the section class B and the fiber class F. We have the intersection numbers

$$B^2 = -2, \quad B \cdot F = 1, \quad F^2 = 0.$$

Hence for all $h \geq 0$ the class

$$\beta_h = B + hF \in H_2(S;\mathbb{Z}) \tag{25}$$

is a primitive and effective curve class of square $\beta_h^2 = 2h-2$.

The projection π and the section s induce maps of Hilbert schemes

$$\pi^{[d]}\colon \mathsf{Hilb}^d(S) \to \mathsf{Hilb}^d(\mathbb{P}^1) = \mathbb{P}^d, \quad s^{[d]}\colon \mathbb{P}^d \to \mathsf{Hilb}^d(S)$$

such that $\pi^{[d]} \circ s^{[d]} = \mathrm{id}_{\mathbb{P}^d}$. The map $s^{[d]}$ is an isomorphism from $\mathsf{Hilb}^d(\mathbb{P}^1)$ to the locus of subschemes of S which are contained in B_0. This gives natural identifications

$$\mathbb{P}^d = \mathsf{Hilb}^d(\mathbb{P}^1) = \mathsf{Hilb}^d(B_0)$$

that we will use sometimes. In unambiguous cases we also write π and s for $\pi^{[d]}$ and $s^{[d]}$, respectively.

2.2.2 Main statement revisited For $d \geq 1$ and cohomology classes $\gamma_1, \dots, \gamma_m \in H^*(\mathsf{Hilb}^d(S); \mathbb{Q})$, define the quantum bracket

$$\langle \gamma_1, \dots, \gamma_m \rangle_q^{\mathsf{Hilb}^d(S)} = \sum_{h \geq 0} \sum_{k \in \mathbb{Z}} y^k q^{h-1} \langle \gamma_1, \dots, \gamma_m \rangle_{\beta_h + kA}^{\mathsf{Hilb}^d(S)},$$

where the bracket on the right-hand side was defined in (4).

Theorem 10 *For all $d \geq 1$,*

$$\big\langle \mathfrak{p}_{-1}(F)^d 1_S, \mathfrak{p}_{-1}(F)^d 1_S \big\rangle_q^{\mathsf{Hilb}^d(S)} = \frac{F(z,\tau)^{2d-2}}{\Delta(\tau)},$$

where $q = e^{2\pi i \tau}$ and $y = -e^{2\pi i z}$.

We begin the proof of Theorem 10 in Section 2.3.

Let $\pi' \colon S' \to \mathbb{P}^1$ be any elliptic K3 surface, and let F' be the class of a fiber of π'. A fiber of the induced Lagrangian fibration

$$\pi'^{[d]} \colon \mathsf{Hilb}^d(S') \to \mathbb{P}^d$$

has class $\mathfrak{p}_{-1}(F')^d 1_S$. Hence Theorem 1 implies Theorem 10. The following lemma shows that conversely Theorem 10 also implies Theorem 1, and hence the claims in both theorems are equivalent.

Lemma 11 *Let S be the fixed Bryan–Leung K3 surface defined in Section 2.2.1, and let $\beta_h = B + hF$ be the curve class (25).*

Let S' be a K3 surface with a primitive curve class β of square $2h-2$, and let $\gamma \in H^2(S', \mathbb{Z})$ be any class with $\beta \cdot \gamma = 1$ and $\gamma^2 = 0$. Then

$$\big\langle \mathfrak{p}_{-1}(\gamma)^d 1_{S'}, \mathfrak{p}_{-1}(\gamma)^d 1_{S'} \big\rangle_{\beta + kA}^{\mathsf{Hilb}^d(S')} = \big\langle \mathfrak{p}_{-1}(F)^d 1_S, \mathfrak{p}_{-1}(F)^d 1_S \big\rangle_{\beta_h + kA}^{\mathsf{Hilb}^d(S)}.$$

Proof of Lemma 11 We will construct an algebraic deformation from S' to the fixed K3 surface S such that β deforms to β_h through classes of Hodge type $(1,1)$, and γ deforms to F. By the deformation invariance of reduced Gromov–Witten invariants, the claim of Lemma 11 follows.

Let $E_8(-1)$ be the negative E_8 lattice, let U be the hyperbolic lattice and consider the K3 lattice

$$\Lambda = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}.$$

Let e, f be a hyperbolic basis for one of the U summands of Λ and let

$$\phi\colon \Lambda \xrightarrow{\cong} H^2(S;\mathbb{Z})$$

be a fixed marking with $\phi(e) = B + F$ and $\phi(f) = F$. We let

$$b_h = e + (h-1)f$$

denote the class corresponding to $\beta_h = B + hF$ under ϕ.

The orthogonal group of Λ is transitive on primitive vectors of the same square; see [14, Lemma 7.8] for references. Hence there exists a marking

$$\phi'\colon \Lambda \xrightarrow{\cong} H^2(S';\mathbb{Z})$$

such that $\phi'(b_h) = \beta$. Let $g = \phi'^{-1}(\gamma) \in \Lambda$ be the vector that corresponds to the class γ under ϕ'. The span

$$\Lambda_0 = \langle g, b_h\rangle \subset \Lambda$$

defines a hyperbolic sublattice of Λ which, by unimodularity, yields the direct sum decomposition

$$\Lambda = \Lambda_0 \oplus \Lambda_0^{\perp}.$$

Because the irreducible unimodular factors of a unimodular lattice are unique up to order, we find

$$\Lambda_0^{\perp} \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 2}.$$

Hence there exists a lattice isomorphism $\sigma\colon \Lambda \to \Lambda$ with $\sigma(b_h) = b_h$ and $\sigma(g) = f$. Replacing ϕ' by $\phi' \circ \sigma^{-1}$, we may therefore assume $\phi'(b_h) = \beta$ and $\phi'(f) = \gamma$.

Since the period domain Ω associated to b_h is connected, there exists a curve inside Ω connecting the period point of S' to the period point of S. Restricting the universal family over Ω to this curve, we obtain a deformation with the desired properties. □

2.2.3 Rational curves in $\mathsf{Hilb}^d(S)$ Let $h \geq 0$ and let k be an integer. We consider rational curves on $\mathsf{Hilb}^d(S)$ in the classes $\beta_h + kA$ and $hF + kA$.

Vertical maps Let $u_1, \dots, u_d \in \mathbb{P}^1$ be points such that

- u_i is not the basepoint of a nodal fiber of $\pi\colon S \to \mathbb{P}^1$ for all i,
- the points $u_1, \dots, u_d$ are pairwise distinct.

Then the fiber of $\pi^{[d]}$ over $u_1 + \cdots + u_d \in \mathsf{Hilb}^d(\mathbb{P}^1)$ is isomorphic to the product of smooth elliptic curves

$$F_{u_1} \times \cdots \times F_{u_d}.$$

The subset of points in $\mathsf{Hilb}^d(\mathbb{P}^1)$ whose preimage under $\pi^{[d]}$ is not of this form is the divisor

$$\mathcal{W} = I(x_1) \cup \cdots \cup I(x_{24}) \cup \Delta_{\mathsf{Hilb}^d(\mathbb{P}^1)} \subset \mathsf{Hilb}^d(\mathbb{P}^1), \tag{26}$$

where $x_1, \dots, x_{24}$ are the basepoints of the nodal fibers of π, $I(x_i)$ is the incidence subscheme and $\Delta_{\mathsf{Hilb}^d(\mathbb{P}^1)}$ is the diagonal; see Section 1.2.2. Since a fiber of $\pi^{[d]}$ over a point $z \in \mathbb{P}^d$ is nonsingular if and only if $z \notin \mathcal{W}$, we call $\mathcal{W}$ the *discriminant* of $\pi^{[d]}$.

Consider a stable map $f\colon C \to \mathsf{Hilb}^d(S)$ of genus 0 and class $hF + kA$. Since the composition

$$\pi^{[d]} \circ f\colon C \to \mathsf{Hilb}^d(\mathbb{P}^1)$$

is mapped to a point, and since nonsingular elliptic curves do not admit nonconstant rational maps, we have the following lemma.

Lemma 12 *Let $f\colon C \to \mathsf{Hilb}^d(S)$ be a nonconstant genus-0 stable map in class $hF + kA$. Then the image of $\pi^{[d]} \circ f$ lies in the discriminant $\mathcal{W}$.*

Nonvertical maps Let $f\colon C \to \mathsf{Hilb}^d(S)$ be a stable genus-0 map in class $f_*[C] = \beta_h + kA$. The composition

$$\pi^{[d]} \circ f\colon C \to \mathbb{P}^d$$

has degree 1 with image a line

$$L \subset \mathbb{P}^d.$$

Let C_0 be the unique irreducible component of C on which $\pi \circ f$ is nonconstant. We call $C_0 \subset C$ the *distinguished component* of C.

Since $C_0 \cong \mathbb{P}^1$, we have a decomposition

$$f|_{C_0} = \phi_0 + \cdots + \phi_r$$

of $f|_{C_0}$ into irreducible maps $\phi_i\colon C_0 \to \mathsf{Hilb}^{d_i}(S)$ where d_i are positive integers such that $d = d_0 + \cdots + d_r$; see Section 1.3.2. By Lemma 9, exactly one of the maps $\pi^{[d_i]} \circ \phi_i$ is nonconstant; we assume this map is ϕ_0.

Lemma 13 *Let $\mathcal{W}$ be the discriminant of $\pi^{[d]}$. If $L \not\subseteq \mathcal{W}$, then*

(i) $d_i = 1$ *for all* $i \in \{1, \dots, r\}$,

(ii) $\phi_i \colon C_0 \to S$ *is constant for all* $i \in \{1, \dots, r\}$,

(iii) $\phi_0 \colon C_0 \to \mathsf{Hilb}^{d_0}(S)$ *is an isomorphism onto a line in* $\mathsf{Hilb}^{d_0}(B_0)$.

Proof Assume $L \not\subseteq \mathcal{W}$.

(i) If $d_i \geq 2$, then $\pi^{[d_i]} \circ \phi_i$ maps C_0 into $\Delta_{\mathsf{Hilb}^{d_i}(\mathbb{P}^1)}$. Hence

$$\pi^{[d]} \circ f = \sum_i \pi^{[d_i]} \circ \phi_i$$

maps C_0 into $\Delta_{\mathsf{Hilb}^d(\mathbb{P}^1)} \subset \mathcal{W}$. Since $L = \pi^{[d]} \circ f(C_0)$, we find $L \subset \mathcal{W}$, which is a contradiction.

(ii) If $\phi_i \colon C_0 \to S$ is nonconstant, then $\pi \circ \phi_i$ maps C_0 to a basepoint of a nodal fiber of $\pi \colon S \to \mathbb{P}^1$. By an argument identical to (i) this implies $L \subset \mathcal{W}$, which is a contradiction. Hence ϕ_i is constant.

(iii) The universal family of curves on the elliptic K3 surface $\pi \colon S \to \mathbb{P}^1$ in class $\beta_h = B + hF$ is the h–dimensional linear system

$$|\beta_h| = \mathsf{Hilb}^h(\mathbb{P}^1) = \mathbb{P}^h.$$

Explicitly, an element $z \in \mathsf{Hilb}^h(\mathbb{P}^1)$ corresponds to the comb curve

$$B_0 + \pi^{-1}(z) \subset S, \tag{27}$$

where $\pi^{-1}(z)$ denotes the fiber of π over the subscheme $z \subset \mathbb{P}^1$.[4]

Let $\mathcal{Z}_d \to \mathsf{Hilb}^d(S)$ be the universal family and consider the fiber diagram

$$\begin{array}{ccccc} \widetilde{C}_0 & \xrightarrow{\ \tilde{f}\ } & \mathcal{Z}_d & \xrightarrow{\ q\ } & S \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} & & \\ C_0 & \xrightarrow{\ f\ } & \mathsf{Hilb}^d(S) & & \end{array}$$

By Lemma 5, the map $f' = q \circ \tilde{f} \colon \widetilde{C}_0 \to S$ is a curve in the linear system $|\beta_{h'}|$ for some $h' \leq h$. Its image is therefore a comb of the form (27).

[4] To see this, let C be a curve in class β_h. Since $\mathrm{Pic}(S)$ is discrete, the restriction of C to a general fiber of π is linear equivalent to the restriction of the zero section B_0. This forces $B_0 \subset C$ so $C = B_0 + \pi^{-1}(z)$ for some $z \subset \mathbb{P}^1$.

Let G_0 be the irreducible component of $\widetilde{C}_0$ such that $\pi \circ f'|_{G_0}$ is nonconstant. The restriction

$$\tilde{p}|_{G_0}\colon G_0 \to C_0 \tag{28}$$

is flat. Since $\pi \circ f'\colon \widetilde{C}_0 \to \mathbb{P}^1$ has degree 1, the curve $\widetilde{C}_0$ has multiplicity 1 at G_0, and the map to the Hilbert scheme of S associated to (28) is equal to ϕ_0.

Since G_0 is reduced and $f'|_{G_0}\colon G_0 \to S$ maps to B_0, the map ϕ_0 maps with degree 1 to $\mathrm{Hilb}^{d_0}(B_0)$. The proof of (iii) is complete. □

The normal bundle of a line Let $s^{[d]}\colon \mathrm{Hilb}^d(\mathbb{P}^1) \hookrightarrow \mathrm{Hilb}^d(S)$ be the section, and consider the normal bundle

$$N = s^{[d]*} T_{\mathrm{Hilb}^d(S)} / T_{\mathrm{Hilb}^d(\mathbb{P}^1)}.$$

Lemma 14 *For every line $L \subset \mathrm{Hilb}^d(\mathbb{P}^1)$,*

$$T_{\mathrm{Hilb}^d(S)}|_L = T_{\mathrm{Hilb}^d(\mathbb{P}^1)}|_L \oplus N|_L$$

with $N|_L = T^\vee_{\mathrm{Hilb}^d(\mathbb{P}^1)}|_L = \mathcal{O}_L(-2) \oplus \mathcal{O}_L(-1)^{\oplus(d-1)}$.

Proof Because the embedding $s^{[d]}\colon \mathrm{Hilb}^d(\mathbb{P}^1) \hookrightarrow \mathrm{Hilb}^d(S)$ has the right inverse $\pi^{[d]}$, the restriction

$$T_{\mathrm{Hilb}^d(S)}|_{\mathrm{Hilb}^d(\mathbb{P}^1)}$$

splits as a direct sum of the tangent and normal bundle of $\mathrm{Hilb}^d(B_0)$.

The vanishing $H^0(\mathbb{P}^d, \Omega^2_{\mathbb{P}^d}) = 0$ implies that the holomorphic symplectic form on $\mathrm{Hilb}^d(S)$ restricts to 0 on $\mathrm{Hilb}^d(\mathbb{P}^1)$ and hence, by nondegeneracy, induces an isomorphism

$$T_{\mathrm{Hilb}^d(\mathbb{P}^1)} \to N^\vee.$$

Since $T_{\mathrm{Hilb}^d(\mathbb{P}^1)}|_L = \mathcal{O}_L(1)^{\oplus(d-1)} \oplus \mathcal{O}_L(2)$, the proof is complete. □

2.3 Analysis of the moduli space

2.3.1 Overview Let S be the fixed elliptic Bryan–Leung K3 surface, let $z_1, z_2 \in \mathrm{Hilb}^d(\mathbb{P}^1)$ be generic points, and for $i \in \{1, 2\}$ let

$$Z_i = \pi^{[d]-1}(z_i) \subset \mathrm{Hilb}^d(S)$$

be the fiber of $\pi^{[d]}$ over z_i. The subscheme Z_i has class $[Z_i] = \mathfrak{p}_{-1}(F)^d 1_S$. Let

$$\mathrm{ev}\colon \overline{M}_{0,2}(\mathsf{Hilb}^d(S), \beta_h + kA) \to \mathsf{Hilb}^d(S) \times \mathsf{Hilb}^d(S)$$

be the evaluation map from the moduli space of genus-0 stable maps in class $\beta_h = B + hF$, and define the moduli space

$$M_Z = M_Z(h,k) = \mathrm{ev}^{-1}(Z_1 \times Z_2)$$

parametrizing maps which are incident to Z_1 and Z_2.

In Section 2.3, we begin the proof of Theorem 10 by studying the moduli space M_Z and its virtual class. First, we prove that M_Z is naturally isomorphic to a product of moduli spaces associated to specific fibers of the elliptic fibration $\pi\colon S \to \mathbb{P}^1$. Second, we show that the virtual class splits as a product of virtual classes on each factor. Both results are summarized in Proposition 16. As a consequence, Theorem 10 is reduced to the evaluation of a series $F^{\mathrm{GW}}(y,q)$ encoding integrals associated to specific fibers of π.

2.3.2 The set-theoretic product Consider a stable map

$$[f\colon C \to \mathsf{Hilb}^d(S), p_1, p_2] \in M_Z$$

with markings $p_1, p_2 \in C$. By definition of M_Z, we have

$$\pi^{[d]}(f(p_1)) = z_1, \quad \pi^{[d]}(f(p_2)) = z_2.$$

Hence, the image of C under $\pi^{[d]} \circ f$ is the unique line

$$L \subset \mathbb{P}^d$$

incident to the points $z_1, z_2 \in \mathbb{P}^d$. Because $z_1, z_2 \in \mathbb{P}^d$ are generic, also L is generic. In particular, since $z_1 \cap z_2 = \varnothing$, we have

$$L \not\subseteq I(x) \quad \text{for all } x \in \mathbb{P}^1. \tag{29}$$

Let C_0 be the distinguished irreducible component of C on which $\pi \circ f$ is nonconstant. By (29), the restriction $f|_{C_0}$ is irreducible, and by Lemma 13(iii), the map $f|_{C_0}$ is an isomorphism onto the embedded line

$$L \subset \mathsf{Hilb}^d(\mathbb{P}^1) \overset{s}{\subset} \mathsf{Hilb}^d(S).$$

We will identify C_0 with L via this isomorphism.

Let $x_1, \dots, x_{24} \in \mathbb{P}^1$ be the basepoints of the nodal fibers of π, and let

$$y_1, \dots, y_{2d-2} \in \mathbb{P}^1$$

be the points such that $2y_i \subset z$ for some $z \in L$. For $x \in \mathbb{P}^1$, let

$$\tilde{x} = I(x) \cap L \in \mathsf{Hilb}^d(\mathbb{P}^1)$$

denote the unique point on L which is incident to x. Then the points

$$\tilde{x}_1, \dots, \tilde{x}_{24}, \tilde{y}_1, \dots, \tilde{y}_{2d-2} \tag{30}$$

are the intersection points of L with the discriminant of $\pi^{[d]}$ defined in (26). Hence, by Lemma 12, components of C can be attached to C_0 only at the points (30). Consider the decomposition

$$C = C_0 \cup A_1 \cup \dots \cup A_{24} \cup B_1 \cup \dots \cup B_{2d-2}, \tag{31}$$

where A_i and B_j are the components of C attached to the points $\tilde{x}_i$ and $\tilde{y}_j$ respectively. We consider the restriction of f to A_i and B_j respectively.

A_i: Let $\tilde{x}_i = x_i + w_1 + \dots + w_{d-1}$ for some points $w_\ell \in \mathbb{P}^1$. By genericity of L, the w_ℓ are basepoints of smooth elliptic fibers. Hence, $f|_{A_i}$ decomposes as

$$f|_{A_i} = \phi + w_1 + \dots + w_{d-1}, \tag{32}$$

where $w_\ell \in \mathbb{P}^1 \subset S$ for all ℓ denote constant maps, and $\phi\colon A_i \to F_{x_i}$ is a map to the i^{th} nodal fiber which sends $\tilde{x}_i$ to the point $s(x_i) \in S$.

B_j: Let $\tilde{y}_j = 2y_j + w_1 + \dots + w_{d-2}$ for some points $w_\ell \in \mathbb{P}^1$. Then $f|_{B_j}$ decomposes as

$$f|_{B_j} = \phi + w_1 + \dots + w_{d-2}, \tag{33}$$

where $\phi\colon B_j \to \mathsf{Hilb}^2(S)$ maps to the fiber $(\pi^{[2]})^{-1}(2y)$ and sends the point $\tilde{y}_j \in L \equiv C_0$ to $s(2y_j)$.

Since L is independent of f, we conclude that the moduli space M_Z is *set-theoretically* a product[5] of moduli spaces of maps of the form $f|_{A_i}$ and $f|_{B_j}$. The next step is to prove the splitting is *scheme-theoretic*.

[5] That is, the set of $\mathbb{C}$–valued points of M_Z is a product.

2.3.3 Deformation theory Let $[f\colon C \to \mathsf{Hilb}^d(S), p_1, p_2] \in M_Z$ be a point and let

$$\begin{array}{ccc} \widehat{C} & \xrightarrow{\widehat{f}} & \mathsf{Hilb}^d(S) \\ {\scriptstyle \widehat{p}_1,\widehat{p}_2}\uparrow\uparrow\downarrow{\scriptstyle p} & & \\ \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2) & & \end{array} \tag{34}$$

be a first-order deformation of f inside M_Z. In particular, p is a flat map, $\widehat{p}_1, \widehat{p}_2$ are sections of p, and $\widehat{f}$ restricts to f at the closed point.

Consider the decomposition (31) and let $\widetilde{x}_i$ for $i = 1, \dots, 24$ and $\widetilde{y}_j$ for $j = 1, \dots, 2d-2$ be the node points $A_i \cap C_0$ and $B_j \cap C_0$, respectively.

Lemma 15 *The deformation (34) does not resolve the nodal points $\widetilde{x}_1, \dots, \widetilde{x}_{24}$ and $\widetilde{y}_1, \dots, \widetilde{y}_{2d-2}$.*

Proof Assume $\widehat{f}$ smooths the node $\widetilde{x}_i$ for some i. Let $\mathcal{Z}_d \to \mathsf{Hilb}^d(S)$ be the universal family and consider the pullback diagram

$$\begin{array}{ccccc} f^*\mathcal{Z}_d = \widetilde{C} & \longrightarrow & \mathcal{Z}_d & \longrightarrow & S \\ \downarrow & & \downarrow & & \\ C & \xrightarrow{f} & \mathsf{Hilb}^d(S) & & \end{array}$$

Let E be the connected component of $f|_{A_i}^*\mathcal{Z}_d$, which defines the nonconstant map ϕ in the decomposition (32), and let $G_0 = f|_{C_0}^*\mathcal{Z}_d$. Then the projection $\widetilde{C} \to C$ is étale at the intersection point $q = G_0 \cap E$.

The deformation $\widehat{f}\colon \widehat{C} \to \mathsf{Hilb}^d(S)$ induces the deformation

$$K = \widehat{f}^*\mathcal{Z}_d \to \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$$

of the curve $\widetilde{C}$. Since $\widehat{C}$ smooths $\widetilde{x}_i$ and $\widetilde{C} \to C$ is étale near q, the deformation K resolves q. Then the natural map $K \to S$ defines a deformation of the curve $\widetilde{C} \to S$ which resolves q. Since $\widetilde{C} \to S$ has class β_h, such a deformation can not exist by the geometry of the linear system $|\beta_h|$. Hence, $\widehat{f}$ does not smooth the node $\widetilde{x}_i$.

Assume $\widehat{f}$ smooths the node $\widetilde{y}_j$ for some j. We follow closely the argument of Graber in [13, page 19]. Let F_{y_j} be the fiber of $\pi\colon S \to \mathbb{P}^1$ over y_j, let

$$D(F_{y_j}) = \{\xi \in \mathsf{Hilb}^d(S) \mid \xi \cap F_{y_j} \neq 0\}$$

be the divisor of subschemes with nonzero intersection with F_{y_j}, and consider the divisor

$$D = \Delta_{\mathsf{Hilb}^d(S)} + D(F_{y_j}).$$

Let C_1 be the irreducible component of C that attaches to C_0 at $q = \tilde{y}_j$, and let C_2 be the union of all irreducible components of B_j except C_1. The curves C_2 and C_1 intersect in a finite number of nodes $\{q_i\}$. The deformation $\hat{f}$ resolves the node q and may also resolve some of the q_i.

The first-order neighborhood $\tilde{C}_1$ of C_1 in the total space of the deformation $\hat{C}$ can be identified with the first-order neighborhood of $\mathbb{P}^1$ in the total space of the bundle $\mathcal{O}(-\ell)$, where $\ell \geq 1$ is the number of nodes on C_1 which are smoothed by $\hat{f}$. Let

$$f' \colon \tilde{C}_1 \to \mathsf{Hilb}^d(S)$$

be the induced map on $\tilde{C}_1$. We consider the case where $f'|_{C_1}$ is a degree $k \geq 1$ map to the exceptional curve at $\tilde{y}_j$. The general case is similar.

Let N be the pullback of $\mathcal{O}(D)$ by $f' \colon \tilde{C}_1 \to \mathsf{Hilb}^d(S)$, and let $s \in H^0(\tilde{C}, N)$ be the pullback of the section of $\mathcal{O}(D)$ defined by D. The bundle N restricts to $\mathcal{O}(-2k)$ on C_1. By [13, page 20], giving N and s is equivalent to giving an element of the vector space

$$\mathrm{Hom}_{\mathcal{O}_{C_1}}(\mathcal{O}(-\ell), f|_{C_1}^* \mathcal{O}(D)),$$

which has dimension $\ell - 2k + 1 \leq \ell - 1$.

The neighborhood $\tilde{C}_1$ intersects C_0 in a double point. Since C_0 intersects the divisor D transversely, s is nonzero on $\tilde{C}_1$. Let $q_1, \dots, q_{\ell-1}$ be the other nodes on C_1 which get resolved by $\hat{f}$. Since $C_2 \subset D$, the section s vanishes at $q_1, \dots, q_{\ell-1}$. For dimension reasons, we find $s = 0$. This contradicts the nonvanishing of s. Hence $\hat{f}$ does not smooth the node $\tilde{y}_j$. □

By Lemma 15, any first-order (and hence any infinitesimal) deformation of

$$[f \colon C \to \mathsf{Hilb}^d(S), p_1, p_2] \in M_Z$$

inside M preserves the decomposition

$$C = C_0 \cup_i A_i \cup_j B_j$$

and therefore induces a deformation of the restriction

$$f|_{C_0} \colon C_0 \xrightarrow{\cong} L \subset \mathsf{Hilb}^d(S). \tag{35}$$

By Lemma 14, every deformation of $L \subset \mathsf{Hilb}^d(S)$ moves the line L in the projective space $\mathsf{Hilb}^d(B_0)$. Since any deformations of f inside M_Z must stay incident to $Z_1, Z_2 \subset \mathsf{Hilb}^d(S)$, we conclude that such deformations induce the constant deformation of (35). The image line $f(C_0)$ stays completely fixed.

2.3.4 The product decomposition For $h > 0$ and for $x \in \mathbb{P}^1$ a basepoint of a nodal fiber of $\pi\colon S \to \mathbb{P}^1$, let

$$M_x^{(\mathrm{N})}(h)$$

be the moduli space of 1–marked genus-0 stable maps to S in class hF which map the marked point to $s(x)$. Hence, $M_x^{(\mathrm{N})}(h)$ parametrizes degree h covers of the nodal fiber F_x. By convention, $M_x^{(\mathrm{N})}(0)$ is taken to be a point.

For $h \geq 0$, $k \in \mathbb{Z}$ and for $y \in \mathbb{P}^1$ a basepoint of a smooth fiber of π, let

$$M_y^{(\mathrm{F})}(h,k) \tag{36}$$

be the moduli space of 1–marked genus-0 stable maps to $\mathsf{Hilb}^2(S)$ in class $hF + kA$ which map the marked point to $s^{[2]}(2y)$. By convention, $M_y^{(\mathrm{F})}(0,0)$ is taken to be a point.

Let T be a connected scheme and consider a family

$$\begin{array}{ccc} C & \xrightarrow{F} & \mathsf{Hilb}^d(S) \\ \downarrow & & \\ T & & \end{array} \tag{37}$$

of stable maps in M_Z. By Lemma 15, the curve C admits a decomposition

$$C = C_0 \cup A_1 \cup \cdots \cup A_{24} \cup B_1 \cup \cdots \cup B_{2d-2},$$

where C_0 is the distinguished component of C and the components A_i and B_j are attached to C_0 at the points $\tilde{x}_i$ and $\tilde{y}_j$, respectively.

The restriction of the family (37) to the components A_i (resp. B_j) defines a family in the moduli space $M_{x_i}^{(\mathrm{N})}(h_{x_i})$ (resp. $M_{y_j}^{(\mathrm{F})}(h_{y_j}, k_{y_j})$) for some h_{x_i} (resp. h_{y_j}, k_{y_j}). Since, by Section 1.3, the line $f(C_0) = L$ has class

$$[L] = B - (d-1)A \in H_2(\mathsf{Hilb}^d(S), \mathbb{Z}),$$

and by the additivity of cohomology classes under decomposing (Lemma 9), we must have $\sum_i h_{x_i} + \sum_j h_{y_j} = h$ and $\sum_j k_{y_j} = k + (d-1)$. Let

$$\Psi\colon M_Z \to \bigsqcup_{\boldsymbol{h},\boldsymbol{k}} \left(\prod_{i=1}^{24} M^{(\mathrm{N})}_{x_i}(h_{x_i}) \times \prod_{j=1}^{2d-2} M^{(\mathrm{F})}_{y_j}(h_{y_j}, k_{y_j}) \right) \tag{38}$$

be the induced map on moduli spaces, where the disjoint union runs over all

$$\begin{aligned} \boldsymbol{h} &= (h_{x_1}, \dots, h_{x_{24}}, h_{y_1}, \dots, h_{y_{2d-2}}) \in (\mathbb{N}^{\geq 0})^{\{x_i, y_j\}}, \\ \boldsymbol{k} &= (k_{y_1}, \dots, k_{y_{2d-2}}) \in \mathbb{Z}^{2d-2} \end{aligned} \tag{39}$$

such that

$$\sum_i h_{x_i} + \sum_j h_{y_j} = h \quad \text{and} \quad \sum_j k_{y_j} = k + (d-1). \tag{40}$$

Since $L \subset \mathsf{Hilb}^d(S)$ is fixed under deformations, we can glue elements of the right-hand side of (38) to C_0 and obtain a map in M_Z. By a direct verification, the induced morphism on moduli spaces is the inverse to Ψ. Hence, Ψ is an isomorphism.

2.3.5 The virtual class Let Z_1, Z_2 be the Lagrangian fibers of $\pi^{[d]}$ defined in Section 2.3.1, and let $Z = Z_1 \times Z_2$. We consider the fiber square

$$\begin{array}{ccc} M_Z & \xrightarrow{\ j\ } & M \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \mathrm{ev}} \\ Z & \xrightarrow{\ i\ } & (\mathsf{Hilb}^d(S))^2 \end{array} \tag{41}$$

where $M = \overline{M}_{0,2}(\mathsf{Hilb}^d(S), \beta_h + kA)$. The map i is the inclusion of a smooth subscheme of codimension $2d$. Hence, the restricted virtual class

$$[M_Z]^{\mathrm{vir}} = i^{!}[M]^{\mathrm{red}} \tag{42}$$

is of dimension 0. By the push-pull formula we have

$$\int_{[M_Z]^{\mathrm{vir}}} 1 = \langle \mathfrak{p}_{-1}(F)^d 1_S, \mathfrak{p}_{-1}(F)^d 1_S \rangle^{\mathsf{Hilb}^d(S)}_{\beta_h + kA}. \tag{43}$$

Let Ψ be the splitting morphism (38). We will show that $\Psi_*[M_Z]^{\mathrm{vir}}$ splits naturally as a product of virtual cycles.

Let $\mathbb{L}_X$ denote the cotangent complex on a space X. Let $E^\bullet \to \mathbb{L}_M$ be the reduced perfect obstruction theory on M, and let $F^\bullet$ be the cone of the map

$$p^* i^* \Omega_{(\mathrm{Hilb}^d(S))^2} \to j^* E^\bullet \oplus p^* \Omega_Z$$

induced by the diagram (41). The cone $F^\bullet$ maps to $\mathbb{L}_{M_Z}$ and defines a perfect obstruction theory on M_Z. By [2, Proposition 5.10], the associated virtual class is $[M_Z]^{\mathrm{vir}}$.

Let $[f\colon C \to \mathrm{Hilb}^d(S), p_1, p_2] \in M_Z$ be a point. For simplicity, we consider all complexes on the level of tangent spaces at the moduli point $[f]$. Let $E_\bullet$ and $F_\bullet$ denote the derived duals of $E^\bullet$ and $F^\bullet$, respectively.

We recall the construction of $E_\bullet$; see [29; 41]. Consider the semiregularity map

$$b\colon R\Gamma(C, f^* T_{\mathrm{Hilb}^d(S)}) \to V[-1], \tag{44}$$

where $V = H^0(\mathrm{Hilb}^d(S), \Omega^2_{\mathrm{Hilb}^d(S)})^\vee$, and recall the ordinary (nonreduced) perfect obstruction theory of M at the point $[f]$,

$$E_\bullet^{\mathrm{vir}} = \mathrm{Cone}\big(R\Gamma(C, \mathbb{T}_C(-p_1 - p_2)) \to R\Gamma(C, f^* T_{\mathrm{Hilb}^d(S)})\big),$$

where $\mathbb{T}_C = \mathbb{L}_C^\vee$ is the tangent complex on C. Then, by the vanishing of the composition

$$R\Gamma(C, \mathbb{T}_C(-p_1 - p_2)) \to R\Gamma(C, f^* T_{\mathrm{Hilb}^d(S)}) \xrightarrow{b} V[-1], \tag{45}$$

the map (44) induces a morphism $\bar{b}\colon E_\bullet^{\mathrm{vir}} \to V[-1]$ with cocone $E_\bullet$.

By a diagram chase, $F_\bullet$ is the cocone of

$$(\bar{b}, d\,\mathrm{ev})\colon E_\bullet^{\mathrm{vir}} \to V[-1] \oplus N_{Z,(z_1,z_2)},$$

where z_1 and z_2 are the basepoints of the Lagrangian fibers Z_1 and Z_2, respectively, $N_{Z,(z_1,z_2)}$ is the normal bundle of Z in $\mathrm{Hilb}^d(S)^2$ at (z_1, z_2), and $d\,\mathrm{ev}$ is the differential of the evaluation map. Since taking the cone and cocone commutes, the complex $F_\bullet$ is therefore the cone of

$$\gamma\colon R\Gamma(C, \mathbb{T}_C(-p_1 - p_2)) \to K, \tag{46}$$

where

$$K = \mathrm{Cocone}\big[(b, d\,\mathrm{ev})\colon R\Gamma(C, f^* T_{\mathrm{Hilb}^d(S)}) \to V[-1] \oplus N_{Z,(z_1,z_2)}\big]. \tag{47}$$

Consider the decomposition

$$C = C_0 \cup A_1 \cup \cdots \cup A_{24} \cup B_1 \cup \cdots \cup B_{2d-2}, \tag{48}$$

where the components A_i and B_j are attached to C_0 at the points $\tilde{x}_i$ and $\tilde{y}_j$, respectively. Tensoring $R\Gamma(C, \mathbb{T}_C(-p_1 - p_2))$ and K against the partial renormalization sequence associated to decomposition (48), we will show that the dependence on L cancels in the cone of (46).

The map $(b, d\,\mathrm{ev})$ fits into the diagram

$$\begin{array}{ccc} R\Gamma(C, f^*T_{\mathsf{Hilb}^d(S)}) & \xrightarrow{u} & R\Gamma(L, f^*T_{\mathsf{Hilb}^d(S)}) \\ \downarrow{\scriptstyle (b,d\,\mathrm{ev})} & & \downarrow{\scriptstyle v=(b,d\,\mathrm{ev})} \\ V[-1] \oplus N_{Z,(z_1,z_2)} & \xrightarrow{(\sigma,\mathrm{id})} & V[-1] \oplus N_{Z,(z_1,z_2)} \end{array} \tag{49}$$

where u is the restriction map and σ is the induced map.[6] By Lemma 14, the cocone of v is $R\Gamma(\mathbb{T}_L(-p_1 - p_2))$.

The partial normalization sequence of C with respect to $\tilde{x}_i$ and $\tilde{y}_j$ is

$$0 \to \mathcal{O}_C \to \mathcal{O}_L \oplus \bigoplus_{D \in \{A_i, B_j\}} \mathcal{O}_D \to \bigoplus_{s \in \{\tilde{x}_i, \tilde{y}_j\}} \mathcal{O}_{C,s} \to 0. \tag{50}$$

Tensoring (50) with $f^*T_{\mathsf{Hilb}^d(S)}$, applying $R\Gamma(\cdot)$ and factoring with (49), we obtain the exact triangle

$$K \to R\Gamma(L, \mathbb{T}_L(-p_1 - p_2)) \oplus \bigoplus_D R\Gamma(D, f^*_{|D} T_{\mathsf{Hilb}^d(S)}) \to \bigoplus_s T_{\mathsf{Hilb}^d(S),s} \to K[1]. \tag{51}$$

For each node $t \in C$, let N_t (resp. T_t) be the tensor product (resp. the direct sum) of the tangent spaces to the branches of C at t. Tensoring (50) with $\mathbb{T}_C(-p_1 - p_2)$ and applying $R\Gamma(\cdot)$, we obtain the exact triangle

$$R\Gamma\mathbb{T}_C(-p_1 - p_2) \to R\Gamma(\mathbb{T}_L(-p_1 - p_2)) \oplus \bigoplus_D R\Gamma(\mathbb{T}_D) \oplus \bigoplus_t N_t[-1] \to \bigoplus_t T_t \to \cdots. \tag{52}$$

By the vanishing of (45) (applied to $C = L$), the sequence (52) maps naturally to (51). Consider the restriction of this map to the summand $R\Gamma(\mathbb{T}_L(-p_1 - p_2))$ which appears

[6] Here σ is the inverse to the natural isomorphism in the other direction induced by the sequence of surjections $H^1(C, \Omega_C) \to \bigoplus H^1(C_i, \Omega_{C_i}) \to H^1(C, \omega_C) \to 0$.

in the second term of (52):

$$\varphi\colon R\Gamma(\mathbb{T}_L(-p_1-p_2)) \to R\Gamma(L,\mathbb{T}_L(-p_1-p_2)) \oplus \bigoplus_D R\Gamma(D, f_{|D}^{*} T_{\mathsf{Hilb}^d(S)}).$$

Then the composition of φ with the projection to $R\Gamma(L,\mathbb{T}_L(-p_1-p_2))$ is the identity. Hence, $F_\bullet = \mathrm{Cone}(\gamma)$ admits the exact sequence

$$F_\bullet \to \bigoplus_D G_D \xrightarrow{\psi} \bigoplus_D H_D \to F_\bullet[1], \tag{53}$$

where D runs over all A_i and B_j, and

$$G_D = \mathrm{Cone}\Big[R\Gamma(\mathbb{T}_D) \oplus \bigoplus_t N_t[-1] \to R\Gamma(D, f_{|D}^{*} T_{\mathsf{Hilb}^d(S)})\Big],$$

$$H_D = \mathrm{Cone}\Big[\bigoplus_t T_t \to \bigoplus_t T_{\mathsf{Hilb}^d(S),t}\Big].$$

Here $t = t(D) = D \cap C_0$ is the attachment point of the component D.

The map ψ in (53) maps the factor G_D to H_D for all D. For $D = A_i$ consider the decomposition

$$f|_{A_i} = \phi + w_1 + \cdots + w_{d-1}.$$

The trivial factors which arise in G_D and H_D from the tangent space of $\mathsf{Hilb}^d(S)$ at the points $w_1,\dots,w_{d-1}$ cancel each other in $\mathrm{Cone}(G_D \to H_D)$. Hence $\mathrm{Cone}(G_D \to H_D)$ only depends on $\phi\colon C \to S$, and therefore only on the image of $[f]$ in the factor $M^{(\mathrm{N})}_{x_i}(h_{x_i})$, where $M^{(\mathrm{N})}_{x_i}(h_{x_i})$ is the moduli space defined in Section 2.3.4. The case $D = B_j$ is similar.

Hence, $F_\bullet$ splits into a sum of complexes pulled back from each factor of the product splitting (38). Since $F_\bullet$ is a perfect obstruction theory on M, the complexes on each factor are perfect obstruction theories. Let

$$[M^{(\mathrm{N})}_{x_i}(h_{x_i})]^{\mathrm{vir}} \quad \text{and} \quad [M^{(\mathrm{F})}_{y_j}(h_{y_j},k_{y_j})]^{\mathrm{vir}}$$

be their virtual classes, respectively. We have proved the following.

Proposition 16 *Let Ψ be the splitting morphism (38). Then Ψ is an isomorphism and we have*

$$\Psi_*[M_Z]^{\mathrm{vir}} = \sum_{\boldsymbol{h},\boldsymbol{k}} \bigg(\prod_{i=1}^{24} [M^{(\mathrm{N})}_{x_i}(h_{x_i})]^{\mathrm{vir}} \times \prod_{j=1}^{2d-2} [M^{(\mathrm{F})}_{y_j}(h_{y_j},k_{y_j})]^{\mathrm{vir}} \bigg),$$

where the sum is over the set (39) satisfying (40).

2.3.6 The series F^{GW} We consider the left-hand side of Theorem 10. By (43), we have

$$\langle \mathfrak{p}_{-1}(F)^d 1_S, \mathfrak{p}_{-1}(F)^d 1_S \rangle_q^{\mathsf{Hilb}^d(S)} = \sum_{h \geq 0} \sum_{k \in \mathbb{Z}} y^k q^{h-1} \int_{[M_Z(h,k)]^{\mathrm{vir}}} 1.$$

By Proposition 16, this equals

$$\begin{aligned}
\sum_{\substack{h\geq 0\\ k\in\mathbb{Z}}} y^k q^{h-1} & \sum_{\substack{(\boldsymbol{h},\boldsymbol{k})\\ \sum_i h_{x_i} + \sum_j h_{y_j} = h \\ \sum_j k_{y_j} = k + (d-1)}} \left(\prod_{i=1}^{24} \int_{[M^{(\mathrm{N})}_{x_i}(h_{x_i})]^{\mathrm{vir}}} 1 \right) \cdot \left(\prod_{j=1}^{2d-2} \int_{[M^{(\mathrm{F})}_{y_j}(h_{y_j}, k_{y_j})]^{\mathrm{vir}}} 1 \right) \\
&= y^{-(d-1)} q^{-1} \left(\prod_{i=1}^{24} \sum_{h_{x_i} \geq 0} q^{h_{x_i}} \int_{[M^{(\mathrm{N})}_{x_i}(h_{x_i})]^{\mathrm{vir}}} 1 \right) \\
&\qquad \times \left(\prod_{j=1}^{2d-2} \sum_{\substack{h_{y_j} \geq 0 \\ k_{y_j} \in \mathbb{Z}}} y^{k_{y_j}} q^{h_{y_j}} \int_{[M^{(\mathrm{F})}_{y_j}(h_{y_j}, k_{y_j})]^{\mathrm{vir}}} 1 \right) \\
&= \left(\prod_{i=1}^{24} \sum_{h\geq 0} q^{h-\frac{1}{24}} \int_{[M^{(\mathrm{N})}_{x_i}(h)]^{\mathrm{vir}}} 1 \right) \cdot \left(\prod_{i=1}^{2d-2} \sum_{\substack{h\geq 0\\ k\in\mathbb{Z}}} q^h y^{k-\frac{1}{2}} \int_{[M^{(\mathrm{F})}_{y_j}(h,k)]^{\mathrm{vir}}} 1 \right).
\end{aligned}$$

The integrals in the first factor were calculated by Bryan and Leung in their proof of the Yau–Zaslow conjecture [3]. The result is

$$\sum_{h\geq 0} q^h \int_{[M^{(\mathrm{N})}_{x_i}(h)]^{\mathrm{vir}}} 1 = \prod_{m \geq 0} \frac{1}{1-q^m}. \tag{54}$$

By deformation invariance, the integrals

$$\int_{[M^{(\mathrm{F})}_{y_j}(h,k)]^{\mathrm{vir}}} 1$$

only depend on h and k. Define the generating series

$$F^{\mathrm{GW}}(y,q) = \sum_{h\geq 0} \sum_{k \in \mathbb{Z}} q^h y^{k-\frac{1}{2}} \int_{[M^{(\mathrm{F})}_{y_j}(h,k)]^{\mathrm{vir}}} 1. \tag{55}$$

By our convention on $M^{(\mathrm{F})}_{y_j}(0,0)$, the $y^{-1/2}q^0$–coefficient of F^{GW} is 1.

Let $\Delta(q) = q\prod_{m\geq 1}(1-q^m)^{24}$ be the modular discriminant $\Delta(\tau)$ considered as a formal expansion in the variable $q = e^{2\pi i\tau}$. We conclude that

$$\langle \mathfrak{p}_{-1}(F)^d 1_S, \mathfrak{p}_{-1}(F)^d 1_S \rangle_q^{\mathsf{Hilb}^d(S)} = \frac{F^{\mathrm{GW}}(y,q)^{2d-2}}{\Delta(q)}.$$

Theorem 10 now follows directly from Theorem 17 below.

2.4 Evaluation of F^{GW} and the Kummer K3

Let F be the theta function which already appeared in Section 0.3,

$$F(z,\tau) = \frac{\vartheta_1(z,\tau)}{\eta^3(\tau)} = (y^{1/2} + y^{-1/2}) \prod_{m\geq 1} \frac{(1+yq^m)(1-y^{-1}q^m)}{(1-q^m)^2},$$

where $q = e^{2\pi i\tau}$ and $y = -e^{2\pi i z}$.

Theorem 17 *Under the variable change $q = e^{2\pi i\tau}$ and $y = -e^{2\pi i z}$,*

$$F^{\mathrm{GW}}(y,q) = F(z,\tau).$$

In Section 2.4 we present a proof of Theorem 17 using the Kummer K3 surface and the Yau–Zaslow formula. An independent proof is given in Section 4 through the geometry of $\mathsf{Hilb}^2(\mathbb{P}^1 \times E)$, where E is an elliptic curve.

The Yau–Zaslow formula was used in the geometry of Kummer K3 surfaces before by S Rose [40] to obtain virtual counts of hyperelliptic curves on abelian surfaces. While the geometry used in [40] is similar to our setting, the closed formula of Theorem 17 in terms of the Jacobi theta function F is new. For example, Theorem 17 yields a new, closed formula for hyperelliptic curve counts on an abelian surface; see [4].

2.4.1 The Kummer K3 Let A be an abelian surface. The *Kummer* of A is the blowup

(56) $$\rho\colon \mathrm{Km}(\mathsf{A}) \to \mathsf{A}/\pm 1$$

of $\mathsf{A}/\pm 1$ along its 16 singular points. It is a smooth projective K3 surface. Alternatively, consider the composition

$$s\colon \mathsf{Hilb}^2(\mathsf{A}) \to \mathrm{Sym}^2(\mathsf{A}) \to \mathsf{A}$$

of the Hilbert–Chow morphism with the addition map. Then $\mathrm{Km}(\mathsf{A})$ is the fiber of s over the identity element $0_{\mathsf{A}} \in \mathsf{A}$:

$$\mathrm{Km}(\mathsf{A}) = s^{-1}(0_{\mathsf{A}}). \tag{57}$$

Let E and E' be generic elliptic curves and let

$$\mathsf{A} = E \times E'.$$

Let $t_1, \dots, t_4$ and $t'_1, \dots, t'_4$ denote the 2–torsion points of E and E', respectively. The *exceptional curves* of $\mathrm{Km}(\mathsf{A})$ are the divisors

$$A_{ij} = \rho^{-1}((t_i, t'_j)) \quad \text{for } i, j = 1, \dots, 4.$$

The projection of A to the factor E induces the elliptic fibration

$$p\colon \mathrm{Km}(\mathsf{A}) \to \mathsf{A}/\pm 1 \to E/\pm 1 = \mathbb{P}^1.$$

Hence, $\mathrm{Km}(\mathsf{A})$ is an elliptically fibered K3 surface. Similarly, we let $p'\colon \mathrm{Km}(\mathsf{A}) \to \mathbb{P}^1$ denote the fibration induced by the projection $\mathsf{A} \to E'$. Since E and E' are generic, the fibration p has exactly four sections

$$s_1, \dots, s_4\colon \mathbb{P}^1 \to \mathrm{Km}(\mathsf{A})$$

corresponding to the torsion points $t'_1, \dots, t'_4$ of E'. We write $B_i \subset \mathrm{Km}(\mathsf{A})$ for the image of s_i, and we let F_x denote the fiber of p over $x \in \mathbb{P}^1$.

Let $y_1, \dots, y_4 \in \mathbb{P}^1$ be the image of the 2–torsion points $t_1, \dots, t_4 \in E$ under $E \to E/\pm 1 = \mathbb{P}^1$. The restriction

$$p\colon \mathrm{Km}(\mathsf{A}) \setminus \{F_{y_1}, \dots, F_{y_4}\} \to \mathbb{P}^1 \setminus \{y_1, \dots, y_4\}$$

is an isotrivial fibration with fiber E'. For $i \in \{1, \dots, 4\}$, the fiber F_{y_i} of p over the points y_i is singular with divisor class

$$F_{y_i} = 2T_i + A_{i1} + \dots + A_{i4},$$

where T_i denotes the image of the section of $p'\colon \mathrm{Km}(\mathsf{A}) \to \mathbb{P}^1$ corresponding to the 2–torsion points t_i. We summarize the notation in Figure 1.

Let F and F' be the classes of fibers of p and p', respectively. We have the intersections

$$F^2 = 0, \quad F \cdot F' = 2, \quad F'^2 = 0$$

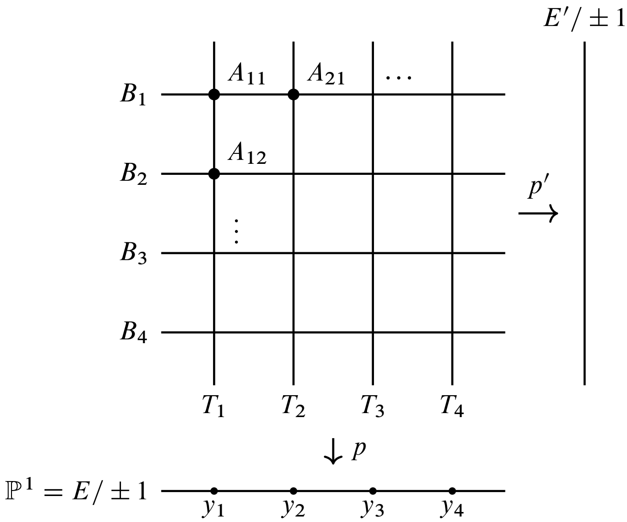

Figure 1: The Kummer K3 of $\mathsf{A} = E \times E'$

and

$$F \cdot A_{ij} = F' \cdot A_{ij} = 0, \quad A_{ij} \cdot A_{k\ell} = -2\delta_{ik}\delta_{j\ell} \quad \text{for all } i,j,k,\ell \in \{1,\dots,4\}.$$

By the relations

$$\begin{aligned} F &= 2T_i + A_{i1} + A_{i2} + A_{i3} + A_{i4}, \\ F' &= 2B_i + A_{1i} + A_{2i} + A_{3i} + A_{4i} \end{aligned} \tag{58}$$

for $i \in \{1,\dots,4\}$, this determines the intersection numbers of all the divisors above.

2.4.2 Rational curves and F^{GW} Let $\beta \in H_2(\mathrm{Km}(\mathsf{A}),\mathbb{Z})$ be an effective curve class and let

$$\langle 1 \rangle_{0,\beta}^{\mathrm{Km}(\mathsf{A})} = \int_{[\overline{M}_0(\mathrm{Km}(\mathsf{A}),\beta)]^{\mathrm{red}}} 1$$

denote the genus-0 Gromov–Witten invariants of $\mathrm{Km}(\mathsf{A})$. For an integer $n \geq 0$ and a tuple $\boldsymbol{k} = (k_{ij})_{i,j=1,\dots,4}$ of half-integers $k_{ij} \in \frac{1}{2}\mathbb{Z}$, define the class

$$\beta_{n,\boldsymbol{k}} = \tfrac{1}{2}F' + \tfrac{1}{2}nF + \sum_{i,j=1}^{4} k_{ij}A_{ij} \in H_2(\mathrm{Km}(\mathsf{A}),\mathbb{Q}).$$

We write $\beta_{n,\boldsymbol{k}} > 0$ if $\beta_{n,\boldsymbol{k}}$ is effective.

Proposition 18 *We have*

$$\sum_{\substack{n,\boldsymbol{k}\\ \beta_{n,\boldsymbol{k}}>0}} \langle 1\rangle^{\mathrm{Km(A)}}_{0,\beta_{n,\boldsymbol{k}}} q^n y^{\sum_{i,j} k_{ij}} = 4\cdot F^{\mathrm{GW}}(y,q)^4,$$

where the sum runs over all $n \geq 0$ *and* $\boldsymbol{k} = (k_{ij})_{i,j} \in \left(\frac{1}{2}\mathbb{Z}\right)^{4\times 4}$ *for which* $\beta_{n,\boldsymbol{k}}$ *is an effective curve class.*

Proof Let $f\colon C \to \mathrm{Km(A)}$ be a genus-0 stable map in class $\beta_{n,\boldsymbol{k}}$. By genericity of E and E', the fibration p has only the sections $B_1,\dots,B_4$. Since $p\circ f$ has degree 1, the image divisor of f is then of the form

$$\mathrm{Im}(f) = B_\ell + D'$$

for some $1\leq \ell\leq 4$ and a divisor D', which is contracted by p. Since the fibration p has fibers isomorphic to E' away from the points $y_1,\dots,y_4\in\mathbb{P}^1$, the divisor D' is supported on the singular fibers F_{y_i}. Hence, there exist nonnegative integers

$$a_i \quad \text{for } i=1,\dots,4 \qquad \text{and} \qquad b_{ij} \quad \text{for } i,j=1,\dots,4$$

such that

$$\mathrm{Im}(f) = B_\ell + \sum_{i=1}^4 a_i T_i + \sum_{i,j=1}^4 b_{ij} A_{ij}.$$

Let C_0 be the component of C which gets mapped by f isomorphically to B_ℓ, and let D_i be the component of C that maps into the fiber F_{y_i}. Then

$$C = C_0 \cup D_1 \cup \cdots \cup D_4, \tag{59}$$

with pairwise disjoint D_i. Under f the intersection points $C_0\cap D_j$ get mapped to $s_\ell(y_j)$, where $s_\ell\colon \mathbb{P}^1 \to \mathrm{Km(A)}$ denotes the ℓ^{th} section of p.

By arguments similar to the proof of Lemma 15 or by the geometry of the linear system $|\beta_{n,\boldsymbol{k}}|$, the nodal points $C_0\cap D_j$ do not smooth under infinitesimal deformations of f. The decomposition (59) is therefore preserved under infinitesimal deformations. This implies that the moduli space $\overline{M}_0(\mathrm{Km(A)},\beta_{n,\boldsymbol{k}})$ admits the decomposition

$$\overline{M}_0(\mathrm{Km(A)},\beta_{n,\boldsymbol{k}}) = \bigsqcup_{\ell=1}^4 \bigsqcup_{n=n_1+\cdots+n_4} \prod_{i=1}^4 M^{(\ell)}_{y_i}\left(n_i, \left(k_{ij}+\tfrac{1}{2}\delta_{j\ell}\right)_j\right), \tag{60}$$

where $M_{y_i}^{(\ell)}(n_i, (k_{ij})_j)$ is the moduli space of stable 1–pointed genus-0 maps to Km(A) in class

$$\tfrac{1}{2}n_i F + \sum_{j=1}^{4} k_{ij} A_{ij}$$

and with marked point mapped to $s_\ell(y_i)$. The term $\frac{1}{2}\delta_{j\ell}$ appears in (60) since

$$B_\ell = \tfrac{1}{2}(F' - A_{1\ell} - A_{2\ell} - A_{3\ell} - A_{4\ell}).$$

For $n_i \geq 0$ and $k_i \in \mathbb{Z}/2$, let

$$M_{y_i}^{(\ell)}(n_i, k_i) = \bigsqcup_{\substack{k_{i1},\dots,k_{i4} \in \mathbb{Z}/2 \\ k_i = k_{i1}+\dots+k_{i4}}} M_{y_i}^{(\ell)}(n_i, (k_{ij})_j) \tag{61}$$

be the moduli space parametrizing stable 1–pointed genus-0 maps to Km(A) in class $\frac{1}{2}n_i F + \sum_j k_{ij} A_{ij}$ for some k_{ij} with $\sum_j k_{ij} = k_i$ and such that the marked points map to $s^\ell(y_i)$.

Let $n \geq 0$ and $k \in \mathbb{Z}/2$ be fixed. Taking the union of (60) over all $\boldsymbol{k}$ such that $k = \sum_{i,j} k_{ij}$, interchanging sum and product and reindexing, we get

$$\bigsqcup_{\boldsymbol{k} : \sum_{i,j} k_{ij} = k} \overline{M}_0(\mathrm{Km(A)}, \beta_{n,\boldsymbol{k}}) = \bigsqcup_{\ell=1}^{4} \bigsqcup_{\substack{n = n_1+\dots+n_4 \\ k+2 = k_1+\dots+k_4}} \prod_{i=1}^{4} M_{y_i}^{(\ell)}(n_i, k_i). \tag{62}$$

Arguments essentially identical to those in Section 2.3.5 imply that the moduli space $M_{y_i}^{(\ell)}(n_i, k_i)$ carries a natural virtual class

$$[M_{y_i}^{(\ell)}(n_i, k_i)]^{\mathrm{vir}} \tag{63}$$

of dimension 0 such that the splitting (62) holds also for virtual classes:

$$\bigsqcup_{\boldsymbol{k} : \sum_{i,j} k_{ij} = k} [\overline{M}_0(\mathrm{Km(A)}, \beta_{n,\boldsymbol{k}})]^{\mathrm{red}} = \bigsqcup_{\ell=1}^{4} \bigsqcup_{\substack{n = n_1+\dots+n_4 \\ k+2 = k_1+\dots+k_4}} \prod_{i=1}^{4} [M_{y_i}^{(\ell)}(n_i, k_i)]^{\mathrm{vir}}. \tag{64}$$

Consider the Bryan–Leung K3 surface $\pi_S \colon S \to \mathbb{P}^1$. Let[7]

$$L \subset \mathsf{Hilb}^2(B)$$

[7] We may restrict here to the Hilbert scheme of two points, since the evaluation of F^{GW} is independent of the number of points.

be a fixed generic line and let $y \in \mathbb{P}^1$ be a point with $2y \in L$. Let

$$M^{(\mathrm{F})}_{S,y}(n,k)$$

be the moduli space parametrizing 1–marked genus-0 stable maps to $\mathsf{Hilb}^2(S)$ in class $nF + kA$, which map the marked point to $s^{[2]}(2y)$; see (36). The subscript S is added to avoid confusion. By Section 2.3.5, the moduli space $M^{(\mathrm{F})}_{S,y}(n,k)$ carries a natural virtual class.

Lemma 19 *We have*

$$\int_{[M^{(\ell)}_{y_i}(n,k)]^{\mathrm{vir}}} 1 = \int_{[M^{(\mathrm{F})}_{S,y}(n,k)]^{\mathrm{vir}}} 1. \tag{65}$$

The lemma is proven below. We finish the proof of Proposition 18. By the decomposition (64),

$$\sum_{n\geq 0} \sum_{\substack{\boldsymbol{k} \\ \beta_{n,\boldsymbol{k}}>0}} \langle 1 \rangle^{\mathrm{Km(A)}}_{0,\beta_{n,\boldsymbol{k}}} q^n y^{\sum_{i,j} k_{ij}} = \sum_{\substack{n\geq 0\\ k\in\mathbb{Z}}} \sum_{\ell=1}^{4} \sum_{\substack{n=n_1+\cdots+n_4\\ k+2=k_1+\cdots+k_4}} \prod_{i=1}^{4} q^{n_i} y^{k_i-\frac{1}{2}} \int_{[M^{(\ell)}_{y_i}(n_i,k_i)]^{\mathrm{vir}}} 1.$$

An application of Lemma 19 then yields

$$\sum_{\ell=1}^{4} \prod_{i=1}^{4} \Bigg(\sum_{\substack{n_i\geq 0\\ k_i\in\mathbb{Z}}} q^{n_i} y^{k_i-\frac{1}{2}} \int_{[M^{(\ell)}_{y_i}(n_i,k_i)]^{\mathrm{vir}}} 1 \Bigg) = 4\cdot (F^{\mathrm{GW}}(y,q))^4.$$

This completes the proof of Proposition 18. □

Proof of Lemma 19 Let $F_y = \pi_S^{-1}(y)$ denote the fiber of π_S over $y \in \mathbb{P}^1$. Consider the deformation of S to the normal cone of F_y,

$$\mathcal{S} = \mathrm{Bl}_{F_y\times 0}(S\times \mathbb{A}^1) \to \mathbb{A}^1,$$

and let $\mathcal{S}^\circ \subset \mathcal{S}$ be the complement of the proper transform of $S\times 0$. The *relative* Hilbert scheme

$$\mathsf{Hilb}^2(\mathcal{S}^\circ/\mathbb{A}^1) \to \mathbb{A}^1 \tag{66}$$

parametrizes length-2 subschemes on the fibers of $\mathcal{S}^\circ \to \mathbb{A}^1$. Let

$$p\colon M' \to \mathbb{A}^1$$

be the moduli space of 1–pointed genus-0 stable maps to $\mathsf{Hilb}^2(\mathcal{S}^\circ/\mathbb{A}^1)$ in class $nF + kA$, with the marked point mapping to the proper transform of $s^{[2]}(2y) \times \mathbb{A}^1$. The fiber of p over $t \neq 0$ is

$$p^{-1}(t) = M^{(\mathrm{F})}_{S,y}(n,k).$$

The fiber over $t = 0$ parametrizes maps to $\mathsf{Hilb}^2(\mathbb{C} \times F_y)$. Since the domain curve has genus 0, these map to a fixed fiber of the natural map

$$\mathsf{Hilb}^2(\mathbb{C} \times F_y) \xrightarrow{\rho} \mathrm{Sym}^2(\mathbb{C} \times F_y) \xrightarrow{+} F_y.$$

We find that $p^{-1}(0)$ parametrizes 1–pointed genus-0 stable maps into a singular D_4 fiber of a trivial elliptic fibration, with given conditions on the class and the marking. Comparing with the construction of $\mathrm{Km}(\mathsf{A})$ via (57) and the definition of $M^{(\ell)}_{y_i}(n_i, k_i)$, one finds

$$p^{-1}(0) \cong M^{(\ell)}_{y_i}(n_i, k_i).$$

The moduli space M' carries the perfect obstruction theory obtained by the construction of Section 2.3 in the relative context. On the fibers over $t \neq 0$ and $t = 0$, the perfect obstruction theory of M' restricts to the perfect obstruction theories of $M^{(\mathrm{F})}_{S,y}(n,k)$ and $M^{(\ell)}_{y_i}(n_i, k_i)$, respectively. Hence, the associated virtual class $[M']^{\mathrm{vir}}$ restricts on the fibers to the previously defined virtual classes:

$$t^![M']^{\mathrm{vir}} = [M^{(\mathrm{F})}_{S,y}(n,k)]^{\mathrm{vir}} \quad \text{for } t \neq 0,$$
$$0^![M']^{\mathrm{vir}} = [M^{(\ell)}_{y_i}(n_i, k_i)]^{\mathrm{vir}}.$$

Since $M' \to \mathbb{A}^1$ is proper, Lemma 19 follows now from the principle of conservation of numbers; see [11, Section 10.2]. □

2.4.3 Effective classes

By Proposition 18, the evaluation of $F^{\mathrm{GW}}(y,q)$ is reduced to the evaluation of the series

$$\sum_{\substack{n,\boldsymbol{k} \\ \beta_{n,\boldsymbol{k}} > 0}} \langle 1 \rangle^{\mathrm{Km}(\mathsf{A})}_{0,\beta_{n,\boldsymbol{k}}} q^n y^{\sum_{i,j} k_{ij}}. \tag{67}$$

Since $\mathrm{Km}(\mathsf{A})$ is a K3 surface, the Yau–Zaslow formula (1) applies to the invariants $\langle 1 \rangle^{\mathrm{Km}(\mathsf{A})}_{\beta}$ when β is effective.[8] The remaining difficulty is to identify precisely the set of effective classes of the form $\beta_{n,\boldsymbol{k}}$.

[8] In fact, the Yau–Zaslow formula applies to all classes $\beta \in H_2(\mathrm{Km}(\mathsf{A}), \mathbb{Z})$ which are of type $(1,1)$ and pair positively with an ample class.

Lemma 20 *Let $n \geq 0$ and $\boldsymbol{k} \in (\mathbb{Z}/2)^{4\times 4}$. If $\beta_{n,\boldsymbol{k}}$ is effective, then there exists a unique $\ell = \ell(n,\boldsymbol{k}) \in \{1,\dots,4\}$ such that*

$$\beta_{n,\boldsymbol{k}} = B_\ell + \sum_{i=1}^{4} a_i T_i + \sum_{i,j=1}^{4} b_{ij} A_{ij}$$

for some integers $a_i \geq 0$ and $b_{ij} \geq 0$.

Proof If $\beta_{n,\boldsymbol{k}}$ is effective, then by the argument in the proof of Proposition 18, there exist nonnegative integers

$$a_i \quad \text{for } i = 1,\dots,4 \qquad \text{and} \qquad b_{ij} \quad \text{for } i,j = 1,\dots,4$$

such that

$$\beta_{n,\boldsymbol{k}} = B_\ell + \sum_{i=1}^{4} a_i T_i + \sum_{i,j=1}^{4} b_{ij} A_{ij}$$

for some $\ell \in \{1,\dots,4\}$. We need to show that ℓ is unique. By (58), we have

$$\beta_{n,\boldsymbol{k}} = \tfrac{1}{2} F' + \tfrac{1}{2} \sum_{i=1}^{4} a_i F + \sum_{i,j=1}^{4} \big(b_{ij} - \tfrac{1}{2} a_i - \tfrac{1}{2}\delta_{j\ell}\big) A_{ij},$$

hence $k_{ij} = b_{ij} - \frac{1}{2} a_{i1} - \frac{1}{2}\delta_{j\ell}$. We find that ℓ is the unique integer such that for every i one of the following holds:

- $k_{ij} \in \mathbb{Z}$ for all $j \neq \ell$ and $k_{i\ell} \notin \mathbb{Z}$, or
- $k_{ij} \notin \mathbb{Z}$ for all $j \neq \ell$ and $k_{i\ell} \in \mathbb{Z}$.

In particular, ℓ is uniquely determined by $\boldsymbol{k}$. □

By the proof of Proposition 18, the contribution from all classes $\beta_{n,\boldsymbol{k}}$ with a given ℓ to the sum (67) is independent of ℓ. Hence, (67) equals

(68) $$4 \cdot \sum_{n,\boldsymbol{k}} \langle 1 \rangle^{\mathrm{Km(A)}}_{0,\beta_{n,\boldsymbol{k}}} q^n y^{\sum_{i,j} k_{ij}},$$

where the sum runs over all $(n,\boldsymbol{k})$ such that $\beta_{n,\boldsymbol{k}}$ is effective *and* $\ell(n,\boldsymbol{k}) = 1$. Hence, we may assume $\ell = 1$ from now on.

It will be useful to rewrite the classes $\beta_{n,\boldsymbol{k}}$ in the basis

(69) $$\{B_1, F\} \cup \{T_i, A_{i2}, A_{i3}, A_{i4} \mid i = 1,\dots,4\}.$$

Consider the class

$$\begin{aligned}\beta_{n,\boldsymbol{k}} &= \tfrac{1}{2}F' + \tfrac{1}{2}nF + \sum_{i,j=1}^{4} k_{ij}A_{ij} \in H_2(\mathrm{Km(A)}, \mathbb{Q}) \\ &= B_1 + \tilde{n}F + \sum_{i=1}^{4}\Bigl(a_i T_i + \sum_{j=2}^{4} b_{ij}A_{ij}\Bigr),\end{aligned}$$

where $(n, \boldsymbol{k})$ and $(\tilde{n}, a_i, b_{ij})$ are related by

$$n = 2\tilde{n} + \sum_i a_i, \qquad k_{i1} = -\tfrac{1}{2}(a_i + 1), \qquad k_{ij} = b_{ij} - \tfrac{1}{2}a_i \quad \text{for } j > 2. \tag{70}$$

Lemma 21 *If $\beta_{n,\boldsymbol{k}}$ is effective, then $\tilde{n}$, a_i, b_{ij} are integers for all i, j.*

Proof If $\beta_{n,\boldsymbol{k}}$ is effective with $\ell(n, \boldsymbol{k}) = 1$, there exist nonnegative integers

$$\tilde{a}_i \quad \text{for } i = 1, \dots, 4 \qquad \text{and} \qquad \tilde{b}_{ij} \quad \text{for } i, j = 1, \dots, 4$$

such that

$$\beta_{n,\boldsymbol{k}} = B_1 + \sum_{i=1}^{4} a_i T_i + \sum_{i,j=1}^{4} b_{ij}A_{ij}.$$

In the basis (69) we obtain

$$\beta_{n,\boldsymbol{k}} = B_1 + \Bigl(\sum_{i=1}^{4}\tilde{b}_{i1}\Bigr)F + \sum_{i=1}^{4}\Bigl((\tilde{a}_i - 2\tilde{b}_{i1})T_i + \sum_{j=2}^{4}(\tilde{b}_{ij} - \tilde{b}_{i1})A_i\Bigr).$$

The claim follows. □

Lemma 22 *If $\tilde{n}$, a_i, b_{ij} are integers and $\beta_{n,\boldsymbol{k}}^2 \geq -2$, then $\beta_{n,\boldsymbol{k}}$ is effective.*

Proof If $\tilde{n}, a_i, b_{ij}$ are integers, then $\beta_{n,\boldsymbol{k}}$ is the class of a divisor D. By Riemann–Roch we have

$$\frac{\chi(\mathcal{O}(D)) + \chi(\mathcal{O}(-D))}{2} = \frac{D^2}{2} + 2,$$

and by Serre duality we have

$$h^0(D) + h^0(-D) \geq \frac{\chi(\mathcal{O}(D)) + \chi(\mathcal{O}(-D))}{2}.$$

Hence, if $\beta_{n,\boldsymbol{k}}^2 = D^2 \geq -2$, then $h^0(D) + h^0(-D) \geq 1$. Since $F \cdot \beta_{n,\boldsymbol{k}} = 1$, we have $h^0(-D) = 0$, and therefore $h^0(D) \geq 1$ and D is effective. □

We are ready to evaluate the series (68).

By Lemma 21 we may replace the sum in (68) by a sum over all integers $\tilde{n} \in \mathbb{Z}$ and all elements

$$x_i = a_i T_i + \sum_{j=2}^{4} b_{ij} A_{ij} \quad \text{for } i = 1, \dots 4$$

such that

(i) $a_i, b_{i2}, b_{i3}, b_{i4}$ are integers for $i \in \{1, \dots, 4\}$,

(ii) $B_1 + \tilde{n} F + \sum_i x_i$ is effective.

Hence, using (70), the series (68) equals

$$4 \cdot \sum_{\tilde{n}} \sum_{x_1, \dots, x_4} q^{2\tilde{n} + \sum_i a_i} y^{-2 + \sum_i \langle x_i, T_i \rangle} \langle 1 \rangle^{\mathrm{Km(A)}}_{0, B_1 + \tilde{n} F + \sum_i x_i}, \tag{71}$$

where the sum runs over all $(\tilde{n}, x_1, \dots, x_4)$ satisfying (i) and (ii) above.

By the Yau–Zaslow formula (1), we have

$$\langle 1 \rangle^{\mathrm{Km(A)}}_{0, B_1 + \tilde{n} F + \sum_i x_i} = \left[\frac{1}{\Delta(\tau)} \right]_{q^{\tilde{n} - 1 + \sum_i \langle x_i, x_i \rangle / 2}}, \tag{72}$$

whenever $B_1 + \tilde{n} F + \sum_i x_i$ is effective; here $[\,\cdot\,]_{q^m}$ denotes the coefficient of q^m. The term (72) vanishes, unless

$$\tilde{n} - 1 + \frac{1}{2} \sum_i \langle x_i, x_i \rangle = \frac{1}{2} \Big(B_1 + \tilde{n} F + \sum_i x_i \Big)^2 \geq -1.$$

When evaluating (71), we may therefore restrict to tuples $(\tilde{n}, x_1, \dots, x_4)$ that also satisfy

(iii) $\big(B_1 + \tilde{n} F + \sum_i x_i \big)^2 \geq -2$.

By Lemma 22, conditions (i) and (iii) together imply condition (ii). In (71) we may therefore sum over tuples $(\tilde{n}, x_1, \dots, x_4)$ satisfying (i) and (iii) alone. Rewriting (iii) as

$$\tilde{n} \geq -\sum_i \tfrac{1}{2} \langle x_i, x_i \rangle$$

and always assuming (i) in the following sums, (71) equals

$$\begin{aligned}
4\cdot \sum_{x_1,\dots,x_4} \sum_{\tilde{n}\geq \sum_i \frac{\langle x_i,x_i\rangle}{-2}} & q^{2\tilde{n}+\sum_i a_i} y^{-2+\sum_i \langle x_i,T_i\rangle} \Big[\frac{1}{\Delta(\tau)}\Big]_{q^{\tilde{n}-1+\sum_i \langle x_i,x_i\rangle/2}} \\
&= 4\cdot \sum_{x_1,\dots,x_4} y^{-2+\sum_i \langle x_i,T_i\rangle} q^{2+\sum_i (a_i-\langle x_i,x_i\rangle)} \\
&\qquad \times \sum_{\tilde{n}\geq \sum_i \frac{\langle x_i,x_i\rangle}{-2}} q^{2\tilde{n}-2+\sum_i \langle x_i,x_i\rangle} \Big[\frac{1}{\Delta(\tau)}\Big]_{q^{\tilde{n}-1+\sum_i \frac{\langle x_i,x_i\rangle}{2}}} \\
&= \frac{4}{\Delta(2\tau)}\cdot \sum_{x_1,\dots,x_4} y^{-2+\sum_i \langle x_i,T_i\rangle} q^{2+\sum_i (a_i-\langle x_i,x_i\rangle)} \\
&= \frac{4}{\Delta(2\tau)}\cdot \prod_{i=1}^{4}\Big(\sum_{x_i} y^{-\frac{1}{2}+\langle x_i,T_i\rangle} q^{\frac{1}{2}+a_i-\langle x_i,x_i\rangle}\Big).
\end{aligned}$$

Consider the D_4 lattice, defined as $\mathbb{Z}^4$ together with the bilinear form

$$(x,y)\mapsto \langle x,y\rangle := x^T M y \quad \text{for } x,y\in\mathbb{Z}^4\times\mathbb{Z}^4,$$

where

$$M = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}.$$

Let $(e_1,\dots,e_4)$ denote the standard basis of $\mathbb{Z}^4$ and let

$$\alpha = 2e_1+e_2+e_3+e_4.$$

Consider the function

$$\Theta(z,\tau) = \sum_{x\in\mathbb{Z}^4} \exp\big(-2\pi i \big\langle x+\tfrac{1}{2}\alpha, ze_1+\tfrac{1}{2}e_1\big\rangle\big)\cdot q^{\langle x+\frac{\alpha}{2}, x+\frac{\alpha}{2}\rangle},$$

where $z\in\mathbb{C}$, $\tau\in\mathbb{H}$ and $q=e^{2\pi i\tau}$. The function $\Theta(z,\tau)$ is a theta function with characteristics associated to the lattice D_4. In particular, $\Theta(z,\tau)$ is a Jacobi form of index $\frac{1}{2}$ and weight 2; see [8, Section 7].[9]

[9] The general form of these theta functions is

$$\Theta_{\mathsf{v}}\begin{bmatrix} A \\ B \end{bmatrix}(z,\tau) = \sum_{x\in\mathbb{Z}^4} q^{\frac{1}{2}\langle x+A,x+A\rangle} \exp\big(2\pi i\cdot\langle x+A, z\cdot\mathsf{v}+B\rangle\big)$$

for characteristics $A,B\in\mathbb{Q}^4$ and a direction vector $\mathsf{v}\in\mathbb{C}^4$. Here $\Theta(z,\tau)=\Theta_{(-e_1)}\big[\begin{smallmatrix}\alpha/2\\ -e_1/2\end{smallmatrix}\big](z,2\tau)$.

Lemma 23 *For every $i \in \{1, \dots, 4\}$,*

$$\sum_{x_i} y^{-\frac{1}{2}+\langle x_i, T_i\rangle} q^{\frac{1}{2}+a_i-\langle x_i, x_i\rangle} = \Theta(z,\tau)$$

under the change of variables $q = e^{2\pi i\tau}$ and $y = -e^{2\pi i z}$.

Proof Let $\mathsf{D}_4(-1)$ denote the lattice $\mathbb{Z}^4$ with intersection form

$$(x,y) \mapsto -x^T M y.$$

The $\mathbb{Z}$–homomorphism defined by

$$e_1 \mapsto T_i, \quad e_2 \mapsto A_{i2}, \quad e_3 \mapsto A_{i3}, \quad e_4 \mapsto A_{i4}$$

is an isomorphism from $\mathsf{D}_4(-1)$ to

$$\big(\mathbb{Z}T_i \oplus \mathbb{Z}A_{i2} \oplus \mathbb{Z}A_{i3} \oplus \mathbb{Z}A_{i4}, \langle\cdot,\cdot\rangle\big),$$

where $\langle\,,\rangle$ denotes the intersection product on $\operatorname{Km}(\mathsf{A})$. Hence

$$\sum_{x_i} y^{-\frac{1}{2}+\langle x_i, T_i\rangle} q^{\frac{1}{2}+a_i-\langle x_i, x_i\rangle} = \sum_{x\in\mathbb{Z}^4} y^{-\frac{1}{2}-\langle x, e_1\rangle} q^{\frac{1}{2}+\langle x,\alpha\rangle+\langle x,x\rangle}.$$

Using the substitution $y = \exp(2\pi i z + \pi i)$, we obtain

$$\sum_{x\in\mathbb{Z}^4} \exp\big(-2\pi i\cdot\big\langle x+\tfrac{1}{2}\alpha, ze_1+\tfrac{1}{2}e_1\big\rangle\big)\cdot q^{\langle x+\frac{\alpha}{2}, x+\frac{\alpha}{2}\rangle} = \Theta(z,\tau). \qquad \square$$

By Lemma 23, we conclude

$$\sum_{\substack{n,\boldsymbol{k}\\ \beta_{n,\boldsymbol{k}}>0}} \langle 1\rangle^{\operatorname{Km}(\mathsf{A})}_{0,\beta_{n,\boldsymbol{k}}} q^n y^{\sum_{i,j} k_{ij}} = \frac{4}{\Delta(2\tau)}\cdot\Theta(z,\tau)^4. \tag{73}$$

2.4.4 The theta function of the D_4 lattice Consider the Dedekind eta function

$$\eta(\tau) = q^{1/24}\prod_{m\geq 1}(1-q^m) \tag{74}$$

and the first Jacobi theta function

$$\vartheta_1(z,\tau) = -iq^{1/8}(p^{1/2}-p^{-1/2})\prod_{m\geq 1}(1-q^m)(1-pq^m)(1-p^{-1}q^m),$$

where $q = e^{2\pi i\tau}$ and $p = e^{2\pi i z}$.

Proposition 24 *We have*

$$\Theta(z,\tau) = \frac{-\vartheta_1(z,\tau)\cdot\eta(2\tau)^6}{\eta(\tau)^3}. \tag{75}$$

The proof of Proposition 24 is given below. We complete the proof of Theorem 17.

Proof of Theorem 17 By Proposition 18, we have

$$4\cdot F^{\mathrm{GW}}(y,q)^4 = \sum_{\substack{n,\boldsymbol{k}\\ \beta_{b,\boldsymbol{k}}>0}} \langle 1\rangle^{\mathrm{Km(A)}}_{0,\beta_{n,\boldsymbol{k}}} q^n y^{\sum_{i,j} k_{ij}}.$$

The evaluation (73) and Proposition 24 yield

$$F^{\mathrm{GW}}(y,q)^4 = \frac{1}{\Delta(2\tau)}\left(\frac{\vartheta_1(z,\tau)\cdot\eta(2\tau)^6}{\eta(\tau)^3}\right)^4.$$

Using $\Delta(\tau)=\eta(\tau)^{24}$ and since by definition (see Section 2.3.6) the $y^{-1/2}q^0$ coefficient of F^{GW} is 1, we conclude

$$F^{\mathrm{GW}}(y,q) = \frac{\vartheta_1(z,\tau)}{\eta^3(\tau)} = F(z,\tau). \qquad \square$$

Proof of Proposition 24 Both sides of (75) are Jacobi forms of weight 2 and index $\frac{1}{2}$ for a certain congruence subgroup of the Jacobi group. The statement would therefore follow by the theory of Jacobi forms [8] after comparing enough coefficients of both sides. For simplicity, we will instead prove the statement directly.

We will work with the variables $q=e^{2\pi i\tau}$ and $p=e^{2\pi i z}$. Consider

$$F(z,\tau) = \frac{\vartheta_1(z,\tau)}{\eta^3(\tau)} = -i(p^{1/2}-p^{-1/2})\prod_{m\geq 1}\frac{(1-pq^m)(1-p^{-1}q^m)}{(1-q^m)^2}. \tag{76}$$

By direct calculation one finds

$$\begin{aligned} F(z+\lambda\tau+\mu,\tau) &= (-1)^{\lambda+\mu}q^{-\lambda/2}p^{-\lambda}K(z,\tau),\\ \Theta(z+\lambda\tau+\mu,\tau) &= (-1)^{\lambda+\mu}q^{-\lambda/2}p^{-\lambda}\Theta(z,\tau). \end{aligned} \tag{77}$$

We have

$$\begin{aligned} \Theta(0,\tau) &= \sum_{x\in\mathbb{Z}^4}\exp\bigl(-2\pi i\bigl\langle x+\tfrac{1}{2}\alpha,\tfrac{1}{2}e_1\bigr\rangle\bigr)q^{\langle x+\frac{\alpha}{2},x+\frac{\alpha}{2}\rangle}\\ &= \sum_{x'\in\mathbb{Z}^4+\frac{\alpha}{2}}\exp(-\pi i\langle x',e_1\rangle)q^{\langle x',x'\rangle}. \end{aligned}$$

Since for every $x' = m + \frac{\alpha}{2}$ with $m \in \mathbb{Z}^4$ one has

$$\exp(-\pi i \langle x', e_1 \rangle) + \exp(-\pi i \langle -x', e_1 \rangle) = -i(-1)^{\langle m, e_1 \rangle} + i(-1)^{-\langle m, e_1 \rangle} = 0,$$

we find $\Theta(0, \tau) = 0$. By (76), we also have $F(0, \tau) = 0$.

Since Θ and F are Jacobi forms of index $\frac{1}{2}$ (see [8, Theorem 1.2]), the point $z = 0$ is the only zero of either in the standard fundamental region. Therefore, the quotient

$$\frac{\Theta(z, \tau)}{F(z, \tau)}$$

is a doubly periodic entire function, and hence a constant in τ. Using the evaluations

$$F\left(\tfrac{1}{2}, \tau\right) = 2 \prod_{m \geq 1} \frac{(1 + q^m)^2}{(1 - q^m)^2} = 2 \frac{\eta(2\tau)^2}{\eta(\tau)^4} \quad \text{and} \quad \Theta\left(\tfrac{1}{2}, \tau\right) = \sum_{x \in \mathbb{Z}^4} (-1) q^{\langle x + \frac{\alpha}{2}, x + \frac{\alpha}{2} \rangle},$$

the statement therefore follows directly from Lemma 25 below. □

Lemma 25 *We have*

$$\sum_{x \in \mathbb{Z}^4} q^{\langle x + \frac{\alpha}{2}, x + \frac{\alpha}{2} \rangle} = 2 \frac{\eta(2\tau)^8}{\eta(\tau)^4}.$$

Proof As a special case of the Jacobi triple product [5], we have

$$2 \frac{\eta(2\tau)^2}{\eta(\tau)} = 2 q^{1/8} \prod_{m \geq 1} \frac{(1 - q^{2m})^2}{(1 - q^m)} = \sum_{m \in \mathbb{Z}} q^{(m + \frac{1}{2})^2 / 2}.$$

For $m = (m_1, \ldots, m_4) \in \mathbb{Z}^4$, let

$$x_m = \left(m_1 + \frac{1}{2}\right) \frac{\alpha}{2} + \left(m_2 + \frac{1}{2}\right) \frac{e_2}{2} + \cdots + \left(m_4 + \frac{1}{2}\right) \frac{e_4}{2}$$

Using that α, e_2, e_3, e_4 are orthogonal, we find

$$16 \frac{\eta(2\tau)^8}{\eta(\tau)^4} = \left(\sum_{m \in \mathbb{Z}} q^{(m + \frac{1}{2})^2 / 2} \right)^4 = \sum_{m \in \mathbb{Z}^4} q^{\langle x_m, x_m \rangle}.$$

We split the sum over $m = (m_1, \ldots, m_4) \in \mathbb{Z}^4$ depending upon whether $m_1 + m_i$ is odd or even for $i = 2, 3, 4$:

$$\sum_{m \in \mathbb{Z}^4} q^{\langle x_m, x_m \rangle} = \sum_{s_2, s_3, s_4 \in \{0,1\}} \sum_{\substack{(m_1, \ldots, m_r) \in \mathbb{Z}^4 \\ m_1 + m_i \equiv s_i \, (2)}} q^{\langle x_m, x_m \rangle}. \tag{78}$$

For every choice of $s_2, s_3, s_4 \in \{0,1\}$, we have

$$\sum_{\substack{(m_1,\dots,m_r)\in\mathbb{Z}^4\\ m_1+m_i\equiv s_i\,(2)}} q^{\langle x_m,x_m\rangle} = \sum_{x\in\mathbb{Z}^4} q^{\langle x+\frac{\beta}{2},x+\frac{\beta}{2}\rangle},$$

where $\beta\in\mathbb{Z}^4$ is a root of the D_4–lattice (ie $\langle\beta,\beta\rangle = 2$). Since the isometry group of D_4 acts transitively on roots,

$$\sum_{x\in\mathbb{Z}^4} q^{\langle x+\frac{\beta}{2},x+\frac{\beta}{2}\rangle} = \sum_{x\in\mathbb{Z}^4} q^{\langle x+\frac{\alpha}{2},x+\frac{\alpha}{2}\rangle}.$$

Inserting this into (78) and dividing by 8 completes the proof. □

3 Evaluation of further Gromov–Witten invariants

3.1 Overview

In Section 3 and Section 4 we prove Theorem 2.

In Section 3.2 we reduce the calculation to a Bryan–Leung K3. We also state one extra evaluation on the Hilbert scheme of two points of a K3 surface, which is required in Section 5. Next, for each case separately, we analyze the moduli space of maps which are incident to the given conditions. In each case, the main result is a splitting statement similar to Proposition 16.

As a result, the proof of Theorem 2 is reduced to the calculation of certain universal contributions associated to single elliptic fibers. These contributions will be determined in Section 4 using the geometry of $\mathsf{Hilb}^2(\mathbb{P}^1\times E)$, where E is an elliptic curve. The strategy is parallel to but more difficult than the evaluation considered in Section 2.3.

3.2 Reduction to the Bryan–Leung K3

Let $\pi\colon S\to\mathbb{P}^1$ be an elliptic K3 surface with a unique section and 24 nodal fibers. Let B and F be the section and fiber classes, respectively, and let

$$\beta_h = B + hF$$

for $h\geq 0$. The quantum bracket $\langle\cdots\rangle_q$ on $\mathsf{Hilb}^d(S)$ for $d\geq 1$ is defined by

$$\langle\gamma_1,\dots,\gamma_m\rangle_q^{\mathsf{Hilb}^d(S)} = \sum_{h\geq 0}\sum_{k\in\mathbb{Z}} y^k q^{h-1}\langle\gamma_1,\dots,\gamma_m\rangle_{\beta_h+kA}^{\mathsf{Hilb}^d(S)},$$

where $\gamma_1, \dots, \gamma_m \in H^*(\mathsf{Hilb}^d(S))$ are cohomology classes. By arguments parallel to those in Section 2.2.2, Theorem 2 is equivalent to the following theorem.

Theorem 26 *Let $P_1, \dots, P_{2d-2} \in S$ be generic points. For $d \geq 2$,*

$$\langle C(F) \rangle_q^{\mathsf{Hilb}^d(S)} = \frac{G(z,\tau)^{d-1}}{\Delta(\tau)},$$

$$\langle A \rangle_q^{\mathsf{Hilb}^d(S)} = -\frac{1}{2}\Big(y\frac{d}{dy}G(z,\tau)\Big)\frac{G(z,\tau)^{d-2}}{\Delta(\tau)},$$

$$\langle I(P_1), \dots, I(P_{2d-2}) \rangle_q^{\mathsf{Hilb}^d(S)} = \frac{1}{d}\binom{2d-2}{d-1}\Big(q\frac{d}{dq}F(z,\tau)\Big)^{2d-2}\frac{1}{\Delta(\tau)}$$

under the variable change $q = e^{2\pi i\tau}$ and $y = -e^{2\pi i z}$.

Later we will require one additional evaluation on $\mathsf{Hilb}^2(S)$. Let $P \in S$ be a generic point and let

$$\mathfrak{p}_{-1}(F)^2 1_S$$

be the class of a generic fiber of $\pi^{[2]}\colon \mathsf{Hilb}^2(S) \to \mathbb{P}^2$.

Theorem 27 *Under the variable change $q = e^{2\pi i\tau}$ and $y = -e^{2\pi i z}$,*

$$\langle \mathfrak{p}_{-1}(F)^2 1_S, I(P) \rangle_q^{\mathsf{Hilb}^2(S)} = \frac{F(z,\tau) \cdot q\frac{d}{dq}F(z,\tau)}{\Delta(\tau)}$$

3.3 Case $\langle C(F) \rangle_q$

We consider the evaluation of $\langle C(F) \rangle_q^{\mathsf{Hilb}^d(S)}$. Let $P_1, \dots, P_{d-1} \in S$ be generic points, let F_0 be a generic fiber of the elliptic fibration $\pi\colon S \to \mathbb{P}^1$, and let

$$Z = F_0[1]P_1[1] \cdots P_{d-1}[1] \subset \mathsf{Hilb}^d(S)$$

be the induced subscheme of class $[Z] = C(F)$, where we used the notation of Section 1.2.2(v). Consider the evaluation map

$$\text{ev}\colon \overline{M}_{0,1}(\mathsf{Hilb}^d(S), \beta_h + kA) \to S, \tag{79}$$

the moduli space parametrizing maps incident to the subscheme Z,

$$M_Z = \text{ev}^{-1}(Z), \tag{80}$$

and an element

$$[f\colon C \to \mathsf{Hilb}^d(S), p] \in M_Z.$$

By Lemma 12, there does not exist a nonconstant genus-0 stable map to $\mathsf{Hilb}^d(S)$ of class $h'F + k'A$ which is incident to Z. Hence, the marking $p \in C$ must lie on the distinguished irreducible component

$$C_0 \subset C$$

on which $\pi^{[d]} \circ f$ is nonconstant. By Lemma 13, the restriction $f|_{C_0}$ is therefore an isomorphism

$$f|_{C_0}\colon C_0 \to B_0[1]P_1[1]\cdots P_{d-1}[1] = I(B_0) \cap I(P_1) \cap \cdots \cap I(P_{d-1}) \subset \mathsf{Hilb}^d(S), \tag{81}$$

where B_0 is the section of $S \to \mathbb{P}^1$. In particular, $f(p) = (F_0 \cap B_0) + \sum_j P_j$. We identify C_0 with its image in $\mathsf{Hilb}^d(S)$.

Let $x_1, \dots, x_{24}$ be the basepoints of the rational nodal fibers of π and let $u_i = \pi(P_i)$ for all i. The image line $L = \pi^{[d]} \circ f(C)$ meets the discriminant locus of $\pi^{[d]}$ in the points

$$x_i + \sum_{j=1}^{d-1} u_j \quad \text{for } i = 1, \dots, 24 \qquad \text{and} \qquad 2u_i + \sum_{j \neq i} u_j \quad \text{for } i = 1, \dots, d-1.$$

By Lemma 12, the curve C is therefore of the form

$$C = C_0 \cup A_1 \cup \cdots \cup A_{24} \cup B_1 \cup \cdots \cup B_{d-1},$$

where the components A_i and B_j are attached to the points

$$x_i + P_1 + \cdots + P_{d-1} \quad \text{and} \quad u_j + P_1 + \cdots + P_{d-1}, \tag{82}$$

respectively. Hence, the moduli space M_Z is set-theoretically a product of spaces parametrizing maps of the form $f|_{A_i}$ and $f|_{B_j}$. We show that the set-theoretic product is scheme-theoretic and the virtual class splits. The argument is similar to Section 2.3.

First, the attachment points (82) do not smooth under infinitesimal deformations: this follows since the projection

$$f^* \mathcal{Z}_d = \widetilde{C} \to C$$

is étale over the points (82) (see the proof of Lemma 15); here $\mathcal{Z}_d \to \mathsf{Hilb}^d(S)$ is the universal family. Therefore, any infinitesimal deformation of f inside M_Z induces a deformation of the image $f(C_0)$. This deformation corresponds to moving the points

$P_1, \dots, P_{d-1}$ in (81), which is impossible since f continues to be incident to Z. Hence, $f(C_0)$ is fixed under infinitesimal deformations.[10]

By a construction parallel to that of Section 2.3.4, we have a splitting map

$$\Psi\colon M_Z \to \bigsqcup_{(\boldsymbol{h},\boldsymbol{k})} \left(\prod_{i=1}^{24} M_{x_i}^{(\mathrm{N})}(h_{x_i}) \times \prod_{j=1}^{d-1} M_{u_j}^{(\mathrm{G})}(h_{y_j}, k_{y_j}) \right), \tag{83}$$

where $M_{x_i}^{(\mathrm{N})}(h_{x_i})$ was defined in Section 2.3.4 and $M_{u_j}^{(\mathrm{G})}(h_{y_j}, k_{y_j})$ is an appropriately defined moduli space; since $f(C_0)$ has class B, the disjoint union in (83) runs over all

$$\begin{aligned} \boldsymbol{h} &= (h_{x_1}, \dots, h_{x_{24}}, h_{u_1}, \dots, h_{u_{d-1}}) \in (\mathbb{N}^{\geq 0})^{\{x_i, u_j\}}, \\ \boldsymbol{k} &= (k_{u_1}, \dots, k_{u_{d-1}}) \in \mathbb{Z}^{d-1} \end{aligned} \tag{84}$$

such that $\sum_i h_{x_i} + \sum_j h_{u_j} = h$ and $\sum_j k_{u_j} = k$. Since $f(C_0)$ is fixed under infinitesimal deformations, the map Ψ is an isomorphism.

Let $[M_Z]^{\mathrm{vir}}$ be the natural virtual class on M_Z. By arguments parallel to Section 2.3.5, the pushforward $\Psi_*[M_Z]^{\mathrm{vir}}$ is a product of virtual classes defined on each factor. Hence, by a calculation identical to Section 2.3.6, $\langle C(F) \rangle_q$ is the product of series corresponding to the points x_i and u_j.

For the points $x_1, \dots, x_{24}$, the contributing factor agrees with the contribution from the nodal fibers in the case of Section 2. It is the series (54). For $u_1, \dots, u_{d-1}$, define the formal series

$$G^{\mathrm{GW}}(y, q) = \sum_{h \geq 0} \sum_{k \in \mathbb{Z}} y^k q^h \int_{[M_{u_j}^{(\mathrm{G})}(h,k)]^{\mathrm{vir}}} 1, \tag{85}$$

where we let $[M_{u_j}^{(\mathrm{G})}(h,k)]^{\mathrm{vir}}$ denote the induced virtual class on $M_{u_j}^{(\mathrm{G})}(h,k)$. We conclude

$$\langle C(F) \rangle_q^{\mathsf{Hilb}^d(S)} = \frac{G^{\mathrm{GW}}(y,q)^{d-1}}{\Delta(q)}. \tag{86}$$

3.4 Case $\langle A \rangle_q$

We consider the evaluation of $\langle A \rangle_q^{\mathsf{Hilb}^d(S)}$. Let $P_0, \dots, P_{d-2} \in S$ be generic points, let

$$Z = P_0[2] P_1[1] \cdots P_{d-2}[1] \subset \mathsf{Hilb}^d(S)$$

[10] Although $f(C_0)$ is fixed under infinitesimal deformations, the point u_j in the attachment point $f(C_0 \cap B_j) = u_j + P_1 + \dots + P_{d-1}$ may move to first order; compare Section 3.6.

be the exceptional curve (of class A) centered at $2P_0 + P_1, \dots + P_{d-2}$, and let

$$M_Z = \mathrm{ev}^{-1}(Z),$$

where ev is the evaluation map (79). We consider an element

$$[f\colon C \to \mathsf{Hilb}^d(S), p] \in M_Z.$$

Let $C_0 \subset C$ be the distinguished component of C on which $\pi^{[d]} \circ f$ is nonconstant, and let C' be the union of all irreducible components of C which map into the fiber

$$(\pi^{[d]})^{-1}(2u_0 + u_1 + \dots + u_{d-2}),$$

where $u_i = \pi(P_i)$. Since $f(C_0)$ cannot meet the exceptional curve Z, the component C' contains the marked point p:

$$p \in C'.$$

The restriction $f|_{C'}$ decomposes into the components

$$f|_{C'} = \phi + P_1 + \dots + P_{d-2},$$

where $\phi\colon C' \to \mathsf{Hilb}^2(S)$ maps into the fiber $\pi^{[2]-1}(2u_0)$ and the P_i are constant maps.

Consider the Hilbert–Chow morphism

$$\rho\colon \mathsf{Hilb}^2(S) \to \mathrm{Sym}^2(S)$$

and the Abel–Jacobi map

$$\mathsf{aj}\colon \mathrm{Sym}^2(F_{u_0}) \to F_{u_0}.$$

Since $\rho(\phi(p)) = 2P_0$, the image of ϕ lies inside the fiber V of

$$\rho^{-1}(\mathrm{Sym}^2(F_{u_0})) \xrightarrow{\rho} \mathrm{Sym}^2(F_{u_0}) \xrightarrow{\mathsf{aj}} F_{u_0}$$

over the point $\mathsf{aj}(2P_0)$. Hence, $f|_{C'}$ maps into the subscheme

$$\widetilde{V} = V + P_1 + \dots + P_{d-2} \subset \mathsf{Hilb}^d(S).$$

The intersection of $\widetilde{V}$ with the divisor $D(B_0) \subset \mathsf{Hilb}^d(S)$ is supported in the reduced point

$$s(u_0) + Q + P_1 + \dots + P_{d-2} \in \mathsf{Hilb}^d(S), \tag{87}$$

where $s\colon \mathbb{P}^1 \to S$ is the section and $Q \in F_{u_0}$ is defined by

$$\mathsf{aj}(s(u_0) + Q) = \mathsf{aj}(2P_0).$$

Since the distinguished component $C_0 \subset C$ must map into $D(B_0)$, the point $f(C_0 \cap C')$ therefore equals (87). Hence, the restriction $f|_{C_0}$ yields an isomorphism

$$f|_{C_0}\colon C_0 \xrightarrow{\cong} B_0[1]\, Q[1]\, P_1[1] \cdots P_{d-2}[1],$$

and we will identify C_0 with its image.

Following the lines of Section 3.3, we find that the domain C is of the form

$$C = C_0 \cup C' \cup A_1 \cup \cdots \cup A_{24} \cup B_1 \cup \cdots \cup B_{d-2},$$

where the components A_i and B_j are attached to the points

$$x_i + Q + P_1 + \cdots + P_{d-2} \quad \text{and} \quad u_j + Q + P_1 + \cdots + P_{d-2},$$

respectively. Hence, M_Z is set-theoretically a product of spaces corresponding to the points

$$u_0,\ u_1,\ \ldots,\ u_{d-2},\ x_1,\ \ldots,\ x_{24}. \tag{88}$$

By arguments parallel to Section 3.3, the moduli scheme M_Z splits scheme-theoretically as a product, and also the virtual class splits. Hence $\langle A \rangle_q$ is a product of series corresponding to the points (88).

For $x_1, \ldots, x_{24}$ the contributing factor is the same as in Section 2.3.6, and for $u_1, \ldots, u_{d-2}$ it is the same as in Section 3.3. Let

$$\widetilde{G}^{\mathrm{GW}}(y,q) \in \mathbb{Q}((y))[[q]] \tag{89}$$

denote the contributing factor from the point u_0. Then we have

$$\langle A \rangle_q^{\mathsf{Hilb}^d(S)} = \frac{G^{\mathrm{GW}}(y,q)^{d-2}\, \widetilde{G}^{\mathrm{GW}}(y,q)}{\Delta(q)}. \tag{90}$$

3.5 Case $\langle I(P_1), \ldots, I(P_{2d-2}) \rangle_q$

Let $P_1, \ldots, P_{2d-2} \in S$ be generic points. In this section, we consider the evaluation of

$$\langle I(P_1), \ldots, I(P_{2d-2}) \rangle_q^{\mathsf{Hilb}^d(S)}. \tag{91}$$

In Section 3.5.1, we discuss the geometry of lines in $\mathsf{Hilb}^d(\mathbb{P}^1)$. In Section 3.5.2, we analyze the moduli space of stable maps incident to $I(P_1), \ldots, I(P_{2d-2})$.

3.5.1 The Grassmannian Let $\mathcal{Z}_d \to \mathsf{Hilb}^d(\mathbb{P}^1)$ be the universal family, and let

$$L \hookrightarrow \mathsf{Hilb}^d(\mathbb{P}^1)$$

be the inclusion of a line such that $L \not\subseteq I(x)$ for all $x \in \mathbb{P}^1$. Consider the fiber diagram

$$\begin{array}{ccccc} \widetilde{L} & \longrightarrow & \mathcal{Z}_d & \longrightarrow & \mathbb{P}^1 \\ \downarrow & & \downarrow & & \\ L & \longrightarrow & \mathsf{Hilb}^d(\mathbb{P}^1) & & \end{array}$$

The curve $\widetilde{L} \subset L \times \mathbb{P}^1$ has bidegree $(d,1)$, and is the graph of the morphism

$$I_L\colon \mathbb{P}^1 \to L, \quad x \mapsto I(x) \cap L. \tag{92}$$

By definition, the subscheme corresponding to a point $y \in L$ is $I_L^{-1}(y)$. Hence, the ramification index of I_L at a point $x \in \mathbb{P}^1$ is the length of $I_L(x)$ (considered as a subscheme of $\mathbb{P}^1$) at x. In particular, for $y \in L$, we have $y \in \Delta_{\mathsf{Hilb}^d(\mathbb{P}^1)}$ if and only if $I_L(x) = y$ for a branch point x of I_L.

Let $R(L) \subset \mathbb{P}^1$ be the ramification divisor of I_L. Since I_L has $2d-2$ branch points counted with multiplicity (or, equivalently, L meets $\Delta_{\mathsf{Hilb}^d(\mathbb{P}^1)}$ with multiplicity $2d-2$), we have

$$R(L) \in \mathsf{Hilb}^{2d-2}(\mathbb{P}^1).$$

Let $G = G(2, d+1)$ be the Grassmannian of lines in $\mathsf{Hilb}^d(\mathbb{P}^1)$. By the construction above relative to G, we obtain a rational map

$$\phi\colon G \dashrightarrow \mathsf{Hilb}^{2d-2}(\mathbb{P}^1), \quad L \mapsto R(L) \tag{93}$$

defined on the open subset of lines $L \in G$ with $L \not\subseteq I(x)$ for all $x \in \mathbb{P}^1$.

The map ϕ will be used in the proof of the following result. For $u \in \mathbb{P}^1$, consider the incidence subscheme

$$I(2u) = \{z \in \mathsf{Hilb}^d(\mathbb{P}^1) \mid 2u \subset z\}.$$

Under the identification $\mathsf{Hilb}^d(\mathbb{P}^1) \equiv \mathbb{P}^d$, the inclusion $I(2u) \subset \mathsf{Hilb}^d(\mathbb{P}^1)$ is a linear subspace of codimension 2. Let

$$\begin{array}{ccl} \mathcal{Z} & \xrightarrow{q} & \mathbb{P}^d = \mathsf{Hilb}^d(\mathbb{P}^1) \\ \downarrow{\scriptstyle p} & & \\ G & & \end{array} \tag{94}$$

be the universal family of G, and let

$$S_u = p(q^{-1}(I(2u))) = \{L \in G \mid L \cap I(2u) \neq \varnothing\} \subset G$$

be the divisor of lines incident to $I(2u)$.

Lemma 28 *Let $u_1, \dots, u_{2d-2} \in \mathbb{P}^1$ be generic points. Then*

$$S_{u_1} \cap \dots \cap S_{u_{2d-2}} \tag{95}$$

is a collection of $\frac{1}{d}\binom{2d-2}{d-1}$ reduced points.

Proof The class of S_u is the Schubert cycle σ_1. By Schubert calculus the expected number of intersection points is

$$\int_G \sigma_1^{2d-2} = \frac{1}{d}\binom{2d-2}{d-1}.$$

It remains to prove that the intersection (95) is transverse.

Given a line $L \subset I(x) \subset \mathsf{Hilb}^d(\mathbb{P}^1)$ for some $x \in \mathbb{P}^1$, there exist at most $2d-1$ different points $v \in \mathbb{P}^1$ with $2v \subset z$ for some $z \in L$. Hence, for every L in (95) we have $L \not\subseteq I(x)$ for all $x \in \mathbb{P}^1$. Therefore, $S_{u_1} \cap \dots \cap S_{u_{2d-2}}$ lies in the domain of ϕ. Then, by construction of ϕ, the intersection (95) is the fiber of ϕ over the point

$$u_1 + \dots + u_{2d-2} \in \mathsf{Hilb}^{2d-2}(\mathbb{P}^1).$$

We will show that ϕ is generically finite. Since $u_1, \dots, u_{2d-2}$ are generic, the fiber over $u_1 + \dots + u_{2d-2}$ is then a set of finitely many reduced points.

We determine an explicit expression for the map ϕ. Let $L \in G$ be a line with $L \not\subseteq I(x)$ for all $x \in \mathbb{P}^1$, let $f, g \in L$ be two distinct points and let x_0, x_1 be coordinates on $\mathbb{P}^1$. We write

$$f = a_n x_0^n + a_{n-1} x_0^{n-1} x_1 + \dots + a_0 x_1^n,$$
$$g = b_n x_0^n + b_{n-1} x_0^{n-1} x_1 + \dots + b_0 x_1^n$$

for coefficients $a_i, b_i \in \mathbb{C}$. The condition $L \not\subseteq I(x)$ for all x is equivalent to f and g having no common zeros. Consider the rational function

$$h(x) = h(x_0/x_1) = \frac{f}{g} = \frac{a_n x^n + \dots + a_0}{b_n x^n + \dots + b_0},$$

where $x = x_0/x_1$. The ramification divisor $R(L)$ is generically the zero locus of the numerator of $h' = (f/g)' = (f'g - fg')/g^2$; in coordinates we have

$$f'g - fg' = \sum_{m=0}^{2d-2} \Big(\sum_{\substack{i+j=m+1 \\ i<j}} (i-j)(a_i b_j - a_j b_i) \Big) x^m.$$

Let $M_{ij} = a_i b_j - a_j b_i$ be the Plücker coordinates on G. Then we conclude

$$\phi(L) = \sum_{m=0}^{2d-2} \Big(\sum_{\substack{i+j=m+1 \\ i<j}} (i-j) M_{ij} \Big) x^m \in \mathsf{Hilb}^{2d-2}(\mathbb{P}^1).$$

By a direct verification, the differential of ϕ at the point with coordinates

$$(a_0, \dots, a_n) = (1, 0, \dots, 0, 1), \quad (b_0, \dots, b_n) = (0, 1, 0, \dots, 0, 1)$$

is an isomorphism. Hence, ϕ is generically finite. □

Let $u_1, \dots, u_{2d-2} \in \mathbb{P}^1$ be generic points. Consider a line

$$L \in S_{u_1} \cap \cdots \cap S_{u_{2d-2}} = \phi^{-1}(u_1 + \cdots + u_{2d-2})$$

and let U_L be the formal neighborhood of L in G. By the proof of Lemma 28, the map

$$\phi\colon G \dashrightarrow \mathsf{Hilb}^{2d-2}(\mathbb{P}^1)$$

is étale near L. Hence, ϕ induces an isomorphism from U_L to

$$\operatorname{Spec}(\widehat{\mathcal{O}}_{\mathsf{Hilb}^d(\mathbb{P}^1), u_1+\cdots+u_{2d-2}}) \equiv \prod_{i=1}^{2d-2} \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathbb{P}^1, u_i}), \tag{96}$$

the formal neighborhood of $\mathsf{Hilb}^d(\mathbb{P}^1)$ at $u_1 + \cdots + u_{2d-2}$. Composing ϕ with the projection to the i^{th} factor of (96), we obtain maps

$$\kappa_i\colon U_L \xrightarrow{\phi} \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathsf{Hilb}^d(\mathbb{P}^1), u_1+\cdots+u_{2d-2}}) \to \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathbb{P}^1, u_i}) \subset \mathbb{P}^1 \tag{97}$$

which parametrize the deformation of the branch points of I_L (defined in (92)).

In the notation of the diagram (94), consider the map

$$q^{-1}(\Delta_{\mathsf{Hilb}^d(\mathbb{P}^1)}) \to G \tag{98}$$

whose fiber over a point $L' \in G$ is the set of intersection points of L' with the diagonal $\Delta_{\mathsf{Hilb}^d(\mathbb{P}^1)}$. Since L is in the fiber of a generically finite map over a generic point, we have

$$L \cap \Delta_{\mathsf{Hilb}^d(\mathbb{P}^1)} = \{\xi_1, \dots, \xi_{2d-2}\}$$

for pairwise disjoint subschemes $\xi_i \in \mathsf{Hilb}^d(\mathbb{P}^1)$ of type (21^{d-2}) with $2u_i \subset \xi_i$. The restriction of (98) to U_L is a $(2d-2)$–sheeted trivial fibration, and hence admits sections

$$v_1, \dots, v_{2d-2} \colon U_L \to q^{-1}(\Delta_{\mathsf{Hilb}^d(\mathbb{P}^1)})|_{U_L}, \tag{99}$$

such that for every i the composition $q \circ v_i$ restricts to ξ_i over the closed point. Moreover, since $q \circ v_i$ is incident to the diagonal and must contain twice the branch point κ_i defined in (97), we have the decomposition

$$q \circ v_i = 2\kappa_i + h_1 + \dots + h_{d-2} \tag{100}$$

for maps $h_1, \dots, h_{d-2} \colon U_L \to \mathbb{P}^1$.

3.5.2 The moduli space Let $P_1, \dots, P_{2d-2} \in S$ be generic points and let $u_i = \pi(P_i)$ for all i. Let

$$\mathrm{ev} \colon \overline{M}_{0,2d-2}(\mathsf{Hilb}^d(S), \beta_h + kA) \to (\mathsf{Hilb}^d(S))^{2d-2}$$

be the evaluation map and let

$$M_Z = \mathrm{ev}^{-1}\big(I(P_1) \times \dots \times I(P_{2d-2})\big)$$

be the moduli space of stable maps incident to $I(P_1), \dots, I(P_{2d-2})$. We consider an element

$$[f \colon C \to \mathsf{Hilb}^d(S), p_1, \dots, p_{2d-2}] \in M_Z.$$

Since $P_i \in f(p_i)$ and P_i is generic, the line $L = \pi(f(C)) \subset \mathsf{Hilb}^d(\mathbb{P}^1)$ is incident to $I(2u_i)$ for all i, and therefore lies in the finite set

$$S_{u_1} \cap \dots \cap S_{u_{2d-2}} \subset G(2, d+1) \tag{101}$$

defined in Section 3.5.1; here $G(2, d+1)$ is the Grassmannian of lines in $\mathbb{P}^d$.

Because the points $u_1, \dots, u_{2d-2}$ are generic, by the proof of Lemma 28, L is also generic. By arguments identical to the case of Section 2.3.2, the map $f|_{C_0} \colon C_0 \to L$ is an isomorphism. We identify C_0 with the image L.

For $x \in \mathbb{P}^1$, let $\widetilde{x} = I(x) \cap L$ be the unique point on L incident to x. The points

$$\widetilde{x}_1, \ \ldots, \ \widetilde{x}_{24}, \ \widetilde{u}_1, \ \ldots, \ \widetilde{u}_{2d-2} \tag{102}$$

are the intersection points of L with the discriminant of $\pi^{[d]}$ defined in (26). Hence, by Lemma 12, the curve C admits the decomposition

$$C = C_0 \cup A_1 \cup \cdots \cup A_{24} \cup B_1 \cup \cdots \cup B_{2d-2},$$

where A_i and B_j are the components of C attached to the points $\widetilde{x}_i$ and $\widetilde{u}_j$, respectively; see also Section 2.3.2.

By Lemma 15, the node points $C_0 \cap A_i$ and $C_0 \cap B_j$ do not smooth under deformations of f inside M_Z. Hence, by the construction of Section 2.3.4, we have a splitting morphism

$$\Psi\colon M_Z \to \bigsqcup_L \bigsqcup_{\boldsymbol{h},\boldsymbol{k}} \biggl(\prod_{i=1}^{24} M^{(\mathrm{N})}_{x_i}(h_{x_i}) \times \prod_{j=1}^{2d-2} M^{(\mathrm{H})}_{u_j}(h_{u_j}, k_{u_j}) \biggr), \tag{103}$$

where $\boldsymbol{h}, \boldsymbol{k}$ runs over the set (39) (with y_j replaced by u_j) satisfying (40), and L runs over the set of lines (101), and where $M^{(\mathrm{H})}_{u_j}(h', k')$ is the moduli space defined as follows.

Consider the evaluation map

$$\mathrm{ev}\colon \overline{M}_{0,2}(\mathsf{Hilb}^2(S), h'F + k'A) \to (\mathsf{Hilb}^2(S))^2$$

and let

$$\mathrm{ev}^{-1}(I(P_j) \times \mathsf{Hilb}^2(B_0)) \tag{104}$$

be the subscheme of maps incident to $I(P_j)$ and $\mathsf{Hilb}^d(B_0)$ at the marked points. We define $M^{(\mathrm{H})}_{u_j}(h', k')$ to be the open and closed component of (104) whose $\mathbb{C}$–points parametrize maps into the fiber $\pi^{[2]-1}(2u_j)$. Using this definition, the map Ψ is well-defined (for example, the intersection point $C_0 \cap B_j$ maps to the second marked point in (104)).

In the case considered in Section 2.3, the image line $L = f(C_0)$ was fixed under infinitesimal deformations. Here, this does not seem to be the case; the line L may move infinitesimally. Nonetheless, the following proposition shows that these deformations are all captured by the image of Ψ.

Proposition 29 *The splitting map (103) is an isomorphism.*

We will require the following lemma, which will be proven later.

Lemma 30 *Let $\phi\colon C \to \mathsf{Hilb}^2(S)$ be a family in $M^{(\mathrm{H})}_{u_j}(h',k')$ over a connected scheme Y:*

$$\begin{array}{ccc} C & \xrightarrow{\phi} & \mathsf{Hilb}^2(S) \\ \downarrow & & \\ Y & & \end{array} \tag{105}$$

Then $\pi^{[2]}\circ\phi$ maps to $\mathsf{Hilb}^2(\mathbb{P}^1)\cap I(u_j)$.

Proof of Proposition 29 We define an inverse to Ψ. Let

$$\big((\phi_i'\colon A_i \to S, q_{x_i})_{i=1,\dots,24}, (\phi_j\colon B_j \to \mathsf{Hilb}^2(S), p_j, q_j)_{j=1,\dots,2d-2}\big) \tag{106}$$

be a family of maps in the right-hand side of (103) over a connected scheme Y. By Lemma 30, $\pi^{[2]}\circ\phi_j\colon B_j \to \mathsf{Hilb}^2(\mathbb{P}^1)$ maps into $I(u_j)\cap\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)}$. Since the intersection of the line $I(u_j)$ and the diagonal $\Delta_{\mathsf{Hilb}^2(S)}$ is infinitesimal, we have the inclusion

$$I(u_j)\cap\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)} \hookrightarrow \operatorname{Spec}(\widehat{\mathcal{O}}_{\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)},2u_j}) = \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathbb{P}^1,u_j}),$$

and therefore the induced map $\iota_j\colon Y \to \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathbb{P}^1,u_i})$ making the diagram

$$\begin{array}{ccc} B_j & & \\ \downarrow & \searrow^{\pi^{[2]}\circ\phi_j} & \\ Y & \xrightarrow{\iota_j} & \operatorname{Spec}(\widehat{\mathcal{O}}_{\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)},2u_j}) = \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathbb{P}^1,u_j}) \end{array}$$

commutative. Let $\ell = (\iota_j)_j\colon Y \to U_L$, where

$$U_L = \prod_{j=1}^{2d-2} \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathbb{P}^1,u_j}) \equiv \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathsf{Hilb}^{2d-2}(\mathbb{P}^1),\sum_i u_i}).$$

Under the generically finite rational map

$$G(2,d+1) \dashrightarrow \mathsf{Hilb}^{2d-2}(\mathbb{P}^1),$$

defined in (93), the formal scheme U_L is isomorphic to the formal neighborhood of $G(2,d+1)$ at the point $[L]$. We identify these neighborhoods under this isomorphism.

Let $\mathcal{Z}_L \to U_L$ be the restriction of the universal family $\mathcal{Z} \to G(2, d+1)$ to U_L. By pullback via ℓ, we obtain a family of lines in $\mathbb{P}^d$ over the scheme Y,

$$\ell^* \mathcal{Z}_L \to Y, \tag{107}$$

together with an induced map

$$\psi\colon \ell^* \mathcal{Z}_L \xrightarrow{\ell} \mathcal{Z}_L \to \mathbb{P}^d \equiv \mathsf{Hilb}^d(B_0) \xrightarrow{s^{[d]}} \mathsf{Hilb}^d(S).$$

We will require sections of $\ell^* \mathcal{Z}_L \to Y$, which allow us to glue the domains of the maps ϕ_i' and ϕ_j to $\ell^* \mathcal{Z}_L$. Consider the sections

$$v_1, \dots, v_{2d-2}\colon Y \to \ell^* \mathcal{Z}_L$$

which are the pullback under ℓ of the sections $v_i\colon U_L \to \mathcal{Z}_L$ defined in (99). By construction, the section $v_i\colon Y \to \ell^* \mathcal{Z}$ parametrizes the points of $\ell^* \mathcal{Z}_L$ which map to the diagonal $\Delta_{\mathsf{Hilb}^d(S)}$ under ψ (in particular, over closed points of Y they map to $I(u_j) \cap L$).

For $j = 1, \dots, 2d-2$, consider the family of maps $\phi_j\colon B_j \to \mathsf{Hilb}^2(S)$,

$$\begin{array}{ccc} B_j & \xrightarrow{\phi_j} & \mathsf{Hilb}^2(S) \\ {\scriptstyle p_j, q_j} \uparrow\uparrow \big\downarrow {\scriptstyle \pi_j} & & \\ Y & & \end{array} \tag{108}$$

where p_j is the marked point mapping to $I(P_j)$, and q_j is the marked point mapping to $\mathsf{Hilb}^2(B_0)$. Let C' be the curve over Y which is obtained by gluing the component B_j to the line $\ell^* \mathbb{Z}_L$ along the points q_j, v_j for all j:

$$C' = \big(\ell^* \mathbb{Z}_L \sqcup B_1 \sqcup \dots \sqcup B_{2d-2}\big) / q_1 \sim v_1, \dots, q_{2d-2} \sim v_{2d-2}.$$

We will define a map $f'\colon C' \to \mathsf{Hilb}^d(S)$.

For all j, let $\kappa_j\colon U_L \to \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathbb{P}^1, u_j}) \subset \mathbb{P}^1$ be the map defined in (97). By construction, we have $\kappa_j \circ \ell = \iota_j$. Hence, by (100), there exist maps $h_1, \dots, h_{d-2}\colon Y \to S$ with

$$\psi \circ v_j = \phi_j \circ q_j + h_1 + \dots + h_{d-2}. \tag{109}$$

Let $\pi_j\colon B_j \to Y$ be the map of the family B_j / Y, and define

$$\widetilde{\phi}_{u_j} = \left(\phi_j + \sum_{i=1}^{n-2} h_i \circ \pi_j \right)\colon B_j \to \mathsf{Hilb}^d(S).$$

Define the map

$$f'\colon C' \to \mathsf{Hilb}^d(S)$$

by $f'|_{C_0} = \psi$ and by $f'|_{B_j} = \widetilde{\phi}_j$ for every j. Equation (109) implies that the map $\widetilde{\phi}_{u_j}$ restricted to q_j agrees with $\psi\colon C' \to \mathsf{Hilb}^d(S)$ restricted to v_j. Hence f' is well-defined.

By a parallel construction, we obtain a canonical gluing of the components A_i to C' together with a gluing of the maps f' and $\phi_i'\colon A_i \to S$. We obtain a family of maps

$$f\colon C \to \mathsf{Hilb}^d(S)$$

over Y, which lies in M_Z and is such that $\Psi(f)$ equals (106). By a direct verification, the induced morphism on the moduli spaces is the desired inverse to Ψ. Hence, Ψ is an isomorphism. □

The remaining steps in the evaluation of (91) are similar to Section 2.3. Using the identification

$$H^0(C_0, f^* T_{\mathsf{Hilb}^d(S)}) = H^0(C_0, T_{C_0}) \oplus \bigoplus_{j=1}^{2d-2} T_{\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)}, \phi_j(q_j)},$$

where $q_j = C_0 \cap B_j$ are the nodes and ϕ_j is as in the proof of Proposition 29, one verifies that the virtual class splits according to the product (103). Hence, the invariant (91) is a product of series associated to the points x_i and u_j. Let

$$H^{\mathrm{GW}}(y,q) = \sum_{h \geq 0} \sum_{k \in \mathbb{Z}} y^{k-\frac{1}{2}} q^h \int_{[M^{(\mathrm{H})}_{u_j}(h,k)]^{\mathrm{vir}}} 1 \in \mathbb{Q}((y^{1/2}))[\![q]\!] \tag{110}$$

be the contribution from the point u_j. By Lemma 28, there are $\frac{1}{d}\binom{2d-2}{d-1}$ lines in the set (101). Hence,

$$\langle I(P_1), \ldots, I(P_{2d-2}) \rangle_q^{\mathsf{Hilb}^d(S)} = \frac{1}{d}\binom{2d-2}{d-1} \frac{H^{\mathrm{GW}}(y,q)^{2d-2}}{\Delta(q)}. \tag{111}$$

Proof of Lemma 30 Since ϕ is incident to $I(P_j)$, the composition $\pi^{[2]} \circ \phi$ maps to $I(u_j)$. Therefore, we only need to show that $\pi^{[2]} \circ \phi$ maps to $\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)}$.

It is enough to consider the case $Y = \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$. Let $f_0\colon C_0 \to \mathsf{Hilb}^2(S)$ be the restriction of f over the closed point of Y, and consider the diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad\phi\quad} & \mathsf{Hilb}^2(S) \\ \downarrow{\scriptstyle \pi_C} & & \downarrow{\scriptstyle \pi^{[2]}} \\ \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2) & \xrightarrow{\ a\ } & \mathsf{Hilb}^2(\mathbb{P}^1) \end{array}$$

where π_C is the given map of the family (105) and a is the induced map. Let s be the section of $\mathcal{O}(\Delta_{\mathbb{P}^1})$ with zero locus $\Delta_{\mathbb{P}^1}$, and assume the pullback $\phi^* s$ is nonzero.

Let Ω_{π_2} be the sheaf of relative differentials of $\pi_2 := \pi^{[2]}$. The composition

$$\phi^*\pi_2^*\pi_{2*}\Omega_{\pi_2} \to \phi^*\Omega_{\pi_2} \xrightarrow{d} \Omega_{\pi_C} \tag{112}$$

factors as

$$\phi^*\pi_2^*\pi_{2*}\Omega_{\pi_2} \to \pi_C^*\pi_{C*}\Omega_{\pi_C} \to \Omega_{\pi_C}. \tag{113}$$

Since the second term in (113) is zero, the map (112) is zero. Hence, d factors as

$$\phi^*\Omega_{\pi_2} \to \phi^*(\Omega_{\pi_2}/\pi_2^*\pi_{2*}\Omega_{\pi_2}) \to \Omega_{\pi_C}. \tag{114}$$

By Lemma 31 below, $\Omega_{\pi_2}/\pi_2^*\pi_{2*}\Omega_{\pi_2}$ is the pushforward of a sheaf supported on $\pi_2^{-1}(\Delta_{\mathbb{P}^1})$. After trivializing $\mathcal{O}(\Delta_{\mathbb{P}^1})$ near $2u_j$, write $\phi^*s = \lambda\epsilon$ for some $\lambda \in \mathbb{C}\setminus\{0\}$. Then, by (114),

$$0 = d(s\cdot\Omega_{\pi_2}) = \lambda\epsilon\cdot d(\Omega_{\pi_2}) \subset \Omega_{\pi_C}.$$

In particular, $db = 0$ for every $b \in \Omega_{\pi_2}$, which does not vanish on $\pi_2^{-1}(\Delta_{\mathbb{P}^1})$. Since $\phi|_C$ is nonzero, this is a contradiction. □

Lemma 31 *Let $x \in \mathbb{P}^1$ be the basepoint of a smooth fiber of $\pi\colon S \to \mathbb{P}^1$. Then there exists a Zariski-open $2x \in U \subset \mathsf{Hilb}^2(\mathbb{P}^1)$ and a map*

$$u\colon \mathcal{O}_U^{\oplus 2} \to \pi_{2*}\Omega_{\pi_2}|_U \tag{115}$$

with cokernel equal to $j_*\mathcal{F}$ for a sheaf $\mathcal{F}$ on $\Delta_{\mathbb{P}^1}\cap U$.

Proof Let U be an open subset of $x \in \mathbb{P}^1$ such that $\pi_*\Omega_\pi|_U$ is trivialized by a section

$$\alpha \in \pi_*\Omega_\pi(U) = \Omega_\pi(S_U),$$

where $S_U = \pi^{-1}(U)$. Consider the open neighborhood $\widetilde{U} = \mathsf{Hilb}^2(U)$ of the point $2x \in \mathsf{Hilb}^2(\mathbb{P}^1)$.

Let $D_U \subset S_U \times S_U$ be the diagonal and consider the $\mathbb{Z}_2$ quotient

$$\mathrm{Bl}_{D_U}(S_U \times S_U) \xrightarrow{/\mathbb{Z}_2} \mathsf{Hilb}^2(S_U) = \pi_2^{-1}(\mathsf{Hilb}^2(U)).$$

For $i \in \{1,2\}$, let

$$q_i \colon \mathrm{Bl}_{D_U}(S_U \times S_U) \to S_U$$

be the composition of the blowdown map with the i^{th} projection. Let t be a coordinate on U and let

$$t_i = q_i^* t \quad \text{and} \quad \alpha_i = q_i^* \alpha$$

for $i = 1, 2$ be the induced global functions and 1–forms on $\mathrm{Bl}_{D_U}(S_U \times S_U)$, respectively. The two 1–forms

$$\alpha_1 + \alpha_2 \quad \text{and} \quad (t_1 - t_2)(\alpha_1 - \alpha_2)$$

are $\mathbb{Z}_2$–invariant and descend to global sections of $\pi_{2*}\Omega_{\pi_2}|U$. Consider the induced map

$$u \colon \mathcal{O}_U^{\oplus 2} \to \pi_{2*}\Omega_{\pi_2}|_U.$$

The map u is an isomorphism away from the diagonal

$$\Delta_\pi \cap \widetilde{U} = V((t_1 - t_2)^2) \subset \widetilde{U}. \tag{116}$$

Hence, it is left to check the statement of the lemma in an infinitesimal neighborhood of (116). Let U' be a small analytic neighborhood of $v \in U$ such that the restriction $\pi_{U'} \colon S_{U'} \to U'$ is analytically isomorphic to the quotient

$$(U' \times \mathbb{C})/\sim \; \to U',$$

where $\sim$ is the equivalence relation

$$(t, z) \sim (t', z') \iff t = t' \text{ and } z - z' \in \Lambda_t,$$

with an analytically varying lattice $\Lambda_t \colon \mathbb{Z}^2 \to \mathbb{C}$. Now, a direct and explicit verification yields the statement of the lemma. □

3.6 Case $\langle \mathfrak{p}_{-1}(F)^2 1_S, I(P)\rangle_q$

Let $F^{\mathrm{GW}}(y,q)$ and $H^{\mathrm{GW}}(y,q)$ be the power series defined in (55) and (110), respectively, let $P \in S$ be a point and let F be the class of a fiber of $\pi\colon S \to \mathbb{P}^1$.

Lemma 32 *We have*

$$\langle \mathfrak{p}_{-1}(F)^2 1_S, I(P)\rangle_q^{\mathsf{Hilb}^2(S)} = \frac{F^{\mathrm{GW}}(y,q)\cdot H^{\mathrm{GW}}(y,q)}{\Delta(q)}.$$

Proof Let F_1, F_2 be fibers of $\pi\colon S \to \mathbb{P}^1$ over generic points $x_1, x_2 \in \mathbb{P}^1$, respectively, and let $P \in S$ be a generic point. Define the subschemes

$$Z_1 = F_1[1]F_2[1] \quad \text{and} \quad Z_2 = I(P).$$

Consider the evaluation map

$$\mathrm{ev}\colon \overline{M}_{0,1}(\mathsf{Hilb}^2(S), \beta_h + kA) \to \mathsf{Hilb}^2(S)$$

from the moduli space of stable maps with *one* marked point, let

$$M_{Z_2} = \mathrm{ev}^{-1}(Z_2),$$

and let

$$M_Z \subset M_{Z_2}$$

be the closed substack of M_{Z_2} of maps which are incident to both Z_1 and Z_2.

Let $[f\colon C \to \mathsf{Hilb}^2(S), p_1] \in M_Z$ be an element, let C_0 be the distinguished component of C on which $\pi^{[2]} \circ f$ is nonzero, and let $L = \pi^{[2]}(f(C_0))$ be the image line. Since $P \in S$ is generic, we have $2v \in L$, where $v = \pi(P)$. Hence L is the line through $2v$ and $u_1 + u_2$, and has the diagonal points

$$L \cap \Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)} = \{2u, 2v\} \tag{117}$$

for some fixed $u \in \mathbb{P}^1 \setminus \{v\}$. By Lemma 13, the restriction $f|_{C_0}$ is therefore an isomorphism onto the embedded line $L \subset \mathsf{Hilb}^2(B_0)$. Using arguments parallel to Section 2.3.2, the moduli space M_Z is *set-theoretically* a product of the moduli space of maps to the nodal fibers, the moduli space $M_u^{(\mathrm{F})}(h',k')$ parametrizing maps over $2u$, and the moduli space $M_v^{(\mathrm{H})}(h'',k'')$ parametrizing maps over $2v$.

Under infinitesimal deformations of $[f\colon C \to \mathsf{Hilb}^2(S)]$ inside M_Z, the line L remains incident to $x_1 + x_2$, but may move to first order at the point $2v$ (see Section 3.5.2);

hence, it may move also at $2u$ to first order. In particular, the moduli space is scheme-theoretically *not* a product of the above moduli spaces. Nevertheless, by degeneration, we will reduce to the case of a scheme-theoretic product. For simplicity, we work on the component of M_Z which parametrizes maps with no component mapping to the nodal fibers of π; the general case follows by completely analogous arguments with an extra $1/\Delta(q)$ factor appearing as a contribution from the nodal fibers.

Let $N \subset M_{Z_2}$ be the *open* locus of maps $f\colon C \to \mathsf{Hilb}^2(S)$ in M_{Z_2} with

$$\pi^{[2]}(f(C)) \cap \Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)} = \{2t, 2v\}$$

for some point $t \in \mathbb{P} \setminus \{x_1, \dots, x_{24}, v\}$. Under deformations of an element $[f] \in N$, the intersection point $2t$ may move freely and independently of v. Hence, we have a splitting *isomorphism*

$$\Psi\colon N \to \bigsqcup_{\substack{h=h_1+h_2\\ k=k_1+k_2-1}} M^{(\mathrm{F})}(h_1,k_1) \times M^{(\mathrm{H})}_v(h_2,k_2), \tag{118}$$

where

- $M^{(\mathrm{F})}(h,k)$ is the moduli space of 1–pointed stable maps to $\mathsf{Hilb}^2(S)$ of genus 0 and class $hF + kA$ such that the marked point is mapped to $s^{[2]}(2t)$ for some $t \in \mathbb{P} \setminus \{x_1, \dots, x_{24}, v\}$, and
- $M^{(\mathrm{H})}_v(h,k)$ is the moduli space defined in Section 3.5.2.

For every decomposition $h = h_1 + h_2$ and $k = k_1 + k_2 - 1$ separately, let

$$M^{(\mathrm{F})}(h_1,k_1) \times M^{(\mathrm{H})}_v(h_2,k_2) \to \Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)} \times \operatorname{Spec}(\widehat{\mathcal{O}}_{\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)},2v})$$

be the product of the compositions of the first evaluation map with $\pi^{[2]}$ on each factor, let

$$\iota\colon V \hookrightarrow \Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)} \times \operatorname{Spec}(\widehat{\mathcal{O}}_{\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)},2v}) \tag{119}$$

be the subscheme parametrizing the intersection points $L \cap \Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)}$ of lines L which are incident to $x_1 + x_2$, and consider the fiber product

$$\begin{array}{ccc} M_{Z,(h_1,h_2,k_1,k_2)} & \longrightarrow & M^{(\mathrm{F})}(h_1,k_1) \times M^{(\mathrm{H})}_v(h_2,k_2) \\ \downarrow & & \downarrow \\ V & \xrightarrow{\ \iota\ } & \Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)} \times \operatorname{Spec}(\widehat{\mathcal{O}}_{\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)},2v}) \end{array} \tag{120}$$

Then, by definition, the splitting isomorphism (118) restricts to an isomorphism

$$\Psi\colon M_Z \to \bigsqcup_{\substack{h=h_1+h_2\\ k=k_1+k_2-1}} M_{Z,(h_1,h_2,k_1,k_2)}.$$

Restricting the natural virtual class on M_{Z_2} to the open locus, we obtain a virtual class $[N]^{\mathrm{vir}}$ of dimension 1. By the arguments of Section 2.3.5,

$$\Psi_*[N]^{\mathrm{vir}} = \sum_{\substack{h=h_1+h_2\\ k=k_1+k_2-1}} [M^{(\mathrm{F})}(h_1,k_1)]^{\mathrm{vir}} \times [M_v^{(\mathrm{H})}(h_2,k_2)]^{\mathrm{vir}}, \tag{121}$$

where $[M^{(\mathrm{F})}(h_1,k_1)]^{\mathrm{vir}}$ is a $\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)}$–relative version of the virtual class considered in Section 2.3.5, and $[M_v^{(\mathrm{H})}(h_2,k_2)]^{\mathrm{vir}}$ is the virtual class constructed in Section 3.5.2. The composition of ι with the projection to the second factor is an isomorphism. Hence ι is a regular embedding and we obtain

$$\begin{aligned}\langle \mathfrak{p}_{-1}(F)^2, I(P)\rangle^{\mathsf{Hilb}^2(S)}_{\beta_h+kA} &= \deg(\Psi_*[M_Z]^{\mathrm{vir}}) \\ &= \sum_{\substack{h=h_1+h_2\\ k=k_1+k_2-1}} \deg \iota^!\big([M^{(\mathrm{F})}(h_1,k_1)]^{\mathrm{vir}} \times [M_v^{(\mathrm{H})}(h_2,k_2)]^{\mathrm{vir}}\big).\end{aligned} \tag{122}$$

We proceed by degenerating the first factor in the product

$$M^{(\mathrm{F})}(h_1,k_1) \times M_v^{(\mathrm{H})}(h_2,k_2),$$

while keeping the second factor fixed. Let

$$\mathcal{S} = \mathrm{Bl}_{F_u \times 0}(S \times \mathbb{A}^1) \to \mathbb{A}^1$$

be a deformation of S to the normal cone of F_u, where u was defined in (117). Let $\mathcal{S}^\circ \subset \mathcal{S}$ be the complement of the proper transform of $S \times 0$ and consider the *relative* Hilbert scheme $\mathsf{Hilb}^2(\mathcal{S}^\circ/\mathbb{A}^1) \to \mathbb{A}^1$, which appeared already in (66). Let

$$p\colon \widetilde{M}^{(\mathrm{F})}(h_1,k_1) \to \mathbb{A}^1 \tag{123}$$

be the moduli space of 1–pointed stable maps to $\mathsf{Hilb}^2(\mathcal{S}^\circ/\mathbb{A}^1)$ of genus 0 and class $h_1 F + k_1 A$, which map the marked point to the closure of

$$(\Delta_{\mathsf{Hilb}^2(B_0)} \setminus \{x_1, \dots, x_{24}, v\}) \times (\mathbb{A}^1 \setminus \{0\}).$$

Over $t \neq 0$, (123) restricts to $M^{(\mathrm{F})}(h_1,k_1)$, while the fiber over 0, denoted

$$M_0^{(\mathrm{F})}(h_1,k_1) = p^{-1}(0),$$

parametrizes maps into the trivial elliptic fibration $\mathsf{Hilb}^2(\mathbb{C}\times E)$ incident to the diagonal $\Delta_{\mathsf{Hilb}^2(\mathbb{C}\times e)}$ for a fixed $e \in E$. Since addition by $\mathbb{C}$ acts on $M_0^{(\mathrm{F})}(h_1,k_1)$ we have the product decomposition

$$M_0^{(\mathrm{F})}(h_1,k_1) = M_{0,\mathrm{fix}}^{(\mathrm{F})}(h_1,k_1) \times \Delta_{\mathsf{Hilb}^2(\mathbb{C}\times e)}, \tag{124}$$

where $M_{0,\mathrm{fix}}^{(\mathrm{F})}(h_1,k_1)$ is a fixed fiber of

$$M_0^{(\mathrm{F})}(h_1,k_1) \to \Delta_{\mathsf{Hilb}^2(\mathbb{C}\times e)}.$$

Consider a deformation of the diagram (120) to $0 \in \mathbb{A}^1$,

$$\begin{array}{ccc} M'_{Z,(h_1,h_2,k_1,k_2)} & \longrightarrow & M_0^{(\mathrm{F})}(h_1,k_1) \times M_v^{(\mathrm{H})}(h_2,k_2) \\ \downarrow & & \downarrow \\ V' & \xrightarrow{\iota'} & \Delta_{\mathsf{Hilb}^2(\mathbb{C}\times E)} \times \mathrm{Spec}(\widehat{\mathcal{O}}_{\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)},2v}) \end{array}$$

where (V',ι') is the fiber over 0 of a deformation of (V,ι) such that the composition with the projection to $\mathrm{Spec}(\widehat{\mathcal{O}}_{\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)},2v})$ remains an isomorphism. By construction, the total space of the deformation

$$M_{Z,(h_1,h_2,k_1,k_2)} \rightsquigarrow M'_{Z,(h_1,h_2,k_1,k_2)}$$

is proper over $\mathbb{A}^1$. Using the product decomposition (124), we find

$$M'_{Z,(h_1,h_2,k_1,k_2)} \cong M_{0,\mathrm{fix}}^{(\mathrm{F})}(h_1,k_1) \times M_v^{(\mathrm{H})}(h_2,k_2).$$

Hence, after degeneration, we are reduced to a scheme-theoretic product. It remains to consider the virtual class.

By the relative construction of Section 2.3.5 the moduli space $\widetilde{M}^{(\mathrm{F})}(h_1,k_1)$ carries a virtual class

$$[\widetilde{M}^{(\mathrm{F})}(h_1,k_1)]^{\mathrm{vir}} \tag{125}$$

which restricts to $[M^{(\mathrm{F})}(h_1,k_1)]^{\mathrm{vir}}$ over $t \neq 0$, while over $t = 0$ we have

$$0^![\widetilde{M}^{(\mathrm{F})}(h_1,k_1)]^{\mathrm{vir}} = \mathrm{pr}_1^*([M_1^{\mathrm{fix}}(h_1,k_1)]^{\mathrm{vir}}). \tag{126}$$

where pr_1 is the projection to the first factor in (124) and $[M_1^{\mathrm{fix}}(h_1,k_1)]^{\mathrm{vir}}$ is the virtual class obtained by the construction of Section 2.3.5. We conclude that

$$\begin{aligned}(127)\quad \deg \iota^!\big([M^{(\mathrm{F})}(h_1,k_1)]^{\mathrm{vir}} \times [M_v^{(\mathrm{H})}(h_2,k_2)]^{\mathrm{vir}}\big) \\ = \deg(\iota')^!\big(\mathrm{pr}_1^*([M_1^{\mathrm{fix}}(h_1,k_1)]^{\mathrm{vir}}) \times [M_v^{(\mathrm{H})}(h_2,k_2)]^{\mathrm{vir}}\big) \\ = \deg([M_1^{\mathrm{fix}}(h_1,k_1)]^{\mathrm{vir}}) \cdot \deg([M_v^{(\mathrm{H})}(h_2,k_2)]^{\mathrm{vir}}).\end{aligned}$$

By definition (see (110)),

$$\deg[M_v^{(\mathrm{H})}(h_2,k_2)]^{\mathrm{vir}} = [H^{\mathrm{GW}}(y,q)]_{q^{h_2}y^{k_2-1/2}},$$

where $[\,\cdot\,]_{q^a y^b}$ denotes the $q^a y^b$ coefficient. The moduli space $M_1^{\mathrm{fix}}(h_1,k_1)$ is isomorphic to the space $M_{y_i}^{(\ell)}(h_1,k_1)$ defined in (61). Since the construction of the virtual classes on both sides agree, the virtual class is the same under this isomorphism. Hence, by Lemma 19,

$$\deg[M_1^{\mathrm{fix}}(h_1,k_1)]^{\mathrm{vir}} = [F^{\mathrm{GW}}(y,q)]_{q^{h_1}y^{k_1-1/2}}.$$

Inserting into (127) yields

$$\begin{aligned}\deg \iota^!\big([M^{(\mathrm{F})}(h_1,k_1)]^{\mathrm{vir}} \times [M_v^{(\mathrm{H})}(h_2,k_2)]^{\mathrm{vir}}\big) \\ = [H^{\mathrm{GW}}(y,q)]_{q^{h_2}y^{k_2-1/2}} \cdot [F^{\mathrm{GW}}(y,q)]_{q^{h_1}y^{k_1-1/2}},\end{aligned}$$

which completes the proof by (122). □

4 The Hilbert scheme of 2 points of $\mathbb{P}^1 \times E$

4.1 Overview

In previous sections we expressed genus-0 Gromov–Witten invariants of the Hilbert scheme of points of an elliptic K3 surface S in terms of universal series which depend only on specific fibers of the fibration $S \to \mathbb{P}^1$. The contributions from nodal fibers were determined before by Bryan and Leung in their proof [3] of the Yau–Zaslow formula (1). The yet-undetermined contributions from smooth fibers, denoted

$$(128)\qquad F^{\mathrm{GW}}(y,q),\ G^{\mathrm{GW}}(y,q),\ \widetilde{G}^{\mathrm{GW}}(y,q),\ H^{\mathrm{GW}}(y,q)$$

in equations (55), (85), (89), (110), respectively, depend only on infinitesimal data near the smooth fibers, and not on the global geometry of the K3 surface. Hence, one may hope to find similar contributions in the Gromov–Witten theory of the Hilbert scheme of points of other elliptic fibrations.

Let E be an elliptic curve with origin $0_E \in E$, and let

$$X = \mathbb{P}^1 \times E$$

be the trivial elliptic fibration. Here, we study the genus-0 Gromov–Witten theory of the Hilbert scheme

$$\mathsf{Hilb}^2(X)$$

and use our results to determine the series (128).

From the view of Gromov–Witten theory, the variety $\mathsf{Hilb}^2(X)$ has two advantages over the Hilbert scheme of two points of an elliptic K3 surface. First, $\mathsf{Hilb}^2(X)$ is not holomorphic symplectic. Therefore, we may use ordinary Gromov–Witten invariants, and in particular the main computational method which exists in genus-0 Gromov–Witten theory: the WDVV equation. Second, we have an additional map

$$\mathsf{Hilb}^2(X) \to \mathsf{Hilb}^2(E)$$

induced by the projection of X to the second factor, which is useful in calculations.

Our study of the Gromov–Witten theory of $\mathsf{Hilb}^2(X)$ will proceed in two independent directions. First, we directly analyze the moduli space of stable maps to $\mathsf{Hilb}^2(X)$ which are incident to certain geometric cycles. Similar to the K3 case, this leads to an explicit expression of generating series of Gromov–Witten invariants of $\mathsf{Hilb}^2(X)$ in terms of the series (128). This is parallel to the study of the Gromov–Witten theory of the Hilbert scheme of points of a K3 surface in Sections 2 and 3.

In a second independent step, we will calculate the Gromov–Witten invariants of $\mathsf{Hilb}^2(X)$ using the WDVV equations and a few explicit calculations of initial data. Then, combining both directions, we are able to solve for the functions (128). This leads to the following result.

Let $F(z,\tau)$ be the Jacobi theta function (5) and, with $y = -e^{2\pi i z}$, let

$$G(z,\tau) = F(z,\tau)^2 \Big(y \frac{d}{dy} \Big)^2 \log F(z,\tau)$$

be the function which appeared in Section 0.4.

Theorem 33 *Under the variable change* $y = -e^{2\pi i z}$ *and* $q = e^{2\pi i \tau}$,

$$\begin{aligned} F^{\mathrm{GW}}(y,q) &= F(z,\tau), \\ G^{\mathrm{GW}}(y,q) &= G(z,\tau), \\ \widetilde{G}^{\mathrm{GW}}(y,q) &= -\frac{1}{2}\Big(y\frac{d}{dq}\Big)G(z,\tau), \\ H^{\mathrm{GW}}(y,q) &= \Big(q\frac{d}{dq}\Big)F(y,q). \end{aligned}$$

The proof of Theorem 33 via the geometry of $\mathsf{Hilb}^2(X)$ is independent of the Kummer K3 geometry studied in Section 2.4. In particular, our approach here yields a second proof of Theorem 17.

4.2 The fiber of $\mathsf{Hilb}^2(\mathbb{P}^1 \times E) \to E$

4.2.1 Definition The projections of $X = \mathbb{P}^1 \times E$ to the first and second factors induce the maps

(129) $$\pi\colon \mathsf{Hilb}^2(X) \to \mathsf{Hilb}^2(\mathbb{P}^1) = \mathbb{P}^2 \quad \text{and} \quad \tau\colon \mathsf{Hilb}^2(X) \to \mathsf{Hilb}^2(E),$$

respectively. Consider the composition

$$\sigma\colon \mathsf{Hilb}^2(X) \xrightarrow{\tau} \mathsf{Hilb}^2(E) \xrightarrow{+} E$$

of τ with the addition map $+\colon \mathsf{Hilb}^2(E) \to E$. Since σ is equivariant with respect to the natural action of E on $\mathsf{Hilb}^2(X)$ by translation, it is an isotrivial fibration with smooth fibers. We let

$$Y = \sigma^{-1}(0_E)$$

be the fiber of σ over the origin $0_E \in E$.

Let $\gamma \in H_2(\mathsf{Hilb}^2(X))$ be an effective curve class and let

$$\overline{M}_{0,m}(\mathsf{Hilb}^2(X), \gamma)$$

be the moduli space of m–pointed stable maps to $\mathsf{Hilb}^2(X)$ of genus 0 and class γ. The map σ induces an isotrivial fibration

$$\sigma\colon \overline{M}_{0,m}(\mathsf{Hilb}^2(X), \gamma) \to E$$

with fiber over 0_E equal to

$$\bigsqcup_{\gamma'} \overline{M}_{0,m}(Y, \gamma'),$$

where the disjoint union runs over all effective curve classes $\gamma' \in H_2(Y;\mathbb{Z})$ with $\iota_*\gamma' = \gamma$; here $\iota\colon Y \to \mathsf{Hilb}^2(X)$ is the inclusion.

For cohomology classes $\gamma_1, \dots, \gamma_m \in H^*(\mathsf{Hilb}^2(X))$, we have

$$\int_{[\overline{M}_{0,m}(\mathsf{Hilb}^2(X),\gamma)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1 \cup [Y]) \cdots \mathrm{ev}_m^*(\gamma_m) = \sum_{\substack{\gamma' \in H_2(Y) \\ \iota_*\gamma' = \gamma}} \int_{[\overline{M}_{0,m}(Y,\gamma')]^{\mathrm{vir}}} (\iota \circ \mathrm{ev}_1)^*(\gamma_1) \cdots (\iota \circ \mathrm{ev}_m)^*(\gamma_m),$$

where we let $[\,\cdot\,]^{\mathrm{vir}}$ denote the virtual class defined by ordinary Gromov–Witten theory. Hence, for calculations related to the Gromov–Witten theory of $\mathsf{Hilb}^2(X)$ we may restrict to the threefold Y.

4.2.2 Cohomology Let $D_X \subset X \times X$ be the diagonal and let

$$\mathrm{Bl}_{D_X}(X \times X) \to \mathsf{Hilb}^2(X) \tag{130}$$

be the $\mathbb{Z}_2$–quotient map which interchanges the factors. Let

$$W = \mathbb{P}^1 \times \mathbb{P}^1 \times E \hookrightarrow X \times X, \quad (x_1, x_2, e) \mapsto (x_1, e, x_2, -e)$$

be the fiber of 0_E under $X \times X \to E \times E \xrightarrow{+} E$ and consider the blowup

$$\rho\colon \widetilde{W} = \mathrm{Bl}_{D_X \cap W} W \to W. \tag{131}$$

Then the restriction of (130) to $\widetilde{W}$ yields the $\mathbb{Z}_2$–quotient map

$$g\colon \widetilde{W} \to \widetilde{W}/\mathbb{Z}_2 = Y. \tag{132}$$

Let $D_{X,1}, \dots, D_{X,4}$ be the components of the intersection

$$D_X \cap W = \{(x_1, x_2, f) \in \mathbb{P}^1 \times \mathbb{P}^1 \times E \mid x_1 = x_2 \text{ and } f = -f\}$$

corresponding to the four 2–torsion points of E, and let

$$E_1, \dots, E_4$$

be the corresponding exceptional divisors of the blowup $\rho\colon \widetilde{W} \to W$. For every i, the restriction of g to E_i is an isomorphism onto its image. Define the cohomology classes

$$\Delta_i = g_*[E_i], \quad A_i = g_*[\rho^{-1}(y_i)]$$

for some $y_i \in D_{X,i}$. We also set

$$\Delta = \Delta_1 + \cdots + \Delta_4, \quad A = \tfrac{1}{4}(A_1 + \cdots + A_4).$$

Let $x_1, x_2 \in \mathbb{P}^1$ and $f \in E$ be points, and define

$$B_1 = g_*[\rho^{-1}(\mathbb{P}^1 \times x_2 \times f)], \qquad B_2 = \tfrac{1}{2} \cdot g_*[\rho^{-1}(x_1 \times x_2 \times E)].$$

Identify the fiber of $\mathsf{Hilb}^2(E) \to E$ over 0_E with $\mathbb{P}^1$, and consider the diagram

(133)
$$\begin{array}{ccc} Y & \xrightarrow{\tau} & \mathbb{P}^1 \\ \downarrow{\scriptstyle \pi} & & \\ \mathbb{P}^2 & & \end{array}$$

induced by the morphisms (129). Let $h \in H^2(\mathbb{P}^2)$ be the class of a line and let $x \in \mathbb{P}^1$ be a point. Define the divisor classes

$$D_1 = [\tau^{-1}(x)], \quad D_2 = \pi^* h.$$

Lemma 34 *The cohomology classes*

(134) $$D_1,\ D_2,\ \Delta_1,\ \dots,\ \Delta_4 \quad (\textit{resp. } B_1,\ B_2,\ A_1,\ \dots,\ A_4)$$

form a basis of $H^2(Y;\mathbb{Q})$ (resp. of $H^4(Y;\mathbb{Q})$).

Proof Since the map g is the quotient map by the finite group $\mathbb{Z}_2$, we have the isomorphism

$$g^*\colon H^*(Y;\mathbb{Q}) \to H^*(\widetilde{W};\mathbb{Q})^{\mathbb{Z}_2},$$

where the right-hand side denotes the $\mathbb{Z}_2$–invariant part of the cohomology of $\widetilde{W}$. The lemma now follows from a direct verification. □

By straightforward calculation, we find the following intersections between the basis elements (134):

$\cdot$	B_1	B_2	A_i
D_1	0	1	0
D_2	1	0	0
Δ_j	0	0	$-2\delta_{ij}$

$\cdot$	D_1	D_2	Δ_i
D_1	0	$2B_1$	0
D_2	$2B_1$	$2B_2$	$2A_i$
Δ_j	0	$2A_j$	$4(A_i - B_1)\delta_{ij}$

Finally, using intersection against test curves, the canonical class of Y is

$$K_Y = -2D_2.$$

4.2.3 Gromov–Witten invariants Let $r, d \geq 0$ be integers and let $\boldsymbol{k} = (k_1, \dots, k_4)$ be a tuple of half-integers $k_i \in \frac{1}{2}\mathbb{Z}$. Define the class

$$\beta_{r,d,\boldsymbol{k}} = rB_1 + dB_2 + k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4.$$

Every algebraic curve in Y has a class of this form.

For cohomology classes $\gamma_1, \dots, \gamma_m \in H^*(Y;\mathbb{Q})$, define the genus-0 potential

$$(135)\quad \langle \gamma_1, \dots, \gamma_l \rangle^Y = \sum_{r,d \geq 0} \sum_{\boldsymbol{k} \in (\frac{1}{2}\mathbb{Z})^4} \zeta^r q^d y^{\sum_i k_i} \int_{[\overline{M}_{0,m}(Y, \beta_{r,d,\boldsymbol{k}})]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \cdots \mathrm{ev}_m^*(\gamma_m),$$

where ζ, y, q are formal variables and the integral on the right-hand side is defined to be 0 whenever $\beta_{r,d,\boldsymbol{k}}$ is not effective.

The virtual class of $\overline{M}_{0,m}(Y, \beta_{r,d,\boldsymbol{k}})$ has dimension $2r + m$. Hence, for homogeneous classes $\gamma_1, \dots, \gamma_m$ of complex degrees $d_1, \dots, d_m$, respectively, satisfying $\sum_i d_i = 2r + m$, only terms with ζ^r contribute to the sum (135). In this case, we often set $\zeta = 1$.

4.2.4 WDVV equations Let $\iota\colon Y \to \mathsf{Hilb}^2(X)$ denote the inclusion and consider the subspace

$$(136)\qquad i^* H^*(\mathsf{Hilb}^2(X);\mathbb{Q}) \subset H^*(Y;\mathbb{Q})$$

of classes pulled back from $\mathsf{Hilb}^2(X)$. The tuple of classes

$$b = (T_i)_{i=1}^8 = (e_Y, D_1, D_2, \Delta, B_1, B_2, A, \omega_Y)$$

forms a basis of (136); here $e_Y = [Y]$ is the fundamental class and ω_Y is the class of a point of Y. Let $(g_{ef})_{e,f}$ with

$$g_{ef} = \langle T_e, T_f \rangle = \int_Y T_e \cup T_f$$

be the intersection matrix of b, and let $(g^{ef})_{e,f}$ be its inverse.

Lemma 35 *Let $\gamma_1, \dots, \gamma_4 \in i^* H^*(\mathsf{Hilb}^2(X);\mathbb{Q})$ be homogeneous classes of complex degrees $d_1, \dots, d_4$, respectively, such that $\sum_i d_i = 5$. Then*

$$(137)\qquad \sum_{e,f=1}^{8} \langle \gamma_1, \gamma_2, T_e \rangle^Y g^{ef} \langle \gamma_3, \gamma_4, T_f \rangle^Y = \sum_{e,f=1}^{8} \langle \gamma_1, \gamma_4, T_e \rangle^Y g^{ef} \langle \gamma_2, \gamma_3, T_f \rangle^Y.$$

Proof The claim follows directly from the classical WDVV equation [12] and direct formal manipulations. □

We reformulate (137) into the form we will use. Let

$$\gamma \in i^* H^2(\mathsf{Hilb}^2(X);\mathbb{Q})$$

be a divisor class and let

$$Q(\zeta, y, q) = \sum_{i,d,k} a_{ikd} \zeta^i y^k q^d$$

be a formal power series. Define the differential operator ∂_γ by

$$\partial_\gamma Q(\zeta, y, q) = \sum_{i,d,k} \left(\int_{iB_1 + dB_2 + kA} \gamma \right) a_{ikd} \zeta^i y^k q^d.$$

Explicitly, we have

$$\partial_{D_1} = q \frac{d}{dq}, \quad \partial_{D_2} = \zeta \frac{d}{d\zeta}, \quad \partial_\Delta = -2y \frac{d}{dy}.$$

Then, for homogeneous classes $\gamma_1, \dots, \gamma_4 \in i^* H^*(\mathsf{Hilb}^2(X);\mathbb{Q})$ of complex degrees $2, 1, 1, 1$, respectively, the left-hand side of (137) equals

$$\text{(138)} \quad \partial_{\gamma_2} \langle \gamma_1, \gamma_3 \cup \gamma_4 \rangle^Y + \partial_{\gamma_4} \partial_{\gamma_3} \langle \gamma_1 \cup \gamma_2 \rangle^Y + \sum_{\substack{T_e \in \{B_1, B_2, A\} \\ T_f \in \{D_1, D_2, \Delta\}}} \partial_{\gamma_2} (\langle \gamma_1, T_e \rangle^Y) g^{ef} \partial_{\gamma_3} \partial_{\gamma_4} \partial_{T_f} \langle 1 \rangle^Y,$$

where we let $\langle 1 \rangle$ denote the Gromov–Witten potential with no insertions. The expression for the right-hand side of (137) is similar.

4.2.5 Relation to the Gromov–Witten theory of $\mathsf{Hilb}^2(K3)$ Recall the power series (128):

$$F^{\mathrm{GW}}(y,q), \ G^{\mathrm{GW}}(y,q), \ \widetilde{G}^{\mathrm{GW}}(y,q), \ H^{\mathrm{GW}}(y,q).$$

Proposition 36 *There exists a power series*

$$\widetilde{H}^{\mathrm{GW}}(y,q) \in \mathbb{Q}((y^{1/2}))[[q]]$$

such that

(i) $$\langle B_2, B_2\rangle^Y = (F^{\mathrm{GW}})^2,$$

(ii) $$\langle \omega_Y\rangle^Y = 2G^{\mathrm{GW}},$$

(iii) $$\langle B_1, B_2\rangle^Y = 2F^{\mathrm{GW}}\cdot H^{\mathrm{GW}} + G^{\mathrm{GW}},$$

(iv) $$\langle A, B_1\rangle^Y = \widetilde{G}^{\mathrm{GW}} + \widetilde{H}^{\mathrm{GW}}\cdot H^{\mathrm{GW}},$$

(v) $$\langle A, B_2\rangle^Y = \tfrac{1}{2}\widetilde{H}^{\mathrm{GW}}\cdot F^{\mathrm{GW}}.$$

Proof Let $d \geq 0$ be an integer, let $\boldsymbol{k} = (k_1, \dots, k_4) \in \left(\frac{1}{2}\mathbb{Z}\right)^4$ be a tuple of half-integers, and let

$$\beta_{d,\boldsymbol{k}} = B_1 + dB_2 + k_1 A_1 + \cdots + k_4 A_4.$$

Consider a stable map $f\colon C \to Y$ of genus 0 and class $\beta_{d,\boldsymbol{k}}$. The composition $\pi \circ f\colon C \to \mathbb{P}^2$ has degree 1 with image a line L. Let C_0 be the component of C on which $\pi \circ f$ is nonconstant.

Let $g\colon \widetilde{W} \to Y$ be the quotient map (132), and consider the fiber diagram

$$\begin{array}{ccccc} \widetilde{C} & \xrightarrow{\tilde{f}} & \widetilde{W} & \xrightarrow{p} & \mathbb{P}^1 \times E \\ \downarrow & & \downarrow g & & \\ C & \xrightarrow{f} & Y & & \end{array}$$

where $p = \mathrm{pr}_{23} \circ \rho$ is the composition of the blowdown map with the projection to the $(2,3)$–factor of $\mathbb{P}^1 \times \mathbb{P}^1 \times E$. Then, parallel to the case of elliptic K3 surfaces, the image of $\widetilde{C}$ under $p \circ \tilde{f}$ is a comb curve

$$B_e + \mathrm{pr}_1^{-1}(z),$$

where B_e is the fiber of the projection $X \to E$ over some point $e \in E$, the map $\mathrm{pr}_1\colon \mathbb{P}^1 \times E \to \mathbb{P}^1$ is the projection to the first factor, and $z \subset \mathbb{P}^1$ is a zero-dimensional subscheme of length d.

Let $G_0 \subset \widetilde{C}$ be the irreducible component which maps with degree 1 to B_e under $p \circ \tilde{f}$. The projection $\widetilde{C} \to C$ induces a flat map

(139) $$G_0 \to C_0.$$

If (139) has degree 2, then, similarly to the arguments of Lemma 13, the restriction $f|_{C_0}$ is an isomorphism onto an embedded line

$$L \subset \mathsf{Hilb}^2(S_e) \subset Y,$$

where $e = -e \in E$ is a 2–torsion point of E. Since $f|_{C_0}$ is irreducible, we have $L \not\subseteq I(x)$ for all $x \in \mathbb{P}^1$. The tangent line to $\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)}$ at $2x$ is $I(x)$ for every $x \in \mathbb{P}^1$. Hence, L intersects the diagonal $\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)}$ in two distinct points.

If (139) has degree 1, the map $f|_{C_0}$ is the sum of two maps $C_0 \to X$. The first of these must map C_0 to a section of $X \to \mathbb{P}^1$, the second must be constant since there are no nonconstant maps to the fiber of $X \to \mathbb{P}^1$. Hence, the restriction $f|_{C_0}$ is an isomorphism onto the embedded line[11]

$$B_e + (x', -e) = g(\rho^{-1}(x' \times B_e \times -e)) \tag{140}$$

for some $x' \in \mathbb{P}^1$ and $e \in E$; here we used the notation (21).

Every irreducible component of C other then C_0 maps into the fiber of

$$\pi\colon Y \to \mathsf{Hilb}^2(\mathbb{P}^1) = \mathbb{P}^2$$

over a diagonal point $2x \in \Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)}$.

Summarizing, the map $f\colon C \to Y$ therefore satisfies one of the following:

(A) The restriction $f|_{C_0}$ is an isomorphism onto a line

$$L \subset \mathsf{Hilb}^2(B_e) \subset Y, \tag{141}$$

where $e \in E$ is a 2–torsion point. The line L intersects the diagonal in the distinct points $2x_1$ and $2x_2$. The curve C has a decomposition

$$C = C_0 \cup C_1 \cup C_2 \tag{142}$$

such that for $i = 1, 2$ the restriction $f|_{C_i}$ maps in the fiber $\pi^{-1}(2x_i)$.

(B) The restriction $f|_{C_0}$ is an isomorphism onto the line (140) for some $x' \in \mathbb{P}^1$ and $e \in E$. The image $f(C_0)$ meets the fiber $\pi^{-1}(\Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)})$ only in the point $(x', e) + (x', -e)$. Hence, the curve C admits the decomposition

$$C = C_0 \cup C_1, \tag{143}$$

where $f|_{C_1}$ maps to the fiber $\pi^{-1}(2x')$.

[11] If e is a 2–torsion point of E, we take the proper transform instead of ρ^{-1} in (140). This case will not appear below.

According to the above cases, we say that $f\colon C\to Y$ is of type (A) or (B).

We consider the different cases of Proposition 36.

Case (i) Let Z_1, Z_2 be generic fibers of the natural map

$$\pi\colon Y\to \mathsf{Hilb}^2(\mathbb{P}^1).$$

The fibers Z_1, Z_2 have class $2B_2$. Let $f\colon C\to Y$ be a stable map of class $\beta_{d,\boldsymbol{k}}$ incident to Z_1 and Z_2. Then f must be of type (A) above, with the line L in (141) uniquely determined by Z_1, Z_2 up the choice of the 2–torsion point $e\in E$. After specifying a 2–torsion point, we are in a case completely parallel to Section 2, except for the existence of the nodal fibers in the K3 case. Following the argument there, we find the contribution from each fixed 2–torsion point to be $(F^{\mathrm{GW}})^2$. Hence,

$$\langle 2B_2, 2B_2\rangle^Y = \bigl|\{e\in E \mid 2e=0\}\bigr|\cdot (F^{\mathrm{GW}})^2 = 4\cdot (F^{\mathrm{GW}})^2.$$

Case (ii) Let $x_1, x_2\in\mathbb{P}^1$ and $e\in E$ be generic, and consider the point

$$y=(x_1,e)+(x_2,-e)\in Y.$$

A stable map $f\colon C\to Y$ of class $\beta_{d,\boldsymbol{k}}$ incident to y must be of type (B) above, with $x'=x_1$ or x_2 in (140). In each case, the calculation proceeds completely analogously to Section 3.3 and yields the contribution G^{GW}. Summing up both cases, we therefore find $\langle y\rangle^Y = 2G^{\mathrm{GW}}$.

Case (iii) Let $x', x_1, x_2\in\mathbb{P}^1$ and $e\in E$ be generic points. Let

$$Z_1 = g(\rho^{-1}(\mathbb{P}^1\times x'\times e)) = (\mathbb{P}^1\times e)+(x',-e) \tag{144}$$

and let Z_2 be the fiber of π over the point x_1+x_2. The cycles Z_1, Z_2 have the cohomology classes $[Z_1]=B_1$ and $[Z_2]=2B_2$, respectively. Let

$$f\colon C\to Y$$

be a 2–marked stable map of genus 0 and class $\beta_{d,\boldsymbol{k}}$ with markings $p_1,p_2\in C$ incident to Z_1, Z_2, respectively. Since $f(p_1)\in Z_1$, we have

$$f(p_1)=(x'',e)+(x',-e)$$

for some $x''\in\mathbb{P}^1$. Since also $f(p_2)\in Z_2$ and e is generic, $x''\in\{x',x_1,x_2\}$.

Assume $x''=x_1$. Then f is of type (B) and the restriction $f|_{C_0}$ is an isomorphism onto the line $\ell=B_e+(x_1,-e)$. The line ℓ meets the cycle Z_2 in the point $(x_2,e)+(x_1,-e)$

and no marked point of C lies on the component C_1 in the splitting (143). Parallel to (ii), the contribution of this case is G^{GW}. The case $x''=x_2$ is identical.

Assume $x''=x'$. Then $\pi(f(p_1))=2x'$. Since $\pi(f(p_2))=x_1+x_2$, we have $\pi(f(p_1))\cap\pi(f(p_2))=\varnothing$. Hence, f is of type (A) and we have the decomposition

$$C=C_0\cup C_1\cup C_2,$$

where $f|_{C_0}$ maps to a line $L\subset \mathsf{Hilb}^2(B_{e'})$ for a 2–torsion point $e'\in E$, the restriction $f|_{C_1}$ maps to $\pi^{-1}(2x')$, and $f|_{C_2}$ maps to the fiber of π over the diagonal point of L which is not $2x'$. We have $p_1\in C_1$ with $f(p_1)\in Z_1$, and $p_2\in C_0$ with $f(p_2)=(x_1,e')+(x_2,-e')$. The contribution from maps to the fiber over $2x'$ matches the contribution H^{GW} considered in Section 3.6. Since there is no marking on C_2, the contribution from maps $f|_{C_2}$ is F^{GW}. For each fixed 2–torsion point $e'\in E$, we therefore find the contribution $F^{\mathrm{GW}}\cdot H^{\mathrm{GW}}$.

In total, we obtain

$$\langle B_1,2B_2\rangle = 2\cdot G^{\mathrm{GW}}+4\cdot F^{\mathrm{GW}}\cdot H^{\mathrm{GW}}.$$

Case (iv) Let $x,x'\in\mathbb{P}^1$ and $e'\in E$ be generic points, and let $e\in E$ be the i^{th} 2–torsion point. Consider the exceptional curve at (x,e),

$$Z_1=g(\rho^{-1}(x,x,e)),$$

and the cycle which appeared in (144) above,

$$Z_2=g(\rho^{-1}(\mathbb{P}^1\times x'\times e'))=(\mathbb{P}^1\times e')+(x',-e').$$

We have $[Z_1]=A_i$ and $[Z_2]=B_1$. Consider a 2–marked stable map $f\colon C\to Y$ of class $\beta_{d,\boldsymbol{k}}$ with markings $p_1,p_2\in C$ incident to Z_1,Z_2, respectively.

If f is of type (B), we must have $\pi(f(p_1))\cap\pi(f(p_2))\neq\varnothing$. Hence, $f(p_2)=(x,e')+(x',-e')$ and the restriction $f|_{C_0}$ is an isomorphism onto

$$\ell=(\rho^{-1}(x\times\mathbb{P}^1\times e'))=B_{(-e')}+(x,e').$$

In the splitting (143), the component C_1 is attached to the component $C_0\equiv\ell$ at $(x,-e')+(x,e')$. Then the contribution here matches precisely the contribution of the point u_0 in the K3 case of Section 3.4; it is $\widetilde{G}^{\mathrm{GW}}$.

Assume f is of type (A). The line L in (141) lies inside $\mathsf{Hilb}^2(B_{e''})$ for some 2–torsion point $e''\in E$. Since e' is generic, $\pi(L)$ is the line through $2x$ and $2x'$. Consider the splitting (142) with C_1 and C_2 mapping to the fibers of π over $2x$ and $2x'$, respectively. The contribution from maps $f|_{C_2}$ over $2x'$ is parallel to Section 3.5.2; it is H^{GW}. Let $\widetilde{H}_0$ (resp. $\widetilde{H}_1$) be the contribution from maps $f|_{C_1}$ over $2x$ if $e''=e$ (resp. if $e''\neq e$). Then, summing up over all 2–torsion points, the total contribution is $\widetilde{H}^{\mathrm{GW}}\cdot H^{\mathrm{GW}}$, where $\widetilde{H}^{\mathrm{GW}}=\widetilde{H}_0+3\widetilde{H}_1$.

Adding up both cases, we obtain $\langle A_i, B_1\rangle^Y=\widetilde{G}^{\mathrm{GW}}+\widetilde{H}^{\mathrm{GW}}\cdot H^{\mathrm{GW}}$.

Case (v) This is identical to the second case of (iv) above, with the difference that the second marked point lies on C_0, not C_2. □

4.3 Calculations

4.3.1 Initial conditions Define the formal power series

$$H=\sum_{d\geq 0}\sum_{k\in\mathbb{Z}} H_{d,k}y^k q^d=\langle B_2,B_2\rangle^Y,$$
$$I=\sum_{d\geq 0}\sum_{k\in\mathbb{Z}} I_{d,k}y^k q^d=\langle \omega_Y\rangle^Y,$$
$$T=\sum_{d\geq 0}\sum_{k\in\mathbb{Z}} T_{d,k}y^k q^d=\langle 1\rangle^Y,$$

where $\langle 1\rangle^Y$ is the Gromov–Witten potential (135) with no insertion, and we have set $\zeta=1$ in (135). We have the following initial conditions.

Proposition 37 *We have*

(i) $T_{0,k}=8/k^3$ *for all* $k\geq 1$,

(ii) $T_{d,-2d}=2/d^3$ *for all* $d\geq 1$,

(iii) $H_{-1,0}=1$,

(iv) $H_{d,k}=0$ *if* $(d=0,k\leq -2)$ *or* $(d>0,k<-2d)$,

(v) $T_{d,k}=0$ *if* $k<-2d$,

(vi) $I_{d,k}=0$ *if* $k<-2d$.

Proof **Case (i)** The moduli space $\overline{M}_0(Y,\sum_i k_iA_i)$ is nonempty only if there exists a $j\in\{1,\dots,4\}$ with $k_i=\delta_{ij}k$ for all i. Hence,

$$T_{0,k}=\sum_{k_1+\dots+k_4=k}\int_{[\overline{M}_0(Y,\sum_i k_iA_i)]^{\mathrm{vir}}}1=\sum_{i=1}^{4}\int_{[\overline{M}_0(Y,kA_i)]^{\mathrm{vir}}}1.$$

Since the term in the last sum is independent of i,

$$T_{0,k} = 4 \int_{[\overline{M}_0(Y,kA_1)]^{\mathrm{vir}}} 1. \tag{145}$$

Let $e \in E$ be the first 2–torsion point, let

$$D_{X,1} = \{(x,x,e) \mid x \in \mathbb{P}^1\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times E$$

and consider the subscheme

$$\Delta_1 = g(\rho^{-1}(D_{X,1}))$$

which appeared in Section 4.2.2. The divisor Δ_1 is isomorphic to the exceptional divisor E_1 of the blowup $\rho\colon \widetilde{W} \to W$; see (131). Hence $\Delta_1 = \mathbb{P}(V)$, where

$$V = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \to \mathbb{P}^1.$$

Under the isomorphism $\Delta_1 = \mathbb{P}(V)$, the map

$$\pi\colon \Delta_1 \to \Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)} \equiv \mathbb{P}^1 \tag{146}$$

is identified with the natural $\mathbb{P}(V) \to \mathbb{P}^1$.

The normal bundle of the exceptional divisor $E_1 \subset \widetilde{W}$ is $\mathcal{O}_{\mathbb{P}(V)}(-1)$. Hence, taking the $\mathbb{Z}_2$–quotient (132) of $\widetilde{W}$, the normal bundle of $\Delta_1 \subset Y$ is

$$N = N_{\Delta_1/Y} = \mathcal{O}_{\mathbb{P}(V)}(-2).$$

For $k \geq 1$, the moduli space

$$M = \overline{M}_0(Y, kA_1)$$

parametrizes maps to the fibers of the fibration (146). Since the normal bundle N of Δ_1 has degree -2 on each fiber, there are no infinitesimal deformations of maps out of Δ_1. Hence, M is isomorphic to $\overline{M}_0(\mathbb{P}(V), d\mathfrak{f})$, where $\mathfrak{f}$ is the class of a fiber of $\mathbb{P}(V)$. In particular, M is smooth of dimension $2k-1$.

By smoothness of M and convexity of $\mathbb{P}(V)$ in class $k\mathfrak{f}$, the virtual class of M is the Euler class of the obstruction bundle Ob with fiber

$$\mathrm{Ob}_f = H^1(C, f^*T_Y)$$

over the moduli point $[f\colon C \to Y] \in M$. The restriction of the tangent bundle T_Y to a fixed fiber A_0 of (146) is

$$T_Y|_{A_0} \cong \mathcal{O}_{A_0}(2) \oplus \mathcal{O}_{A_0} \oplus \mathcal{O}_{A_0}(-2).$$

Hence,

$$\mathrm{Ob}_f = H^1(C, f^*T_Y) = H^1(C, f^*N).$$

Consider the relative Euler sequence of $p\colon \mathbb{P}(V) \to \mathbb{P}^1$,

$$0 \to \Omega_p \to p^*V \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{\mathbb{P}(V)} \to 0. \tag{147}$$

By direct calculation, $\Omega_p = \mathcal{O}_{\mathbb{P}(V)}(-2) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(-2)$. Hence, twisting (147) by $p^*\mathcal{O}_{\mathbb{P}^1}(2)$, we obtain the sequence

$$0 \to N \to p^*V(2) \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to p^*\mathcal{O}_{\mathbb{P}^1}(2) \to 0. \tag{148}$$

Let $q\colon \mathcal{C} \to M$ be the universal curve and let $f\colon \mathcal{C} \to \Delta_1 \subset Y$ be the universal map. Pulling back (148) by f, pushing forward by q, and taking cohomology, we obtain the exact sequence

$$0 \to R^0q_*f^*p^*\mathcal{O}_{\mathbb{P}^1}(2) \to R^1q_*f^*N \to R^1q_*f^*p^*V(2) \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to 0. \tag{149}$$

The bundle $R^1q_*f^*N$ is the obstruction bundle Ob, and

$$R^0q_*f^*p^*\mathcal{O}_{\mathbb{P}^1}(2) = q_*q^*p'^*\mathcal{O}_{\mathbb{P}^1}(2) = p'^*\mathcal{O}_{\mathbb{P}^1}(2),$$

where $p'\colon M \to \mathbb{P}^1$ is the map induced by $p\colon \mathbb{P}(V) \to \mathbb{P}^1$. We find

$$c_1(p'^*\mathcal{O}_{\mathbb{P}^1}(2)) = 2p'^*\omega_{\mathbb{P}^1},$$

where $\omega_{\mathbb{P}^1}$ is the class of a point on $\mathbb{P}^1$. Taking everything together, we have

$$\begin{aligned} \int_{[\overline{M}_0(Y,kA_1)]^{\mathrm{vir}}} 1 &= \int_M e(R^1q_*f^*N) \\ &= \int_M c_1(p'^*\mathcal{O}_{\mathbb{P}^1}(2)) c_{2k-2}(R^1q_*f^*p^*V(2) \otimes \mathcal{O}_{\mathbb{P}(V)}(-1)) \\ &= 2\int_{M_x} c_{2k-2}(R^1q_*f^*p^*V(2) \otimes \mathcal{O}_{\mathbb{P}(V)}(-1))|_{M_x}, \end{aligned} \tag{150}$$

where $M_x = \overline{M}_0(\mathbb{P}^1, k)$ is the fiber of $p'\colon M \to \mathbb{P}^1$ over some $x \in \mathbb{P}^1$. Since

$$p^*V(2) \otimes \mathcal{O}_{\mathbb{P}(V)}(-1)|_{p^{-1}(x)} = \mathcal{O}_{\mathbb{P}(V)_x}(-1) \oplus \mathcal{O}_{\mathbb{P}(V)_x}(-1),$$

the term (150) equals $2\int_{\overline{M}_{0,0}(\mathbb{P}^1,k)} c_{2k-2}(\mathcal{E})$, where $\mathcal{E}$ is the bundle with fiber

$$H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \oplus H^1(C, f^*\mathcal{O}_{\mathbb{P}^1}(-1))$$

over a moduli point $[f\colon C \to \mathbb{P}^1] \in M_x$. Hence, using the Aspinwall–Morrison formula [17, Section 27.5] the term (150) is

$$\int_{[\overline{M}_0(Y,kA_1)]^{\mathrm{vir}}} 1 = 2 \cdot \int_{\overline{M}_{0,0}(\mathbb{P}^1,k)} c_{2k-2}(\mathcal{E}) = \frac{2}{k^3}.$$

Combining with (145), the proof of case (i) is complete.

Cases (ii) and (v) Let $f\colon C \to Y$ be a stable map of genus 0 and class $dB_2 + \sum k_i A_j$. Then f maps into the fiber of

$$\pi\colon Y \to \mathsf{Hilb}^2(\mathbb{P}^1)$$

over some diagonal point $2x \in \Delta_{\mathsf{Hilb}^2(\mathbb{P}^1)}$. The reduced locus of such a fiber is the union

$$\Sigma_x \cup A_{x,e_1} \cup \cdots \cup A_{x,e_4} \tag{151}$$

where $e_1, \dots, e_4 \in E$ are the 2–torsion points of E,

$$A_{x,e} = g(\rho^{-1}(x \times x \times e))$$

is the exceptional curve of $\mathsf{Hilb}^2(X)$ at $(x,e) \in X$, and Σ_x is the fiber of the addition map $\mathsf{Hilb}^2(F_x) \to F_x = E$ over the origin 0_E. Hence,

$$f_*[C] = a[\Sigma_x] + \sum_i b_i [A_{x,e_i}]$$

for some $a, b_1, \dots, b_4 \geq 0$. Since $[A_{x,e_i}] = A_i$ and

$$[\Sigma_x] = B_2 - \tfrac{1}{2}(A_1 + A_2 + A_3 + A_4),$$

we must have $d = a$ and therefore

$$f_*[C] = dB_2 + \sum_i (b_i - d/2) A_i.$$

Since $b_i \geq 0$ for all i, we find $\sum_i k_i \geq -2d$, with equality if and only if $k_i = -d/2$ for all i. This proves (v) and shows

$$T_{d,-2d} = \int_{[\overline{M}_0(Y, dB_2 - \sum_i (d/2)A_i)]^{\mathrm{vir}}} 1. \tag{152}$$

Moreover, if $f\colon C \to Y$ has class $dB_2 - \sum_i (d/2) A_i$, it is a degree-d cover of the curve Σ_x for some x.

We evaluate the integral (152). Let Z' be the proper transform of

$$\mathbb{P}^1 \times E \hookrightarrow W, \ (x,e) \mapsto (x,x,e)$$

under the blowup map $\rho\colon \widetilde{W} \to W$, and let

$$Z = g(Z') = Z'/\mathbb{Z}_2 \subset Y$$

be its image under $g\colon \widetilde{W} \to Y$. The projection map $\mathrm{pr}_{1,3} \circ \rho\colon Z' \to \mathbb{P}^1 \times E$ descends by $\mathbb{Z}_2$–quotient to the isomorphism

$$(\tau_{|Z}, \pi_{|Z})\colon Z \to \mathbb{P}^1 \times \mathbb{P}^1, \tag{153}$$

where $\tau\colon Y \to \mathbb{P}^1$ is the morphism defined in (133). Under the isomorphism (153), the curve Σ_x equals $\mathbb{P}^1 \times x$. Since moreover the normal bundle of $Z \subset Y$ has degree -2 on Σ_x, we find

$$\overline{M}_0(Y, dB_2 - 2dA) \cong \overline{M}_0(\mathbb{P}^1, d) \times \mathbb{P}^1.$$

The normal bundle $Z \subset Y$ is the direct sum

$$N = N_{Z/Y} = \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(-2)$$

for some a. We determine a. Under the isomorphism (153), the curve

$$R = x \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$$

corresponds to the diagonal in a generic fiber of $\tau\colon Y \to \mathbb{P}^1$. The generic fiber of τ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, hence $c_1(N) \cdot R = 2$ and $a = 2$. The result now follows by an argument parallel to (i).

Case (iii) This follows directly from the proof of Case (i) of Proposition 36 since the line in (141) has class $B_1 - A_i$ for some i.

Case (iv) Let $f\colon C \to Y$ be a stable map of genus 0 and class $\beta_{d,\boldsymbol{k}}$ incident to the cycles Z_1, Z_2 of the proof of Case (i) of Proposition 36. Then there exists an irreducible component $C_0 \subset C$ which maps isomorphically to the line L considered in (141). We have $[L] = B_1 - A_i$ for some i.

Since all irreducible components of C except for C_0 gets mapped under f to curves of the form Σ_x or $A_{x,i}$, we have

$$\begin{aligned} f_*[C] = \beta_{d,\boldsymbol{k}} &= [L] + d[\Sigma_x] + \sum_j b_j A_j \\ &= B_1 + dB_2 + \sum_j (-d/2 - \delta_{ij} + b_j) \end{aligned}$$

for some $b_1, \dots, b_4 \geq 0$. If $d = 0$ we find $k = \sum_i k_i \geq -1$. If $d > 0$, then f maps to at least one curve of the form Σ_x with nonzero degree. Since L and Σ_x are disjoint, we must have $b_j > 0$ for some j. This shows $k = \sum_j k_j \geq -2d$.

Case (vi) This case follows by an argument parallel to Case (iv). □

4.3.2 The system of equations Let d/dq and d/dy be the formal differentiation operators with respect to q and y, respectively. We will use the notation

$$\partial_\tau = q\frac{d}{dq} \quad \text{and} \quad \partial_z = y\frac{d}{dy}.$$

The WDVV equation (137), applied to the cohomology insertions

$$\xi = (\gamma_1, \dots, \gamma_4)$$

specified below, yields the following relations:

(154)
$$\begin{aligned}
\xi = (B_2, D_2, D_2, \Delta) &\implies & \langle B_2, A\rangle &= -\tfrac{1}{2}\partial_z(H),\\
\xi = (B_2, D_2, D_2, D_1) &\implies & \langle B_1, B_2\rangle &= \partial_\tau H + \tfrac{1}{2} I,\\
\xi = (A, D_2, D_2, \Delta) &\implies & \langle A, A\rangle &= \tfrac{1}{4}\partial_z^2 H - \tfrac{1}{4} I,\\
\xi = (A, D_2, D_2, D_1) &\implies & \langle B_1, A\rangle &= -\tfrac{1}{2}\partial_z\partial_\tau H,\\
\xi = (B_1, D_2, D_2, \Delta) &\implies & \langle B_1, A\rangle - \tfrac{1}{4}\partial_z I &= -\tfrac{1}{2}\partial_z\langle B_1, B_2\rangle,\\
\xi = (B_1, D_2, D_2, D_1) &\implies & 2\langle B_1, B_1\rangle + \partial_\tau I &= 2\partial_\tau\langle B_1, B_2\rangle\\
&& \iff \langle B_1, B_1\rangle &= \partial_\tau^2 H.
\end{aligned}$$

Using (154) and the WDVV equations (137) with insertions ξ further yields:

(W1) $\xi = (B_2, D_1, D_1, D_2)$:

$$0 = 2\partial_\tau^2 H + 2\partial_\tau I - H \cdot \partial_\tau^3 T + \tfrac{1}{2}\partial_z H \cdot \partial_z \partial_\tau^2 T;$$

(W2) $\xi = (B_2, \Delta, \Delta, D_2)$:

$$0 = 2\partial_z^2 H + 4\partial_\tau H + 2I - H \cdot \partial_z^2\partial_\tau T + \tfrac{1}{2}\partial_z H \cdot (4 + \partial_z^3 T);$$

(W3) $\xi = (B_2, \Delta, \Delta, D_1)$:

$$\begin{aligned}0 = 4\partial_\tau^2 H + 2\partial_\tau I - \partial_z^2 I + \tfrac{1}{2}\partial_z\partial_\tau H \cdot (4 + \partial_z^3 T) - \partial_\tau H \cdot \partial_z^2\partial_\tau T\\
- \tfrac{1}{2}\partial_z^2 H \cdot \partial_z^2\partial_\tau T + \partial_z H \cdot \partial_z\partial_\tau^2 T;\end{aligned}$$

(W4) $\xi = (A, \Delta, \Delta, D_2)$:

$$0 = -8\partial_z\partial_\tau H - 4\partial_z^3 H + 8\partial_z I + 2\partial_z H \cdot \partial_z^2\partial_\tau T - \partial_z^2 H \cdot (4 + \partial_z^3 T) + I \cdot (4 + \partial_z^3 T);$$

(W5) $\xi = (A, \Delta, D_1, D_1)$:

$$0 = -2\partial_\tau^2 I + \tfrac{1}{2}\partial_z^2\partial_\tau H \cdot \partial_z^2\partial_\tau T - \partial_z\partial_\tau H \cdot \partial_z\partial_\tau^2 T - \tfrac{1}{2}\partial_z^3 H \cdot \partial_z\partial_\tau^2 T + \partial_z^2 H \cdot \partial_\tau^3 T - \tfrac{1}{2}\partial_\tau I \cdot \partial_z^2\partial_\tau T + \tfrac{1}{2}\partial_z I \cdot \partial_z\partial_\tau^2 T;$$

(W6) $\xi = (B_1, D_1, D_1, D_2)$:

$$0 = 2\partial_\tau^3 H - \partial_\tau^2 I - \partial_\tau H \cdot \partial_\tau^3 T - \tfrac{1}{2} I \cdot \partial_\tau^3 T + \tfrac{1}{2}\partial_z\partial_\tau H \cdot \partial_z\partial_\tau^2 T.$$

4.3.3 Nondegeneracy of the equations

Proposition 38 *The initial conditions of Proposition 37 and the equations* **(W1)–(W6)** *together determine* $H_{d,k}$, $I_{d,k}$, $T_{d,k}$ *for all* d *and* k.

Proof of Proposition 38 For all d, k, taking the coefficient of $q^d y^k$ in equations **(W1)–(W6)** yields:

(W1) $$2d^2 H_{d,k} + 2d I_{d,k} = \sum_{j,l} (d-l)^2 \big((d-l) - \tfrac{1}{2} j(k-j)\big) H_{l,j} T_{d-l,k-j};$$

(W2) $$(2k(k+1) + 4d) H_{d,k} + 2I_{d,k} = \sum_{j,l} (k-j)^2 \big((d-l) - \tfrac{1}{2} j(k-j)\big) H_{l,j} T_{d-l,k-j};$$

(W3) $$2d(2d+k) H_{d,k} + (2d - k^2) I_{d,k} = -\sum_{j,l} (k-j)\big(j(d-l) - l(k-j)\big)\big((d-l) - \tfrac{1}{2} j(k-j)\big) H_{l,j} T_{d-l,k-j};$$

(W4) $$(2k+1) I_{d,k} - k(k^2 + k + 2d) H_{d,k} = -\frac{1}{2}\sum_{j,l} (k-j)^2 \big((j(d-l) - \tfrac{1}{2}(k-j)) H_{l,j} + \tfrac{1}{2}(k-j) I_{l,j}\big) T_{d-l,k-j};$$

(W5) $$2d^2 I_{d,k} = \sum_{j,l} (d-l)\big(j(d-l) - l(k-j)\big) \cdot \big(j(d-l) H_{l,j} - \tfrac{1}{2} j^2 (k-j) H_{l,j} + \tfrac{1}{2}(k-j) I_{l,j}\big) T_{d-l,k-j};$$

(W6) $$2d^3 H_{d,k} - d^2 I_{d,k} = \sum_{j,l} (d-l)^2 \cdot \big((d-l)(l H_{l,j} + \tfrac{1}{2} I_{l,j}) - \tfrac{1}{2} j l (k-j) H_{l,j}\big) T_{d-l,k-j}.$$

Claim 1 The initial conditions and **(W1)–(W6)** determine $H_{0,k}$, $I_{0,k}$, $T_{0,k}$ for all k, except for $H_{0,0}$.

Proof of Claim 1 The values $T_{0,k}$ are determined by the initial conditions. Consider the equation **(W2)** for $(d,k)=(0,0)$. Plugging in $(d,k)=(0,0)$ and using $H_{0,-1}=1$, $T_{0,1}=8$, we find $I_{0,0}=2$.

Let $d=0$ and $k>0$, and assume we know the values $H_{0,j}, I_{0,j}$ for all $j<k$ except for $H_{0,0}$. Then equations **(W3)** and **(W4)** read

$$-4k^2 I_{0,k} + (\text{known terms}) = 0,$$
$$b - 4k^2(k+1)H_{0,k} + (\text{known terms}) = 0.$$

Hence $I_{0,k}$ and $H_{0,k}$ are also uniquely determined. By induction, the proof of Claim 1 is complete. □

Let $d>0$. We argue by induction. Calculating the first values of $H_{0,k}$, $I_{0,k}$ and $T_{0,k}$, and plugging them into equations **(W1)–(W6)** for $(d,k)=(1,-2)$ and $(d,k)=(1,-1)$, we find by direct calculation that the values

$$H_{0,0},\ H_{1,-2},\ H_{1,-1},\ I_{1,-2},\ I_{1,-1},\ T_{1,-1},\ T_{1,0}$$

are determined.

Now let $(d=1,\ k\geq 0)$ or $(d>1,\ k\geq -2d)$, and assume we know the values $H_{l,j}$, $I_{l,j}$, $T_{l,j}$ for all $l<d$, $j\leq k+2(d-l)$ and for all $l=d$, $j<k$. Also assume that we know $T_{d,k}$. The proof of Proposition 38 follows now from the following claim.

Claim 2 The values $H_{d,k}, I_{d,k}, T_{d,k+1}$ are determined.

Proof of Claim 2 Solving for the terms $H_{d,k}$, $I_{d,k}$, $T_{d,k+1}$ in the equations **(W1)**, **(W6)** and **(W5)**, we obtain

$$\text{(W1)} \qquad 2d^2 H_{d,k} + 2d I_{d,k} - d^2\big(d+\tfrac{1}{2}(k+1)\big)T_{d,k+1} = (\text{known terms}),$$
$$\text{(W6)} \qquad 2d^3 H_{d,k} - d^2 I_{d,k} = (\text{known terms}),$$
$$\text{(W5)} \qquad -2I_{d,k} + \big(d+\tfrac{1}{2}(k+1)\big)T_{d,k+1} = (\text{known terms}),$$

where in the last line we divided by d^2. These equations in matrix form read

$$\begin{pmatrix} 2d & 2 & -d\big(d+\frac{1}{2}(k+1)\big) \\ 2d & -1 & 0 \\ 0 & -2 & d+\frac{1}{2}(k+1) \end{pmatrix} \cdot \begin{pmatrix} H_{d,k} \\ I_{d,k} \\ T_{d,k+1} \end{pmatrix} = (\text{known terms}).$$

The matrix on the left-hand side has determinant $(2d-3)(k+2d+1)d$. It vanishes if $d=\frac{3}{2}$ or $k=-2d-1$ or $d=0$. By assumption, each of these cases was excluded. Hence the values $H_{d,k}$, $I_{d,k}$, $T_{d,k+1}$ are uniquely determined. □

Remark We have selected very particular WDVV equations for Y above. Using additional equations, one may show that the values

$$H_{0,-1} = 1, \quad T_{0,0} = 0, \quad T_{0,1} = 8, \quad T_{1,-2} = 2$$

together with the vanishings of Proposition 37(iv)–(vi) suffice to determine the series H, I, T.

4.3.4 Solution of the equations Let $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ and consider the actual variables

$$y = -e^{2\pi i z} \quad \text{and} \quad q = e^{2\pi i \tau}. \tag{155}$$

Let $F(z,\tau)$ and $G(z,\tau)$ be the functions (5) and (8), respectively.

Theorem 39 *We have*

$$\begin{aligned} H &= F(z,\tau)^2, \\ I &= 2G(z,\tau), \\ T &= 8\sum_{k\geq 1} \frac{1}{k^3} y^k + 12 \sum_{k,n\geq 1} \frac{1}{k^3} q^{kn} \\ &\quad + 8 \sum_{k,n\geq 1} \frac{1}{k^3}(y^k + y^{-k})q^{kn} + 2 \sum_{k,n\geq 1} \frac{1}{k^3}(y^{2k} + y^{-2k})q^{(2n-1)k} \end{aligned}$$

under the variable change $y = -e^{2\pi i z}$ *and* $q = e^{2\pi i \tau}$.

Proof By Proposition 38, it suffices to show that the functions defined in the statement of Theorem 39 satisfy the initial conditions of Proposition 37 and the WDVV equations **(W1)–(W6)**. By a direct check, the initial conditions are satisfied. We consider the WDVV equations.

For the scope of this proof, define $H = F(z,\tau)^2$ and $I = 2G(z,\tau)$ and

$$\begin{aligned} T = 8\sum_{k\geq 1} \frac{1}{k^3} y^k &+ 12 \sum_{k,n\geq 1} \frac{1}{k^3} q^{kn} \\ &+ 8 \sum_{k,n\geq 1} \frac{1}{k^3}(y^k + y^{-k})q^{kn} + 2 \sum_{k,n\geq 1} \frac{1}{k^3}(y^{2k} + y^{-2k})q^{(2n-1)k} \end{aligned}$$

considered as functions in z and τ under the variable change (155). We show these functions satisfy the equations **(W1)–(W6)**.

For a function $A(z,\tau)$, we write

$$A^{\bullet} = \partial_z A := \frac{1}{2\pi i}\frac{\partial A}{\partial z} = y\frac{d}{dy}A, \quad A' = \partial_\tau A := \frac{1}{2\pi i}\frac{\partial A}{\partial \tau} = q\frac{d}{dq}A$$

for the differentials of A with respect to z and τ, respectively.

For $n \geq 1$, define the deformed Eisenstein series [33]

$$J_{2,n}(z,\tau) = \delta_{n,1}\frac{y}{y-1} + B_n - n\sum_{k,r\geq 1} r^{n-1}(y^k + (-1)^n y^{-k})q^{kr},$$
$$J_{3,n}(z,\tau) = -B_n\Big(1-\frac{1}{2^{n-1}}\Big) - n\sum_{k,r\geq 1}\big(r-\tfrac{1}{2}\big)^{n-1}(y^k + (-1)^n y^{-k})q^{k(r-\frac{1}{2})},$$

where B_n are the Bernoulli numbers $\big(\text{with } B_1 = -\frac{1}{2}\big)$ and we used the variable change (155). We also let

$$\begin{aligned} G_n(z,\tau) &= J_{4,n}(2z,2\tau) \\ &= -B_n\Big(1-\frac{1}{2^{n-1}}\Big) - n\sum_{k,r\geq 1}\big(r-\tfrac{1}{2}\big)^{n-1}(y^{2k} + (-1)^n y^{-2k})q^{k(2r-1)}. \end{aligned}$$

Then we have

$$\begin{aligned} \partial_z^3 T &= -4 - 8J_{2,1} - 16G_1, \\ \partial_z^2\partial_\tau T &= -4J_{2,2} - 8G_2, \\ \partial_z\partial_\tau^2 T &= -\tfrac{8}{3}J_{2,3} - \tfrac{16}{3}G_3, \\ \partial_\tau^3 T &= -2J_{2,4} - 4G_4 + \tfrac{1}{20}E_4. \end{aligned} \tag{156}$$

Since $T(z,\tau)$ appears only as a third derivative in the equations **(W1)–(W6)**, we may trade it for a deformed Eisenstein series using (156).

The first theta function $\vartheta_1(z,\tau)$ satisfies the heat equation

$$\partial_z^2\vartheta_1 = 2\partial_\tau\vartheta_1,$$

which implies that $F = F(z,\tau) = \vartheta_1(z,\tau)/\eta^3(\tau)$ satisfies

$$\partial_\tau F = \tfrac{1}{2}\partial_z^2 F - \tfrac{1}{8}E_2(\tau)F, \tag{157}$$

where $E_2(\tau) = 1 - 24\sum_{d\geq 1}\sum_{k|d} kq^d$ is the second Eisenstein series. With a small calculation, we obtain the relation

$$I = 4\partial_\tau(H) - \partial_z^2(H) + E_2\cdot H. \tag{158}$$

Hence, using (156) and (158), we may replace in the equations **(W1)–(W6)** the function T with a deformed Eisenstein series and I with terms involving only H and E_2. Hence, we are left with a system of partial differential equations between the square of the Jacobi theta function F, deformed Eisenstein series and classical modular forms.

These new equations may now be checked directly by methods of complex analysis, as follows. Divide each equation by H; derive how the quotients

$$\frac{H^{k\bullet}}{H} \quad \text{and} \quad \frac{H^{k\prime}}{H}$$

(with $H^{k\bullet}$ and $H^{k\prime}$ the k^{th} derivatives of H with respect to z and τ, respectively) transform under the variable change

$$(z,\tau) \mapsto (z+\lambda\tau+\mu,\tau) \quad \text{for } \lambda,\mu \in \mathbb{Z};$$

using the periodicity properties of the deformed Eisenstein series proven in [33], show that each equation is doubly periodic in z; calculate all appearing poles using the expansions of the deformed Eisenstein series in [33]; prove all appearing poles cancel; finally prove that the constant term is 0 by evaluating at $z=\frac{1}{2}$. Using this procedure, the proof reduces to a long but standard calculation. □

4.3.5 Proof of Theorem 33 We will identify functions in (z,τ) with their expansions in y,q under the variable change (155). By Proposition 36, the definition of H in Section 4.3.1, and Theorem 39, we have

$$(F^{\mathrm{GW}})^2 = \langle B_2, B_2\rangle^Y = H = F(z,\tau)^2,$$

which implies

$$F^{\mathrm{GW}}(y,q) = \pm F(z,\tau). \tag{159}$$

By definition (55), the $y^{-1/2}q^0$–coefficient of $F^{\mathrm{GW}}(y,q)$ is 1. Hence, there is a positive sign in (159), and we have equality. This proves the first equation of Theorem 33. The case $G^{\mathrm{GW}} = G$ is parallel.

Finally, the two remaining cases follow directly from Proposition 36, the relations (154) and Theorem 39. This completes the proof of Theorem 33.

5 Quantum cohomology

5.1 Overview

Let S be a K3 surface. In Section 5.2 we recall basic facts about the Fock space

$$\mathcal{F}(S) = \bigoplus_{d \geq 0} H^*(\mathsf{Hilb}^d(S); \mathbb{Q}).$$

In Section 5.3 we define a 2–point quantum operator $\mathcal{E}^{\mathsf{Hilb}}$, which encodes the quantum multiplication with a divisor class. In Section 5.4 we introduce natural operators $\mathcal{E}^{(r)}$ acting on $\mathcal{F}(S)$. In Section 5.5, we state a series of conjectures which link $\mathcal{E}^{(r)}$ to the operator $\mathcal{E}^{\mathsf{Hilb}}$. In Section 5.6 we present several example calculations and prove our conjectures in the case of $\mathsf{Hilb}^2(S)$. Here, we also discuss the relationship of the K3 surface case to the case of $\mathcal{A}_1$–resolution studied by Maulik and Oblomkov in [27].

5.2 The Fock space

The Fock space of the K3 surface S,

$$\mathcal{F}(S) = \bigoplus_{d \geq 0} \mathcal{F}_d(S) = \bigoplus_{d \geq 0} H^*(\mathsf{Hilb}^d(S), \mathbb{Q}), \tag{160}$$

is naturally bigraded with the $(d,k)^{\text{th}}$ summand given by

$$\mathcal{F}_d^k(S) = H^{2(k+d)}(\mathsf{Hilb}^d(S), \mathbb{Q}).$$

For a bihomogeneous element $\mu \in \mathcal{F}_d^k(S)$, we let

$$|\mu| = d, \quad k(\mu) = k.$$

The Fock space $\mathcal{F}(S)$ carries a natural scalar product $\langle \cdot | \cdot \rangle$ defined by declaring the direct sum (160) orthogonal and setting

$$\langle \mu | \nu \rangle = \int_{\mathsf{Hilb}^d(S)} \mu \cup \nu$$

for $\mu, \nu \in H^*(\mathsf{Hilb}^d(S), \mathbb{Q})$. If $\alpha, \alpha' \in H^*(S, \mathbb{Q})$ we also write

$$\langle \alpha, \alpha' \rangle = \int_S \alpha \cup \alpha'.$$

If μ, ν are bihomogeneous, then $\langle \mu | \nu \rangle$ is nonvanishing only if

$$|\mu| = |\nu| \quad \text{and} \quad k(\mu) + k(\nu) = 0.$$

For all $\alpha \in H^*(S,\mathbb{Q})$ and $m \neq 0$, the Nakajima operators $\mathfrak{p}_m(\alpha)$ act on $\mathcal{F}(S)$ bihomogeneously of bidegree $(-m, k(\alpha))$:

$$\mathfrak{p}_m(\alpha)\colon \mathcal{F}_d^k \to \mathcal{F}_{d-m}^{k+k(\alpha)}.$$

The commutation relations

$$[\mathfrak{p}_m(\alpha), \mathfrak{p}_n(\beta)] = -m\delta_{m+n,0}\langle \alpha, \beta\rangle \operatorname{id}_{\mathcal{F}(S)} \tag{161}$$

are satisfied for all $\alpha, \beta \in H^*(S)$ and all $m, n \in \mathbb{Z} \setminus \{0\}$.

The inclusion of the diagonal $S \subset S^m$ induces a map

$$\tau_{*m}\colon H^*(S,\mathbb{Q}) \to H^*(S^m,\mathbb{Q}) \cong H^*(S,\mathbb{Q})^{\otimes m}.$$

For $\tau_* = \tau_{*2}$, we have

$$\tau_*(\alpha) = \sum_{i,j} g^{ij}(\alpha \cup \gamma_i) \otimes \gamma_j,$$

where $\{\gamma_i\}_i$ is a basis of $H^*(S)$ and g^{ij} is the inverse of the intersection matrix $g_{ij} = \langle \gamma_i, \gamma_j\rangle$.

For $\gamma \in H^*(S,\mathbb{Q})$ and $n \in \mathbb{Z}$ define the Virasoro operator

$$L_n(\gamma) = -\frac{1}{2}\sum_{k\in\mathbb{Z}} : \mathfrak{p}_k\mathfrak{p}_{n-k} : \tau_*(\gamma),$$

where $: -- :$ is the normal ordered product [22] and we used

$$\mathfrak{p}_k\mathfrak{p}_l \cdot \alpha \otimes \beta = \mathfrak{p}_k(\alpha)\mathfrak{p}_l(\beta).$$

We are particularly interested in the degree-0 Virasoro operator

$$\begin{aligned} L_0(\gamma) &= -\frac{1}{2}\sum_{k\in\mathbb{Z}\setminus 0} : \mathfrak{p}_k\mathfrak{p}_{-k} : \tau_*(\gamma) \\ &= -\sum_{k\geq 1}\sum_{i,j} g^{ij}\mathfrak{p}_{-k}(\gamma_i \cup \gamma)\mathfrak{p}_k(\gamma_j). \end{aligned}$$

The operator $L_0(\gamma)$ is characterized by the commutator relations

$$\left[\mathfrak{p}_k(\alpha), L_0(\gamma)\right] = k\mathfrak{p}_k(\alpha \cup \gamma).$$

Let $e \in H^*(S)$ denote the unit. The restriction of $L_0(\gamma)$ to $\mathcal{F}_d(S)$,

$$L_0(\gamma)|_{\mathcal{F}_d(S)}\colon H^*(\mathsf{Hilb}^d(S),\mathbb{Q}) \to H^*(\mathsf{Hilb}^d(S),\mathbb{Q}),$$

is the cup product by the class

$$D(\gamma) = \frac{1}{(d-1)!}\mathfrak{p}_{-1}(\gamma)\mathfrak{p}_{-1}(e)^{d-1} \in H^*(\mathsf{Hilb}^d(S), \mathbb{Q}) \tag{162}$$

of subschemes incident to γ; see [21]. In the special case $\gamma = e$, the operator $L_0 = L_0(e)$ is the *energy operator*,

$$L_0|_{\mathcal{F}_d(S)} = d \cdot \mathrm{id}_{\mathcal{F}_d(S)}\,. \tag{163}$$

Finally, define Lehn's diagonal operator [21]

$$\partial = -\frac{1}{2}\sum_{i,j\geq 1}(\mathfrak{p}_{-i}\mathfrak{p}_{-j}\mathfrak{p}_{i+j} + \mathfrak{p}_i\mathfrak{p}_j\mathfrak{p}_{-(i+j)})\tau_{3*}([S]).$$

For $d \geq 2$, the operator ∂ acts on $\mathcal{F}_d(S)$ by cup product with $-\frac{1}{2}\Delta_{\mathsf{Hilb}^d(S)}$, where

$$\Delta_{\mathsf{Hilb}^d(S)} = \frac{1}{(d-2)!}\mathfrak{p}_{-2}(e)\mathfrak{p}_{-1}(e)^{d-2}$$

is the class of the diagonal in $\mathsf{Hilb}^d(S)$.

5.3 The WDVV equation

Let S be an elliptic K3 surface with a section. Let B and F be the section and fiber classes, respectively, and let

$$\beta_h = B + hF.$$

For $d \geq 1$ and cohomology classes $\gamma_1, \dots, \gamma_m \in H^*(\mathsf{Hilb}^d(S); \mathbb{Q})$, define the quantum bracket

$$\langle \gamma_1, \dots, \gamma_m \rangle_q^{\mathsf{Hilb}^d(S)} = \sum_{h\geq 0}\sum_{k\in\mathbb{Z}} y^k q^{h-1} \langle \gamma_1, \dots, \gamma_m \rangle_{\beta_h + kA}^{\mathsf{Hilb}^d(S)},$$

where the bracket on the right-hand side was defined in (4).

Define the 2–point quantum operator

$$\mathcal{E}^{\mathsf{Hilb}}\colon \mathcal{F}(S) \otimes \mathbb{Q}((y))((q)) \to \mathcal{F}(S) \otimes \mathbb{Q}((y))((q))$$

by the following two conditions:

- for all homogeneous $a, b \in \mathcal{F}(S)$,

$$\langle a \mid \mathcal{E}^{\mathsf{Hilb}} b\rangle = \begin{cases} \langle a, b\rangle_q & \text{if } |a| = |b|, \\ 0 & \text{otherwise;} \end{cases}$$

- $\mathcal{E}^{\mathsf{Hilb}}$ is linear over $\mathbb{Q}((y))((q))$.

Since $\overline{M}_{0,2}(\mathsf{Hilb}^d(S),\alpha)$ has reduced virtual dimension $2d$, the operator $\mathcal{E}^{\mathsf{Hilb}}$ is self-adjoint of bidegree $(0,0)$.

For $d\geq 0$, consider a divisor class

$$D\in H^2(\mathsf{Hilb}^d(S)),$$

and the operator of primitive quantum multiplication[12] with D,

$$\mathsf{M}_D\colon a\mapsto D*a$$

for all $a\in\mathcal{F}_d(S)\otimes\mathbb{Q}((y))((q))\otimes\mathbb{Q}[\hbar]/\hbar^2$. If

$$D=D(\gamma)\text{ for some }\gamma\in H^2(S)\qquad\text{or}\qquad D=-\tfrac{1}{2}\Delta_{\mathsf{Hilb}^d(S)},$$

by the divisor axiom we have

$$\begin{aligned}\mathsf{M}_{D(\gamma)}|_{\mathcal{F}_d(S)}&=\bigl(L_0(\gamma)+\hbar\mathfrak{p}_0(\gamma)\mathcal{E}^{\mathsf{Hilb}}\bigr)\big|_{\mathcal{F}_d(S)},\\ -\tfrac{1}{2}\mathsf{M}_{\Delta_{\mathsf{Hilb}^d(S)}}|_{\mathcal{F}_d(S)}&=\Bigl(\partial+\hbar y\frac{d}{dy}\mathcal{E}^{\mathsf{Hilb}}\Bigr)\Big|_{\mathcal{F}_d(S)},\end{aligned}$$

where d/dy is formal differentiation with respect to the variable y, and $\mathfrak{p}_0(\gamma)$ for $\gamma\in H^*(S)$ is the degree-0 Nakajima operator defined by the following conditions:[13]

$$\begin{aligned}&[\mathfrak{p}_0(\gamma),\mathfrak{p}_m(\gamma')]=0 &&\text{for all }\gamma'\in H^*(S),\ m\in\mathbb{Z},\\ &\mathfrak{p}_0(\gamma)q^{h-1}y^k1_S=\langle\gamma,\beta_h\rangle q^{h-1}y^k1_S &&\text{for all }h,k.\end{aligned}\tag{164}$$

Since the classes $D(\gamma)$ and $\Delta_{\mathsf{Hilb}^d(S)}$ span $H^2(\mathsf{Hilb}^d(S)$, the operator $\mathcal{E}^{\mathsf{Hilb}}$ therefore determines quantum multiplication M_D for every divisor class D.

Let $D_1,D_2\in H^2(\mathsf{Hilb}^d(S),\mathbb{Q})$ be divisor classes. By associativity and commutativity of quantum multiplication, we have

$$D_1*(D_2*a)=D_2*(D_1*a)\tag{165}$$

for all $a\in\mathcal{F}_d(S)$. After specializing D_1 and D_2, we obtain the main commutator relations for the operator $\mathcal{E}^{\mathsf{Hilb}}$:

For all $\gamma,\gamma'\in H^2(S,\mathbb{Q})$, after restriction to $\mathcal{F}(S)$, we have

$$\begin{aligned}\mathfrak{p}_0(\gamma)[\mathcal{E}^{\mathsf{Hilb}},L_0(\gamma')]&=\mathfrak{p}_0(\gamma')[\mathcal{E}^{\mathsf{Hilb}},L_0(\gamma)],\\ \mathfrak{p}_0(\gamma)[\mathcal{E}^{\mathsf{Hilb}},\partial]&=y\frac{d}{dy}[\mathcal{E}^{\mathsf{Hilb}},L_0(\gamma)].\end{aligned}\tag{166}$$

[12] Quantum multiplication is defined in (11).

[13] The definition precisely matches the action of the extended Heisenberg algebra $\langle\mathfrak{p}_k(\gamma),\ k\in\mathbb{Z}\rangle$ on the full Fock space $\mathcal{F}(S)\otimes\mathbb{Q}[H^*(S,\mathbb{Q})]$ under the embedding $q^{h-1}\mapsto q^{B+hF}$; see [19, Section 6.1].

The equalities (166) hold only after restricting to $\mathcal{F}(S)$. In both cases, the extension of these equations to $\mathcal{F}(S) \otimes \mathbb{Q}((y))((q))$ does *not* hold, since $\mathfrak{p}_0(\gamma)$ is not q–linear, and $y(d/dy)$ is not y–linear.

The equations (166) show that the commutator of $\mathcal{E}^{\mathrm{Hilb}}$ with a divisor intersection operator is essentially independent of the divisor.

5.4 The operators $\mathcal{E}^{(r)}$

For all $(m,\ell) \in \mathbb{Z}^2 \setminus \{0\}$ consider fixed formal power series

$$\varphi_{m,\ell}(y,q) \in \mathbb{C}((y^{1/2}))[\![q]\!] \tag{167}$$

which satisfy the symmetries

$$\begin{aligned} \varphi_{m,\ell} &= -\varphi_{-m,-\ell}, \\ \ell\varphi_{m,\ell} &= m\varphi_{\ell,m}. \end{aligned} \tag{168}$$

Let $\Delta(q) = q\prod_{m\geq 1}(1-q^m)^{24}$ be the modular discriminant and let

$$F(y,q) = (y^{1/2}+y^{-1/2})\prod_{m\geq 1}\frac{(1+yq^m)(1+y^{-1}q^m)}{(1-q^m)^2}$$

be the Jacobi theta function which appeared in Section 0.3, considered as formal power series in q and y in the region $|q|<1$.

Depending on the functions (167), define for all $r \in \mathbb{Z}$ operators

$$\mathcal{E}^{(r)}\colon \mathcal{F}(S)\otimes\mathbb{C}((y^{1/2}))((q)) \to \mathcal{F}(S)\otimes\mathbb{C}((y^{1/2}))((q)) \tag{169}$$

by the following recursion relations:

Relation 1 For all $r \geq 0$,

$$\mathcal{E}^{(r)}|_{\mathcal{F}_0(S)\otimes\mathbb{C}((y^{1/2}))((q))} = \frac{\delta_{0r}}{F(y,q)^2\Delta(q)}\cdot \mathrm{id}_{\mathcal{F}_0(S)\otimes\mathbb{C}((y^{1/2}))((q))}.$$

Relation 2 For all $m \neq 0$, $r \in \mathbb{Z}$ and homogeneous $\gamma \in H^*(S)$,

$$[\mathfrak{p}_m(\gamma),\mathcal{E}^{(r)}] = \sum_{\ell\in\mathbb{Z}}\frac{\ell^{k(\gamma)}}{m^{k(\gamma)}} :\mathfrak{p}_\ell(\gamma)\mathcal{E}^{(r+m-\ell)}: \varphi_{m,\ell}(y,q),$$

where $k(\gamma)$ denotes the shifted complex cohomological degree of γ,

$$\gamma \in H^{2(k(\gamma)+1)}(S;\mathbb{Q}),$$

and $:-\!-:$ is a variant of the normal ordered product defined by

$$:\mathfrak{p}_\ell(\gamma)\mathcal{E}^{(k)}: \;:= \begin{cases} \mathfrak{p}_\ell(\gamma)\mathcal{E}^{(k)} & \text{if } \ell \leq 0, \\ \mathcal{E}^{(k)}\mathfrak{p}_\ell(\gamma) & \text{if } \ell > 0. \end{cases}$$

By definition, the operator $\mathcal{E}^{(r)}$ is homogeneous of bidegree $(-r, 0)$; it is y–linear, but *not* q–linear.

Lemma 40 *The operators $\mathcal{E}^{(r)}$ for $r \in \mathbb{Z}$ are well-defined.*

Proof By induction, Relations 1 and 2 uniquely determine the operators $\mathcal{E}^{(r)}$. It remains to show that the Nakajima commutator relations (161) are preserved by $\mathcal{E}^{(r)}$. Hence, we need to show

$$\big[[\mathfrak{p}_m(\alpha), \mathfrak{p}_n(\beta)], \mathcal{E}^{(r)}\big] = \big[-m\delta_{m+n,0}\langle \alpha, \beta\rangle \operatorname{id}_{\mathcal{F}(S)}, \mathcal{E}^{(r)}\big] = 0$$

for all homogeneous $\alpha, \beta \in H^*(S)$ and all $m, n \in \mathbb{Z} \setminus \{0\}$. We have

$$\big[[\mathfrak{p}_m(\alpha), \mathfrak{p}_n(\beta)], \mathcal{E}^{(r)}\big] = \big[\mathfrak{p}_m(\alpha), [\mathfrak{p}_n(\beta), \mathcal{E}^{(r)}]\big] - \big[\mathfrak{p}_n(\beta), [\mathfrak{p}_m(\alpha), \mathcal{E}^{(r)}]\big]. \tag{170}$$

Using Relation 2, we obtain

$$\begin{aligned} &\big[\mathfrak{p}_m(\alpha), [\mathfrak{p}_n(\beta), \mathcal{E}^{(r)}]\big] \\ &\quad= \Big[\mathfrak{p}_m(\alpha), \sum_{\ell \in \mathbb{Z}} \frac{\ell^{k(\beta)}}{n^{k(\beta)}} :\mathfrak{p}_\ell(\beta)\mathcal{E}^{(r+m-\ell)}: \varphi_{m,\ell}(y,q)\Big] \\ &\quad= \frac{(-m)^{k(\beta)+1}}{n^{k(\beta)}} \langle \alpha, \beta\rangle \mathcal{E}^{(r+n+m)} \varphi_{n,-m} \\ &\qquad+ \sum_{\ell, \ell' \in \mathbb{Z}} \frac{\ell^{k(\beta)}(\ell')^{k(\alpha)}}{n^{k(\beta)} m^{k(\alpha)}} :\mathfrak{p}_\ell(\beta)\big(:\mathfrak{p}_{\ell'}(\alpha)\mathcal{E}^{(r+n+m-\ell-\ell')}:\big): \varphi_{m,\ell'}\varphi_{n,\ell}. \end{aligned} \tag{171}$$

Similarly, we have

$$\begin{aligned} \big[\mathfrak{p}_n(\beta), [\mathfrak{p}_m(\alpha), \mathcal{E}^{(r)}]\big] &= \frac{(-n)^{k(\alpha)+1}}{m^{k(\alpha)}} \langle \alpha, \beta\rangle \mathcal{E}^{(r+n+m)} \varphi_{m,-n} \\ &\quad+ \sum_{\ell, \ell' \in \mathbb{Z}} \frac{\ell^{k(\beta)}(\ell')^{k(\alpha)}}{n^{k(\beta)} m^{k(\alpha)}} :\mathfrak{p}_{\ell'}(\alpha)\big(:\mathfrak{p}_\ell(\beta)\mathcal{E}^{(r+n+m-\ell-\ell')}:\big): \varphi_{m,\ell'}\varphi_{n,\ell}. \end{aligned} \tag{172}$$

Since for all $\ell, \ell' \in \mathbb{Z}$ we have

$$:\mathfrak{p}_\ell(\beta)\big(:\mathfrak{p}_{\ell'}(\alpha)\mathcal{E}^{(r+n+m-\ell-\ell')}:\big): \;=\; :\mathfrak{p}_{\ell'}(\alpha)\big(:\mathfrak{p}_\ell(\beta)\mathcal{E}^{(r+n+m-\ell-\ell')}:\big):$$

the second terms in (171) and (172) agree. Hence, (170) equals

$$\langle\alpha,\beta\rangle\mathcal{E}^{(r+m+n)}\left\{\frac{(-m)^{k(\beta)+1}}{n^{k(\beta)}}\varphi_{n,-m}-\frac{(-n)^{k(\alpha)+1}}{m^{k(\alpha)}}\varphi_{m,-n}\right\}. \tag{173}$$

If $\langle\alpha,\beta\rangle=0$ we are done, hence we may assume otherwise. Then, for degree reasons, $k(\alpha)=-k(\beta)$. Using the symmetries (168), we find

$$\varphi_{m,-n}=-\frac{m}{n}\varphi_{-n,m}=\frac{m}{n}\varphi_{n,-m}.$$

Inserting both equations into (173), this yields

$$\langle\alpha,\beta\rangle\mathcal{E}^{(r+m+n)}\varphi_{n,-m}\left\{-\frac{m^{-k(\alpha)+1}}{n^{-k(\alpha)}}+\frac{n^{k(\alpha)+1}}{m^{k(\alpha)}}\cdot\frac{m}{n}\right\}=0.$$ □

5.5 Conjectures

Let $G(y,q)$ be the formal expansion in the variables y,q of the function $G(z,\tau)$ which appeared in Section 0.4,

$$\begin{aligned}G(y,q)&=F(y,q)^2\Big(y\frac{d}{dy}\Big)^2\log(F(y,q))\\&=F(y,q)^2\cdot\left\{\frac{y}{(1+y)^2}-\sum_{d\geq 1}\sum_{m|d}m\big((-y)^{-m}+(-y)^m\big)q^d\right\}.\end{aligned}$$

Conjecture A *There exist unique series $\varphi_{m,\ell}$ for $(m,\ell)\in\mathbb{Z}^2\setminus\{0\}$ such that the following hold:*

(i) *The symmetries (168) are satisfied.*

(ii) *The initial conditions*

$$\varphi_{1,1}=G(y,q)-1,\quad \varphi_{1,0}=-i\cdot F(y,q),\quad \varphi_{1,-1}=-\frac{1}{2}q\frac{d}{dq}(F(y,q)^2)$$

hold, where $i=\sqrt{-1}$ is the imaginary unit.

(iii) *If $\mathcal{E}^{(r)}$ for $r\in\mathbb{Z}$ are the operators (169) defined by the functions $\varphi_{m,\ell}$, then $\mathcal{E}^{(0)}$ satisfies, after restriction to $\mathcal{F}(S)$, the WDVV equations*

$$\begin{aligned}\mathfrak{p}_0(\gamma)[\mathcal{E}^{(0)},L_0(\gamma')]&=\mathfrak{p}_0(\gamma')[\mathcal{E}^{(0)},L_0(\gamma)],\\ \mathfrak{p}_0(\gamma)[\mathcal{E}^{(0)},\partial]&=y\frac{d}{dy}[\mathcal{E}^{(0)},L_0(\gamma)]\end{aligned} \tag{174}$$

for all $\gamma,\gamma'\in H^2(S,\mathbb{Q})$.

Conjecture A is a purely algebraic nondegeneracy statement for the WDVV equations (174). It has been checked numerically on $\mathcal{F}_d(S)$ for all $d \leq 5$. The first values of the series $\varphi_{m,\ell}$ are given in Section B.0.3. For the remainder of Section 5, we *assume* Conjecture A to be true, and we let $\mathcal{E}^{(r)}$ denote the operators defined by the (hence unique) functions $\varphi_{m,\ell}$ satisfying (i)–(iii) above. Since Conjecture A has been shown to be true for $\mathcal{F}_d(S)$ for all $d \leq 5$, the restriction of $\mathcal{E}^{(0)}$ to the subspace $\bigoplus_{d\leq 5} \mathcal{F}_d(S)$ is well-defined unconditionally.

The following conjecture relates $\mathcal{E}^{(0)}$ to the quantum operator $\mathcal{E}^{\mathsf{Hilb}}$. Let L_0 be the energy operator (163). Define the operator

$$G(y,q)^{L_0}\colon \mathcal{F}(S) \otimes \mathbb{Q}((y))((q)) \to \mathcal{F}(S) \otimes \mathbb{Q}((y))((q))$$

by the assignment

$$G(y,q)^{L_0}(\mu) = G(y,q)^{|\mu|} \cdot \mu$$

for any homogeneous $\mu \in \mathcal{F}(S)$.

Conjecture B *After restriction to* $\mathcal{F}(S)$,

$$\mathcal{E}^{\mathsf{Hilb}} = \mathcal{E}^{(0)} - \frac{1}{F(y,q)^2 \Delta(q)} G(y,q)^{L_0}. \tag{175}$$

Combining Conjectures A and B, we obtain an algorithmic procedure to determine the 2–point quantum bracket $\langle \cdot , \cdot \rangle_q$. The equality of Conjecture B is conjectured to hold only after restriction to $\mathcal{F}(S)$. The extension of (175) to $\mathcal{F}(S) \otimes \mathbb{Q}((y))((q))$ is clearly false: the operators $\mathcal{E}^{\mathsf{Hilb}}$ and $G^{L_0}/(F^2\Delta)$ are q–linear by definition, but $\mathcal{E}^{(0)}$ is not.

Let QJac be the ring of holomorphic quasi-Jacobi forms defined in Appendix B, and let

$$\mathrm{QJac} = \bigoplus_{m \geq 0} \bigoplus_{k \geq -2m} \mathrm{QJac}_{k,m}$$

be the natural bigrading of QJac by index m and weight k, where m runs over all nonnegative half-integers $\frac{1}{2}\mathbb{Z}^{\geq 0}$.

Conjecture C *For every* $(m,\ell) \in \mathbb{Z}^2 \setminus \{0\}$, *the series*

$$\varphi_{m,\ell} + \mathrm{sgn}(m)\delta_{m\ell}$$

is a quasi-Jacobi form of index $\frac{1}{2}(|m| + |\ell|)$ *and weight* $-\delta_{0\ell}$.

Define a new degree function $\underline{\deg}$ on $H^*(S)$ by the assignment

$$\underline{\deg}(\gamma) = \begin{cases} -1 & \text{if } \gamma \in \mathbb{Q}F, \\ 1 & \text{if } \gamma \in \mathbb{Q}(B+F), \\ 0 & \text{if } \gamma \in \{F, B+F\}^{\perp}, \end{cases}$$

where the orthogonal complement $\{F, B+F\}^{\perp}$ is defined with respect to the inner product $\langle \cdot, \cdot \rangle$.

Lemma* 41 *Assume that Conjectures A and C hold. Let $\gamma_i, \tilde{\gamma}_i \in H^*(S)$ be $\underline{\deg}$–homogeneous classes, and let*

(176) $$\mu = \prod_i \mathfrak{p}_{-m_i}(\gamma_i) 1_S, \quad \nu = \prod_j \mathfrak{p}_{-n_j}(\tilde{\gamma}_j) 1_S$$

be cohomology classes of $\mathsf{Hilb}^m(S)$ and $\mathsf{Hilb}^n(S)$, respectively. Then

$$\langle \mu \mid \mathcal{E}^{(n-m)} \nu \rangle = \frac{\Phi}{F(y,q)^2 \Delta(q)}$$

for a quasi-Jacobi form $\Phi \in \mathrm{QJac}$ of index $\frac{1}{2}(|m| + |n|)$ and weight

$$\sum_i \underline{\deg}(\gamma_i) + \sum_j \underline{\deg}(\gamma_j').$$

Proof of Lemma* 41 By a straightforward induction on $|\mu| + |\nu|$. □

Let $\mu, \nu \in H^*(\mathsf{Hilb}^d(S))$. By Lemma* 41 and Conjecture B, we have

(177) $$\langle \mu, \nu \rangle_q = \frac{\varphi(y,q)}{F(y,q)^2 \Delta(q)}$$

for a quasi-Jacobi form $\varphi(y,q)$. Since $F(y,q)$ has a simple zero at $z=0$, we expect the function (177) to have a pole of order 2 at $z=0$. Numerical experiments (Conjecture J) or deformation invariance[14] suggest that the series $\langle \mu, \nu \rangle_q$ is nonetheless holomorphic at $z=0$. Combining everything, we obtain the following prediction.

Lemma* 42 *Assume Conjectures A, B, C, J hold. Let $\mu, \nu \in H^*(\mathsf{Hilb}^d(S))$ be cohomology classes of the form (176). Then*

$$\langle \mu, \nu \rangle_q^{\mathsf{Hilb}^d(S)} = \frac{\Phi(y,q)}{\Delta(q)}$$

[14] See [34] for a discussion of the monodromy action by deformations of $\mathsf{Hilb}^d(S)$ in the moduli space of irreducible holomorphic-symplectic varieties.

for a quasi-Jacobi form $\Phi(y,q)$ *of index* $d-1$ *and weight*

$$2+\sum_i \underline{\deg}(\gamma_i)+\sum_j \underline{\deg}(\gamma_j').$$

5.6 Examples

5.6.1 The higher-dimensional Yau–Zaslow formula **(i)** Let F be the fiber of the elliptic fibration $\pi\colon S\to\mathbb{P}^1$. Then the calculation

$$\begin{aligned}
\Big\langle \mathfrak{p}_{-1}(F)^d 1_S \,\Big|\, \Big(\mathcal{E}^{(0)}-\frac{1}{F^2\Delta}G^{L_0}\Big)\mathfrak{p}_{-1}(F)^d 1_S\Big\rangle
&= \big\langle \mathfrak{p}_{-1}(F)^d 1_S \mid \mathcal{E}^{(0)}\mathfrak{p}_{-1}(F)^d 1_S\big\rangle \\
&= (-1)^d\big\langle 1_S \mid \mathfrak{p}_1(F)^d \mathcal{E}^{(0)}\mathfrak{p}_{-1}(F)^d 1_S\big\rangle \\
&= (-1)^d\big\langle 1_S \mid \mathfrak{p}_0(F)^d \mathcal{E}^{(d)}\varphi_{1,0}^d\mathfrak{p}_{-1}(F)^d 1_S\big\rangle \\
&= (-1)^d\big\langle 1_S \mid \mathfrak{p}_0(F)^{2d} \mathcal{E}^{(0)}(-1)^d\varphi_{1,0}^d\varphi_{-1,0}^d 1_S\big\rangle \\
&= \frac{\varphi_{1,0}^d\varphi_{-1,0}^d}{F(y,q)^2\Delta(q)} \\
&= \frac{F(y,q)^{2d-2}}{\Delta(q)}
\end{aligned}$$

shows Conjecture B to be in agreement with Theorem 1; here we have used $\mathfrak{p}_0(F)=1$.

(ii) Let B be the class of the section of $\pi\colon S\to\mathbb{P}^1$ and consider the class

$$W=B+F.$$

We have $\langle W,W\rangle=0$ and $\langle W,\beta_h\rangle=h-1$. Hence, $\mathfrak{p}_0(W)$ acts as $q(d/dq)$ on functions in q. We have

$$\begin{aligned}
\Big\langle \mathfrak{p}_{-1}(W)^d 1_S \,\Big|\, \Big(\mathcal{E}^{(0)}-\frac{1}{F^2\Delta}G^{L_0}\Big)\mathfrak{p}_{-1}(W)^d 1_S\Big\rangle
&= \big\langle \mathfrak{p}_{-1}(W)^d 1_S \mid \mathcal{E}^{(0)}\mathfrak{p}_{-1}(W)^d 1_S\big\rangle \\
&= (-1)^d\big\langle 1_S \mid \mathfrak{p}_0(W)^d \mathcal{E}^{(d)}\varphi_{1,0}^d\mathfrak{p}_{-1}(W)^d 1_S\big\rangle \\
&= \big\langle 1_S \mid \mathfrak{p}_0(W)^{2d} \mathcal{E}^{(0)}\varphi_{1,0}^d\varphi_{-1,0}^d 1_S\big\rangle \\
&= \Big(q\frac{d}{dq}\Big)^{2d}\Big(\frac{\varphi_{1,0}^d\varphi_{-1,0}^d}{F(y,q)^2\Delta(q)}\Big) \\
&= \Big(q\frac{d}{dq}\Big)^{2d}\Big(\frac{F(y,q)^{2d-2}}{\Delta(q)}\Big).
\end{aligned}$$

5.6.2 Further Gromov–Witten invariants **(i)** Let $\omega \in H^4(S;\mathbb{Z})$ be the class of a point. For $d \geq 1$, let

$$C(F) = \mathfrak{p}_{-1}(F)\mathfrak{p}_{-1}(\omega)^{d-1}1_S \in H_2(\mathsf{Hilb}^2(S),\mathbb{Z})$$

and

$$D(F) = \mathfrak{p}_{-1}(F)\mathfrak{p}_{-1}(e)^{d-1}1_S \in H^2(\mathsf{Hilb}^2(S),\mathbb{Z}).$$

Then

$$\begin{aligned}\Big\langle C(F) \,\Big|\, \Big(\mathcal{E}^{(0)} - \frac{1}{F^2\Delta}G^{L_0}\Big)D(F)\Big\rangle \\ &= \frac{1}{(d-1)!}\big\langle \mathfrak{p}_{-1}(F)\mathfrak{p}_{-1}(\omega)^{d-1}1_S \mid \mathcal{E}^{(0)}\mathfrak{p}_{-1}(F)\mathfrak{p}_{-1}(e)^{d-1}1_S\big\rangle \\ &= \frac{1}{(d-1)!}\big\langle \mathfrak{p}_{-1}(\omega)^{d-1}1_S \mid \mathcal{E}^{(0)}\varphi_{1,0}\varphi_{-1,0}\mathfrak{p}_{-1}(e)^{d-1}1_S\big\rangle \\ &= \frac{(-1)^{d-1}}{(d-1)!}\big\langle 1_S \mid \mathcal{E}^{(0)}\varphi_{1,0}\varphi_{-1,0}(\varphi_{1,1}+1)^{d-1}\mathfrak{p}_1(\omega)^{d-1}\mathfrak{p}_{-1}(e)^{d-1}1_S\big\rangle \\ &= \frac{\varphi_{1,0}\varphi_{-1,0}(\varphi_{1,1}+1)^{d-1}}{F(y,q)^2\Delta(q)} \\ &= \frac{G(y,q)^{d-1}}{\Delta(q)}.\end{aligned}$$

By the divisor equation and the fact that $\langle D(F), \beta_h + kA\rangle = 1$ for all h, k, Conjecture B is in full agreement with the first equation of Theorem 26.

(ii) Let $A = \mathfrak{p}_{-2}(\omega)\mathfrak{p}_{-1}(\omega)^{d-2}1_S$ be the class of an exceptional curve. Then

$$\begin{aligned}\Big\langle A \,\Big|\, \Big(\mathcal{E}^{(0)} - \frac{1}{F^2\Delta}G^{L_0}\Big)D(F)\Big\rangle \\ &= \frac{(-1)^d}{(d-1)!}\big\langle 1_S \mid \mathfrak{p}_2(\omega)\mathcal{E}^{(0)}\mathfrak{p}_1(\omega)^{d-2}\mathfrak{p}_{-1}(F)\mathfrak{p}_{-1}(e)^{d-1}(\varphi_{1,1}+1)^{d-2}1_S\big\rangle \\ &= \frac{(-1)^d}{(d-1)!}\big\langle 1_S \mid \tfrac{1}{2}\mathcal{E}^{(1)}\mathfrak{p}_1(\omega)^{d-1}\mathfrak{p}_{-1}(F)\mathfrak{p}_{-1}(e)^{d-1}\varphi_{2,1}(\varphi_{1,1}+1)^{d-2}1_S\big\rangle \\ &= -\tfrac{1}{2}\big\langle 1_S \mid \mathcal{E}^{(1)}\mathfrak{p}_{-1}(F)\varphi_{2,1}(\varphi_{1,1}+1)^{d-2}\big\rangle \\ &= -\frac{1}{2}\frac{(-\varphi_{-1,0})\varphi_{2,1}(\varphi_{1,1}+1)^{d-2}}{F^2(y,q)\Delta} \\ &= -\frac{1}{2}\frac{\big(y\frac{d}{dy}G\big)\cdot G^{d-2}}{\Delta}.\end{aligned}$$

Hence, again, Conjecture B is in full agreement with the second equation of Theorem 26.

(iii) For a point $P \in S$, the incidence subscheme

$$I(P) = \{\xi \in \mathsf{Hilb}^2(S) \mid P \in \xi\}$$

has class $[I(P)] = \mathfrak{p}_{-1}(\omega)\mathfrak{p}_{-1}(e)1_S$. We calculate

$$\begin{aligned}
&\Big\langle I(P) \,\Big|\, \Big(\mathcal{E}^{(0)} - \frac{1}{F^2\Delta} G^{L_0}\Big) I(P) \Big\rangle \\
&\qquad = -\big\langle \mathfrak{p}_{-1}(e)1_S \mid \mathfrak{p}_1(\omega)\mathcal{E}^{(0)} I(P)\big\rangle - \frac{G^2}{F^2\Delta} \\
&\qquad = -\big\langle \mathfrak{p}_{-1}(e)1_S \mid \big(\mathcal{E}^{(0)}\mathfrak{p}_1(\omega)(\varphi_{1,1}+1) - \mathfrak{p}_{-1}(\omega)\mathcal{E}^{(2)}\varphi_{1,-1}\big) I(P)\big\rangle - \frac{G^2}{F^2\Delta} \\
&\qquad = \big\langle \mathfrak{p}_{-1}(e)1_S \mid \mathcal{E}^{(0)}\mathfrak{p}_{-1}(\omega)(\varphi_{1,1}+1)1_S\big\rangle \\
&\qquad\qquad + \big\langle 1_S \mid \mathcal{E}^{(2)}\mathfrak{p}_{-1}(\omega)\mathfrak{p}_{-1}(e)\varphi_{1,-1}1_S\big\rangle - \frac{G^2}{F^2\Delta} \\
&\qquad = \frac{(\varphi_{1,1}+1)^2}{F^2\Delta} + \frac{-\varphi_{-1,1}\varphi_{1,-1}}{F^2\Delta} - \frac{G^2}{F^2\Delta} \\
&\qquad = \frac{\big(q\frac{d}{dq}F\big)^2}{\Delta(q)}.
\end{aligned}$$

Hence, Conjecture B agrees with the third equation of Theorem 26, case $d = 2$.

(iv) For a point $P \in S$, we have

$$\begin{aligned}
\Big\langle \mathfrak{p}_{-1}(F)^2 1_S \,\Big|\, \Big(\mathcal{E}^{(0)} - \frac{1}{F^2\Delta}G^{L_0}\Big) I(P)\Big\rangle &= -\big\langle 1_S \mid \mathcal{E}^{(2)}\varphi_{1,0}^2 I(P)\big\rangle \\
&= \frac{-\varphi_{1,0}^2\varphi_{-1,1}}{F^2\Delta} \\
&= \frac{F(y,q)\cdot q\frac{d}{dq}F(y,q)}{\Delta(\tau)}.
\end{aligned}$$

Hence, Conjecture B is in agreement with Theorem 27.

5.6.3 The Hilbert scheme of two points We check Conjectures A, B, C, J for $\mathsf{Hilb}^2(S)$. Conjecture A is seen to hold for $\mathsf{Hilb}^2(S)$ by direct calculation. The corresponding functions $\varphi_{m,\ell}$ are given in Section B.0.3. This implies Conjecture C by inspection. Conjectures B and J hold by the following result.

Theorem 43 *For all* $\mu, \nu \in H^*(\mathsf{Hilb}^2(S))$,

$$\langle \mu, \nu\rangle_q = \Big\langle \mu \,\Big|\, \Big(\mathcal{E}^{(0)} - \frac{G^{L_0}}{F^2\Delta}\Big)\nu\Big\rangle.$$

Theorem 44 *Let $\mu, \nu \in H^*(\mathsf{Hilb}^2(S))$ be cohomology classes of the form (176). Then*

$$\langle \mu, \nu \rangle_q = \frac{\Phi}{\Delta(q)}$$

for a quasi-Jacobi form Φ of index 1 and weight

$$2 + \sum_i \underline{\deg}(\gamma_i) + \sum_j \underline{\deg}(\gamma'_j).$$

By Sections 5.6.1 and 5.6.2 above, Theorem 43 holds in the cases considered in Theorems 10, 26 and 27. Applying the WDVV equation (164) successively to these base cases, one evaluates the bracket $\langle \mu, \nu \rangle_q$ for all $\mu, \nu \in H^*(\mathsf{Hilb}^2(S))$ in finitely many steps. This implies Theorem 43 since for $\mathsf{Hilb}^2(S)$ the WDVV equation also holds for $\mathcal{E}^{(0)} - G^{L_0}/(F^2\Delta)$. Theorem 44 now follows from direct inspection.

Proof of Proposition 4 By degenerating $(E, 0)$ to the nodal elliptic curve and using the divisor equation, we may rewrite $\mathsf{H}_d(y, q)$ as

$$\mathsf{H}_d(y, q) = \sum_{k \in \mathbb{Z}} \sum_{h \geq 0} y^k q^{h-1} \int_{[\overline{M}_{0,2}(\mathsf{Hilb}^d(S), \beta_h + kA)]^{\mathrm{red}}} (\mathrm{ev}_1 \times \mathrm{ev}_2)^* [\Delta^{[d]}], \tag{178}$$

where $[\Delta^{[d]}] \in H^{2d}(\mathsf{Hilb}^d(S) \times \mathsf{Hilb}^d(S))$ is the diagonal class. The proposition now follows from calculating the right-hand side of (178) using Theorem 43. □

5.6.4 The $\mathcal{A}_1$ resolution Let $[q^{-1}]$ be the operator that extracts the q^{-1} coefficient, and let

$$\mathcal{E}_B^{\mathsf{Hilb}} = [q^{-1}]\mathcal{E}^{\mathsf{Hilb}}$$

be the restriction of $\mathcal{E}^{\mathsf{Hilb}}$ to the case of the section class B. The corresponding local case was considered before in [26; 27].

Define operators $\mathcal{E}_B^{(r)}$ by the relations

$$\langle 1_S \mid \mathcal{E}_B^{(r)} 1_S \rangle = \frac{y}{(1+y)^2} \delta_{0r},$$
$$[\mathfrak{p}_m(\gamma), \mathcal{E}_B^{(r)}] = \langle \gamma, B \rangle ((-y)^{-m/2} - (-y)^{m/2}) \mathcal{E}_B^{(r+m)}$$

for all $m \neq 0$ and all $\gamma \in H^*(S)$; see [27, Section 5.1]. Translating the results of [26; 27] to the K3 surface leads to the following evaluation.

Theorem 45 (Maulik, Oblomkov) *After restriction to* $\mathcal{F}(S)$,

$$\mathcal{E}_B^{\mathsf{Hilb}} + \frac{y}{(1+y)^2}\,\mathrm{id}_{\mathcal{F}(S)} = \mathcal{E}_B^{(0)}.$$

By the numerical values of Section B.0.3, we expect the expansions

$$\begin{aligned} \varphi_{m,0} &= \big((-y)^{-m/2} - (-y)^{m/2}\big) + O(q) && \text{for all } m \neq 0, \\ \varphi_{m,\ell} &= O(q) && \text{for all } m \in \mathbb{Z}, \ell \neq 0. \end{aligned}$$

Because of the equality

$$[q^{-1}]\frac{G^{L_0}}{F^2\Delta} = \frac{y}{(1+y)^2}\mathrm{id}_{\mathcal{F}(S)},$$

we find Conjectures A and B in agreement with Theorem 45.

Appendix A The reduced WDVV equation

Let $\overline{M}_{0,4}$ be the moduli space of stable genus-0 curves with four marked points. The boundary of $\overline{M}_{0,4}$ is the union of the divisors

$$D(12|34),\ D(14|23),\ D(13|24) \tag{179}$$

corresponding to a broken curve with the respective prescribed splitting of the marked points. Since $\overline{M}_{0,4}$ is isomorphic to $\mathbb{P}^1$, any two of the divisors (179) are rationally equivalent.

Let Y be a smooth projective variety and let $\overline{M}_{0,n}(Y,\beta)$ be the moduli space of stable maps to Y of genus 0 and class β. Let

$$\pi\colon \overline{M}_{0,n}(Y,\beta) \to \overline{M}_{0,4}$$

be the map that forgets all but the last four points. The pullback of the boundary divisors (179) under π defines rationally equivalent divisors on $\overline{M}_{0,n}(Y,\beta)$. The intersection of these divisors with curve classes obtained from the virtual class yields relations among Gromov–Witten invariants of Y, namely the WDVV equations [12]. We derive the precise form of these equations for reduced Gromov–Witten theory. For simplicity, we restrict to the case $n=4$.

Let Y be a holomorphic symplectic variety and let

$$\langle \gamma_1, \dots, \gamma_n \rangle_\beta^{\mathrm{red}} = \int_{[\overline{M}_{0,n}(Y,\beta)]^{\mathrm{red}}} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n)$$

denote the *reduced* Gromov–Witten invariants of Y of genus 0 and class $\beta \in H_2(Y;\mathbb{Z})$ with primary insertions $\gamma_1, \dots, \gamma_m \in H^*(Y)$.

Proposition 46 *Let $\gamma_1, \dots, \gamma_4 \in H^{2*}(Y;\mathbb{Q})$ be cohomology classes with*

$$\sum_i \deg(\gamma_i) = \operatorname{vdim} \overline{M}_{0,4}(Y,\beta) - 1 = \dim Y + 1,$$

where $\deg(\gamma_i)$ denotes the complex degree of γ_i. Then

$$\langle \gamma_1, \gamma_2, \gamma_3 \cup \gamma_4 \rangle_\beta^{\mathrm{red}} + \langle \gamma_1 \cup \gamma_2, \gamma_3, \gamma_4 \rangle_\beta^{\mathrm{red}} = \langle \gamma_1, \gamma_4, \gamma_2 \cup \gamma_3 \rangle_\beta^{\mathrm{red}} + \langle \gamma_1 \cup \gamma_4, \gamma_2, \gamma_3 \rangle_\beta^{\mathrm{red}}.$$

Proof Consider the fiber of π over $D(12|34)$,

$$D = \pi^{-1}(D(12|34)).$$

The intersection of D with the class

$$\left(\prod_{i=1}^4 \mathrm{ev}_i^*(\gamma_i)\right) \cap [\overline{M}_{0,4}(Y,\beta)]^{\mathrm{red}} \tag{180}$$

splits into a sum of integrals over the product

$$M' = \overline{M}_{0,3}(Y,\beta_1) \times \overline{M}_{0,3}(Y,\beta_2),$$

for all effective decompositions $\beta = \beta_1 + \beta_2$.

The reduced virtual class $[\overline{M}_{0,4}(Y,\beta)]^{\mathrm{red}}$ restricts to M' as the sum of

$$(\mathrm{ev}_3 \times \mathrm{ev}_3)^* \Delta_Y \cap [\overline{M}_{0,3}(Y,\beta_1)]^{\mathrm{red}} \times [\overline{M}_{0,3}(Y,\beta_2)]^{\mathrm{ord}}$$

with the same term, except for "red" and "red" interchanged; here

$$\Delta_Y \in H^{2\dim Y}(Y \times Y;\mathbb{Z})$$

is the class of the diagonal and $[\,\cdot\,]^{\mathrm{ord}}$ denotes the ordinary virtual class.

Since $[\overline{M}_{0,3}(Y,\beta)]^{\mathrm{ord}} = 0$ unless $\beta = 0$, we find

$$\begin{aligned}\int_{[\overline{M}_{0,4}(Y,\beta)]^{\mathrm{red}}} D \cup \prod_i \gamma_i &\\ = \sum_{e,f} \langle \gamma_1, \gamma_2, T_e \rangle_\beta^{\mathrm{red}} g^{ef} \langle \gamma_3, \gamma_4, T_f \rangle_0^{\mathrm{ord}} &+ \langle \gamma_1, \gamma_2, T_e \rangle_0^{\mathrm{ord}} g^{ef} \langle \gamma_3, \gamma_4, T_f \rangle_\beta^{\mathrm{red}} \\ = \langle \gamma_1, \gamma_2, \gamma_3 \cup \gamma_4 \rangle_\beta^{\mathrm{red}} &+ \langle \gamma_1 \cup \gamma_2, \gamma_3, \gamma_4 \rangle_\beta^{\mathrm{red}},\end{aligned} \tag{181}$$

where $\{T_e\}_e$ is a basis of $H^*(Y;\mathbb{Z})$ and $(g^{ef})_{e,f}$ is the inverse of the intersection matrix $g_{ef} = \int_Y T_e \cup T_f$.

After comparing (181) with the integral of (180) over the pullback of $D(14|23)$, the proof of Proposition 46 is complete. □

We may use the previous proposition to define reduced quantum cohomology. Let $\hbar$ be a formal parameter with $\hbar^2 = 0$. Let Eff_Y be the cone of effective curve class on Y, and for any $\beta \in \mathrm{Eff}_Y$ let q^β be the corresponding element in the semigroup algebra $\mathbb{Q}[\mathrm{Eff}_Y]$. Define the *reduced* quantum product $*$ on

$$H^*(Y;\mathbb{Q}) \otimes \mathbb{Q}[\![\mathrm{Eff}_Y]\!] \otimes \mathbb{Q}[\hbar]/\hbar^2$$

by

$$\langle \gamma_1 * \gamma_2, \gamma_3 \rangle = \langle \gamma_1 \cup \gamma_2, \gamma_3 \rangle + \hbar \sum_{\beta > 0} q^\beta \langle \gamma_1, \gamma_2, \gamma_3 \rangle_\beta^{\mathrm{red}}$$

for all $a, b, c \in H^*(Y)$, where $\langle \gamma_1, \gamma_2 \rangle = \int_Y \gamma_1 \cup \gamma_2$ is the standard inner product on $H^*(Y;\mathbb{Q})$ and β runs over all nonzero effective curve classes of Y. Then Proposition 46 implies that $*$ is associative.

Appendix B Quasi-Jacobi forms

B.0.1 Overview We give a short concrete definition of quasi-Jacobi forms, and list the first values of $\varphi_{k,m}$. Quasi-Jacobi forms arise more generally as constant terms of almost-holomorphic Jacobi forms; see [25] for an introduction.[15] Our definition here is closely related to Libgober's definition. If f is a quasi-Jacobi form (in our sense) of index m, then f/F^{2m} is a quasi-Jacobi form in the sense of [25].

B.0.2 Definition Let $(z,\tau) \in \mathbb{C} \times \mathbb{H}$, and let $y = -p = -e^{2\pi i z}$ and $q = e^{2\pi i \tau}$. For all expansions below, we will work in the region $|y| < 1$.

Consider the Jacobi theta functions

$$F(z,\tau) = \frac{\vartheta_1(z,\tau)}{\eta^3(\tau)} = (y^{1/2} + y^{-1/2}) \prod_{m \geq 1} \frac{(1 + yq^m)(1 + y^{-1}q^m)}{(1 - q^m)^2},$$

the logarithmic derivative

$$J_1(z,\tau) = y \frac{d}{dy} \log F(y,q) = \frac{y}{1+y} - \frac{1}{2} - \sum_{d \geq 1} \sum_{m|d} ((-y)^m - (-y)^{-m}) q^d,$$

[15] The authors would like to thank A Libgober for pointing out [25].

	$F(z,\tau)$	$E_{2k}(\tau)$	$J_1(z,\tau)$	$\wp(z,\tau)$	$\wp^\bullet(z,\tau)$
pole order at $z=0$	0	0	1	2	3
weight	-1	$2k$	1	2	3
index	$1/2$	0	0	0	0

Table 2: Weight and pole order at $z=0$

the Weierstrass elliptic function

$$\wp(z,\tau) = \frac{1}{12} - \frac{y}{(1+y)^2} + \sum_{d\geq 1}\sum_{m|d} m((-y)^m - 2 + (-y)^{-m})q^d, \tag{182}$$

the derivative

$$\wp^\bullet(z,\tau) = y\frac{d}{dy}\wp(z,\tau) = \frac{y(y-1)}{(1+y)^3} + \sum_{d\geq 1}\sum_{m|d} m^2((-y)^m - (-y)^{-m})q^d,$$

and for $k\geq 1$ the Eisenstein series

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}}\sum_{d\geq 1}\Bigl(\sum_{m|d} m^{2k-1}\Bigr)q^d, \tag{183}$$

where B_{2k} are the Bernoulli numbers. Define the free polynomial algebra

$$\mathsf{V} = \mathbb{C}\bigl[F(z,\tau), E_2(\tau), E_4(\tau), J_1(z,\tau), \wp(z,\tau), \wp^\bullet(z,\tau)\bigr].$$

Define the weight and index of the generators of V by Table 2. Here, we list also their pole order at $z=0$ for later use. The grading on the generators induces a natural bigrading on V by weight k and index m,

$$\mathsf{V} = \bigoplus_{m\in(\frac{1}{2}\mathbb{Z})_{\geq 0}}\bigoplus_{k\in\mathbb{Z}} \mathsf{V}_{k,m},$$

where m runs over all nonnegative half-integers.

In the variable z, the functions

$$E_{2k}(\tau),\ J_1(z,\tau),\ \wp(z,\tau),\ \wp^\bullet(z,\tau) \tag{184}$$

can have a pole in the fundamental region

$$\{x+y\tau \mid 0\leq x,y<1\} \tag{185}$$

only at $z=0$. The function $F(z,\tau)$ has a simple zero at $z=0$ and no other zeros (or poles) in the fundamental region (185).

Definition 47 Let m be a nonnegative half-integer and let $k \in \mathbb{Z}$. A function

$$f(z,\tau) \in \mathsf{V}_{k,m}$$

which is holomorphic at $z=0$ for generic τ is called a *quasi-Jacobi form* of weight k and index m.

The subring $\mathrm{QJac} \subset \mathsf{V}$ of quasi-Jacobi forms is graded by index m and weight k,

$$\mathrm{QJac} = \bigoplus_{m \geq 0} \bigoplus_{k \geq -2m} \mathrm{QJac}_{k,m},$$

with finite-dimensional summands $\mathrm{QJac}_{k,m}$.

By the classical relation

$$(\wp^\bullet(z))^2 = 4\wp(z)^3 - \tfrac{1}{12}E_4(\tau)\wp(z) + \tfrac{1}{216}E_6(\tau),$$

we have $E_6(\tau) \in \mathsf{V}$ and therefore $E_6(\tau) \in \mathrm{QJac}$. Hence, QJac contains the ring of quasimodular forms $\mathbb{C}[E_2, E_4, E_6]$. Since the functions

$$\varphi_{-2,1} = -F(z,\tau)^2, \quad \varphi_{0,1} = -12F(z,\tau)^2\wp(z,\tau)$$

lie both in QJac, it follows from [8, Theorem 9.3] that QJac also contains the ring of weak Jacobi forms.

Lemma 48 *The ring* QJac *is closed under differentiation by* z *and* τ*.*

Proof We write

$$\partial_\tau = \frac{1}{2\pi i}\frac{\partial}{\partial \tau} = q\frac{d}{dq} \quad \text{and} \quad \partial_z = \frac{1}{2\pi i}\frac{\partial}{\partial z} = y\frac{d}{dy}$$

for differentiation with respect to τ and z, respectively. The lemma now directly follows from the relations

$$\begin{aligned}
\partial_\tau(F) &= F \cdot \left(\tfrac{1}{2}J_1^2 - \tfrac{1}{2}\wp - \tfrac{1}{12}E_2\right), & \partial_z(F) &= J_1 \cdot F, \\
\partial_\tau(J_1) &= J_1 \cdot \left(\tfrac{1}{12}E_2 - \wp\right) - \tfrac{1}{2}\wp^\bullet, & \partial_z(J_1) &= -\wp + \tfrac{1}{12}E_2, \\
\partial_\tau(\wp) &= 2\wp^2 + \tfrac{1}{6}\wp E_2 + J_1\wp^\bullet - \tfrac{1}{36}E_4, & \partial_z(\wp) &= \wp^\bullet, \\
\partial_\tau(\wp^\bullet) &= 6J_1\wp^2 - \tfrac{1}{24}J_1E_4 + 3\wp\wp^\bullet + \tfrac{1}{4}E_2\wp^\bullet, & \partial_z(\wp^\bullet) &= 6\wp^2 - \tfrac{1}{24}E_4.
\end{aligned}$$

□

B.0.3 Numerical values We present the first values of the functions $\varphi_{m,\ell}$ satisfying the conditions of Conjecture A of Section 5.5. Let $K = iF$, where $i = \sqrt{-1}$. Then

$$\begin{aligned}
\varphi_{1,-1} &= K^2\big(\tfrac12 J_1^2 - \tfrac12 \wp - \tfrac1{12}E_2\big),\\
\varphi_{1,0} &= -K,\\
\varphi_{1,1} &= K^2\big(\wp - \tfrac1{12}E_2\big),
\end{aligned}$$

$$\begin{aligned}
\varphi_{2,-2} &= 2K^4\big(J_1^4 - 2J_1^2\wp - \tfrac1{12}J_1^2E_2 - \tfrac12 J_1\wp^\bullet\big),\\
\varphi_{2,-1} &= 2K^3\big(\tfrac23 J_1^3 - J_1\wp - \tfrac1{12}J_1E_2 - \tfrac16\wp^\bullet\big),\\
\varphi_{2,0} &= -2\cdot J_1\cdot K^2,\\
\varphi_{2,1} &= 2K^3\cdot\big(J_1\wp - \tfrac1{12}J_1E_2 + \tfrac12\wp^\bullet\big),\\
\varphi_{2,2}+1 &= 2K^4\cdot\big(J_1^2\wp - \tfrac1{12}J_1^2E_2 + \tfrac32\wp^2 + J_1\wp^\bullet - \tfrac1{96}E_4\big),
\end{aligned}$$

$$\begin{aligned}
\varphi_{3,-2} &= 3K^5\cdot\big(\tfrac95 J_1^5 - \tfrac92 J_1^3\wp - \tfrac18 J_1^3E_2 + \tfrac12 J_1\wp^2\\
&\qquad + \tfrac1{24}J_1\wp E_2 - \tfrac54 J_1^2\wp^\bullet + \tfrac1{180}J_1E_4 + \tfrac3{20}\wp\wp^\bullet\big),\\
\varphi_{3,-1} &= 3K^4\cdot\big(\tfrac98 J_1^4 - \tfrac94 J_1^2\wp - \tfrac18 J_1^2E_2 + \tfrac18\wp^2 + \tfrac1{24}\wp E_2 - \tfrac12 J_1\wp^\bullet + \tfrac1{288}E_4\big),\\
\varphi_{3,0} &= K^3\cdot\big(-\tfrac92 J_1^2 + \tfrac32\wp\big),\\
\varphi_{3,1} &= 3K^4\cdot\big(\tfrac32 J_1^2\wp - \tfrac18 J_1^2E_2 + \tfrac12\wp^2 + \tfrac1{24}\wp E_2 + J_1\wp^\bullet - \tfrac1{144}E_4\big),\\
\varphi_{3,2} &= 3K^5\cdot\big(\tfrac32 J_1^3\wp - \tfrac18 J_1^3E_2 + \tfrac72 J_1\wp^2\\
&\qquad + \tfrac1{24}J_1\wp E_2 + \tfrac74 J_1^2\wp^\bullet - \tfrac1{36}J_1E_4 + \tfrac34\wp\cdot\wp^\bullet\big),\\
\varphi_{3,3}+1 &= 3K^6\cdot\big(\tfrac94 J_1^4\wp - \tfrac3{16}J_1^4E_2 + \tfrac{15}2 J_1^2\wp^2 + \tfrac18 J_1^2\wp E_2 + 3J_1^3\wp^\bullet + \tfrac54\wp^3\\
&\qquad - \tfrac1{48}\wp^2E_2\tfrac1{16}J_1^2E_4 + 3J_1\wp\cdot\wp^\bullet - \tfrac1{144}\wp E_4 + \tfrac13(\wp^\bullet)^2\big),\\
\varphi_{4,0} &= K^4\cdot\big(-\tfrac{32}3 J_1^3 + 8J_1\wp + \tfrac23\wp^\bullet\big).
\end{aligned}$$

In the variables

$$q = e^{2\pi i\tau} \quad\text{and}\quad s = (-y)^{1/2} = e^{\pi i z}$$

the first coefficients of the functions above are

$$\begin{aligned}
\varphi_{1,-1} &= (-s^{-4}+4s^{-2}-6+4s^2-s^4)q + O(q^2),\\
\varphi_{1,0} &= (s^{-1}-s) + (-s^{-3}+3s^{-1}-3s+s^3)q + O(q^2),\\
\varphi_{1,1} &= (s^{-4}-4s^{-2}+6-4s^2+s^4)q + O(q^2),
\end{aligned}$$

$$\begin{aligned}
\varphi_{2,-2} &= (-2s^{-6}+4s^{-4}+2s^{-2}-8+2s^2+4s^4-2s^6)q+O(q^2),\\
\varphi_{2,-1} &= (-2s^{-5}+6s^{-3}-4s^{-1}-4s+6s^3-2s^5)q+O(q^2),\\
\varphi_{2,-0} &= (s^{-2}-s^2)+(-4s^{-4}+8s^{-2}-8s^2+4s^4)q+O(q^2),\\
\varphi_{2,1} &= (2s^{-5}-6s^{-3}+4s^{-1}+4s-6s^3+2s^5)q+O(q^2),\\
\varphi_{2,2}+1 &= 1+(2s^{-6}-4s^{-4}-2s^{-2}+8-2s^2-4s^4+2s^6)q+O(q^2),
\end{aligned}$$

$$\begin{aligned}
\varphi_{3,-2} &= (-3s^{-7}+6s^{-5}-3s^{-1}-3s+6s^5-3s^7)q+O(q^2),\\
\varphi_{3,-1} &= (-3s^{-6}+9s^{-4}-9s^{-2}+6-9s^2+9s^4-3s^6)q+O(q^2),\\
\varphi_{3,0} &= (s^{-3}-s^3)+(-9s^{-5}+18s^{-3}-9s^{-1}+9s-18s^3+9s^5)q+O(q^2),\\
\varphi_{3,1} &= (3s^{-6}-9s^{-4}+9s^{-2}-6+9s^2-9s^4+3s^6)q+O(q^2),\\
\varphi_{3,2} &= (3s^{-7}-6s^{-5}+3s^{-1}+3s-6s^5+3s^7)q+O(q^2),\\
\varphi_{3,3}+1 &= 1+(3s^{-8}-6s^{-6}+3s^{-4}-6s^{-2}+12-6s^2+3s^4-6s^6+3s^8)q+O(q^2),
\end{aligned}$$

$$\varphi_{4,0} = (s^{-4}-s^4)+(-16s^{-6}+32s^{-4}-16s^{-2}+16s^2-32s^4+16s^6)q+O(q^2).$$

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