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Equivariant characteristic classes of external and symmetric products of varieties

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We obtain refined generating series formulae for equivariant characteristic classes of external and symmetric products of singular complex quasiprojective varieties. More concretely, we study equivariant versions of Todd, Chern and Hirzebruch classes for singular spaces, with values in delocalized Borel–Moore homology of external and symmetric products. As a byproduct, we recover our previous characteristic class formulae for symmetric products and obtain new equivariant generalizations of these results, in particular also in the context of twisting by representations of the symmetric group.

55S15, 57R20; 20C30

1 Introduction

We obtain refined generating series formulae for equivariant characteristic classes of external and symmetric products of singular complex quasiprojective varieties, generalizing our previous results for symmetric products from Cappell, Maxim, Schürmann, Shaneson and Yokura [12].

1.1 Equivariant characteristic classes

All spaces in this paper are assumed to be complex quasiprojective, though many constructions also apply to other categories of spaces with a finite group action (eg compact complex analytic manifolds or varieties over any base field of characteristic zero). For such a variety X, consider an algebraic action $G \times X \to X$ by a finite group G, with quotient map $\pi\colon X \to X' := X/G$. For any $g \in G$, we let X^g denote the corresponding fixed point set.

We let $\mathrm{cat}^G(X)$ be a category of G–equivariant objects on X in the underlying category $\mathrm{cat}(X)$ (eg see Cappell, Maxim, Schürmann and Shaneson [11; 27]), which in this paper refers to one of the following examples: coherent sheaves $Coh(X)$, algebraically

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constructible sheaves of complex vector spaces $Constr(X)$, and (algebraic) mixed Hodge modules $MHM(X)$ on X. We denote by $K_0(\mathrm{cat}^G(X))$ the corresponding Grothendieck groups of these $\mathbb{Q}$–linear abelian categories. We will also work with the relative Grothendieck group $K_0^G(\mathrm{var}/X)$ of G–equivariant quasiprojective varieties over X, defined by using the scissor relation as in [11]. Let $H_*(X)$ denote the even-degree Borel–Moore homology $H_{\mathrm{ev}}^{\mathrm{BM}}(X) \otimes R$ with coefficients in a commutative $\mathbb{C}$–algebra R (or $\mathbb{Q}$–algebra if G is a symmetric group). Note that $H_*(-)$ is functorial for all proper maps, with a compatible cross-product $\boxtimes$.

Let

$$\mathrm{cl}_*(-;g)\colon K_0(\mathrm{cat}^G(X)) \to H_*(X^g)$$

be one of the following *equivariant characteristic class transformation of Lefschetz type* — see Section 5.1:

(i) The Lefschetz–Riemann–Roch transformation of Baum, Fulton and Quart [6] and Moonen [30],

$$\mathrm{td}_*(-;g)\colon K_0(Coh^G(X)) \to H_*(X^g),$$

with $R = \mathbb{C}$ (or $R = \mathbb{Q}$ if G is a symmetric group).

(ii) The localized Chern class transformation of Schürmann [37],

$$c_*(-;g)\colon K_0(Constr^G(X)) \to H_*(X^g),$$

with $R = \mathbb{C}$ (or $R = \mathbb{Q}$ if G is a symmetric group).

(iii) The motivic version of the (unnormalized) Atiyah–Singer class transformation of Cappell, Maxim, Schürmann and Shaneson [11],

$$T_{y*}(-;g)\colon K_0^G(\mathrm{var}/X) \to H_*(X^g),$$

with $R = \mathbb{C}[y]$ (or $R = \mathbb{Q}[y]$ if G is a symmetric group).

(iv) The mixed Hodge module version of the (unnormalized) Atiyah–Singer class transformation of [11],

$$T_{y*}(-;g)\colon K_0(MHM^G(X)) \to H_*(X^g),$$

with $R = \mathbb{C}[y^{\pm 1}]$ (or $R = \mathbb{Q}[y^{\pm 1}]$ if G is a symmetric group).

These class transformations are covariant functorial for G–equivariant proper maps and cross-products $\boxtimes$. Over a point space, they reduce to a certain g–trace (as explained in Section 5.1). For a subgroup K of G, with $g \in K$, these transformations $\mathrm{cl}_*(-;g)$

of Lefschetz type commute with the obvious restriction functor Res_K^G. Moreover, $\mathrm{cl}_*(-;g)$ depends only on the action of the cyclic subgroup generated by g. In particular, if $g = \mathrm{id}_G$ is the identity of G, we can take K to be the identity subgroup $\{\mathrm{id}_G\}$ with Res_K^G the forgetful functor

$$\mathit{For}\colon K_0(\mathrm{cat}^G(X)) \to K_0(\mathrm{cat}(X)),$$

so that $\mathrm{cl}_*(-;\mathrm{id}_G) = \mathrm{cl}_*(-)$ fits with a corresponding nonequivariant characteristic class, which in the above examples are:

(i) The Todd class transformation td_* of Baum, Fulton and MacPherson [5] appearing in the Riemann–Roch theorem for singular varieties.

(ii) The MacPherson–Chern class transformation c_* of MacPherson [25].

(iii) The motivic version of the (unnormalized) Hirzebruch class transformation T_{y*} of Brasselet, Schürmann and Yokura [9].

(iv) The mixed Hodge module version of the (unnormalized) Hirzebruch class transformation T_{y*} of [9]; see also Schürmann [38].

The disjoint union $IX := \bigsqcup_{g\in G} X^g$ (which is also called the inertia space of the G–space X) admits an induced G–action by $h\colon X^g \to X^{hgh^{-1}}$ such that the canonical map

$$i\colon IX = \bigsqcup_{g\in G} X^g \to X$$

defined by the inclusions of fixed point sets becomes G–equivariant. Therefore, G acts in a natural way on $\bigoplus_{g\in G} H_*(X^g) = H_*(IX)$ by conjugation.

Definition 1.1 The *(delocalized) G–equivariant homology of X* is the G–invariant subgroup

$$H_*^G(X) := (H_*(IX))^G = \left(\bigoplus_{g\in G} H_*(X^g)\right)^G. \tag{1}$$

This theory is functorial for proper G–maps and induced cross-products $\boxtimes$.

This notion is different (except for free actions) from the equivariant Borel–Moore homology $H^G_{\mathrm{BM},2*}(X)\otimes R$ defined by the Borel construction. In fact, since G is finite and R is a $\mathbb{Q}$–algebra, one has

$$H^G_{\mathrm{BM},2*}(X)\otimes R \simeq (H^{\mathrm{BM}}_{2*}(X)\otimes R)^G \simeq H^{\mathrm{BM}}_{2*}(X/G)\otimes R, \tag{2}$$

which is just a direct summand of $H_*^G(X)$ corresponding to the identity element of G, denoted by $H^G_{\mathrm{id},*}(X)$. For example, if G acts *trivially* on X (eg X is a point), then

$$H_*^G(X) \simeq H_*(X) \otimes C(G),$$

where $C(G)$ denotes the free abelian group of $\mathbb{Z}$–valued class functions on G (ie functions which are constant on the conjugacy classes of G). Note also that

$$H_*^G(X) = (H_*(IX))^G \simeq H_*(IX/G). \tag{3}$$

Definition 1.2 For any of the above Lefschetz-type characteristic class transformations $\mathrm{cl}_*(-;g)$, we define a corresponding G–equivariant class transformation (with T^G_{y*} the G–equivariant Hirzebruch class transformation)

$$\mathrm{cl}_*^G\colon K_0(\mathrm{cat}^G(X)) \to H_*^G(X)$$

by

$$\mathrm{cl}_*^G(-) := \bigoplus_{g\in G} \mathrm{cl}_*(-;g) \in \Bigl(\bigoplus_{g\in G} H_*(X^g)\Bigr)^G.$$

The G–invariance of the class $\mathrm{cl}_*^G(-)$ is a consequence of the *conjugacy invariance* of the Lefschetz-type characteristic class $\mathrm{cl}_*(-;g)$; see Cappell, Maxim, Schürmann and Shaneson [11, Section 5.3]. Note that the summand $\mathrm{cl}_*(-;\mathrm{id}) \in (H_*(X))^G$ corresponding to the identity element of G is just the nonequivariant characteristic class, which for equivariant coefficients is invariant under the G–action by functoriality. Under the identification (2), this class also agrees (for our finite group G) with the corresponding (naive) equivariant characteristic class defined in terms of the Borel construction, eg for $\mathrm{cl}_* = \mathrm{td}_*$, this is the equivariant Riemann–Roch transformation of Edidin and Graham [15]; and for $\mathrm{cl}_* = c_*$, this is the equivariant Chern class transformation of Ohmoto [32; 33].

The above transformation $\mathrm{cl}_*^G(-)$ has the same properties as the Lefschetz-type transformations $\mathrm{cl}_*(-;g)$, eg functoriality for proper push-downs, restrictions to subgroups, and multiplicativity for exterior products.

Remark 1.3 The G–equivariant characteristic classes defined here for $\mathrm{cl}_* = \mathrm{td}_*$, T_{y*} agree, up to the normalization factor $1/|G|$, with the corresponding notions introduced in [11].

1.2 Generating series formulae

Now let Z be a quasiprojective variety, and denote by $Z^{(n)} := Z^n/\Sigma_n$ its n^{th} symmetric product (ie the quotient of Z^n by the natural permutation action of the symmetric group Σ_n on n elements), with $\pi_n\colon Z^n \to Z^{(n)}$ the natural projection map. The standard approach for computing invariants of the symmetric products $Z^{(n)}$ is to collect the respective invariants of all symmetric products in a generating series, and then compute the latter solely in terms of invariants of Z; eg see Cappell, Maxim, Schürmann, Shaneson and Yokura [12] and the references therein. In this paper, we obtain generalizations of results of [12], formulated in terms of equivariant characteristic classes of external products and symmetric products of varieties.

To a given object $\mathcal{F} \in \mathrm{cat}(Z)$ in a category as above, ie coherent or constructible sheaves, or mixed Hodge modules on Z (or morphisms $f\colon Y \to Z$ in the motivic context), we attach new objects as follows (see [12], Maxim, Saito and Schürmann [27; 26] for details):

(a) The Σ_n–equivariant object $\mathcal{F}^{\boxtimes n} \in \mathrm{cat}^{\Sigma_n}(Z^n)$ on the cartesian product Z^n (eg $f^n\colon Y^n \to Z^n$ in the motivic context).

(b) The Σ_n–equivariant object $\pi_{n*}\mathcal{F}^{\boxtimes n} \in \mathrm{cat}^{\Sigma_n}(Z^{(n)})$ on the symmetric product $Z^{(n)}$ (eg the Σ_n–equivariant map $Y^n \to Z^{(n)}$ in the motivic context).

(c) The following nonequivariant objects in $\mathrm{cat}(Z^{(n)})$:

   (1) The n^{th} *symmetric power* object $\mathcal{F}^{(n)} := (\pi_{n*}\mathcal{F}^{\boxtimes n})^{\Sigma_n}$ on $Z^{(n)}$, defined by using the projector $(-)^{\Sigma_n}$ onto the Σ_n–invariant part (or the map $f^{(n)}\colon Y^{(n)} \to Z^{(n)}$ induced by dividing out the Σ_n–action in the motivic context).

   (2) The n^{th} *alternating power* object $\mathcal{F}^{\{n\}} := (\pi_{n*}\mathcal{F}^{\boxtimes n})^{\mathrm{sign}-\Sigma_n}$ on $Z^{(n)}$, defined by using the alternating projector $(-)^{\mathrm{sign}-\Sigma_n}$ onto the sign-invariant part. (This construction does not apply in the motivic context.)

   (3) $\mathit{For}(\pi_{n*}\mathcal{F}^{\boxtimes n})$, which is obtained by forgetting the Σ_n–action on $\pi_{n*}\mathcal{F}^{\boxtimes n} \in \mathrm{cat}^{\Sigma_n}(Z^{(n)})$ (eg the induced map $Y^n \to Z^{(n)}$ in the motivic context).

These constructions and all of the following results also apply to suitable bounded complexes (eg the constant Hodge module complex $\mathbb{Q}^H_Z$); see Remark 5.12 for details.

The main goal of this paper is to compute generating series formulae for the (equivariant) characteristic classes of these new coefficients only in terms of the original characteristic class $\mathrm{cl}_*(\mathcal{F})$. These generating series take values in a corresponding commutative graded $\mathbb{Q}$–algebra $\mathbb{H}^{\Sigma}_*(Z)$, $\mathbb{P}\mathbb{H}^{\Sigma}_*(Z)$ or $\mathbb{P}\mathbb{H}_*(Z)$, and are formulated with the help of certain operators which transport homology classes from Z into these corresponding commutative graded $\mathbb{Q}$–algebras. In each of three situations (a)-(c) above, these algebras of Pontrjagin type and operators are described explicitly as follows:

(a)
$$\mathbb{H}^{\Sigma}_*(Z) := \bigoplus_{n\geq 0} H^{\Sigma_n}_*(Z^n)\cdot t^n,$$

with *creation operator* $\mathfrak{a}_r$ defined by: if $\sigma_r = (r)$ is an r–cycle in Σ_r, then $\mathfrak{a}_r$ is the composition

$$\mathfrak{a}_r\colon H_*(Z) \xrightarrow{\cdot r} H_*(Z) \cong H_*((Z^r)^{\sigma_r})^{Z_{\Sigma_r}(\sigma_r)} \hookrightarrow H^{\Sigma_r}_*(Z^r),$$

where $\langle\sigma_r\rangle = Z_{\Sigma_r}(\sigma_r)$ acts trivially on $H_*((Z^r)^{\sigma_r})$.

Note that the direct summand

$$\mathbb{H}^{\Sigma}_{\mathrm{id},*}(Z) := \bigoplus_{n\geq 0} H^{\Sigma_n}_{\mathrm{id},*}(Z^n)\cdot t^n \subset \mathbb{H}^{\Sigma}_*(Z)$$

corresponding to the identity component is a subring, and the projection of $\mathbb{H}^{\Sigma}_*(Z)$ onto the subring $\mathbb{H}^{\Sigma}_{\mathrm{id},*}(Z)$ kills all the creation operators except $\mathfrak{a}_1 = \mathrm{id}_{H_*(Z)}$.

(b)
$$\begin{aligned}\mathbb{P}\mathbb{H}^{\Sigma}_*(Z) &:= \bigoplus_{n\geq 0} H^{\Sigma_n}_*(Z^{(n)})\cdot t^n\\ &\simeq \bigoplus_{n\geq 0} (H_*(Z^{(n)})\otimes C(\Sigma_n))\cdot t^n \hookrightarrow \mathbb{P}\mathbb{H}_*(Z)\otimes\mathbb{Q}[p_i \mid i\geq 1],\end{aligned}$$

with corresponding operator $p_r\cdot d_{r*}\colon H_*(Z)\to H_*(Z^{(r)})\otimes\mathbb{Q}[p_i \mid i\geq 1]$, where $d_r := \pi_r\circ\Delta_r\colon Z\to Z^{(r)}$ is the composition of the natural projection $\pi_r\colon Z^r\to Z^{(r)}$ with the diagonal embedding $\Delta_r\colon Z\to Z^r$. The algebra inclusion above is induced from the Frobenius character

$$\mathrm{ch}_F\colon C(\Sigma)\otimes\mathbb{Q} := \bigoplus_n C(\Sigma_n)\otimes\mathbb{Q} \xrightarrow{\simeq} \mathbb{Q}[p_i \mid i\geq 1] =: \Lambda\otimes\mathbb{Q}$$

to the graded ring of $\mathbb{Q}$–valued symmetric functions in infinitely many variables x_m ($m\in\mathbb{N}$), with $p_i := \sum_m x_m^i$ the i^{th} power sum function; see Macdonald [24, Chapter I, Section 7], and $\mathbb{P}\mathbb{H}_*(Z)$ defined as below.

(c)
$$\mathbb{P}\mathbb{H}_*(Z) := \bigoplus_{n\geq 0} H_*(Z^{(n)})\cdot t^n,$$

with corresponding operator $d_{r*}\colon H_*(Z)\to H_*(Z^{(r)})$.

Moreover, the creation operator $\mathfrak{a}_r$ satisfies the identity

$$\pi_{r*} \circ \mathfrak{a}_r = p_r \cdot d_{r*},$$

justifying the multiplication by r in its definition.

The main characteristic class formula of this paper is contained in:

Theorem 1.4 *The following generating series formula holds in the commutative graded $\mathbb{Q}$–algebra $\mathbb{H}_*^{\Sigma}(Z) := \bigoplus_{n\geq 0} H_*^{\Sigma_n}(Z^n) \cdot t^n$ if cl_* is either td_*, c_* or T_{-y*}:*

$$\sum_{n\geq 0} \mathrm{cl}_*^{\Sigma_n}(\mathcal{F}^{\boxtimes n}) \cdot t^n = \exp\left(\sum_{r\geq 1} \mathfrak{a}_r(\Psi_r(\mathrm{cl}_*(\mathcal{F}))) \cdot \frac{t^r}{r} \right), \tag{4}$$

where Ψ_r denotes the homological Adams operation defined by

$$\Psi_r = \begin{cases} \mathrm{id} & \textit{if } \mathrm{cl}_* = c_*, \\ \cdot 1/r^i \ \textit{ on } H_{2i}^{\mathrm{BM}}(Z)\otimes\mathbb{Q} & \textit{if } \mathrm{cl}_*, = \mathrm{td}_* \\ \cdot 1/r^i \ \textit{ on } H_{2i}^{\mathrm{BM}}(Z)\otimes\mathbb{Q} \textit{ and } y \mapsto y^r & \textit{if } \mathrm{cl}_* = T_{-y*}. \end{cases}$$

In particular, by projecting onto the identity component, we get

$$\sum_{n\geq 0} \mathrm{cl}_*(\mathcal{F}^{\boxtimes n}; \mathrm{id}) \cdot t^n = \exp(t \cdot \mathrm{cl}_*(\mathcal{F})) \in \mathbb{H}_{\mathrm{id},*}^{\Sigma}(Z). \tag{5}$$

For the rest of this introduction, cl_* denotes any of the classes td_*, c_* or T_{-y*}. The proof of Theorem 1.4 is purely formal, based on the multiplicativity and conjugacy invariance of the Lefschetz-type characteristic classes $\mathrm{cl}_*(-; g)$, together with the following key *localization* formula from Cappell, Maxim, Schürmann, Shaneson and Yokura [12, Lemmas 3.3, 3.6 and 3.10]:

$$\mathrm{cl}_*(\mathcal{F}^{\boxtimes r}; \sigma_r) = \Psi_r \mathrm{cl}_*(\mathcal{F}), \tag{6}$$

under the identification $(Z^r)^{\sigma_r} \simeq Z$. For this localization formula in the context of Hirzebruch classes, it is important to work with the parameter $-y$ and the unnormalized versions of Hirzebruch classes and their respective equivariant analogues; see [12]. In fact, formula (4) is a special case of an abstract generating series formula (37), which holds for any functor H (covariant for isomorphisms) with a compatible commutative, associative cross-product $\boxtimes$, with a unit $1_{\mathrm{pt}} \in H(\mathrm{pt})$. The above-mentioned abstract formula (37) codifies the combinatorics of the action of the symmetric groups Σ_n, and it should be regarded as a far-reaching generalization of the well-known identity of symmetric functions (eg see the proof of Macdonald [24, (2.14)])

$$\sum_{n\geq 0} h_n t^n = \exp\Big(\sum_{r\geq 1} p_r \cdot \frac{t^r}{r}\Big),$$

with h_n the n^{th} complete symmetric function. Other applications of the abstract generating series formula (37) in the framework of orbifold cohomology and localized K–theory are explained in Section 3.1. In this way, we reprove and generalize some results from Qin and Wang [34] and Wang [40], respectively. Moreover, in Section 4, we give another application of (37) to canonical constructible functions and orbifold-type Chern classes of symmetric products, reproving some results of Ohmoto [33].

Remark 1.5 In the motivic context, the exponentiation map

$$K_0(\mathrm{var}/Z) \to \bigoplus_{n\geq 0} K_0^{\Sigma_n}(\mathrm{var}/Z^n)\cdot t^n, \quad [f\colon X\to Z]\mapsto \sum_{n\geq 0}[f^n\colon X^n \to Z^n]\cdot t^n, \tag{7}$$

is well-defined as in Bergh [7], and should be regarded as an equivariant analogue of the (relative) Kapranov zeta function used in [12] and Maxim and Schürmann [27]. In fact, the latter can be recovered from (7) by pushing down to the symmetric products (resp. to a point), and taking the quotients by the Σ_n–action.

By pushing formula (4) down to the symmetric products, we obtain by functoriality the following result:

Corollary 1.6 *The following generating series formula holds in the commutative graded $\mathbb{Q}$–algebra $\mathbb{PH}_*^{\Sigma}(Z) := \bigoplus_{n\geq 0} H_*^{\Sigma_n}(Z^{(n)})\cdot t^n \hookrightarrow \mathbb{PH}_*(Z)\otimes \mathbb{Q}[p_i \mid i\geq 1]$:*

$$\sum_{n\geq 0} \mathrm{cl}_*^{\Sigma_n}(\pi_{n*}\mathcal{F}^{\boxtimes n})\cdot t^n = \exp\Big(\sum_{r\geq 1} p_r\cdot d_{r*}(\psi_r(\mathrm{cl}_*(\mathcal{F})))\cdot \frac{t^r}{r}\Big). \tag{8}$$

This should be regarded as a characteristic class version of Getzler [17, Proposition 5.4]. In particular, if Z is projective, then by taking degrees, we get in Section 5.3 generating series formulae for the characters of virtual Σ_n–representations of $H^*(Z^n;\mathcal{F}^{\boxtimes n})$, that is,

$$\sum_{n\geq 0} \mathrm{tr}_{\Sigma_n}(Z^n;\mathcal{F}^{\boxtimes n})\cdot t^n = \exp\Big(\sum_{r\geq 1} p_r\cdot \chi(H^*(Z,\mathcal{F}))\cdot \frac{t^r}{r}\Big) \in \mathbb{Q}[p_i \mid i\geq 1][\![t]\!] \tag{9}$$

for $\mathcal{F}$ a coherent or constructible sheaf and with χ denoting the corresponding Euler characteristic, and

$$\sum_{n\geq 0} \mathrm{tr}_{\Sigma_n}(Z^n; \mathcal{M}^{\boxtimes n}) \cdot t^n = \exp\left(\sum_{r\geq 1} p_r \cdot \chi_{-y^r}(H^*(Z,\mathcal{M})) \cdot \frac{t^r}{r} \right) \in \mathbb{Q}[y^{\pm 1}, p_i \mid i \geq 1][\![t]\!] \tag{10}$$

for $\mathcal{M}$ a mixed Hodge module on Z, and with $\chi_y(H^*(Z,\mathcal{M}))$ the corresponding χ_y–polynomial.

By specializing all the p_i to the value 1 — which corresponds to the use of the projectors $(-)^{\Sigma_n}$ — formula (8) reduces to the main result of [12], namely:

Corollary 1.7 *The following generating series formula holds in the Pontrjagin ring* $\mathbb{PH}_*(Z) := \bigoplus_{n\geq 0} H_*(Z^{(n)}) \cdot t^n$*:*

$$\sum_{n\geq 0} \mathrm{cl}_*(\mathcal{F}^{(n)}) \cdot t^n = \exp\left(\sum_{r\geq 1} d_{r*}(\psi_r(\mathrm{cl}_*(\mathcal{F}))) \cdot \frac{t^r}{r} \right). \tag{11}$$

In particular, if Z is projective, we recover the degree formulae from [12], which can now also be derived from (9) and (10) by specializing all the p_i to 1.

Corollary 1.6 also has other important applications. For example, by specializing the p_i to the value $\mathrm{sign}(\sigma_i) = (-1)^{i-1}$ (which corresponds to the use of the alternating projectors $(-)^{\mathrm{sign}-\Sigma_n}$), formula (8) reduces to:

Corollary 1.8 *The following generating series formula holds in the Pontrjagin ring* $\mathbb{PH}_*(Z)$*:*

$$\sum_{n\geq 0} \mathrm{cl}_*(\mathcal{F}^{\{n\}}) \cdot t^n = \exp\left(-\sum_{r\geq 1} d_{r*}(\psi_r(\mathrm{cl}_*(\mathcal{F}))) \cdot \frac{(-t)^r}{r} \right). \tag{12}$$

In particular, if Z is projective, we recover special cases of the main formulae from Maxim and Schürmann [27, Corollary 1.5], which can now be also derived from (9) and (10) by specializing the p_i to $(-1)^{i-1}$. For example, if $\mathrm{cl}_* = T_{-y*}$ and $\mathcal{F} = \mathbb{Q}_Z^H$, we recover the generating series formula for the degrees

$$\deg(T_{-y*}(\mathbb{Q}_Z^{H\,\{n\}})) = \chi_{-y}\big([H_c^*(B(Z,n), \epsilon_n)]\big),$$

where $B(Z,n) \subset Z^{(n)}$ is the *configuration space* of unordered n–tuples of distinct points in Z, and ϵ_n is the rank-one local system on $B(Z,n)$ corresponding to a sign representation of $\pi_1(B(Z,n))$ as in [27, page 293]; compare also with Gorsky [18, Example 3b] and Getzler [17, Corollary 5.7].

Note also that the specialization $p_1 \mapsto 1$ and $p_i \mapsto 0$ for all $i \geq 2$ corresponds to the evaluation homomorphisms (for all $n \in \mathbb{N}$)

$$\frac{1}{n!}\,\mathrm{ev}_{\mathrm{id}} = \frac{1}{n!}\,\mathrm{Res}_{\mathrm{id}}^{\Sigma_n}\colon\ H_*^{\Sigma_n}(Z^{(n)}) \to H_*(Z^{(n)}).$$

Then, by forgetting the Σ_n–action on $\pi_{n*}\mathcal{F}^{\boxtimes n}$, Corollary 1.6 specializes to the following result:

Corollary 1.9 *The following exponential generating series formula holds in the Pontrjagin ring $\mathbb{PH}_*(Z)$:*

$$\sum_{n\geq 0} \mathrm{cl}_*(\pi_{n*}\mathcal{F}^{\boxtimes n}) \cdot \frac{t^n}{n!} = \exp(t \cdot \mathrm{cl}_*(\mathcal{F})). \tag{13}$$

The above corollary also follows from formula (5), after a suitable renormalization of the product structure of $\mathbb{H}^{\Sigma}_{\mathrm{id},*}(Z)$ in order to make the pushforward

$$\pi_* := \bigoplus \pi_{n,*}\colon\ \mathbb{H}^{\Sigma}_{\mathrm{id},*}(Z) \to \mathbb{PH}_*(Z)$$

into a ring homomorphism; see Section 4 for details.

In particular, if Z is projective, by taking degrees we get exponential generating series formulae for the Euler characteristic and χ_y–polynomial of $H^*(Z^n, \mathcal{F}^{\boxtimes})$. For example, if $\mathrm{cl}_* = T_{-y*}$ and $\mathcal{F} = \mathcal{M}$ is a mixed Hodge module on Z, we get

$$\sum_{n\geq 0} \chi_{-y}(Z^n, \mathcal{M}^{\boxtimes n}) \cdot \frac{t^n}{n!} = \exp(t \cdot \chi_{-y}(Z, \mathcal{M})), \tag{14}$$

which also follows directly from the Künneth formula; eg see Maxim, Saito and Schürmann [26].

1.3 Twisting by Σ_n–representations

Additionally, for a fixed n, one can consider the coefficient of t^n in the generating series (4) for the (equivariant) characteristic classes of all exterior powers $\mathcal{F}^{\boxtimes n} \in \mathrm{cat}^{\Sigma_n}(Z^n)$. Moreover, in this case, one can twist the equivariant coefficients $\mathcal{F}^{\boxtimes n}$ by a (finite-dimensional) rational Σ_n–representation V, and compute the corresponding equivariant characteristic classes of Lefschetz type (see Remark 5.3)

$$\mathrm{cl}_*(V \otimes \mathcal{F}^{\boxtimes n}; \sigma) = \mathrm{trace}_\sigma(V) \cdot \mathrm{cl}_*(\mathcal{F}^{\boxtimes n}; \sigma) \tag{15}$$

for $\sigma \in \Sigma_n$. By pushing down to the symmetric product $Z^{(n)}$ along the natural map $\pi_n\colon Z^n \to Z^{(n)}$, we then get by the projection formula the following identity in $H_*^{\Sigma_n}(Z^{(n)}) \cong H_*(Z^{(n)}) \otimes C(\Sigma_n) \hookrightarrow H_*(Z^{(n)}) \otimes \mathbb{Q}[p_i \mid i \geq 1]$:

$$\mathrm{cl}_*^{\Sigma_n}(\pi_{n*}(V \otimes \mathcal{F}^{\boxtimes n})) = \sum_{\lambda=(k_1,k_2,\ldots)\dashv n} \frac{p_\lambda}{z_\lambda} \chi_\lambda(V) \cdot \bigodot_{r\geq 1} \big(d_{r*}(\psi_r(\mathrm{cl}_*(\mathcal{F})))\big)^{k_r}. \tag{16}$$

Here, the symbol $\lambda = (k_1, k_2, \ldots) \dashv n$ denotes the partition $\lambda = (k_1, k_2, \ldots)$ of n corresponding to a conjugacy class of an element $\sigma \in \Sigma_n$ (ie $\sum_r rk_r = n$). We also denote by $z_\lambda := \prod_{r\geq 1} r^{k_r} \cdot k_r!$ the order of the stabilizer of σ, by $\chi_\lambda(V) = \mathrm{trace}_\sigma(V)$ the corresponding trace, and we set $p_\lambda := \prod_{r\geq 1} p_r^{k_r}$. Finally, $\bigodot$ denotes the Pontrjagin-type product, as defined in Section 4.

If Z is projective, by taking the degree in formula (16) we have the following character formulae generalizing (9) and (10):

(i) For $\mathcal{F}$ a coherent or constructible sheaf, we get

$$\mathrm{tr}_{\Sigma_n}(H^*(Z^n; V \otimes \mathcal{F}^{\boxtimes n})) = \sum_{\lambda\dashv n} \frac{p_\lambda}{z_\lambda} \chi_\lambda(V) \cdot \chi(H^*(Z;\mathcal{F}))^{\ell(\lambda)}, \tag{17}$$

with χ denoting the corresponding Euler characteristic, and for a partition $\lambda = (k_1, k_2, \ldots)$ of n we let $\ell(\lambda) := k_1 + k_2 + \cdots$ be the length of λ.

(ii) For $\mathcal{M}$ a mixed Hodge module on Z, we get

$$\begin{aligned}\mathrm{tr}_{\Sigma_n}&(H^*(Z^n; V \otimes \mathcal{M}^{\boxtimes n})) \\ &= \sum_{\lambda=(k_1,k_2,\ldots)\dashv n} \frac{p_\lambda}{z_\lambda} \chi_\lambda(V) \cdot \prod_{r\geq 1} \big(\chi_{-y^r}(H^*(Z;\mathcal{M}))\big)^{k_r},\end{aligned} \tag{18}$$

with $\chi_y(H^*(Z,\mathcal{M}))$ the corresponding χ_y–polynomial.

The formula (16) is a generalization of Corollary 1.6, which one gets back for V the trivial representation. Furthermore, by specializing all the p_i in (16) to the value 1 (which corresponds to the use of the projectors $(-)^{\Sigma_n}$), one obtains the following identity in $H_*(Z^{(n)})$:

$$\mathrm{cl}_*((\pi_{n*}(V \otimes \mathcal{F}^{\boxtimes n}))^{\Sigma_n}) = \sum_{\lambda=(k_1,k_2,\ldots)\dashv n} \frac{1}{z_\lambda} \chi_\lambda(V) \cdot \bigodot_{r\geq 1} \big(d_{r*}(\psi_r(\mathrm{cl}_*(\mathcal{F})))\big)^{k_r}. \tag{19}$$

Note that by letting V be the trivial (resp. sign) representation, formula (19) reduces to Corollary 1.7 (resp. Corollary 1.8). Another important special case of (19) is obtained by choosing $V = \mathrm{Ind}_K^{\Sigma_n}(\mathrm{triv})$, the representation induced from the trivial representation of a subgroup K of Σ_n, with

$$\begin{aligned}(\pi_{n*}(V \otimes \mathcal{F}^{\boxtimes n}))^{\Sigma_n} &\simeq (\mathrm{Ind}_K^{\Sigma_n}(\mathrm{triv}) \otimes \pi_{n*}(\mathcal{F}^{\boxtimes n}))^{\Sigma_n} \\ &\simeq (\pi_{n*}(\mathcal{F}^{\boxtimes n}))^K \simeq \pi'_*\big((\pi_*(\mathcal{F}^{\boxtimes n}))^K\big).\end{aligned}$$

Here $\pi\colon Z^n \to Z^n/K$ and $\pi'\colon Z^n/K \to Z^{(n)}$ are the projections factoring π_n. In this case, formula (19) calculates the characteristic class

$$\mathrm{cl}_*((\pi_{n*}(\mathcal{F}^{\boxtimes n}))^K) = \pi'_*\mathrm{cl}_*\big((\pi_*(\mathcal{F}^{\boxtimes n}))^K\big).$$

In particular, if Z is projective and we consider the constant Hodge module $\mathcal{F} = \mathbb{Q}_Z^H$, we get at the degree level the following formula for the χ_y–polynomial of the quotient Z^n/K:

$$\chi_{-y}(Z^n/K) = \sum_{\lambda=(k_1,k_2,\dots)\dashv n} \frac{1}{z_\lambda}\chi_\lambda(\mathrm{Ind}_K^{\Sigma_n}(\mathrm{triv})) \cdot \prod_{r\geq 1} \chi_{-y^r}(Z)^{k_r}. \tag{20}$$

The corresponding Euler characteristic formula, obtained for $y = 1$, is also a special case of Macdonald's formula (see [23, page 567]) for the corresponding Poincaré polynomial.

Finally, by letting $V = V_\mu \simeq V_\mu^*$ be the (self-dual) irreducible representation of Σ_n corresponding to a partition μ of n, the coefficients

$$(\pi_{n*}(V_\mu \otimes \mathcal{F}^{\boxtimes n}))^{\Sigma_n} \simeq (V_\mu \otimes \pi_{n*}(\mathcal{F}^{\boxtimes n}))^{\Sigma_n} =: S_\mu(\pi_{n*}\mathcal{F}^{\boxtimes n})$$

of the left-hand side of (19) calculate the corresponding Schur functor of $\pi_{n*}\mathcal{F}^{\boxtimes n}$ as an element in $\mathrm{cat}^{\Sigma_n}(Z^{(n)})$, with

$$\pi_{n*}\mathcal{F}^{\boxtimes n} \simeq \sum_{\mu\dashv n} V_\mu \otimes S_\mu(\pi_{n*}\mathcal{F}^{\boxtimes n}) \in \mathrm{cat}^{\Sigma_n}(Z^{(n)}); \tag{21}$$

eg see Remark 5.9. These Schur functors generalize the symmetric and alternating powers of $\mathcal{F}$, which correspond to the trivial and sign representation, respectively. Note that, by using (21), we get an alternative description of the equivariant classes

$$\mathrm{cl}_*^{\Sigma_n}(\pi_{n*}\mathcal{F}^{\boxtimes n}) \in H_*^{\Sigma_n}(Z^{(n)}) \cong H_*(Z^{(n)}) \otimes C(\Sigma_n) \hookrightarrow H_*(Z^{(n)}) \otimes \mathbb{Q}[p_i \mid i \geq 1]$$

in terms of the Schur functions $s_\mu := \mathrm{ch}_F(V_\mu) \in \mathbb{Q}[p_i \mid i \geq 1]$ — see Macdonald [24, Chapter 1, Sections 3 and 7]:

$$\mathrm{cl}_*^{\Sigma_n}(\pi_{n*}\mathcal{F}^{\boxtimes n}) = \sum_{\mu\dashv n} s_\mu \cdot \mathrm{cl}_*(S_\mu(\pi_{n*}\mathcal{F}^{\boxtimes n})), \tag{22}$$

with $\mathrm{cl}_*(S_\mu(\pi_{n*}\mathcal{F}^{\boxtimes n}))$ computed as in (19).

As a concrete example, for Z pure-dimensional with coefficients given by the intersection cohomology Hodge module IC^H_Z on Z, the corresponding Schur functor S_μ of $\pi_{n*}\mathrm{IC}^H_{Z^n}$ is given by the twisted intersection cohomology Hodge module $S_\mu(\pi_{n*}\mathrm{IC}^H_{Z^n}) = \mathrm{IC}^H_{Z^{(n)}}(V_\mu)$ with twisted coefficients corresponding to the local system on the configuration space $B(Z,n)$ of unordered n–tuples of distinct points in Z, induced from V_μ by the group homomorphism $\pi_1(B(Z,n)) \to \Sigma_n$ (compare Maxim and Schürmann [27, page 293] and Meinhardt and Reineke [29, Proposition 3.5]). For Z projective and pure-dimensional, by taking the degrees in (19) for the present choice of coefficients IC^H_Z and representation V_μ, we obtain the following identity for the χ_y–polynomial of the twisted intersection cohomology:

$$\chi_{-y}\big(H^*(Z^{(n)};\mathrm{IC}^H_{Z^{(n)}}(V_\mu))\big) = \sum_{\lambda=(k_1,k_2,\dots)\dashv n} \frac{1}{z_\lambda}\chi_\lambda(V_\mu)\cdot \prod_{r\geq 1} \chi_{-y^r}(H^*(Z;\mathrm{IC}^H_Z))^{k_r}. \tag{23}$$

Note that results like (20) or (23) cannot be deduced only from the nonequivariant study of symmetric products as in [12].

We conclude the introduction with a brief discussion of potential applications of our results.

First, the techniques developed here have also been applied by the authors to the study of cohomology representations of external and symmetric products (see Maxim and Schürmann [28]), generalizing our previous results from [27].

Secondly, we plan to employ the results of this paper for the study of Hilbert schemes of points on quasiprojective manifolds. In fact, our prior work on symmetric products from [12] has already been used for the study of (pushforwards under the Hilbert–Chow morphism of) characteristic classes of Hilbert schemes of points on smooth quasiprojective varieties; see [10]. But for smooth surfaces, the results of [10] may be improved via the McKay correspondence — see Krug [22] and Scala [35; 36] — by using the stronger equivariant results of the present paper. In addition, the present work can also be used for obtaining generating series formulae for the singular Todd classes $\mathrm{td}_*(\mathcal{F}^{[n]})$ of tautological sheaves $\mathcal{F}^{[n]}$ (associated to a given $\mathcal{F} \in Coh(Z)$) on the Hilbert scheme $Z^{[n]}$ of n points on a smooth quasiprojective algebraic surface Z. For degree formulae in this context, see eg Wang and Zhou [42].

One can also use similar techniques for the study of equivariant characteristic classes of the Σ_n–equivariant Fulton–MacPherson (and other similar) compactifications of configurations spaces of points on a smooth variety Z of any dimension. Here, again, the equivariant results of this paper (and in particular, the twisting by a representation) are needed, while the nonequivariant results from [12] are not sufficient. For equivariant degree formulae in this context, see eg Getzler [17].

Finally, techniques and results of this paper can be extended to the context of actions of wreath products $G_n := G \wr \Sigma_n = G^n \rtimes \Sigma_n$ on external powers $\mathcal{F}^{\boxtimes n} \in \mathrm{cat}^{G_n}(Z^n)$ of a given object $\mathcal{F} \in \mathrm{cat}^G(Z)$ on a G–space Z, provided that the corresponding key localization formula analogous to (6) holds. This will be the object of future work by the authors. Results of this type for Chern classes already appeared in Ohmoto [33], while for degree versions, see eg Qin, Wang and Zhou [34; 40; 41; 43].

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2 Delocalized equivariant theories

In this section, we introduce the notion of *delocalized equivariant theory* of a G–space X (with G a finite group) associated to a covariant functor H with compatible cross-product. We also describe the corresponding *restriction* and *induction* functors, which will play an essential role in the subsequent sections of the paper.

In the classical context of the usual (co)homology functor, such delocalized theories have been defined in [3; 4], as well as in an unpublished paper of Segal, where they were used for obtaining Riemann–Roch-type theorems. For analogues in the context of Deligne–Mumford stacks, see also [14; 39]. The corresponding orbifold index theorem was developed in [21], by using for the first time the G–equivariant cohomology of the inertia space IX (as in (3)) of a smooth G–space X, described in terms of differential forms. The corresponding restriction and induction functors were also studied in this classical context in [34; 43].

For simplicity, all spaces in this paper are assumed to be complex quasiprojective, though many constructions in this section apply to other categories of spaces with a finite group action (eg topological spaces or varieties over any base field). For such a variety X, consider an algebraic action $G \times X \to X$ by a finite group G. For any $g \in G$, we let X^g denote the corresponding fixed point set. Let H be a covariant (with respect to isomorphisms) functor to abelian groups, with a compatible cross-product $\boxtimes$ ($\mathbb{Z}$–linear in each variable) which is commutative, associative and with a unit $1_{\mathrm{pt}} \in H(\mathrm{pt})$. As main examples used in this paper, we consider the following, with R a commutative ring with unit (eg $R = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$):

(i) The even-degree Borel–Moore homology $H^{\mathrm{BM}}_{\mathrm{ev}}(X) \otimes R$ of X with coefficients in R.

(ii) Chow groups $\mathrm{CH}_*(X) \otimes R$ with R–coefficients.

(iii) The Grothendieck group of coherent sheaves $K_0(Coh(X)) \otimes R$ with R–coefficients.

Another possible choice would be the usual R–homology in even degrees, $H_{\mathrm{ev}}(X) \otimes R$. Since in this section we only need functoriality with respect to isomorphisms, we could also work with cohomological theories, such as the even-degree (compactly supported) R–cohomology $H^{\mathrm{ev}}_{(c)}(X) \otimes R$ or the Grothendieck group of algebraic vector bundles $K^0(X) \otimes R$ with R–coefficients, as used in [34; 43]. In this case, the corresponding covariant transformation g_*, as used in this paper, is given by $(g^*)^{-1}$, the inverse of the induced pullback under g. If X is smooth, this fits with the following Poincaré duality isomorphisms:

$$\begin{aligned} H_{\mathrm{ev}}(X) \otimes R &\cong H^{\mathrm{ev}}_c(X) \otimes R, \\ H^{\mathrm{BM}}_{\mathrm{ev}}(X) \otimes R &\cong H^{\mathrm{ev}}(X) \otimes R, \\ K_0(Coh(X)) \otimes R &\simeq K^0(X) \otimes R. \end{aligned} \tag{24}$$

The disjoint union $\bigsqcup_{g \in G} X^g$ admits an induced G–action by $h\colon X^g \to X^{hgh^{-1}}$ such that the canonical map

$$i\colon \bigsqcup_{g \in G} X^g \to X$$

defined by the inclusions of fixed-point sets becomes G–equivariant. Therefore, G acts in a natural way on $\bigoplus_{g \in G} H(X^g)$.

Definition 2.1 The *delocalized G–equivariant theory of X associated to H* is the G–invariant subgroup of $\bigoplus_{g\in G} H(X^g)$, namely,

$$H^G(X) := \Big(\bigoplus_{g\in G} H(X^g)\Big)^G. \tag{25}$$

This theory is functorial for proper G–maps (or G–equivariant isomorphisms).

Remark 2.2 An equivalent interpretation of this delocalized G–equivariant theory $H^G(X)$ of a G–space X can be obtained by breaking the summation on the right-hand side of (25) into conjugacy classes, ie

$$H^G(X) = \bigoplus_{(g)\in G_*} \Big(\bigoplus_{[h]\in G/Z_G(g)} h_*(H(X^g)^{Z_G(g)})\Big) \cong \bigoplus_{(g)\in G_*} H(X^g)^{Z_G(g)}, \tag{26}$$

where G_* denotes the set of all conjugacy classes of G, and $Z_G(g)$ is the centralizer of $g\in G$.

Remark 2.3 If X is smooth, then also all fixed-point sets X^g are smooth, so the classical Poincaré duality isomorphisms (24) induce similar duality isomorphisms

$$H^G_*(X) \cong H^*_G(X)$$

between the corresponding delocalized equivariant (co)homology theories.

Remark 2.4 If G acts *trivially* on X (eg X is a point), then

$$H^G(X) \cong H(X)\otimes C(G), \tag{27}$$

where $C(G)$ denotes the free abelian group of $\mathbb{Z}$–valued class functions on G (ie functions which are constant on the conjugacy classes of G).

Remark 2.5 If G is an *abelian* group, then

$$H^G(X) = \bigoplus_{g\in G} H(X^g)^G. \tag{28}$$

Let us next describe two functors which will be used later.

Definition 2.6 (restriction functor) Let X be a G–space, as before. For a subgroup K of G, the *restriction functor* Res^G_K from G to K is the group homomorphism

$$\mathrm{Res}^G_K\colon H^G(X)\to H^K(X)$$

induced by restricting to the G–invariant part the projection

$$\bigoplus_{g\in G} H(X^g) \to \bigoplus_{g\in K} H(X^g).$$

Clearly, Res_K^G is transitive with respect to subgroups, with Res_G^G the identity homomorphism. In terms of fixed-point sets of conjugacy classes, ie with respect to the isomorphisms

$$H^G(X) \cong \bigoplus_{(g)\in G_*} H(X^g)^{Z_G(g)}, \quad H^K(X) \cong \bigoplus_{(k)\in K_*} H(X^k)^{Z_K(k)},$$

the restriction factor can be described explicitly as follows (compare with [43, page 4]): If an element $g \in G$ is not conjugate by elements in G to any element in K, then $\mathrm{Res}_K^G|_{H(X^g)^{Z_G(g)}} = 0$. Otherwise, assume that g is conjugate by elements in G to $k_1, \dots, k_s \in K$ which have mutually different conjugacy classes in K; then $H(X^g)^{Z_G(g)} \cong H(X^{k_i})^{Z_G(k_i)}$ for $i = 1, \dots, s$, and $\mathrm{Res}_K^G|_{H(X^g)^{Z_G(g)}}$ is given by the direct sum of inclusions $H(X^{k_i})^{Z_G(k_i)} \hookrightarrow H(X^{k_i})^{Z_K(k_i)}$.

The following induction functor will be used in Section 4 in the definition of Pontrjagin-type products.

Definition 2.7 (induction functor) For a G–space X as before and K a subgroup of G, the induction Ind_K^G from K to G is the group homomorphism (compare with [34, page 9])

$$\mathrm{Ind}_K^G = \sum_{[g]\in G/K} g_*(-)\colon H^K(X) \to H^G(X), \tag{29}$$

where the summation is over K–cosets of G. In particular, on a G–invariant class (ie in the image of the restriction functor Res_K^G) this induction map is just multiplication by the index $[G:K]$ of K in G. Note that Ind_K^G is transitive for subgroups of G, with Ind_G^G the identity homomorphism. In terms of fixed-point sets of conjugacy classes, this induction is given as follows (compare with [43, page 4]): for any conjugacy class (k) in K which intersects the conjugacy class (g) in G, we have

$$\mathrm{Ind}_K^G = \sum_{[h]\in Z_G(k)/Z_K(k)} h_*(-)\colon H(X^k)^{Z_K(k)} \to H(X^k)^{Z_G(k)} \cong H(X^g)^{Z_G(g)}, \tag{30}$$

so on a G–invariant class this is just multiplication by the index $[Z_G(k) : Z_K(k)]$.

Remark 2.8 In terms of the above induction functors, the identification (26) is given by

$$\bigoplus_{(g)\in G_*} \mathrm{Ind}^G_{Z_G(g)} \colon \bigoplus_{(g)\in G_*} H(X^g)^{Z_G(g)} \to H^G(X),$$

where $\mathrm{Ind}^G_{Z_G(g)} \colon H(X^g)^{Z_G(g)} \to H^G(X)$ is the restriction of $\mathrm{Ind}^G_{Z_G(g)}$ to the direct summand

$$H(X^g)^{Z_G(g)} \hookrightarrow H^{Z_G(g)}(X)$$

coming from the $Z_G(g)$–equivariant direct summand $H(X^g) \subset \bigoplus_{h\in Z_G(g)} H_*(X^h)$.

Remark 2.9 Over a point space, the above functors reduce in many cases to the classical restriction and induction functors from the representation theory of finite groups.

2.1 Compatibilities with cross-product

Assume G acts on X, with $g\in G$ and $K\subset G$ a subgroup, and similarly for G' acting on X', with $g'\in G'$ and $K'\subset G'$ a subgroup. Then $(X\times X')^{g\times g'} = X^g \times X'^{g'}$, $Z_{G\times G'}(g\times g') = Z_G(g)\times Z_{G'}(g')$, as well as $G\times G'/K\times K' = G/K\times G'/K'$, and similarly for the quotient of centralizers as above.

Then all products $\boxtimes\colon H(X^g)\times H(X'^{g'}) \to H(X^g\times X'^{g'})$ induce, by the functoriality of $\boxtimes$, a corresponding commutative and associative cross-product

$$\boxtimes\colon H^G(X)\times H^{G'}(X') \to H^{G\times G'}(X\times X')$$

(with unit $1_{\mathrm{pt}} \in H^{\{\mathrm{id}\}}(\mathrm{pt})$, with $\{\mathrm{id}\}$ denoting the trivial group). Moreover, this product is compatible with the restriction and induction functors, ie

$$\mathrm{Ind}^{G\times G'}_{K\times K'}(-\boxtimes -) = \mathrm{Ind}^G_K(-)\boxtimes \mathrm{Ind}^{G'}_{K'}(-) \tag{31}$$

and

$$\mathrm{Res}^{G\times G'}_{K\times K'}(-\boxtimes -) = \mathrm{Res}^G_K(-)\boxtimes \mathrm{Res}^{G'}_{K'}(-). \tag{32}$$

Finally, the above facts about cross-product and restriction functors can be used to define a pairing

$$C(G)\times H^G(X) \dot{\to} H^G(X)$$

by

$$H^G(\mathrm{pt})\times H^G(X) \xrightarrow{\boxtimes} H^{G\times G}(\mathrm{pt}\times X) \xrightarrow{\mathrm{Res}} H^G(X),$$

with $\mathrm{pt} \times X \cong X$ and Res denoting the restriction functor for the diagonal subgroup $G \hookrightarrow G \times G$.

Remark 2.10 The distinguished unit element $\mathrm{id} \in G$ gives the direct summand

$$H^G_{\mathrm{id}}(X) := H(X)^G \subset H^G(X),$$

ie the G–invariant subgroup $H(X)^G$ of $H(X)$. This direct summand is compatible with restriction, induction and induced cross-products. If the functor H is also covariantly functorial for closed embeddings, we get a pushforward for the closed fixed point inclusions $i_g\colon X^g \hookrightarrow X$, ie

$$i_{g*}\colon H(X^g) \to H(X),$$

and a group homomorphism

$$\mathrm{sum}_G := \sum_g i_{g*}\colon H^G(X) \to H^G_{\mathrm{id}}(X) = H(X)^G \subset H(X).$$

Note that this homomorphism commutes with induction and cross-products.

Remark 2.11 If X is in addition smooth, the induction and restriction functors, as well as their compatibilities with cross-products, are also compatible with Poincaré duality for (co)homology as in Remark 2.3.

3 Generating series for symmetric group actions on external products

In this section, we describe a very general generating series formula for symmetric group actions on external products, which should be regarded as a far-reaching generalization of a well-known identity of symmetric functions. In Section 3.1, we give applications of this abstract generating series formula in the context of orbifold cohomology and localized K–theory.

Let Z be a quasiprojective variety, with the symmetric group Σ_n acting on the cartesian product Z^n of $n \geq 0$ copies of Z by the natural permutation action. For our generating series formula, it is important to look at all groups $H^{\Sigma_n}(Z^n)$ simultaneously. Let

$$\mathbb{H}^{\Sigma}(Z) := \bigoplus_{n \geq 0} H^{\Sigma_n}(Z^n) \cdot t^n \tag{33}$$

be the commutative graded $\mathbb{Z}$–algebra (with unit) with product

$$\odot := \mathrm{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}}(\cdot \boxtimes \cdot)$$

induced from the external product by induction. Here, $\bigoplus_{n\geq 0} H^{\Sigma_n}(Z^n)$ becomes a commutative graded ring with product $\odot$, and we view the completion $\mathbb{H}^{\Sigma}(Z)$ as a subring of the formal power series ring $\bigoplus_{n\geq 0} H^{\Sigma_n}(Z^n)[\![t]\!]$.

The algebra $\mathbb{H}^{\Sigma}(Z)$ is, in addition, endowed with *creation operators*

$$\mathfrak{a}_r \colon H(Z) \to H^{\Sigma_r}(Z^r)$$

for $r \geq 1$, which allow us to transport elements from $H(Z)$ to the delocalized groups $H^{\Sigma_r}(Z^r)$. These are defined as follows: if $\sigma_r = (r)$ is an r–cycle in Σ_r, then $\mathfrak{a}_r$ is the composition

$$\mathfrak{a}_r \colon H(Z) \xrightarrow{\cdot r} H(Z) \cong H((Z^r)^{\sigma_r})^{Z_{\Sigma_r}(\sigma_r)} \hookrightarrow H^{\Sigma_r}(Z^r),$$

where $\langle \sigma_r \rangle = Z_{\Sigma_r}(\sigma_r)$ acts trivially on $(Z^r)^{\sigma_r}$, and therefore also on $H((Z^r)^{\sigma_r})$. The role of multiplication by r in the definition of creation operator will become clear later on, eg in the proof of Theorem 3.1 below. The creation operator $\mathfrak{a}_r$ can be rewritten as

$$\mathfrak{a}_r := r \cdot \mathrm{Ind}_{\langle \sigma_r \rangle}^{\Sigma_r} \circ i_r,$$

with

$$i_r \colon H(Z) \simeq H((Z^r)^{\sigma_r})^{\langle \sigma_r \rangle} \subset H^{<\sigma_r>}(Z^r).$$

Here, the last inclusion is just a direct summand, because $\langle \sigma_r \rangle$ is abelian. In the following we omit to mention i_r explicitly.

Let $\sigma \in \Sigma_n$ have cycle partition $\lambda = (k_1, k_2, \dots)$, ie k_r is the number of length-r cycles in σ and $n = \sum_r r \cdot k_r$. Then

$$(Z^n)^{\sigma} \simeq \prod_r ((Z^r)^{\sigma_r})^{k_r} \simeq \prod_r \Delta_r(Z)^{k_r} \simeq Z^{k_1+k_2+\cdots},$$

where σ_r denotes, as above, a cycle of length r in Σ_n, and $\Delta_r(Z)$ is the diagonal in Z^r, ie the image of the diagonal map $\Delta_r \colon Z \to Z^r$.

Let us now choose a sequence $\underline{b} = (b_1, b_2, \dots)$ of elements $b_r \in H(Z)$ for $r \geq 1$, and associate to a conjugacy class represented by $\sigma \in \Sigma_n$ of type $(k_1, k_2, \dots)$ the element $\underline{b}^{(\sigma)} \in H^{\Sigma_n}(X^n)$ corresponding to

$$\boxtimes_r (b_r)^{\boxtimes k_r} \in H\Bigl(\prod_r Z^{k_r}\Bigr) \simeq H((Z^n)^{\sigma}),$$

as will be explained below. Recall that $Z_{\Sigma_n}(\sigma)$ is a product over r of semidirect products of Σ_{k_r} with $\langle\sigma_r\rangle^{k_r}$, that is,

$$Z_{\Sigma_n}(\sigma) \cong \prod_r \Sigma_{k_r} \ltimes \mathbb{Z}_r^{k_r} \tag{34}$$

(with σ_r denoting as before an r–cycle). The group $\mathbb{Z}_r^{k_r} \cong \langle\sigma_r\rangle^{k_r}$ acts trivially on Z^{k_r}, whereas Σ_{k_r} permutes the corresponding Z–factors of Z^{k_r} (compare [43, page 8]). By commutativity and associativity of the cross-product $\boxtimes$, it follows that $\boxtimes_r (b_r)^{\boxtimes k_r}$ is invariant under $Z_{\Sigma_n}(\sigma)$, so it indeed defines an element

$$\underline{b}^{(\sigma)} = \mathrm{Ind}_{Z_{\Sigma_n}(\sigma)}^{\Sigma_n}\big(\boxtimes_r (b_r)^{\boxtimes k_r}\big) \in H^{\Sigma_n}(Z^n), \tag{35}$$

with induction defined as in Remark 2.8. Moreover, for $\sigma \in \Sigma_n$ and $\sigma' \in \Sigma_m$, we have

$$\underline{b}^{(\sigma)} \odot \underline{b}^{(\sigma')} = \underline{b}^{(\sigma\times\sigma')} \in H^{\Sigma_{n+m}}(Z^{n+m}). \tag{36}$$

In what follows, we assume that the functor H takes values in R–modules, with R a commutative $\mathbb{Q}$–algebra (otherwise, work with $\mathbb{H}^\Sigma(Z)\otimes R$). It follows that $\mathbb{H}^\Sigma(Z)$ is also a commutative graded $\mathbb{Q}$–algebra. Note that one can also switch between covariant and contravariant notions, eg between homology and cohomology by Poincaré duality, if X is smooth.

The main result of this section is the following generating series formula:

Theorem 3.1 *With the above notations, the following generating series formula holds in the $\mathbb{Q}$–algebra $\mathbb{H}^\Sigma(Z)$:*

$$\sum_{n\geq 0}\Big(\sum_{(\sigma)\in(\Sigma_n)_*} \underline{b}^{(\sigma)}\Big)\cdot t^n = \exp\Big(\sum_{r\geq 1} \mathfrak{a}_r(b_r)\cdot \frac{t^r}{r}\Big), \tag{37}$$

where $(\Sigma_n)_*$ denotes the set of conjugacy classes of Σ_n.

Proof We have the following string of equalities in the $\mathbb{Q}$–algebra $(\mathbb{H}^\Sigma(Z),\odot)$:

$$\begin{aligned}\exp\Big(\sum_{r=1}^{\infty} x_r \frac{t^r}{r}\Big) &= \prod_{r=1}^{\infty} \exp\Big(x_r \frac{t^r}{r}\Big)\\ &= \prod_{r=1}^{\infty}\sum_{k_r=0}^{\infty}\Big(x_r\frac{t^r}{r}\Big)^{k_r}\frac{1}{k_r!}\\ &= \sum_{N\geq 0}\sum_{k_1,\dots,k_N}\frac{x_1^{k_1}\cdots x_N^{k_N}}{k_1!\cdots k_N!}\prod_{r=1}^{N}\Big(\frac{t^r}{r}\Big)^{k_r}\end{aligned} \tag{38}$$

$$
\begin{aligned}
&= \sum_{N \geq 0} \sum_{k_1,\dots,k_N} \frac{x_1^{k_1} \cdots x_N^{k_N}}{k_1! \cdots k_N!} \frac{t^{k_1+2k_2+\cdots+Nk_N}}{1^{k_1} \cdots N^{k_N}} \\
&= \sum_{m=0}^{\infty} t^m \sum_{k_1+2k_2+\cdots+Nk_N=m} \frac{x_1^{k_1} \cdots x_N^{k_N}}{k_1! \cdots k_N! \, 1^{k_1} \cdots N^{k_N}} \\
&= \sum_{m=0}^{\infty} t^m \sum_{k_1+2k_2+\cdots+Nk_N=m} \prod_{r=1}^{N} \frac{x_r^{k_r}}{k_r! \, r^{k_r}}.
\end{aligned}
$$

Note that the sum over $k_1 + 2k_2 + \cdots + Nk_N = m$ corresponds to a summation over the cycle classes (σ) in Σ_m given by $\prod_r {\sigma_r}^{k_r}$ for $\sigma_r = (r)$ an r–cycle in Σ_r. In our case, we take

$$
x_r = \mathfrak{a}_r(b_r) = \mathrm{Ind}_{\langle \sigma_r \rangle}^{\Sigma_r}(r \cdot b_r).
$$

All products (and powers) above are with respect to the multiplication $\odot$ in $\mathbb{H}^{\Sigma}(Z)$, which is defined via cross-product and induction. In particular,

$$
\prod_{r=1}^{N} \frac{(x_r)^{k_r}}{r^{k_r}} = \mathrm{Ind}_{\prod_r (\Sigma_r)^{k_r}}^{\Sigma_m} \left(\boxtimes_{r=1}^{N} \frac{(x_r)^{\boxtimes k_r}}{r^{k_r}} \right).
$$

Moreover, by using the compatibility of induction with $\boxtimes$, $\mathbb{Z}$–linearity and transitivity, we have

$$
\begin{aligned}
\mathrm{Ind}_{\prod_r (\Sigma_r)^{k_r}}^{\Sigma_m} \left(\boxtimes_{r=1}^{N} \frac{(x_r)^{\boxtimes k_r}}{r^{k_r}} \right) &= \mathrm{Ind}_{\prod_r \langle \sigma_r \rangle^{k_r}}^{\Sigma_m} \left(\boxtimes_r (b_r)^{\boxtimes k_r} \right) \\
&= \mathrm{Ind}_{Z_{\Sigma_m}(\sigma)}^{\Sigma_m} \circ \mathrm{Ind}_{\prod_r \langle \sigma_r \rangle^{k_r}}^{Z_{\Sigma_m}(\sigma)} \left(\boxtimes_r (b_r)^{\boxtimes k_r} \right),
\end{aligned}
$$

where, as before, σ is a representative of the cycle type $(k_1, k_2, \dots)$. But, as already mentioned, $Z_{\Sigma_m(\sigma)}$ acts trivially on $\boxtimes_r (b_r)^{\boxtimes k_r}$, so $\mathrm{Ind}_{\prod_r \langle \sigma_r \rangle^{k_r}}^{Z_{\Sigma_m}(\sigma)}$ is just multiplication by the index $\left[Z_{\Sigma_m(\sigma)} : \prod_r \langle \sigma_r \rangle^{k_r} \right] = \prod_r k_r!$.

Altogether, we get

$$
\prod_{r=1}^{N} \frac{(x_r)^{k_r}}{k_r! \, r^{k_r}} = \underline{b}^{(\sigma)},
$$

which finishes the proof. □

3.1 Examples

Let us now explain some special cases of Theorem 3.1 in the cohomological language, which in some situations are already available in the literature. Our main applications, to equivariant characteristic classes for singular spaces, will be given later on, in Section 5.3, after we develop the necessary background.

3.1.1 Orbifold cohomology Here we work with $H(X) := H^{\mathrm{ev}}(X) \otimes \mathbb{Q}$, the (even-degree) rational cohomology functor. For X smooth, our notion of $H^G(X)$ corresponds to the even-degree orbifold cohomology $H^{2*}_{\mathrm{orb}}(X/G)$, as used for example in [34].

For Z a quasiprojective complex variety and for a given $\gamma \in H(Z)$, let $b_r := \gamma$ for all $r \geq 1$. Then $\underline{b}^{(\sigma)}$ corresponds to $\gamma^{\boxtimes \ell(\sigma)} \in H((Z^n)^\sigma)$ for $\sigma \in \Sigma_n$ of cycle type $(k_1, k_2, \dots)$, and $\ell(\sigma) := \sum_r k_r$ the length of the partition associated to σ. Following [34], we set

$$\eta_n(\gamma) := \sum_{(\sigma) \in (\Sigma_n)_*} \mathrm{Ind}^{\Sigma_n}_{Z_{\Sigma_n(\sigma)}}(\gamma^{\boxtimes \ell(\sigma)}) = \sum_{(\sigma) \in (\Sigma_n)_*} \underline{b}^{(\sigma)}. \tag{39}$$

Then our formula (37) specializes to the following result:

$$\sum_{n \geq 0} \eta_n(\gamma) \cdot t^n = \exp\left(\sum_{r \geq 1} \mathfrak{a}_r(\gamma) \cdot \frac{t^r}{r} \right). \tag{40}$$

For X smooth, this fits with the formula stated after Definition 3.2 in [34]. However, our proof is purely formal, so it applies to any topological space, as well as to algebraic varieties with rational Chow groups for H.

3.1.2 Localized K–theory Here we work with $H(X) := K^0(X) \otimes \mathbb{C}$, the complexified Grothendieck group of algebraic vector bundles on X. For a finite group G acting algebraically on a quasiprojective complex variety X, we define a localization map of Lefschetz type,

$$L^G\colon K^0_G(X) \to H^G(X),$$

on the Grothendieck group of G–equivariant algebraic vector bundles as a direct sum of transformations

$$\begin{aligned} L(g)\colon K^0_G(X) \to K^0_{\langle g \rangle}(X^g) \simeq K^0(X^g) \otimes \mathit{Rep}_{\mathbb{C}}(\langle g \rangle) \to K^0(X^g) \otimes \mathbb{C}, \\ [W] \mapsto [W|_{X^g}] \mapsto \mathrm{trace}_g([W|_{X^g}]), \end{aligned}$$

where the last map is induced by taking the trace against $g \in G$. Here, $\mathit{Rep}_{\mathbb{C}}(\langle g \rangle)$ is the Grothendieck group of complex representations of the group $\langle g \rangle$, and the isomorphism in the above definition holds since $\langle g \rangle$ acts trivially on the fixed-point set X^g (eg this fact follows from [13, (1.3.4)]; compare also [17]). Note that, by construction, L^G commutes with cross-products as in Section 2.1

For Z a quasiprojective complex variety and an algebraic vector bundle V on Z, we get the Σ_n–equivariant vector bundle $V^{\boxtimes n}$ on Z^n. Let $\sigma \in \Sigma_n$ be of cycle type $(k_1, k_2, \dots)$. Then, by the multiplicativity of $L(\sigma)$, we get

$$L(\sigma) = \boxtimes_r L(\sigma_r)^{\boxtimes k_r}.$$

So it suffices to understand the transformations $L(\sigma_r)$ for all r–cycles, with $r \geq 1$.

For $\sigma_n = (n)$ an n–cycle, we have that

$$L(\sigma_n)[V^{\boxtimes n}] = \psi_n(V) \in K^0(Z) \otimes \mathbb{Q} \tag{41}$$

under the identification $(Z^n)^{\sigma_n} = \Delta_n(Z) \simeq Z$, with

$$\Delta_n^*(V^{\boxtimes n}) = V^{\otimes n}.$$

Here ψ_n denotes the n^{th} Adams operation defined by Atiyah [1] in the topological context and eg Nori [31, Lemma 3.2] in the algebraic geometric context. Note that we can work here with rational coefficients, since characters of symmetric groups are integer-valued.

If we choose $b_r := \psi_r(V)$ for all $r \geq 1$, our main formula (37) specializes to the generating series identity

$$\sum_{n \geq 0} L^{\Sigma_n}(V^{\boxtimes n}) \cdot t^n = \exp\biggl(\sum_{r \geq 1} \mathfrak{a}_r(\psi_r(V)) \cdot \frac{t^r}{r} \biggr), \tag{42}$$

since, by *multiplicativity* and *conjugacy-invariance* of $L(\sigma)$, we have that

$$L^{\Sigma_n}(V^{\boxtimes n}) = \sum_{(\sigma) \in (\Sigma_n)_*} \operatorname{Ind}_{Z_{\Sigma_n}(\sigma)}^{\Sigma_n} \bigl(\boxtimes_r L(\sigma_r)^{\boxtimes k_r} \bigr) = \sum_{(\sigma) \in (\Sigma_n)_*} \underline{b}^{(\sigma)}.$$

The same proof applies in the topological context, for topological K–theory, in which case we obtain a special case (for G the identity group) of Proposition 4 of [40]. Note that [40] uses the identification $L^G \otimes \mathbb{C}$: $K_G^0(X) \otimes \mathbb{C} \simeq H^G(X)$.

Similarly, one can work with algebraic varieties over any base field of characteristic zero, with $L(g)$ the corresponding Lefschetz transformation of Baum, Fulton and Quart [6].

3.1.3 Localized Grothendieck groups of constructible sheaves Here we work with

$$H(X) := K_0(Constr(X)) \otimes \mathbb{C},$$

the complexified Grothendieck group of (algebraically) constructible sheaves of complex vector spaces. For a finite group G acting algebraically on a quasiprojective complex variety X, we define a localization map of Lefschetz type

$$L^G\colon K_0^G(Constr(X)) \to H^G(X)$$

on the Grothendieck group of G–equivariant (algebraically) constructible sheaves of complex vector spaces as a direct sum of transformations

$$\begin{aligned} L(g)\colon K_0^G(Constr(X)) \to K_0^{\langle g\rangle}(Constr(X^g)) \simeq K_0(Constr(X^g)) \otimes Rep_{\mathbb{C}}(\langle g\rangle) \\ \to K_0(Constr(X^g)) \otimes \mathbb{C}, \\ [\mathcal{F}] \mapsto [\mathcal{F}|_{X^g}] \mapsto \mathrm{trace}_g([\mathcal{F}|_{X^g}]), \end{aligned}$$

where the last map is induced, as before, by taking the trace against $g \in G$. The isomorphism in the above definition holds since $\langle g\rangle$ acts trivially on the fixed-point set X^g and $Constr(X^g)$ is an abelian $\mathbb{C}$–linear category (eg see as before [13, (1.3.4)]; compare also [17]). Note that, by construction, L^G commutes as before with cross-products, as in Section 2.1

For Z a quasiprojective complex variety and an (algebraically) constructible sheaf $\mathcal{F}$ on Z, we get the Σ_n–equivariant (algebraically) constructible sheaf $\mathcal{F}^{\boxtimes n}$ on Z^n. For $\sigma_n = (n)$ an n–cycle, we have that

$$L(\sigma_n)[\mathcal{F}^{\boxtimes n}] = \psi_n(V) \in K_0(Constr(Z)) \otimes \mathbb{Q} \tag{43}$$

under the identification $(Z^n)^{\sigma_n} = \Delta_n(Z) \simeq Z$, with

$$\Delta_n^*(\mathcal{F}^{\boxtimes n}) = \mathcal{F}^{\otimes n}.$$

Here, ψ_n denotes the n^{th} Adams operation corresponding to the pre-lambda ring structure on $K_0(Constr(Z))$ induced from the symmetric monoidal tensor product $\otimes$ of constructible sheaves, as in [27, Lemma 2.1]. As before, we can work here with rational coefficients.

If we choose $b_r := \psi_r(\mathcal{F})$ for all $r \geq 1$, our main formula (37) specializes as above to the generating series identity

$$\sum_{n\geq 0} L^{\Sigma_n}(\mathcal{F}^{\boxtimes n}) \cdot t^n = \exp\left(\sum_{r\geq 1} \mathfrak{a}_r(\psi_r(\mathcal{F})) \cdot \frac{t^r}{r}\right). \tag{44}$$

3.1.4 Frobenius character Specializing to a point space X, the above localized theories for vector bundles and constructible sheaves reduce to the classical character theory of a finite group G,

$$\mathrm{tr}_G\colon Rep_{\mathbb{C}}(G) \to C(G)\otimes \mathbb{C}, \quad [V]\mapsto \{\mathrm{trace}_g(V), g\in G\},$$

with $Rep_{\mathbb{C}}(G)$ the Grothendieck group of complex representations of G and $C(G)$ the free abelian group of $\mathbb{Z}$–valued class functions on G. The trace, trace_g, is of course multiplicative and conjugacy-invariant.

For symmetric groups, we can work again with rational coefficients, and get an algebra homomorphism

$$\mathrm{tr}_\Sigma\colon Rep_{\mathbb{C}}(\Sigma) := \bigoplus_n Rep_{\mathbb{C}}(\Sigma_n) \to C(\Sigma)\otimes\mathbb{Q} := \bigoplus_n C(\Sigma_n)\otimes\mathbb{Q}$$

with respect to the classical induction product: $\odot := \mathrm{Ind}_{\Sigma_n\times\Sigma_m}^{\Sigma_{n+m}}(\cdot\boxtimes\cdot)$ for representations and characters; see eg [24, Chapter I, Section 7]. This homomorphism can be composed with the *Frobenius character*

$$\mathrm{ch}_F\colon C(\Sigma)\otimes\mathbb{Q} := \bigoplus_n C(\Sigma_n)\otimes\mathbb{Q} \xrightarrow{\simeq} \mathbb{Q}[p_i \mid i\geq 1] =: \Lambda\otimes\mathbb{Q}$$

to the graded ring of $\mathbb{Q}$–valued symmetric functions in infinitely many variables x_m for $m\in\mathbb{N}$, with $p_i := \sum_m x_m^i$ the i^{th} power sum function. On $C(\Sigma_n)\otimes\mathbb{Q}$, ch_F is defined by

$$\mathrm{ch}_F(f) := \frac{1}{n!}\sum_{\sigma\in\Sigma_n} f(\sigma)\psi(\sigma), \tag{45}$$

with

$$\psi(\sigma) := \prod_r p_r^{k_r}$$

for σ of cycle type $(k_1, k_2, \dots)$; eg see [24, Chapter I, Section 7]. For example, if f is the indicator function of the conjugacy class of the n–cycle σ_n in Σ_n, then $\mathrm{ch}_F(f) = \frac{1}{n}p_n$ since $n = |Z_{\Sigma_n}(\sigma_n)|$. In particular, the creation operator $\mathfrak{a}_r\colon \mathbb{Q}\to\mathbb{Q}[p_i\mid i\geq 1] = \Lambda\otimes\mathbb{Q}$ is (up to the Frobenius isomorphism) given by multiplication with p_r, which also motivates the use of multiplication by r in the definition of our creation operator in Section 3.

If we choose $b_r := 1\in\mathbb{Q}$ for all $r\geq 1$, our main formula (37) specializes to the well-known identity of symmetric functions (eg see the proof of [24, (2.14)])

$$H(t) := \sum_{n\geq 0} h_n t^n = \exp\Bigl(\sum_{r\geq 1} p_r\cdot\frac{t^r}{r}\Bigr), \tag{46}$$

with $h_n = \mathrm{ch}_F(1_{\Sigma_n})$ the n^{th} complete symmetric function (see [24, page 113]) and $1_{\Sigma_n} := \mathrm{tr}_{\Sigma_n}(\mathrm{triv}_n)$ the identity character of the trivial representation triv_n of Σ_n.

4 (Equivariant) Pontrjagin rings for symmetric products

Let Z be a quasiprojective variety, and denote by $Z^{(n)}$ its n^{th} symmetric product, ie the quotient of the product Z^n of n copies of Z by the natural action of the symmetric group on n elements, Σ_n. Let $\pi_n\colon Z^n \to Z^{(n)}$ denote the natural projection map.

In this section, the functor H from Section 2 is required in addition to be covariant (at least) for *finite maps* such as π_n or the closed embedding $i_\sigma\colon (Z^n)^\sigma \hookrightarrow Z^n$ for $\sigma \in \Sigma_n$. We will carry over the assumption that H takes values in R–modules, with R a commutative $\mathbb{Q}$–algebra.

Besides $\mathbb{H}^\Sigma(Z) := \bigoplus_{n\geq 0} H^{\Sigma_n}(Z^n)\cdot t^n$, here we consider other structures of commutative graded $\mathbb{Q}$–algebra with units, defined in terms of symmetric or external products of Z:

(a) On

$$\mathbb{PH}(Z) := \bigoplus_{n\geq 0} H(Z^{(n)})\cdot t^n = \prod_{n\geq 0} H(Z^{(n)})$$

there is the Pontrjagin ring structure, with multiplication $\odot$ induced from the maps

$$Z^{(n)} \times Z^{(m)} \to Z^{(m+n)};$$

see [12, Definition 1.1] for more details. Here, $\bigoplus_{n\geq 0} H(Z^{(n)})$ becomes a commutative graded ring with product $\odot$, and we view the completion $\mathbb{PH}(Z)$ as a subring of the formal power series ring $\bigoplus_{n\geq 0} H(Z^{(n)})[\![t]\!]$.

(b) On

$$\mathbb{PH}^\Sigma(Z) := \bigoplus_{n\geq 0} H^{\Sigma_n}(Z^{(n)})\cdot t^n \cong \bigoplus_{n\geq 0} (H(Z^{(n)}) \otimes C(\Sigma_n) \otimes \mathbb{Q})\cdot t^n \hookrightarrow \mathbb{PH}(Z) \otimes (C(\Sigma) \otimes \mathbb{Q})$$

there is a product induced from that of the Pontrjagin product in the H–factor and the induction product for class functions. Via the Frobenius character identification

$$\mathrm{ch}_F\colon C(\Sigma) \otimes \mathbb{Q} \simeq \mathbb{Q}[p_i \mid i \geq 1],$$

we can also view $\mathbb{P}\mathbb{H}^{\Sigma}(Z)$ as a graded subalgebra of $\mathbb{P}\mathbb{H}(Z) \otimes \mathbb{Q}[p_i \mid i \geq 1]$, and with i^{th} power sum p_i regarded as a degree-i variable.

(c) By Remark 2.10, the direct summand

$$\mathbb{H}^{\Sigma}_{\mathrm{id}}(Z) := \bigoplus_{n \geq 0} H^{\Sigma_n}_{\mathrm{id}}(Z^n) \cdot t^n \subset \mathbb{H}^{\Sigma}(Z)$$

corresponding to the identity component is a subring, so that

$$\bigoplus_{n \geq 0} \mathrm{sum}_{\Sigma_n} \colon \mathbb{H}^{\Sigma}(Z) \to \mathbb{H}^{\Sigma}_{\mathrm{id}}(Z)$$

is a ring homomorphism. With respect to the Frobenius homomorphism, it is more natural to use the averaging homomorphisms $\mathrm{av}_n := \frac{1}{n!} \mathrm{sum}_{\Sigma_n} \colon H^{\Sigma_n}(Z^n) \to H^{\Sigma_n}_{\mathrm{id}}(Z^n)$. Then the graded group homomorphism

$$\mathrm{av} := \bigoplus \mathrm{av}_n \colon \mathbb{H}^{\Sigma}(Z) \to \mathbb{H}^{\Sigma}_{\mathrm{id}}(Z)$$

becomes a graded algebra homomorphism if we introduce on $\mathbb{H}^{\Sigma}_{\mathrm{id}}(Z)$ the *twisted product*

$$\tilde{\odot} := \frac{n!m!}{(n+m)!} \odot \colon H^{\Sigma_n}_{\mathrm{id}}(Z^n) \times H^{\Sigma_m}_{\mathrm{id}}(Z^m) \to H^{\Sigma_{n+m}}_{\mathrm{id}}(Z^{n+m}).$$

With this twisted product, we also have a Frobenius-type ring homomorphism

$$\mathrm{av}_F \colon \mathbb{H}^{\Sigma}(Z) \to \mathbb{H}^{\Sigma}_{\mathrm{id}}(Z) \otimes \mathbb{Q}[p_i \mid i \geq 1]$$

given by

$$\frac{1}{n!} \sum_{\sigma \in \Sigma_n} i_{\sigma *} \cdot \psi(\sigma) \colon H^{\Sigma_n}(Z^n) \to H^{\Sigma_n}_{\mathrm{id}}(Z^n) \otimes \mathbb{Q}[p_i \mid i \geq 1],$$

with $\psi(\sigma)$ as in the Frobenius homomorphism ch_F of (45).

These structures are related by homomorphisms of commutative graded $\mathbb{Q}$–algebras, fitting into the commutative diagram

$$\begin{tikzcd}
\mathbb{H}^{\Sigma}(Z) := \bigoplus_{n \geq 0} H^{\Sigma_n}(Z^n) \cdot t^n \arrow[r, "\mathrm{av}_F"] \arrow[d, "\pi_* = \bigoplus_n \pi_{n*}"'] & \mathbb{H}^{\Sigma}_{\mathrm{id}}(Z) \otimes \mathbb{Q}[p_i \mid i \geq 1] \arrow[d, "\pi_* \otimes \mathrm{id}"] \\
\mathbb{P}\mathbb{H}^{\Sigma}(Z) := \bigoplus_{n \geq 0} H^{\Sigma_n}(Z^{(n)}) \cdot t^n \arrow[r] \arrow[d, "\bigoplus_n (1/|\Sigma_n|) \sum_\sigma \mathrm{ev}_\sigma"'] & \mathbb{P}\mathbb{H}(Z) \otimes \mathbb{Q}[p_i \mid i \geq 1] \arrow[d, "p_i = 1"] \\
\mathbb{P}\mathbb{H}(Z) := \bigoplus_{n \geq 0} H(Z^{(n)}) \cdot t^n \arrow[r, "="] & \mathbb{P}\mathbb{H}(Z)
\end{tikzcd} \tag{47}$$

with $\mathrm{ev}_\sigma \colon C(\Sigma_n) \otimes \mathbb{Q} \to \mathbb{Q}$ the evaluation map at $\sigma \in \Sigma_n$.

Let $d_r := \pi_r \circ \Delta_r$ be the composition $d_r\colon Z \to Z^r \to Z^{(r)}$ of the natural projection $\pi_r\colon Z^r \to Z^{(r)}$ with the diagonal embedding $\Delta_r\colon Z \to Z^r$. Then the creation operator $\mathfrak{a}_r$ satisfies the identities

$$\pi_{r*} \circ \mathfrak{a}_r = p_r \cdot d_{r*}, \quad \mathrm{av}_F \circ \mathfrak{a}_r = p_r \cdot \Delta_{r*} \quad \text{and} \quad \mathrm{av} \circ \mathfrak{a}_r = \Delta_{r*}. \tag{48}$$

This generalizes the corresponding relation between $\mathfrak{a}_r$ and p_r discussed at the end of Section 3.1.4.

Remark 4.1 Under the assumptions from the beginning of this section, the commutative diagram (47) is functorial in Z for finite maps. If, moreover, the functor H and the cross-product are functorial for proper morphisms, then (47) is also functorial for such morphisms. In particular, for Z compact, we can push down our generating series formulae (such as (37)) to a point to obtain (equivariant) degree formulae. Finally, the diagram (47) is compatible with natural transformations of such functors.

4.1 Example: constructible functions and orbifold Chern classes

In this example, we explain how our main result of Theorem 3.1 can be used to reprove Ohmoto's generating series identities for canonical constructible functions [33, Proposition 3.9] and orbifold Chern classes [33, Theorem 1.1]. These are generalized class versions of the celebrated Hirzebruch–Höfer (or Atiyah–Segal) formula for the orbifold Euler characteristic of symmetric products. Recall that for a (compact) G–space X as before, the orbifold Euler characteristic $\chi(X,G)$ is defined as

$$\chi(X,G) = \frac{1}{|G|} \sum_{gh=hg} \chi(X^g \cap X^h) = \sum_{(g)\in G_*} \chi(X^g/Z_G(g)) = \chi(IX/G), \tag{49}$$

with $IX = \bigsqcup_{g\in G} X^g$ the inertia space as in the introduction.

Let H be the functor $F(-)$ of $\mathbb{Q}$–valued algebraically constructible functions, which is covariant for all morphisms, and with a compatible cross-product. Following Ohmoto's notations, for a fixed group A, let $j_r(A)$ be the number of index-r subgroups of A, which is assumed to be finite for all r. In the notation of Theorem 3.1, let

$$b_r := j_r(A) \cdot 1_Z \in F(Z).$$

Then

$$\underline{b}^{(\sigma)} = \mathrm{Ind}_{Z_{\Sigma_n}(\sigma)}^{\Sigma_n}\bigl(\boxtimes_r (j_r(A)\cdot 1_Z)^{\boxtimes k_r}\bigr) \in F^{\Sigma_n}(Z^n),$$

and the element

$$\mathbb{1}^{A}_{Z^n;\Sigma_n} := \sum_{(\sigma)\in(\Sigma_n)_*} \underline{b}^{(\sigma)} \in F^{\Sigma_n}(Z^n)$$

appearing on the left-hand side of (37) is a delocalized version of Ohmoto's canonical constructible function $\mathbb{1}^{(A)}_{Z^n;\Sigma_n}$ of [33, Definition 2.2], in the sense that

$$\mathrm{av}_n(\mathbb{1}^{A}_{Z^n;\Sigma_n}) = \mathbb{1}^{(A)}_{Z^n;\Sigma_n} \in F^{\Sigma_n}_{\mathrm{id}}(Z^n).$$

This identification follows from [33, Lemma 3.4]. (Note that Ohmoto's product in [33] corresponds to our twisted product $\widetilde{\odot}$.)

Let us illustrate what these distinguished constructible functions are in the cases $A = \mathbb{Z}$ and $A = \mathbb{Z}^2$ (for other examples, see [33]):

(a) If $A = \mathbb{Z}$, then

$$\mathbb{1}^{\mathbb{Z}}_{Z^n;\Sigma_n} = \bigoplus_{\sigma\in\Sigma_n} 1_{(Z^n)^\sigma} \in \Big(\bigoplus_{\sigma\in\Sigma_n} F((Z^n)^\sigma)\Big)^{\Sigma_n}$$

and

$$\mathbb{1}^{(\mathbb{Z})}_{Z^n;\Sigma_n} = \frac{1}{n!}\sum_{\sigma\in\Sigma_n} 1_{(Z^n)^\sigma} \in F(Z^n)^{\Sigma_n}.$$

(b) If $A = \mathbb{Z}^2$, then

$$\mathbb{1}^{(\mathbb{Z}^2)}_{Z^n;\Sigma_n} = \frac{1}{n!}\sum_{\sigma\sigma'=\sigma'\sigma} 1_{(Z^n)^\sigma\cap(Z^n)^{\sigma'}} \in F(Z^n)^{\Sigma_n},$$

as shown in [33]. On the other hand, the combinatorics used in [33] only applies to $F(Z^n)^{\Sigma_n}$, but not to the delocalized theory

$$\Big(\bigoplus_{\sigma\in\Sigma_n} F((Z^n)^\sigma)\Big)^{\Sigma_n}.$$

Nevertheless, the function $\mathbb{1}^{\mathbb{Z}^2}_{Z^n;\Sigma_n}$ is a canonical lift of $\mathbb{1}^{(\mathbb{Z}^2)}_{Z^n;\Sigma_n}$ with respect to the averaging av_n.

Then our main Theorem 3.1 yields the identity

$$\sum_{n\ge 0} \mathbb{1}^{A}_{Z^n;\Sigma_n}\cdot t^n = \exp\Big(\sum_{r\ge 1}\frac{j_r(A)}{r}t^r\cdot \mathfrak{a}_r(1_Z)\Big) \in \mathbb{F}^{\Sigma}(Z). \tag{50}$$

By applying to (50) the ring homomorphism av: $(\mathbb{F}^{\Sigma}(Z), \odot) \to (\mathbb{F}^{\Sigma}_{\mathrm{id}}(Z), \tilde{\odot})$, we recover, by (48), Ohmoto's generating series formula [33, Proposition 3.9]

(51) $$\sum_{n\geq 0} \mathbb{1}^{(A)}_{Z^n;\Sigma_n} \cdot t^n = \exp\biggl(\sum_{r\geq 1} \frac{j_r(A)}{r} t^r \cdot \Delta_{r*}(1_Z)\biggr).$$

Recall now that MacPherson's Chern class transformation (with rational coefficients) $c_*\colon F(-) \to H_*(-) := H^{\mathrm{BM}}_{\mathrm{ev}}(-) \otimes \mathbb{Q}$ commutes with proper pushforward and cross-products. Applying c_* to (50), we obtain a new generating series

(52) $$\sum_{n\geq 0} c_*(\mathbb{1}^{A}_{Z^n;\Sigma_n}) \cdot t^n = \exp\biggl(\sum_{r\geq 1} \frac{j_r(A)}{r} t^r \cdot \mathfrak{a}_r(c_*(1_Z))\biggr) \in \mathbb{H}^{\Sigma}(Z),$$

with $c_*(\mathbb{1}^{A}_{Z^n;\Sigma_n}) \in H^{\Sigma_n}(Z^n)$ a delocalized version of Ohmoto's orbifold Chern class $c_*(\mathbb{1}^{(A)}_{Z^n;\Sigma_n}) \in H^{\Sigma_n}_{\mathrm{id}}(Z^n)$. Ohmoto's formula [33, Theorem 1.1] for orbifold Chern classes of symmetric products is obtained by applying c_* to (51), or equivalently, by applying av to (52).

If Z is projective, by taking the degrees $\deg c_*(\mathbb{1}^{A}_{Z^n;\Sigma_n}) = \deg c_*(\mathbb{1}^{(A)}_{Z^n;\Sigma_n})$ of the above characteristic class formulae, one recovers generating series for orbifold-type Euler characteristics; see [33] for details and examples.

Finally, note that for $A = \mathbb{Z}$ (hence $j_r(A) = 1$ for all r), we recover a special case (for $\mathrm{cl}_* = c_*$ and $\mathcal{F} = \mathbb{Q}_Z$) of Theorem 1.4 from the introduction, via the identification

$$c^{\Sigma_n}_*(\mathbb{Q}^{\boxtimes n}_Z) = \bigoplus_{\sigma\in\Sigma_n} c_*(1_{(Z^n)^\sigma}) = c_*(\mathbb{1}^{\mathbb{Z}}_{Z^n;\Sigma_n}) \in \biggl(\bigoplus_{\sigma\in\Sigma_n} H_*((Z^n)^\sigma)\biggr)^{\Sigma_n}.$$

The corresponding degree formula for Z compact is the classical Euler characteristic formula

$$\chi(Z^n/\Sigma_n) = \frac{1}{n!} \sum_{\sigma\in\Sigma_n} \chi((Z^n)^\sigma).$$

Similarly, for $A = \mathbb{Z}^2$ and Z compact, one recovers at the degree level the orbifold Euler characteristic $\chi(Z^n, \Sigma_n)$ as in (49).

Remark 4.2 Instead of fixing the group A and the coefficients $b_r = j_r(A)\cdot 1_Z \in F(Z)$, one could also start with $b_r = 1_Z$ for all r. Applying the Frobenius-type homomorphism av_F to the corresponding identity derived from our Theorem 3.1, we recover the above results in a uniform way by specializing p_r for a given group A to $p_r = j_r(A)$ for all r.

Remark 4.3 The delocalized equivariant homology $H_*^G(X) = H_*(IX/G)$ of the G–space X and related invariants (eg Euler characteristic, Hodge numbers) can be used in two different ways:

(a) For the study of (equivariant) invariants of the quotient space X/G (resp. the G–space X), via fixed-point contributions; this is the context studied in this paper via characteristic classes of Lefschetz-type.

(b) For studying (equivariant) orbifold-type invariants of X/G, defined as the corresponding (equivariant) invariants of the quotient space IX/G (resp. the inertia space IX). A de Rham-type description of the orbifold cohomology is already implicit in [21], but see also [16] for a more recent version adapted to compact group actions. Generating series for the corresponding orbifold Euler characteristic and orbifold Hirzebruch genus of symmetric products have been obtained, eg in [34; 40; 41; 43]. Orbifold Chern class versions have been systematically studied in [33], whereas orbifold elliptic classes are considered in relation to the McKay correspondence in [8]. Note that the orbifold elliptic classes specialize to orbifold Hirzebruch as well as orbifold Todd classes, which are only implicitly studied in [8].

Let us finally note that in order to study these orbifold-type invariants via techniques of the present paper one has to apply the Lefschetz-type Riemann–Roch transformations to the inertia space IX (as opposed to the G–space X, as in this paper). This in turn changes the underlying combinatorics, and it is only for constructible functions that the corresponding invariants can be directly deduced from our abstract generating series formula (37), as indicated above.

5 Generating series for (equivariant) characteristic classes

In this section, we specialize our abstract generating series formula (37) in the framework of characteristic classes of singular varieties.

5.1 Characteristic classes of Lefschetz type

For a complex quasiprojective variety X, with an algebraic action $G \times X \to X$ of a finite group G, let $\pi\colon X \to X' := X/G$ be the quotient map. We denote generically by $\mathrm{cat}^G(X)$ a category of G–equivariant objects on X in the underlying category $\mathrm{cat}(X)$; see [27; 11]. From now on, $H(X) := H_*(X)$ will be $H_{\mathrm{ev}}^{\mathrm{BM}}(X) \otimes R$, the even-degree Borel–Moore homology of X with R–coefficients for R a commutative $\mathbb{C}$–algebra, or $\mathbb{Q}$–algebra if G is a symmetric group. Note that $H(-)$ is functorial for all proper maps, with a compatible cross-product (as used in the previous section).

Definition 5.1 An *equivariant characteristic class transformation of Lefschetz type* is a transformation

$$\mathrm{cl}_*(-;g)\colon K_0(\mathrm{cat}^G(X)) \to H_*(X^g)$$

such that the following properties are satisfied:

(i) $\mathrm{cl}_*(-;g)$ is covariant functorial for G–equivariant proper maps.

(ii) $\mathrm{cl}_*(-;g)$ is multiplicative under cross-product $\boxtimes$.

(iii) if X is a point space and $\mathrm{cat}(\mathrm{pt})$ is an abelian $\mathbb{C}$–linear category, then the category $\mathit{Vect}_{\mathbb{C}}(G)$ of (finite-dimensional) complex G–representations is a subcategory of $\mathrm{cat}^G(\mathrm{pt})$ and $\mathrm{cl}_*(-;g)$ is a certain g–trace (as shall be explained later on), with $\mathrm{cl}_*(-;g) = \mathrm{trace}_g$ on $\mathit{Rep}_{\mathbb{C}}(G)$.

(iv) if G acts trivially on X and $\mathrm{cat}(X)$ is an abelian $\mathbb{C}$–linear category, then

$$K_0(\mathrm{cat}^G(X)) \simeq K_0(\mathrm{cat}(X)) \otimes \mathit{Rep}_{\mathbb{C}}(G)$$

via the Schur functor decomposition as in (21), and

$$\mathrm{cl}_*(-;g) = \mathrm{cl}_*(-) \otimes \mathrm{trace}_g, \tag{53}$$

with $\mathrm{cl}_*(-)$ the corresponding nonequivariant characteristic class, as explained below. If $G = \Sigma_n$ is a symmetric group, it is enough to assume that $\mathrm{cat}(X)$ is an abelian $\mathbb{Q}$–linear category, with the category $\mathit{Vect}_{\mathbb{Q}}(\Sigma_n)$ of rational Σ_n–representations a subcategory of $\mathrm{cat}^{\Sigma_n}(\mathrm{pt})$.

Remark 5.2 For a subgroup K of G, with $g \in K$, we assume that such a transformation $\mathrm{cl}_*(-;g)$ of Lefschetz type commutes with the restriction functor Res^G_K. Then $\mathrm{cl}_*(-;g)$ depends only on the action of the cyclic subgroup generated by g. In particular, if $g = \mathrm{id}_G$ is the identity of G, we can take K the identity subgroup $\{\mathrm{id}_G\}$ with $\mathrm{Res}^G_{\{\mathrm{id}_G\}}$ the forgetful functor

$$\mathit{For}\colon K_0(\mathrm{cat}^G(X)) \to K_0(\mathrm{cat}(X)),$$

so that $\mathrm{cl}_*(-;\mathrm{id}_G) = \mathrm{cl}_*(-)$ fits with a corresponding nonequivariant characteristic class.

Remark 5.3 The above assumptions about cross-product and restriction functors can be used to define a pairing

$$Vect_{\mathbb{C}}(G) \times \mathrm{cat}^G(X) \xrightarrow{\otimes} \mathrm{cat}^G(X)$$

by

$$\mathrm{cat}^G(\mathrm{pt}) \times \mathrm{cat}^G(X) \xrightarrow{\boxtimes} \mathrm{cat}^{G\times G}(\mathrm{pt}\times X) \xrightarrow{\mathrm{Res}} \mathrm{cat}^G(X),$$

with $\mathrm{pt}\times X \cong X$ and Res denoting the restriction functor for the diagonal subgroup $G \hookrightarrow G\times G$. This induces a pairing

$$Rep_{\mathbb{C}}(G) \times K_0(\mathrm{cat}^G(X)) \xrightarrow{\otimes} K_0(\mathrm{cat}^G(X))$$

on the corresponding Grothendieck groups such that

$$\mathrm{cl}_*(V\otimes \mathcal{F}; g) = \mathrm{trace}_g(V)\cdot \mathrm{cl}_*(\mathcal{F}; g) \tag{54}$$

for V a G–representation and $\mathcal{F} \in \mathrm{cat}^G(X)$. If G is the symmetric group, then the above holds also for rational representations.

Let us give some examples of equivariant characteristic class transformations of Lefschetz type.

Example 5.4 (Todd classes) Let X be a quasiprojective G–variety, and denote by $K_0(Coh^G(X))$ the Grothendieck group of the abelian category $Coh^G(X)$ of G–equivariant coherent algebraic sheaves on X. For each $g\in G$, the Lefschetz–Riemann–Roch transformation [6; 30]

$$\mathrm{td}_*(-;g)\colon K_0(Coh^G(X)) \to H_*(X^g) \tag{55}$$

is of Lefschetz type with $R := \mathbb{C}$ (or $R := \mathbb{Q}$ if G is a symmetric group). Moreover, $\mathrm{td}_*(-;\mathrm{id}_G)$ is the complexified (nonequivariant) Todd class transformation td_* of Baum, Fulton and MacPherson [5]. Over a point space, the transformation $\mathrm{td}_*(-;g)$ reduces to the g–trace on the corresponding G–equivariant vector space. In particular, if X is projective, by pushing down to a point we recover the equivariant holomorphic Euler characteristic, ie for $\mathcal{F} \in Coh^G(X)$ the following degree formula holds:

$$\chi_a(X,\mathcal{F};g) := \sum_i (-1)^i \,\mathrm{trace}(g|H^i(X;\mathcal{F})) = \int_{[X^g]} \mathrm{td}_*([\mathcal{F}];g).$$

Example 5.5 (Chern classes) Let $K_0(Constr^G(X))$ be the Grothendieck group of the abelian category $Constr^G(X)$ of algebraically constructible G–equivariant sheaves of complex vector spaces on X. Then the localized Chern class transformation [37, Example 1.3.2]

$$c_*(-;g) := c_*(\mathrm{tr}_g(-|_{X^g}))\colon K_0(\mathit{Constr}^G(X)) \to H_*(X^g)$$

is of Lefschetz type with $R := \mathbb{C}$ (or $R := \mathbb{Q}$ if G is a symmetric group). Here $c_*(-)$ is the Chern–MacPherson class transformation [25] and $\mathrm{tr}_g(-|_{X^g})$ is the group homomorphism which, for $\mathcal{F} \in \mathit{Constr}^G(X)$, assigns to $[\mathcal{F}] \in K_0(\mathit{Constr}^G(X))$ the constructible function on X^g defined by

$$x \mapsto \mathrm{trace}(g|\mathcal{F}_x).$$

(Note that for $x \in X^g$, g acts on the finite-dimensional stalk $\mathcal{F}_x$ for a constructible G–equivariant sheaf $\mathcal{F}$.) For the identity element, the transformation $c_*(-;g)$ reduces to the complexification of MacPherson's Chern class transformation. It also follows by definition that if X is a point space, then $c_*(-;g)$ reduces to the g–trace on the corresponding G–equivariant vector space. In particular, if X is projective, by pushing down to a point we recover the equivariant Euler characteristic, ie for $\mathcal{F} \in \mathit{Constr}^G(X)$ the following degree formula holds:

$$\chi(X,\mathcal{F};g) := \sum_i (-1)^i \,\mathrm{trace}(g|H^i(X;\mathcal{F})) = \int_{[X^g]} c_*([\mathcal{F}];g).$$

Example 5.6 ((unnormalized) Atiyah–Singer classes, mixed Hodge module version) Let $K_0(\mathit{MHM}^G(X))$ be the Grothendieck group of G–equivariant (algebraic) mixed Hodge modules. The (unnormalized) Atiyah–Singer class transformation of [11],

$$T_{y*}(-;g)\colon K_0(\mathit{MHM}^G(X)) \to H_*(X^g),$$

is of Lefschetz type with $R = \mathbb{C}[y^{\pm 1}]$ (or $R := \mathbb{Q}[y^{\pm 1}]$ if G is a symmetric group). In this case, $\mathit{MHM}(X)$ is only a $\mathbb{Q}$–linear abelian category, so for the isomorphism of Grothendieck groups in property (3) of Definition 5.1 one should assume that G is a symmetric group (or work with the Grothendieck group of the underlying $\mathbb{C}$–linear exact category of filtered holonomic $\mathcal{D}$–modules). For the identity element of G, this reduces to the mixed Hodge module version of the (unnormalized) Hirzebruch class transformation of Brasselet, Schürmann and Yokura [9; 38]. Over a point space, $T_{y*}(-;g)$ coincides with the equivariant $\chi_y(-;g)$–genus ring homomorphism

$$\chi_y(-;g)\colon K_0^G(\mathit{MHS}^p) \to \mathbb{C}[y^{\pm 1}]$$

defined on the Grothendieck group of the category $G\text{–}MHS^p$ of G–equivariant (graded) polarizable mixed Hodge structures by

$$\chi_y([H];g) := \sum_p \operatorname{trace}(g|\operatorname{Gr}_F^p(H\otimes\mathbb{C}))\cdot(-y)^p$$

for $F^\bullet$ the Hodge filtration on $H \in G\text{–}MHS^p$. Here we use the identification $MHM^G(\mathrm{pt}) \simeq G\text{–}MHS^p$ of G–equivariant mixed Hodge modules over a point space with G–equivariant (graded) polarizable mixed Hodge structures, so that the category of (finite-dimensional) rational G–representations is a subcategory of $G\text{–}MHS^p$ (viewed as mixed Hodge structure of pure type $(0,0)$). In particular, if X is projective and $\mathcal{M} \in MHM^G(X)$, by pushing down to a point we recover the equivariant twisted Hodge genus, ie the following degree formula holds (see [11, Proposition 4.7] for $\mathcal{M}$ the intersection cohomology mixed Hodge module):

$$\chi_y(X,\mathcal{M};g) := \sum_{i,p}(-1)^i \operatorname{trace}(g|\operatorname{Gr}_F^p H^i(X;\mathcal{M})\otimes\mathbb{C})\cdot(-y)^p = \int_{[X^g]} T_{y*}(\mathcal{M};g).$$

Remark 5.7 The motivic version of the Atiyah–Singer class transformation, as mentioned in the introduction, can be deduced from the corresponding mixed Hodge module version through the natural transformation (eg see [11])

$$\chi^G_{\mathrm{Hdg}}\colon K_0^G(\mathrm{var}/X) \to K_0(MHM^G(X))$$

mapping $[\mathrm{id}_X]$ to the class of the constant Hodge module (complex) $[\mathbb{Q}_X^H]$. This transformation commutes with push-downs, cross-products and restriction functors. Then the (unnormalized) motivic Atiyah–Singer class transformation is the composition

$$T_{y*}(-;g)\colon K_0^G(\mathrm{var}/X) \xrightarrow{\chi^G_{\mathrm{Hdg}}} K_0(MHM^G(X)) \xrightarrow{T_{y*}(-;g)} H_*(X^g).$$

In particular, the Atiyah–Singer class of X is defined as

$$T_{y*}(X;g) := T_{y*}([\mathrm{id}_X];g) = T_{y*}([\mathbb{Q}_X^H];g).$$

Let us explain some of the above examples in the simplest situation when X is *smooth* (see [11] for complete details). Then X^g is also smooth, and we denote by T_{X^g} and N_{X^g} its tangent and normal bundles in X. In this case, the homological Lefschetz-type transformations correspond under Poincaré duality (and for suitable "smooth" coefficients in $\mathrm{cat}^G(X)$) to similar cohomological transformations, as explained below.

Todd classes Let $K^0_G(X)$ be the Grothendieck group of algebraic G–vector bundles, and note that the natural map $K^0_G(X) \to K_0(Coh^G(X))$ is an isomorphism. Let ch^* and td^* denote the Chern character and the Todd class in cohomology. The Lefschetz–Riemann–Roch transformation is then given by: for V an algebraic G–vector bundle on X,

$$\mathrm{td}_*(V;g) = \mathrm{ch}^*(g)(V|_{X^g}) \cap \left(\frac{\mathrm{td}^*(T_{X^g})}{\mathrm{ch}^*(g)(\Lambda_{-1} N^*_{X^g})} \cap [X^g] \right). \tag{56}$$

Here, $N^*_{X^g}$ denotes the dual of the normal bundle of X^g, and for a vector bundle E we let $\Lambda_{-1}(E) := \sum_i (-1)^i \Lambda^i E$. Moreover, the equivariant Chern character

$$\mathrm{ch}^*(g)(-)\colon K^0_G(X^g) \to H^*(X^g)$$

is defined as follows: for $W \in K^0_G(X^g)$ we let

$$\mathrm{ch}^*(g)(W) := \sum_\chi \chi(g) \cdot \mathrm{ch}^*(W_\chi)$$

for $W \simeq \bigoplus_\chi W_\chi$ the (finite) decomposition of W into subbundles W_χ on which g acts by a character $\chi\colon \langle g \rangle \to \mathbb{C}^*$. Note that $\mathrm{ch}^*(g)(V|_{X^g})$ is just the complexified Chern character of $L(g)(V)$, with $L(g)$ the Lefschetz-type transformation of Section 3.1.2. If X is projective, by taking degrees in formula (56) we obtain the Atiyah–Singer holomorphic Lefschetz formula from [2; 20].

Atiyah–Singer classes, mixed Hodge module version Let X be smooth with an algebraic G–action, together with a G–equivariant "good" variation $\mathcal{L}$ of rational mixed Hodge structures (ie graded polarizable, admissible and with quasiunipotent monodromy at infinity). This corresponds to a (shifted) smooth G–equivariant mixed Hodge module. Let $\mathcal{V} := \mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_X$ be the flat G–equivariant bundle whose sheaf of horizontal sections is $\mathcal{L} \otimes \mathbb{C}$. The bundle $\mathcal{V}$ comes equipped with a decreasing (Hodge) filtration (compatible with the G–action) by holomorphic subbundles $\mathcal{F}^p\mathcal{V}$. Note that since we work with a "good" variation, each $\mathcal{F}^p\mathcal{V}$ underlies (by GAGA) a unique complex algebraic G–vector bundle. Let

$$\chi_y(\mathcal{V}) := \sum_p [\mathrm{Gr}^p_{\mathcal{F}}\mathcal{V}] \cdot (-y)^p \in K^0_G(X)[y, y^{-1}]$$

be the χ_y–characteristic of $\mathcal{V}$. Then

$$T_{y_*}(X, \mathcal{L}; g) = \mathrm{ch}^*(g)(\chi_y(\mathcal{V})|_{X^g}) \cap T_{y*}(X; g), \tag{57}$$

with $T_{y*}(X;g) := \sum_{i \geq 0} \mathrm{td}_*([\Omega^i_X]; g) \cdot y^i$ the Atiyah–Singer class of X.

Our generating series results for these characteristic class transformations, as discussed in the next sections, will, however, be valid for any quasiprojective complex variety X (possibly singular) and all coefficients in $\mathrm{cat}^G(X)$.

5.2 Delocalized equivariant characteristic classes

Let X be a (possibly singular) quasiprojective variety acted upon by a finite group G of algebraic automorphisms.

From now on, we use the symbol cl_* to denote any of the characteristic classes c_*, td_* and T_{y*}, respectively, with their corresponding equivariant versions of Lefschetz type, $\mathrm{cl}_*(-;g)\colon K_0(\mathrm{cat}^G(X)) \to H_*(X^g)$, as discussed in the previous section.

Definition 5.8 For any of the above Lefschetz-type characteristic class transformations $\mathrm{cl}_*(-;g)$, we define a corresponding G–equivariant class transformation

$$\mathrm{cl}_*^G\colon K_0(\mathrm{cat}^G(X)) \to H_*^G(X)$$

by

$$\mathrm{cl}_*^G(-) := \bigoplus_{g\in G} \mathrm{cl}_*(-;g) = \bigoplus_{(g)} \mathrm{Ind}_{Z_G(g)}^G(\mathrm{cl}_*(-;g)) \in \Big(\bigoplus_{g\in G} H_*(X^g)\Big)^G,$$

with induction as in Remark 2.8.

Note that the G–invariance of the class $\mathrm{cl}_*^G(-)$ is a consequence of the *conjugacy-invariance* of $\mathrm{cl}_*(-;g)$, proved in [11, Section 5.3]. This also explains the equality of the two descriptions of $\mathrm{cl}_*^G(-)$.

The above transformation $\mathrm{cl}_*^G(-)$ has the same properties as the Lefschetz-type transformations $\mathrm{cl}_*(-;g)$, eg functoriality for proper push-downs, restrictions to subgroups, and multiplicativity for exterior products.

If X is projective, then by pushing $\mathrm{cl}_*^G(-)$ down to a point, the degree

$$\deg(\mathrm{cl}_*^G(-)) = \mathrm{tr}_G(X,-) := \mathrm{tr}_G(H^*(X,-)) \in C(G)\otimes R$$

is the character tr_G of the corresponding virtual cohomology representation

$$\sum_i (-1)^i [H^i(X,-)] \in Rep_{\mathbb{C}}(G)\otimes R \simeq C(G)\otimes R.$$

In addition, if $\mathrm{cl}_*^G = T_{y*}^G$ is the G–equivariant Hirzebruch class, then

$$\deg\bigl(T_{y*}^G([\mathcal{M}])\bigr) = \mathrm{tr}_G(X,\mathcal{M}) := \sum_p \mathrm{tr}_G(\mathrm{Gr}_F^p H^*(X,\mathcal{M}))(-y)^p$$

for $\mathcal{M} \in \mathit{MHM}(X)$.

If G acts trivially on X, then there is a functor

$$[-]^G\colon K_0(\mathrm{cat}^G(X)) \to K_0(\mathrm{cat}(X))$$

defined by taking G–invariants, which in the case of any of the $\mathbb{Q}$–linear abelian categories $\mathit{Coh}(X)$, $\mathit{Constr}(X)$ and $\mathit{MHM}(X)$ is induced from the exact projector

$$(-)^G := \frac{1}{|G|}\sum_{g\in G}\mu_g\colon \mathrm{cat}^G(X) \to \mathrm{cat}(X).$$

Here $\mu_g\colon \mathcal{F} \to g_*\mathcal{F}$ (with $g \in G$) is the isomorphism of the G–action on $\mathcal{F} \in \mathrm{cat}^G(X)$; see [11; 27] for details.

Remark 5.9 For a given G–representation V, by using the pairing of Remark 5.3 one can define *Schur functors* $S_V\colon \mathrm{cat}^G(X) \to \mathrm{cat}(X)$ by $S_V(\mathcal{F}) := (V \otimes \mathcal{F})^G$. Here we assume that G acts trivially on X. This notion agrees with the abstract categorical definition from [13; 19].

For the Grothendieck group $K_0^G(\mathrm{var}/X)$ of G–varieties over X, the functor

$$[-]^G\colon K_0^G(\mathrm{var}/X) \to K_0(\mathrm{var}/X)$$

is given by $[Y \to X] \mapsto [Y/G \to X]$. This is a well-defined functor since our equivariant Grothendieck group $K_0^G(\mathrm{var}/X)$ from [11] only uses the scissor relation. Note that the transformation

$$\chi_{\mathrm{Hdg}}^G\colon K_0^G(\mathrm{var}/X) \to K_0(\mathit{MHM}^G(X))$$

relating the two versions of the Atiyah–Singer transformation, commutes with $[-]^G$ since, with $\pi\colon Y \to Y/G$ denoting the quotient map, we have that (see [11, Lemma 5.3], but see also [26, Remark 2.4])

$$\mathbb{Q}_{Y/G}^H = (\pi_*\mathbb{Q}_Y^H)^G \in D^b\mathit{MHM}(Y/G).$$

Then if G acts trivially on X, the following *averaging property* holds by the definition of the projector $(-)^G$ together with (53) (compare also with [11, Section 5.3], [12, Section 3]):

$$
\begin{tikzcd}
K_0(\mathrm{cat}^G(X)) \arrow[r, "\mathrm{cl}_*^G"] \arrow[d, "{[-]^G}"'] & H_*^G(X) \cong H_*(X) \otimes C(G) \arrow[d, "(1/|G|)\sum_{g\in G}\mathrm{ev}_g"] \\
K_0(\mathrm{cat}(X)) \arrow[r, "\mathrm{cl}_*"] & H_*(X)
\end{tikzcd}
\tag{58}
$$

where ev_g is the evaluation at $g \in G$ of class functions on G.

5.3 Proof of the main theorem, Theorem 1.4, and its applications

In this section, we explain how to deduce our main result, Theorem 1.4, for equivariant characteristic classes of external products of varieties from the abstract generating series formula (37). We also explain the various specializations of Theorem 1.4, as formulated in the introduction.

Let Z be a quasiprojective variety, with a given object $\mathcal{F} \in \mathrm{cat}(Z)$ as in the introduction. We use as before the symbol

$$\mathrm{cl}_*\colon K_0(\mathrm{cat}(Z)) \to H_*(Z) = H_{2*}^{\mathrm{BM}}(Z) \otimes R$$

to denote any of the characteristic classes

- $\mathrm{td}_*\colon K_0(\mathit{Coh}(Z)) \to H_{2*}^{\mathrm{BM}}(Z) \otimes \mathbb{Q}$,
- $c_*\colon K_0(\mathit{Constr}(Z)) \to H_{2*}^{\mathrm{BM}}(Z) \otimes \mathbb{Q}$,
- $T_{y*}\colon K_0(\mathrm{var}/Z) \to H_{2*}^{\mathrm{BM}}(Z) \otimes \mathbb{Q}[y]$,
- $T_{y*}\colon K_0(\mathit{MHM}(Z)) \to H_{2*}^{\mathrm{BM}}(Z) \otimes \mathbb{Q}[y^{\pm 1}]$,

with their corresponding localized and delocalized equivariant versions $\mathrm{cl}_*(-;g)$ and $\mathrm{cl}_*^G(-)$, as discussed in the previous two subsections.

The following properties will allow us to further specialize our main generating series formula (37) in the context of characteristic classes.

It follows from [12] that $\mathrm{cl}_*(-;g)$ satisfies the following *multiplicativity property* (see [12, Lemmas 3.2, 3.5 and 3.9]):

Lemma 5.10 (multiplicativity) *If $\sigma \in \Sigma_n$ has cycle-type $(k_1, k_2, \dots)$, ie k_r is the number of r–cycles in σ and $n = \sum_r r \cdot k_r$, then*

$$\mathrm{cl}_*(\mathcal{F}^{\boxtimes n};\sigma) = \boxtimes_r (\mathrm{cl}_*(\mathcal{F}^{\boxtimes r};\sigma_r))^{k_r} \in H_*((Z^n)^\sigma)^{Z_{\Sigma_n}(\sigma)} \subset H_*((Z^n)^\sigma), \tag{59}$$

with σ_r denoting an r–cycle in Σ_r.

Moreover, the following *localization* result holds — see [12, Lemmas 3.3, 3.6 and 3.10]:

Lemma 5.11 (localization) *Under the identification* $(Z^r)^{\sigma_r} \simeq Z$,

$$\mathrm{cl}_*(\mathcal{F}^{\boxtimes r}; \sigma_r) = \Psi_r \mathrm{cl}_*(\mathcal{F}), \tag{60}$$

where Ψ_r *denotes the homological Adams operation defined by*

$$\Psi_r = \begin{cases} \mathrm{id} & \text{if } \mathrm{cl}_* = c_*, \\ \cdot 1/r^i \ \text{ on } H_{2i}^{\mathrm{BM}}(Z) \otimes \mathbb{Q} & \text{if } \mathrm{cl}_* = \mathrm{td}_*, \\ \cdot 1/r^i \ \text{ on } H_{2i}^{\mathrm{BM}}(Z) \otimes \mathbb{Q} \ \text{ and } \ y \mapsto y^r & \text{if } \mathrm{cl}_* = T_{-y*}. \end{cases}$$

We can now explain how to use the abstract generating series formula (37) to derive Theorem 1.4 from the introduction.

Proof of Theorem 1.4 For any $r \geq 1$, let

$$b_r := \mathrm{cl}_*(\mathcal{F}^{\boxtimes r}; \sigma_r) \in H_*(Z).$$

By multiplicativity (see Lemma 5.10) and conjugacy-invariance of $\mathrm{cl}_*(-;\sigma)$, it follows that

$$\mathrm{cl}_*^{\Sigma_n}(\mathcal{F}^{\boxtimes n}) = \sum_{(\sigma) \in (\Sigma_n)_*} \mathrm{Ind}_{Z_{\Sigma_n(\sigma)}}^{\Sigma_n} \big(\boxtimes_r \mathrm{cl}_*(\mathcal{F}^{\boxtimes r}; \sigma_r)^{\boxtimes k_r} \big) = \sum_{(\sigma) \in (\Sigma_n)_*} \underline{b}^{(\sigma)}.$$

Then (4) follows from our main formula (37) together with the localization formula (60). □

Let us now apply $\pi_* := \bigoplus_n \pi_{n*}$ to formula (4). Then, by using functoriality and the corresponding $\mathbb{Q}$–algebra homomorphism $\pi_*\colon \mathbb{H}_*^{\Sigma}(Z) \to \mathbb{PH}_*^{\Sigma}(Z)$ of (47), we obtain by the first identity of (48) a proof of Corollary 1.6 from the introduction.

The averaging property (58) together with the $\mathbb{Q}$–algebra evaluation homomorphism $\mathbb{PH}_*(Z) \otimes \mathbb{Q}[p_i \mid i \geq 1] \to \mathbb{PH}_*(Z)$, $p_i \mapsto 1$ (for all i), as in (47), now yields a proof of Corollary 1.7 from the introduction, which recovers the main result of [12].

We want to emphasize that Corollary 1.6 has other important applications, as already mentioned in the introduction. For example, it specializes to Corollary 1.8 by using the $\mathbb{Q}$–algebra evaluation homomorphism

$$\mathbb{PH}_*(Z) \otimes \mathbb{Q}[p_i \mid i \geq 1] \to \mathbb{PH}_*(Z), \quad p_i \mapsto \mathrm{sign}(\sigma_i) = (-1)^{i-1}$$

(for all i), together with the commutative diagram:

$$\begin{tikzcd} K_0(\mathrm{cat}^{\Sigma_n}(Z^{(n)})) \arrow[r, "\mathrm{cl}_*^{\Sigma_n}"] \arrow[d, "{[-]^{\mathrm{sign}-\Sigma_n}}"'] & H_*^{\Sigma_n}(Z^{(n)}) \cong H_*(Z^{(n)}) \otimes C(\Sigma_n) \arrow[d, "(1/n!)\sum_{\sigma\in\Sigma_n}\mathrm{sign}(\sigma)\cdot\mathrm{ev}_\sigma"] \\ K_0(\mathrm{cat}(Z^{(n)})) \arrow[r, "\mathrm{cl}_*"] & H_*(Z^{(n)}) \end{tikzcd} \tag{61}$$

Finally, formula (16) from the introduction follows by combining the multiplicativity of (54), together with the identification of the coefficient of t^n in the explicit expansion (as in the proof of Theorem 3.1) of the exponential on the right-hand side of (8), that is,

$$\mathrm{cl}_*^{\Sigma_n}(\pi_{n*}\mathcal{F}^{\boxtimes n}) = \sum_{\lambda=(k_1,k_2,\ldots)\dashv n} \frac{p_\lambda}{z_\lambda}\cdot \bigodot_{r\geq 1}\big(d_{r*}(\psi_r(\mathrm{cl}_*(\mathcal{F})))\big)^{k_r}. \tag{62}$$

Remark 5.12 All these results and arguments also apply to a bounded complex $\mathcal{F}$ in $D^b_{\mathrm{coh}}(Z)$ or $D^b_c(Z)$, with coherent or constructible cohomology, respectively, as well as to bounded complexes (such as $\mathbb{Q}^H_Z$) in $D^b(MHM(Z))$. Then $\mathcal{F}^{\boxtimes n}$ and $\pi_{n*}\mathcal{F}^{\boxtimes n}$ become *weakly equivariant Σ_n–complexes* (as in [11, Appendix]), which still have well-defined Grothendieck classes $[\mathcal{F}^{\boxtimes n}] \in K_0(\mathrm{cat}^{\Sigma_n}(Z^n))$, respectively, $[\pi_{n*}\mathcal{F}^{\boxtimes n}] \in K_0(\mathrm{cat}^{\Sigma_n}(Z^{(n)}))$. Moreover, the definition of the symmetric and alternating power objects via the projector $[-]^{\Sigma_n}$ and $[-]^{\mathrm{sign}-\Sigma_n}$, respectively, still works as above since the corresponding derived categories are $\mathbb{Q}$–linear Karoubian (see [12; 27] for more details).

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