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Complete minimal surfaces densely lying in arbitrary domains of $\mathbb{R}^n$

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In this paper we prove that, given an open Riemann surface M and an integer $n \geq 3$, the set of complete conformal minimal immersions $M \to \mathbb{R}^n$ with $\overline{X(M)} = \mathbb{R}^n$ forms a dense subset in the space of all conformal minimal immersions $M \to \mathbb{R}^n$ endowed with the compact-open topology. Moreover, we show that every domain in $\mathbb{R}^n$ contains complete minimal surfaces which are dense on it and have arbitrary orientable topology (possibly infinite); we also provide such surfaces whose complex structure is any given bordered Riemann surface.

Our method of proof can be adapted to give analogous results for nonorientable minimal surfaces in $\mathbb{R}^n$ $(n \geq 3)$, complex curves in $\mathbb{C}^n$ $(n \geq 2)$, holomorphic null curves in $\mathbb{C}^n$ $(n \geq 3)$, and holomorphic Legendrian curves in $\mathbb{C}^{2n+1}$ $(n \in \mathbb{N})$.

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1 Introduction and main results

The existence of complete minimal surfaces densely lying in $\mathbb{R}^3$ is well known. The first example of such, due to Rosenberg, was obtained by Schwarzian reflection on a fundamental domain, is simply connected, and has bounded curvature. Later, Gálvez and Mira [24] found complete dense simply connected minimal surfaces in $\mathbb{R}^3$, in explicit coordinates, as solutions to certain Björling problems. Finally, López [30] constructed complete dense minimal surfaces in $\mathbb{R}^3$ with weak finite total curvature, arbitrary genus, and parabolic conformal type; so far, these are the only known examples with nontrivial topology. In a parallel line of results, Andrade [13] gave an example of a complete simply connected minimal surface in $\mathbb{R}^3$ which is not dense in the whole space but whose closure has nonempty interior. It is therefore a natural question whether a given domain in $\mathbb{R}^3$ contains complete minimal surfaces which are dense on it; as far as the authors' knowledge extends, no domain is known to enjoy this property besides $\mathbb{R}^3$ itself.

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The aim of this paper is to answer the above question by showing a general existence result for complete dense minimal surfaces in *any* given domain $D \subset \mathbb{R}^n$ for arbitrary dimension $n \geq 3$. We provide such surfaces with *arbitrary orientable topology and flux map*; moreover, if $n \geq 5$ we give examples with no self-intersections. Furthermore, if $D = \mathbb{R}^n$ then we construct such surfaces not only with arbitrary topology but also with *arbitrary complex structure*. To be precise, our first main result may be stated as follows.

Theorem 1.1 *Let $D \subset \mathbb{R}^n$ $(n \geq 3)$ be a domain, M be an open Riemann surface, $\mathfrak{p}\colon H_1(M;\mathbb{Z}) \to \mathbb{R}^n$ be a group homomorphism, $K \subset M$ be a smoothly bounded Runge compact domain, and $X\colon K \to \mathbb{R}^n$ be a conformal minimal immersion of class $\mathscr{C}^1(K)$. Assume that $X(K) \subset D$ and that the flux map $\mathrm{Flux}_X\colon H_1(K;\mathbb{Z}) \to \mathbb{R}^n$ of X satisfies $\mathrm{Flux}_X(\gamma) = \mathfrak{p}(\gamma)$ for all closed curves $\gamma \subset K$.*

Then, for any $\epsilon > 0$, there are a domain $\Omega \subset M$ and a complete conformal minimal immersion $Y\colon \Omega \to \mathbb{R}^n$ satisfying the following properties:

(I) *$K \subset \Omega$ and Ω is a deformation retract of M and homeomorphic to M.*

(II) *$\|Y - X\|_{1,K} < \epsilon$.*

(III) *$\mathrm{Flux}_Y(\gamma) = \mathfrak{p}(\gamma)$ for all closed curves $\gamma \subset \Omega$.*

(IV) *$Y(\Omega) \subset D$ and the closure satisfies $\overline{Y(\Omega)} = \overline{D}$.*

(V) *Y is one-to-one if $n \geq 5$.*

Furthermore, if $D = \mathbb{R}^n$ we may choose $\Omega = M$.

Theorem 1.1 gives the first examples of complete dense minimal surfaces in $\mathbb{R}^n$ for $n > 3$. Notice that the density of $Y(M)$ in D does not allow the immersions $Y\colon \Omega \to D$ in the theorem to be proper maps.

We emphasize that, while certainly wild, complete dense minimal surfaces in $\mathbb{R}^n$ $(n \geq 3)$ are surprisingly abundant. Indeed, if we denote by $\mathrm{CMI}(M,\mathbb{R}^n)$ the space of all conformal minimal immersions of a given open Riemann surface M into $\mathbb{R}^n$ (which is nonempty by the results in Alarcón and López [10]), Theorem 1.1 ensures that *those conformal minimal immersions $M \to \mathbb{R}^n$ which are complete and have dense image form a dense subset of* $\mathrm{CMI}(M,\mathbb{R}^n)$ *with respect to the compact-open topology.*

It is also worth mentioning at this point that it is not hard to find dense minimal surfaces in $\mathbb{R}^n$ for any $n \geq 3$. Indeed, solving the Björling problem for any real analytic regular dense curve in $\mathbb{R}^n$ and any tangent plane distribution along it gives such a surface; we thank Pablo Mira for providing us with this simple argument. Obviously, this method only produces simply connected examples and does not guarantee their completeness.

As will become apparent later in this introduction, constructing *complete* dense minimal surfaces in $\mathbb{R}^n$, *prescribing their topology and even their complex structure*, is a much more arduous task which requires a number of powerful and sophisticated tools of the theory that have been developed only recently.

It is well known that a general domain $D \subset \mathbb{R}^n$ does not contain minimal surfaces with arbitrary complex structure. Indeed, if for instance D is relatively compact then it only admits minimal surfaces of *hyperbolic* conformal type (see Farkas and Kra [17, page 179]). We also prove in this paper that every domain $D \subset \mathbb{R}^n$ contains complete minimal surfaces which are dense on it and whose complex structure is any given bordered Riemann surface.

Theorem 1.2 *Let* $D \subset \mathbb{R}^n$ $(n \geq 3)$ *be a domain and* $\overline{M} = M \cup bM$ *be a compact bordered Riemann surface. Every conformal minimal immersion* $X\colon \overline{M} \to \mathbb{R}^n$ *of class* $\mathscr{C}^1(\overline{M})$*, with* $X(\overline{M}) \subset D$*, may be approximated uniformly on compact subsets of* $M = \overline{M} \setminus bM$ *by complete conformal minimal immersions* $Y\colon M \to \mathbb{R}^n$ *assuming values in* D *and such that* $\overline{Y(M)} = \overline{D}$ *and* $\mathrm{Flux}_Y = \mathrm{Flux}_X$*. Moreover, if* $n \geq 5$ *then the approximating immersions* Y *can be chosen to be one-to-one.*

Recall that a *compact bordered Riemann surface* is a compact Riemann surface $\overline{M}$ with nonempty boundary $bM \subset \overline{M}$ consisting of finitely many pairwise disjoint smooth Jordan curves. The interior $M = \overline{M} \setminus bM$ of $\overline{M}$ is called a *bordered Riemann surface*. By a *conformal minimal immersion* $\overline{M} \to \mathbb{R}^n$ *of class* $\mathscr{C}^1(\overline{M})$ we mean a map of class $\mathscr{C}^1(\overline{M})$ whose restriction to M is a conformal minimal immersion.

We shall prove Theorems 1.1 and 1.2 in Section 3. The main tools in our method of proof come from the strong connection between minimal surfaces in $\mathbb{R}^n$ and complex analysis; in particular, *Oka theory* (see the note by Lárusson [28] and the surveys by Forstnerič and Lárusson [21], Forstnerič [19], and Kutzschebauch [27] for an introduction to this theory, and the monograph by Forstnerič [18] for a comprehensive treatment; see eg Alarcón and Forstnerič [4; 6] or Alarcón, Forstnerič, and López [7] for a discussion of the interplay between minimal surfaces and Oka manifolds). To be more precise, our proof relies on a *Runge–Mergelyan type approximation theorem* for conformal minimal immersions of open Riemann surfaces into $\mathbb{R}^n$ (see Alarcón and López [10] for $n = 3$ and Alarcón, Forstnerič, and López [7] for arbitrary dimension), a *general position theorem* for conformal minimal surfaces in $\mathbb{R}^n$ for $n \geq 5$ (see [7]), and the existence of approximate solutions to certain *Riemann–Hilbert type boundary value problems* for conformal minimal surfaces in $\mathbb{R}^n$ where the complex structure of the central surface is a compact bordered Riemann surface (see Alarcón and Forstnerič [5] for $n = 3$ and Alarcón, Drinovec Drnovšek, Forstnerič, and López [1] for $n \geq 3$). Actually, the Riemann–Hilbert method is not explicitly applied in the present paper but it plays a fundamental role in the proof of [1, Lemma 4.1], which we use in a strong way. Furthermore, our method of proof also exploits the technique by Forstnerič and Wold [23] for *exposing boundary points on a bordered Riemann surface*, which pertains to Riemann surface theory.

All the above-mentioned tools are also available for some other families of surfaces which are the focus of interest, namely, *nonorientable minimal surfaces* in $\mathbb{R}^n$ for $n \geq 3$, *complex curves* in the complex Euclidean spaces $\mathbb{C}^n$ for $n \geq 2$, *holomorphic null curves* in $\mathbb{C}^n$ for $n \geq 3$, and *holomorphic Legendrian curves* in $\mathbb{C}^{2n+1}$ for $n \in \mathbb{N}$. Thus, our methods easily adapt to give results analogous to Theorems 1.1 and 1.2 in all these geometric contexts; we motivate, state, and discuss some of them in Section 4.

2 Preliminaries

Given $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, we denote by $|\cdot|$, $\mathrm{dist}(\cdot, \cdot)$, and $\mathrm{length}(\cdot)$ the Euclidean norm, distance, and length in $\mathbb{R}^n$, respectively. Given a set $A \subset \mathbb{R}^n$ we denote by $\overline{A}$ the topological closure of A in $\mathbb{R}^n$.

If K is a compact topological space and $f\colon K \to \mathbb{R}^n$ is a continuous map, we denote by

$$\|f\|_{0,K} := \max\{|f(p)| : p \in K\}$$

the maximum norm of f on K. If K is a subset of a Riemann surface M, then for any $r \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ we denote by

$$\|f\|_{r,K}$$

the standard $\mathscr{C}^r$–norm of a function $f\colon K \to \mathbb{R}^n$ of class $\mathscr{C}^r(K)$, where the derivatives are measured with respect to a fixed Riemannian metric on M (the precise choice of the metric will not be important).

Given a smooth connected surface S (possibly with nonempty boundary) and a smooth immersion $X\colon S \to \mathbb{R}^n$ $(n \geq 3)$, we denote by

$$\mathrm{dist}_X\colon S \times S \to \mathbb{R}_+ = [0, +\infty)$$

the Riemannian distance induced on S by the Euclidean metric of $\mathbb{R}^n$ via X:

$$\mathrm{dist}_X(p, q) = \inf\{\mathrm{length}(X(\gamma)) : \gamma \subset S \text{ any arc connecting } p \text{ and } q\} \quad \text{for } p, q \in S.$$

Likewise, if $K \subset S$ is a relatively compact subset we define

$$\mathrm{dist}_X(p,K) := \inf\{\mathrm{dist}_X(p,q) : q \in K\} \quad \text{for } p \in S.$$

An immersed open surface $X\colon S \to \mathbb{R}^n$ $(n \geq 3)$ is said to be *complete* if the image by X of any proper path $\gamma\colon [0,1) \to S$ has infinite Euclidean length; this is equivalent to the Riemannian metric induced on S by the Euclidean metric of $\mathbb{R}^n$ via X being complete.

Let M be an open Riemann surface and $n \geq 3$ be an integer. A conformal (ie angle-preserving) immersion $X = (X_1, \dots, X_n)\colon M \to \mathbb{R}^n$ is *minimal* (ie X has everywhere-vanishing mean curvature vector) if, and only if, X is a harmonic map in the classical sense: $\triangle X = 0$. Denoting by ∂ the $\mathbb{C}$–linear part of the exterior differential $d = \partial + \overline{\partial}$ on M (here $\overline{\partial}$ is the $\mathbb{C}$–antilinear part of d), the 1–form $\partial X = (\partial X_1, \dots, \partial X_n)$ with values in $\mathbb{C}^n$ is holomorphic, has no zeros, and satisfies $\sum_{j=1}^n (\partial X_j)^2 = 0$ everywhere on M. It follows that the real part $\Re(\partial X)$ is an exact real 1–form on M. On the other hand, the *flux map* (or simply the *flux*) of X is defined as the group homomorphism

$$\mathrm{Flux}_X\colon H_1(M;\mathbb{Z}) \to \mathbb{R}^n$$

of the first homology group $H_1(M;\mathbb{Z})$ of M with integer coefficients, given by

$$\mathrm{Flux}_X(\gamma) = \int_\gamma \Im(\partial X) = -\mathrm{i}\int_\gamma \partial X \quad \text{for } \gamma \in H_1(M;\mathbb{Z}),$$

where $\Im$ denotes the imaginary part and $\mathrm{i} := \sqrt{-1}$. We refer eg to Osserman's monograph [32] for a standard reference on minimal surface theory.

A compact subset $K \subset M$ is said to be *Runge* (also called *holomorphically convex* or $\mathcal{O}(M)$*–convex*) if its complement $M \setminus K$ has no relatively compact connected components on M; by the Runge–Mergelyan theorem [33; 31; 14] this is equivalent to that every continuous function $K \to \mathbb{C}$, holomorphic in the interior $\mathring{K}$, may be approximated uniformly on K by holomorphic functions $M \to \mathbb{C}$.

A *compact bordered Riemann surface* is a compact Riemann surface $\overline{M}$ with nonempty boundary $bM \subset \overline{M}$ consisting of finitely many pairwise disjoint smooth Jordan curves; its interior $M = \overline{M} \setminus bM$ is called a *bordered Riemann surface*. It is classical that every compact bordered Riemann surface $\overline{M}$ is diffeomorphic to a smoothly bounded compact domain in an open Riemann surface. By a *conformal minimal immersion of class* $\mathscr{C}^1(\overline{M})$ of a compact bordered Riemann surface $\overline{M}$ into $\mathbb{R}^n$, we mean a map $\overline{M} \to \mathbb{R}^n$ of class $\mathscr{C}^1(\overline{M})$ whose restriction to M is a conformal minimal immersion.

3 Proofs of the main results

In this section we prove Theorems 1.1 and 1.2; both will follow from a recursive application of the following approximation result.

Lemma 3.1 *Let $D \subset \mathbb{R}^n$ ($n \geq 3$) be a domain, $\overline{M} = M \cup bM$ be a compact bordered Riemann surface, and $X\colon \overline{M} \to \mathbb{R}^n$ be a conformal minimal immersion of class $\mathscr{C}^1(\overline{M})$ such that*

$$X(\overline{M}) \subset D.$$

Given a compact domain $K \subset M$, points $p_0 \in \mathring{K}$ and $x_1, \dots, x_k \in D$ ($k \in \mathbb{N}$), and numbers $\epsilon > 0$ and $\lambda > 0$, there is a conformal minimal immersion $Y\colon \overline{M} \to \mathbb{R}^n$ of class $\mathscr{C}^1(\overline{M})$ satisfying the following conditions:

- (i) $Y(\overline{M}) \subset D$.
- (ii) $\|Y - X\|_{1,K} < \epsilon$.
- (iii) $\operatorname{dist}(x_j, Y(\overline{M})) < \epsilon$ *for all* $j \in \{1, \dots, k\}$.
- (iv) $\operatorname{Flux}_Y = \operatorname{Flux}_X$.
- (v) $\operatorname{dist}_Y(p_0, bM) > \lambda$.

We will prove Lemma 3.1 in Section 3.3; we first proceed with the proofs of the main results of the paper.

3.1 Proof of Theorem 1.1 assuming Lemma 3.1

Let $D \subset \mathbb{R}^n$, M, $\mathfrak{p}\colon H_1(M;\mathbb{Z}) \to \mathbb{R}^n$, $K \subset M$, $X\colon K \to \mathbb{R}^n$, and $\epsilon > 0$ be as in the statement of Theorem 1.1.

Set $M_0 := K$ and choose an exhaustion of M by connected Runge compact domains $\{M_j\}_{j \in \mathbb{N}}$ such that the Euler characteristic $\chi(M_j \setminus \mathring{M}_{j-1})$ is -1 or 0 for all $j \in \mathbb{N}$ and

(1) $$M_0 \Subset M_1 \Subset \cdots \Subset \bigcup_{j \in \mathbb{Z}_+} M_j = M.$$

Existence of such is well known; see for instance [11, Lemma 4.2] for a simple proof.

Fix a countable subset $C = \{z_j\}_{j \in \mathbb{N}} \subset D$ with

(2) $$\overline{C} = \overline{D}.$$

Set $N_0 := M_0 = K$, $Y_0 := X$, and if $n \geq 5$ assume without loss of generality that Y_0 is an embedding (as we may in view of [7, Theorem 1.1]). Also fix a point $p_0 \in \mathring{N}_0$.

Take a sequence of positive real numbers $\{\epsilon_j\}_{j\in\mathbb{N}} \searrow 0$ which will be specified later.

We shall recursively construct a sequence $\{N_j, Y_j\}_{j\in\mathbb{N}}$ of smoothly bounded Runge compact domains $N_j \subset M$ and conformal minimal immersions $Y_j\colon N_j \to \mathbb{R}^n$ of class $\mathscr{C}^1(N_j)$ satisfying the following properties for all $j \in \mathbb{N}$:

(a$_j$) $Y_j(N_j) \subset D$.

(b$_j$) $N_j \subset M_j$ and N_j is a strong deformation retract of M_j.

(c$_j$) $\|Y_j - Y_{j-1}\|_{1,N_{j-1}} < \epsilon_j$.

(d$_j$) $\mathrm{dist}(z_k, Y_j(N_j)) < \epsilon_j$ for all $k \in \{1, \dots, j\}$.

(e$_j$) $\mathrm{dist}_{Y_j}(p_0, bN_j) > j$.

(f$_j$) $\mathrm{Flux}_{Y_j}(\gamma) = \mathfrak{p}(\gamma)$ for all closed curves $\gamma \subset N_j$.

(g$_j$) If $D = \mathbb{R}^n$ then $N_j = M_j$.

(h$_j$) If $n \geq 5$ then Y_j is an embedding.

Observe that condition (a$_j$) always holds in the case $D = \mathbb{R}^n$.

Assume for a moment that we have already constructed such a sequence and let us show that if each $\epsilon_j > 0$ in the recursive procedure is chosen sufficiently small (in terms of the geometry of Y_{j-1}) then the sequence $\{Y_j\}_{j\in\mathbb{N}}$ converges uniformly on compact subsets of

$$\Omega := \bigcup_{j\in\mathbb{N}} N_j \subset M \tag{3}$$

to a conformal minimal immersion

$$Y := \lim_{j\to+\infty} Y_j\colon \Omega \to \mathbb{R}^n$$

satisfying the conclusion of the theorem. Indeed, first of all notice that the properties (b$_j$) and (g$_j$), along with (1) and (3), ensure condition (I) in the statement of the theorem and that $\Omega = M$ if $D = \mathbb{R}^n$. Now, choosing the ϵ_j so that

$$\sum_{j\in\mathbb{N}} \epsilon_j < \epsilon, \tag{4}$$

we have in view of (c$_j$) that the limit map Y exists and satisfies condition (II). Furthermore, if the sequence $\{\epsilon_j\}_{j\in\mathbb{N}}$ decreases to zero fast enough then, by Harnack's theorem, Y is a conformal minimal immersion. Likewise, by (c$_j$), (e$_j$), and (f$_j$), we have that Y is complete and satisfies (III) whenever each $\epsilon_j > 0$ is small enough.

Let us now check condition (IV). For the first part observe that the properties (a_j) ensure that $Y(\Omega) \subset \overline{D}$; let us show that $Y(\Omega) \cap bD = \varnothing$. For that, we choose

$$\epsilon_j < \frac{1}{j^2} \operatorname{dist}(Y_{j-1}(N_{j-1}), bD) \quad \text{for all } j \in \mathbb{N}. \tag{5}$$

Notice that the term on the right-hand side of the above inequality is positive by (a_j), and hence such an $\epsilon_j > 0$ exists. Take $p \in \Omega$ and let us show that $\operatorname{dist}(Y(p), bD) > 0$; this will ensure that $Y(\Omega) \subset D$. Choose $j_0 \in \mathbb{N}$ such that $p \in N_{j-1}$ for all $j \geq j_0$. Then

$$\begin{aligned}\operatorname{dist}(Y_{j-1}(p), bD) &\leq |Y_{j-1}(p) - Y_j(p)| + \operatorname{dist}(Y_j(p), bD)\\ &\overset{(c_j)}{<} \epsilon_j + \operatorname{dist}(Y_j(p), bD)\\ &\overset{(5)}{<} \frac{1}{j^2} \operatorname{dist}(Y_{j-1}(p), bD) + \operatorname{dist}(Y_j(p), bD).\end{aligned}$$

Thus, $\operatorname{dist}(Y_j(p), bD) \geq (1 - 1/j^2) \operatorname{dist}(Y_{j-1}(p), bD)$ for all $j \geq j_0$, and so

$$\operatorname{dist}(Y_{j_0+i}(p), bD) \geq \operatorname{dist}(Y_{j_0}(p), bD) \prod_{j=j_0+1}^{j_0+i} \left(1 - \frac{1}{j^2}\right) \quad \text{for all } i \in \mathbb{N}.$$

Taking limits in the above inequality as $i \to +\infty$ we obtain

$$\operatorname{dist}(Y(p), bD) \geq \tfrac{1}{2} \operatorname{dist}(Y_{j_0}(p), bD) > 0,$$

where the latter inequality is ensured by (a_{j_0}); take into account that $Y_{j_0}(N_{j_0})$ is compact. This shows that $Y(\Omega) \subset D$.

In order to check the second part of condition (IV) pick a point $z \in \overline{D}$ and a positive number $\delta > 0$ and let us show that $\operatorname{dist}(z, Y(\Omega)) < \delta$; this will imply that $\overline{Y(\Omega)} = \overline{D}$. Indeed, in view of (2) there exists $j_0 \in \mathbb{N}$ such that the point $z_{j_0} \in C \subset D$ meets

$$|z_{j_0} - z| < \delta/3. \tag{6}$$

Moreover, since $\{\epsilon_j\} \searrow 0$, there exists $j_1 \in \mathbb{N}$ such that $\epsilon_{j_1} < \delta/3$, and so the properties (d_j) guarantee that for any $j \geq j_1$,

$$\operatorname{dist}(z_k, Y_j(N_j)) < \delta/3 \quad \text{for all } k \leq j. \tag{7}$$

Finally, (4) ensures the existence of $j_2 \in \mathbb{N}$ such that $\sum_{k=j_2}^{\infty} \epsilon_k < \delta/3$, and hence, the properties (c_j) imply that for all $j > j_2$,

$$\|Y - Y_j\|_{1,N_j} < \delta/3. \tag{8}$$

Combining (6), (7), and (8) we obtain that, for any $j > \max\{j_0, j_1, j_2\}$,

$$\begin{aligned}\operatorname{dist}(z, Y(\Omega)) &\le |z - z_{j_0}| + \operatorname{dist}(z_{j_0}, Y(\Omega)) \\ &\overset{(3)}{\le} |z - z_{j_0}| + \operatorname{dist}(z_{j_0}, Y(N_j)) \\ &\le |z - z_{j_0}| + \operatorname{dist}(z_{j_0}, Y_j(N_j)) + \|Y_j - Y\|_{1,N_j} < \delta.\end{aligned}$$

This proves that $Y(\Omega)$ is dense on $\overline{D}$ and hence condition (IV).

Finally, assume $n \ge 5$ and let us show the limit map $Y\colon \Omega \to \mathbb{R}^n$ is one-to-one provided that the positive numbers $\{\epsilon_j\}_{j\in\mathbb{N}}$ are taken sufficiently small. It suffices to choose

$$\epsilon_j < \frac{1}{2j^2} \inf\{|Y_{j-1}(p) - Y_{j-1}(q)| : p, q \in N_{j-1}, \mathsf{d}(p,q) > 1/j\}, \tag{9}$$

where $\mathsf{d}(\cdot\,,\,\cdot)$ is any fixed Riemannian distance on M. Indeed, pick points $p, q \in \Omega$, $p \neq q$, and let us check $Y(p) \neq Y(q)$. Choose $j_0 \in \mathbb{N}$ large enough that $p, q \in N_{j-1}$ and $\mathsf{d}(p,q) > 1/j$ for all $j \ge j_0$; such a j_0 exists in view of (b$_j$) and (3). Then

$$\begin{aligned}|Y_{j-1}(p) - Y_{j-1}(q)| &\le |Y_{j-1}(p) - Y_j(p)| + |Y_j(p) - Y_j(q)| + |Y_j(q) - Y_{j-1}(q)| \\ &\overset{(c_j)}{<} 2\epsilon_j + |Y_j(p) - Y_j(q)| \\ &\overset{(9)}{<} \frac{1}{j^2}|Y_{j-1}(p) - Y_{j-1}(q)| + |Y_j(p) - Y_j(q)|.\end{aligned}$$

As above, this gives $|Y_j(p) - Y_j(q)| \ge (1 - 1/j^2)|Y_{j-1}(p) - Y_{j-1}(q)|$ for all $j \ge j_0$, and hence

$$|Y_{j_0+i}(p) - Y_{j_0+i}(q)| \ge |Y_{j_0}(p) - Y_{j_0}(q)| \prod_{j=j_0+1}^{j_0+i} \Big(1 - \frac{1}{j^2}\Big) \quad \text{for all } i \in \mathbb{N}.$$

Taking limits we obtain that

$$|Y(p) - Y(q)| \ge \tfrac{1}{2}|Y_{j_0}(p) - Y_{j_0}(q)| > 0,$$

where the latter inequality follows from (h$_{j_0}$). This implies that Y is one-to-one, proving condition (V) in the statement of the theorem.

To complete the proof it remains to construct the sequence $\{N_j, Y_j\}_{j\in\mathbb{N}}$ satisfying the required properties. We proceed in a recursive way. The basis of the induction is given by the pair (N_0, Y_0) which clearly meets properties (a$_0$), (b$_0$), (e$_0$), (f$_0$), (g$_0$), and (h$_0$); whereas (c$_0$) and (d$_0$) are vacuous. For the inductive step assume that we have (N_{j-1}, Y_{j-1}) satisfying (a$_{j-1}$)–(h$_{j-1}$) and let us construct (N_j, Y_j) enjoying the corresponding properties. We distinguish two different cases depending on the Euler characteristic of $M_j \setminus \mathring{M}_{j-1}$.

Noncritical case $\chi(M_j \setminus \mathring{M}_{j-1}) = 0$ By the Mergelyan theorem for conformal minimal immersions (see [7, Theorem 5.3]) we may assume without loss of generality that Y_{j-1} extends, with the same name, to a conformal minimal immersion $M \to \mathbb{R}^n$ with

(10) $$\mathrm{Flux}_{Y_{j-1}} = \mathfrak{p}.$$

Next, we choose $N_j \subset M_j$ as any smoothly bounded compact neighborhood of N_{j-1} such that

(11) $$Y_{j-1}(N_j) \subset D$$

and that N_{j-1} is a strong deformation retract of N_j; such an N_j exists in view of (a_{j-1}). Because $\chi(M_j \setminus \mathring{M}_{j-1}) = 0$, it follows that N_j is a strong deformation retract of M_j as well. This proves (b_j). If $D = \mathbb{R}^n$ then we choose, as we may since (11) is always satisfied, $N_j = M_j$, ensuring condition (g_j).

Now, in view of (11), we may apply Lemma 3.1 to the domain D, the compact bordered Riemann surface N_j, the conformal minimal immersion $Y_{j-1}\colon N_j \to D \subset \mathbb{R}^n$ of class $\mathscr{C}^1(N_j)$, the compact domain $N_{j-1} \subset \mathring{N}_j$, the points $p_0 \in \mathring{K} \subset \mathring{N}_{j-1}$, the points $z_1, \dots, z_j \in D$, and the positive numbers ϵ_j and $j > 0$. This provides a conformal minimal immersion $Y_j\colon N_j \to \mathbb{R}^n$ of class $\mathscr{C}^1(N_j)$ enjoying the following properties:

- (i) $Y_j(N_j) \subset D$.
- (ii) $\|Y_j - Y_{j-1}\|_{1,N_{j-1}} < \epsilon_j$.
- (iii) $\mathrm{dist}(z_k, Y_j(N_j)) < \epsilon_j$ for all $k \in \{1, \dots, j\}$.
- (iv) $\mathrm{Flux}_{Y_j}(\gamma) = \mathrm{Flux}_{Y_{j-1}}(\gamma)$ for all closed curves $\gamma \subset N_j$.
- (v) $\mathrm{dist}_{Y_j}(p_0, bN_j) > j$.

Furthermore, we may assume by [7, Theorem 1.1] that

- (vi) if $n \geq 5$ then Y_j is an embedding.

We claim that (N_j, Y_j) meets conditions (a_j)–(h_j). Indeed, (b_j) and (g_j) are already ensured. On the other hand, conditions (a_j), (c_j), (d_j), (e_j), and (h_j) coincide with (i), (ii), (iii), (v), and (vi), respectively, whereas (f_j) is implied by (iv) and (10). This concludes the proof of the inductive step in the noncritical case.

Critical case $\chi(M_j \setminus \mathring{M}_{j-1}) = -1$ In this case there is a smooth Jordan arc $\alpha \subset \mathring{M}_j \setminus \mathring{N}_{j-1}$, with its two endpoints in bN_{j-1} and otherwise disjoint from N_{j-1}, such that

$$S := N_{j-1} \cup \alpha \subset \mathring{M}_j$$

is a Runge *admissible* subset in M in the sense of [7, Definition 5.1] and a strong deformation retract of M_j. Fix a nowhere-vanishing holomorphic 1–form θ on M (such always exists by the Oka–Grauert principle; see Forstnerič [18, Theorem 5.3.1]; for an alternative proof see Alarcón, Fernández, and López [2, Proof of Theorem 4.2]). Next, consider a *generalized conformal minimal immersion* $(\widetilde{Y}, f\theta)$ *on* S in the sense of [7, Definition 5.2] such that

$$\widetilde{Y}|_{N_{j-1}} = Y_{j-1}, \quad \widetilde{Y}(\alpha) \subset D, \quad \text{and} \quad \int_\gamma f\theta = \imath\mathfrak{p}(\gamma) \text{ for all closed curves } \gamma \text{ in } S.$$

Such trivially exists in view of (a_{j-1}), (f_{j-1}), and the path-connectedness of D. By [7, Theorem 5.3] we may approximate $\widetilde{Y}$ in the $\mathscr{C}^1(S)$–topology by conformal minimal immersions $\widetilde{Y}_{j-1}\colon M \to \mathbb{R}^n$ having $\mathfrak{p}$ as flux map and being embeddings if $n \geq 5$. For any close enough such approximation $\widetilde{Y}_{j-1}$ of $\widetilde{Y}$ there exists a compact neighborhood N'_{j-1} of S in $\mathring{M}_j$ such that $N'_{j-1} \subset M$ is a smoothly bounded Runge compact domain, S is a strong deformation retract of N'_{j-1}, and $\widetilde{Y}_{j-1}$ formally meets conditions (a_{j-1})–(h_{j-1}) besides (g_{j-1}). It follows that the Euler characteristic $\chi(M_j \setminus \mathring{N}'_{j-1})$ equals 0, which reduces the proof of the inductive step to the noncritical case.

This concludes the recursive construction of the sequence $\{N_j, Y_j\}_{j\in\mathbb{N}}$ with the desired properties, and hence the proof of the theorem.

3.2 Proof of Theorem 1.2 assuming Lemma 3.1

Let $K_0 \subset M$ be a smoothly bounded compact subset and let $\epsilon > 0$. To prove the theorem it suffices to find a complete conformal minimal immersion $Y\colon M \to \mathbb{R}^n$ such that the following conditions are satisfied:

(a) $\|Y - X\|_{1,K_0} < \epsilon$.

(b) $\mathrm{Flux}_Y = \mathrm{Flux}_X$.

(c) $Y(M) \subset D$ and $\overline{Y(M)} = \overline{D}$.

(d) If $n \geq 5$ then Y is one-to-one.

Up to enlarging K_0 if necessary, we may assume that K_0 is a strong deformation retract of $\overline{M}$. Pick any countable subset $C = \{z_j\}_{j\in\mathbb{N}}$ of D such that

$$\overline{C} = \overline{D}. \tag{12}$$

Fix a point $p_0 \in \mathring{K}_0 \neq \varnothing$ and choose a sequence of positive numbers $\{\epsilon_j\}_{j\in\mathbb{N}} \searrow 0$ that will be specified later. Set $Y_0 := X\colon \overline{M} \to D \subset \mathbb{R}^n$ and, if $n \geq 5$, assume without loss of generality that Y_0 is an embedding (see [7, Theorem 1.1]). We shall inductively construct a sequence $\{K_j, Y_j\}_{j\in\mathbb{N}}$ of smoothly bounded compact domains

$$K_0 \Subset K_1 \Subset K_2 \Subset \cdots \Subset \bigcup_{j\in\mathbb{N}} K_j = M \tag{13}$$

and conformal minimal immersions $\{Y_j\colon \overline{M} \to \mathbb{R}^n\}_{j\in\mathbb{N}}$ of class $\mathscr{C}^1(\overline{M})$, satisfying the following properties for all $j \in \mathbb{N}$:

(I$_j$) $Y_j(\overline{M}) \subset D$.

(II$_j$) $\|Y_j - Y_{j-1}\|_{1,K_{j-1}} < \epsilon_j$.

(III$_j$) $\mathrm{dist}(z_k, Y_j(K_j)) < \epsilon_j$ for all $k \in \{1,\dots,j\}$.

(IV$_j$) $\mathrm{Flux}_{Y_j}(\gamma) = \mathrm{Flux}_{Y_{j-1}}(\gamma)$ for all closed curves $\gamma \subset M$.

(V$_j$) $\mathrm{dist}_{Y_j}(p_0, bK_j) > j$.

(VI$_j$) If $n \geq 5$ then Y_j is an embedding.

We construct the sequence in an inductive procedure similar to the one in the proof of Theorem 1.1. The basis of the induction is accomplished by the pair (K_0, Y_0) which clearly satisfies (I$_0$), (V$_0$), and (VI$_0$); conditions (II$_0$), (III$_0$), and (IV$_0$) are vacuous. For the inductive step we assume that we already have (K_{j-1}, Y_{j-1}) satisfying (I$_{j-1}$)–(V$_{j-1}$). By (I$_{j-1}$) we may apply Lemma 3.1 to the conformal minimal immersion Y_{j-1}, the compact domain K_{j-1}, the point $p_0 \in \mathring{K}_0 \subset \mathring{K}_{j-1}$, the points $z_1,\dots,z_j \in D$, and the positive numbers $\epsilon_j > 0$ and $j > 0$, obtaining a conformal minimal immersion $Y_j\colon \overline{M} \to \mathbb{R}^n$ of class $\mathscr{C}^1(\overline{M})$ satisfying the following properties:

(i) $Y_j(\overline{M}) \subset D$.

(ii) $\|Y_j - Y_{j-1}\|_{1,K_{j-1}} < \epsilon_j$.

(iii) $\mathrm{dist}(z_k, Y(\overline{M})) < \epsilon_j$ for all $k \in \{1,\dots,j\}$.

(iv) $\mathrm{Flux}_{Y_j} = \mathrm{Flux}_{Y_{j-1}}$.

(v) $\mathrm{dist}_{Y_j}(p_0, bM) > j$.

Further, by [7, Theorem 1.1] we may assume that

(vi) if $n \geq 5$ then Y_j is an embedding.

Conditions (I$_j$), (II$_j$), (IV$_j$), and (VI$_j$) coincide with (i), (ii), (iv), and (vi). Finally, since the inequalities in (iii) and (v) are both strict, conditions (III$_j$) and (V$_j$) hold for any large enough smoothly bounded compact domain $K_j \subset M$ being a strong deformation retract of $\overline{M}$. At each step in the recursive construction, we choose such a K_j containing K_{j-1} in its interior and being large enough that (13) is satisfied. This completes the inductive step and concludes the construction of the sequence $\{K_j, Y_j\}_{j\in\mathbb{N}}$ satisfying conditions (I_j)–(VI_j).

We claim that choosing the number $\epsilon_j > 0$ sufficiently small (depending on the geometry of Y_{j-1}) at each step in the recursive construction, the sequence $\{Y_j\}_{j\in\mathbb{N}}$ converges uniformly on compact subsets in M to a limit map

$$Y := \lim_{j\to\infty} Y_j \colon M \to \mathbb{R}^n$$

which satisfies conditions (a)–(d). Indeed, reasoning as in the proof of Theorem 1.1, (II_j) ensures that the limit map Y is a conformal minimal immersion and meets (a). On the other hand, (IV_j) implies (b); (V_j) and (II_j) guarantee the completeness of Y; (c) follows from (I_j), (II_j), and (III_j); and properties (II_j) and (VI_j) give condition (d). This completes the proof.

3.3 Proof of Lemma 3.1

Without loss of generality we may assume that $k = 1$; the general case follows from a standard finite recursive application of this particular one. Define $x := x_1$.

We may also assume without loss of generality that $\overline{M}$ is a smoothly bounded compact domain in an open Riemann surface $\mathcal{R}$. Pick a point $p \in bM$ and a smooth embedded arc $\gamma \subset \mathcal{R}\setminus M$ having p as an endpoint, being otherwise disjoint from $\overline{M}$, and such that

$$S := \overline{M} \cup \gamma$$

is a Runge *admissible* subset of $\mathcal{R}$ in the sense of [7, Definition 5.1]. Let $q \in \mathcal{R} \setminus \overline{M}$ denote the other endpoint of γ.

Fix a nowhere-vanishing holomorphic 1–form θ on $\mathcal{R}$. Consider a *generalized conformal minimal immersion* $(\widetilde{X}, f\theta)$ *on* S in the sense of [7, Definition 5.2] such that the $\mathscr{C}^1(S)$–map $\widetilde{X}\colon S \to \mathbb{R}^n$ satisfies the following properties:

(A) $\widetilde{X}|_{\overline{M}} = X$.

(B) $\widetilde{X}|_\gamma \subset D$.

(C) $\widetilde{X}(q) = x$.

Existence of such is trivial; recall that $X(\overline{M}) \subset D$ and that D is path-connected.

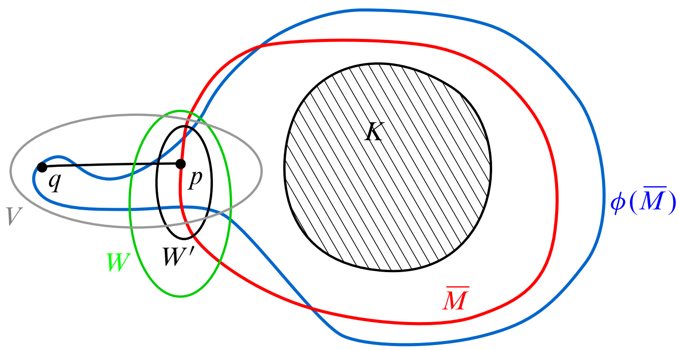

Figure 1: The diffeomorphism $\phi\colon \overline{M} \to \phi(\overline{M}) \subset U$

Fix a constant $\delta > 0$ to be specified later.

The Runge–Mergelyan theorem for conformal minimal immersions [7, Theorem 5.3] provides a conformal minimal immersion $\widetilde{Y}\colon \mathcal{R} \to \mathbb{R}^n$ such that

(D) $\|\widetilde{Y} - \widetilde{X}\|_{1,S} < \delta$, and

(E) $\mathrm{Flux}_{\widetilde{Y}}(\alpha) = \mathrm{Flux}_{\widetilde{X}}(\alpha)$ for all closed curves $\alpha \subset M$.

Since X assumes values in D, properties (A) and (B) ensure that $\widetilde{X}(S) \subset D$, and hence, choosing $\delta > 0$ sufficiently small, (D) guarantees the existence of a small open neighborhood U of S in $\mathcal{R}$ such that

$$\widetilde{Y}(U) \subset D. \tag{14}$$

Next we use the method of exposing boundary points on a compact bordered Riemann surface. Choose small open neighborhoods $W' \Subset W \Subset U \setminus K$ and $V \Subset U$ of p and γ in U, respectively. By Forstnerič and Wold [23, Theorem 2.3] (see also Forstnerič [18, Theorem 8.8.1]) there exists a smooth diffeomorphism

$$\phi\colon \overline{M} \to \phi(\overline{M}) \subset U \tag{15}$$

satisfying the following properties (see Figure 1):

(F) $\phi\colon M \to \phi(M)$ is a biholomorphism.

(G) ϕ is δ–close to the identity in the $\mathscr{C}^1$–norm on $\overline{M} \setminus W'$.

(H) $\phi(p) = q \in b\phi(\overline{M})$ and $\phi(\overline{M} \cap W') \subset W \cup V$.

We claim that the conformal minimal immersion $\widetilde{Y} \circ \phi\colon \overline{M} \to \mathbb{R}^n$ of class $\mathscr{C}^1(\overline{M})$ formally satisfies conditions (i)–(iv) in the statement of the lemma provided that $\delta > 0$

is chosen sufficiently small. Indeed, by (14) and (15) we have that $\widetilde{Y}(\phi(\overline{M})) \subset D$, proving (i). On the other hand, since $K \subset \overline{M} \setminus W'$, properties (G), (D), and (A) give that $\|\widetilde{Y} \circ \phi - X\|_{1,K} < \epsilon$, whenever $\delta > 0$ is small enough, which ensures condition (ii). Finally, properties (H), (D), and (C) guarantee (iii) for any $\delta < \epsilon$, whereas (F), (E), and (A) imply (iv).

Finally, [1, Lemma 4.1] enables us to approximate the immersion $\widetilde{Y} \circ \phi \colon \overline{M} \to \mathbb{R}^n$ in the $\mathscr{C}^0(\overline{M})$–topology, and hence in the $\mathscr{C}^1(K)$–topology, by conformal minimal immersions $Y \colon \overline{M} \to \mathbb{R}^n$ of class $\mathscr{C}^1(\overline{M})$ satisfying (v) and $\mathrm{Flux}_Y = \mathrm{Flux}_{\widetilde{Y} \circ \phi}$; the latter ensures (iv). Clearly, any close enough such approximation Y of $\widetilde{Y} \circ \phi$ still satisfies conditions (i), (ii), and (iii). This concludes the proof of Lemma 3.1.

The proofs of Theorems 1.1 and 1.2 are now complete.

4 Analogous results for other families of surfaces

As we already pointed out in the introduction of this paper, all the tools required in the proofs of Theorems 1.1 and 1.2 (ie the Runge–Mergelyan approximation, the general position result, and the Riemann–Hilbert method) are also available for some other interesting objects; namely, nonorientable minimal surfaces, complex curves, and holomorphic null and Legendrian curves. Therefore, our techniques easily adapt to give analogous results to Theorems 1.1 and 1.2 for all these families of surfaces; we shall now discuss some of them, leaving the details of the proofs to the interested reader.

4.1 Nonorientable minimal surfaces in $\mathbb{R}^n$

These surfaces appeared in the very origin of minimal surface theory (we refer to the seminal paper by Lie [29] from 1878) and there is a large literature devoted to their study. *Conformal nonorientable minimal surfaces* in $\mathbb{R}^n$ for $n \geq 3$ are characterized as the images of conformal minimal immersions $X \colon M \to \mathbb{R}^n$ such that $X \circ \mathfrak{I} = X$, where $\mathfrak{I} \colon M \to M$ is an antiholomorphic involution without fixed points on an open Riemann surface M. For such an immersion we have that

$$\mathrm{Flux}_X(\mathfrak{I}_* \gamma) = -\mathrm{Flux}_X(\gamma) \quad \text{for all } \gamma \in H_1(M; \mathbb{Z}). \tag{16}$$

Recently Alarcón, Forstnerič, and López introduced in [8] new complex analytic techniques in the study of nonorientable minimal surfaces in $\mathbb{R}^n$; in particular, they provided all the required tools in our method of proof (see also Alarcón and López [12] for the

Runge–Mergelyan approximation in dimension 3). As happens in the orientable case, the general position of nonorientable minimal surfaces is embedded in $\mathbb{R}^n$ for all $n \geq 5$. Thus, completely analogous results to Theorems 1.1 and 1.2 may be proved in the nonorientable framework under the necessary condition (16) on the flux map.

4.2 Complex curves in $\mathbb{C}^n$

All the above-mentioned tools are classical for holomorphic immersions of open Riemann surfaces into $\mathbb{C}^n$ for $n \geq 2$, with being embedded the general position for $n \geq 3$. We refer to Bishop [14] for the Runge–Mergelyan approximation (see also Runge [33] and Mergelyan [31]) and to Drinovec Drnovšek and Forstnerič [15] and Alarcón and Forstnerič [3; 6] for the Riemann–Hilbert method (see also the introduction of Drinovec Drnovšek and Forstnerič [16] for a survey on this subject).

For example, by following the proof of Theorem 1.1 one may show the following result:

Theorem 4.1 *Let M be an open Riemann surface. The set of complete holomorphic immersions $M \to \mathbb{C}^n$ $(n \geq 2)$ with dense images forms a dense subset in the set $\mathscr{O}(M, \mathbb{C}^n)$ of all holomorphic functions $M \to \mathbb{C}^n$ endowed with the compact-open topology. Furthermore, if $n \geq 3$ then the set of all complete holomorphic one-to-one immersions $M \to \mathbb{C}^n$ with dense images is also dense in $\mathscr{O}(M, \mathbb{C}^n)$.*

We emphasize that the novelty of Theorem 4.1 is that it concerns *complete* immersions; obviously, the set of all holomorphic immersions $M \to \mathbb{C}^n$ is much larger than the subset consisting of the complete ones. Indeed, without completeness, there are many general such results in the literature. For instance, if we consider the space $\mathscr{O}(S, Z)$ of all holomorphic maps of a *Stein manifold* S (we refer to Gunning and Rossi [25] and Hörmander [26] for the theory of Stein manifolds) into an *Oka manifold* Z, endowed with the compact-open topology, then the basic Oka property with approximation and interpolation (see Forstnerič [18, Theorem 5.4.4]) easily implies that those maps in $\mathscr{O}(S, Z)$ having dense image form a dense subset; further, if $\dim Z \geq 2 \dim S$ (respectively, $\dim Z \geq 2 \dim S + 1$) then, by general position (see [18, Theorem 7.9.1 and Corollary 7.9.3]), the subset of immersions (respectively, one-to-one immersions) with dense image is also dense in $\mathscr{O}(S, Z)$. On the other hand, if $\dim S \geq \dim Z$ then there are strongly dominating *surjective* holomorphic maps $S \to Z$ (see Forstnerič [20]).

Along the same lines, Forstnerič and Winkelmann proved in [22] that, for any connected complex manifold Z (not necessarily Oka), the set of all holomorphic maps of the unit disk $\mathbb{D} \subset \mathbb{C}$ into Z with dense images is dense in $\mathcal{O}(\mathbb{D}, Z)$; see also Winkelmann [34] for a previous partial result in this direction.

4.3 Holomorphic null curves in $\mathbb{C}^n$

These are holomorphic immersions $F = (F_1, \dots, F_n)\colon M \to \mathbb{C}^n$ $(n \geq 3)$ of an open Riemann surface M into $\mathbb{C}^n$ which are directed by the null quadric

$$\mathfrak{A} = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : z_1^2 + \cdots + z_n^2 = 0\},$$

equivalently, which satisfy the nullity condition

$$(dF_1)^2 + \cdots + (dF_n)^2 = 0 \quad \text{everywhere on } M.$$

Notice that the punctured null quadric $\mathfrak{A}_* = \mathfrak{A} \setminus \{0\}$ is an Oka manifold (see Alarcón and Forstnerič [4, Example 4.4]). These curves are closely related to minimal surfaces in $\mathbb{R}^n$ since the real and the imaginary part of a null curve $M \to \mathbb{C}^n$ are flux-vanishing conformal minimal immersions $M \to \mathbb{R}^n$ (see eg Osserman [32]). The tools required to prove analogous results to Theorems 1.1 and 1.2 for holomorphic null curves have been provided recently in Alarcón and López [10], Alarcón and Forstnerič [4; 5], and Alarcón, Drinovec Drnovšek, and Forstnerič [1]. In this framework, the general position is embedded for $n \geq 3$.

4.4 Holomorphic Legendrian curves in $\mathbb{C}^{2n+1}$

These are holomorphic immersions $F{=}(X_1, Y_1, \dots, X_n, Y_n, Z)\colon M \to \mathbb{C}^{2n+1}$ $(n \in \mathbb{N})$ of an open Riemann surface M into $\mathbb{C}^{2n+1}$ which are tangent to the standard holomorphic contact structure of $\mathbb{C}^{2n+1}$, equivalently, such that

$$dZ + \sum_{j=1}^{n} X_j\, dY_j = 0 \quad \text{everywhere on } M.$$

All the needed tools in this case were furnished by Alarcón, Forstnerič, and López in [9], with being embedded the general position for all $n \in \mathbb{N}$. Holomorphic Legendrian curves are complex analogues of real Legendrian curves in $\mathbb{R}^{2n+1}$, which play an important role in differential geometry, in particular, in contact geometry.

Recall a *complex contact manifold* is a complex manifold W of odd dimension $2n{+}1{\geq}3$ endowed with a *holomorphic contact structure* $\mathscr{L}$; the latter is a holomorphic vector subbundle of complex codimension one in the tangent bundle TW such that every point $p \in W$ admits an open neighborhood $U \subset W$ in which $\mathscr{L}|_U = \ker \eta$ for a holomorphic 1–form η on U satisfying $\eta \wedge (d\eta)^n \neq 0$ everywhere on U. A holomorphic immersion $F\colon M \to W$ is said to be *Legendrian* if it is everywhere tangent to the contact structure:

$$dF_p(T_p M) \subset \mathscr{L}_{F(p)} \quad \text{for all } p \in M.$$

By Darboux's theorem (see [9, Theorem A.2]) every complex contact manifold $(W, \mathscr{L})$ of dimension $2n+1$ is locally contactomorphic to $\mathbb{C}^{2n+1}$ endowed with its standard holomorphic contact structure. Thus, as a direct consequence of the results analogous to Theorems 1.1 and 1.2 for Legendrian curves in $\mathbb{C}^{2n+1}$ one easily obtains:

Corollary 4.2 *Let $(W, \mathscr{L})$ be a complex contact manifold. Every point $p \in W$ admits an open neighborhood $U \subset W$ with the following property: Given a domain $V \Subset U$ there are holomorphic Legendrian one-to-one immersions $M \to V$ which are dense on V and are complete with respect to every Riemannian metric in W, where M is either a given bordered Riemann surface or some complex structure on any given smooth orientable connected open surface.*

The proof of the above corollary follows the one of [9, Corollary 1.3]; we refer the reader there for the details. It remains as an open question whether every complex contact manifold, endowed with a Riemannian metric, admits complete dense complex Legendrian curves.

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