Geometry & Topology 22 (2018) 591–644

Intrinsic structure of minimal discs in metric spaces

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We study the intrinsic structure of parametric minimal discs in metric spaces admitting a quadratic isoperimetric inequality. We associate to each minimal disc a compact, geodesic metric space whose geometric, topological, and analytic properties are controlled by the isoperimetric inequality. Its geometry can be used to control the shapes of all curves and therefore the geometry and topology of the original metric space. The class of spaces arising in this way as intrinsic minimal discs is a natural generalization of the class of Ahlfors regular discs, well-studied in analysis on metric spaces.

49Q05, 53A10, 53C23

1 Introduction

1.1 Motivation

A smooth minimal surface in a Riemannian manifold has vanishing mean curvature. The Gauss equation then forces the intrinsic curvature of the minimal surface to be no larger than that of the ambient space. This has strong implications on the intrinsic geometry and thus on the local and global shape of the minimal surface. Given a smooth Jordan curve Γ in a Riemannian manifold M of bounded geometry, the classical solution of the Plateau problem (Douglas [8], Radó [35], Morrey [29]) provides a minimal disc $u\colon D \to M$ spanned by Γ. This solution, which is a priori constructed in a Sobolev space, turns out to be a smooth map. Moreover, it is an immersion up to finitely many *branch points*. In the absence of branch points, the minimal disc with its induced Riemannian metric turns out to be a smooth 2–dimensional Riemannian manifold Z with boundary given by Γ. The map u factors as $u = \bar{u} \circ P$, where $P\colon \bar{D} \to Z$ is a diffeomorphism and $\bar{u}\colon Z \to M$ is a Riemannian immersion, and thus $\bar{u}$ preserves the length of all curves. Moreover, the Gauss equation and other implications of minimality provide restrictions on the geometry of Z.

The aim of the present article is to investigate the intrinsic geometry of minimal discs in the much broader setting of metric spaces with quadratic isoperimetric inequality and

Published: 6 November 2017 DOI: 10.2140/gt.2018.22.591

to find structures analogous to those observed in the setting of Riemannian manifolds. Given any *minimal disc* $u\colon \bar{D} \to X$ in such a metric space X we would like to find a nice metric space Z whose properties reflect geometric properties of X, and such that u factorizes into a homeomorphism from $\bar{D}$ to Z and a length-preserving immersion $Z \to X$. One cannot achieve this in full, as examples will demonstrate, but *almost* as our results will show. The geometric properties of the space Z control the shape of the Jordan curve Γ and therefore the geometry of X. Results obtained in this paper are used in Lytchak and Wenger [22] to prove an isoperimetric characterization of upper curvature bounds, in Lytchak, Wenger and Young [27] to study topological and asymptotic properties of spaces with quadratic isoperimetric inequality and in Lytchak and Wenger [25] to find natural parametrizations of Ahlfors 2–regular discs and their generalizations.

1.2 Setting and construction

In Lytchak and Wenger [24], we have found a solution to the classical Plateau problem in any proper metric space X. Given any Jordan curve $\Gamma \subset X$ one would like to find a disc bounded by Γ with minimal (parametrized Hausdorff) area. As in the classical situation, it is natural to look for a solution in the set $\Lambda(\Gamma, X)$ of all Sobolev maps $v \in W^{1,2}(D, X)$, whose trace $\operatorname{tr}(v)$ is a weakly monotone reparametrization of the curve Γ. If X is a proper metric space and $\Lambda(\Gamma, X)$ is not empty then there indeed exists a map u with minimal area $\operatorname{Area}(u)$ in $\Lambda(\Gamma, X)$, as we showed in [24]. We found a special area minimizer $u \in \Lambda(\Gamma, X)$ which moreover has minimal (Reshetnyak) energy $E_+^2(u)$ among all area minimizers in $\Lambda(\Gamma, X)$. We will call such a map u a *minimal disc* or *solution of the Plateau problem for* (Γ, X).

If no restrictions are imposed on X, then a solution u of the Plateau problem can be as irregular as any Sobolev map. The situation changes under the natural assumption that in X any Lipschitz curve $\gamma\colon S^1 \to X$ of length $l \le l_0$ bounds a (Sobolev) disc $v\colon D \to X$ of area $A \le Cl^2$, where C, $l_0 > 0$ are fixed constants. We say that X admits a (C, l_0)–isoperimetric inequality (for the Hausdorff area). Many geometrically significant spaces like compact Lipschitz manifolds, compact spaces with one-sided curvature bounds, Banach spaces and many others satisfy this assumption.

Under this isoperimetric assumption the classical "a priori Hölder estimates" apply, and any solution $u \in W^{1,2}(D, X)$ of the Plateau problem for (Γ, X) turns out to have a continuously extendible representative $u\colon \bar{D} \to X$. The continuity of the solution u makes it, from an almost everywhere defined map, a geometric and topological object. We borrow the recipe for the construction of the space Z from the smooth situation. As to any continuous map, one can associate to u an *intrinsic pseudodistance* $d_u\colon \bar{D}\times\bar{D}\to[0,\infty]$ by

$$d_u(z_1,z_2) := \inf\{\text{length of } u\circ\gamma \mid \gamma \text{ curve in } \bar{D} \text{ between } z_1 \text{ and } z_2\}.$$

It turns out (see Theorem 1.1) that d_u is finite-valued. Hence, identifying points on $\bar{D}$ with d_u–distance 0 from each other we obtain a metric space $Z=Z_u$ which we will call the *intrinsic minimal disc associated with* u. Since Z_u arises from $\bar{D}$ by an identification of some points we have a canonical projection $P\colon \bar{D}\to Z$.

1.3 Properties of the intrinsic minimal disc

We fix the following general setting for the whole subsection:

- X is a complete metric space admitting a (C,l_0)–isoperimetric inequality.
- $\Gamma\subset X$ is a Jordan curve of finite length.
- $u\colon \bar{D}\to X$ is a solution of the Plateau problem for (Γ,X).
- $d_u\colon \bar{D}\times\bar{D}\to[0,\infty)$ is the pseudodistance induced by u.
- $Z=Z_u$ is the intrinsic minimal disc associated with u.
- $P\colon \bar{D}\to Z$ is the induced canonical projection.

Morrey's proof of the Hölder regularity of minimal discs, generalized in Lytchak and Wenger [24] to metric spaces, leads to:

Theorem 1.1 *In the general setting and notations introduced above, the pseudodistance d_u assumes only finite values and is continuous. The metric space $Z=Z_u$ obtained from the pseudometric d_u is compact and geodesic. The canonical projection $P\colon \bar{D}\to Z$ is continuous. The map $u\colon \bar{D}\to X$ has a canonical factorization $u=\bar{u}\circ P$, where $\bar{u}\colon Z\to X$ is a 1–Lipschitz map. For any curve γ in $\bar{D}$ the lengths of $P\circ\gamma$ and $u\circ\gamma$ coincide, and thus $\bar{u}$ preserves the length of $P\circ\gamma$.*

We are going to discuss the topological, geometric, and analytic properties of the constructed space Z.

Theorem 1.2 *The intrinsic minimal disc Z is homeomorphic to $\bar{D}$. The Hausdorff area $\mathcal{H}^2(Z)$ and the length $\ell(\partial Z)$ of the boundary circle are finite. The domain $\Omega\subset Z$ enclosed by any Jordan curve of length $l<l_0$ in Z satisfies*

$$\mathcal{H}^2(\Omega)\le C\cdot l^2. \tag{1-1}$$

The isoperimetric property of the topological disc Z has strong implications: lower bound on the area growth, linear local contractibility and the existence of controlled decompositions into subsets of small diameter. For the sake of simplicity, we formulate these consequences here only in the case of short boundary curves Γ, referring the reader to Section 8 for the general case.

Corollary 1.3 *Let Z be as above and assume that the length l of the boundary circle ∂Z is at most l_0. Then:*

(i) *For every $z \in Z$ and any $r \leq d(\partial Z, z)$ the area of the ball $B(z,r)$ of radius r around z is bounded by*

$$\mathcal{H}^2(B(z,r)) \geq \left(\frac{\pi}{4}\right)^2 \cdot \frac{1}{4C} \cdot r^2. \tag{1-2}$$

(ii) *For any $z \in Z$ and $r > 0$ the ball $B(z,r)$ in Z is contractible inside the ball $B(z,(8C+1)\cdot r)$.*

(iii) *There exists a constant $M = M(C) > 0$ such that the following holds true. For every $n > 0$ there exists a finite, connected graph $\partial Z \subset G_n \subset Z$ such that any component of $Z \setminus G_n$ is a disc of diameter at most l/n and such that the number of these components is at most $M \cdot n^2$.*

Theorem 1.2 and Corollary 1.3 may be considered as weak analogues of the statement that the curvature of a minimal surface is not larger than that of the ambient space. Indeed, on a smooth surface the isoperimetric inequality is closely related to upper bounds on the curvature and lower bounds on the injectivity radius, a circumstance which will be analyzed in depth in Lytchak and Wenger [22]. Similarly, lower bounds on the volume of balls are well known to be related to the curvature bounds of the manifold, for instance, by the theorem of Bishop and Gromov. We will deduce in [22] from Corollary 1.3 that Z inherits from X upper curvature bounds in the sense of Alexandrov; see Petrunin [31] and Petrunin and Stadler [33].

Corollary 1.3 shows that Z is metrically very similar to the Euclidean disc. According to (i) and (ii) of Corollary 1.3 and to Bonk and Kleiner [4] and Wildrick [40], the space Z is locally *quasisymmetric* to the unit disc if areas of balls in Z have a quadratic *upper* bound in terms of the radius. However, this need not be the case in general. Indeed, Z might arise from $\bar{D} \subset \mathbb{R}^2$ by collapsing a closed ball $B \subset \bar{D}$ to a point; see Example 11.3. The decomposition result (iii) is a topological-geometric version of a similar discrete statement in groups with quadratic Dehn function proved by Papasoglu in [30]. It immediately implies that the set of isometry classes of spaces Z arising in

Theorem 1.2 is a relatively compact set with respect to the Gromov–Hausdorff topology, once C and l_0 are fixed and Γ has length at most l_0.

We emphasize that Corollary 1.3 follows only from the assumptions that Z is a geodesic 2–dimensional disc with the isoperimetric property (1-1). This observation is used in Lytchak and Wenger [25] as the starting point of further investigations of all such discs Z, also shedding new light on the well-investigated theory of Ahlfors 2–regular discs; see Bonk and Kleiner [4] and Rajala [36].

The topological and analytic properties of the map $P\colon \bar{D} \to Z$ are summarized in the next theorem.

Theorem 1.4 *The canonical projection $P\colon \bar{D} \to Z$ is a uniform limit of homeomorphisms $P_i\colon \bar{D} \to Z$. Moreover:*

(i) $P \in \Lambda(\partial Z, Z) \subset W^{1,2}(D, Z)$.

(ii) *$P\colon D \to Z$ is contained in $W^{1,p}_{\mathrm{loc}}(D, Z)$ for some $p > 2$ depending on C.*

(iii) *$P\colon D \to Z$ is locally α–Hölder with $\alpha = \frac{\pi}{4} \cdot \frac{1}{4\pi C}$.*

(iv) *$\mathcal{H}^2(P(V)) = \mathrm{Area}(P|_V) = \mathrm{Area}(u|_V)$ for all open subsets $V \subset D$.*

The claim that P is a uniform limit of homeomorphisms comes as close as possible to the statement that $P\colon \bar{D} \to Z$ is a diffeomorphism in the classical smooth case. Even in the smooth case, if branch points are present, the natural map $P\colon \bar{D} \to Z$ is not bi-Lipschitz. However, in the smooth case, branch points are isolated and this forces the map P to be a homeomorphism. In our general setting, the set of "branch points" does not have to be discrete. Indeed, the map u (and then also P) may send an open subset of $\bar{D}$ to a single point of X (respectively of Z), as in Example 11.3 already mentioned above.

Inequality (1-2) and the constant α in (iii) of Theorem 1.4 are optimal at most up to the factors $\left(\frac{\pi}{4}\right)^2$ and $\frac{\pi}{4}$, respectively, as the example of a cone over a short circle Γ shows; see Example 11.1. This factor is related to the coarea formula in normed planes. It can be replaced by 1 if only Euclidean norms appear as metric differentials of u, for instance, in spaces with upper or lower curvature bounds; see Section 3.7. Another possibility to get an optimal factor is to use instead of $\mathcal{H}^2$ another definition of area; see Section 1.4.

The next theorem describes the map $\bar{u}\colon Z \to X$ as an *almost everywhere infinitesimal isometry* and thus as an "almost Riemannian immersion". In particular, $\bar{u}$ preserves $\mathcal{H}^2$ up to multiplicities.

Theorem 1.5 *Let $\overline{u}\colon Z \to X$ be the canonical map of the minimal disc Z to X described in Theorem 1.1. There exists a decomposition $Z = S \cup \bigcup_{1\le i<\infty} K_i$ with compact K_i and $\mathcal{H}^2(S) = 0$ such that the restrictions $\overline{u}\colon K_i \to \overline{u}(K_i)$ of the 1–Lipschitz map $\overline{u}$ are bi-Lipschitz. Moreover, for any $1 \le i < \infty$ and any $x \in K_i$ we have*

$$\lim_{y\to x, y\in K_i} \frac{d_Z(x,y)}{d_X(\overline{u}(x),\overline{u}(y))} = 1.$$

The map $\overline{u}$ sends ∂Z to Γ by an arclength-preserving homeomorphism.

If Γ is a *chord-arc curve*, and thus bi-Lipschitz equivalent to S^1, then one can control the regularity of u, P and Z uniformly up to the boundary, as is often the case in the investigation of the Plateau problem:

Theorem 1.6 *Assume in addition that Γ is a chord-arc curve. Then $P \in W^{1,p}(D,Z)$ for some $p > 2$. In particular, $P\colon \bar{D} \to Z$ is globally $\left(1-\frac{2}{p}\right)$–Hölder continuous. Moreover, there exists $\delta > 0$ such that for all $z \in Z$ and all $0 \le r < \delta$ we have $\mathcal{H}^2(B(z,r)) \ge \delta \cdot r^2$.*

The exponent p and the noncollapsing number δ are bounded in terms of C, l_0 and the bi-Lipschitz constant of Γ.

1.4 Area minimizers for different areas

There are several natural ways of measuring the area of 2–rectifiable subsets in general metric spaces, beyond the Hausdorff area used in the results above. Any choice of a definition of area μ in the sense of convex geometry (see Section 2 of this paper, Lytchak and Wenger [26] and Alvarez Paiva and Thompson [1]) provides a natural way to assign the μ–area $\mathrm{Area}_\mu(u)$ to any Sobolev disc $u\colon D \to X$. Among such definitions of area the most important ones are the (Busemann)–Hausdorff area $\mathcal{H}^2$, the Holmes–Thompson area μ^{ht}, the (Benson)–Gromov m^*–measure and (Ivanov's) inscribed Riemannian area μ^i. By Lytchak and Wenger [24], for any quasiconvex definition of area μ (for instance, for the four examples above) one can find a minimizer of the μ–area in any nonempty set $\Lambda(\Gamma, X)$, whenever Γ is a Jordan curve in a proper metric space X. As in the case of the Hausdorff measure, we can find a map u with minimal Reshetnyak energy $E_+^2(u)$ among all such minimizers. We call such a map $u \in \Lambda(\Gamma, X)$ a *μ–minimal map*. Quasiconvexity of μ is essential for the existence of μ–minimal maps, but does not play a role in the regularity questions discussed in [24] and here.

If X admits a (C, l_0)–isoperimetric inequality for the definition of area μ (see Section 4 and [24]), and if $u \in \Lambda(\Gamma, X)$ is μ–minimal, then again u has a continuous representative $u\colon \bar{D} \to X$. All results above apply to this more general setting. We just need to mention that the constructed intrinsic minimal disc $Z = Z_u$ is a countably 2–rectifiable set. The only difference from the special case of the Hausdorff area discussed above is that the constant $\frac{\pi}{4}$ appearing in (1-2) and in (iii) of Theorem 1.4, is replaced by a constant $q(\mu) \in \left[\frac{1}{2}, 1\right]$ depending on the definition of area μ. This constant $q(\mu)$ is maximal for the inscribed Riemannian area μ^i, that is, $q(\mu^i) = 1$, making the corresponding statements of Theorem 1.2 and Theorem 1.4 optimal in this case.

All definitions of area coincide for all Sobolev maps with values in X if the space X has the so-called property (ET) discussed in [24]. This is the case for many geometrically significant spaces such as spaces with one-sided curvature bounds in the sense of Alexandrov, sub-Riemannian manifolds or infinitesimally Hilbertian spaces with lower Ricci curvature bounds. Thus if X satisfies the property (ET), one can replace the factor $\frac{\pi}{4}$ appearing in (1-2) and in (iii) of Theorem 1.4 by $1 = q(\mu^i)$.

Moreover, if X satisfies property (ET) then any minimal disc $u \in \Lambda(\Gamma, X)$ is conformal and the map $P \in W^{1,2}(D, Z)$ is conformal as well. Without property (ET) the map P is $\sqrt{2}$–quasiconformal and the constant $\sqrt{2}$ is optimal. We emphasize that the quasiconformality and conformality are understood here in the infinitesimal almost everywhere sense (Section 3.7 and [24]) and do not imply that P is a homeomorphism.

1.5 Absolute minimal fillings

Our results apply to the problem of finding a disc realizing the infimum of Gromov's restricted minimal filling area problem; see S V Ivanov [15]. Let μ be a definition of area. Let (Γ, d_0) be a metric space bi-Lipschitz equivalent to the unit circle S^1. The restricted filling μ–area of Γ is defined as

$$m_\mu(\Gamma) := \inf\{\mu(M)\},$$

where M runs over all smooth Finsler metrics on the disc $\bar{D}$ such that for the induced distance function on ∂M one has a 1–Lipschitz homeomorphism $\partial M \to \Gamma$. Using the solution of the absolute Plateau problem in Lytchak and Wenger [24] and the results in the present paper we get:

Theorem 1.7 *Let (Γ, d_0) be a bi-Lipschitz circle and let μ be a quasiconvex definition of area. Then the restricted filling area $m_\mu(\Gamma)$ equals the Sobolev filling area defined as*

$$m_{\mu,\mathrm{Sob}}(\Gamma) = \inf\{\mathrm{Area}_\mu(u) : Y \textit{ complete}, \Gamma \subset Y, u \in \Lambda(\Gamma, Y)\}.$$

There exists a compact, geodesic, countably 2*–rectifiable metric space* Z *homeomorphic to* $\bar{D}$ *such that* $\mu(Z) = m_\mu(\Gamma)$*, and there exists a map* $P\colon \bar{D} \to Z$ *such that the conclusions of Theorems 1.2, 1.4 and 1.6 hold true with* $C = \frac{1}{2\pi}$ *and* $l_0 = \infty$ *and with the constant* $\frac{\pi}{4}$ *replaced by* $q(\mu)$*. Moreover, there exists a* 1*–Lipschitz arclength-preserving homeomorphism* $(\partial Z, d_Z) \to (\Gamma, d_0)$*.*

1.6 A useful technical result

We mention a technical achievement of the paper. The *geometrically obvious* fact that the restriction of a minimal disc to an open subdisc is again a minimal disc is indeed nontrivial because of two problems: the boundary of the subdisc might be wild, and even if it is smooth, the restriction of u to this boundary might be very far from being Lipschitz continuous. Both problems are solved in Section 4. The main implication for the present paper is the following seemingly obvious but technically nontrivial statement.

Proposition 1.8 *Let* X *admit a* (C, l_0)*–isoperimetric inequality for the definition of area* μ*. Let* $u\colon \bar{D} \to X$ *be a solution of the Plateau problem as above. Let* $T \subset \bar{D}$ *be a Jordan curve with Jordan domain* O*. If the curve* $u|_T$ *has finite length* $l \le l_0$ *then* $\mathrm{Area}_\mu(u|_O) \le C l^2$*.*

If the boundary curve T and the restriction of u to T are sufficiently regular, the proof of Proposition 1.8 is simple [24, Lemma 8.6].

1.7 Possible variations of the construction

It is possible to define the intrinsic metric structure for a minimal disc $u\colon \bar{D} \to X$ in a slightly different way. Namely, we can restrict the set of curves γ in the definition of the pseudometric d_u to any of the following families of curves: rectifiable, piecewise bi-Lipschitz, or piecewise smooth curves in $\bar{D}$. Unlike the smooth situation, the arising pseudometric and thus the associated metric space Z may depend on the choice of the family, even if u is Lipschitz continuous; see Petrunin [32, Section 4] for related discussions. However, for any of these choices of the family of curves and the corresponding associated metric space Z, all the theorems stated above remain valid. The last statement of Theorem 1.1 is then only true for curves γ in the corresponding family. The proofs remain the same; see Section 9.3 for some remarks.

It is possible to view the space Z (and the variants of Z constructed via different families of curves as above) from another classical perspective. The map u induces an L^2–field $z \mapsto \operatorname{ap}\operatorname{md} u_z$ of seminorms on D, the analogue of the pullback of the Riemannian metric; see Section 3.3. The length of *almost* any curve $\gamma \subset D$ with respect to the pseudodistance d_u can be computed by the same formula as in Finsler geometry, using this measurable field of seminorms. Thus, the space Z is *almost* defined by the approximate metric differentials of u.

1.8 Structure of the paper

Sections 2, 3, 7 and 4 consist of preliminaries and preparations. A reader familiar with the subject may skip these sections. In Section 2 we collect preliminaries from metric geometry, including definitions of area, as well as area and coarea formulas for rectifiable sets. In Section 3 we collect some basics about Sobolev maps from domains in $\mathbb{R}^2$ to metric spaces. In Section 7 we recall several statements from classical 2–dimensional topology related to the Jordan curve theorem. In Section 4 we deal with fillings of badly parametrized curves and gluings of Sobolev maps on nonregular domains, preparing for the proof of Proposition 1.8.

In Section 5 we recall the existence and regularity results for minimal discs from Lytchak and Wenger [24] and prove Proposition 1.8. We slightly reformulate the regularity statements from [24], emphasizing that the Hölder continuity is controlled via an estimate of the lengths of some image curves, thus giving us control over the intrinsic structure of the minimal disc. In Section 6 we fix a minimal disc $u\colon \bar{D} \to X$, associate to u a metric space Z as in Section 1.3 above and prove Theorem 1.1. We observe that the approximate metric differentials of u and P coincide at almost all points of D. In particular, this shows that P is as regular as u. This includes all statements of Theorem 1.4 except the first and main one that P is a uniform limit of homeomorphisms. Moreover, it implies that Z is a countably 2–rectifiable space. What remains to be controlled are the topological and isoperimetric properties of Z. In Section 8 we prove the topological and isoperimetric properties of the space Z stated in Theorem 1.2 and Corollary 1.3, using classical results from 2–dimensional topology. In Section 9 we collect everything proven so far and finish the proof of the main results. In Section 10 we discuss the absolute filling problem and prove Theorem 1.7. In the last section we collect examples mentioned above and some natural questions about the structure of minimal discs.

Acknowledgements We would like to thank Robert Young for a discussion on [30] which inspired (iii) of Corollary 1.3 above. We thank Heiko von der Mosel, Anton Petrunin, Stephan Stadler and the referee for helpful comments and discussions.

Lytchak was partially supported by the DFG Grant SPP 2026. Wenger was partially supported by Swiss National Science Foundation Grants 153599 and 165848.

2 Preliminaries

2.1 Basic notation

The following notation will be used throughout the paper. The Euclidean norm of a vector $v \in \mathbb{R}^n$ is denoted by $|v|$. We denote the open unit disc in $\mathbb{R}^2$ by D. A *domain* will always mean an open, bounded, connected subset of $\mathbb{R}^2$.

Metric spaces appearing in this paper will be assumed complete. A metric space is called proper if its closed bounded subsets are compact. We will denote distances in a metric space X by d or d_X. Let $X=(X,d)$ be a metric space. The open ball in X of radius r and center $x_0 \in X$ is denoted by

$$B(x_0,r)=B_X(x_0,r)=\{x\in X : d(x_0,x)<r\}.$$

A Jordan curve in X is a subset $\Gamma \subset X$ which is homeomorphic to S^1. Given a Jordan curve $\Gamma \subset X$, a continuous map $c\colon S^1 \to X$ is called a weakly monotone parametrization of Γ if c is the uniform limit of homeomorphisms $c_i\colon S^1 \to \Gamma$. For $m \geq 0$, the m–dimensional Hausdorff measure on X is denoted by $\mathcal{H}^m = \mathcal{H}^m_X$. The normalizing constant is chosen in such a way that on Euclidean $\mathbb{R}^m$ the Hausdorff measure $\mathcal{H}^m$ equals the Lebesgue measure $\mathcal{L}^m$. By $\mathfrak{S}_2$ we denote the proper metric space of seminorms on $\mathbb{R}^2$ with the distance given by $d_{\mathfrak{S}_2}(s,s') = \max_{v\in S^1}\{|s(v)-s'(v)|\}$.

2.2 Rectifiable curves

Let $X=(X,d)$ be a metric space. The *length* of a (continuous) curve $c\colon I \to X$, defined on an interval $I \subset \mathbb{R}$, is given by

$$\ell_X(c) := \sup\left\{\sum_{i=1}^{k} d(c(t_i),c(t_{i+1})) : t_i \in I,\ t_1 < \cdots < t_{k+1}\right\}. \tag{2-1}$$

The definition extends to continuous curves defined on S^1. A continuous curve of finite length is called *rectifiable*.

If $c\colon I=[a,b]\to X$ is a rectifiable curve of length l then the length function of c is the continuous monotone map $s\colon I=[a,b]\to[0,l]$ given by $s(t)=\ell(c|_{[a,t]})$. The curve c is parametrized by arclength if $s\colon I\to[0,l]$ is an isometry. The curve c has the form $c_0\circ s$, where $c_0\colon[0,l]\to X$ is the arclength parametrization of c.

A *geodesic* is an isometric embedding of an interval. A space X is called *a geodesic space* if any pair of points in X is connected by a geodesic. A space X is a *length space* if for all $x,y\in X$ the distance $d(x,y)$ coincides with $\inf\{\ell_X(c)\}$, where c runs over the set of all curves connecting x and y. A proper length space is a geodesic space by the Hopf–Rinow theorem. A rectifiable curve c is called *absolutely continuous* if it sends subsets of $\mathcal{H}^1$–measure 0 in $\mathbb{R}$ to subsets of $\mathcal{H}^1$–measure 0 in X. Equivalently, the length function s of c is contained in the Sobolev space $W^{1,1}(I)$. In this case we have $\ell(c)=\int_I s'(t)\,dt$. Moreover, for almost all $t\in I$, the value $s'(t)$ is the *metric differential* of c at t, and thus

$$s'(t)=\lim_{\epsilon\to 0}\frac{d(c(t),c(t+\epsilon))}{|\epsilon|}. \tag{2-2}$$

For a Borel function $f\colon c\to[0,\infty]$ we set as usual

$$\int_c f:=\int_0^l f(c_0(t))\,dt.$$

A rectifiable curve $c\colon I\to X$ with length parametrization $s\colon I\to\mathbb{R}$ is in the Sobolev space $W^{1,p}(I,X)$, $1\le p<\infty$, if and only if s is in the classical space $W^{1,p}(I,\mathbb{R})$. A concatenation of Sobolev curves is a Sobolev curve. If T is a metric space homeomorphic to an interval or a circle and $u\colon T\to X$ is a continuous map, we can unambiguously talk about the length of the curve $u(T)$, since it does not depend on the special parametrization of T by an interval. If T is a metric space bi-Lipschitz equivalent to an interval or a circle and $u\colon T\to X$ a continuous map to a metric space X we say that u is in the Sobolev class $W^{1,2}(T,X)$ if for the arclength parametrization $c\colon I\to T$ of T we have $u\circ c\in W^{1,2}(I,X)$.

A continuous curve $c\colon I\to X$ is called a piecewise bi-Lipschitz curve if there exists a partition of I into a finite number of subintervals such that the restriction of c to each subinterval is a bi-Lipschitz map.

A Jordan curve Γ is a *chord-arc* curve if Γ is bi-Lipschitz homeomorphic to S^1. By [38], a (connected, bounded) domain $\Omega\subset\mathbb{R}^2$ is a *Lipschitz domain* if and only if Ω is a bounded component of $R^2\setminus T$, where $T\subset R^2$ is a finite union of pairwise disjoint chord-arc curves.

2.3 Definitions of area

While there is an essentially unique natural way to measure areas of Riemannian surfaces, there are many different ways to measure areas of Finsler surfaces, some of them more appropriate for different questions. We refer the reader to [15; 3; 26; 1] for more information.

A definition of area μ assigns a multiple μ_V of $\mathcal{H}^2$ on any 2–dimensional normed space V, such that natural monotonicity assumptions are fulfilled; see [1]. In particular, it assigns the number $\boldsymbol{J}^\mu(s)$, *the* μ*–Jacobian*, to any seminorm s on $\mathbb{R}^2$ in the following way. By definition, $\boldsymbol{J}^\mu(s)=0$ if the seminorm is not a norm. If s is a norm then $\boldsymbol{J}^\mu(s)$ equals the $\mu_{(\mathbb{R}^2,s)}$–area of the unit Euclidean square in $\mathbb{R}^2$. Indeed, a choice of a definition of area μ is equivalent to a choice of a *Jacobian* $\boldsymbol{J}^\mu\colon \mathfrak{S}_2\to[0,\infty)$ which satisfies natural transformation and monotonicity conditions; see [26, Section 2.3].

Any two definitions of area differ at most by a factor of 2. The largest definition of area is the inscribed Riemannian definition of area μ^i, introduced in [15]. Other prominent examples are the Busemann definition $\mathcal{H}^2$, the Holmes–Thompson definition μ^{ht}, Gromov's *mass**–definition m^*. We refer the reader to [1; 15] for a thorough discussion of these examples and of the whole subject and to [26] for a detailed description of the corresponding Jacobians.

For a definition of area μ, the number $q(\mu)\in\left[\frac{1}{2},1\right]$ appearing in the main theorems of this paper (see Section 1.4) is defined to be the maximal number q such that $\mu_V\geq q\cdot\mu^i_V$ holds true on any normed plane V. Thus $q(\mu)$ equals the infimum of the quotients $\boldsymbol{J}^\mu(s)/\boldsymbol{J}^{\mu^i}(s)$ taken over all norms s. In particular (see [26]),

$$q(\mu^i)=1,\quad q(\mathcal{H}^2)=\frac{\pi}{4},\quad q(\mu^{\mathrm{ht}})=\frac{2}{\pi}.$$

2.4 Lipschitz maps and rectifiable sets

Given a measurable subset $K\subset\mathbb{R}^2$ and a Lipschitz map $f\colon K\to X$ to a metric space X, we say that a seminorm $s\colon\mathbb{R}^2\to[0,\infty)$ is the metric differential of f at the point $z\in K$ and denote it by $\mathrm{md}_z f$ if

$$\lim_{y\to z}\frac{d(f(z),f(y))-s(y-z)}{|y-z|}=0. \tag{2-3}$$

Any Lipschitz map defined on a measurable subset $K\subset\mathbb{R}^2$ has a uniquely defined metric differential at almost every point [18]. Moreover, for any $\epsilon>0$, the set K can be decomposed as a disjoint union $K=S\cup\bigcup_{1\leq i<\infty}K_i$ such that the following holds true. The set S is the union of a set of $\mathcal{H}^2$–area 0 and the set of all points at which the metric differential is not a norm. The sets K_i are compact. For any i, the restriction $f\colon K_i \to f(K_i) \subset X$ is $(1+\epsilon)$–bi-Lipschitz, if K_i is endowed with the distance induced by the norm $s_i = \mathrm{md}_z f$, for an arbitrary $z \in K_i$. Finally, $\mathcal{H}^2(f(S))$ is 0; see [18].

Recall that a metric space X is called *countably* 2*–rectifiable* if up to a subset $S \subset X$ of $\mathcal{H}^2$–measure 0, X is a countable union of Lipschitz images $f_i(K_i)$ of compact subsets of $K_i \subset \mathbb{R}^2$. The above decomposition result shows that up to a set of $\mathcal{H}^2$–area 0, any rectifiable set is a disjoint union of pieces which are arbitrarily bi-Lipschitz close to compact subsets of normed planes [18]. From this it follows that any definition of area μ provides a measure μ_X on any countably 2–rectifiable set X uniquely determined by the following properties. The measure μ_X is absolutely continuous with respect to $\mathcal{H}^2_X$, on Borel subsets of normed planes μ_X is defined as above and, finally, any 1–Lipschitz map between rectifiable sets does not increase the μ–area; see [15].

The decomposition result above yields a way to compute the μ–area of the image of a Lipschitz map and thus of any rectifiable set; see [18, Theorem 7; 15]:

Lemma 2.1 *Let $K \subset \mathbb{R}^2$ be measurable, let $f\colon K \to X$ be a Lipschitz map with $Y = f(K)$ and let μ be a definition of area. Let $N\colon Y \to [1,\infty]$ be the multiplicity function $N(y) = \#\{z \in K : f(z) = y\}$. Then*

$$\int_Y N(y)\, d\mu_Y(y) = \int_K \boldsymbol{J}^{\mu}(\mathrm{md}_z f)\, dz.$$

2.5 Coarea inequality with respect to μ

If X is a measurable subset of $\mathbb{R}^2$ and $f\colon X \to \mathbb{R}$ a 1–Lipschitz function then the classical coarea formula [10, Theorem 3.2.22] implies

$$\mathcal{H}^2(X) \geq \int_{\mathbb{R}} \mathcal{H}^1(f^{-1}(t))\, dt. \tag{2-4}$$

In the realm of metric spaces one needs to insert an (optimal) factor of $\frac{\pi}{4}$ on the right side; see [10, Theorems 2.10.25, 2.10.26]:

Lemma 2.2 *For any proper metric space Y, any Borel subset $X \subset Y$ with finite $\mathcal{H}^2(X)$ and any* 1*–Lipschitz function $f\colon X \to \mathbb{R}$ we have*

$$\mathcal{H}^2(X) \geq \frac{\pi}{4} \cdot \int_{\mathbb{R}} \mathcal{H}^1(f^{-1}(t))\, dt.$$

Note that the factor $\frac{\pi}{4}$ coincides with the number $q(\mathcal{H}^2)$ introduced in Section 2.3.

Thus under the assumption that X is countably 2–rectifiable the lemma above is a special case of the following:

Lemma 2.3 *Let X be a countably 2–rectifiable set and let $f\colon X \to \mathbb{R}$ be a 1–Lipschitz function. Then*
$$\mu(X) \geq q(\mu) \cdot \int_{\mathbb{R}} \mathcal{H}^1(f^{-1}(t))\, dt.$$

Proof By definition of $q(\mu)$, we have $q(\mu)^{-1} \cdot \mu \geq \mu^i$, where μ^i is the inscribed Riemannian area. Hence we only need to show

$$\mu^i(X) \geq \int_{\mathbb{R}} \mathcal{H}^1(f^{-1}(t))\, dt. \tag{2-5}$$

Rademacher's theorem [18] and the decomposition of X into small pieces approximated by pieces of normed spaces as in Section 2.4 show that it suffices to prove the result in the case that X is a subset of a normed plane $(\mathbb{R}^2, s)$ and $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a linear map; see [2, Section 9]. Let s_0 be the Euclidean norm whose unit ball is the Löwner ellipsoid of the unit ball of s. Then $s_0 \geq s$; hence $f\colon (\mathbb{R}^2, s_0) \to \mathbb{R}$ is still 1–Lipschitz. Moreover, by the definition of the inscribed Riemannian area, the μ^i–area on $(\mathbb{R}^2, s)$ coincides with the Lebesgue area of the Euclidean plane $(\mathbb{R}^2, s_0)$. Thus (2-5) follows from (2-4). □

Remark 2.4 The coarea factor in Lemma 2.3 is optimal for some definitions of area μ, but not for all. It can be shown, that the optimal μ–dependent coarea constant in Lemma 2.3 equals $4/v$, where v is the μ–area of the unit ball in the plane $(\mathbb{R}^2, s_\infty)$ with the sup-norm. Lemma 2.3 is optimal for the inscribed Riemannian, Hausdorff and Holmes–Thompson but not for the Benson definition of area.

3 Sobolev maps

3.1 Exceptional families of curves

We refer the reader to [13] for more details. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $1 < p < \infty$ be given. A family $\mathcal{E}$ of curves in Ω is called *p–exceptional* if there exists a Borel function $f\colon \Omega \to [0, \infty]$ such that $f \in L^p(\Omega)$ and $\int_\gamma f = \infty$ for all rectifiable curves γ in the family $\mathcal{E}$. Note that the set of all nonrectifiable curves is exceptional by definition. We say that a property holds true for *p–almost every curve* if the set of curves for which the property fails is p–exceptional.

A countable union of p–exceptional families is p–exceptional. If S is a subset of Ω

with $\mathcal{H}^2(S)=0$ then for p–almost all curves γ we have $\mathcal{H}^1(\gamma\cap S)=0$. By Fubini's theorem, a set of nonconstant curves in Ω parallel to a given line l is p–exceptional for some and then for any $p\in(1,\infty)$ if and only if the projection of the set of curves to the orthogonal line $l^\perp$ has $\mathcal{H}^1$–measure 0.

3.2 Generalities on Sobolev maps

We assume some experience with Sobolev maps with values in complete metric spaces X and refer the reader to [13; 24]. In this paper we consider only Sobolev maps defined on open bounded domains $\Omega\subset\mathbb{R}^2$, intervals and circles. In [24] we worked with the Sobolev spaces $W^{1,p}(\Omega,X)$ as defined in [19]. For the present work it is more natural to stick to the Newton–Sobolev spaces as defined in [13]. Recall that both notions are equivalent [13, Theorem 7.1.20; 37]. More precisely, every map in the Newton–Sobolev space $N^{1,p}(\Omega,X)$ is contained in $W^{1,p}(\Omega,X)$ and any element $u\in W^{1,p}(\Omega,X)$ has a representative in $N^{1,p}(\Omega,X)$, uniquely defined up to some p–exceptional subset. Most maps appearing in this paper are continuous, and a continuous map $u\in W^{1,p}(\Omega,X)$ is automatically in $N^{1,p}(\Omega,X)$. Thus the difference is not visible in the cases important in this paper. Therefore, we will freely interchange between $N^{1,p}$ and $W^{1,p}$.

The space $L^p(\Omega,X)$ consists of those measurable and essentially separably valued maps $u\colon\Omega\to X$ for which the composition $f\circ u$ with the distance function f to some point in X is in the classical space $L^p(\Omega)$. A map $u\colon\Omega\to X$ is in the Newton–Sobolev space $N^{1,p}(\Omega,X)$ if $u\in L^p(\Omega,X)$ and if there exists a Borel function $\rho\in L^p(\Omega)$, such that for p–almost all curves $\gamma\colon I\to\Omega$ the composition $u\circ\gamma$ is a continuous curve in X and the following inequality holds true:

$$\int_\gamma \rho \geq \ell_X(u\circ\gamma). \tag{3-1}$$

Up to sets of measure 0, there exists a uniquely defined minimal function ρ_u satisfying the condition above. It is called the *generalized gradient* of u. The integral $\int_\Omega \rho_u^p(z)\,dz$ coincides with the Reshetnyak energy $E_+^p(u)$ of u [13, Theorem 7.1.20; 24, Section 4]. Here we take this equality as the definition of $E_+^p(u)$.

3.3 Approximate metric differentials

Let $u\in W^{1,p}(\Omega,X)$ be as above. Then u has an *approximate metric differential* at almost every point $z\in\Omega$. This approximate metric differential is a seminorm s on $\mathbb{R}^2$, denoted by $\operatorname{ap}\operatorname{md}u_z$, which satisfies (2-3), where lim is replaced by the approximate limit ap lim. We refer the reader to [17; 24, Section 4] and recall here only the following two structural results that a posteriori could be taken as the definition of the approximate metric differential. The field of seminorms $z \mapsto \operatorname{ap}\operatorname{md} u_z$ is a measurable map contained in $L^p(\Omega, \mathfrak{S}_2)$, and thus, changing this map on a subset of measure 0, we may assume that $z \mapsto \operatorname{ap}\operatorname{md} u_z$ is an everywhere defined Borel map. There is a countable, disjoint decomposition $\Omega = S \cup \bigcup_{1\le i<\infty} K_i$ into a set S of zero measure and compact subsets K_i such that the following holds true. The restriction of u to any K_i is Lipschitz continuous, the metric differential of the restriction $u\colon K_i \to X$ exists at any $z \in K_i$ and coincides with $\operatorname{ap}\operatorname{md} u_z$.

3.4 Length of almost all curves

Let $u \in N^{1,p}(\Omega, X)$. Then for p–almost all rectifiable curves $\gamma\colon I \to \Omega$ parametrized by arclength, the composition $u \circ \gamma$ is absolutely continuous [13, Proposition 6.3.2]. Using approximate metric differentials we can compute the length of almost all curves by the usual formula:

Lemma 3.1 *Let $u \in N^{1,p}(\Omega, X)$ be given. Then for p–almost all rectifiable curves $\gamma\colon I \to \Omega$ parametrized by arclength we have*

$$\ell_X(u\circ\gamma) = \int_I \operatorname{ap}\operatorname{md} u_{\gamma(t)}(\dot\gamma(t))\, dt. \tag{3-2}$$

Proof Fix a Borel representative of the map $z \mapsto \operatorname{ap}\operatorname{md} u_z$. For any $\gamma\colon I \to \Omega$ parametrized by arclength the integrand on the right-hand side of (3-2) is measurable, and therefore the right-hand side is well defined. Consider the decomposition $\Omega = S \cup \bigcup_{1\le i<\infty} K_i$ described above, such that S has measure 0. Moreover, the sets K_i are compact, and the restrictions $u\colon K_i \to X$ are Lipschitz continuous and have metric differentials at all points. Finally, these metric differentials coincide with $\operatorname{ap}\operatorname{md} u_z$ at all $z \in K_i$. The set of curves $\gamma\colon I \to \Omega$ whose intersection with S has nonzero $\mathcal{H}^1$–measure is p–exceptional. The lemma follows, once we have shown (3-2) for all $\gamma\colon I \to \Omega$ outside this p–exceptional set such that $u\circ\gamma$ is absolutely continuous.

For any γ as above, set $I_i = \gamma^{-1}(K_i) \subset I$ and let J_i be the set of all Lebesgue points of I_i in I, at which γ has a differential and the absolutely continuous curve $u\circ\gamma$ has a metric differential. By assumption on γ, the union $J = \bigcup J_i$ has full measure in I. On the other hand, for any $t \in J$ the metric differential $\operatorname{md}_t(u\circ\gamma)(1)$ must coincide with $\operatorname{ap}\operatorname{md} u_{\gamma(t)}(\dot\gamma(t))$. Thus, integrating this equality and using (2-2) we obtain (3-2). □

We deduce as consequences:

Corollary 3.2 *Let $X^{\pm}$ be complete metric spaces. Let $u^{\pm} \in L^p(\Omega, X^{\pm})$ be two maps. Assume that for p–almost all curves $\gamma \subset \Omega$ the compositions $u^{\pm} \circ \gamma$ are continuous and*

$$\ell_{X^+}(u^+ \circ \gamma) = \ell_{X^-}(u^- \circ \gamma). \tag{3-3}$$

Then $u^+ \in N^{1,p}(\Omega, X^+)$ if and only if $u^- \in N^{1,p}(\Omega, X^-)$. In this case, the approximate metric differentials $\operatorname{ap\,md} u^{\pm}$ *of $u^{\pm}$ coincide almost everywhere.*

Proof The first claim follows directly from the definition of the Newton–Sobolev spaces $N^{1,p}$.

Fix a unit vector $v \in \mathbb{R}^2$. Then for almost every $z \in \Omega$ we have (3-3) and (3-2) for all sufficiently short segments γ centered at z and in direction v. The Lebesgue differentiation theorem thus implies that $\operatorname{ap\,md} u_z^+(v) = \operatorname{ap\,md} u_z^-(v)$ for almost every $z \in \Omega$. Applying this to a countable dense set of directions v implies that the measurable fields of seminorms $\operatorname{ap\,md} u^{\pm}$ are almost everywhere equal. □

3.5 Traces and gluings

Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain with boundary $\partial\Omega$ and let $u \in W^{1,p}(\Omega, X)$ be given. Then u has a well-defined trace $\operatorname{tr} u \in L^p(\partial\Omega, X)$ with the following property; see [19]. For the distance function $f\colon X \to \mathbb{R}$ to any point $x \in X$, we have $\operatorname{tr}(f \circ u) = f \circ \operatorname{tr} u \in L^p(\partial\Omega)$, where on the left-hand side the usual trace of Sobolev real-valued functions is considered.

Let a curve $S \subset \Omega$ separate the Lipschitz domain Ω into two Lipschitz subdomains $\Omega^{\pm}$. If $u^{\pm} \in W^{1,p}(\Omega^{\pm}, X)$ have the same trace on S then $u^{\pm}$ define together a map $u \in W^{1,p}(\Omega, X)$; see [19].

3.6 Area of Sobolev maps

Let μ be a definition of area and consider the corresponding Jacobian $\boldsymbol{J}^{\mu}\colon \mathfrak{S}_2 \to [0,\infty)$; see Section 2.3. For $u \in W^{1,2}(\Omega, X)$ the μ–area of u is defined by

$$\operatorname{Area}_{\mu}(u) := \int_{\Omega} \boldsymbol{J}^{\mu}(\operatorname{ap\,md} u_z)\, dz. \tag{3-4}$$

The number $\operatorname{Area}_{\mu}(u)$ is finite and satisfies $\operatorname{Area}_{\mu}(u) \le E_+^2(u)$ [24, Lemma 7.2]. In view of the area formula for Lipschitz maps this is a natural extension of the parametrized μ–area to Sobolev maps.

Recall that for any $p > 2$, any map $u \in N^{1,p}_{\mathrm{loc}}(\Omega, X)$ is continuous and has *Lusin's property (N)* and thus sends $\mathcal{H}^2$–zero sets to $\mathcal{H}^2$–zero sets. From the decomposition of Ω into parts on which u is Lipschitz and a set of measure 0, we deduce that the image $u(\Omega)$ is countably 2–rectifiable and of finite Hausdorff area. More precisely we deduce from Lemma 2.1 (see [17]):

Lemma 3.3 *Let $u \in N^{1,2}(\Omega, X)$ be a continuous map with Lusin's property (N). Let K be a measurable subset of Ω and $Y = u(K)$. Let $N\colon Y \to [0,\infty]$ be the multiplicity function $N(y) = \#\{z \in \Omega : u(z) = y\}$. Then the following holds true:*

$$\int_Y N(y)\, d\mu_Y(y) = \int_K \boldsymbol{J}^{\mu}(\operatorname{ap}\operatorname{md} u_z)\, dz.$$

3.7 Special infinitesimal structure

A seminorm $s \in \mathfrak{S}_2$ is *Q–quasiconformal* for some $Q \geq 1$ if for all $v, w \in S^1$ the inequality $s(v) \leq Q \cdot s(w)$ holds true. A quasiconformal seminorm is either a norm or the 0–seminorm. A map $u \in W^{1,2}(\Omega, X)$ is called Q–quasiconformal if the seminorms $\operatorname{ap}\operatorname{md} u_z$ are Q–quasiconformal for almost every $z \in \Omega$. For $Q = 1$ we call such maps conformal.

For every definition of area μ and every Q–quasiconformal map $u \in W^{1,2}(\Omega, X)$,

$$\operatorname{Area}_{\mu}(u) \geq Q^{-2} \cdot E_+^2(u). \tag{3-5}$$

The inequality above is the basic ingredient for most regularity results in [24]. The $\sqrt{2}$–quasiconformality of a map $u \in W^{1,2}(D, X)$ can be guaranteed by the following lemma [26, Lemma 4.1]; see also [24, Theorem 1.2]. This lemma also strengthens (3-5) replacing the constant $Q^{-2} = \frac{1}{2}$ by the μ–depending constant $q(\mu) \in \left[\frac{1}{2}, 1\right]$.

Lemma 3.4 *Let $u \in W^{1,2}(D, X)$ be such that $E_+^2(u) \leq E_+^2(u \circ \phi)$ for all bi-Lipschitz homeomorphisms $\phi\colon \bar{D} \to \bar{D}$. Then u is $\sqrt{2}$–quasiconformal. Moreover, for any definition of area μ and any subdomain $\Omega \subset D$ we have*

$$\operatorname{Area}_{\mu}(u|_{\Omega}) \geq q(\mu) \cdot E_+^2(u|_{\Omega}).$$

A space X satisfies property (ET) if for any map $u \in W^{1,2}(D, X)$ the seminorm $\operatorname{ap}\operatorname{md} u_z$ is either degenerate or comes from some Euclidean product at almost every point $z \in D$. We refer the reader to [24] for a discussion of property (ET) and the classes of spaces satisfying it; see also Section 1.4. We just recall here that if X satisfies property (ET) then for any $u \in W^{1,2}(D, X)$ the μ–area of u does not depend on the definition of area μ. Moreover, in this case any map u satisfying the assumption of

Lemma 3.4 is conformal, and therefore the equality $\mathrm{Area}_\mu(u|_\Omega) = E_+^2(u|_\Omega)$ holds for any subdomain Ω of D.

3.8 Lengths of circles

We will need a variation of the classical Courant–Lebesgue lemma; see [24, Lemma 7.3]. Let X be a complete metric space and let $u \in N^{1,2}(D, X)$. Let $x \in \bar{D}$ be an arbitrary point. For $t \le 1$ denote by $S_t = S_t(x)$ the set $S_t(x) = \{z \in D : |z - x| = t\}$, which is either a circle or a circular arc. For almost all $t \le 1$ the restriction $u\colon S_t \to X$ is absolutely continuous. Denote by l_t the length of $u\colon S_t \to X$.

Lemma 3.5 *In the above notations let $1 > r > 0$ be given and set $e_r = E_+^2(u|_{B(x,r)\cap D})$. Then there exists a set S of positive measure in the interval $\left[\frac{2}{3}r, r\right]$, such that for any $t \in S$ we have $l_t^2 \le 6\pi \cdot e_r$.*

Proof For the generalized gradient ρ_u of u, we integrate ρ_u^2 in polar coordinates around x, use Hölder's inequality and (3-1):

$$e_r = \int_0^r \left(\int_{S_t} \rho_u^2\right) dt \ge \int_0^r \frac{1}{2\pi t} \cdot \left(\int_{S_t} \rho_u\right)^2 dt \ge \frac{1}{2\pi r} \int_0^r l_t^2\, dt. \tag{3-6}$$

If $l_t^2 > 6\pi \cdot e_r$ for almost all $t \in \left[\frac{2}{3}r, r\right]$ then $e_r > \frac{1}{2\pi r} \cdot \frac{r}{3} \cdot 6\pi \cdot e_r$, which is absurd. This proves the claim. □

4 Quadratic isoperimetric inequality

4.1 Equal traces in irregular domains

For a bounded domain $\Omega \subset \mathbb{R}^2$, we denote by $W_0^{1,2}(\Omega)$ the closure of the set of all smooth functions with compact support in Ω with respect to the usual norm in the Sobolev space $W^{1,2}(\Omega, \mathbb{R})$. By continuity, for any $u \in W_0^{1,2}(\Omega)$ the extension of u by 0 outside of Ω defines a function in $W^{1,2}(\mathbb{R}^2, \mathbb{R})$. We can now define *equality of traces* even for irregular domains, when traces are not defined.

Definition 4.1 Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let X be a complete metric space. We say that $f_1, f_2 \in W^{1,2}(\Omega, \mathbb{R})$ have equal traces if $f_1 - f_2 \in W_0^{1,2}(\Omega)$. We say that $u_1, u_2 \in W^{1,2}(\Omega, X)$ have equal traces if for every $x \in X$ the compositions of u_1 and u_2 with the distance function to x have equal traces.

If Ω is a Lipschitz domain then a function $u \in W^{1,2}(\Omega, \mathbb{R})$ is contained in $W_0^{1,2}(\Omega)$ if and only if $\mathrm{tr}\, u\colon \partial\Omega \to \mathbb{R}$ equals 0 almost everywhere on $\partial\Omega$. Therefore, for maps $u_{1,2} \in W^{1,2}(\Omega, X)$ on a Lipschitz domain Ω the maps $u_{1,2}$ have equal traces in the sense of the definition above if and only if $\operatorname{tr} u_1 = \operatorname{tr} u_2$ almost everywhere on $\partial\Omega$.

We have the following extension of the classical gluing statement:

Lemma 4.2 *Let $V \subset \Omega$ be bounded domains in $\mathbb{R}^2$. Let $u \in W^{1,2}(\Omega, X)$ and $v \in W^{1,2}(V, X)$ be maps into a complete metric space X. Assume that v and $u|_V$ have equal traces. Then the map $\hat{u}$ which equals u on $\Omega \setminus V$ and v on V is contained in $W^{1,2}(\Omega, X)$. Moreover, $\hat{u}$ and u have equal traces. Finally, for any definition of area μ, we have $\operatorname{Area}_\mu(\hat{u}) - \operatorname{Area}_\mu(u) = \operatorname{Area}_\mu(v) - \operatorname{Area}_\mu(u|_V)$.*

Proof If $\hat{u} \in W^{1,2}(\Omega, X)$ then the approximate metric differentials of $\hat{u}$ and u must coincide at almost all points of $\Omega \setminus V$. Thus the last claim about the difference of areas follows from the definition (3-4) of the area of Sobolev maps. In order to establish the first two claims it suffices to prove the corresponding statements for compositions of u and v with distance functions to points $x \in X$. Therefore, we may assume $X = \mathbb{R}$. In this case the difference of u and $\hat{u}$ equals 0 outside of V and equals $u - v$ on V. Thus $u - \hat{u} \in W^{1,2}(\Omega, \mathbb{R})$; hence $\hat{u} \in W^{1,2}(\Omega, \mathbb{R})$. Moreover, $u - \hat{u}$ is a limit in $W^{1,2}(\Omega, \mathbb{R})$ of a sequence of smooth functions with support contained in V; hence $\hat{u}$ and u have equal traces. □

4.2 Conformal changes

Since the Reshetnyak energy is conformally invariant, it is possible to control the pullbacks of Sobolev maps by conformal diffeomorphisms even if they are not bi-Lipschitz.

Lemma 4.3 *Let $F\colon \Omega_1 \to \Omega_2$ be a conformal diffeomorphism between bounded domains in $\mathbb{R}^2$. Let X be a complete metric space and $u \in N^{1,2}(\Omega_2, X)$. Then $v = u \circ F\colon \Omega_1 \to X$ is measurable and essentially separably valued. The map v is contained in $N^{1,2}(\Omega_1, X)$ if and only if $v \in L^2(\Omega_1, X)$. In that case we have $E_+^2(u) = E_+^2(v)$ and $\operatorname{Area}_\mu(u) = \operatorname{Area}_\mu(v)$ for any definition of area μ.*

Proof Since F is locally bi-Lipschitz, the composition with F defines a bijection $N^{1,2}_{\mathrm{loc}}(\Omega_2, X) \to N^{1,2}_{\mathrm{loc}}(\Omega_1, X)$. In particular, $v\colon \Omega_1 \to X$ is measurable and essentially separably valued. The map $F\colon \Omega_1 \to \Omega_2$ preserves 2–exceptional families of curves. Let $\rho \in L^2(\Omega_2)$ be the minimal generalized gradient of u. Consider $\hat{\rho} \in L^2(\Omega_1)$ defined by $\hat{\rho}^2(z) := \rho^2(F(z)) \cdot |\det DF(z)|$. Then for any $O \subset \bar{O} \subset \Omega_1$, the function $\hat{\rho}$ is the minimal generalized gradient of $v|_O \in W^{1,2}(O, X)$; see [24, Lemma 6.4]. Therefore, $\hat{\rho} \in L^2(\Omega)$ is the minimal generalized gradient of $u \circ F$ in the sense of (3-1), since this is true for all subdomains $O \subset \bar{O} \subset \Omega_1$. Thus $v \in N^{1,2}(\Omega_1, X)$ if and only if $v \in L^2(\Omega_1, X)$. The last equality statement for energies and areas follows from the corresponding statements for the restrictions to subdomains $O \subset \bar{O} \subset \Omega_1$ [24, Lemma 6.4]. □

The condition $u \circ F \in L^2(\Omega_1, X)$ is automatically fulfilled if the (essential) image of u is contained in a bounded set, in particular, if u has a continuous extension to $\overline{\Omega}_2$. In general, $u \circ F$ need not be contained in $L^2(\Omega_1, X)$. However, this condition turns out to depend only on the traces:

Lemma 4.4 *Let $F \colon \Omega_1 \to \Omega_2$ be a conformal diffeomorphism between bounded domains in $\mathbb{R}^2$. Let $u^\pm$ be two maps in $N^{1,2}(\Omega_2, X)$ which have equal traces. If $u^+ \circ F \in N^{1,2}(\Omega_1, X)$ then $u^- \circ F \in N^{1,2}(\Omega_1, X)$ and the compositions $u^\pm \circ F$ have equal traces.*

Proof Taking the composition with distance functions to points $x \in X$, we may assume that $X = \mathbb{R}$. Hence, considering the difference $u^+ - u^-$, it suffices to prove that $v \circ F \in W_0^{1,2}(\Omega_1)$ for any $v \in W_0^{1,2}(\Omega_2)$. For any smooth function v with compact support in Ω_2 the composition $v \circ F$ is smooth, has compact support in Ω_1 and the same energy as v. Since v has compact support in Ω_1 the L^2–norm of $v \circ F$ is bounded by $K \cdot E_+^2(v \circ F)$ for some constant $K = K(\Omega_1)$, by the Sobolev inequality. This shows that the composition with F defines a continuous map $W_0^{1,2}(\Omega_2) \to W_0^{1,2}(\Omega_1)$ and finishes the proof. □

We infer that restrictions of area minimizers to subdiscs minimize the area as well:

Corollary 4.5 *Let the image of $u \in N^{1,2}(D, X)$ be contained in a bounded set. Let $F \colon D \to O$ be a conformal diffeomorphism onto a subdomain $O \subset D$. Then $v := u \circ F \in N^{1,2}(D, X)$. If u has minimal μ–area among all maps in $N^{1,2}(D, X)$ with the same trace as u then v has minimal μ–area among all maps in $N^{1,2}(D, X)$ with the same trace as v.*

Proof From Lemma 4.3 and the subsequent remark we deduce $v \in N^{1,2}(D, X)$. Assume that $v_+ \in N^{1,2}(D, X)$ has the same trace and smaller μ–area than v. Then $w := v \circ F^{-1}$ is contained in $N^{1,2}(O, X)$ and $u|_O$ and w have equal traces by Lemma 4.4. Moreover, w has smaller μ–area than the corresponding restriction of u. Now, define $\hat{u}$ to equal u on $D \setminus O$ and equal w on O. The corresponding map $\hat{u}$ is in $W^{1,2}(D, X)$, has the same trace as u and smaller μ–area, by Lemma 4.2. This contradicts the minimality assumption. □

4.3 Quadratic isoperimetric inequality

Let μ be a fixed definition of area. A space X admits a (C, l_0)–isoperimetric inequality with respect to μ, if for any *Lipschitz* curve $\gamma\colon S^1 \to X$ of length $l \le l_0$ there exists some $u \in W^{1,2}(D, X)$ with trace γ and $\mathrm{Area}_\mu(u) \le C l^2$. Then X admits a $(2C, l_0)$–isoperimetric inequality with respect to any other definition of area μ'. However, μ–minimal and μ'–minimal discs may be completely different; see [24, Proposition 11.6].

In order to avoid many reparametrizations we state the following:

Lemma 4.6 *Let X admit a (C, l_0)–isoperimetric inequality with respect to μ. Let T be a bi-Lipschitz Jordan curve in $\mathbb{R}^2$ and let Ω be the Jordan domain bounded by T. If $\gamma \in W^{1,2}(T, X)$ is a curve of length $l \le l_0$ then there exists a map $u \in W^{1,2}(\Omega, X)$ with $\mathrm{tr}\, u = \gamma$ and such that $\mathrm{Area}_\mu(u) \le C \cdot l^2$.*

Proof The domain Ω is bi-Lipschitz homeomorphic to $\bar{D}$ [38]. Since compositions with bi-Lipschitz homeomorphisms preserve the class of Sobolev maps, lengths and areas, the statement follows from the corresponding statement in the case $T = S^1$. In the case $T = S^1$ the statement was proved in [24, Lemma 8.5]. □

4.4 Filling area of curves

Let $\gamma\colon S^1 \to X$ be a continuous curve in a complete metric space X. We define the *filling area of γ in X with respect to μ* as

$$\mathrm{Fill}(\gamma) = \mathrm{Fill}_{X,\mu}(\gamma) := \inf\{\mathrm{Area}_\mu(u) : u \in W^{1,2}(D, X), \mathrm{tr}(u) = \gamma\}.$$

We let I denote the unit interval and A denote the annulus $S^1 \times I$. Any Sobolev map $v \in W^{1,2}(A, X)$ has as a trace a map defined on the boundary of A which consists of two copies of S^1. Thus, the trace of v consists of two maps $\gamma_{0,1} \in L^2(S^1, X)$ and we say that u is a Sobolev annulus connecting γ_0 with γ_1. Assume that the Sobolev annulus v connects two continuous curves γ_0 and γ_1. Then gluing v to a disc u arising in the definition of the filling area and reparametrizing the arising map we deduce $|\mathrm{Fill}(\gamma_0) - \mathrm{Fill}(\gamma_1)| \le \mathrm{Area}_\mu(v)$; see [23, Section 3.2]. Any continuous curve $\gamma \in W^{1,2}(S^1, X)$ can be connected by a Sobolev annulus contained in the image of γ to a constant speed parametrization γ_0 of γ. Thus, $\mathrm{Fill}(\gamma) = \mathrm{Fill}(\gamma_0)$ [23, Lemma 2.6].

If $\gamma\colon S^1 \to X$ is any rectifiable curve with constant speed parametrization γ_0, then $\mathrm{Fill}(\gamma)$ might be infinite, while $\mathrm{Fill}(\gamma_0)$ is finite. Even if $X = \mathbb{R}^2$ there exist *absolutely* *continuous* curves $\gamma\colon S^1 \to \mathbb{R}^2$ which do not bound any Sobolev disc at all, since they do not belong to the fractional Sobolev space $W^{\frac{1}{2},2}(S^1,\mathbb{R}^2)$; see [21; 6].

Finally, $\mathrm{Fill}_{X,\mu}(\gamma) \le C l^2$ holds for any Lipschitz curve $\gamma\colon S^1 \to X$ of length $l \le l_0$ if and only if X admits the (C', l_0)–isoperimetric inequality with respect to μ for all $C' > C$.

4.5 Filling irregular curves

We are going to prove:

Theorem 4.7 *Let X be a space with a (C, l_0)–isoperimetric inequality. Assume that $\gamma\colon S^1 \to X$ is a curve of length $l \le l_0$ which is the trace of a map $u \in W^{1,2}(D, X)$. Then* $\mathrm{Fill}(\gamma) \le C l^2$.

This statement is not obvious even if X is Euclidean space $\mathbb{R}^n$. However, in this case it is a consequence of the following analytic fact; see [7, page 283]. For the harmonic map $v\colon D \to \mathbb{R}^n$ with trace γ, the length of images of concentric circles is nondecreasing as a function of the radius. Thus one can find a small filling of γ by taking a small annulus out of v which connects γ to a smooth curve η of length not larger than l and then fill this smooth curve η. In general spaces X, it might very well happen, that all curves outside of γ are much longer than γ. Instead, we prove the following lemma which immediately implies Theorem 4.7:

Lemma 4.8 *Let X be a space with a (C, l_0)–isoperimetric inequality. Let the rectifiable curve $\gamma\colon S^1 \to X$ be the trace of a map $u \in W^{1,2}(D, X)$. Then there exist Sobolev annuli $\hat{u} \in W^{1,2}(A, X)$ of arbitrarily small area which connect γ and its arclength parametrization γ_0. In particular,* $\mathrm{Fill}_{X,\mu}(\gamma) = \mathrm{Fill}_{X,\mu}(\gamma_0)$. *If X is proper then the annuli $\hat{u}$ can be chosen to be continuous and to satisfy Lusin's property (N).*

Proof We may assume $u \in N^{1,2}(D, X)$. We choose a very small δ which will control the area of $\hat{u}$. Then we choose a small $\rho = \sin(2\pi/n)$ such that the restriction of u to the ρ–neighborhood of S^1 in D has energy at most δ. We choose equidistant points $p_1, \dots, p_n$ on S^1 with pairwise Euclidean distance ρ. If ρ is small enough then the length of γ on segments between consecutive points p_i, p_{i+1} is smaller than δ.

Denote by E_i the energy of the restriction of u to $B(p_i, \rho) \cap D$. Use Lemma 3.5 to find some $r_i \in \left(\frac{2}{3}\rho, \rho\right)$, such that the following holds true. The restriction of u to the distance circle c_i of radius r_i around p_i in D is a continuous curve in $W^{1,2}(c_i, X)$, and the length l_i of $u \circ c_i$ satisfies $l_i^2 < 6\pi \cdot E_i$.

We define B_i to be the ball $B(p_i, r_i)$. By construction, the boundary η of $\cup B_i$ in D is a bi-Lipschitz Jordan curve, and the restriction of u to η is in $W^{1,2}(\eta, X)$. Let Ω denote the annulus in D between η and S^1 and note that Ω is bi-Lipschitz to A. It suffices to find a map $v \in W^{1,2}(\Omega, X)$ of area going to 0 with δ, whose traces coincide with $u|_\eta$ and γ_0, respectively. Then after a bi-Lipschitz identification of Ω with A, we can glue both annuli v and $u|_\Omega$ along η to obtain the desired annulus $\hat{u}$ between γ and γ_0.

In order to construct v we proceed as follows. The domain Ω is subdivided by the circular arcs c_i in $2n$ Lipschitz discs T_j. The boundary of any T_j consists of two or three parts of consecutive circles c_i and a part I_j of S^1. By our assumption on ρ, any restriction $\gamma\colon I_j \to X$ has length m_j smaller than 2δ. We now consider the curve $k_j\colon \partial T_j \to X$ which coincides with u on the parts of the circles c_i and whose restriction to I_j parametrizes the corresponding part of γ proportionally to arclength. Thus k_j is just a reparametrization of $u\colon \partial T_j \to X$ and has the same length $a_j = \ell(k_j)$ as the restriction of u to ∂T_j.

By our choice of the circles c_i, all the curves k_j are Sobolev curves $k_j \in W^{1,2}(\partial T_j, X)$. Moreover, if ρ has been chosen small enough, the length of any k_j does not exceed l_0. Lemma 4.6 provides a map $v_j \in W^{1,2}(T_j, X)$ whose trace is k_j and whose μ–area is at most $C \cdot a_j^2$. Whenever two domains T_j and T_l have a common part of the boundary, the traces of v_j and v_l on this common part coincide (with the restriction of u). Therefore, gluing the v_j together we obtain a well-defined map $v \in W^{1,2}(\Omega, X)$ which coincides with v_j on T_j for all j. By construction, the trace of v coincides with the restriction of u on η. On the outer circle S^1, the trace of v is a Lipschitz continuous reparametrization γ_1 of γ_0. Thus we can attach to v another annulus of zero area connecting γ_1 with γ_0 (Section 4.4) to obtain an annulus with the required trace. Therefore, it remains to prove that the μ–area of the constructed annulus v goes to 0 as δ goes to 0.

By construction, the μ–area of v is at most $C \cdot \sum_{j=1}^{2n} a_j^2$. The length a_j of $k_j = u \circ \partial T_j$ has a contribution m_j of a part of the boundary circle S^1. The rest of k_j consists of two or three parts of circles c_i. We estimate those parts by the whole lengths l_i of the corresponding circular arcs $u \circ c_i$. Parts of each circle c_i appear as boundaries of at most five different domains T_j. This implies

$$\sum_{j=1}^{2n} a_j^2 \le K \cdot \left(\sum_{j=1}^{2n} m_j^2 + \sum_{i=1}^{n} l_i^2 \right) \quad \text{for some universal constant } K.$$

Using the bound $l_i^2 < 6\pi \cdot E_i$ we obtain

$$\mathrm{Area}_\mu(v) \le K_1 \cdot \left(\sum_{j=1}^{2n} m_j^2 + \sum_{i=1}^{n} E_i \right) \quad \text{for some constant } K_1 \text{ depending on } C.$$

Since the balls B_i intersect at most pairwise, the sum $\sum_{i=1}^n E_i$ is bounded by $2 \cdot E_+^2(u|_\Omega) \le 2\delta$. On the other hand $\sum_{j=1}^{2n} m_j$ is the finite length l of γ; hence $\sum_{j=1}^{2n} m_j^2$ becomes arbitrarily small once δ has been chosen small enough. This finishes the proof that the constructed annulus v has arbitrarily small area, once δ is chosen small enough.

If X is proper, we may first replace u by the harmonic map with the same trace; see [23]. Then u is continuous on $\bar{D}$ and satisfies Lusin's property (N). Also the fillings v_j can be chosen to be continuous and satisfy Lusin's property (N); see [23]. Then the constructed annulus v is continuous as well and satisfies Lusin's property (N). Therefore, also the annulus $\hat{u}$ obtained from a gluing of v and u will be continuous and satisfy Lusin's property (N). □

5 Solutions of the Plateau problem

5.1 Setting

Let μ be a definition of area. Let X be a complete metric space which admits a (C, l_0)–isoperimetric inequality for the area μ. Let Γ be a rectifiable Jordan curve in X. Denote as always in this paper by $\Lambda(\Gamma, X)$ the set of maps $u \in W^{1,2}(D, X)$, whose trace is a weakly monotone parametrization of Γ. Let $u \in \Lambda(\Gamma, X)$ be a solution of the Plateau problem for the curve Γ with respect to μ. Thus $\mathrm{Area}_\mu(u)$ is minimal in $\Lambda(\Gamma, X)$, and u has minimal Reshetnyak energy $E_+^2(u)$ among all minimizers of the μ–area in $\Lambda(\Gamma, X)$. By [24, Theorem 9.1], u has a unique representative which continuously extends to $\bar{D}$. From now on we will fix this representative $u\colon \bar{D} \to X$.

The map u is $\sqrt{2}$–quasiconformal, and it is conformal if X satisfies property (ET); see Lemma 3.4 and the subsequent paragraph. By [24, Theorem 1.4], there is a number $p > 2$ depending only on the constant C such that u is contained in $W^{1,p}_{\mathrm{loc}}(D, X)$. Moreover, if Γ is a chord-arc curve then one can improve the integrability globally: there is some $p > 2$ depending on C and the bi-Lipschitz constant of Γ such that $u \in W^{1,p}(D, X)$ [23, Theorem 1.3]; see also [24, Theorem 1.4 (iii)]. The last results and the Sobolev embedding theorems [24, Proposition 3.3] imply that u is locally Hölder continuous on D, respectively globally Hölder continuous on $\bar{D}$, with the exponent $1 - \frac{2}{p}$.

5.2 Restrictions to subdomains

The next result generalizes Proposition 1.8.

Proposition 5.1 *Let $u\colon \bar{D} \to X$ be a solution of the Plateau problem as above. Let $T \subset \bar{D}$ be a Jordan curve with Jordan domain O. Assume that $u|_T$ is a curve of finite length l and let $\gamma_0\colon S^1 \to X$ be a parametrization of $u|_T$ proportional to arclength. Then $\mathrm{Area}_\mu(u|_O) \le \mathrm{Fill}_{X,\mu}(\gamma_0)$. In particular, if $l \le l_0$ then $\mathrm{Area}_\mu(u|_O) \le C l^2$.*

Proof Fix a conformal diffeomorphism $F\colon D \to O$. By Corollary 4.5, the map $v = u \circ F$ is contained in $N^{1,2}(D, X)$ and has minimal area among all maps with the same trace. Since F extends to a homeomorphism $F\colon \bar{D} \to \bar{O}$ by the theorem of Carathéodory, v extends to a continuous map on $\bar{D}$. Thus the trace of v coincides with $u \circ F|_{S^1}$. Therefore, the trace of v is a rectifiable curve of length l, and γ_0 is a reparametrization of this trace. By the minimality of v and Corollary 4.5, $\mathrm{Area}_\mu(v) = \mathrm{Area}_\mu(u|_O)$ is the filling area of the curve $u \circ F|_{S^1}$. From Lemma 4.8 we deduce $\mathrm{Area}_\mu(u|_O) \le \mathrm{Fill}_{X,\mu}(\gamma_0)$. □

5.3 Intrinsic regularity

All regularity results in [24] are based on the estimate of lengths of some curves in the image of u. This estimate goes back to Morrey's classical proof of the a priori Hölder continuity [29] and is proven in [24, Proposition 8.7], in the present setting of metric spaces with quadratic isoperimetric inequalities. As shown in [26, Theorem 4.5], the Hölder exponent from [24, Proposition 8.7], can be slightly improved using Lemma 3.4.

Lemma 5.2 *For any $1 > \delta > 0$ and $A > 0$ there is some $L = L(C, l_0, \delta, A) > 0$ such that the following holds true whenever the μ–area of the μ–minimal disc u is at most A. For any $z_1, z_2 \in B(0, \delta)$ there is a piecewise affine curve γ from z_1 to z_2 such that*

$$\ell_X(u \circ \gamma) \le L \cdot |z_1 - z_2|^\alpha,$$

where $\alpha = q(\mu) \cdot \frac{1}{4\pi C}$.

In fact, similarly to [24, Proposition 8.7], the above result applies to the slightly more general situation where the μ–minimality of u is replaced by the following slightly weaker assumption: the map u minimizes the μ–area among all maps with the same trace as u, and u satisfies the conclusion of Lemma 3.4. This extension is needed only in the proof of Lemma 5.3 below. In fact, the slightly smaller Hölder exponent used in [24, Proposition 8.7] suffices for the conclusion of Lemma 5.3.

5.4 Boundary continuity

For regular curves, the following result is implicitly contained in the proof of the boundary regularity in [24, Section 9]. We sketch the proof; see [24] for details.

Lemma 5.3 *For any $\epsilon > 0$ there exists some $s > 0$ depending on C, l_0 and ϵ with the following property. Let $u \in \Lambda(\Gamma, X)$ be a solution of the Plateau problem as above. Let T be a Jordan curve in $\bar{D}$ with Jordan domain Ω. Assume that the restriction of u to Ω has μ–area at most s. Then for any $p \in \Omega$ there exists a curve $\eta \subset \Omega$ connecting p with T such that $\ell_X(u \circ \eta) < \epsilon$.*

Proof Let $0 < s < 1$ be small enough, to be determined later. Choose a conformal map $\phi\colon D \to \Omega$ such that $\phi(0) = p$. Denote by ϕ its continuous extension to a homeomorphism from $\bar{D} \to \bar{\Omega}$. As we have seen in Corollary 4.5, the composition $\hat{u} := u \circ \phi$ is contained in $W^{1,2}(D, X)$ and has minimal μ–area among all maps in $W^{1,2}(D, X)$ with the same trace as $\hat{u}$. Since ϕ is conformal it preserves the μ–area and the Reshetnyak energy on all subdomains. Thus the composition $\hat{u} = u \circ \phi\colon \bar{D} \to X$ satisfies the conclusions of Lemma 3.4. As has been explained after Lemma 5.2, the conclusion of Lemma 5.2 applies to the map $\hat{u}$. Thus, there exists some $1 > \delta > 0$ depending only on C, l_0 and ϵ with the following property. For any point $\theta \in S^1$ there exists some curve γ_θ in D connecting 0 with $\delta \cdot \theta$ such that $\ell_X(\hat{u} \circ \gamma_\theta) < \epsilon/2$.

The restriction of $\hat{u}$ to the annulus $A = \{z : \delta < |z| < 1\}$ is $\sqrt{2}$–quasiconformal and satisfies $E_+^2(\hat{u}|_A) \le 2\,\mathrm{Area}_\mu(\hat{u}|_A) \le 2s$. Denote by η_θ the radial curve connecting $\delta \cdot \theta$ and θ and by ρ the minimal generalized gradient of $\hat{u}$. Integrating in polar coordinates, using the Hölder inequality and (3-1), we deduce

$$2s \ge E_+^2(\hat{u}|_A) = \int_A \rho^2(z)\,dz \ge \delta \int_{S^1} \left(\int_{\eta_\theta} \rho^2 \right) d\theta \ge \delta \int_{S^1} \ell_X(\hat{u} \circ \eta_\theta)^2.$$

Thus, if s is small enough we find some $\theta \in S^1$ with $\ell_X(\hat{u} \circ \eta_\theta) < \epsilon/2$.

Now the concatenation $\gamma = \gamma_\theta * \eta_\theta$ connects the origin 0 with $\theta \in \partial D$ and $\epsilon > \ell_X(\hat{u} \circ \gamma)$. Thus we obtain the required curve η as $\eta = \phi \circ \gamma$. □

We can now deduce:

Corollary 5.4 *Let $u \in \Lambda(\Gamma, X)$ be a solution of the Plateau problem as above. Then for any $\epsilon > 0$ there exists some $\epsilon > r > 0$ with the following property. For any pair of points $x, y \in \bar{D}$ with $|x - y| < r$ there is some curve γ connecting these points inside $B(x, \epsilon)$ and such that $\ell_X(u \circ \gamma) < \epsilon$.*

Proof Using Lemma 5.3 it suffices to prove the following claim:

For any $\delta > 0$ there exists some $r > 0$ such that any $x, y \in \bar{D}$ with $|x - y| < r$ are contained in the closure $\overline{\Omega}$ of a convex domain $\Omega \subset D$ such that the quantities $\mathrm{Area}_\mu(u|_\Omega)$, $\mathrm{diam}(\Omega)$ *and* $\ell_X(u|_{\partial\Omega})$ *are bounded from above by* δ.

The claim is proven by taking $\overline{\Omega}$ to be the closure of the ball $B(x, t) \subset \bar{D}$ of an appropriate radius $t \in (r, \sqrt{r})$ around x. Indeed, if r is small enough then the diameter of any such ball is certainly smaller than δ. The boundary $T := \partial\Omega \subset \bar{D}$ of such balls consists of the distance circle $S_t = S_t(x)$ and, possibly, an interval $T^b = \overline{\Omega} \cap \partial D$ in the boundary circle ∂D. Since $u\colon \partial D \to \Gamma$ is rectifiable, the image $u(T^b)$ has arbitrarily small length, if the diameter of T^b is small enough. On the other hand, $\ell_X(u \circ S_t)$ is controlled by the Courant–Lebesgue lemma [24, Lemma 7.3]; see also Lemma 3.5 above. Therefore, $\ell_X(u \circ T)$ becomes arbitrarily small if r is small enough and t is chosen by the Courant–Lebesgue lemma. Finally, an upper bound on $\mathrm{Area}_\mu(u|_\Omega)$ follows from the upper bound on $\ell_X(u \circ T)$ and Proposition 5.1. □

6 Metric and analytic properties of the space Z

6.1 Setting, notations and basic properties

Throughout this section, we fix a definition of area μ and constants C, $l_0 > 0$. We fix a complete metric space X admitting a (C, l_0)–isoperimetric inequality. We fix a Jordan curve Γ of finite length in X. Finally, we let $u\colon \bar{D} \to X$ be a solution of the Plateau problem for (Γ, X). Consider the pseudodistance $d_u\colon \bar{D} \times \bar{D} \to [0, \infty]$ defined as in the introduction by

$$d_u(z_1, z_2) := \inf\{\ell_X(u \circ \gamma) : \gamma \subset \bar{D},\ \gamma \text{ connects } z_1 \text{ and } z_2\}.$$

Now the statement of Corollary 5.4 immediately implies:

Lemma 6.1 *For any $\epsilon > 0$ there exists some $r > 0$ such that for $x, y \in \bar{D}$ which satisfy $|x - y| < r$, we have $d_u(x, y) \le \epsilon$.*

As a direct consequence of the triangle inequality we deduce that the pseudometric $d_u\colon \bar{D} \times \bar{D} \to [0, \infty]$ is finite-valued and continuous. Consider the equivalence relation on $\bar{D}$ identifying pairs of points $z_1, z_2 \in \bar{D}$ with $d_u(z_1, z_2) = 0$. Let Z be the set of equivalence classes of points in $\bar{D}$. We obtain a canonical surjective projection $P\colon \bar{D} \to Z$. The pseudometric d_u defines a metric d_Z on the set Z. The continuity of d_u implies that $P\colon \bar{D} \to Z$ is continuous. Thus Z is a compact metric space. For any $z_1, z_2 \in \bar{D}$ we have $d_u(z_1, z_2) \geq d_X(u(z_1), u(z_2))$. Therefore, u admits a unique factorization $u = \bar{u} \circ P$ with a unique $\bar{u}\colon Z \to X$. Moreover, $\bar{u}$ is 1–Lipschitz. Hence we have already verified most statements in the following direct generalization of Theorem 1.1.

Proposition 6.2 *The pseudodistance d_u assumes only finite values and is continuous. The metric space Z associated with the pseudometric d_u is compact and geodesic, and the canonical projection $P\colon \bar{D} \to Z$ is continuous. The map $u\colon \bar{D} \to X$ has a canonical factorization $u = \bar{u} \circ P$, where $\bar{u}\colon Z \to X$ is a 1–Lipschitz map. For any curve $\gamma\colon I \to \bar{D}$ we have $\ell_X(u \circ \gamma) = \ell_Z(P \circ \gamma)$; hence $\bar{u}$ preserves the length of $P \circ \gamma$.*

Proof It remains to prove the last equality and that Z is a geodesic space. Thus let $\gamma\colon I \to \bar{D}$ be given. Since $u = \bar{u} \circ P$ and $\bar{u}$ is 1–Lipschitz, we have $\ell_X(u \circ \gamma) \leq \ell_Z(P \circ \gamma)$. On the other hand, for any $[t, t'] \subset I$ we have $d_Z(P \circ \gamma(t), P \circ \gamma(t')) \leq \ell_X(u \circ \gamma|_{[t,t']})$. Thus the reverse inequality follows directly from the definition of length (2-1).

For any $p_1 = P(z_1)$ and $p_2 = P(z_2) \in Z$, the definition of $d_Z(p_1, p_2) = d_u(z_1, z_2)$ together with the equality of lengths proved above shows that Z is a length space. Since Z is compact, the Hopf–Rinow theorem shows that Z is geodesic. □

Lemma 6.3 *Any fiber $P^{-1}(q)$ is a connected subset of $\bar{D}$.*

Proof The set $K = P^{-1}(q)$ is closed, hence compact. If it is not connected we find a decomposition $K = K_1 \cup K_2$ such that $d(K_1, K_2) > 0$. Let S be the compact set of all points in $\bar{D}$ which are at the same distance from K_1 and K_2. Choose $k_1 \in K_1$ and $k_2 \in K_2$. By continuity and compactness, the function $x \mapsto d_u(k_1, x)$ assumes a minimum ϵ on S. Since S does not intersect K, we have $\epsilon > 0$. By definition of d_u, we find a curve γ connecting k_1 and k_2 with $\ell_X(u \circ \gamma) < \epsilon$. This curve must intersect the set S at some point p. We deduce $d_u(p, k_1) \leq \ell_X(u \circ \gamma) < \epsilon$. This contradiction finishes the proof. □

Since $u\colon S^1 \to \Gamma$ is rectifiable the curve $P\colon S^1 \to Z$ is rectifiable as well. Since the restrictions of u and P to any subarc of S^1 have equal lengths and since $u|_{S^1}$ a weakly monotone parametrization of the Jordan curve Γ, we conclude:

Lemma 6.4 *The restriction $P\colon S^1 \to Z$ is a weakly monotone parametrization of a rectifiable Jordan curve Γ'. The restriction $\bar{u}\colon \Gamma' \to \Gamma$ is an arclength-preserving homeomorphism.*

6.2 Analytic properties

From Lemma 5.2 we infer:

Lemma 6.5 *The restriction $P\colon D \to Z$ is locally α–Hölder continuous with $\alpha = q(\mu)\cdot\frac{1}{4\pi C}$.*

Since $\ell_X(u\circ\gamma) = \ell_Z(P\circ\gamma)$ for all curves γ in $\bar{D}$ we deduce from Corollary 3.2 and the corresponding property of u:

Lemma 6.6 *The map $P\colon D \to Z$ is in the Sobolev class $W^{1,2}(D,Z)$ and in the local Sobolev class $W^{1,p}_{\mathrm{loc}}$ for some $p > 2$ depending on C. The approximate metric differentials of P and u coincide at almost all points $z \in D$. In particular, the restrictions of u and P to any subdomain $O \subset D$ have equal μ–area and equal energy.*

From Lemma 6.6 and the infinitesimal properties of u we get:

Corollary 6.7 *The map $P\colon D \to Z$ is $\sqrt{2}$–quasiconformal. If the space X satisfies property (ET) then $P\colon D \to Z$ is conformal.*

From Proposition 5.1, Lemma 6.6 and the last statement in Proposition 6.2 we directly deduce:

Lemma 6.8 *Let $T \subset \bar{D}$ be a Jordan curve with Jordan domain Ω. If $\ell_Z(P|_T) \le l_0$ then $\mathrm{Area}_\mu(P|_\Omega) \le C\cdot\ell_Z(P|_T)^2$.*

Since $P \in W^{1,p}_{\mathrm{loc}}(D,Z)$, for some $p > 2$, the map P satisfies Lusin's property (N). Thus, $P(D)$ is countably 2–rectifiable and has finite $\mathcal{H}^2$–area; see Section 3.6. Since $Z = P(D)\cup\Gamma'$ and $\mathcal{H}^2(\Gamma') = 0$ we obtain:

Lemma 6.9 *The space Z is countably 2–rectifiable and $\mathcal{H}^2(Z) < \infty$.*

We consider the multiplicity function $N\colon Z \to [1,\infty]$ which appears in the area formula and is defined by $N(z) = \#\{x \in \bar{D} : P(x) = z\}$. By Lemma 6.3, the fibers of the map P are connected, and thus the function N can only assume the values 1 and ∞. From Lemma 3.3 we deduce that for $\mathcal{H}^2$–almost all points $z \in Z$ the value $N(z)$ is exactly 1. Another application of the area formula in Lemma 3.3 now gives us:

Lemma 6.10 *For any open subset $V \subset Z$ we have*

$$\mu_Z(V) = \mathrm{Area}_\mu(P|_{P^{-1}(V)\cap D}).$$

The measure μ_Z and hence $\mathcal{H}^2$ has the whole set Z as its support:

Lemma 6.11 *For every $z_0 \in Z$ and every $r > 0$ we have*

$$\mu_Z(B(z_0, r)) > 0.$$

Proof Otherwise, we find some $z_0 = P(x_0)$ and some $r > 0$ with $\mu_Z(B(z_0, r)) = 0$. The set $\Omega := P^{-1}(B(z_0, r))$ is open in $\bar{D}$. It consists of all points which can be connected to x_0 by some curve γ with $\ell_X(u \circ \gamma) < r$. Therefore, Ω is connected. Hence $\Omega \cap D = \Omega \setminus \partial D$ is connected as well. From Lemma 6.10 we deduce $\mathrm{Area}_\mu(P|_{\Omega \cap D}) = 0$. Since P is quasiconformal, the restriction of P to $\Omega \cap D$ has vanishing energy. Since $\Omega \cap D$ is connected and P continuous, we infer that P is constant on $\Omega \cap D$. By continuity, $B(z_0, r) = P(\Omega) = \{z_0\}$. Since Z is geodesic this implies that $Z = \{z_0\}$. This is impossible since $\Gamma' = P(\Gamma)$ is a Jordan curve in Z. □

7 Topological preliminaries

In this section we collect some well-known statements in 2–dimensional topology and provide minor variants of these statements. A reader with some experience in this area may proceed directly to the next section.

7.1 Jordan curve theorem

By the theorem of Jordan, any Jordan curve $\Gamma \subset S^2$ divides S^2 into two domains. These domains are (homeomorphic to) open discs with boundary Γ and their closures are homeomorphic to $\bar{D}$.

By a theorem of Rado, these domains depend "continuously" on the Jordan curve in the following sense [34, Theorem 2.11]. Let a Jordan curve Γ be fixed, let O be one of the corresponding domains and $p \in O$ an arbitrary point. Then for any $\epsilon > 0$ there is some $\delta > 0$ with the following property. If $f\colon \Gamma \to S^2$ is a homeomorphism onto the image Jordan curve Γ' such that $d(f(x), x) < \delta$ for all $x \in \Gamma$ then there exists a homeomorphism $F\colon \bar{O} \to \bar{O}'$ with $d(F(x), x) < \epsilon$ for all $x \in \bar{O}$. Here we denote by O' the Jordan domain of Γ' which contains the point p.

Given disjoint subsets A, B, C of a topological space Y, we say that A *separates* B from C if any connected subset S of Y which contains points of B and C must intersect A. A *Peano continuum* is a compact, connected, locally connected metric space. Any Peano continuum is locally path connected. We will need the following simple observation.

Lemma 7.1 *Let Y be a simply connected Peano continuum. Let K be a compact subset of Y which separates two points x and y in Y. Then K contains a minimal compact subset K' which still separates x and y. Moreover, K' is connected.*

Proof If K_j is a chain of decreasing compact subsets separating x and y then their intersection separates x and y as well. Indeed, any K_j intersects any curve connecting x and y, hence so does $K' = \bigcap K_j$ by compactness.

By Zorn's lemma, there exists a minimal compact subset K' of K which separates x and y. If K' is not connected then it can be written as the nontrivial disjoint union of compact subsets $K' = K_1 \cup K_2$. Since Y is simply connected, it has trivial first homology group $H_1(Y, \mathbb{Z})$. The exactness of the Mayer–Vietoris sequence in homology implies the injectivity of the canonical map

$$H_0(Y \setminus K', \mathbb{Z}) \to H_0(Y \setminus K_1, \mathbb{Z}) \oplus H_0(Y \setminus K_2, \mathbb{Z}).$$

Thus, if points x and y define the same element in $H_0(Y \setminus K_1, \mathbb{Z})$ and in $H_0(Y \setminus K_2, \mathbb{Z})$ then they define the same element in $H_0(Y \setminus K', \mathbb{Z})$, hence are in the same component of $Y \setminus K'$. Thus, either K_1 or K_2 must separate x and y, in contradiction with the minimality of K'. □

We cite the following result from [39, Theorem IV.6.7]:

Lemma 7.2 *Let K be a Peano continuum in S^2 which separates two points x and y. Then K contains a Jordan curve which still separates x and y.*

We will need:

Lemma 7.3 *Let $K \subset S^2$ be closed and connected. Then any component U of the complement $S^2 \setminus K$ is homeomorphic to a disc.*

Proof Otherwise we find a Jordan curve T in U which is noncontractible in U. Then K must contain at least one point in both Jordan domains defined by T in S^2. Then K cannot be connected. □

For the proof of the following result we refer the reader to [11, Corollary 2B]:

Lemma 7.4 *Let K be a compact, connected metric space with finite $\mathcal{H}^1(K)$. Then K is a Peano continuum.*

Taking Lemma 7.4 and Lemma 7.2 we arrive at:

Corollary 7.5 *Let Y be a compact metric space homeomorphic to S^2. Let $K \subset Y$ be a compact subset which separates two points $x, y \in Y$. If $l = \mathcal{H}^1(K) < \infty$ then K contains a rectifiable Jordan curve of length at most l which still separates x and y.*

We can now deduce a corresponding separating result in discs:

Corollary 7.6 *Let Z be a metric space homeomorphic to $\bar{D}$. Let K be a compact subset of Z with finite $\mathcal{H}^1(K)$. If K separates points $x, y \in Z$ then K contains a minimal compact subset T separating x and y. Either T is a Jordan curve which intersects ∂Z in at most one point or T is homeomorphic to a compact interval which intersects ∂Z exactly at its endpoints.*

Proof We may assume without loss of generality that $x, y \notin \partial Z$. By Lemma 7.1, we find a minimal compact $T \subset K$ separating x and y, and T is connected. The set $T \setminus \partial Z$ separates x and y in the open disc $Z \setminus \partial Z$. For any subset T_0 of $T \setminus \partial Z$ which separates x and y in $Z \setminus \partial Z$, the closure $\bar{T}_0$ of T_0 in Z separates x and y in Z; hence it coincides with T by minimality. Thus $T \setminus \partial Z$ does not contain any proper closed subset separating x and y in $Z \setminus \partial Z$. Using a Mayer–Vietoris sequence argument as in the proof of Lemma 7.1 we deduce that $T \setminus \partial Z$ is connected. Summarizing, we see that $T \setminus \partial Z$ is connected and dense in T.

Using Lemma 7.4 we see that T is a Peano continuum. Consider the doubling Y of Z along ∂Z, and let T^+ be the union of T and ∂Z. The compact set T^+ separates x and y in Y. If T does not intersect ∂Z then T and ∂Z are connected components of T^+. Since ∂Z does not separate x and y, we deduce that T separates x and y in Y. By Corollary 7.5 and minimality, T is a Jordan curve in this case.

Assume from now on that T intersects ∂Z. Then T^+ is a Peano continuum, as a connected union of the Peano continua T and ∂Z. By Lemma 7.2, we find a Jordan curve $T^- \subset T^+$ which still separates x and y in Y.

The set $T^- \setminus \partial Z \subset T$ separates x and y in the open disc $Z \setminus \partial Z$. By the discussion at the beginning of the proof, $T^- \setminus \partial Z = T \setminus \partial Z$. Thus the connected set $T \setminus \partial Z$ is an open subset of the Jordan curve T^-. We deduce that T is a connected subset of the Jordan curve T^-. Moreover, either $T = T^-$ and T intersects ∂Z in exactly one point, or T is homeomorphic to a compact interval and $T \cap \partial Z$ consists of the two endpoints of the interval. □

By induction we can derive an extension of Corollary 7.6 to sets of finitely many points:

Lemma 7.7 *Let Z be a metric space homeomorphic to $\bar{D}$. Let F be a finite set $F = \{p_1, \dots, p_m\} \subset Z \setminus \partial Z$. Let $K \subset Z \setminus F$ be a compact subset which separates any pair of points of F. Then K contains a minimal compact subset K_0 which separates any pair of points of F in Z. If $\mathcal{H}^1(K) < \infty$ then $K_0 \cup \partial Z$ is homeomorphic to a finite graph. Moreover, $Z \setminus (K_0 \cup \partial Z)$ has exactly m connected components.*

Proof The existence of a minimal set K_0 follows as in the case of two points in Lemma 7.1. Thus we may assume that $K = K_0$ has finite $\mathcal{H}^1$–measure and need to prove that $K \cup \partial Z$ is a finite graph, whose complement has exactly m components.

We proceed by induction on m. If $m = 2$, then the claim follows from Corollary 7.6. Assume that $m > 2$ and the result is true for all $m' < m$. For any pair of distinct points $p_i, p_j \in F$ the set K contains a minimal compact subset K_{ij} separating p_i from p_j. By the minimality of K, we get $K = \bigcup_{1 \le i < j \le m} K_{ij}$.

Assume that some of the sets K_{ij} intersect ∂Z. Then any such K_{ij} is a simple arc or a Jordan curve, and it divides Z into two closed discs $Y^{\pm}$ with common boundary (as subsets of Z) given by K_{ij}. Then we can apply the inductive hypothesis to the intersections of F and K with those discs. This implies that $Y^{\pm} \setminus (K \cup \partial Y^{\pm})$ has as many components as points in $F \cap Y^{\pm}$. Moreover, the union of $\partial Y^{\pm}$ and $Y^{\pm} \cap K$ is a finite graph. It follows that K is a finite graph as well, and that $Z \setminus (K \cup \partial Z)$ has exactly m connected components.

If, on the other hand, none of the sets K_{ij} intersects ∂Z, then any K_{ij} is a Jordan curve and K is disjoint from ∂Z. We embed Z into its double Y homeomorphic to S^2. The sphere Y is divided by the Jordan curve K_{12} into two closed discs $Y^{\pm}$, and we can apply the inductive hypothesis to $Y^{\pm}$, $F^{\pm} = F \cap Y^{\pm}$ and $K^{\pm} := K \cap Y^{\pm}$. As above, we deduce that K is a finite graph and that $Y \setminus K$ has exactly m connected components, each of them containing exactly one point of F. Since K does not intersect ∂Z, the union $K \cup \partial Z$ is again a finite graph. Moreover, the complement of Z in Y is an open disc O contained in one component U of $Y \setminus K$. Then U contains ∂Z, and $U \setminus O = U \cap Z$ is connected. We deduce that $Z \setminus (K \cup \partial Z)$ has exactly m components. □

7.2 Cell-like maps

The following definitions and statements can be found in [12, page 97]; see also [9].

Definition 7.8 A compact space is called cell-like if it admits an embedding into the Hilbert cube Q in which it is null-homotopic in every neighborhood of itself. A continuous surjection $f\colon X \to Y$ between metric spaces X and Y is called cell-like if $f^{-1}(q)$ is cell-like for every $q \in Y$.

Let X and Y be compact metric spaces of finite topological dimension and $f\colon X \to Y$ a continuous surjection. If $f\colon X \to Y$ is cell-like and X is an absolute neighborhood retract then so is Y. If X and Y are absolute neighborhood retracts then f is cell-like if and only if for every open set $U \subset Y$ the restriction $f\colon f^{-1}(U) \to U$ is a homotopy equivalence.

Basic examples of cell-like sets are contractible sets. Any cell-like subset of S^1 is a closed interval. In S^2 the situation is slightly more complicated but it is still very well understood:

Example 7.9 A compact subset $K \subset S^2$ is cell-like if and only if K and $S^2 \setminus K$ are connected.

The most important class of cell-like maps between absolute neighborhood retracts is given by uniform limits of homeomorphisms. Sometimes, all cell-like maps are of this type. The next example is a direct consequence of the above characterizations of cell-like subsets of S^1:

Example 7.10 Let Y be a compact metric space. A continuous surjection $f\colon S^1 \to Y$ is cell-like if and only if Y is homeomorphic to S^1 and f is a weakly monotone parametrization of Y.

In the 2–dimensional case the corresponding result is a milestone in classical geometric topology and goes back to Moore.

Theorem 7.11 *Let X be a compact 2–dimensional manifold without boundary and let $f\colon X \to Y$ be a cell-like map. Then Y is homeomorphic to X and f is a uniform limit of homeomorphisms.*

In order to recognize the topology of our minimal disc we will need a similar result for manifolds with boundary. Unfortunately, we could not find a reference and, therefore, provide the proof of the following consequence of Theorem 7.11:

Corollary 7.12 *Let Z be a compact metric space and $\varphi\colon \bar{D} \to Z$ a cell-like map. Then Z is a contractible and locally contractible space which is homeomorphic to the* *complement of some open topological disc O in S^2. As a subset of S^2, the boundary of Z is exactly $\varphi(S^1)$.*

If the restriction $\varphi|_{S^1}\colon S^1 \to \varphi(S^1)$ is cell-like then φ is a uniform limit of homeomorphisms $\varphi_i\colon \bar{D} \to Z$. In particular, Z is homeomorphic to $\bar{D}$ in this case.

Proof Denote by Y the space obtained by attaching a copy $\bar{D}'$ of $\bar{D}$ to Z along the map $\varphi|_{S^1}$. Denote by $\iota\colon \bar{D}' \to Y$ the natural projection. View S^2 as the union of $\bar{D}$ and $\bar{D}'$, glued along S^1. Define $f\colon S^2 \to Y$ by $f = \varphi$ on $\bar{D}$ and $f = \iota$ on $\bar{D}'$. Then f is well defined and cell-like. Therefore, by Theorem 7.11, the space Y is homeomorphic to S^2, and Z is homeomorphic to the complement of the image of D' in the sphere S^2. This shows that Z is homeomorphic to the complement of some open topological disc in Y. Moreover, as a subset of Y, the boundary of Z is $\varphi(S^1)$. Since φ is cell-like, $\bar{D}$ is a 2–dimensional absolute retract and Z has dimension at most 2, it follows that Z is an absolute neighborhood retract and, in particular, locally contractible. Moreover, since φ is a homotopy equivalence it follows that Z is contractible. This proves the first statement.

If $\varphi|_{S^1}$ is cell-like as a map to $\varphi(S^1)$ it follows that $\varphi(S^1)$ is a Jordan curve by Example 7.10. Since Y is homeomorphic to S^2 the Schoenflies theorem shows that $\varphi(S^1)$ divides Y into two domains Ω_1 and Ω_2 such that $\bar{\Omega}_1$ and $\bar{\Omega}_2$ are homeomorphic to $\bar{D}$. Clearly, one of the two domains is exactly Z, viewed as a subset of Y. Thus Z is homeomorphic to $\bar{D}$.

By Theorem 7.11, the map $f\colon S^2 \to Y$ is a uniform limit of homeomorphisms $f_i\colon S^2 \to Y$. Then $Z_i = f_i(\bar{D})$ and $Z = f(\bar{D})$ are closed discs in the sphere $Y = S^2$, and f_i converges uniformly to f. To obtain homeomorphisms $\varphi_i\colon \bar{D} \to Z$ we just need to change f_i by a homeomorphism $\psi_i\colon Z_i \to Z$ which is close to the identity. But the existence of such ψ_i follows from the theorem of Rado, mentioned in Section 7.1. □

7.3 A curve cutting lemma

In order to find a suitable Jordan curve inside some noninjective curve we will use the following observation only in the case of a punctured disc Y.

Lemma 7.13 *Let Y be a locally contractible metric space. Let the curve $\gamma\colon S^1 \to Y$ be noncontractible in Y. Then there exists a weakly monotone parametrization $\eta\colon S^1 \to Y$ of a Jordan curve $T \subset \gamma(S^1)$ which is noncontractible in Y and such that for every continuous map $F\colon Y \to X$ to another metric space X we have $\ell_X(F \circ \eta) \le \ell_X(F \circ \gamma)$.*

Proof Consider the set $\mathcal{G}$ of all curves $\eta\colon S^1 \to Y$ with the following property. If $\eta(t) \neq \gamma(t)$ for some $t \in S^1$ then t is an inner point of an interval on which η is constant. For any maximal interval, on which $\eta \in \mathcal{G}$ is constant, the boundary points of I are mapped by γ to the same point in Y. The curve γ is contained in $\mathcal{G}$. The family $\mathcal{G}$ of curves is equicontinuous: the modulus of continuity of γ gives also a modulus of continuity for any $\eta \in \mathcal{G}$.

Denote by $\mathcal{G}^+$ the set of all noncontractible curves $\eta \in \mathcal{G}$. For any $\eta \in \mathcal{G}$, let $O(\eta) \subset S^1$ be the open set of points around which η is locally constant. We claim that there exists some $\eta_0 \in \mathcal{G}^+$ for which $O(\eta_0)$ is maximal among all $\{O(\eta) : \eta \in \mathcal{G}^+\}$. Assume that $\eta_i \in \mathcal{G}^+$ is a sequence such that $O(\eta_i) \subset O(\eta_{i+1})$ for all i. The curves η_i are equicontinuous curves in the compact set $\gamma(S^1) \subset Y$. By the Arzelà–Ascoli theorem, we find a subsequence η_j converging uniformly to a curve $\eta_0\colon S^1 \to Y$. By the local contractibility of Y, the curve η_j is homotopic to η_0 for j large enough, and thus η_0 is noncontractible. If $\eta_0(t) \neq \gamma(t)$ then $t \in O(\eta_j)$ for all j large enough. We deduce that $\eta_0 \in \mathcal{G}$ and that $O(\eta_0)$ contains all subsets $O(\eta_j)$. An application of Zorn's lemma finishes the proof of the claim.

If η_0 is not a weakly monotone parametrization of a Jordan curve we find some $t, t' \in S^1$ such that $\eta_0(t) = \eta_0(t')$ but η_0 is not constant on any of the two intervals $I^\pm$ of S^1 bounded by t, t'. Let $\eta^\pm$ be the curve that coincides with η_0 on $I^\pm$ and is constant on the complementary interval $I^\mp$. By definition, $\eta^\pm$ are contained in $\mathcal{G}$ and their constancy sets are strictly larger than $O(\eta_0)$. By the maximality of $O(\eta_0)$, we deduce that $\eta^\pm \notin \mathcal{G}^+$. Thus $\eta^\pm$ are contractible curves. But up to a reparametrization, η_0 is a concatenation of η^+ and η^-. Thus η_0 is contractible, in contradiction with $\eta_0 \in \mathcal{G}^+$. This contradiction shows η_0 is a weakly monotone parametrization of a Jordan curve.

By the definition of length, the inequality $\ell_X(F \circ \eta) \leq \ell_X(F \circ \gamma)$ holds true for any continuous map $F\colon Y \to X$ and any $\eta \in \mathcal{G}$. □

8 Topological and isoperimetric properties of Z

We proceed using the notation from Section 6.

8.1 Topology

With our topological preparations we are in position to describe the topology of Z.

Theorem 8.1 *The space Z is homeomorphic to $\bar{D}$, and $P\colon \bar{D} \to Z$ is a uniform limit of homeomorphisms.*

Proof The restriction $P\colon \partial D \to \Gamma' \subset Z$ is weakly monotone by Lemma 6.4, hence cell-like by Example 7.10. By Corollary 7.12, it suffices to prove that $P\colon \bar{D} \to Z$ is cell-like. Thus we need to prove that for any $q \in Z$ the preimage $K = P^{-1}(q)$ is a cell-like subset of $\bar{D}$.

By Lemma 6.3, the set K is connected. Consider $\bar{D}$ as the lower hemisphere of S^2. By Example 7.9, it is enough to prove that $S^2 \setminus K$ is connected. Assume otherwise. Then there exists at least one component O of $S^2 \setminus K$ which does not intersect the closed upper hemisphere; hence O is contained in D. By Lemma 7.3, O is homeomorphic to a disc, since K is connected. By the Riemann mapping theorem, we find a conformal diffeomorphism $F\colon D \to O$. By Corollary 4.5, the composition $v = u \circ F$ is contained in $W^{1,2}(D, X)$ and has minimal μ–area among all maps with the same trace as v. We claim that $\operatorname{tr} v$ is a constant curve. By construction, $u(K)$ is a single point $p = \bar{u}(q)$. For any sequence $z_j \in D$ converging to ∂D the points $F(z_j)$ subconverge to some point in K. Therefore, the sequence $v(z_j) = u \circ F(z_j)$ converges to the point p. This proves the claim.

The constant curve $\operatorname{tr} v\colon S^1 \to X$ can be filled by the constant disc. By the minimality of $\operatorname{Area}_\mu(v)$, we deduce that v has zero area. But $\operatorname{Area}_\mu(v) = \operatorname{Area}_\mu(u|_O)$, which is nonzero, since u is quasiconformal and u is nonconstant on O. This contradiction finishes the proof. □

8.2 Isoperimetric inequality

We can approximate arbitrary Jordan curves in Z by P–images of Jordan curves in $\bar{D}$ and use Proposition 5.1 to control the isoperimetric properties of Z:

Theorem 8.2 *Every Jordan curve T in Z bounds a unique open disc $\Omega \subset Z$. Furthermore, if $\ell_Z(T) < l_0$ then*

$$\mu_Z(\Omega) \le C \ell_Z(T)^2. \tag{8-1}$$

Proof Existence and uniqueness of Ω is a consequence of the Jordan curve theorem and Theorem 8.1. Since P is a cell-like map, $P\colon P^{-1}(O) \to O$ is a homotopy equivalence, for any open subset $O \subset Z$. In particular, $P^{-1}(\Omega) \subset D$ is contractible, hence an open disc. In order to estimate the area of Ω we fix a small $\epsilon > 0$ with $\ell_Z(T) + \epsilon < l_0$. We fix some open disc $U \subset \Omega$, such that $\bar{U} \subset \Omega$ is homeomorphic to $\bar{D}$ and

$$\mu_Z(U) \ge \mu_Z(\Omega) - \epsilon.$$

Set $V := P^{-1}(U)$. Then V is contractible, hence homeomorphic to D.

Fix a homeomorphism $\gamma\colon S^1 \to T$. Choose $\delta > 0$ so small that the open δ–neighborhood $N(T,\delta)$ of T in Z does not intersect U and such that every ball of radius δ based at a point of T is contractible in $Z \setminus U$. Let $\{t_0, t_1, \dots, t_k, t_{k+1} = t_0\}$ be a partition of S^1 such that

$$\gamma([t_i, t_{i+1}]) \subset B_Z(\gamma(t_i), \delta/2)$$

for every i. Choose $x_i \in \bar{D}$ with $P(x_i) = \gamma(t_i)$. By the definition of the metric in Z, there exists a curve γ_i in $\bar{D}$ from x_i to x_{i+1} such that

$$\ell_Z(P \circ \gamma_i) < \min\left\{\frac{\delta}{2}, d_Z(\gamma(t_i), \gamma(t_{i+1})) + \frac{\epsilon}{k+1}\right\}.$$

It follows that γ_i does not intersect V. Let $\tilde{\gamma}$ be the concatenation of the curves γ_i for $i = 0, \dots, k$. Then $\tilde{\gamma}$ is a closed curve and

$$\ell_Z(P \circ \tilde{\gamma}) < \ell_Z(\gamma) + \epsilon < l_0.$$

Moreover, $\tilde{\gamma}$ does not intersect V and $P \circ \tilde{\gamma}$ is homotopic to γ in $Z \setminus U$. In particular, $\tilde{\gamma}$ is not null-homotopic in $\bar{D} \setminus V$. By Lemma 7.13, there exists a noncontractible Jordan curve η in $\bar{D} \setminus V$ with $\ell_Z(P \circ \eta) \le \ell_Z(P \circ \tilde{\gamma})$. It follows that the Jordan domain enclosed by η in $\bar{D}$ contains V. Hence, Lemma 6.10 and Lemma 6.8 imply

$$\mu_Z(\Omega) - \epsilon \le \mu_Z(U) = \mathrm{Area}_\mu(P|_V) \le C\,\ell_Z(P \circ \eta)^2 \le C(\ell_Z(\gamma) + \epsilon)^2.$$

Since $\epsilon > 0$ was arbitrary this yields (8-1). □

Remark 8.3 The proof of Theorem 8.2 shows the following slightly stronger statement. Let T be any Jordan curve of finite length l in Z with Jordan domain $\Omega \subset Z$. Then for every $\epsilon > 0$ there exists a Jordan curve $\eta\colon S^1 \to \bar{D}$ such that $\ell_X(u \circ \eta) < l + \epsilon$ and such that $\mathrm{Fill}_{X,\mu}(u \circ \eta) + \epsilon \ge \mu_Z(\Omega)$. Moreover, reparametrizing η if needed and using Lemma 4.8 we may assume that $u \circ \eta$ is Lipschitz continuous.

Remark 8.4 All subsequent results of this section are derived only using Theorem 8.2 and the coarea inequality, Lemma 2.3. If we just assume that a metric space Z satisfies the conclusions of Theorem 1.2 and do not assume that Z is countably 2–rectifiable, the proofs below remain valid once every reference to Lemma 2.3 is replaced by a reference to Lemma 2.2. We obtain as conclusions all theorems below with μ_Z replaced by $\mathcal{H}^2$ and $q(\mu_Z)$ replaced by $q(\mathcal{H}^2) = \frac{\pi}{4}$.

8.3 Area growth

In the sequel we will denote the Jordan curve $\Gamma' = P(S^1) \subset Z$ by ∂Z, since it is the boundary circle of the topological disc Z. It is well known that isoperimetric inequalities often imply lower bounds on volume growth:

Theorem 8.5 *Let $z_0 \in Z$. Then*

$$\mu_Z(B(z_0, r)) \geq \min\{q(\mu)^2 \cdot \tfrac{1}{4C} \cdot r^2, C l_0^2\} \tag{8-2}$$

for every $0 \leq r < d_Z(z_0, \partial Z)$.

Proof Fix any $z_0 \in Z \setminus \partial Z$. Assume that there exists some $r_0 > 0$, which we fix from now on, such that (8-2) does not hold. We set $b(r) := \mu_Z(B(z_0, r))$ for $r \leq r_0$. Then the ball $B(z_0, r_0)$ does not intersect ∂Z, and $b(r_0) < C l_0^2$. Thus $b(r) < C l_0^2$ for all $r \leq r_0$. Let $h\colon Z \to \mathbb{R}$ be the 1–Lipschitz distance function defined by $h(z) := d_Z(z, z_0)$. Applying the coarea inequality (Lemma 2.3) to the restriction $h\colon B(z_0, r) \to \mathbb{R}$ we deduce

$$b(r) \geq q(\mu) \cdot \int_0^r \mathcal{H}^1(h^{-1}(s))\, ds \tag{8-3}$$

for all $r \leq r_0$. In particular, the compact set $S_r = h^{-1}(r)$ has finite $\mathcal{H}^1(S_r)$ for almost every $0 \leq r \leq r_0$.

We denote by $F(r)$ the right-hand side of (8-3). Then the function $F\colon [0, r_0] \to \mathbb{R}$ is absolutely continuous and

$$F'(r) = q(\mu) \cdot \mathcal{H}^1(S_r) \tag{8-4}$$

for almost all $0 < r \leq r_0$. We claim

$$b(r) \leq C \cdot \mathcal{H}^1(S_r)^2 \tag{8-5}$$

for all $r \leq r_0$. Indeed, fix an embedding of Z into S^2. The compact subset S_r separates z_0 from every point $z \in Z$ with $d_Z(z, z_0) > r$, hence from any point q on ∂Z, which we fix now. On the other hand, the ball $B(z_0, r)$ is connected, since the metric on Z is intrinsic. If $\mathcal{H}^1(S_r) = \infty$ then (8-5) is valid. On the other hand, if $\mathcal{H}^1(S_r)$ is finite we can apply Corollary 7.5 and find a Jordan curve $T \subset S_r$ which still separates z_0 from q. Then p and therefore the whole ball $B(z_0, r)$ must be contained in the Jordan domain of T. Since $\ell_Z(T) \leq \mathcal{H}^1(S_r)$ we deduce (8-5) from Theorem 8.2.

Taking (8-3), (8-5) and (8-4) we deduce for almost all $r \leq r_0$ the inequality

$$F(r) \leq b(r) \leq C \cdot q(\mu)^{-2} \cdot [F'(r)]^2. \tag{8-6}$$

By Lemma 6.11, we have $F(r) > 0$ for all $r > 0$. Thus integrating (8-6) yields $b(r_0) \geq F(r_0) \geq q(\mu)^2 \cdot \left(\frac{1}{4C}\right) \cdot r_0^2$, in contradiction with our assumption. This contradiction finishes the proof. □

8.4 Linear local contractibility

The isoperimetric inequality and the lower bound on the area growth of balls imply uniform linear local contractibility of the space Z:

Theorem 8.6 *Let Z be as above. For any $0 < r < l_0/2$ and any $z \in Z$, the ball $B(z,r)$ in Z is contractible inside the ball $B(z,(8C+1)\cdot r)$.*

Proof Fix a point $z \in Z$ and consider the open connected sets $O = B(z,r)$ and $U = B(z,(8C+1)\cdot r)$, where $r < l_0/2$. Assume that O is not contractible in U. Note that O has trivial higher homotopy groups as does any open subset of the disc. Thus we find a curve $\gamma\colon S^1 \to O$ which is noncontractible in U. Since O is locally contractible, we may replace parts of γ of small diameter by short geodesic segments. Thus we may assume that γ is a concatenation of short geodesics $\gamma = \gamma_1 * \gamma_2 * \cdots * \gamma_k$. Connect the starting and ending points of γ_i with the point z by geodesics $\eta_i^{\pm}$. We may assume $\eta_{i+1}^- = \eta_i^+$ for all i. For any $1 \leq i \leq k$, these geodesics $\eta_i^{\pm}$ together with γ_i provide a closed piecewise geodesic curve c_i of length smaller than $2r$. Moreover, γ is homotopic to the concatenation $c_1 * c_2 * \cdots * c_k$. Thus one of the curves c_i is noncontractible in U. Hence we may assume $\ell_Z(\gamma) < 2r < l_0$. Using Lemma 7.13 we may further assume that γ is a Jordan curve.

Consider the Jordan domain Ω of γ in Z and deduce from Theorem 8.2 that $\mu_Z(\Omega) < 4Cr^2$. Since γ is contractible in $\overline{\Omega}$, we find a point $y \in \Omega \setminus U$; hence $d_Z(z,y) \geq (8C+1)r$. By the triangle inequality, the distance $d_Z(y,\gamma)$ from y to the curve γ is larger than $8Cr$. The connected open ball $B(y,8Cr)$ does not intersect γ; hence it does not intersect ∂Z and is completely contained in Ω. In particular, its area is bounded from above by the area of Ω, which is at most $4Cr^2 < Cl_0^2$. From Theorem 8.5 and using $q(\mu) \geq \frac{1}{2}$, we deduce $\mu_Z(\Omega) \geq q(\mu)^2 \cdot \left(\frac{1}{4C}\right) \cdot (8Cr)^2 \geq 4Cr^2$.

This contradiction finishes the proof. □

Using that Z is a closed disc we directly deduce:

Corollary 8.7 *Let V be a subset of Z homeomorphic to a closed disc. If the boundary circle ∂V of V has diameter $r < l_0/2$ then V has diameter at most $(8C+1)\cdot r$.*

Finally, note that the assumption $r < l_0/2$ in Theorem 8.6 was only needed in one step, namely to assure that the area of some domain in Z does not exceed $C l_0^2$, a condition which is automatically satisfied if $l = \partial Z < l_0$. Thus no bound on r is needed in this case. In particular, Corollary 8.7 can be applied to the whole disc $V = Z$, using that ∂Z has diameter at most $l/2$ in this case. Thus we have:

Corollary 8.8 *If ∂Z has length l smaller than l_0 then Z has diameter at most $\left(4C + \frac{1}{2}\right) \cdot l$. Moreover, no bounds on r are needed in Theorem 8.6 and Corollary 8.7.*

8.5 Equicompactness

The area growth of balls implies that the number of disjoint balls of a given radius in Z can be bounded in terms of the area of Z. More precisely:

Theorem 8.9 *Let Z be as above. Set $A = \mu(Z)$, and set $l = \ell_Z(\partial Z)$. For any integer $n > l/(8C l_0)$ there exists some (l/n)–dense subset F_n in Z with at most $2n + 64C \cdot (A/l^2) \cdot n^2$ elements.*

Proof Let U_n denote the $(l/(2n))$–tubular neighborhood of ∂Z in Z and $V_n = Z \setminus U_n$. Let F_n^+ denote a maximal (l/n)–separated subset in V_n and let F_n^- denote a maximal $(l/(2n))$–separated subset in ∂Z. Then F_n^+ is (l/n)–dense in V_n and F_n^- is (l/n)–dense in U_n. Hence the union F_n of F_n^+ and F_n^- is (l/n)–dense in Z. Since ∂Z is a 1–Lipschitz image of the circle of length l, the set F_n^- has at most $2n$ elements.

On the other hand, for any $p \in F_n^+$ the ball $B_p := B(p, l/(2n))$ does not intersect ∂Z. From Theorem 8.5, the general estimate $q(\mu) \geq \frac{1}{2}$, and the assumption on n we infer

$$\mu(B_p) \geq \min\left\{\frac{1}{16C} \cdot \frac{l^2}{4n^2}, C l_0^2\right\} = \frac{l^2}{64C \cdot n^2}. \tag{8-7}$$

Moreover, all these balls B_p are disjoint. Thus, the number of elements in F_n^+ times the right-hand side of (8-7) is not larger than A. Hence F_n^+ has at most $64C \cdot (A/l^2) \cdot n^2$ elements. This finishes the proof. □

If $l < l_0$ then $A/l^2 \leq C$ and, moreover, no bound on n is needed to conclude (8-7). Therefore:

Corollary 8.10 *Assume that the length l of ∂Z is smaller than l_0. For any integer n the set Z contains some (l/n)–dense subset F_n with at most $2n + 64C^2 n^2$ elements.*

8.6 Decomposition of Z by a graph

The following result is a topological version of a similar discrete statement proved in [30] for curves in groups with quadratic isoperimetric inequality.

Theorem 8.11 *Let Z be as above. There exists a constant M depending on l_0 and the upper bounds of C, the area A of Z, and the length l of ∂Z such that the following holds true. For any integer n there exists a finite connected graph $\partial Z \subset G \subset Z$ such that $Z \setminus G$ has at most $M \cdot n^2$ components and such that any of these components is a topological disc of diameter at most l/n.*

Proof It suffices to prove the result for all n which satisfy $n \geq l/(8Cl_0)$ and $4l/n < l_0/2$. By Theorem 8.9, we find an (l/n)–dense subset $F = F_n$ in Z with elements $p_1, \dots, p_m$ for some $m \leq M_1 \cdot n^2$ for some M_1 depending on the upper bounds of C, l and A. Since $Z \setminus \partial Z$ is dense in Z, we may assume that F is contained in the open disc $Z \setminus \partial Z$.

The idea is now to consider the *Voronoi domains* defined by the set F. However, we need a few minor modifications. For each $1 \leq i \leq m$, let $\hat{f}_i \colon Z \to \mathbb{R}$ be the distance function to the point p_i. Since the functions $\hat{f}_i$ are 1–Lipschitz, we deduce from Lemma 2.3 that for almost all $\epsilon > 0$ the fiber $(\hat{f}_i - \hat{f}_j)^{-1}(\epsilon)$ has finite $\mathcal{H}^1$–measure, for any $1 \leq i, j \leq m$.

Thus we find arbitrarily small $\epsilon > 0$ such that $(\hat{f}_j - \hat{f}_i)^{-1}(k\epsilon)$ has finite $\mathcal{H}^1$–measure for any $1 \leq i, j \leq m$ and any $1 \leq k \leq m$. We fix such ϵ satisfying

$$4m \cdot \epsilon \leq \inf\{d(p_i, p_j), j \neq i\} \leq \frac{2l}{n}.$$

Consider the modified distance functions $f_i \colon Z \to \mathbb{R}$ given by $f_i(z) := d(p_i, z) + i \cdot \epsilon$. By construction, for any $i \neq j$ the set S_{ij} of points z with $f_i(z) = f_j(z)$ has finite $\mathcal{H}^1$–measure.

Let U_i be the set of points $z \in Z$ with $f_i(z) < f_j(z)$ for all $j \neq i$. The sets U_i are open. By assumption on ϵ, we have $p_i \in U_i$. The functions f_j are 1–Lipschitz and decrease with velocity 1 on any geodesic connecting z with p_j. Therefore, for any $z \in U_i$ any geodesic from z to p_i is entirely contained in U_i. In particular, U_i is connected. Since F is (l/n)–dense in Z, and by the smallness of ϵ, any point $z \in U_i$ has distance at most $2l/n$ to p_i. Therefore, the diameter of U_i is at most $4l/n$.

Set $K = Z \setminus \left(\bigcup_{i=1}^m U_i\right)$ and note that $K \subset \bigcup_{1\leq i<j\leq m} S_{ij}$. Then K is a compact subset of Z, which has finite $\mathcal{H}^1$–measure and separates points p_i pairwise. The complement $Z\setminus K$ has exactly m connected components U_i. Let K_0 denote a minimal compact subset of K which still separates the points from F pairwise. We deduce from Lemma 7.7 that $K_1 = K_0 \cup \partial Z$ is a finite graph, whose complement $Z \setminus K_1$ has exactly m connected components $W_1, \dots, W_m$ containing the corresponding points p_i. Since U_i is connected, we obtain $U_i \subset W_i$. Since K_1 is nowhere dense in Z, the sets U_i are dense in W_i for all i. Thus the diameter of any W_i is also bounded by $4l/n$.

Let now G be the connected component of ∂Z in K_1. This is a finite connected graph, which is open in K_1. Any component V of $Z \setminus G$ must intersect at least one of the components W_j; hence V must contain this component W_j in this case. We deduce that $Z \setminus G$ has at most m components.

It remains to control the size of these possibly larger components of $Z \setminus G$. By Lemma 7.3, any component V of $Z \setminus G$ is homeomorphic to an open disc. We claim that the boundary ∂V of this disc has diameter at most $4l/n$. Indeed, some neighborhood O of ∂V in $\bar{V}$ intersects K_1 only in ∂V. Hence, choosing such a connected neighborhood O, we deduce that $O \setminus \partial V$ is contained in one of the components W_j. Therefore its diameter is bounded by $4l/n$. We deduce the same bound for $\bar{O}$, hence for ∂V. From Corollary 8.7 we infer that V has diameter at most $(8C+1)\cdot 4l/n$.

We set $N = n/(4\cdot(8C+1))$. Then the diameter of any component of $Z \setminus G$ is at most l/N. Moreover, $Z \setminus G$ has at most $m \leq M_1 \cdot n^2 = M \cdot N^2$ components for a constant M depending only on M_1 and C, hence only on the upper bounds of l, A and C. □

Note again that if the length l of ∂Z is smaller than l_0 then one does not need any additional assumption on n in the first lines of the above proof. Recall that $C \geq \frac{1}{8\pi}$ [24, Corollary 1.6]. Thus, we can estimate $4(8C+1)$ by $k\cdot C$ and $2n + 64C^2\cdot n^2$ by $kC^2\cdot n^2$ for a universal constant k. Thus, from Corollary 8.10 and the last lines of the proof of Theorem 8.11 we obtain:

Corollary 8.12 *If the length l of the boundary curve ∂Z is smaller than l_0 then the constant M in Theorem 8.11 can be chosen to be $\widehat{M}\cdot C^4$ for some universal constant $\widehat{M}$.*

9 Collecting the harvest

We now provide the proofs of the main theorems stated in the introduction. We formulate them for general definitions of area μ and continue to use the notations of the previous sections.

9.1 General case

We begin with a generalization of Theorem 1.2. Recall that $\frac{\pi}{4}$ in Theorem 1.2 is equal to $q(\mathcal{H}^2)$.

Theorem 9.1 *The metric space Z is homeomorphic to $\bar{D}$. It is countably 2–rectifiable with finite $\mathcal{H}^2(Z)$. For any Jordan curve η in Z of length $l < l_0$, the domain Ω of the disc Z enclosed by η satisfies*

(9-1) $$\mu_Z(\Omega) \le C \cdot l^2.$$

Proof The space Z is homeomorphic to $\bar{D}$ by Theorem 8.1, and it is countably 2–rectifiable with finite $\mathcal{H}^2(Z)$ by Lemma 6.9. The isoperimetric property (9-1) is exactly Theorem 8.2. □

The results in Corollary 1.3 have been all proven in the more general form stated in Section 8. Namely, under the assumptions of Corollary 1.3, the $\mu{=}\mathcal{H}^2$–area of Z is less than the critical value $C l_0^2$. The area growth (i) of Corollary 1.3 is exactly the statement of Theorem 8.5. The uniform local contractibility (ii) is contained in Theorem 8.6 and Corollary 8.8. Finally, the decomposition statement (iii) of Corollary 1.3 is contained in Theorem 8.11 and Corollary 8.12.

The next theorem generalizes Theorem 1.4.

Theorem 9.2 *The canonical projection $P\colon \bar{D} \to Z$ is a uniform limit of homeomorphisms $P_i\colon \bar{D} \to Z$. Moreover:*

- (i) $P \in \Lambda(\partial Z, Z) \subset W^{1,2}(D, Z)$.
- (ii) *$P\colon D \to Z$ is contained in $W^{1,p}_{\mathrm{loc}}(D, Z)$ for some $p > 2$ depending on C.*
- (iii) *$P\colon D \to Z$ is locally α–Hölder with $\alpha = q(\mu) \cdot \frac{1}{4\pi C}$.*
- (iv) *$\mu_Z(P(V)) = \mathrm{Area}_\mu(P|_V) = \mathrm{Area}_\mu(u|_V)$ for all open subsets $V \subset D$.*
- (v) *The map P is $\sqrt{2}$–quasiconformal. If X has property (ET) then P is conformal.*

Proof The first statement was proved in Theorem 8.1. In Lemma 6.6 we showed that $P \in W^{1,2}(D,Z)$ and (ii). Since P restricts to a weakly monotone parametrization $P\colon \partial D \to \partial Z$ by Lemma 6.4, we deduce $P \in \Lambda(\partial Z, Z)$. The statement of (iii) is contained in Lemma 6.5. The second equality of (iv) is contained in Lemma 6.6. The first equality follows from the definition of the μ–area (3-4) and the fact that the multiplicity function N appearing in the area formula Lemma 3.3 equals 1 almost everywhere on Z, as proven after Lemma 6.9. Statement (v) is contained in Corollary 6.7. □

Now we turn to the generalization of Theorem 1.5.

Theorem 9.3 *For every $\epsilon > 0$ there exists a decomposition $Z = S \cup \bigcup_{1\leq i<\infty} K_i$ with compact K_i and $\mu_Z(S) = 0$ such that the restrictions $\bar{u}\colon K_i \to \bar{u}(K_i)$ of the 1–Lipschitz map $\bar{u}$ are $(1+\epsilon)$–bi-Lipschitz. Moreover, for any $1 \leq i < \infty$ and any $x \in K_i$ we have*

(9-2) $$\lim_{y\to x, y\in K_i} \frac{d_Z(x,y)}{d_X(\bar{u}(x), \bar{u}(y))} = 1.$$

Finally, $\bar{u}\colon \partial Z \to \Gamma$ is an arclength-preserving homeomorphism.

Proof Since the map $P\colon \bar{D} \to Z$ satisfies Lusin's property (N), we combine Section 3.3 and Section 2.4 and obtain a disjoint decomposition $\bar{D} = S_0 \cup \bigcup_{1\leq i<\infty} L_i$ with the following properties. The subsets L_i are compact and $\mathcal{H}^2(P(S_0)) = 0$. The restriction $P\colon L_i \to P(L_i)$ is a bi-Lipschitz map, which has a metric differential at each point. Moreover, this metric differential is a norm at each point $z \in L_i$ and coincides with the approximate metric differential $\operatorname{ap}\operatorname{md} P_z$. Finally, if one considers L_i with the metric determined by any of these norms $\operatorname{ap}\operatorname{md} u_z$ then the map $P\colon L_i \to P(L_i)$ is $(1+\epsilon)$–bi-Lipschitz.

Since $u = \bar{u} \circ P$, the restriction $u\colon L_i \to u(L_i)$ is a Lipschitz map. By Lemma 6.6, at almost all points of L_i its metric differential coincides with $\operatorname{ap}\operatorname{md} u_z = \operatorname{ap}\operatorname{md} P_z$. Decompose every L_i into a negligible set and countably many compact sets on which u is $(1+\epsilon)$–bi-Lipschitz with respect to the appropriate norm. Taking all these negligible subsets together into a set S_1 with $\mathcal{H}^2(S_1) = 0$, we obtain a decomposition $\bar{D} = S_0 \cup S_1 \cup \bigcup_{1\leq j<\infty} M_j$, with S_0 from above, such that the following holds true. The sets M_j are compact, the restrictions of u and P to M_j have equal metric differentials at all points of M_j. Moreover, $u\colon M_j \to u(M_j)$ and $P\colon M_j \to P(M_j)$ are $(1+\epsilon)$–bi-Lipschitz if M_j is equipped with the norm $\operatorname{ap}\operatorname{md} u_z$ for some $z \in M_j$.

Therefore, for the compact set $K_j = P(M_j)$, the restriction $\bar{u}\colon K_j \to u(M_j)$ is $(1+\epsilon)^2$–bi-Lipschitz. Moreover, since P and u have the same metric differentials at all $z \in M_j$, we get (9-2). We note that $\mu_Z(P(S_0)) = 0$ by construction and $\mu_Z(P(S_1)) = 0$ since P has Lusin's property (N). Thus, with $S = P(S_0 \cup S_1)$ we can write $Z = S \cup \bigcup_{1 \leq j < \infty} K_j$, so that $\mu_Z(S) = 0$ and the restriction of $\bar{u}$ to any K_j has all the required properties. It can happen that this union is not disjoint. Then we make it disjoint by a further subdivision, noting that any compact subset of K_j has the property required in the statement of the theorem.

The last statement is contained in Lemma 6.4. □

9.2 The chord-arc case

We are generalizing Theorem 1.6 now.

Theorem 9.4 *Assume in addition that Γ is a chord-arc curve. Then $P \in W^{1,p}(D, Z)$ for some $p > 2$ depending on C and the bi-Lipschitz constant L of some parametrization $S^1 \to \Gamma$. In particular, $P\colon \bar{D} \to Z$ is globally $\left(1 - \frac{2}{p}\right)$–Hölder continuous.*

There exists $\delta > 0$, depending only on C, l_0 and L, such that for all $z_0 \in Z$ and all $0 \leq r \leq \delta$ we have

$$\mu_Z(B(z_0, r)) \geq \delta \cdot r^2. \tag{9-3}$$

Proof There exists some $p > 2$ depending only on C and L such that $u \in W^{1,p}(D, X)$, by [23, Theorem 3.1]. By Lemma 6.6, we get $P \in W^{1,p}(D, Z)$.

It remains to prove (9-3). We fix a sufficiently small δ, to be determined later and proceed in analogy with the proof of Theorem 8.5. We may assume $\delta^3 < C l_0^2$. Consider an arbitrary $z_0 \in Z$ and set $b(r) = \mu_Z(B(z_0, r))$. We consider the distance function $h\colon Z \to \mathbb{R}$ from the point z_0 and the corresponding level sets S_r, the distance spheres around z_0. Finally, we fix a point $q \in \partial Z$ with maximal distance on ∂Z from z_0 and note that $d_Z(z_0, q) > \delta$ if $2\delta < \operatorname{diam}(\partial Z)$.

If for some $r_0 < \delta$ we have $b(r_0) \geq C l_0^2$ then (9-3) holds true for all $r_0 \leq r \leq \delta$. Arguing as in the proof of Theorem 8.5, we only need to find some constant $k > 0$, such that for all $r < \delta$ with $b(r) < C l_0^2$ the following inequality holds true:

$$k \cdot \mathcal{H}^1(S_r)^2 \geq b(r). \tag{9-4}$$

The inequality is trivially fulfilled if $\mathcal{H}^1(S_r) = \infty$. For all $0 < r \le \delta$, the set S_r separates p from q. For all r with finite $\mathcal{H}^1(S_r)$, we apply Corollary 7.6 and find a subset γ of S_r still separating p from q such that one of the following two possibilities holds true. Either γ is a Jordan curve or γ is a simple curve connecting two points on ∂Z and not intersecting ∂Z in any other points. If γ is a Jordan curve, then the same argument as in the proof of Theorem 8.5 gives us $C \cdot \mathcal{H}^1(S_r)^2 \ge C \cdot \mathcal{H}^1(\gamma)^2 \ge b(r)$. Otherwise, the shorter part of ∂Z between the two points of intersection with the simple curve γ has length bounded from above by $L \cdot \ell_Z(\gamma)$, by the chord-arc condition. Moreover, if δ has been chosen small enough, this shorter part of ∂Z does not contain the point q. Therefore, the Jordan curve $\hat{\gamma}$ consisting of γ and the piece of ∂Z we have found has length bounded from above by $(1+L) \cdot \ell_Z(\gamma)$. Moreover, the closure of the Jordan domain of $\hat{\gamma}$ contains z_0, hence the whole ball $B(z_0, r)$ by construction. Now we apply the isoperimetric inequality Theorem 8.2 to the curve $\hat{\gamma}$ to deduce that $C \cdot \mathcal{H}^1(\hat{\gamma})^2 \ge b(r)$. Since $(1+L) \cdot \mathcal{H}^1(S_r) \ge (1+L) \cdot \mathcal{H}^1(\gamma) \ge \ell_Z(\hat{\gamma})$ we obtain the desired inequality (9-4) with $k = C \cdot (1+L)^2$. □

9.3 Different choices of the family of curves

As already mentioned in Section 1.7, all of our results concerning the space Z remain valid if in the definition of the pseudometric d_u one uses rectifiable, piecewise bi-Lipschitz, or piecewise smooth curves instead of continuous curves. Taking into account the following observations, the proofs remain literally the same as the ones given above. Firstly, Corollary 3.2 remains valid if (3-3) is only assumed for p–almost all piecewise smooth curves. Secondly, the curves constructed in Lemma 5.3 and Corollary 5.4 are piecewise smooth. Thirdly, in Proposition 6.2, the last equation remains valid for all curves γ in the chosen family.

10 The absolute minimal filling

10.1 The proof of Theorem 1.7

Let us fix a quasiconvex definition of area μ and a bi-Lipschitz circle Γ. We consider an isometric embedding of Γ into its injective hull $i\colon \Gamma \to Y$. Concerning the definition and properties of injective metric spaces and injective hulls we refer the reader to [20; 14]. Recall Y satisfies the (C,∞)–isoperimetric inequality for any μ and $C = \frac{1}{2\pi}$ [24, Lemma 10.3].

As proved in [24, Corollary 10.4], the Sobolev filling area

$$m_{\mu,\mathrm{Sob}}(\Gamma) := \inf\{\mathrm{Area}_\mu(u) : G \text{ complete}, \Gamma \subset G, u \in \Lambda(\Gamma, G)\}$$

is realized by a solution u of the Plateau problem for (Γ, Y) with respect to μ. Denoting by $\gamma_0\colon S^1 \to \Gamma$ a parametrization proportional to arclength, we deduce

$$m_{\mu,\mathrm{Sob}}(\Gamma) = \inf\{\mathrm{Area}_\mu(u) : u \in \Lambda(\Gamma, Y), \mathrm{tr}(u) = \gamma_0\} \tag{10-1}$$

from Lemma 4.8; see also [23, Corollary 3.3].

In order to compare the Sobolev filling area $m_{\mu,\mathrm{Sob}}(\Gamma)$ with Gromov's restricted filling area $m_\mu(\Gamma)$, we recall the following result of Ivanov proven in [15]. The restricted filling area $m_\mu(\Gamma)$ is the infimum over all μ–areas of Lipschitz maps $v\colon \bar{D} \to G$ into some metric space G containing Γ, such that the restriction of v to S^1 is a bi-Lipschitz parametrization of Γ. Since any such map v is a Sobolev map in $\Lambda(\Gamma, G)$, we get

$$m_{\mu,\mathrm{Sob}}(\Gamma) \le m_\mu(\Gamma).$$

The reverse inequality is a direct consequence of (10-1) and the following lemma, whose proof is essentially contained in [13, Theorem 8.2.1].

Lemma 10.1 *Let Y be an injective metric space and let $u \in W^{1,2}(D, Y)$ be such that the trace $\mathrm{tr}(u)\colon S^1 \to Y$ is Lipschitz continuous. Then for every $\epsilon > 0$ there exists a Lipschitz map $v\colon \bar{D} \to V$ with $\mathrm{tr}(v) = \mathrm{tr}(u)$ and $\mathrm{Area}_\mu(v) < \mathrm{Area}_\mu(u) + \epsilon$.*

Proof We denote by B the ball $B(0,2) \subset \mathbb{R}^2$. We extend u to a map $\hat{u} \in W^{1,2}(B, Y)$ by $\hat{u}(rz) = u(z)$ for $r > 1$ and $z \in S^1$. By assumption, the map $\hat{u}$ is Lipschitz continuous on $B \setminus D$ and $\mathrm{Area}_\mu(\hat{u}) = \mathrm{Area}_\mu(u)$.

Fix $\epsilon > 0$. As in the proof of [13, Theorem 8.2.1], there exist a sufficiently large $t > 0$ and a set $E_t \subset B$ such that $\hat{u}\colon B \setminus E_t \to Y$ is t–Lipschitz continuous. Moreover, the Lebesgue measure of E_t is at most ϵ/t^2. Finally, by the construction in [13, Theorem 8.2.1], for sufficiently large $t > 0$ the set E_t is contained in the ball $B\big(0, \frac{3}{2}\big)$. Since Y is injective, we find some t–Lipschitz extension $v\colon B \to Y$ of $\hat{u}|_{B\setminus E_t}$.

Since v and $\hat{u}$ coincide on $B \setminus E_t$ and since $\mathrm{Area}_\mu(v|_{E_t}) \le t^2 \cdot (\epsilon/t^2) = \epsilon$ it follows that $\mathrm{Area}_\mu(v) - \mathrm{Area}_\mu(u) \le \epsilon$. Moreover, after rescaling the ball B so that v is defined on D we clearly have $\mathrm{tr}\, v = \mathrm{tr}\, u$. □

Remark 10.2 The statement of Lemma 10.1 (and its proof, up to minor modifications) remains valid if Y is 1–Lipschitz connected up to some scale; see [27, Proposition 3.1].

The remainder of Theorem 1.7 is a consequence of the previous results. Indeed, consider our μ–minimal map $u \in \Lambda(\Gamma, Y)$ with $m_{\mu,\mathrm{Sob}}(\Gamma) = \mathrm{Area}_\mu(u)$. Consider its unique continuous extension $u\colon \bar{D} \to Y$ and the intrinsic metric space Z defined via the pseudodistance d_u on $\bar{D}$. Consider the corresponding projection $P\colon \bar{D} \to Z$. The space Z is compact, geodesic and homeomorphic to $\bar{D}$ by Lemma 6.3. The remaining statements of Theorem 1.7 are direct consequences of Theorem 9.1, Theorem 9.2, Theorem 9.3 and Theorem 9.4.

10.2 Absolute minimizers

Following [15; 16] we say that a geodesic metric space M bi-Lipschitz homeomorphic to $\bar{D}$ is an *absolute minimal filling (of its boundary with respect to the definition of area μ)* if $\mu(M) = m_\mu(\partial M)$. By Theorem 1.7, this implies $m_{\mu,\mathrm{Sob}}(\partial M) = \mu(M)$. The following classes of spaces are important examples of absolute minimal fillings.

Example 10.3 Let V be a 2–dimensional normed space and let M be any closed subset of V bi-Lipschitz homeomorphic to $\bar{D}$. Then M with its induced intrinsic metric is an absolute minimal filling with respect to any quasiconvex μ. Indeed, consider the injective hull W of V. Then W is a Banach space which contains V as a linear subspace [14]. By the definition of quasiconvexity, $\mu(M)$ equals the infimum of μ–areas of Lipschitz discs in W which have ∂M as their boundaries. By the injectivity of W, this implies $\mu(M) = m_\mu(\partial M)$.

Example 10.4 Let M be a 2–dimensional smooth Finsler manifold homeomorphic to $\bar{D}$. If all local geodesics in M are globally minimizing then M is an absolute minimal filling with respect to the Holmes–Thompson definition of area μ^{ht}; see [16].

For any bi-Lipschitz circle Γ, Theorem 1.7 provides a generalized minimal filling of Γ, which may be slightly less regular than a bi-Lipschitz disc. We hope to investigate further properties of such generalized minimal fillings in a continuation of this paper; see Question 11.7.

11 Examples and questions

The first example is well known; see [28].

Example 11.1 Let X be the Euclidean cone over a circle Γ of length $2\pi\alpha$ with $0 < \alpha \leq 1$. The space has property (ET) and admits a (C, l_0)–isoperimetric inequality for any l_0 and $C = 1/(4\pi\alpha)$, for any definition of area μ. The unique solution of the

Plateau problem for the curve Γ is α–Hölder continuous, but not β–Hölder continuous for any $\beta > \alpha$. The arising space Z coincides with X and P coincides with u. The balls of radius $r < 1$ around the origin have area $\alpha \cdot \pi r^2 = \left(\frac{1}{4C}\right) r^2$.

In the last example, the map P is not Lipschitz continuous for $\alpha < 1$, but Z is still bi-Lipschitz equivalent to $\bar{D}$. Moreover, for $\alpha \to 1$ the isoperimetric constant tends to the critical value $\frac{1}{4\pi}$. Nevertheless, the solution of the Plateau problem need not be Lipschitz continuous. Therefore, the answer to the next question must involve very fine invariants of spaces.

Question 11.2 What conditions, apart from upper curvature bounds, imply that solutions of the Plateau problems are Lipschitz continuous? Under which conditions is Z bi-Lipschitz homeomorphic to a disc?

Natural examples to study in connection with the last question might be Finsler manifolds and Riemannian manifolds with nonsmooth metrics. The next example mentioned in the introduction is a typical counterexample for many results valid in the smooth case.

Example 11.3 Choose a compact metric ball $T \subset D$ and consider the quotient metric space $X = \bar{D}/T$ with the quotient metric; see [5, Definition 3.1.12]. Then X is a geodesic space, homeomorphic to a disc, and X is flat outside a single thick point, the image of T. It follows from the Euclidean isoperimetric inequality that X admits a (C,∞)–isoperimetric inequality with optimal constant $C = \frac{1}{2\pi}$. The "worst" curves, enclosing the maximal area, are projections of tiny circles which meet the boundary of T orthogonally.

The canonical projection $u\colon \bar{D} \to X$ is a conformal solution of the Plateau problem. The metric space Z coincides with X and u coincides with P. The minimal disc u has the set T as the set of "branch points". Moreover, small balls around nonthick points in $Z = X$ have quadratic area growth and small balls around the thick point have a linear area growth. Thus, the Hausdorff area is not a doubling measure on Z. In particular, Z is not bi-Lipschitz homeomorphic to a disc.

In view of the nature of this example, the isoperimetric constant $\frac{1}{2\pi}$ might be the critical value for constructions of this type. Note that this value $C = \frac{1}{2\pi}$ is also very interesting in view of the absolute Plateau problem.

Question 11.4 Can the set of branch points of a solution of the Plateau problem be large if the isoperimetric constant C is smaller than $\frac{1}{2\pi}$? Can the map P be noninjective in this case?

The fibers of P, which are a priori allowed by the statement that P is a uniform limit of homeomorphisms, can be arbitrary cell-like sets, for instance any continuous simple arc. We do not know if such general fibers can indeed occur.

Question 11.5 Can fibers of the canonical map P be noncontractible? Are such fibers always Lipschitz retracts?

The following question is closely related to the previous one. By Theorem 1.1, the question has an affirmative answer if we find controlled approximations of any curve in Z by P–images of curves in $\bar{D}$.

Question 11.6 Does the map $\bar{u}\colon Z \to X$ preserve the lengths of all curves in Z?

Finally, we do not know to what extent the conclusions about absolute minimizers are optimal.

Question 11.7 Are solutions of the absolute Plateau problem Lipschitz continuous? What can be said about their geometry?

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Proposed: John Lott — Received: 1 December 2016
Seconded: Tobias H Colding, Bruce Kleiner — Revised: 7 April 2017