msp

Geometry & Topology 22 (2018) 645–691

The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link

ALEXEI OBLOMKOV
JACOB RASMUSSEN
VIVEK SHENDE
APPENDIX BY EUGENE GORSKY

We conjecture an expression for the dimensions of the Khovanov–Rozansky HOMFLY homology groups of the link of a plane curve singularity in terms of the weight polynomials of Hilbert schemes of points scheme-theoretically supported on the singularity. The conjecture specializes to our previous conjecture (2012) relating the HOMFLY polynomial to the Euler numbers of the same spaces upon setting $t=-1$. By generalizing results of Piontkowski on the structure of compactified Jacobians to the case of Hilbert schemes of points, we give an explicit prediction of the HOMFLY homology of a (k,n) torus knot as a certain sum over diagrams.

The Hilbert scheme series corresponding to the summand of the HOMFLY homology with minimal "a" grading can be recovered from the perverse filtration on the cohomology of the compactified Jacobian. In the case of (k,n) torus knots, this space furnishes the unique finite-dimensional simple representation of the rational spherical Cherednik algebra with central character k/n. Up to a conjectural identification of the perverse filtration with a previously introduced filtration, the work of Haiman and Gordon and Stafford gives formulas for the Hilbert scheme series when $k=mn+1$.

14H20, 14N35; 57M27

1 Overview

Let X be the germ of a complex plane curve singularity. Its topological properties are captured by its *link*, the intersection of a representative of X with the boundary of a small ball surrounding the singularity; see Arnol'd, Guseĭn-Zade and Varchenko [5] and Milnor [52].

We previously conjectured [56] that the HOMFLY polynomial of the link is recovered from the Euler characteristics of certain moduli spaces associated to the singularity.

Published: 22 January 2018 DOI: 10.2140/gt.2018.22.645

Specifically, let the HOMFLY polynomial $\overline{\boldsymbol{P}}$ be normalized by the skein relation

$$a\,\overline{\boldsymbol{P}}(\nearrow) - a^{-1}\,\overline{\boldsymbol{P}}(\nearrow) = (q-q^{-1})\,\overline{\boldsymbol{P}}(\,)\,(\,), \tag{1}$$

$$a - a^{-1} = (q-q^{-1})\,\overline{\boldsymbol{P}}(\bigcirc). \tag{2}$$

We write $X^{[n]}$ for the Hilbert scheme of n points on X. We define an incidence variety

$$X^{[l]} \times X^{[l+m]} \supset X^{[l\leq l+m]} := \{(I,J) \mid I \supset J \supset M \cdot J\},$$

where M is the maximal ideal at the central point.

Conjecture 1 [56] *Let X be the germ of a plane curve singularity, with Milnor number μ. Then*

$$\left(\frac{a}{q}\right)^{\mu-1} \sum_{l,m} q^{2l}(-a^2)^m \chi(X^{[l\leq l+m]}) = \overline{\boldsymbol{P}}(\text{link of } X).$$

The object of the present article is to promote this to a homological conjecture. On the right-hand side, we replace the HOMFLY polynomial with the Poincaré polynomial of the triply graded HOMFLY homology of Khovanov and Rozansky [45]. This has several slight variants; we discuss what is called by Rasmussen [60] the unreduced homology, and denoted by $\overline{\mathscr{H}}^{i,j,k}(K)$. It is infinite-dimensional, though finite in each graded piece. We write its graded dimension as

$$\overline{\mathscr{P}} = \sum_{i,j,k} a^i q^j t^k \overline{\mathscr{H}}^{i,j,k}(K).$$

We discuss our grading conventions for $\overline{\mathscr{H}}$ at the end of this section; for the moment, let us say that the homological grading t is chosen as in Dunfield, Gukov and Rasmussen [20], rather than in [45], so that $\overline{\boldsymbol{P}} = \overline{\mathscr{P}}|_{t=-1}$.

Recall that the cohomology of a complex algebraic variety admits a weight filtration W, in terms of which one may form the *weight polynomial*[1]

$$\mathfrak{w}(X) = \sum_{j,k} (-1)^{j+k} t^k \operatorname{Gr}_W^k(\mathrm{H}_c^j(X)).$$

The weight polynomial is characterized by two properties; first, it agrees with the Poincaré polynomial if X is a proper smooth variety, and second, it factors through

[1] Other authors prefer the terms Serre polynomial, virtual Poincaré polynomial, and E–polynomial. Its existence was conjectured by Serre, and follows from Deligne's theory of weights and mixed Hodge structures [16; 17; 18; 19]; for some discussion see Danilov and Khovanskii [15] and Durfee [22].

the Grothendieck ring of varieties. That is, for Y closed in X, we have

$$\mathfrak{w}(X-Y)=\mathfrak{w}(X)-\mathfrak{w}(Y).$$

Conjecture 2 *Let X be the germ of a plane curve singularity, with Milnor number μ. Then*

$$\Big(\frac{a}{q}\Big)^{\mu-1}\sum_{l,m}q^{2l}a^{2m}t^{m^2}\mathfrak{w}(X^{[l\le l+m]})=\overline{\mathscr{P}}(\text{link of } X).$$

Throughout we write $\overline{\mathscr{P}}_{\mathrm{alg}}$ for the left-hand side of Conjecture 2. As in Oblomkov and Shende [56] there is a useful equivalent formulation obtained by pushforward along the forgetful map $X^{[l\le l+m]}\to X^{[l]}$. By Nakayama's lemma, the fibers are $\mathrm{Grass}(\mathbb{C}^m\subset\mathbb{C}^r)$ over the locus $X^{[l]}_r\subset X^{[l]}$ parametrizing subschemes whose ideals require r generators. The weight polynomial of this Grassmannian is given by the q–binomial coefficient $\binom{r}{m}_{t^2}$. Thus, by the identity

$$\sum_{m=0}^{r}\binom{r}{m}_{t^2}a^{2m}t^{m^2}=\prod_{k=1}^{r}(1+t^{2k-1}a^2),$$

we may rewrite

$$\sum_{l,m}q^{2l}a^{2m}t^{m^2}\mathfrak{w}(X^{[l\le l+m]})=\sum_{l,r}q^{2l}\mathfrak{w}(X^{[l]}_r)\prod_{k=1}^{r}(1+t^{2k-1}a^2).$$

The series $\overline{\mathscr{P}}_{\mathrm{alg}}$ enjoys the following symmetry and rationality properties:

Proposition 3 *Let X have b branches and contribute δ to the arithmetic genus. Then the expression $(q^{-1}-q)^b\,\overline{\mathscr{P}}_{\mathrm{alg}}$ is a Laurent polynomial in q with terms of degrees between -2δ and 2δ. Moreover, it is invariant under $q\to 1/(qt)$. The number of different powers of a^2 appearing is at most one more than the multiplicity of the singularity.*

The polynomiality is known to hold for $\overline{\mathscr{P}}$ — for unibranch singularities the scaling factor corresponds to taking reduced HOMFLY homology — and the invariance was predicted in [20], but remains conjectural. In the specialization of $\overline{\mathscr{P}}$ to the HOMFLY polynomial, the symmetry $q\to -1/q$ is manifest in the skein relation. The bound on the degrees of a which appear corresponds to the fact that a singularity admits a braid presentation in which the number of strands is equal to the multiplicity.

It may be viewed as a defect that $\overline{\mathscr{P}}_{\mathrm{alg}}$ is assembled from the cohomologies of many different spaces. In fact, the coefficient of the lowest degree power of a,

$$\overline{\mathscr{P}}^{\mathrm{min}}_{\mathrm{alg}} := q^{1-\mu} \sum_{l=0}^{\infty} q^{2l} \mathfrak{w}(X^{[l]}),$$

may be recovered from a single space, namely the compactified Jacobian. As the previous proposition suggests, it is convenient to consider

$$\mathscr{P}^{\mathrm{min}}_{\mathrm{alg}} := (q^{-1}-q)^b \, \overline{\mathscr{P}}^{\mathrm{min}}_{\mathrm{alg}},$$

where b is the number of analytic local branches of the singularity. If C is a rational curve with X as its unique singularity, then we have

$$\mathscr{P}^{\mathrm{min}}_{\mathrm{alg}} = q^{-2g}(1-q^2)(1-q^2t^2) \sum_{l=0}^{\infty} q^{2l} \mathfrak{w}(C^{[l]}).$$

Let $\bar{J}^l(C)$ be the moduli space of rank one, degree l, torsion-free sheaves on C; it is integral of dimension g and locally a complete intersection; see Altman, Iarrobino and Kleiman [4; 3]. The choice of a degree l line bundle identifies $\bar{J}^l(C)$ with $\bar{J}^0(C)$. We henceforth suppress the index for $\bar{J}(C) := \bar{J}^0(C)$, which we term the *compactified Jacobian* of C. There is a map $C^{[l]} \to \bar{J}^l(C)$ given by sending a subscheme to the dual of the ideal sheaf cutting it out. For $l \gg 0$ this map is a projective space bundle and so $\mathfrak{w}(C^{[l]})$ is determined by $\mathfrak{w}(\bar{J}(C))$. In fact, work of Maulik and Yun [49], or of Migliorini and the third author [51], shows that we can recover the entire series $\mathscr{P}^{\mathrm{min}}_{\mathrm{alg}}$ from the perverse filtration on the cohomology of $\bar{J}_0(C)$. Specifically, according to Fantechi, Göttsche and van Straten [25] there is a deformation $\pi\colon \mathcal{C} \to B$ such that the total space $\bar{\mathcal{J}}$ of the relative compactified Jacobian $\bar{\mathcal{J}} \to B$ is nonsingular. According to Beĭlinson, Bernstein and Deligne [6] there is a decomposition $\mathrm{R}\pi_* \mathbb{C}_{\bar{\mathcal{J}}}[g + \dim B] = \bigoplus \mathcal{F}^i[-i]$, where the $\mathcal{F}_i$ are perverse sheaves and $i = -g, \dots, g$, where g is genus of the generic fiber of the map π. We write ${}^p\mathrm{H}^i(\bar{\mathcal{J}}_b) := \mathcal{F}^i|_b[-\dim B]$; this is a complex of vector spaces carrying a weight filtration. It can be shown that this does not depend on the family $\mathcal{C}$, and that moreover:

Proposition 4 [49; 51] $\quad \mathscr{P}^{\mathrm{min}}_{\mathrm{alg}} = \sum_{i=-g}^{g} q^{2i} \mathfrak{w}\big({}^p\mathrm{H}^i(\bar{J}^0(C))\big)$.

Suppose X is a unibranch singularity, and let K be its link. Let $\mathscr{H}^{\mathrm{min}}(K)$ denote the part of the *reduced* HOMFLY homology with minimal a–grading ($= \mu$). Then in

combination with Proposition 4, Conjecture 2 would imply that

$$\mathscr{H}^{\min}(K) \cong \mathrm{H}^*(\bar{J}^0(X)).$$

The homological grading on the left-hand side is identified with the weight grading on the right, and the q–grading on the left with the perverse filtration on the right.

To calculate $\overline{\mathscr{P}}_{\mathrm{alg}}$, it is necessary to work out the deformation theory of ideals and nested ideals inside the local ring of functions for the singularity in question. In the case when the singularity is described by a single Puiseux pair, eg if it is of the form $y^k = x^n$, we can reduce the calculation to (nontrivial) combinatorics. The argument is similar to Piontkowski's calculation [59] of the stratification of the compactified Jacobian of a rational curve with this singularity.

Let $\Gamma_{k,n} = \{ak+bn \mid a,b \in \mathbb{Z}_{\geq 0}\}$ be the semigroup of degrees of elements of $\mathbb{C}[\![t^n, t^k]\!]$. We say $\mathfrak{i} \subset \Gamma_{k,n}$ is a semigroup ideal if it is closed under adding elements from $\Gamma_{k,n}$.

Theorem 5 *Let $X_{k,n}$ be the germ of a singularity whose link is a (k,n) torus knot* (that is, $k,n>0$ and $(k,n)=1$), *and let $\Gamma = \Gamma_{k,n}$. Then $X_{k,n}^{[l\leq l+m]}$ is stratified by linear spaces enumerated by nested pairs of semigroup ideals $\mathfrak{j} \supset \mathfrak{i} \supset \mathfrak{j} + \{ak+bn \mid a,b \in \mathbb{Z}_{>0}\}$ such that $\#(\Gamma \setminus \mathfrak{j}) = l$ and $\#(\mathfrak{j} \setminus \mathfrak{i}) = m$.* (We write $N(\mathfrak{j} \supset \mathfrak{i})$ for the dimension of this linear space.)

Fix $\mathfrak{j} \supset \mathfrak{i}$ and let $\{\gamma_1, \dots, \gamma_r\}$ be the unique minimal subset of $\mathfrak{j}$ which generates it as a Γ–module. Let $\sigma\mathfrak{j}$ be the set of all elements of $\mathfrak{j}$ with more than one expression of the form $j+\gamma$ with $j \in \mathfrak{j}$ and $\gamma \in \Gamma$. Then $\sigma\mathfrak{j}$ is again a semigroup ideal requiring r generators, say $s_1, \dots, s_r$, and

$$N(\mathfrak{j} \supset \mathfrak{i}) = \sum_{\gamma_i \in \mathfrak{i}} \#(\Gamma_{>\gamma_i} \setminus \mathfrak{i}) + \sum_{\gamma_i \notin \mathfrak{i}} \#(\Gamma_{>\gamma_i} \setminus \mathfrak{j}) - \sum_{a=1}^{r} \#(\Gamma_{>s_a} \setminus \mathfrak{i}).$$

Remark The existence of this stratification implies that there are no cancellations among monomials of $\overline{\mathscr{P}}_{\mathrm{alg}}(X_{k,n})$ when setting $t=-1$. It is an interesting question whether the analogous statement holds for all algebraic knots.

Using the formulas above, $\overline{\mathscr{P}}_{\mathrm{alg}}$ can in principle be computed by summing up the contributions; the computation is finite because for any $\mathfrak{i} \subset c + \mathbb{N} \subset \Gamma_{m,n}$, we have $N(\mathfrak{i}) = N(\mathfrak{i}+1) - X_{n,k}^{[*]}$ is stratified by the linear spaces which are naturally labeled by the monomial ideals $\mathfrak{i}$ and $N(\mathfrak{i})$ is its dimension — and similarly for the nested case. In a certain limit, the formulas simplify:

Proposition 6 *Write $X_{n,k}$ for the germ at the origin of $y^n = x^k$ and $X_{n,\infty}$ for that of $y^n = 0$. Then*

$$\sum_{l,m} q^{2l} a^{2m} t^{m^2} \mathfrak{w}(X_{n,k}^{[l \le l+m]}) = \sum_{l,m} q^{2l} a^{2m} t^{m^2} \mathfrak{w}(X_{n,\infty}^{[l \le l+m]}) + O(q^{2k})$$

and we calculate

$$\sum_{l,m} q^{2l} a^{2m} t^{m^2} \mathfrak{w}(X_{n,\infty}^{[l \le l+m]}) = \prod_{i=1}^{n} \frac{1 + a^2 q^{2i-2} t^{2i-1}}{1 - q^{2i} t^{2i-2}}.$$

This computation matches the formula for the "stable superpolynomial" of torus knots conjectured in [20].

To understand $\overline{\mathscr{P}}_{\mathrm{alg}}(X_{k,n})$ in general, it is profitable to consider the reformulation in terms of the compactified Jacobian. For the unibranch singularities $x^k = y^n$, the K–theory of the compactified Jacobian is known — see Varagnolo and Vasserot [62] — to furnish a representation of the spherical Cherednik algebra of rank n and central charge k/n. It will be shown elsewhere that the *rational* spherical Cherednik algebra acts on its cohomology; see Oblomkov and Yun [57]. Moreover, the homological grading (t) and the perverse filtration (q) have representation-theoretic meanings.

In the case $k = mn + 1$, the graded dimensions of conjecturally equivalent filtrations can be readily calculated in a different geometric incarnation of the Cherednik algebra; see Haiman [39] and Gordon and Stafford [28; 29]. We obtain a formula expressing $\mathscr{P}_{\mathrm{alg}}^{\min}(X_{mn+1,n})$ as a sum over partitions of n. For a partition $\lambda \vdash n$ and a box x in the diagram of λ, we write $a(x)$ and $l(x)$ for the arm and leg, and $a'(x)$ and $l'(x)$ for the coarm and coleg. We write λ' for the dual partition, and $\kappa(\lambda) = \sum \binom{\lambda'_i}{2}$. We have the following formula:

Conjecture 7 *Let $T_1 = q^2$ and $T^2 = 1/q^2 t^2$. Then $t^{-\mu} \mathscr{P}_{\mathrm{alg}}^{\min}(X_{mn+1,n})$ is given by the formula*

$$\sum_{\lambda \vdash n} T_1^{m\kappa(\lambda)} T_2^{m\kappa(\lambda')} \frac{(1-T_1)(1-T_2) \prod_{x \in \lambda \setminus \{(0,0)\}} (1 - T_1^{l'(x)} T_2^{a'(x)})}{\prod_{x \in \lambda} (1 - T_1^{1+l(x)} T_2^{-a(x)})(1 - T_1^{-l(x)} T_2^{1+a(x)})} \sum_{x \in \lambda} T_1^{l'(x)} T_2^{a'(x)}.$$

In a subsequent article [34], we suggest how all of $\overline{\mathscr{P}}(X_{k,n})$ (rather than just $\overline{\mathscr{P}}^{\min}$) may be recovered from the analogous representation of the rational Cherednik algebra (rather than just the spherical part). This leads to the following conjecture:

Conjecture 8 *Let $T_1=q^2$, $T_2=1/(q^2t^2)$ and $A=a^2t$. Then $(at)^{-\mu}\mathscr{P}_{\mathrm{alg}}(X_{mn+1,n})$ is given by the formula*

$$\sum_{\lambda\vdash n} T_1^{m\kappa(\lambda)}T_2^{m\kappa(\lambda')} \times\frac{(1-T_1)(1-T_2)\prod_{x\in\lambda\setminus\{(0,0)\}}(1-T_1^{l'(x)}T_2^{a'(x)})(1+AT_1^{-l'(x)}T_2^{-a'(x)})}{\prod_{x\in\lambda}(1-T_1^{1+l(x)}T_2^{-a(x)})(1-T_1^{-l(x)}T_2^{1+a(x)})} \times\sum_{x\in\lambda}T_1^{l'(x)}T_2^{a'(x)}.$$

Formulas of this sort were first conjectured by Gorsky [30] in the case of $T(n,n+1)$, and subsequently in the physics literature by Dunin-Barkowski, Mironov, Morozov, Sleptsov and Smirnov [21], and Aganagic and Shakirov [2] for $T(n,nm+1)$.

Grading conventions for $\overline{\mathscr{H}}$

Our normalization of the HOMFLY homology is the one used in [20], rather than that of Khovanov and Rozansky [45] or Rasmussen [60]. Specifically, our main interest is in the group $\overline{\mathscr{H}}(K)$, which in the terminology of [60] is the *unreduced* HOMFLY homology. It satisfies

$$\overline{\mathscr{H}}(K)\cong\mathscr{H}(K)\otimes \mathrm{H}^*(S^1)\otimes\mathbb{Q}[X],$$

where the group $\mathscr{H}(K)$ is the *reduced* HOMFLY homology.

We normalize the homological (t) grading on $\mathscr{H}(K)$ so as to coincide with the homological grading on reduced Khovanov homology under the spectral sequence of [60]; for example, the Poincaré polynomial of $\mathscr{H}$ of the positive trefoil is given by

$$a^2q^{-2}t^0+a^4q^0t^3+a^2q^2t^2.$$

The Poincaré polynomials of the reduced and unreduced homologies are related by

$$\overline{\mathscr{P}}(K)=\frac{a^{-1}+at}{q^{-1}-q}\mathscr{P}(K).$$

The Poincaré polynomial with respect to the homological grading (s) of [45] may be obtained by substituting $t=s^{-1}$ and $a^2=a^2s$ in $\overline{\mathscr{P}}$. This reflects the fact that the homological grading on $\mathrm{sl}(n)$ homology is obtained as a linear combination of the a and s gradings [60], and the fact that $\mathrm{sl}(2)$ homology is dual to Khovanov homology.

Latest developments

Since the first version of this paper appeared on arXiv, this area of research have seen a lot of important developments. It would take another paper to properly discuss all of the new results and connections between them, so here we just list a few references which we find most relevant.

Abel, Elias and Hogancamp [1; 24; 41; 42] developed a completely new method of computing Khovanov–Rozansky homology for a large class of knots and links, in particular, for all $(n, mn \pm 1)$ torus knots. These computations confirm Conjectures 7 and 8.

Conjectures 23 and 24 were deeply generalized by Oblomkov and Rozansky [54; 55], and by Gorsky, Rasmussen and Neguț [32; 33]. In particular, the sheaves $F_{k/n}$ were explicitly constructed and the coefficients $g_{r/n}(\lambda)$ were explicitly computed for all k, n and r in [32]. Conjecture 24 was reformulated as a combinatorial "rational shuffle conjecture" there, and it was later proved by Carlsson and Mellit [13; 50]. The construction of the analogues of $F_{k/n}$ for general knots and links appeared in Gorsky, Neguț and Rasmussen [33] and Oblomkov and Rozansky [54; 55].

Acknowledgements We thank Emanuel Diaconescu, Pavel Etingof, Lothar Göttsche, Ian Grojnowski, Mikhail Khovanov, Ivan Losev, Luca Migliorini, Rahul Pandharipande, Ivan Smith, Cumrun Vafa, Ben Webster, Geordie Williamson and Zhiwei Yun for helpful discussions. Oblomkov is supported by Sloan Foundation and NSF. Rasmussen would like to thank the Simons Center for Geometry and Physics for its hospitality and support while much of this work was done. Shende is supported by the Simons Foundation.

2 Hilbert schemes and Jacobians

In this section, we develop some general facts about the series $\overline{\mathscr{P}}_{\mathrm{alg}}$: its rationality and symmetry properties, behavior under blowups, and relation to the cohomology of the compactified Jacobian. For the most part, these are straightforward consequences of previous work.

2.1 Rationality and symmetry

In this section, we prove Proposition 3. We begin by reviewing [56, Section 4; 58, Appendix B], presenting them now in the Grothendieck ring of varieties. This is the

ring generated by the classes $[V]$ of varieties V, the sum and product coming from disjoint union and direct product, respectively. The classes are subject to the relation $[V \setminus Z] = [V] - [Z]$ for Z a closed subvariety of V. If $\phi\colon V \to A$ is a constructible function, we write $[V, \phi] := \sum_{a \in A} a \cdot [\phi^{-1}(a)]$.

It is convenient to rewrite slightly the series $\overline{\mathscr{P}}_{\mathrm{alg}}$. Consider the locus $X_r^{[l]} \subset X^{[l]}$ parametrizing subschemes whose ideals require r generators. By Nakayama's lemma, the projection $X^{[l \le l+m]} \to X^{[l]}$ has fibers $\mathrm{Grass}(\mathbb{C}^r \subset \mathbb{C}^m)$ over $X_r^{[l]}$. Thus we have

$$\sum_{l,m} q^{2l} a^{2m} t^{m^2} [X^{[l \le l+m]}] = \sum_{l,r} q^{2l} [X_r^{[l]}] \sum_m a^{2m} t^{m^2} [G(r,m)] = \sum_l q^{2l} [X^{[l]}, \Phi],$$

where the constructible function Φ on $X^{[l]}$ takes the value $\sum_m a^{2m} t^{m^2} [G(r,m)]$ at a point corresponding to an ideal requiring r generators.

We pass to a complete curve C with a unique singularity at p, at which its germ is X. There is a stratified map $C^{[l]} \to \coprod_{l' \le l} X^{[l']}$ which forgets points away from the singularity; we extend Φ to $C^{[l]}$ by pullback along this map. Then by a standard stratification argument we have

$$\sum_l q^{2l} [X^{[l]}, \Phi] = \frac{\sum_l q^{2l} [C^{[l]}, \Phi]}{(1-q^2)^b \sum_l q^{2l} [\widetilde{C}^{[l]}]},$$

where $\widetilde{C}$ is the normalization of C.

Essentially by definition, Φ depends only on the isomorphism class as a sheaf of the ideal sheaf of the subscheme, and does not change when this sheaf is tensored by a line bundle. Moreover, it is shown in [56] (using the planarity of the singularities and applying the Auslander–Buchsbaum theorem) that Φ of a sheaf and its dual agree. In other words, it satisfies the hypothesis of the following lemma, whose statement and proof are modeled on Lemma 3.13 of [58].

Lemma 9 *Let C be a Gorenstein curve of arithmetic genus g. Let ϕ be a constructible function on the moduli space $\bar{J}(C)$ of rank one, torsion-free sheaves. Assume that for any rank one, torsion-free sheaf $\mathcal{F}$ and any line bundle L, we have $\phi(\mathcal{F}^*) = \phi(\mathcal{F}) = \phi(\mathcal{F} \otimes L)$. Denote also by ϕ the function induced on $C^{[n]}$ by composition with the Abel–Jacobi map.*

Then there exist classes $N_h(C, \phi)$ in the Grothendieck group of varieties (with coefficients in the ring in which ϕ takes values) *such that*

$$(1-q)(1-q\mathbb{L})\sum_{n=0}^{\infty} q^n [C^{[n]},\phi] = \sum_{h=0}^{g} N_h(C,\phi)\cdot(1-q)^h(1-q\mathbb{L})^h q^{g-h},$$

where $\mathbb{L}=[\mathbb{A}^1]$ is the class of the affine line. In particular, denoting by $Z_C(q)$ the quantity on either side of the above formula, we have $Z_C(q) = (q^2\mathbb{L})^g Z_C(1/q\mathbb{L})$.

Proof The following useful properties of Gorenstein curves may be found in an article of Hartshorne [40]. Let C be a Gorenstein curve, and let F be a torsion-free sheaf on C. Write F^* for $\mathcal{H}om(F,\mathcal{O}_C)$. Then $\mathcal{E}xt^{\geq 1}(F,\mathcal{O}_C)=0$ and $F=(F^*)^*$. Serre duality holds in the form $\mathrm{H}^i(F)=\mathrm{H}^{1-i}(F^*\otimes\omega_C)^*$. For F rank one and torsion-free, define its degree $d(F):=\chi(F)-\chi(\mathcal{O}_C)$. This satisfies $d(F)=-d(F^*)$, and, for L any line bundle, $d(F\otimes L)=d(F)+d(L)$.

We proceed with the proof of the lemma. Fix a degree 1 line bundle $\mathcal{O}(1)$ on C. We map $C^{[d]}\to \bar{J}^0(C)$ by associating the ideal $I\subset\mathcal{O}_C$ to the sheaf

$$I^* = \mathcal{H}om(I,\mathcal{O}_C)\otimes\mathcal{O}(-d);$$

the fiber is $\mathbb{P}(\mathrm{H}^0(C,I^*))$. For $\mathcal{F}$ a rank one, degree zero, torsion-free sheaf, we write the Hilbert function as $h_{\mathcal{F}}(d)=\dim \mathrm{H}^0(C,\mathcal{F}\otimes\mathcal{O}(d))$. Then since, over the strata with constant Hilbert function, the map from the Hilbert schemes to the compactified Jacobian is the projectivization of a vector bundle, we have the equality

$$\sum_{d=0}^{\infty} q^d [C^{[d]},\phi] = \sum_{a\in A}\sum_{h} a[\{\mathcal{F}\mid h_{\mathcal{F}}=h,\ \phi(\mathcal{F})=a\}]\sum_{d=0}^{\infty} q^d[\mathbb{P}^{h(d)-1}].$$

Fix $h=h_{\mathcal{F}}$ for some $\mathcal{F}$. Evidently h is supported in $[0,\infty)$, and, by Riemann–Roch and Serre duality, is equal to $d+1-g$ in $(2g-2,\infty)$. Inside $[0,2g-2]$, it increases by either 0 or 1 at each step. Let $\phi_\pm(h)=\{d\mid 2h(d-1)-h(d-2)-h(d)=\pm 1\}$; evidently $\phi_-\subset[0,2g]$ and $\phi_+\subset[1,2g-1]$, and

$$Z_h(q) := (1-q)(1-q\mathbb{L})\sum_{d=0}^{\infty} q^d[\mathbb{P}^{h(d)-1}] = \sum_{d\in\phi_-(\mathcal{F})} q^d\mathbb{L}^{h(d)-1} - \sum_{d\in\phi_+(\mathcal{F})} q^d\mathbb{L}^{h(d-1)}.$$

This is a polynomial in q of degree at most $2g$, hence so is $Z_C(q)$.

Let $\mathcal{G}=\mathcal{F}^*\otimes\omega_C\otimes\mathcal{O}(2-2g)$ and $h^\vee=h_{\mathcal{G}}$. By Serre duality and Riemann–Roch, $h^\vee(d)=h(2g-2-d)+d+1-g$, so in particular, $d\in\phi_\pm(h^\vee)$ if and only if $2g-d\in\phi_\pm(h)$. It follows that $q^{2g}\mathbb{L}^g Z_h(1/q\mathbb{L}) = Z_{h^\vee}(q)$. As $Z_C(q)=\sum_h[\{\mathcal{F}\mid h_{\mathcal{F}}=h\}]Z_h(q)$, we obtain the final stated equality. □

Proof of Proposition 3 Take the curve C to be rational, recall that $\mu = 2\delta + 1 - b$ [52], and note that in Lemma 9 we used the variable q rather than q^2. Then we see that there exist classes N_h in the Grothendieck ring of varieties (with coefficients in $\mathbb{Z}[a,t]$) such that

$$\overline{\mathscr{P}}_{\mathrm{alg}} = \mathfrak{w}\left(a^{2\delta-b}(q^{-1}-q)^{-b}\sum_{h=0}^{\delta} N_h(C,\phi)\cdot(q^{-1}-q)^h(q^{-1}-q\mathbb{L})^h\right).$$

From this expression, we see immediately that $(q^{-1}-q)^b\,\overline{\mathscr{P}}_{\mathrm{alg}}$ is a Laurent polynomial in q with coefficients between $q^{-2\delta}$ and $q^{2\delta}$ which is invariant under $q \to 1/qt$. Finally note the degree of N_h in a^2 is bounded by the multiplicity of the singularity, as this is the maximal number of generators which any ideal will require [11, Exercise 4.6.16]. This completes the proof of Proposition 3. □

Remark When $b = 1$, the link of X is a knot, and the product $(q-q^{-1})\overline{\mathscr{P}}_{\mathrm{alg}}(X)$ corresponds under Conjecture 2 to $a^{-1}+at$ times the Poincaré polynomial of $\mathscr{H}(K)$. In particular, all of its terms are positive. For $b > 1$, the quantity $(q-q^{-1})^b\,\overline{\mathscr{P}}_{\mathrm{alg}}(X)$ is somewhat less natural from the point of view of the HOMFLY homology. For a two-component link, $(q-q^{-1})^2\,\overline{\mathscr{P}}_{\mathrm{alg}}(X)$ will typically have negative terms, so it cannot coincide with the Poincaré polynomial of the completely reduced homology considered in [60].

2.2 Blowups

If X is the germ of a plane curve singularity, we can blow up X at the central point to obtain the germ of a new singularity $\widetilde{X}$. The effect of this operation on the link of the singularity is well-known. If m is the multiplicity of X, we can write the link of X as the closure of a m–strand braid σ. Then the link of $\widetilde{X}$ is the closure of the braid $\sigma\Delta^{-2}$, where Δ^{-2} denotes a full left-handed twist on m strands.

Now let σ be an arbitrary braid with closure $\overline{\sigma}$, and let n and N be the minimum and maximum powers of a appearing in $\boldsymbol{P}(\overline{\sigma})$. The Morton–Franks–Williams inequality says that $w-m \le n \le N \le w+m$, where w and m denote the writhe and number of braid strands in m, respectively. Let $\boldsymbol{P}^{\min}$ and $\boldsymbol{P}^{\max}$ be the coefficients of a^{w-m} and a^{w+m} in $\boldsymbol{P}(\overline{\sigma})$. Then, according to a theorem of Kalman [43], $\boldsymbol{P}^{\min}(\overline{\sigma\Delta^{-2}}) = \boldsymbol{P}^{\max}(\overline{\sigma})$. For algebraic knots, an analogous statement holds at the level of $\overline{\mathscr{P}}_{\mathrm{alg}}$:

Proposition 10 *If X is the germ of a unibranch plane curve singularity with multiplicity m, then $\overline{\mathscr{P}}^{\max}_{\mathrm{alg}}(X) = t^{m^2}\,\overline{\mathscr{P}}^{\min}_{\mathrm{alg}}(\widetilde{X})$.*

Proof It is a classical result that $\mu(X) = \mu(\widetilde{X}) + m(m-1)$. Thus the proposition is equivalent to saying that

$$\sum_{l=0}^{\infty} q^{2l}\, \mathfrak{w}(X^{[l \le l+m]}) = q^{m(m-1)} \sum_{l=0}^{\infty} q^{2l}\, \mathfrak{w}(\widetilde{X}^{[l]}).$$

We will show that $X^{[l+m(m-1)/2 \le l+m(m+1)/2]} \cong \widetilde{X}^{[l]}$.

Let us write $\mathcal{O}_X = \mathbb{C}[\![x,y]\!]/f(x,y)$. By the Weierstrass preparation theorem and unibranchness we may choose variables so that

$$f(x,y) = y^m + y^{m-1}x^2 f_1(x) + y^{m-2}x^3 f_2(x) + \cdots + x^{m+1} f_m(x)$$

Then $x^{-m} f\left(x, x.\frac{y}{x}\right) \in \mathbb{C}\left[\!\left[x, \frac{y}{x}\right]\!\right]$, and $\mathcal{O}_{\widetilde{X}} = \mathbb{C}\left[\!\left[x, \frac{y}{x}\right]\!\right]/x^{-m} f\left(x, x\cdot\frac{y}{x}\right)$. In particular, $\mathcal{O}_{\widetilde{X}}/\mathcal{O}_X$ is generated as an $\mathcal{O}_X$–module by $1, \frac{y}{x}, \ldots, \left(\frac{y}{x}\right)^{m-1}$. Therefore,

$$x^{m-1}\mathcal{O}_{\widetilde{X}} = (x^{m-1}, x^{m-2}y, \ldots, y^{m-1}) \subset \mathcal{O}_X.$$

As $f \in (x,y)^{m-1}$, we have

$$\dim \mathcal{O}_X/(x,y)^{m-1} = \dim \mathbb{C}[\![x,y]\!]/(x,y)^{m-1} = \tfrac{1}{2}m(m-1).$$

Moreover, for any $J \subset \mathcal{O}_{\widetilde{X}}$, we have

$$\dim \mathcal{O}_X/x^{m-1}J = \dim \mathcal{O}_X/x^{m-1}\mathcal{O}_{\widetilde{X}} + \dim \mathcal{O}_{\widetilde{X}}/J = \tfrac{1}{2}m(m-1) + \dim \mathcal{O}_{\widetilde{X}}/J.$$

An identical argument shows $\dim \mathcal{O}_X/x^m J = \frac{1}{2}m(m+1) + \dim \mathcal{O}_{\widetilde{X}}/J$. Finally, $x^m J \subset (x,y)x^{m-1}J$. In fact, these are equal since $\frac{y}{x}J \subset J$.

Therefore, we define a map $\Phi\colon \widetilde{X}^{[l]} \to X^{[l+m(m-1)/2 \le l+m(m+1)/2]}$ by $\Phi(J) = (x^{m-1}J, x^m J)$. The map is injective because x is not a zero divisor.

To see that Φ is surjective, suppose we are given $\mathcal{O}_X$–ideals $\mathcal{O}_X \supset J \supset I \supset (x,y)J$ with $\dim(J/I) = m$. As J is a *free* $\mathbb{C}[\![x]\!]$–module of rank m, $\dim_{\mathbb{C}} J/xJ = m$ and therefore $I = xJ$. On the other hand, since X has multiplicity m, we certainly have $\dim_{\mathbb{C}} J/(x,y)J \le m$. We conclude $xJ = (x,y)J$. In particular, $yJ \subset xJ$; therefore $\frac{y}{x}J \subset J$, ie J is an $\mathcal{O}_{\widetilde{X}}$–module. It is elementary to show that *any* $\mathcal{O}_{\widetilde{X}}$–submodule $J \subset \mathcal{O}_X$ must satisfy $J \subset x^{m-1}\mathcal{O}_{\widetilde{X}}$. □

Remark The result holds for nonunibranch singularities as well; the proof is more technical and will appear elsewhere.

2.3 Relation with the compactified Jacobian

In this subsection, we provide background and context for the statement of Proposition 4. Recall that for a *smooth* curve C, the Hilbert schemes $C^{[n]}$ are just symmetric products, and as such their cohomology may be computed by taking S_n invariants: $\mathrm{H}^*(C^{[n]},\mathbb{C}) = \mathrm{H}^*(C^n,\mathbb{C})^{S_n} = (\mathrm{H}^*(C,\mathbb{C})^{\otimes n})^{S_n}$. On the other hand, $\mathrm{H}^i(J(C),\mathbb{C}) = \bigwedge^i \mathrm{H}^1(C,\mathbb{C})$. This leads to the following formula of Macdonald [48]:

$$(3)\qquad \sum_{d=0}^{\infty}\sum_{i=0}^{2d} q^{2d}\mathrm{H}^i(C^{[d]},\mathbb{C}) = \frac{\sum_{i=0}^{2g} q^{2i}\bigwedge^i(\mathrm{H}^1(C,\mathbb{C}))}{(1-q^2)(1-q^2\mathbb{C}(-1))} = \frac{\sum_{i=0}^{2g} q^{2i}\mathrm{H}^i(J(C),\mathbb{C})}{(1-q^2)(1-q^2\mathbb{C}(-1))}.$$

The Tate twists in the denominator are necessary to make this an equality of Hodge structures. Since all spaces are smooth and compact, taking weight polynomials amounts to replacing $\mathbb{C}(-1)$ by t^2 and $\mathrm{H}^i(\cdot)$ by $t^i \dim \mathrm{H}^i(\cdot)$.

Proposition 4 says that an analogous formula holds for a *singular* plane curve C, but we must take into account the perverse filtration on $\mathrm{H}^*(\bar{J}^0(C))$. More precisely, let ${}^p\mathrm{H}^i(\bar{J}^0(C))$ be the i^{th} associated graded piece of the perverse filtration on $\mathrm{H}^*(\bar{J}^0(C))$. Then the main result of [49; 51] is that

$$(4)\qquad (1-q^2)(1-q^2t^2)\sum_{d=0}^{\infty} q^{2d}\mathfrak{w}(C^{[d]}) = \sum_{i=0}^{2g} q^{2i}\mathfrak{w}\big({}^p\mathrm{H}^i(\bar{J}^0(C))\big).$$

The $q \to 1/qt$ symmetry of the left-hand side proven in Proposition 3 manifests on the right-hand side as (relative) Poincaré duality.

We recall the definition of the perverse filtration for the interested reader. Let $\pi\colon \mathcal{C} \to B$ be a family of curves over a smooth base, with the general fiber smooth and some special singular fiber $C = \mathcal{C}_b$ we are interested in. Let $\pi^J\colon \mathcal{J} \to B$ be the relative compactified Jacobian. As shown in [25], there exist families such that $\mathcal{J}$ is smooth; fix any such. Then from the decomposition theorem of Beĭlinson, Bernstein, and Deligne [6] we learn that

$$\mathrm{R}\pi^J_*\mathbb{C}[g+\dim B] = \bigoplus ({}^p\mathrm{R}^{g+i}\pi^J_*\mathbb{C}[g+\dim B])[-i].$$

Passing to the central fiber, we write

$${}^p\mathrm{H}^i(\mathcal{J}_b) := ({}^p\mathrm{R}^{g+i}\pi^J_*\mathbb{C}[g+\dim B])_b[-g-i].$$

These are complexes of mixed Hodge modules, which should be Tate-twisted to ensure $\mathrm{H}^j(\mathcal{J}_b) = \bigoplus \mathrm{H}^j({}^p\mathrm{H}^i(\mathcal{J}_b))$.[2]

It is sensible to take weight polynomials of the ${}^p\mathrm{H}^i(\mathcal{J}_b)$. It is shown in [49; 51] that

$$(1-q^2)(1-q^2t^2)\sum_{d=0}^{\infty}\sum_{i=0}^{2d} q^{2d}\,\mathfrak{w}(\mathcal{C}_b^{[d]}) = \sum_{i=-g}^{g} q^{2i}\,\mathfrak{w}({}^p\mathrm{H}^i(\mathcal{J}_b)). \tag{5}$$

As the left-hand side did not depend on the family, we learn a posteriori the same for the right-hand side. Equation (5) is proven by showing that no summand of $\mathrm{R}\pi_*^J\mathbb{C}[g+\dim B]$ and $\mathrm{R}\pi_*^{[d]}\mathbb{C}[d+\dim B]$ is supported in positive codimension, and thus we can check on the generic point where the assertion reduces to (3). For the Jacobians this follows from Ngô's support theorem [53]; the geometric content of [49; 51] was to establish the same for the Hilbert schemes.

3 Equations for Hilbert schemes

In this section we prove that the Hilbert schemes and nested Hilbert schemes for unibranch singularities with a single Puiseux pair (eg $x^k = y^n$ with k and n coprime) admit a stratification by cells which admit bijective morphisms from various $\mathbb{A}^N$. We give explicitly the dimensions N in terms of certain combinatorial data. Our approach is adapted from the methods of Piontkowski [59].

3.1 Semigroups, stratifications and syzygies

Let X be the germ of a unibranch plane curve singularity with complete local ring $\mathcal{O}_X \subset \mathbb{C}[\![t]\!]$. We have the valuation ord: $\mathbb{C}[\![t]\!]\setminus\{0\} \to \mathbb{N}$, given by $\lambda t^k + O(t^{k+1}) \mapsto k$. The set $\Gamma := \mathrm{ord}(\mathcal{O}_X\setminus\{0\})$ is a semigroup. Given an ideal $J \subset \mathcal{O}_X$, the set $\mathrm{ord}(J\setminus\{0\})$ is a Γ–module, which we call the *symbol* of J. We will study the geometry of the moduli space of ideals with given symbol,

$$X^{[\mathfrak{j}]} := \{J \subset \mathcal{O}_X \mid \mathrm{ord}(J) = \mathfrak{j}\}.$$

Let $\mathbb{C}[\Gamma] = \mathbb{C}[t^i \mid i \in \Gamma]$. For a Γ–submodule $\mathfrak{j} \subset \mathbb{Z}$, we write $\mathfrak{j} = (t^j \mid j \in \mathfrak{j})\mathbb{C}[\Gamma]$ for the associated monomial ideal, and $c(\mathfrak{j}) := \min\{j \in \mathfrak{j} \mid j + \mathbb{N} \subset \mathfrak{j}\}$ for its conductor.

[2] This direct sum decomposition is not canonical, but it does come from a canonical filtration; these matters are irrelevant here.

We choose a basis of $\mathcal{O}_X$ compatible with the monomial basis of $\mathbb{C}[\Gamma]$:

$$\phi_i = t^i + \sum_{j>i} a_{ij} t^j, \quad i \in \Gamma.$$

Evidently the multiplication matrix in this basis is upper triangular:

$$\phi_\alpha \cdot \phi_\beta = \phi_{\alpha+\beta} + \sum_{\gamma > \alpha+\beta} C^\gamma_{\alpha,\beta} \phi_\gamma \tag{6}$$

Below we use the map

$$\mathrm{gr}\colon \mathcal{O}_X \to \mathbb{C}[\Gamma], \quad \mathrm{gr}(f) := t^{\mathrm{ord}(f)}([t^{\mathrm{ord}(f)}]f),$$

where $([t^i]f)$ denotes the coefficient of t^i in f. We say $f \in \mathcal{O}_X$ lifts $at^i \in \mathbb{C}[\Gamma]$ if and only if $\mathrm{gr}(f) = at^i$.

Suppose we are given an ideal J with symbol $\mathfrak{j}$. For any element $j \in \mathfrak{j}$ there is a unique lift $\tau_j \in J$ of the form

$$\tau_j = \phi_j + \sum_{k \in \Gamma_{>j} \setminus \mathfrak{j}} \lambda_j^{k-j} \phi_k,$$

where $\lambda_j^{k-j} \in \mathbb{C}$ This observation motivates us to study the following map. Fix generators $\gamma_1, \dots, \gamma_n$ of $\mathfrak{j}$. Let $\mathrm{Gen} = \mathrm{Spec}\, \mathbb{C}[\lambda_j^{k-\gamma_j} \mid k \in \Gamma_{>\gamma_j} \setminus \mathfrak{j}]$ be an affine space of dimension $N = \sum_j |\Gamma_{>\gamma_j} \setminus \mathfrak{j}|$. Then we define deformations of the generators

$$\tau_{\gamma_j}(\lambda^\bullet_\bullet) = \phi_{\gamma_j} + \sum_{k \in \Gamma_{>\gamma_j} \setminus \mathfrak{j}} \lambda_j^{k-\gamma_j} \phi_k$$

and an "exponential" map

$$\mathrm{Exp}_\gamma\colon \mathrm{Gen} \to \coprod X^{[n]}, \quad \lambda \mapsto (\tau_{\gamma_1}, \dots, \tau_{\gamma_n}).$$

Note that $\mathrm{Exp}_\gamma(\lambda)$ may have different colengths at different λ; taking a flattening stratification shows Exp_γ is constructible. It is easy to see from [56, Theorem 27] that $\mathrm{Exp}_\gamma^{-1}(X^{[\#\Gamma\setminus\mathfrak{j}]}) = \mathrm{Exp}_\gamma^{-1}(X^{[\mathfrak{j}]})$ and that the map restricts to a bijective morphism $\mathrm{Exp}_\gamma\colon \mathrm{Exp}_\gamma^{-1}(X^{[\mathfrak{j}]}) \to X^{[\mathfrak{j}]}$. From now on, we tacitly identify $X^{[\mathfrak{j}]}$ with $\mathrm{Exp}_\gamma^{-1}(X^{[\mathfrak{j}]}) \subset \mathrm{Gen}$. We illustrate the behavior at points $\lambda \in \mathrm{Gen} \setminus \mathrm{Exp}_\gamma^{-1}(X^{[\mathfrak{j}]})$:

Example Let $\mathcal{O}_X = \mathbb{C}[\![t^3, t^7]\!]$ and $\mathfrak{j} = \langle 6, 10\rangle$. Then $(t^6 + t^7, t^{10}) \in \mathrm{Gen}$ is generated by elements of orders 6 and 10, and $14 \notin \langle 6, 10\rangle$. On the other hand, $\mathrm{ord}(t^6 + t^7, t^{10}) = \langle 6, 10, 14\rangle$ because $t^7(t^6 + t^7) - t^3(t^{10}) = t^{14}$, so $(t^6 + t^7, t^{10}) \notin \mathrm{Exp}_\gamma^{-1}(X^{[\mathfrak{j}]})$.

That is, although the orders of the generators of the ideal $\mathrm{Exp}(\lambda)$ generate $\mathfrak{j}$, it can and does happen that some $\mathcal{O}_X$–linear combination of the generators has order $\notin \mathfrak{j}$. To prevent this from happening one needs to control the syzygies (relations between generators).

The choice of generators γ_i of the ideal $\mathfrak{j}$ determines a surjection $G\colon \mathbb{C}[\Gamma]^{\oplus n} \to \mathfrak{j}$. Extend this to a presentation

$$0 \leftarrow \mathbb{C}[\Gamma]/\mathfrak{j} \leftarrow \mathbb{C}[\Gamma] \xleftarrow{G} \mathbb{C}[\Gamma]^{\oplus n} \xleftarrow{S} \mathbb{C}[\Gamma]^{\oplus m}, \tag{7}$$

where the matrix of syzygies $S = (\vec{s}_1, \dots, \vec{s}_m)$ is homogenous, in the sense that $(\vec{s}_i)_j = u_i^j t^{\sigma_i - \gamma_j}$ for some constants $u_i^j \in \mathbb{C}$. We call σ_i the order of the syzygy $\vec{s}_i$. We regard G as a row vector, and $\vec{s}_i$ as the columns of S.

The choice of $\lambda \in \mathrm{Gen}$ determines a lift $\mathcal{G}_\lambda \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_X)$ of G by the formula

$$(\mathcal{G}_\lambda)_j := \tau_{\gamma_j}(\lambda_\bullet^\bullet) = \phi_{\gamma_j} + \sum_{k \in \Gamma_{>\gamma_j} \setminus \mathfrak{j}} \lambda_j^{k-\gamma_j} \phi_k.$$

We define ord and gr on $\mathcal{O}_X^{\oplus n}$; note that these *do not* act entrywise:

$$\mathrm{ord}\colon \mathcal{O}_X^{\oplus n} \to \mathbb{Z}, \quad \mathrm{ord}(v) = \min\{\mathrm{ord}(v_j) + \gamma_j\},$$
$$\mathrm{gr}\colon \mathcal{O}_X^{\oplus n} \to \mathbb{C}[\Gamma]^n, \quad \mathrm{gr}(v)_j := t^{d-\gamma_j}[t^{d-\gamma_j}]v_j, \quad d = \mathrm{ord}(v).$$

We define $\mathrm{gr}\colon \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus m}, \mathcal{O}_X^{\oplus n}) \to \mathrm{Hom}_{\mathbb{C}[\Gamma]}(\mathbb{C}[\Gamma]^{\oplus m}, \mathbb{C}[\Gamma]^{\oplus n})$ column by column by the formula $\mathrm{gr}(s_1, \dots, s_m) = (\mathrm{gr}\, s_1, \dots, \mathrm{gr}\, s_m)$.

Lemma 11 *Fix $\lambda \in \mathrm{Gen}$. The following are equivalent:*

1. *There exists $\mathcal{S} \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus m}, \mathcal{O}_X^{\oplus n})$ such that $\mathrm{gr}(\mathcal{S}) = S$ and every entry of $\mathcal{G}_\lambda \circ \mathcal{S}$ has order at least $c(\mathfrak{j})$.*
2. *There exists $\widetilde{\mathcal{S}} \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus m}, \mathcal{O}_X^{\oplus n})$ such that $\mathrm{gr}(\widetilde{\mathcal{S}}) = S$ and $\mathcal{G}_\lambda \circ \widetilde{\mathcal{S}} = 0$.*
3. *$\lambda \in X^{[\mathfrak{j}]}$.*

Proof (1) $\Longrightarrow$ (2) From a column s of $\mathcal{S}$, we will produce a column $\tilde{s}$ of $\widetilde{\mathcal{S}}$. By the hypothesis each term of $\mathcal{G}_\lambda s$ is in $\mathfrak{j}$. Thus we can write $\mathcal{G}_\lambda s = \sum f_i \tau_{\gamma_i}$ for some f_i. Then set $\tilde{s} = s - (f_1, \dots, f_n)$. The converse (2) $\Longrightarrow$ (1) is obvious.

(3) $\Longrightarrow$ (2) Let $\vec{s}$ be a column of S; then $\vec{s}_j = c_j t^{l-\gamma_j}$ for some constants c_i such that $\sum c_j = 0$. Let us define a first approximation s^l by $(s^l)_j = c_j \phi_{l-\gamma_i}$; evidently $\mathrm{gr}(s^l) = \vec{s}$ and each term of $\mathcal{G}_\lambda s^l$ has order greater than l. By hypothesis, these terms

have order in $\mathfrak{j}_{>l}$. Assume now we have found s^r such that $\mathrm{gr}(s^r)=\vec{s}$ and each term of $\mathcal{G}_\lambda s^r$ has order in $\mathfrak{j}_{>r}$. Then we can write $\mathrm{gr}(\mathcal{G}_\lambda s^r)=\sum_{j=1}^n b_j t^{d-\gamma_j}t^{\gamma_j}$, where $d=\mathrm{ord}(\mathcal{G}_\lambda s^r)$ and $b_j t^{d-\gamma_j}\in\mathbb{C}[\Gamma]$ for $j=1,\dots,n$. The induction step is given by the formula $s^{r+1}=s^r-(b_1\phi_{d-\gamma_1},\dots,b_n\phi_{d-\gamma_n})$.

(2) $\Longrightarrow$ (3) Assume $\mathrm{Exp}_\gamma(\lambda)\notin X^{[\mathfrak{j}]}$. Then there is $\varphi\in\mathcal{O}_X^{\oplus n}$ such that $\mathrm{ord}(\mathcal{G}_\lambda(\varphi))\notin\mathfrak{j}$. Observe this implies $G(\mathrm{gr}(\varphi_1),\dots,\mathrm{gr}(\varphi_n))=0$. Since $\mathrm{gr}(\varphi)$ lies in the kernel of G, it is a linear combination of the columns of S. Take ψ to be the standard lift of this linear combination. Therefore we found $\psi\in\mathcal{O}_X^{\oplus n}$ such that $\mathcal{G}_\lambda(\psi)=0$ and $\mathrm{gr}(\psi)=\mathrm{gr}(\varphi)$. We have $\mathrm{ord}(\varphi-\psi)>\mathrm{ord}(\varphi)$ and $\mathcal{G}_\lambda(\varphi-\psi)=\mathcal{G}_\lambda(\varphi)$. Continuing this process, we may produce an element $\widetilde{\varphi}$ of arbitrarily high order with $\mathrm{ord}(\mathcal{G}_\lambda(\varphi))\notin\mathfrak{j}$. However, $\mathrm{ord}(\mathcal{G}_\lambda(\widetilde{\varphi}))>\mathrm{ord}(\widetilde{\varphi})$, so once $\widetilde{\varphi}>c(\mathfrak{j})$ we find a contradiction. □

Let $\widetilde{\mathrm{Syz}}$ be the (infinite-dimensional) space parametrizing possible syzygies $\mathcal{S}$ with $\mathrm{gr}(\mathcal{S})=S$. From the lemma it follows that $X^{[\mathfrak{j}]}$ is the image in Gen of the variety in $\mathcal{X}\subset\mathrm{Gen}\times\widetilde{\mathrm{Syz}}$ cut out by the equation $\mathcal{G}\circ\mathcal{S}=0$. We now describe a finite-dimensional affine slice $\mathrm{Syz}\subset\widetilde{\mathrm{Syz}}$ such that $\mathcal{X}\cap\mathrm{Syz}=X^{[\mathfrak{j}]}$. For each $s\in\mathfrak{j}_{\le c(\mathfrak{j})}$ fix a decomposition $s=\gamma_{g(s)}+\rho(s)$ for $\rho(s)\in\Gamma$.

Let Syz be the affine space with coordinates $v_{is}^{s-\sigma_i}$, where $i=1,\dots,m$ and $c(\mathfrak{j})>s>\sigma_i$. To a point v in Syz we assign an $n\times m$ matrix with entries

$$(\mathcal{S}_v)_i^j=u_i^j\phi_{\sigma_i-\gamma_j}+\sum_{\substack{c(\mathfrak{j})>s>\sigma_i\\ g(s)=j}} v_{is}^{s-\sigma_i}\phi_{s-\gamma_j}.$$

Proposition 12 *The subvariety of* $\mathrm{Gen}\times\mathrm{Syz}$ *defined by the equation* $\mathcal{G}_\lambda\circ\mathcal{S}_v=O(t^{c(\mathfrak{j})})$ *maps bijectively onto* $X^{[\mathfrak{j}]}$.

Proof By Lemma 11, we must show that if any matrix $\mathcal{S}\in\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus m},\mathcal{O}_X^{\oplus n})$ satisfies $\mathrm{gr}\,\mathcal{S}=S$ and $\mathcal{G}_\lambda\circ\mathcal{S}=0$, then there is a unique matrix of the above form such that $\mathcal{G}_\lambda\circ\mathcal{S}_v=O(t^{c(\mathfrak{j})})$.

First we check uniqueness. Suppose given λ, v and v' such that $\mathcal{G}_\lambda\circ\mathcal{S}_v=\mathcal{G}_\lambda\circ\mathcal{S}_{v'}=0 \bmod t^{c(\mathfrak{j})}$. On the one hand, the columns of $\mathrm{gr}(\mathcal{S}_{v'}-\mathcal{S}_v)$ are in the subspace $W_\mathfrak{j}'\subset W_\mathfrak{j}=\bigoplus_j\mathbb{C}[\Gamma]_{<c(\mathfrak{j})-\gamma_j}$ spanned by elements of the form $t^{\rho(s)}e_{g(s)}$, where $e_{g(s)}$ is the unit vector in the $g(s)^{\text{th}}$ summand. By inspection, $\ker G\cap W_\mathfrak{j}'=0$. On the other hand, the columns of $\mathrm{gr}(\mathcal{S}_{v'}-\mathcal{S}_v)$ are necessarily in the kernel of G.

For existence, it suffices to observe that $G(W_{\mathrm{j}}') = G(W_{\mathrm{j}}) = \mathrm{j}_{<c(\mathrm{j})}$. Indeed, we can compute S_ν by induction: let $\{j_1 < j_2 < \cdots < j_N\} = \mathrm{j} \cap [0, c(\mathrm{j})]$ and suppose we have found ν_k such that

$$v_k := \mathcal{G}_\lambda \circ \mathcal{S}_{\nu_k} = 0 \mod t^{j_k}.$$

By assumption $\lambda \in X^{[\mathrm{j}]}$, so the entries of the vector v_k have orders which are elements of j. Hence, by the construction of W_{j}' there is $\nu' \in \mathrm{Syz}$ such that $gr(\mathcal{G}_\lambda \circ \mathcal{S}_{\nu'}) = \mathrm{gr}(v_k)$. Thus for $\nu_{k+1} := \nu_k - \nu'$ we have

$$\mathcal{G}_\lambda \circ \mathcal{S}_{\nu_{k+1}} = 0 \mod t^{j_{k+1}}. \qquad \square$$

For future use we write the matrix entries $(\mathcal{G}_\lambda \circ \mathcal{S}_\nu)_i = \sum_j (\mathcal{G}_\lambda)_j (\mathcal{S}_\nu)_i^j$ explicitly:

$$\sum_j \Bigg(u_i^j \phi_{\sigma_i - \gamma_j} \phi_{\gamma_j} + \sum_{\substack{c(\mathrm{j}) - \gamma_j > s > \sigma_i \\ g(s) = j}} \nu_{is}^{s-\sigma_i} \phi_{s-\gamma_j} \phi_{\gamma_j} + \sum_{k \in \Gamma_{>\gamma_j} \setminus \mathrm{j}} u_i^j \lambda_j^{k-\gamma_j} \phi_{\sigma_i - \gamma_j} \phi_k + \sum_{\substack{c(\mathrm{j}) - \gamma_j > s > \sigma_i \\ g(s) = j \\ k \in \Gamma_{>\gamma_j} \setminus \mathrm{j}}} \nu_{is}^{s-\sigma_i} \lambda_j^{k-\gamma_j} \phi_{s-\gamma_j} \phi_k \Bigg)$$

Let $\mathcal{I} \subset \mathbb{C}[\lambda_\bullet^\bullet, \nu_{\bullet,\bullet}^\bullet]$ be the ideal defining the Hilbert scheme $X^{[\mathrm{j}]}$, ie the ideal generated by the entries of $\mathcal{G}_\lambda \circ \mathcal{S}_\nu$. Then we write $X^{[\mathrm{j}]}_{<r} \subset \mathrm{Spec}\, \mathbb{C}[\lambda_\bullet^{<r}, \nu_{\bullet,\bullet}^{<r}]$ for the subscheme cut out by the ideal $\mathcal{I}^{<r} := \mathcal{I} \cap \mathbb{C}[\lambda_\bullet^{<r}, \nu_{\bullet,\bullet}^{<r}]$.

Expand $(\mathcal{G}_\lambda \circ \mathcal{S}_\nu)_i$ in the basis ϕ_k, and denote by Eq_i^r the coefficient of $\phi_{r+\sigma_i}$. Implicitly Eq_i^r does not occur if $r + \sigma_i \notin \Gamma$ or $r + \sigma_i \geq c(\mathrm{j})$. Accounting for the upper triangularity of the basis ϕ_k, we see the nontrivial equations Eq_i^r are of the form

$$L_i^r + \text{terms in } \lambda^{<r} \text{ and } \nu^{<r},$$
$$L_i^r := \delta_{\mathrm{j} \cap (\sigma_i, c(\mathrm{j}))} (r + \sigma_i) \nu_{i, r+\sigma_i}^r + \sum_{j=1}^n \delta_{\Gamma \setminus \mathrm{j}} (r + \gamma_j) u_i^j \lambda_j^r.$$

As $\mathcal{I}^{<r+1} = (\mathcal{I}^{<r}, \mathrm{Eq}_1^r, \ldots, \mathrm{Eq}_m^r)$, the space $X^{[\mathrm{j}]}_{<r+1}$ is cut out of $X^{[\mathrm{j}]}_{<r} \times \mathrm{Spec}\, \mathbb{C}[\lambda^r, \nu^r]$ by the ideal $(\mathrm{Eq}_1^r, \ldots, \mathrm{Eq}_m^r)$. We write

$$\pi_r \colon X^{[\mathrm{j}]}_{<r+1} \to X^{[\mathrm{j}]}_{<r}$$

for the projection. Once r is greater than the conductor $c(\mathrm{j})$, we have $X^{[\mathrm{j}]}_{<r} = X^{[\mathrm{j}]}$ by the equivalence of (1) and (2) of Lemma 11. Thus it remains to study the fibers in the sequence

$$X^{[\mathrm{j}]} \xrightarrow{\pi_{c(\mathrm{j})}} X^{[\mathrm{j}]}_{<c(\mathrm{j})} \xrightarrow{\pi_{c(\mathrm{j})-1}} \cdots \xrightarrow{\pi_0} X^{[\mathrm{j}]}_{<0} = \mathrm{pt}.$$

For general singularities, the projections are not surjective and the fibers are hard to control. However, when the linear forms L_i^r are independent, the fibers are affine spaces of constant dimension and the projections are surjective. We show next that this independence holds when the singularity is unibranch with a single Puiseux pair, and compute the dimensions of fibers.

3.2 $\Gamma = \langle n, k \rangle$

We now restrict ourselves to singularities X with semigroup Γ generated by two relatively prime integers, n and k. The prototypical example is $x^n = y^k$, but in fact such singularities vary with moduli [64]. We will show in these cases that the Hilbert schemes are stratified by affine cells.

We often picture Γ in terms of the coordinate plane, in which we write $ni + kj$ in the unit square with bottom-left coordinate (i, j). As every element $m \in \Gamma$ admits a unique presentation in the form $m = ak + bn$ with $0 \le a < n$, each occurs exactly once in the semi-infinite strip of height n in the first quadrant. Ideals correspond to Young diagrams (or staircases) contained in the strip containing at most k columns of height strictly between 0 and n. For example, we assign the following staircase to $(t^{21}, t^{23}, t^{24}) \subset \mathbb{C}[\![t^4, t^5]\!]$ (bold numbers correspond to monomials in the ideal):

15	19	**23**	**27**	**31**	**35**	**39**	**43**
10	14	18	22	**26**	**30**	**34**	**38**
5	9	13	17	**21**	**25**	**29**	**33**
0	4	8	12	16	20	**24**	**28**

To see the generators and syzygies it is better to draw the infinite staircase of which the elements in the ideal are above and those not in the ideal are below. For example, in the above case we get:

20	24	28	32	**36**	**40**	**44**	**48**	**52**	**56**
15	19	23	27	**31**	**35**	**39**	**43**	**47**	**51**
10	14	18	22	**26**	**30**	**34**	**38**	**42**	**46**
5	9	13	17	**21**	**25**	**29**	**33**	**37**	**41**
0	4	8	12	16	20	**24**	**28**	**32**	**36**
−5	−1	3	7	11	15	19	**23**	**27**	**31**
−10	−6	−2	2	6	10	14	18	22	**26**
−15	−11	−7	−3	1	5	9	13	17	**21**

The generators of the ideal are the concave corners of the staircase. We order them $(\gamma_1, \dots, \gamma_m)$ in such a way that if one began ascending the staircase at γ_1 then one

would encounter them in order. For example, in the above we may take $\gamma_1 = 24$, $\gamma_2 = 21$ and $\gamma_3 = 23$. Moreover, we take the labels modulo m, ie $\gamma_k = \gamma_{m+k}$. We write σ_i for the outside convex corner encountered between γ_i and γ_{i+1}. That is, writing $\gamma_i = a_i k + b_i n$ with $0 \le a_i < p$, we define $\sigma_i = a_{i+1} q + b_i p$. In the example above, we have $\sigma_1 = 29$, $\sigma_2 = 31$ and $\sigma_3 = 28$. It is obvious from the pictorial description that the module of syzygies of $(t^{\gamma_1}, \dots, t^{\gamma_m})$ is generated over $\mathbb{C}[\Gamma]$ by the elements $\omega_i = e_i t^{\sigma_i - \gamma_i} - e_{i+1} t^{\sigma_i - \gamma_{i+1}}$ for $i = 1, \dots, m$.

Theorem 13 *For $\mathrm{j} = (\gamma_1, \dots, \gamma_m) \subset \Gamma = \langle n, k \rangle$, the scheme $X^{[\mathrm{j}]}$ admits a bijective morphism from $\mathbb{A}^{N(\mathrm{j})}$, where*

$$N(\mathrm{j}) = \sum_i |\Gamma_{>\gamma_i} \setminus \mathrm{j}| - \sum_i |\Gamma_{>\sigma_i} \setminus \mathrm{j}|.$$

Proof We will study the maps $X^{[\mathrm{j}]}_{<r+1} \to X^{[\mathrm{j}]}_{<r}$. We have seen that $X^{[\mathrm{j}]}_{<r+1}$ is cut out of $\operatorname{Spec} \mathbb{C}[\lambda^r_\bullet, \nu^r_{\bullet,\bullet}]$ by some equations Eq^r_i, where by our description of the syzygies $i = 1, \dots, r$. We have written L^r_i for the linear term of Eq^r_i. Once we show that L^r_i are linearly independent, it will follow that the zero locus of the Eq^r_i is isomorphic to the zero locus of the L^r_i. Thus $X^{[\mathrm{j}]}_{<r+1}$ is a vector bundle over $X^{[\mathrm{j}]}_{<r}$ of fiber dimension

$$\begin{aligned}\lambda^m_\bullet + \#\nu^r_{\bullet,\bullet} - \#\,\mathrm{Eq}^r_i &= \sum_{i=1}^m \delta_{\Gamma\setminus \mathrm{j}}(r+\gamma_i) + \sum_{i=1}^m \delta_{\mathrm{j}\cap(0,c(\mathrm{j}))}(r+\sigma_i) - \sum_{i=1}^m \delta_{\Gamma\cap(0,c(\mathrm{j}))}(r+\sigma_i) \\ &= \sum_{i=1}^m \delta_{\Gamma\setminus \mathrm{j}}(r+\gamma_i) - \sum_{i=1}^m \delta_{\Gamma\setminus \mathrm{j}}(r+\sigma_i).\end{aligned}$$

Summing over r gives the claimed value of $N(\mathrm{j})$.

It remains to prove the linear independence of the forms L^r_i. If $r + \sigma_i \notin \mathrm{j} \cap (\sigma_i, c)$ then L^r_i does not depend on variables $\nu^r_{\bullet,\bullet}$. On the other hand, if $r + \sigma_i \in \mathrm{j} \cap (\sigma_i, c)$ then L^r_i does depend on $\nu^r_{\bullet,\bullet}$. Moreover, the linear forms L^r_i, $r + \sigma_i \in \mathrm{j} \cap (\sigma_i, c)$ are mutually linearly independent and linear span of these forms does not contain a nontrivial linear form independent of $\nu^m_{\bullet,\bullet}$. Thus to finish the proof we need to show that the linear forms

$$L^r_i,\ r + \sigma_i \in \Gamma \setminus \mathrm{j}$$

are linearly independent. The easiest proof of this statement uses a pictorial interpretation of the syzygies and the linear forms L^r_i which we explain below.

Suppose we are given i and $r > 0$ such that $r + \sigma_i \in \Gamma \setminus \mathrm{j}$. From our description of the syzygies we see that L^r_i is of the form

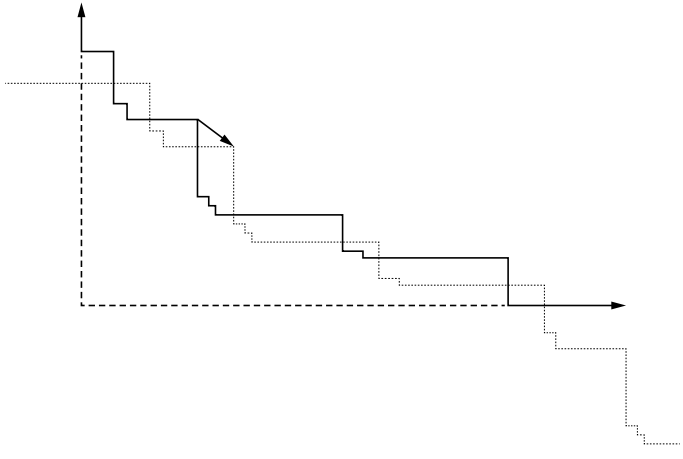
$$\lambda_{i+1}^r \delta_{\Gamma\setminus \mathrm{j}}(\gamma_{i+1}+r) - \lambda_i^r \delta_{\Gamma\setminus \mathrm{j}}(\gamma_i + r).$$

We can visualize the nontrivial coefficients of the L_i^r in the following way. Begin with an $m \times m$ matrix A with $A_{ii} = -1$ and $A_{i+1,i} = 1$ (where the latter equation is to be interpreted mod m). If $r + \sigma_i \notin \Gamma \setminus \mathrm{j}$, we delete the i^{th} column of A. Finally, we replace some of the nonzero coefficients with 0, depending on the value of $\delta_{\Gamma\setminus \mathrm{j}}(\gamma_i + r)$.

To interpret the condition $r + \sigma_i \in \Gamma \setminus \mathrm{j}$, write $r = \alpha k + \beta n$ and let v be the vector (α, β). Consider the new infinite staircase obtained by translating the original staircase by v. Then $r + \sigma_i \in \Gamma \setminus \mathrm{j}$ if and only if the translate of the convex corner corresponding to σ_i shifts to a square below the original staircase and above the (rather simpler) staircase defined by the elements of Γ.

Now if $r + \sigma_i \in \Gamma \setminus \mathrm{j}$ for all i, the entire translate of the infinite staircase by v would lie below the infinite staircase. This would imply $\mathrm{j} \subset \mathrm{j} + r$, which is impossible, since $r > 0$. Thus there is at least one deleted column in the matrix A. Without loss of generality, we may renumber the γ_i so that the final column is deleted. The remaining matrix is supported on the diagonal and on the off-diagonal just below it. After deleting additional columns, the remaining matrix will be block diagonal. The number of blocks will be the number of "runs" of consecutive undeleted columns, and the size of each block will be $(i+1) \times i$, where i is the length of the run.

More geometrically, each block corresponds to a connected component of the intersection of the shifted infinite staircase with the region below the original staircase and above the Γ–staircase. For example, in the case of the picture below, the matrix consists of three blocks of sizes $i = 1, 2, 1$:



Now we need to show that the diagonal blocks are of maximal rank. There are four types of diagonal block, depending on how the path exits the finite staircase. These four types are shown in the figure below:

As in the previous figure, the fine dotted line marks the translated copy of the infinite staircase, the heavy solid line is the finite staircase, and the heavy dashed lines mark the boundary of the first quadrant. In the first three cases the corresponding diagonal blocks are of maximal rank but in the last case the matrix is degenerate. To finish the proof, we must check that the last case does not occur. Indeed, if it did, we see from the picture that the fine dotted curve would have a unique connected component between the heavy dotted curve and the bold curve. Thus the fine dotted curve would lie under the bold curve, but we showed above that this is impossible. □

Remark The above argument actually shows that $X^{[\mathfrak{j}]} \simeq \mathbb{A}^{N(\mathfrak{j})}$. Indeed, in the proof we show that $\pi\colon X^{[\mathfrak{j}]}_{<r+1} \to X^{[\mathfrak{j}]}_{<r}$ is a smooth affine fibration. But by construction the fibration is a subfibration of the trivial fibration and the linear part of the equations defining the subfibration does not depend on the point on $X^{[\mathfrak{j}]}_{<r}$. Thus π is a trivial fibration.

3.3 Nested Hilbert schemes

We turn to the study of the nested Hilbert schemes. Let M be the maximal ideal of $\mathcal{O}_X$. Recall that $X^{[a\leq b]} \subset X^{[a]} \times X^{[b]}$ was defined to be the locus

$$\{(J, I) \in X^{[a]} \times X^{[b]} \mid J \supset I \supset MJ\}.$$

This space admits a stratification by type of semigroup-module. Indeed let $X^{[\mathfrak{j}\supset\mathfrak{i}]} \subset X^{[\mathfrak{j}]} \times X^{[\mathfrak{i}]}$ be the locus of $\{(J, I) \in X^{[\mathfrak{j}]} \times X^{[\mathfrak{i}]} \mid J \supset I \supset MJ\}$. Evidently

$$X^{[a\leq b]} = \coprod_{\substack{\#\Gamma\setminus\mathfrak{j}=a\\ \#\Gamma\setminus\mathfrak{i}=b}} X^{[\mathfrak{j}\supset\mathfrak{i}]}$$

For the source of the nested version of the "exponential" map, we take an affine space Gen with coordinates $\lambda_i^{k-\gamma_i}$, where $k \in \Gamma_{>\gamma_i}\setminus\mathfrak{j}$ if $\gamma_i \notin \mathfrak{i}$ and $k \in \Gamma_{>\gamma_i}\setminus\mathfrak{i}$ if $\gamma_i \in \mathfrak{i}$. We

define, as before,

$$\tau_{\gamma_j}(\lambda_\bullet^\bullet) := \phi_{\gamma_j} + \sum_s \lambda_j^{s-\gamma_j}\phi_{s-\gamma_j}$$

and set

$$J(\lambda_\bullet^\bullet) = (\tau_{\gamma_i}(\lambda_\bullet^\bullet)),$$
$$I(\lambda_\bullet^\bullet) = (\tau_{\gamma_i}(\lambda_\bullet^\bullet) \mid \gamma_i \in \mathfrak{i}) + MJ(\lambda_\bullet^\bullet).$$

Finally we define $\mathrm{Exp}(\lambda_\bullet^\bullet) = (J(\lambda_\bullet^\bullet), I(\lambda_\bullet^\bullet))$. This constructible map induces a bijective morphism $\mathrm{Exp}_\gamma^{-1}(X^{[\mathfrak{j}\supset\mathfrak{i}]}) \to X^{[\mathfrak{j}\supset\mathfrak{i}]}$. The locus $\mathrm{Exp}_\gamma^{-1}(X^{[\mathfrak{j}\supset\mathfrak{i}]})$ is characterized by requiring $\mathrm{ord}(J(\lambda)) = \mathfrak{j}$ and $\mathrm{ord}(I(\lambda)) = \mathfrak{i}$, which is equivalent to requiring that the syzygies of $\underline{\mathfrak{j}}$ and $\underline{\mathfrak{i}}$ lift to $J(\lambda)$ and $I(\lambda)$ respectively. In fact, because by construction $I(\lambda) \supset MJ(\lambda)$, we see that

$$\mathrm{ord}(J(\lambda)) = \mathrm{ord}(J(\lambda) \setminus MJ(\lambda)) \cup \mathrm{ord}(MJ(\lambda)) \subset \mathfrak{j} \cup \mathrm{ord}(I(\lambda)).$$

Therefore it suffices to check that $\mathrm{ord}(I(\lambda)) = \mathfrak{i}$, or in other words we need only concern ourselves with the syzygies of $\mathfrak{i}$.

Rather than continue a general treatment, we restrict to the case when $\Gamma = \langle n, k\rangle$, in which the syzygies are easier to describe.

Theorem 14 *Let $\mathfrak{j} = (\gamma_1, \dots, \gamma_m) \subset \Gamma = \langle n, k\rangle$ and $\mathfrak{j} \supset \mathfrak{i} \supset \mathfrak{j} + \mathfrak{m}$. Let σ_i be the degrees of the syzygies of $\mathfrak{j}$. Then $X^{[\mathfrak{j}\supset\mathfrak{i}]}$ admits a bijective morphism from $\mathbb{A}^{N(\mathfrak{j}\supset\mathfrak{i})}$, where*

$$N(\mathfrak{j} \supset \mathfrak{i}) = \sum_{\gamma_i \notin \mathfrak{i}} |\Gamma_{>\gamma_i} \setminus \mathfrak{j}| + \sum_{\gamma_i \in \mathfrak{i}} |\Gamma_{>\gamma_i} \setminus \mathfrak{i}| - \sum_{i=1}^{m} |\Gamma_{>\sigma_i} \setminus \mathfrak{i}|.$$

Proof We must determine the locus in Gen in which the syzygies of $\mathfrak{i}$ lift. Here we drop our numbering convention of the generators of $\mathfrak{j}$ and instead number them $\gamma_1, \dots, \gamma_l, \gamma_{l+1}, \dots, \gamma_m$ so that $\mathfrak{j} \setminus \mathfrak{i} = \{\gamma_1, \dots, \gamma_l\}$. Then a not necessarily minimal set of generators for $\mathfrak{i}$ is $\{\gamma_1 + n, \gamma_1 + k, \dots, \gamma_l + n, \gamma_l + k\} \cup \{\gamma_{l+1}, \dots, \gamma_m\}$. From this, for instance from the pictorial description, it is easy to see that the degrees of the minimal syzygies are among $\sigma_1, \dots, \sigma_m, \gamma_1 + n + k, \dots, \gamma_l + n + k$, where the σ_i are the syzygies of $\mathfrak{j}$. Note that the syzygies of degree $\gamma_i + p + q$ between the generators of degree $\gamma_i + n$ and $\gamma_i + k$ always lift to $I(\lambda)$. Indeed, if f_i is the generator of degree γ_i, then the generators of degrees $\gamma_i + n$ and $\gamma_i + k$ are just $\phi_n f_i$ and $\phi_k f_i$, thus the syzygy in question is $\phi_n(\phi_k f_i) - \phi_k(\phi_n f_i) = 0$.

We denote the generators of $\mathfrak{i}$ as follows:

$$\epsilon_{2i-1} = \gamma_i + n, \quad \epsilon_{2i} = \gamma_i + k \quad \text{for } i = 1, \dots, l, \qquad \epsilon_{i+l} = \gamma_i \quad \text{for } i = l+1, \dots, m.$$

We have reduced the problem of determining when $\lambda \in X^{[\mathfrak{j}\supset\mathfrak{i}]}$ to determining when the syzygies of the form $(\vec{s}_i)_j = u_i^j t^{\sigma_i - \epsilon_j}$ for $j = 1, \dots, m+l$ and $i = 1, \dots, m$ lift. These are generated by the syzygies in which only two entries u_i^j are nonzero.

As in the unnested case, we introduce a subspace Syz of all possible such syzygies, so that $X^{[\mathfrak{j}\supset\mathfrak{i}]}$ will be cut out of $\mathrm{Gen} \times \mathrm{Syz}$ by explicit equations. Thus fix, for each $s \in \mathfrak{i}$, a splitting $s = \epsilon_{g(s)} + \rho(s)$ with $\rho(s) \in \Gamma$. Let Syz be an affine space with coordinates $\nu_{is}^{s-\sigma_i}$, where $i = 1, \dots, m$ and $c(\mathfrak{i}) > s > \sigma_i$. To a point ν in Syz we assign an $(m+l) \times m$ matrix with entries

$$(\mathcal{S}_\nu)_i^j = u_i^j \phi_{\sigma_i - \epsilon_j} + \sum_{\substack{c(\mathfrak{i}) > s > \sigma_i \\ g(s) = j}} \nu_{is}^{s-\sigma_i} \phi_{s-\epsilon_j}.$$

By an argument similar to that in the unnested case, $X^{[\mathfrak{j}\supset\mathfrak{i}]}$ is cut out of $\mathrm{Gen} \times \mathrm{Syz}$ by the equation $\mathcal{G}_\lambda \circ \mathcal{S}_\nu = 0 \bmod t^{c(\mathfrak{i})}$. Denoting by $\mathcal{I}$ the ideal generated by the matrix entries of $\mathcal{G}_\lambda \circ \mathcal{S}_\nu$, we may again define the successive approximations $X_{<r}^{[\mathfrak{j}\supset\mathfrak{i}]} \subset \mathbb{C}[\lambda_\bullet^{<r}, \nu_{\bullet,\bullet}^{<r}]$ as the locus cut out by $\mathcal{I} \cap \mathbb{C}[\lambda_\bullet^{<r}, \nu_{\bullet,\bullet}^{<r}]$. Then $X_{<r+1}^{[\mathfrak{j}\supset\mathfrak{i}]}$ is cut out of $X_{<r}^{[\mathfrak{j}\supset\mathfrak{i}]} \times \operatorname{Spec} \mathbb{C}[\lambda_\bullet^{<r}, \nu_{\bullet,\bullet}^{<r}]$ by the coefficients Eq_i^r of $\phi_{r+\sigma_i}$ in the matrix entries of $\mathcal{G}_\lambda \circ \mathcal{S}_\nu$. The terms depending on $\lambda_\bullet^r$ and $\nu_{\bullet,\bullet}^r$ are linear, and we denote them by L_i^r:

$$\mathrm{Eq}_i^r = L_i^r + \text{terms in } \lambda^{<r} \text{ and } \nu^{<r},$$

$$L_i^r = \delta_{\mathfrak{i}\cap(\sigma_i, c(\mathfrak{j}))}(r+\sigma_i)\nu_{i,r+\sigma_i}^r + \sum_{s=1}^{l}(u_i^{2s-1} + u_i^{2s})\delta_{\Gamma\setminus\mathfrak{j}}(r+\gamma_s)\lambda_s^r + \sum_{s=l+1}^{m} u_i^{l+s}\delta_{\Gamma\setminus\mathfrak{i}}(r+\gamma_s)\lambda_s^r; \tag{8}$$

here $\mathrm{Eq}_i^r = 0$ if $r + \sigma_i \notin \Gamma$ or $r + \sigma_i \geq c(\mathfrak{i})$.

As in the proof of Theorem 13, from the shape of the forms L_i^r we see that we only need to show that the linear forms

$$L_i^r,\ r + \sigma_i \in \Gamma \setminus \mathfrak{i}$$

are linearly independent. From the structure of the coefficients u_j^i we see that for $r + \sigma_i \in \Gamma \setminus \mathfrak{i}$ the linear form L_i^r is equal to

$$\sum_{s=1}^{l}(u_i^{2s-1} + u_i^{2s})\delta_{\Gamma\setminus\mathfrak{j}}(r+\gamma_s)\lambda_s^r + \sum_{s=l+1}^{m} u_i^{l+s}\delta_{\Gamma\setminus\mathfrak{i}}(r+\gamma_s)\lambda_s^r.$$

At this point, it is convenient to reorder the generators of $\mathfrak{j}$ consecutively along the staircase, as we did in the unnested case. Once we have done so, we see that the matrix

of coefficients of the L_i^r is obtained as in the unnested case: we start with the same matrix A, delete some columns (corresponding to those $\sigma_i + r \notin \Gamma \setminus \mathfrak{i}$), and then set some coefficients to zero. The set of coefficients which is set to zero is smaller than that in the unnested case, since the support of $\delta_{\Gamma\setminus\mathfrak{i}}$ is larger than that of $\delta_{\Gamma\setminus\mathfrak{i}}$.

Hence, to compute the rank of the space spanned by the linear forms L_i^r for $r+\sigma_i \in \Gamma\setminus\mathfrak{i}$, one has to analyze connected components of the intersection of the periodic path associated to $\mathfrak{j}$ and shifted by a vector v (as in Theorem 13) with the area under the path associated to $\mathfrak{i}$ and above the path associated to Γ. The path $\mathfrak{j}+r$ cannot lie completely under the path for $\mathfrak{i}$: if it did, it is easy to see that we would have $\mathfrak{j}+r \subset \mathfrak{j}$. For $r>0$, we have already shown this is impossible. The remainder of the argument proceeds exactly as in the proof of Theorem 13.

Thus we conclude that $\pi_r\colon X_{<r+1}^{[\mathfrak{j}\supset\mathfrak{i}]} \to X_{<r}^{[\mathfrak{j}\supset\mathfrak{i}]}$ is a vector bundle with fibers of dimension $\#\lambda_\bullet^r + \#\nu_{\bullet,\bullet}^r - \#\operatorname{Eq}_i^r$. By construction,

$$\#\lambda_\bullet^r = \sum_{i=1}^{l} \delta_{\Gamma\setminus\mathfrak{j}}(r+\gamma_i) + \sum_{i=l+1}^{m} \delta_{\Gamma\setminus\mathfrak{i}}(r+\gamma_i),$$

$$\#\nu_{\bullet,\bullet}^r = \sum_{i=1}^{m} \delta_{\mathfrak{i}\cap(0,c(\mathfrak{i}))}(r+\sigma_i),$$

while on the other hand the nontrivial Eq_i^r impose

$$\#\operatorname{Eq}_i^r = \sum_{i=1}^{m} \delta_{\Gamma\cap(0,c(\mathfrak{i}))}(r+\sigma_i)$$

independent conditions. Summing over these terms and summing over m gives the stated dimension. □

4 Examples

In this section we verify that the predictions of Conjecture 2 agree with previously known or conjectured values of the HOMFLY homology for torus knots. We first consider the "stable" HOMFLY homology, whose Poincaré polynomial is defined by

$$\overline{\mathscr{P}}(T(n,\infty)) = \lim_{k\to\infty} \left(\frac{q}{a}\right)^{\mu-1} \overline{\mathscr{P}}(T(n,k)).$$

This limit exists by a theorem of Stošić [61]. It was conjectured in [20] that

$$\overline{\mathscr{P}}(T(n,\infty)) = \prod_{i=1}^{n} \frac{1+a^2q^{2i-2}t^{2i-1}}{1-q^{2i}t^{2i-2}}.$$

Proposition 6 from the introduction says that the analogous statement holds for $\overline{\mathscr{P}}_{\mathrm{alg}}$.

Proof of Proposition 6 The assertion $\overline{\mathscr{P}}_{\mathrm{alg}}(X_{n,\infty}) = \overline{\mathscr{P}}_{\mathrm{alg}}(X_{n,k}) + O(q^{2k})$ follows immediately from the fact that all ideals parametrized by the spaces $X_{n,\infty}^{[l\le l+m]}$ and $X_{n,k}^{[l\le l+m]}$ for $l<k$ contain the ideal $(x,y)^k$. But modulo this ideal the equations $y^n = x^k$ and $y^n = 0$ are identical, so for $l<k$ we have $X_{n,\infty}^{[l\le l+m]} = X_{n,k}^{[l\le l+m]}$.

Recall that $X_{n,k}^{[l\le l+m]}$ is a union of affine spaces corresponding to staircases with some marked external corners sitting in the semi-infinite strip of height n in the first quadrant. There is a condition on which staircases occur, but any fixed staircase will contribute for all sufficiently large k. By inspection of the formula in Theorem 13, the contributions converge as well.

Let $S = (n^{s_n}, (n-1)^{s_{n-1}}, \dots, 1^{s_1})$ be a staircase with s_i columns of height i. For any subset $\Sigma \subset \{1,\dots,n\}$, we form Σ' by subtracting 1 from each element of Σ. Write $S \cup \Sigma$ for the staircase with additional columns with heights from Σ, and similarly $S \cup \Sigma'$. Every admissible nested pair of semigroup ideals $\mathfrak{j} \supset \mathfrak{i}$ has staircase of the form $S \cup \Sigma \supset S \cup \Sigma'$ for some S and Σ; to recover S from the staircases of $\mathfrak{j}$ and $\mathfrak{i}$, delete every column in which those staircases differ.

It remains to sum the contributions of the staircases. When $k+n^2$ is greater than the number of boxes in the staircase, one calculates from Theorem 13 that

$$\mathrm{Cont}_S = q^{2\sum i s_i} t^{2\sum (i-1)s_i} \prod_{i=1}^{n} (1+a^2q^{2i-2}t^{2i-1}).$$

Summing over staircases gives the stated formula. □

When combined with the symmetry of $\overline{\mathscr{P}}_{\mathrm{alg}}$, Proposition 6 is enough to determine $\overline{\mathscr{P}}_{\mathrm{alg}}(X_{2,n})$ and $\overline{\mathscr{P}}_{\mathrm{alg}}(X_{3,n})$.

Corollary 15 *We have*

$$\frac{1-q^2}{1+a^2t}\left(\frac{q}{a}\right)^{2k-1} \overline{\mathscr{P}}_{\mathrm{alg}}(X_{2,2k+1}) = \frac{1+a^2q^2t^3}{1-q^4t^2} - q^{4k}t^{2k}\left(\frac{q^4t^2+a^2q^2t^3}{1-q^4t^2}\right),$$

$$\frac{1-q^2}{1+a^2t}\left(\frac{q}{a}\right)^{2k-2} \overline{\mathscr{P}}_{\mathrm{alg}}(X_{2,2k}) = \frac{1+a^2q^2t^3}{1-q^4t^2} + q^{4k}t^{2k}\left(\frac{q^2+a^2t}{1-q^4t^2}\right)\left(\frac{1-q^2t^2}{1-q^2}\right).$$

Corollary 16 *We have*

$$\frac{1-q^2}{1+a^2t}\left(\frac{q}{a}\right)^{6k-1}\overline{\mathscr{P}}_{\mathrm{alg}}(X_{3,3k+1})$$
$$=\frac{(1+a^2q^2t^3)(1+a^2q^4t^5)}{(1-q^4t^2)(1-q^6t^4)}-q^{2+6k}t^{2+4k}\frac{(q^2+a^2t)(1+a^2q^2t^3)(1+q^2t^2+q^4t^2)}{(1-q^6t^2)(1-q^6t^4)}$$
$$+q^{4+12k}t^{4+6k}\frac{(q^2+a^2t)(q^4+a^2t)}{(1-q^6t^2)(1-q^4t^2)},$$
$$\frac{1-q^2}{1+a^2t}\left(\frac{q}{a}\right)^{6k+1}\overline{\mathscr{P}}_{\mathrm{alg}}(X_{3,3k+2})$$
$$=\frac{(1+a^2q^2t^3)(1+a^2q^4t^5)}{(1-q^4t^2)(1-q^6t^4)}-q^{4+6k}t^{4+4k}\frac{(q^2+a^2t)(1+a^2q^2t^3)(1+q^2+q^4t^2)}{(1-q^6t^2)(1-q^6t^4)}$$
$$+q^{8+12k}t^{6+6k}\frac{(q^2+a^2t)(q^4+a^2t)}{(1-q^6t^2)(1-q^4t^2)}.$$

Proof We recall that $\delta(X_{2,n})=\frac{1}{2}(n-1)$ and $\delta(X_{3,n})=n-1$. Thus, by Proposition 6, the equality in each case holds modulo $q^{2\delta+2}$. By inspection, the right-hand side is a polynomial in q of degree 4δ; Proposition 3 implies the same for the left-hand side. It remains only to observe that the symmetry imposed by Proposition 3 on the left-hand side also holds on the right. □

In the $(2,n)$ case, these formulas match the known value of $\overline{\mathscr{P}}$ [44]; in the $(3,n)$ case, the resulting formula matches a conjectural formula for $\overline{\mathscr{P}}$ advanced in [20].

5 Torus knots and Cherednik algebras

In this section, we make some conjectures about the structure of $\overline{\mathscr{P}}_{\mathrm{alg}}(X_{n,k})$. The main thrust of these conjectures is that the HOMFLY homology of torus knots should be related to the representation theory of rational Cherednik algebras. We will explore this idea more fully in a subsequent paper [34]. Here, we focus on the problem of writing explicit formulas for $\overline{\mathscr{P}}_{\mathrm{alg}}(X_{n,k})$.

For the moment, we restrict attention to the unnested Hilbert scheme series $\overline{\mathscr{P}}^{\min}_{\mathrm{alg}}$. As we explained in Section 2, $\overline{\mathscr{P}}^{\min}_{\mathrm{alg}}(X)$ admits an alternate description in terms of the perverse filtration on the cohomology of the compactified Jacobian of a complete rational curve with a unique singularity of type X. When X is unibranch of the form $y^n=x^k$, it can be shown using the techniques of Z. Yun's thesis that the cohomology of

the compactified Jacobian furnishes the unique simple representation of $\mathcal{H}_n^{\mathrm{sph}}(k/n)$, the spherical rational Cherednik algebra of type A_{n-1} and central character k/n [63; 57].[3] It is moreover the case that the perverse filtration on $\mathrm{H}^*(\bar{J})$ is compatible with the natural bigrading on $\mathcal{H}_n^{\mathrm{sph}}(k/n)$; ie it is a good filtration in the sense of Gordon and Stafford [28; 29]. An a priori different good filtration on $\mathcal{H}_n^{\mathrm{sph}}(k/n)$ can be constructed by means of results of Calaque, Enriquez and Etingof [12] (see details below). We conjecture these filtrations have at least the same associated graded dimensions. When $n = mk + 1$, a formula for the character of the filtration of [12] is established by the work of Gordon and Stafford [28; 29], which gives a prediction for the lowest-order coefficient of a in the superpolynomial of the $(n, mn + 1)$ torus knot.

5.1 Filtrations on rational Cherednik algebras

We recall some relevant definitions. The symmetric group S_n acts on the free algebra $\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle$ by permuting the variables.

Definition 17 For any complex number ξ, the rational Cherednik algebra $\mathcal{H}_n(\xi)$ is the quotient of the algebra $\mathbb{C}[S_n] \ltimes \mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle$ by the relations

$$\sum_{i=1}^{n} x_i = 0 = \sum_{i=1}^{n} y_i, \quad [x_i, x_j] = 0 = [y_i, y_j],$$
$$[x_i, y_j] = \frac{1}{n} - \xi s_{ij} \quad \text{for } i \neq j.$$

Using the symmetrizer

$$e = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$$

we form the spherical rational Cherednik algebra $\mathcal{H}_n^{\mathrm{sph}}(\xi) := e\mathcal{H}_n(\xi)e$.

We fix the following grading on $\mathcal{H}_n(\xi)$ and $\mathcal{H}_n^{\mathrm{sph}}(\xi)$:

$$\deg(x_i) = 1, \quad \deg(y_i) = -1, \quad \deg(\sigma) = 0 \quad \text{for } \sigma \in S_n.$$

The grading is equivalently given by the eigenvalue of the operator $[h, \cdot]$, where $h = \sum_i x_i y_i$.

[3] Varagnolo and Vasserot have shown that the equivariant K–theory of this space admits the action of the double affine Hecke algebra, and presumably an application of their methods in Borel–Moore homology would yield a construction of its trigonometric degeneration. We however have been unable to use their approach to construct the rational Cherednik algebra representation geometrically.

In studying the representation theory of $\mathcal{H}_n(\xi)$, it is natural to restrict to the category $\mathcal{O}$ of $\mathcal{H}_n(\xi)$–modules which are locally nilpotent under the action of operators y_i and decompose into a direct sum of finite-dimensional generalized eigenspaces of h. We recall from [9; 7; 8; 27] some basic facts about this abelian category. The simple objects are enumerated by Young diagrams of size n. Denoting by $L_\xi(\mu)$ the representation corresponding to the Young diagram μ, the subspace annihilated by the y_i furnishes the irreducible representation of S_n corresponding to μ. For $\xi > 0$, $L_\xi(\mu)$ is finite-dimensional if and only if μ corresponds to the trivial representation and $\xi = k/n$ with $k \in \mathbb{Z}$ and $(k,n) = 1$. We denote this module by L_ξ. One may analogously define the category of $\mathcal{O}^{\mathrm{sph}}$ of $\mathcal{H}_n^{\mathrm{sph}}(\xi)$–modules; the map $M \mapsto eM$ is an equivalence between $\mathcal{O}$ and $\mathcal{O}^{\mathrm{sph}}$ for all ξ, except rational ξ from $(-1,0)$ with denominator smaller than or equal to n [10].

The element h gives a $\mathbb{Z}$–grading to elements M of $\mathcal{O}$ or $\mathcal{O}^{\mathrm{sph}}$,

$$M_i = \{m \in M \mid h(m) = im\}.$$

Definition 18 [28; 29] Let M be a module in $\mathcal{O}$ (resp. $\mathcal{O}^{\mathrm{sph}}$). A filtration

$$\Lambda: \quad \Lambda^{i,j}M \subset \Lambda^{i+1,j}M, \quad \Lambda^{i,j}M \subset \Lambda^{i,j+1}M$$

is good if

$$z \cdot M_{ij} \subset M_{i+k,j+l} \quad \text{for any } z \text{ with degree } k \text{ in the } x_i \text{ and degree } l \text{ in the } y_i,$$

$$M_k = \bigoplus_{i-j=k} M_k \cap \Lambda^{i,j}M,$$

$$M_k \cap \Lambda^{i,j}M = 0 \quad \text{if } i-j < k,$$

and the corresponding associated graded module is a finitely generated $\mathbb{Z}\times\mathbb{Z}$–graded module over $\mathbb{C}[S_n] \ltimes \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]$ (resp. $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{S_n}$).

The procedure of [57] realizing $L_{k/n}$ as the cohomology of the compactified Jacobian of a curve with singularity $X^n = Y^k$ matches the *grading* on $L_{k/n}$ to the associated graded structure of the perverse filtration on the Jacobian; the process of taking associated graded destroys the homological grading on the Jacobian, but leaves a (good) filtration on $L_{k/n}$. Another, a priori different, good filtration is given by the following construction of [12]:

Proposition 19 *The representation $eL_{k/n}$ of $\mathcal{H}_n^{\mathrm{sph}}(k/n)$ has a good filtration.*

Proof First, let us notice that $e\mathcal{H}_n(k/n)e-\mathcal{H}_n(k/n)$ bimodule $e\mathcal{H}_n(k/n)$ has a good filtration. Since this bimodule provides a Morita equivalence between $e\mathcal{H}_n(k/n)e-$modules and $\mathcal{H}_n(k/n)-$modules, it is enough to construct a good filtration on the spherical modules.

According to [12, Theorem 9.8], there is an isomorphism of vector spaces $eL_{k/n} \cong eL_{n/k}$, which identifies $\sum_{i=1}^{k} x_i^a y_i^b \in \mathcal{H}_k^{\mathrm{sph}}(n/k)$ and $\sum_{i=1}^{n} x_i^a y_i^b \in \mathcal{H}_n^{\mathrm{sph}}(k/n)$ for all positive a and b as endomorphisms of this vector space. The spherical subalgebra is spanned by these elements, hence this isomorphism carries good filtrations to good filtrations.

As explained in [7],

$$eL_{k/n+1} = eL_{k/n} \otimes_{e\mathcal{H}_n(k/n)e} P_{k/n}, \quad \text{with } P_{k/n} := e\mathcal{H}_n\Big(\frac{k}{n}+1\Big)\delta e,$$

where $\delta := \prod_{i<j}(x_i - x_j)$. In particular, it is shown in [7] that $P_{k/n}$ has the structure of an $(e\mathcal{H}_n(k/n)e-e\mathcal{H}_n(k/n+1)e)-$bimodule. Thus, if $eL_{k/n}$ has a good filtration, the above construction induces a good filtration on $eL_{k/n+1}$.

Thus by the Euclidean algorithm, we may construct a good filtration on any $L_{k/n}$ starting from a good filtration on L_1. This latter module is one-dimensional, so we may give it the trivial filtration $\Lambda^{0,0}L_1 = 0$ and $\Lambda^{>0,>0} = L_1$. □

Conjecture 20 *The homological filtration and the filtration of Proposition 19 agree.*

Remark Let $\overline{\mathcal{H}}_n(k/n)$ be the quotient of $\mathbb{C}[S_n] \ltimes \mathbb{C}[\overline{x}_1, \dots, \overline{x}_n, \overline{y}_1, \dots, \overline{y}_n]$ by the relations

$$\begin{aligned} [\overline{x}_i, \overline{x}_j] &= [\overline{y}_i, \overline{y}_j] = 0, \\ [\overline{y}_i, \overline{x}_j] &= k s_{ij} \quad \text{for } i \neq j, \\ [\overline{y}_i, \overline{x}_i] &= \frac{1}{n-1} - k \sum_{j \neq i} s_{ij}. \end{aligned}$$

Let $\overline{X} := \sum_i \overline{x}_i$ and $\overline{Y} := \sum_i \overline{y}_i$. Then $\mathbb{C}[\overline{Y}, \overline{X}]$ is isomorphic to the algebra of differential operators on $\mathbb{C}$, and $\mathcal{H}_n(k/n)$ is embedded via

$$x_i \mapsto \overline{x}_i - \frac{1}{n}\overline{X}, \quad y_i \mapsto \overline{y}_i - \frac{1}{n}\overline{Y}.$$

Moreover, $\mathcal{H}_n(k/n)$ commutes with $\overline{X}$ and $\overline{Y}$, so $\overline{\mathcal{H}}_n(k/n) = \mathcal{H}_n(k/n) \times \mathbb{C}[\overline{Y}, \overline{X}]$. Given an $\mathcal{H}_n(k/n)-$module M, we can produce an $\overline{\mathcal{H}}_n(k/n)-$module just by tensoring over $\mathbb{C}$ with $\mathbb{C}[\overline{X}]$. It is natural to expect that an action of $\overline{\mathcal{H}}_n(k/n)$ may

be constructed on the Hilbert scheme homologies $\bigoplus_{i,n} \mathrm{H}^i(X^{[n]})$ as an algebra of correspondences, compatibly with the action of $\mathcal{H}_n(k/n)$ on the cohomology of the compactified Jacobian.

5.2 Results of Haiman [39] and Gordon and Stafford [28; 29]

Let $\mathrm{Hilb}(n)$ be the locus in the Hilbert scheme of n points in $\mathbb{C}^2$ with center of mass at the origin. Note that $\mathbb{C}^2 \times \mathrm{Hilb}(n)$ is the usual Hilbert scheme of n points in $\mathbb{C}^2$. There is a functor

$$\Phi\colon e\mathcal{H}_n(\xi)e\text{–module } M \text{ with a good filtration } \Lambda \mapsto S_{\Lambda,M} \in \mathrm{Coh}(\mathrm{Hilb}(n)),$$

The sheaf $S_{\Lambda,M}$ may depend on the filtration, though its support does not.

The functors Φ interact well with the shift functors. Let $\mathcal{T}$ be the tautological rank n sheaf on $\mathrm{Hilb}(n)$ whose fiber at a given point is the structure sheaf of the corresponding subscheme. Then

$$\Phi\Big(M \otimes_{e\mathcal{H}_n(k/n)e} e\mathcal{H}_n\Big(\frac{k}{n}+1\Big)\delta e\Big) = \Phi(M)\otimes \det \mathcal{T}. \tag{9}$$

In addition, if $L_{1/n}$ is equipped with the trivial filtration, and $\mathcal{Z} \subset \mathrm{Hilb}(n)$ is the subvariety parametrizing schemes supported at the origin, then

$$\Phi(L_{1/n}) = \mathcal{O}_{\mathcal{Z}}. \tag{10}$$

Combining with vanishing results of Haiman gives:

Theorem 21 [29] *Consider the module $eL_{m+1/n}$ of $e\mathcal{H}_n(m+1/n)e$ equipped with the filtration of Proposition 19. Then*

$$\Phi(eL_{m+1/n}) = \mathcal{O}_{\mathcal{Z}} \otimes (\det \mathcal{T})^{\otimes m}.$$

Moreover, if T_1 and T_2 are the equivariant characters of $(\mathbb{C}^*)^2$,

$$\sum_{i,j} T_1^i T_2^j \dim \mathrm{gr}_\Lambda^{i,j}\, eL_{m+1/n} = \chi_{\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{O}_{\mathcal{Z}} \otimes (\det \mathcal{T})^{\otimes m}).$$

The equivariant character is computed [39] by localization in equivariant K–theory. As the calculation is in any case restricted to $\mathcal{Z}$, we may compute on the usual Hilbert scheme $(\mathbb{C}^2)^{[n]}$ rather than $\mathrm{Hilb}(n)$. The fixed points are enumerated by partitions λ of n. For a box x in the diagram of a partition λ, we use the standard notations $a(x)$ and $l(x)$ for its arm and leg, and a' and l' for the coarm and coleg. We write λ' for

the dual partition to λ, and $\kappa(\lambda) := \sum_{x\in\lambda} l'(x) = \sum \binom{\lambda_i'}{2}$. The equivariant weights are

$$\mathcal{T}_\lambda = \sum_{x\in\lambda} T_1^{l'(x)} T_2^{a'(x)}, \tag{11}$$

$$\det \mathcal{T}_\lambda = T_1^{\kappa(\lambda)} T_2^{\kappa(\lambda')}, \tag{12}$$

$$T(\mathbb{C}^2)^{[n]}_\lambda = \prod_{x\in\lambda} (1 - T_1^{1+l(x)} T_2^{-a(x)})(1 - T_1^{-l(x)} T_2^{1+a(x)}), \tag{13}$$

$$\mathcal{O}_{\mathcal{Z}}|_\lambda = \mathcal{T}_\lambda (1-T_1)(1-T_2) \prod_{x\in\lambda\setminus\{(0,0)\}} (1 - T_1^{l'(x)} T_2^{a'(x)}), \tag{14}$$

and so in all $\chi_{\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{O}_{\mathcal{Z}} \otimes (\det \mathcal{T})^{\otimes m})$ is given by the formula

$$\sum_{\lambda\vdash n} T_1^{m\kappa(\lambda)} T_2^{m\kappa(\lambda')} \frac{(1-T_1)(1-T_2)\prod_{x\in\lambda\setminus\{(0,0)\}}(1-T_1^{l'(x)}T_2^{a'(x)})}{\prod_{x\in\lambda}(1-T_1^{1+l(x)}T_2^{-a(x)})(1-T_1^{-l(x)}T_2^{1+a(x)})} \times \sum_{x\in\lambda} T_1^{l'(x)} T_2^{a'(x)}. \tag{15}$$

5.3 Formulas for torus knots

The precise relation between the filtration in the construction of [57] and the filtration of [28; 29] is such that the equivariant variables T_1 and T_2 are related to the perverse and homological q and t as follows:

$$T_1 := q^2, \quad T_2 := \frac{1}{q^2 t^2}. \tag{16}$$

Combining Proposition 4 and Conjecture 20 we expect:

Conjecture 22 $(at)^{-\mu} \mathscr{P}^{\min}_{\mathrm{alg}}(X_{k,n}) = \sum_{i,j} T_1^i T_2^j \dim \mathrm{gr}_\Lambda^{i,j}\, eL_{k/n}$.

From the above results of Haiman [39] and Gordon and Stafford [28; 29], we expect more explicitly that $(at)^{-\mu} \mathscr{P}^{\min}_{\mathrm{alg}}(X_{mn+1,n})$ is given by (15). For general k/n, one might hope that the sheaf $F_{k/n}$ constructed by Gordon and Stafford [28] is $\mathbb{C}^*\times\mathbb{C}^*$–equivariant and that, moreover, we again have

$$\sum_{i,j} T_1^i T_2^j \dim \mathrm{gr}\, \Lambda^{i,j}(L_{k/n}) = \chi_{\mathbb{C}^*\times\mathbb{C}^*}(F_{k/n}).$$

This conjecture can be extended to describe the entire polynomial $\mathscr{P}_{\mathrm{alg}}(X_{n,k})$. Recall that $\mathcal{T}$ splits as $\mathcal{T}' \bigoplus \mathcal{O}_{\mathrm{Hilb}}$. We write $A := a^2 t$ and

$$\Lambda^* A\mathcal{T}'^* := \sum_{i=0}^{n-1} A^i \Lambda^i \mathcal{T}'^*.$$

Conjecture 23 *For any coprime (k,n), there exists a $\mathbb{C}^*\times\mathbb{C}^*$–equivariant sheaf $F_{k/n}$ such that $(at)^{-\mu}\mathscr{P}_{\mathrm{alg}}(X_{k,n}) = \chi_{\mathbb{C}^*\times\mathbb{C}^*}(F_{k/n}\otimes\Lambda^* A\mathcal{T}'^*)$.*

Remark In [34] it will be clarified that Conjecture 23 amounts to an assertion that the coefficients of a in $\mathcal{P}_{\mathrm{alg}}(X_{n,k})$ correspond to representations of hyperspherical rational Cherednik algebras. The assertion above is consistent with $F_{k/n}\otimes\det\mathcal{T} = F_{(k+n)/n}$.

The weights of $\Lambda^* A\mathcal{T}'^*$ are

$$(\Lambda^* A\mathcal{T}'^*)_\lambda = \prod_{x\in\lambda\setminus\{(0,0)\}} (1 + AT_1^{-l'(x)}T_2^{-a'(x)}).$$

Thus, when $k = mn+1$ we obtain Conjecture 8 from the introduction. For general (k,n) we lack an explicit description of the sheaf $F_{k,n}$, but computer experiments with the combinatorial formula for the Poincaré polynomial of the Hilbert scheme suggest:

Conjecture 24 *There exist $g_{r/n}(\lambda)$ such that*

$$(at)^{-\mu}\mathscr{P}_{\mathrm{alg}}(X_{mn+r,n}) = \sum_{|\lambda|=n} \frac{g_{r/n}(\lambda)}{T(\mathbb{C}^2)^{[n]}_\lambda}(1-T_1)(1-T_2)(\Lambda^*\mathcal{T}')_\lambda(\Lambda^* A\mathcal{T}'^*)_\lambda(\det\mathcal{T})^m_\lambda$$

with the following properties:

$$\delta_n \cdot g_{r/n}(\lambda) \in \mathbb{Z}[T_1^{\pm1}, T_2^{\pm1}], \tag{17}$$

$$g_{r/n}(\lambda') = g_{r/n}(\lambda)|_{T_1=T_2,\, T_2=T_1}, \tag{18}$$

$$g_{r/n}(\lambda) = g_{(n-r)/n}(\lambda)|_{T_1=1/T_1,\, T_2=1/T_2} T_1^{\kappa(\lambda)}T_2^{\kappa(\lambda')}, \tag{19}$$

$$g_{1/n}(\lambda) = \sum_{x\in\lambda} T_1^{l(x)}T_2^{a(x)}, \tag{20}$$

where $\delta_n = \prod_{0<i,j<n}(1-T_1^iT_2^j)$.

The conjecture is confirmed by computer experiment up to $n<9$. The last two formulas combined produce explicit formulas for the superpolynomial of $T(n,mn-1)$ analogous to those for $T(n,mn+1)$.

Our calculation of the stable superpolynomial in Proposition 6, together with the symmetry of (18), imply that

$$g_{r/n}((1^n)) = T_1^{(n-1)(r-1)/2}\frac{T_1^n-1}{T_1-1}, \quad g_{r/n}((n)) = T_2^{(n-1)(r-1)/2}\frac{T_2^n-1}{T_2-1}.$$

Computer calculations suggest the following formulas, which we have checked for $n<8$:

$$\begin{aligned} g_{r/n}(2,1^{n-2}) &= ([n-r]_{T_1} + T_2[r]_{1/T_1})T_1^{(n-1)(r-1)/2},\\ g_{r/n}(3,1^{n-3}) &= T_1^{(r-1)(n-1)/2}[(n-2r)\vee 0]_{T_1} + T_1^{(r-1)(n-3)/2}T_2[(n-r)\wedge r]_{T_1}\\ &\quad + T_1^{(n-r-1)(n-3)/2}T_2^2[(n-r)\wedge r]_{1/T_1}\\ &\quad + T_1^{(n-r-1)(n-1)/2-n+2}T_2^3[(2r-n)\vee 0]_{1/T_1}, \end{aligned}$$

where $[n]_t := (t^n-1)/(t-1)$, $a\wedge b := \min(a,b)$ and $a\vee b := \max(a,b)$.

For $n=5$, the following data together with the symmetries discussed above is enough to determine $g_{r/5}(\lambda)$ for all r and λ:

r	$g_{r/5}(2^2,1)$
1	$(1+T_1+T_1^2)+(1+T_1)T_2$
2	$(T_1^2+T_1^3)+(T_1+T_1^2)T_2+T_2^2$
3	$T_1^4+(T_1^2+T_1^3)T_2+(T_1+T_1^2)T_2^2$
4	$(T_1^3+T_1^4)T_2+(T_1^2+T_1^3+T_1^4)T_2^2$

The predicted answers become increasingly complicated as n grows. Below we show the answers in the case $r=3$ and $n=7$ (formulas for higher n and different r are available by request to the authors):

$$g_{3/7}(22111) = \frac{T_1^2\big(T_1^7+2T_1^6+T_1^5(T_2+1)+T_1^4(2T_2+1)+2T_1^3T_2+T_1^2(T_2^2+T_2)+T_1T_2^2+T_2^2\big)}{1+T_1^2},$$

$$g_{3/7}(2221) = \frac{\begin{aligned}&T_1^2\big(T_1^6T_2+T_1^5(T_2^2+T_2-1)+T_1^4(2T_2^2-1)+T_1^3(T_2^3+T_2^2-T_2)\\&\quad +T_1^2(T_2^3-T_2^2-T_2)-2T_1T_2^2+(T_2^4-T_2^3-T_2^2)\big)\end{aligned}}{T_2^2T_1-1},$$

$$g_{3/7}(3211)=\frac{\begin{array}{r}T_1\big(T_1^6(T_2+1)+T_1^5(T_2^2+3T_2+1)+T_1^4(T_2^3+4T_2^2+3T_2)\\ +T_1^3(4T_2^3+6T_2^2+T_2)+T_1^2(T2^4+5T_2^3+3T_2^2)\\ +T_1(2T_2^4+3T_2^3+T_2^2)+T_2^4\big)\end{array}}{(T_2+1)(T_1^2+T_1+1)},$$

$$g_{3/7}(322)=\frac{\begin{array}{r}T_1^6(T_2^2+T_2-1)+T_1^5(T_2^3+2T_2^2)+T_1^4(T_2^4+2T_2^3-T_2^2-T_2)\\ +T_1^3(T_2^4-T_2^3-2T_2^2)+T_1^2(T_2^5-2T_2^3-T_2^2)-T_1T_2^4\end{array}}{T_1^2T_2-1},$$

$$g_{3/7}(4111)=T_1^3(T_2+1)+T_1^2T_2+T_1(T_2^3+T_2^2+T_2)+T_2^3.$$

Appendix: Combinatorics of HOMFLY homology
by Eugene Gorsky

This appendix relates the combinatorics of the cells in the Hilbert scheme of a plane curve singularity with one Puiseaux pair to the existing results [35; 36; 23] on the combinatorics of diagonal harmonics and DAHA representations. This connection justifies some of the conjectures made in [30], where a relation between (q,t)–Catalan numbers and torus knot homology was proposed.

As an outcome of this combinatorial study, the authors wrote a computer program computing the polynomials $\mathscr{P}_{\mathrm{alg}}(T(k,n))$ for any k and n. These polynomials agree with all conjectured [2; 14; 21] formulas for the superpolynomials of torus knots. The output of the program is available by request to the authors.

A.1 Reformulation of Theorem 5

Let us recall the setup of Theorem 5.

One has a semigroup Γ generated by two coprime integers k and n, a semigroup ideal $\mathfrak{i}$ with generators $i_1,\dots,i_r$ and syzygies $s_1,\dots,s_r$, and a semigroup ideal $\mathfrak{j}=\mathfrak{i}\setminus\{i_1,\dots i_m\}$.

Then by Theorem 5 the dimension of the cell in the Hilbert scheme of a curve singularity with semigroup Γ labeled by the ideal $\mathfrak{i}$ equals

$$N(\mathfrak{i})=\sum_{a=1}^{r}\#(\Gamma_{>i_a}\setminus\mathfrak{i})-\sum_{a=1}^{r}\#(\Gamma_{>s_a}\setminus\mathfrak{i}),$$

while the dimension of the cell in the nested Hilbert scheme labeled by the pair $\mathfrak{i} \supset \mathfrak{j}$ equals

$$N(\mathfrak{i} \supset \mathfrak{j}) = \sum_{a=1}^{m} \#(\Gamma_{>i_a} \setminus \mathfrak{i}) + \sum_{a=m+1}^{r} \#(\Gamma_{>i_a} \setminus \mathfrak{j}) - \sum_{a=1}^{r} \#(\Gamma_{>s_a} \setminus \mathfrak{j}).$$

Let us relate the values of $N(\mathfrak{i})$ and $N(\mathfrak{i} \supset \mathfrak{j})$.

Lemma A.1 *The following identity holds:*

$$N(\mathfrak{i} \supset \mathfrak{j}) - N(\mathfrak{i}) = \sum_{a=m+1}^{r} \sum_{b=1}^{m} \chi(i_a < i_b) - \sum_{a=1}^{r} \sum_{b=1}^{m} \chi(s_a < i_b).$$

Proof Note that

$$N(\mathfrak{i} \supset \mathfrak{j}) - N(\mathfrak{i}) = \sum_{a=m+1}^{r} [\#(\Gamma_{>i_a} \setminus \mathfrak{j}) - \#(\Gamma_{>i_a} \setminus \mathfrak{i})] - \sum_{a=1}^{r} [\#(\Gamma_{>s_a} \setminus \mathfrak{j}) - \#(\Gamma_{>s_a} \setminus \mathfrak{i})].$$

It remains to remark that for every $x \in \Gamma$ one has

$$\#(\Gamma_{>x} \setminus \mathfrak{j}) - \#(\Gamma_{>x} \setminus \mathfrak{i}) = \sum_{a=1}^{m} \chi(x < i_a). \qquad \square$$

Theorem A.2 *Let*

$$\beta_b(\mathfrak{i}) := \sum_{a=1}^{r} \chi(i_a < i_b) - \sum_{a=1}^{r} \chi(s_a < i_b).$$

Then

$$\text{(A-1)} \qquad \sum_{m=0}^{r} z^m q^{\binom{m}{2}} \sum_{\#(\mathfrak{i} \setminus \mathfrak{j}) = m} q^{N(\mathfrak{i} \supset \mathfrak{j})} = q^{N(\mathfrak{i})} \prod_{b=1}^{r} (1 + z q^{\beta_b(\mathfrak{i})}).$$

Proof Note that, by Lemma A.1,

$$\begin{aligned}
&\binom{m}{2} + N(\mathfrak{i} \supset \mathfrak{j}) \\
&\quad = \sum_{a=1}^{m} \sum_{b=1}^{m} \chi(i_a < i_b) + N(\mathfrak{i}) + \sum_{a=m+1}^{r} \sum_{b=1}^{m} \chi(i_a < i_b) - \sum_{a=1}^{r} \sum_{b=1}^{m} \chi(s_a < i_b) \\
&\quad = N(\mathfrak{i}) + \sum_{a=1}^{r} \sum_{b=1}^{m} \chi(i_a < i_b) - \sum_{a=1}^{r} \sum_{b=1}^{m} \chi(s_a < i_b) \\
&\quad = N(\mathfrak{i}) + \sum_{b=1}^{m} \beta_b(\mathfrak{i}),
\end{aligned}$$

therefore

$$\sum_{m=0}^{r} z^m q^{\binom{m}{2}} \sum_{\#(\mathfrak{i}\setminus\mathfrak{j})=m} q^{N(\mathfrak{i}\supset\mathfrak{j})} = \sum_{m} \sum_{b_1,\dots,b_m} z^m q^{N(\mathfrak{i})+\beta_{b_1}(\mathfrak{i})+\cdots+\beta_{b_m}(\mathfrak{i})}$$
$$= q^{N(\mathfrak{i})} \prod_{b=1}^{r} (1 + z q^{\beta_b(\mathfrak{i})}). \qquad \square$$

The geometric meaning of Theorem A.2 is not known to the authors. However, it is quite useful for the computations with the nested Hilbert scheme.

Definition A.3 Let us call a number x an n–generator of $\mathfrak{i}$ if $x \in \mathfrak{i}$, but $x - n \notin \mathfrak{i}$.

Lemma A.4 *The number $\beta_b(\mathfrak{i})$ equals the number of n–generators of $\mathfrak{i}$ on $]i_b - k, i_b]$.*

Proof Note that if u is a n–generator of $\mathfrak{i}$ then there exist a unique (k,n)–generator i_a and a unique syzygy s_c such that $i_a = u - l_1 k$ and $s_c = u + l_2 k$. All elements of the form $i_a + lk$ are n–generators for $0 \le l < l_1 + l_2$. Now the statement follows from the equation

$$\chi(i_a < i_b) - \chi(s_a < i_b) = \sum_{l=0}^{l_1+l_2-1} \big(\chi(i_a + lk < i_b) - \chi(i_a + (l+1)k < i_b)\big)$$
$$= \sum_{l=0}^{l_1+l_2-1} \chi(i_a + lk \in]i_b - k, i_b]). \qquad \square$$

For semigroup ideals $\mathfrak{i} \supset \mathfrak{j}$ let us define $l(\mathfrak{i}) = \#(\Gamma \setminus \mathfrak{i})$ and $m(\mathfrak{i},\mathfrak{j}) = \#(\mathfrak{i} \setminus \mathfrak{j})$. Recall that the following formula is a corollary of Theorem 5:

$$\overline{\mathscr{P}}_{\mathrm{alg}}(T(p,q)) = \Big(\frac{a}{q}\Big)^{\mu(C)-1} \sum_{\mathfrak{i}\supset\mathfrak{j}} q^{2l} a^{2m} t^{m^2+2N(\mathfrak{i}\supset\mathfrak{j})}.$$

Here $\mu(C) = (k-1)(n-1)$ is the Milnor number of the corresponding singularity.

Applying Theorem A.2, we obtain the following result:

Corollary A.5 *We have*

$$\text{(A-2)} \qquad \overline{\mathscr{P}}_{\mathrm{alg}}(T(k,n)) = \Big(\frac{a}{q}\Big)^{\mu(C)-1} \sum_{\mathfrak{i}} q^{2l(\mathfrak{i})} t^{2N(\mathfrak{i})} \prod_{m=1}^{r} (1 + t^{2\beta_m(\mathfrak{i})-1} a^2).$$

A.2 Compactified Jacobian

In what follows we need some detailed information on the structure of the compactified Jacobian of a singularity with semigroup Γ, which will allow us to give a conjectural reformulation of the results of [49; 51].

Definition A.6 [59] Let Δ be a 0–normalized Γ–semimodule, ie $0 \in \Delta \subset \mathbb{Z}_{\geq 0}$, and $\Delta + \Gamma \subset \Delta$. The dimension of Δ is defined as

$$\dim \Delta = \sum_{j=0}^{n-1} \#([a_j, a_j + k[\setminus \Delta),$$

where $(0 = a_0 < a_1 < \cdots < a_{n-1})$ are the n–generators of Δ.

Theorem A.7 [59] *The Jacobi factor of the singularity with semigroup Γ admits the natural cell decomposition with affine cells C_Δ. The cells are parametrized by the 0–normalized Γ–semimodules Δ, and the dimension of the cell equals* $\dim \Delta$.

We will parametrize these cells by certain Young diagrams. Consider the $n \times k$ rectangle R and draw the diagonal from the top-left to the bottom-right corner.

Definition A.8 Let $R_+ \subset R$ be the subset consisting of boxes which lie under the left-top to right-bottom diagonal.

Label the boxes of R and around with integers, so that the shift by 1 up subtracts n, and the shift by 1 to the right subtracts k. We normalize these numbers so that kn is in the box $(0,0)$ (note that this box is not in the rectangle R, as we start enumerating boxes from 1). In other words, the numbers are given by the linear function $f(x,y) = kn - kx - ny$. One can see that the labels of the boxes of R_+ are positive, while all other numbers in R are negative. Moreover, numbers in the boxes of R_+ are exactly the numbers from the complement $\mathbb{Z}_{\geq 0} \setminus \Gamma$, and each such number appears only once in R_+. In particular, the area of R_+ is equal to $\delta = \frac{1}{2}(k-1)(n-1)$.

Definition A.9 [31] For a 0–normalized Γ–semimodule Δ, let $D(\Delta)$ denote the set of boxes with labels belonging to $\Delta \setminus \Gamma$.

Definition A.10 [47] Let D be a Young diagram, with $c \in D$. Let $a(c)$ and $l(c)$ denote the lengths of the arm and leg for c. For each real nonnegative x define

$$h_x^+(D) = \#\left\{ c \in D \;\middle|\; \frac{a(c)}{l(c)+1} \leq x < \frac{a(c)+1}{l(c)} \right\}.$$

The following theorem is the main result of [31].

Theorem A.11 *The dimensions of cells can be expressed through the h^+ statistic:*

$$\dim C_\Delta = \tfrac{1}{2}(k-1)(n-1) - h^+_{n/k}(D(\Delta)).$$

Conjecture A.12 *One can match the following generating functions for the Hilbert scheme of points and the compactified Jacobian:*

$$\text{(A-3)} \qquad \overline{\mathscr{P}}_{\mathrm{alg}}(a=0,q,t) = \sum_{\mathrm{i}} q^{2l(\mathrm{i})} t^{2N(\mathrm{i})} = \frac{1}{1-q^2} \sum_{D} q^{2|D|+2h^+_{n/k}(D)} t^{2|D|},$$

$$\text{(A-4)} \qquad \mathscr{P}_{\mathrm{alg}}(a=0,q,t) = \sum_{D} q^{2|D|+2h^+_{n/k}(D)} t^{2|D|}.$$

Remark This conjecture is expected to be the combinatorial counterpart of the generalized Macdonald formula (5), proved in [49; 51]. Namely, $|D(\Delta)| = \#(\Delta \setminus \Gamma)$ is expected to be related to the perverse filtration on the cohomology of the compactified Jacobian.

To formulate the analogous conjecture for the nested Hilbert scheme, we have to define the analogues of β–statistic for the admissible diagrams. Roughly speaking, we consider the complement to a diagram D as an ideal whose generators correspond to the SE corners of D and syzygies correspond to the ES corners of D. To get the corresponding semigroup ideal one has to replace a number x by $kn-k-n-x$, thus reversing the order.

Definition A.13 Consider a diagram D corresponding to a semigroup module Δ. Let P_m denote the numbers in the SE corners, and Q_i denote the numbers in the ES corners. Then

$$\beta(P_m) = \sum_i \chi(P_i > P_m) - \sum_i \chi(Q_i > P_m).$$

Example Consider a semigroup generated by 5 and 6, and a module

$$\Delta = \{0,1,2,5,6,\ldots\}.$$

Its diagram has the form:

4	**−2**			
9	**3**		**−9**	
14	8	2	**−4**	**−10**
19	13	7	1	**−5**

We have

$$\{P_i\} = \{-5, -4, 3, 4\}, \quad \{Q_j\} = \{-10, -9, -2\}.$$

Therefore,

$$\beta(-5) = 3 - 1 = 2, \quad \beta(-4) = 2 - 1 = 1, \quad \beta(3) = 1, \quad \beta(4) = 0.$$

Conjecture A.14 *One can match the following generating functions for the Hilbert scheme of points and the compactified Jacobian:*

$$\text{(A-5)} \quad \overline{\mathscr{P}}_{\mathrm{alg}}(a,q,t) = \sum_{\mathrm{i}} q^{2l(\mathrm{i})} t^{2N(\mathrm{i})} \prod_{m=1}^{r} (1 + t^{2\beta_m(\mathrm{i})-1} a^2)$$
$$= \frac{1+a^2 t}{1-q^2} \sum_{D} q^{2|D|+2h^+_{n/k}(D)} t^{2|D|} \prod_{m=1}^{r} (1 + a^2 q^{-2\beta(P_m)} t),$$

$$\text{(A-6)} \quad \mathscr{P}_{\mathrm{alg}}(a,q,t) = \sum_{D} q^{2|D|+2h^+_{n/k}(D)} t^{2|D|} \prod_{m=1}^{r} (1 + a^2 q^{-2\beta(P_m)} t).$$

A.3 Comparison of combinatorial statistics

Let us denote the right-hand side of Conjecture A.14 by $\mathscr{P}_{\mathrm{DAHA}}(T(n, mn+1))$. The bivariate polynomial

$$C_n^{(m)}(q,t) := \mathscr{P}_{\mathrm{DAHA}}(T(n, mn+1),\, a = 0)$$

was introduced by A Garsia and M Haiman [26] in connection with the conjectures of [38] on the structure of the module of diagonal harmonics, eventually proved in [39]. In a special case, the polynomials $C_n(q,t) := C_n^{(1)}(q,t)$ are called the (q,t)–Catalan numbers.

When $(p,q) = (n, n+1)$, the statistic $h^+_{(n+1)/n}(D)$ is also called $\mathrm{dinv}(D)$. Therefore, Theorem A.11 can be reformulated for this case as

$$\dim \Delta_D = \binom{n}{2} - \mathrm{dinv}(D).$$

Theorem A.15 [37] *The (q,t)–Catalan numbers admit the description*

$$C_n(q,t) = \sum_D q^{\mathrm{dinv}(D)} t^{\binom{n}{2}-|D|}. \tag{A-7}$$

Modulo Conjecture A.12, we obtain the identity

$$\mathscr{P}_{\mathrm{alg}}(T(n,n+1), a=0) = \mathscr{P}_{\mathrm{DAHA}}(T(n,n+1),\, a=0).$$

It has been conjectured in [46] that the analogue of (A-7) holds for the $(n, mn+1)$ case:

$$C_n^{(m)}(q,t) = \sum_D q^{h^+_{(mn+1)/n}(D)} t^{m\binom{n}{2}-|D|}. \tag{A-8}$$

Modulo this conjecture and Conjecture A.12, the identity

$$\mathscr{P}_{\mathrm{alg}}(T(n,mn+1),\, a=0) = \mathscr{P}_{\mathrm{DAHA}}(T(n,mn+1),\, a=0)$$

holds as well.

The combinatorial statistics for higher a–levels for the $(n,n+1)$ case were proposed in [23]. For their definition we will use a combinatorial bijection on Dyck paths, described in [37].

Definition A.16 Let Δ be an $(n,n+1)$–semimodule, and let $a_0,\dots,a_{n-1}$ be its n–generators. Define a Young diagram $G(\Delta)$ with columns $g(a_0),\dots,g(a_{n-1})$.

The following result describes the properties of the map G.

Theorem A.17 [31] *The following statements hold in the $(n,n+1)$ case:*

(1) *For any Δ, the diagram $G(\Delta)$ is below the diagonal.*

(2) *The correspondence between Δ and $G(\Delta)$ is bijective.*

(3) *This bijection coincides with the bijection from [37] exchanging dinv and area statistics with area and bounce statistics.*

Remark We conjecture that the map G is bijective in general.

Definition A.18 Let $b_i = n-1-i-g(a_i)$ be the number of cells in column i between the diagram $G(\Delta)$ and the diagonal.

The following theorem was conjectured in [23] and proved in [36].

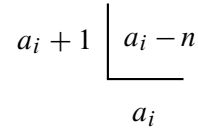
Theorem A.19 [36] *We have*

$$\mathscr{P}_{\mathrm{DAHA}}(T(n,n+1)) = \sum_D q^{\mathrm{dinv}(D)} t^{\binom{n}{2}-|D|} \prod_{b_i > b_{i+1}} (1 + a^2 q^{-b_i} t).$$

Two following lemmas show that this combinatorial formula is equivalent to (A-6).

Lemma A.20 *Let a_i and a_{i+1} be two consecutive n–generators of a Γ–semimodule Δ with a diagram D. Then the following statements are equivalent:*

$$b_i > b_{i+1} \iff g(a_i) = g(a_{i+1}) \iff a_i + 1 \in \Delta \iff a_i - n \text{ is an SE corner of } D.$$



Proof Since $g(a_i) \geq g(a_{i+1})$, one can check that

$$b_i > b_{i+1} \iff g(a_i) = g(a_{i+1}).$$

Let c be the maximal number such that $[a_i, c] \in \Delta$; suppose that $a_i < c < a_{i+1}$. Since c is not a n–generator, $c - n \in \Delta$ implies $c + 1 \in \Delta$, a contradiction; therefore $c = a_i$ or $c = a_{i+1}$.

In the first case let d be the maximal number such that $[a_i + 1, d] \cup \Delta = \varnothing$, then

$$d + 1 \in \Delta \implies d + n + 1 \in \Delta \implies g(a_{i+1}) > g(a_i),$$

a contradiction; therefore $c = a_{i+1}$ and $[a_i, a_{i+1}] \subset \Delta$. □

Lemma A.21 *The following relation holds:*

$$\beta(a_i - n) = b_i.$$

Proof By definition, $g(a_i) = \#([a_i, a_i + n[\setminus \Delta)$, so

$$n - g(a_i) = \#([a_i, a_i + n[\cap \Delta).$$

For every $j < i$ there exists a unique element of the form $a_j + kn$ on $[a_i, a_i + n]$, which are not n–generators. The remaining $n - 1 - i - g(a_i)$ elements are n–generators of Δ, hence the desired relation follows from Lemma A.4. □

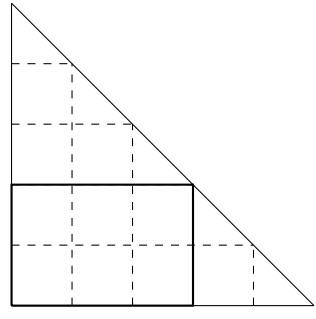
Example Let us return to the example on page 683:

4	**−2**			
9	**3**		**−9**	
14	8	2	**−4**	**−10**
19	13	7	1	**−5**

We showed that

$$\beta(-5) = 3 - 1 = 2, \quad \beta(-4) = 2 - 1 = 1, \quad \beta(3) = 1, \quad \beta(4) = 0.$$

The 5–generators corresponding to the internal corners are 0, 1 and 8. Since $g(0) = g(1) = g(2) = 2$, the diagram $G(\Delta)$ looks like



On the diagram $G(\Delta)$, we count $b(0) = 2$, $b(1) = 1$ and $b(8) = 1$.

Finally, since $|D| = 8$ and $h^+_{6/5} = 10 - |G(D)| = 4$, the contribution of this Γ–semimodule in (A-6) equals

$$q^{24}t^{16}(1 + a^2q^{-2}t)^2(1 + a^2q^{-4}t).$$

References

[1] **M Abel**, **M Hogancamp**, *Categorified Young symmetrizers and stable homology of torus links, II*, Selecta Math. 23 (2017) 1739–1801 MR

[2] **M Aganagic**, **S Shakirov**, *Knot homology from refined Chern–Simons theory*, preprint (2011) arXiv

[3] **A B Altman**, **A Iarrobino**, **S L Kleiman**, *Irreducibility of the compactified Jacobian*, from "Real and complex singularities" (P Holm, editor), Sijthoff and Noordhoff, Alphen aan den Rijn, Netherlands (1977) 1–12 MR

[4] **A B Altman**, **S L Kleiman**, *Compactifying the Picard scheme*, Adv. in Math. 35 (1980) 50–112 MR

[5] **V I Arnol'd**, **S M Guseĭn-Zade**, **A N Varchenko**, *Singularities of differentiable maps, II*, Monographs in Mathematics 83, Birkhäuser, Boston (1988) MR

[6] **A A Beĭlinson**, **J Bernstein**, **P Deligne**, *Faisceaux pervers*, from "Analysis and topology on singular spaces, I", Astérisque 100, Soc. Math. France, Paris (1982) 5–171 MR

[7] **Y Berest**, **P Etingof**, **V Ginzburg**, *Cherednik algebras and differential operators on quasi-invariants*, Duke Math. J. 118 (2003) 279–337 MR

[8] **Y Berest**, **P Etingof**, **V Ginzburg**, *Finite-dimensional representations of rational Cherednik algebras*, Int. Math. Res. Not. 2003 (2003) 1053–1088 MR

[9] **Y Berest**, **P Etingof**, **V Ginzburg**, *Morita equivalence of Cherednik algebras*, J. Reine Angew. Math. 568 (2004) 81–98 MR

[10] **R Bezrukavnikov**, **P Etingof**, *Parabolic induction and restriction functors for rational Cherednik algebras*, Selecta Math. 14 (2009) 397–425 MR

[11] **W Bruns**, **J Herzog**, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics 39, Cambridge Univ. Press (1993) MR

[12] **D Calaque**, **B Enriquez**, **P Etingof**, *Universal KZB equations: the elliptic case*, from "Algebra, arithmetic, and geometry: in honor of Yu I Manin, I" (Y Tschinkel, Y Zarhin, editors), Progr. Math. 269, Birkhäuser, Boston (2009) 165–266 MR

[13] **E Carlsson**, **A Mellit**, *A proof of the shuffle conjecture*, preprint (2015) arXiv

[14] **I Cherednik**, *Jones polynomials of torus knots via DAHA*, Int. Math. Res. Not. 2013 (2013) 5366–5425 MR

[15] **V I Danilov**, **A G Khovanskii**, *Newton polyhedra and an algorithm for computing Hodge–Deligne numbers*, Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986) 925–945 MR In Russian; translated in Math. USSR-Izv. 29 (1987) 279-298

[16] **P Deligne**, *Théorie de Hodge, I*, from "Actes du Congrès International des Mathématiciens" (M Berger, J Dieudonné, J Leray, J-L Lions, P Malliavin, J-P Serre, editors), volume 1, Gauthier-Villars, Paris (1971) 425–430 MR

[17] **P Deligne**, *Théorie de Hodge, II*, Inst. Hautes Études Sci. Publ. Math. 40 (1971) 5–57 MR

[18] **P Deligne**, *Théorie de Hodge, III*, Inst. Hautes Études Sci. Publ. Math. 44 (1974) 5–77 MR

[19] **P Deligne**, *Poids dans la cohomologie des variétés algébriques*, from "Proceedings of the International Congress of Mathematicians" (R D James, editor), volume 1, Canad. Math. Congress, Montreal (1975) 79–85 MR

[20] **N M Dunfield**, **S Gukov**, **J Rasmussen**, *The superpolynomial for knot homologies*, Experiment. Math. 15 (2006) 129–159 MR

[21] **P Dunin-Barkowski**, **A Mironov**, **A Morozov**, **A Sleptsov**, **A Smirnov**, *Superpolynomials for torus knots from evolution induced by cut-and-join operators*, J. High Energy Phys. (2013) art. id. 021 MR

[22] **A H Durfee**, *Algebraic varieties which are a disjoint union of subvarieties*, from "Geometry and topology" (C McCrory, T Shifrin, editors), Lecture Notes in Pure and Appl. Math. 105, Dekker, New York (1987) 99–102 MR

[23] **E S Egge**, **J Haglund**, **K Killpatrick**, **D Kremer**, *A Schröder generalization of Haglund's statistic on Catalan paths*, Electron. J. Combin. 10 (2003) art. id. R16 MR

[24] **B Elias**, **M Hogancamp**, *On the computation of torus link homology*, preprint (2016) arXiv

[25] **B Fantechi**, **L Göttsche**, **D van Straten**, *Euler number of the compactified Jacobian and multiplicity of rational curves*, J. Algebraic Geom. 8 (1999) 115–133 MR

[26] **A M Garsia**, **M Haiman**, *A remarkable q,t–Catalan sequence and q–Lagrange inversion*, J. Algebraic Combin. 5 (1996) 191–244 MR

[27] **V Ginzburg**, **N Guay**, **E Opdam**, **R Rouquier**, *On the category $\mathcal{O}$ for rational Cherednik algebras*, Invent. Math. 154 (2003) 617–651 MR

[28] **I Gordon**, **J T Stafford**, *Rational Cherednik algebras and Hilbert schemes*, Adv. Math. 198 (2005) 222–274 MR

[29] **I Gordon**, **J T Stafford**, *Rational Cherednik algebras and Hilbert schemes, II: Representations and sheaves*, Duke Math. J. 132 (2006) 73–135 MR

[30] **E Gorsky**, *q,t–Catalan numbers and knot homology*, from "Zeta functions in algebra and geometry" (A Campillo, G Cardona, A Melle-Hernández, W Veys, editors), Contemp. Math. 566, Amer. Math. Soc., Providence, RI (2012) 213–232 MR

[31] **E Gorsky**, **M Mazin**, *Compactified Jacobians and q,t–Catalan numbers, I*, J. Combin. Theory Ser. A 120 (2013) 49–63 MR

[32] **E Gorsky**, **A Neguț**, *Refined knot invariants and Hilbert schemes*, J. Math. Pures Appl. 104 (2015) 403–435 MR

[33] **E Gorsky**, **A Neguț**, **J Rasmussen**, *Flag Hilbert schemes, colored projectors and Khovanov–Rozansky homology*, preprint (2016) arXiv

[34] **E Gorsky**, **A Oblomkov**, **J Rasmussen**, **V Shende**, *Torus knots and the rational DAHA*, Duke Math. J. 163 (2014) 2709–2794 MR

[35] **J Haglund**, *Conjectured statistics for the q,t–Catalan numbers*, Adv. Math. 175 (2003) 319–334 MR

[36] **J Haglund**, *A proof of the q,t–Schröder conjecture*, Int. Math. Res. Not. 2004 (2004) 525–560 MR

[37] **J Haglund**, *The q,t–Catalan numbers and the space of diagonal harmonics*, University Lecture Series 41, Amer. Math. Soc., Providence, RI (2008) MR

[38] **M D Haiman**, *Conjectures on the quotient ring by diagonal invariants*, J. Algebraic Combin. 3 (1994) 17–76 MR

[39] **M Haiman**, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. 149 (2002) 371–407 MR

[40] **R Hartshorne**, *Generalized divisors on Gorenstein curves and a theorem of Noether*, J. Math. Kyoto Univ. 26 (1986) 375–386 MR

[41] **M Hogancamp**, *Categorified Young symmetrizers and stable homology of torus links*, preprint (2015) arXiv

[42] **M Hogancamp**, *Khovanov–Rozansky homology and higher Catalan sequences*, preprint (2017) arXiv

[43] **T Kálmán**, *Meridian twisting of closed braids and the Homfly polynomial*, Math. Proc. Cambridge Philos. Soc. 146 (2009) 649–660 MR

[44] **M Khovanov**, *Triply-graded link homology and Hochschild homology of Soergel bimodules*, Internat. J. Math. 18 (2007) 869–885 MR

[45] **M Khovanov**, **L Rozansky**, *Matrix factorizations and link homology, II*, Geom. Topol. 12 (2008) 1387–1425 MR

[46] **N A Loehr**, *Conjectured statistics for the higher q,t–Catalan sequences*, Electron. J. Combin. 12 (2005) art. id. R9 MR

[47] **N A Loehr**, **G S Warrington**, *A continuous family of partition statistics equidistributed with length*, J. Combin. Theory Ser. A 116 (2009) 379–403 MR

[48] **I G Macdonald**, *Symmetric products of an algebraic curve*, Topology 1 (1962) 319–343 MR

[49] **D Maulik**, **Z Yun**, *Macdonald formula for curves with planar singularities*, J. Reine Angew. Math. 694 (2014) 27–48 MR

[50] **A Mellit**, *Toric braids and (m,n)–parking functions*, preprint (2016) arXiv

[51] **L Migliorini**, **V Shende**, *A support theorem for Hilbert schemes of planar curves*, J. Eur. Math. Soc. 15 (2013) 2353–2367 MR

[52] **J Milnor**, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies 61, Princeton Univ. Press (1968) MR

[53] **B C Ngô**, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. 111 (2010) 1–169 MR

[54] **A Oblomkov**, **L Rozansky**, *Knot homology and sheaves on the Hilbert scheme of points on the plane*, preprint (2016) arXiv

[55] **A Oblomkov**, **L Rozansky**, *Affine braid group, JM elements and knot homology*, preprint (2017) arXiv

[56] **A Oblomkov**, **V Shende**, *The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link*, Duke Math. J. 161 (2012) 1277–1303 MR

[57] **A Oblomkov**, **Z Yun**, *Geometric representations of graded and rational Cherednik algebras*, Adv. Math. 292 (2016) 601–706 MR

[58] **R Pandharipande**, **R P Thomas**, *Stable pairs and BPS invariants*, J. Amer. Math. Soc. 23 (2010) 267–297 MR

[59] **J Piontkowski**, *Topology of the compactified Jacobians of singular curves*, Math. Z. 255 (2007) 195–226 MR

[60] **J Rasmussen**, *Some differentials on Khovanov–Rozansky homology*, Geom. Topol. 19 (2015) 3031–3104 MR

[61] **M Stošić**, *Homological thickness and stability of torus knots*, Algebr. Geom. Topol. 7 (2007) 261–284 MR

[62] **M Varagnolo**, **E Vasserot**, *Finite-dimensional representations of DAHA and affine Springer fibers: the spherical case*, Duke Math. J. 147 (2009) 439–540 MR

[63] **Z Yun**, *Global Springer theory*, Adv. Math. 228 (2011) 266–328 MR

[64] **O Zariski**, *Le problème des modules pour les branches planes*, 2nd edition, Hermann, Paris (1986) MR

Department of Mathematics and Statistics, University of Massachusetts
Amherst, MA, United States

DPMMS, University of Cambridge
Cambridge, United Kingdom

Department of Mathematics, University of California
Berkeley, CA, United States

Department of Mathematics, University of California
Davis, CA, United States

oblomkov@math.umass.edu, jar60@dpmms.cam.ac.uk,
vivek@math.berkeley.edu, egorskiy@math.ucdavis.edu

Proposed: Peter Ozsváth
Seconded: Lothar Göttsche, Gang Tian
Received: 14 September 2012
Accepted: 20 April 2017