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Detecting periodic elements in higher topological Hochschild homology

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Given a commutative ring spectrum R, let $\Lambda_X R$ be the Loday functor constructed by Brun, Carlson and Dundas. Given a prime $p \geq 5$, we calculate $\pi_*(\Lambda_{S^n} H\mathbb{F}_p)$ and $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$ for $n \leq p$, and use these results to deduce that v_{n-1} in the $(n-1)^{\text{st}}$ connective Morava K-theory of $(\Lambda_{T^n} H\mathbb{F}_p)^{hT^n}$ is nonzero and detected in the homotopy fixed-point spectral sequence by an explicit element, whose class we name the Rognes class.

To facilitate these calculations, we introduce multifold Hopf algebras. Each axis circle in T^n gives rise to a Hopf algebra structure on $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$, and the way these Hopf algebra structures interact is encoded with a multifold Hopf algebra structure. This structure puts several restrictions on the possible algebra structures on $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$ and is a vital tool in the calculations above.

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1 Introduction

Topological Hochschild homology of an orthogonal commutative ring spectrum R can be defined as the tensor $S^1 \otimes R$ in the category of orthogonal commutative ring spectra; see McClure, Schwänzl and Vogt [25], and Mandell and May [22]. Several people have put a lot of effort into computing the homotopy groups of topological Hochschild homology of various ring spectra. In [8], Bökstedt calculates the homotopy groups of the THH of the Eilenberg–Mac Lane spectra $H\mathbb{F}_p$ and $H\mathbb{Z}$; in [26], McClure and Staffeldt calculate the mod p homotopy groups of the THH of the Adams summand ℓ; in [3], Ausoni calculates the mod v_1 homotopy groups of the THH of connective complex K-theory and in [1], Angeltveit, Hill and Lawson calculate the integral homotopy groups of $\mathrm{THH}(\ell)$ and the 2–local homotopy groups of $\mathrm{THH}(ko)$.

Let X be a space, and write $\Lambda_X R$ for the Loday functor, defined in Definition 2.1, first defined for Γ–spaces by Brun, Carlsson and Dundas [10] and then defined for orthogonal spectra by Stolz [31] and Brun, Dundas and Stolz [11]. If G is a compact

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group, and X is a G–space, then $\Lambda_X R$ is a G–spectrum which is G–equivariant equivalent to $R \otimes X$, the categorical tensor, when using the $\mathbb{S}$–model structure from [11]. We will be interested in the case when both X and G are tori.

We write $L_*(X)$ for the graded ring $\pi_*(\Lambda_X H\mathbb{F}_p)$. Iterated topological Hochschild homology of $H\mathbb{F}_p$ is then isomorphic to $L_*(T^n)$, where T^n is the n–fold pointed torus.

We calculate $L_*(T^n)$ using the bar spectral sequences associated with the cofibration given by attaching the top cell in T^n. A first step is thus to calculate $L_*(S^n)$.

Let p be an odd prime, $B_0 = \mathbb{F}_p$, $B_1 = P(\mu)$ the polynomial algebra over $\mathbb{F}_p$ on a generator of degree 2, and for $n \geq 2$, we recursively define $B_n = \operatorname{Tor}^{B_{n-1}}(\mathbb{F}_p, \mathbb{F}_p)$, where the grading in B_n is given by the total grading in Tor.

In Theorem 3.6, we prove:

Theorem 1.1 *When $n \leq 2p$, there is an $\mathbb{F}_p$–Hopf algebra isomorphism*

$$L_*(S^n) \cong B_n.$$

Let $\boldsymbol{n}$ denote the set $\{1, \dots, n\}$ of natural numbers.

In Theorem 6.2, we prove:

Theorem 1.2 *Given $1 \leq n \leq p$ when $p \geq 5$ and $1 \leq n \leq 2$ when $p = 3$, there is a graded $\mathbb{F}_p$–algebra isomorphism*

$$L_*(T^n) \cong \bigotimes_{U \subseteq \boldsymbol{n}} B_{|U|}.$$

The fold and pinch maps on each circle factor in T^n produces n different $L_*(T^{n-1})$–Hopf algebra structures on $L_*(T^n)$.

Let $V(\boldsymbol{n})$ be the category with objects given by subsets of $2^{\boldsymbol{n}}$ and morphisms from U to V given by subsets of $U \cap V$, where composition is intersection. Define the functor $\mathcal{L}\colon V(\boldsymbol{n}) \to \mathrm{CRings}$, where CRings is the category of commutative rings, by mapping $U \subset \boldsymbol{n}$ to $\mathcal{L}(U) = L_*(T^U)$, and sending a morphism $W \subset U \cap V$ to $L_*(T^U \to T^W \to T^V)$, where the first map is projection and the second is inclusion.

The functor $\mathcal{L}$ has the structure of a multifold Hopf algebra as introduced in Section 4. This follows from the fact that the all maps from T^n to $(S^1 \vee S^1)^{\times n}$ given by pinching each circle in T^n once are homotopic. The notion of simultaneously primitive elements is a generalization of primitive elements to multifold Hopf algebras; they are elements which are primitive in all the Hopf algebra structures.

The calculation of $L_*(T^n)$ is a double induction proof on the dimension of T^n and the degree of $L_*(T^n)$. Similarly to Hopf algebras, simultaneously primitive elements limit the possible nonzero differentials in the bar spectral sequence calculating $L_*(T^n)$, and using this it is shown to collapse at the E^2–page, giving the $\mathbb{F}_p$–module structure. Furthermore, the possible algebra structures on $L_*(T^n)$ are limited by the simultaneously primitive elements helping us identifying the $\mathbb{F}_p$–algebra structure.

The redshift conjecture predicts that under favorable circumstances, algebraic K-theory increases "telescopic complexity". See the introduction to Ausoni and Rognes [4] and [5] and Baas, Dundas and Rognes [6], and also Rognes [29] for a wider perspective. The simplest sequence of examples that should display this phenomenon for all complexities is the following: for a given prime p, the iterated algebraic K-theory $K^{(n)}(\mathbb{F}_p)$ should have telescopic complexity $n-1$. In particular, if $k(n-1)$ is the $(n-1)^{\mathrm{st}}$ connective Morava K-theory with coefficient ring $k(n-1)_* = \mathbb{F}_p[v_{n-1}]$, the redshift conjecture predicts that the element $v_{n-1} \in k(n-1)_* K^{(n)}(\mathbb{F}_p)$ is not a zero divisor.

Most of the evidence for redshift stems from trace methods: according to Bökstedt, Hsiang and Madsen [9], the trace $K(A) \to \mathrm{THH}(A) = \Lambda_{S^1}(A)$ factors through the inclusion of fixed points under the action of subgroups of S^1. Since these fixed points provide a very close approximation to $K(A)$ (see Dundas, Goodwillie and McCarthy [15]), it is reasonable to hope that chromatic behavior for $K^{(n)}(\mathbb{F}_p)$ is reflected in similar behavior for fixed points of the iterates $\mathrm{THH}^{(n)}(\mathbb{F}_p) = \Lambda_{T^n} H\mathbb{F}_p$. See the introduction to Carlsson, Douglas and Dundas [13] for more details.

Theorem 7.9 below is an indication that this is indeed so. While we are not able to establish that v_{n-1} is not a zero divisor, we are able to show it is nonzero in an unprecedented range.

Theorem 1.3 *Let $p \geq 5$ and $1 \leq n \leq p$ or $p = 3$ and $1 \leq n \leq 2$. The unit*

$$k(n-1)_* \xrightarrow{v_{n-1}} k(n-1)_*(\Lambda_{T^n} H\mathbb{F}_p)^{hT^n}$$

maps v_{n-1} to a nontrivial class.

We call this class *the Rognes class*.

The proof shows a specific element in the homotopy fixed-point spectral sequence, called the Rognes element, is not hit by any differential, is a cycle and is the image of v_{n-1}. The only differential that might hit it is the d^2–differential, which is induced by the various circle actions on the factors in T^n. This is not possible due to there being n circle factors, but only $n-1$ odd-dimensional generators $\overline{\tau}_i$ in $k(n-1)_*$ of degree less than the degree of v_{n-1}.

It should be possible to generalize the calculation of $L_*(T^n)$ to a calculation of the mod p homotopy groups $V(0)_*(\Lambda_{T^n} H\mathbb{Z})$ and possibly to the mod v_1 homotopy groups $V(1)_*(\Lambda_{T^n}\ell)$ in some range which depends on p.

Organization Section 2 recalls some results about spectra and the Loday functor, and in Section 3, we explicitly calculate $L_*(S^n)$ for $n \leq 2p$. Section 4 introduces multifold Hopf algebras, and in Section 5, we prove that the structure of a multifold Hopf algebra restricts the possible coalgebra structures that can appear in $L_*(T^n)$. In Section 6, we calculate $L_*(T^n)$ for $n \leq p$ when $p \geq 5$ and for $n \leq 2$ when $p = 3$.

In Section 7, we show that there is an element in the second column of the homotopy fixed-point spectral sequence calculating $k(n-1)_*((\Lambda_{T^n} H\mathbb{F}_p)^{hT^n})$ that is a cycle and not a boundary, and represents v_{n-1}.

The rest of the sections contain several technical results which have been moved out of the main sections to improve the flow of the arguments.

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2 The Loday functor

We will work in the category of orthogonal spectra. See Hill, Hopkins and Ravenel [19] for details. Mandell, May, Schwede and Shipley [23] prove that the category of orthogonal commutative ring spectra is enriched over topological spaces and is tensored and cotensored.

We recall the definition of the Loday functor given in Definition 4.3.9 of [11]. The Loday functor was originally stated for Γ–spaces in [10], but restated in simpler terms for orthogonal ring spectra in Martin Stoltz's thesis [31], on which [11] is heavily based. We will use the $\mathbb{S}$–model structure on orthogonal spectra from [11] as this gives us nice equivariant properties as explained below Definition 2.1.

Definition 2.1 Given a space X and a commutative ring spectrum R, we define

$$\Lambda_X R = R \otimes X,$$

where $R \otimes X$ is the categorical tensor in orthogonal spectra.

If G is a compact Lie group and X is a G–space, then G acts on $R \otimes X$ through the action on X. When using the $\mathbb{S}$–model structure, this is the action we are interested in, the one used to calculate, among other things, TC, topological cyclic homology in the case of $G = S^1$.

When X is a simplicial space, Lemma 4.3.10 in [11] gives us a natural isomorphism $\Lambda_{|X|} R \cong |\Lambda_X R|$, where the realization on the right is in orthogonal spectra.

Proposition 2.2 *The Loday functor has the following properties:*

(1) *A weak equivalence $X \to Y$ of simplicial sets induces a weak equivalence $\Lambda_X R \to \Lambda_Y R$.*
(2) *Given a simplicial set Y, there is a natural equivalence $\Lambda_X(\Lambda_Y R) \simeq \Lambda_{X\times Y} R$.*
(3) *Given a cofibration $L \to X$ and a map $L \to K$ between simplicial sets, there is an equivalence $\Lambda_{X \amalg_L K} R \simeq \Lambda_X R \wedge_{\Lambda_L R} \Lambda_K R$.*

Proof Statements 1 and 2 follow from the definition of the Loday functor, and statement 3 follows from the fact that the tensor commutes with colimits. □

The following two Hopf algebra structures will be essential for our calculations.

Proposition 2.3 *Let $n \geq 1$, let R be a commutative ring spectrum, and assume that $\pi_*(\Lambda_{S^n} R)$ is flat as a $\pi_*(R)$–module. Then $\pi_*(\Lambda_{S^n} R)$ is an $\pi_*(R)$–Hopf algebra with unit and counit induced by choosing a base point in S^n and by collapsing S^n to a point, respectively. The multiplication and coproduct are induced by the fold map $\nabla\colon S^n \vee S^n \to S^n$ and the pinch map $\psi\colon S^n \to S^n \vee S^n$ given by collapsing a chosen equator through the basepoint, respectively, and the conjugation map is induced by the reflection map $-\mathrm{id}\colon S^n \to S^n$.*

Proof By Proposition 2.2(3), $\Lambda_{S^n \vee S^n} R \simeq \Lambda_{S^n} R \wedge_R \Lambda_{S^n} R$, and since $\pi_*(\Lambda_{S^n} R)$ is flat as a $\pi_*(R)$–module, $\pi_*(\Lambda_{S^n} R \wedge_R \Lambda_{S^n} R) \cong \pi_*(\Lambda_{S^n} R) \wedge_{\pi_*(R)} \pi_*(\Lambda_{S^n} R)$ by Proposition 8.2. That the various diagrams in the definition of a $\pi_*(R)$–Hopf algebra commute now follows from commutativity of the corresponding diagrams on the level of simplicial sets. □

Given a map of spaces $f\colon X \to Y$, we will write f for both the induced maps $\Lambda_f H\mathbb{F}_p\colon \Lambda_X H\mathbb{F}_p \to \Lambda_Y H\mathbb{F}_p$ and $L_*(f)\colon L_*(X) \to L_*(Y)$ when there is no room for confusion.

Proposition 2.4 *Let U be a finite set and let T^U be the U–fold torus. For each $u \in U$, if $L_*(T^U)$ is flat as an $L_*(T^{U\setminus u})$–module, then $(L_*(T^U), L_*(T^{U\setminus u}))$ is a commutative Hopf algebra where we have*

1. *multiplication induced by the fold map $T^U \amalg_{T^{U\setminus u}} T^U \cong T^{U\setminus u} \times (S^1 \vee S^1) \to T^U$,*
2. *the coproduct induced by the pinch map $S^1 \to S^1 \vee S^1$ on the u^{th} circle in T^U,*
3. *the unit map induced by the inclusion $U \setminus u \to U$, and*
4. *the counit map induced by collapsing the u^{th} circle in T^U to its basepoint.*

Proof Since $\Lambda_{T^U} H\mathbb{F}_p \cong \Lambda_{S^1} \Lambda_{T^{U\setminus u}} H\mathbb{F}_p$, this follows from Proposition 2.3. □

We now set the notation for two maps which are used throughout the article.

Definition 2.5 Let X be a simplicial set, R a commutative ring spectrum and $\bigvee_{x\in X} R$ the $|X|$–fold wedge sum of R indexed by the elements in X. Each element $x \in X$ induces a map $\Lambda_x R \to \Lambda_X R$, and these maps assemble to a natural map

$$\omega_X\colon X_+ \wedge R \cong \bigvee_{x\in X} R \cong \bigvee_{x\in X} \Lambda_x R \to \Lambda_X R.$$

Let Y be a simplicial set. Composing $\omega_X\colon X_+ \wedge \Lambda_Y R \to \Lambda_{X\times Y} R$ with the map induced by the quotient map $X \times Y \to X \times Y/(X \vee Y) \cong X \wedge Y$ yields a natural map

$$\widehat{\omega}_X\colon X_+ \wedge \Lambda_Y R \to \Lambda_{X\wedge Y} R.$$

The map ω_X was first constructed in Section 5 of [25].

Definition 2.6 Composing the maps ω_{S^1} and $\widehat{\omega}_{S^1}$ with a chosen stable splitting $S^1_+ \simeq S^1 \vee S^0$ induces maps in homotopy

$$\begin{aligned} \pi_*(S^1 \wedge R) &\cong \quad H_*(S^1) \otimes \pi_*(R) \quad \to \pi_*(\Lambda_{S^1} R),\\ \pi_*(S^1 \wedge \Lambda_Y R) &\cong H_*(S^1) \otimes \pi_*(\Lambda_Y R) \to \pi_*(\Lambda_{S^1\wedge Y} R). \end{aligned}$$

We define maps

$$\begin{aligned} \sigma\colon \pi_*(R) &\to \pi_*(\Lambda_{S^1} R),\\ \sigma\colon \pi_*(\Lambda_Y R) &\to \pi_*(\Lambda_{S^1\wedge Y} R), \end{aligned}$$

by mapping $z \in \pi_*(R)$ and $y \in \pi_*(\Lambda_Y R)$ to the image of $[S^1] \otimes z$ and $[S^1] \otimes y$ under ω_{S^1} and $\widehat{\omega}_{S^1}$, where $[S^1]$ is a chosen generator of $\widetilde{H}_1(S^1)$.

The following statement is well known and is proven by Angeltveit and Rognes for homology, but the same proof works for homotopy.

Proposition 2.7 [2, Proposition 5.10] *Let R be a commutative ring spectrum. Then $\sigma\colon \pi_*(R) \to \pi_*(\Lambda_{S^1} R)$ is a graded derivation of degree 1; ie for $x, y \in \pi_* R$,*

$$\sigma(xy) = \sigma(x)y + (-1)^{|x|} x\sigma(y).$$

Similarly, the composite $\sigma\colon \pi_*(\Lambda_{S^1} R) \to \pi_*(\Lambda_{S^1\times S^1} R) \to \pi_*(\Lambda_{S^1} R)$, where the last map is induced by the multiplication in S^1, is also a derivation.

Proposition 2.8 *Let R be a commutative ring spectrum, and assume that $\pi_*(\Lambda_{S^1} R)$ is flat as a $\pi_*(R)$–module. If z is in $\pi_*(R)$, then $\sigma(z)$ is primitive in the $\pi_*(R)$–Hopf algebra $\pi_*(\Lambda_{S^1} R)$.*

Proof The diagram

$$\begin{tikzcd} S^1_+ \wedge R \arrow[r, "\omega"] \arrow[d, "\psi_+\wedge \mathrm{id}"] & \Lambda_{S^1} R \arrow[d, "\Lambda_\psi R"] \\ (S^1 \vee S^1)_+ \wedge R \arrow[r, "\omega"] & \Lambda_{S^1\vee S^1} R \end{tikzcd} \tag{2.9}$$

commutes. Hence, $\psi(\sigma(z)) = \sigma(z) \otimes 1 + 1 \otimes \sigma(z)$. □

3 Calculating the homotopy groups of $\Lambda_{S^n} H\mathbb{F}_p$

In this section, we will calculate $L_*(S^n)$ when $n \leq 2p$ and p is odd. First we describe an $\mathbb{F}_p$–Hopf algebra B_n, and then we show that $L_*(S^n) \cong B_n$.

The calculations of B_n were first done in the calculations of the Eilenberg–Mac Lane spaces for $\mathbb{F}$ by Cartan [14].

Definition 3.1 Given the letters μ, ϱ, ϱ^k and φ^k for $k \geq 0$, define an *admissible word* to be a word such that:

(1) it ends with the letter μ;

(2) if μ is preceded by a letter, it must be ϱ;

(3) if ϱ is preceded by a letter, it must be ϱ^k;

(4) if ϱ^k or φ^k is preceded by a letter, it must be ϱ or φ^l for some $l \geq 0$.

We define a *monic word* to be an admissible word that begins with one of the letters ϱ, ϱ^0, φ^0 or μ.

We define the degree of μ to be 2 and recursively define the degree of an admissible word by the rules

$$|\varrho x| = 1 + |x|,$$
$$|\varrho^k x| = p^k(1 + |x|),$$
$$|\varphi^k x| = p^k(2 + p|x|).$$

An example of an admissible word of length 6 is $\varrho\varphi^m\varphi^l\varrho^k\varrho\mu$.

Lemma 3.2 (1) *An admissible word of length at least 3 always ends with the letter combination $\varrho^k\varrho\mu$.*

(2) *There are at most $\frac{1}{2}(n-1)$ occurrences of the letter ϱ in an admissible word of even degree of length n.*

(3) *Every admissible word of length n has degree at least $n+1$.*

(4) *All admissible words of odd degree begin with the letter ϱ.*

(5) *Let $0 \le k < p$. A monic word of degree $2k$ modulo $2p$ is either equal to $(\varrho^0\varrho)^{k-1}\mu$, or it starts with the letter combination $(\varrho^0\varrho)^{k-1}\varphi^0$ or $(\varrho^0\varrho)^k$. A monic word of degree $2k+1$ modulo $2p$ is either equal to $\varrho(\varrho^0\varrho)^{k-1}\mu$, or it starts with the letter combination $\varrho(\varrho^0\varrho)^{k-1}\varphi^0$ or $\varrho(\varrho^0\varrho)^k$.*

Proof All but the last statement is obvious. The last statement follows from the observation that the degree of a word starting with φ^l or $\varrho^l\varrho$ is 0 modulo $2p$ for $l \ge 1$, and the degree of a word starting with φ^0 is 2 modulo $2p$. □

Let R be a commutative ring, and let x and y be of even and odd degree, respectively. We let $P_R(x)$ be the polynomial ring over R and let $E_R(y)$ be the exterior algebra over R. When R is clear from the setup, we will often leave it out of the notation and write $P_p(x) = P(x)/(x^p)$ for the truncated polynomial ring. Furthermore, we let $\Gamma(x)$ be the divided power algebra over R which, as an R–module, is generated by the elements $\gamma_i(x)$ in degrees $i|x|$ for $i \ge 0$, with R–algebra structure given by $\gamma_i(x)\gamma_j(x) = \binom{i+j}{j}\gamma_{i+j}(x)$ and R–coalgebra structure given by $\psi(\gamma_k(x)) = \sum_{i+j=k}\gamma_i(x)\otimes\gamma_j(x)$.

Definition 3.3 We define B_1 to be the polynomial $\mathbb{F}_p$–Hopf algebra $P(\mu)$ with $|\mu| = 2$. Given $n \ge 2$, we define the $\mathbb{F}_p$–Hopf algebra B_n to be equal to the tensor product of exterior algebras on all monic words of length n of odd degrees, and divided power algebras on all monic words of length n of even degrees. For the divided power algebra structure on B_n, we will write $\varrho^k x = \gamma_{p^k}(\varrho^0 x)$ and $\varphi^k x = \gamma_{p^k}(\varphi^0 x)$, where x is an admissible word of length $n-1$ and of odd and even degree, respectively.

For example, the monic words of length 4 are $\varrho\varrho^k\varrho\mu$ and $\varphi^0\varrho^k\varrho\mu$ for $k \geq 0$. Hence, $B_4 = \bigotimes_{k\geq 0}(E(\varrho\varrho^k\varrho\mu) \otimes \Gamma(\varphi\varrho^k\varrho\mu))$.

Proposition 3.4 *When $n \geq 2$, there is an isomorphism of $\mathbb{F}_p$–Hopf algebras*

$$B_n \cong \mathrm{Tor}^{B_{n-1}}(\mathbb{F}_p, \mathbb{F}_p).$$

The map $\sigma\colon B_{n-1} \to B_n$ is determined by $\sigma(x) = \varrho x$ and $\sigma(x) = \varrho^0 x$ when x is an admissible word of even and odd degree, respectively.

Proof This is classical. For a proof, see McCleary [24, Proposition 7.24]. □

Before we calculate $L(S^n)$, we state a technical lemma which is needed in the proof. Given a graded module A, we will write A_i for the part in degree i.

Lemma 3.5 *Let $P(B_n)$ be the submodule of primitive elements in B_n. If $2 \leq n \leq 2p$, then $P(B_n)_{2pi-1} = P(B_n)_{2pi} = 0$ for all $i \geq 2$.*

Proof In a divided power algebra $\Gamma(x)$, the only primitive elements are nonzero scalar multiples of $\gamma_1(x)$, so by the graded version of Milnor and Moore's [28, Proposition 3.12] and Proposition 3.4, the primitive elements in B_n are linear combinations of monic words of length n.

We will show that the shortest monic word in degree 0 modulo $2p$, and of degree greater than $2p$, has length $2p+2$.

By Lemma 3.2(5), a monic word of degree 0 modulo $2p$ must either be equal to $(\varrho^0\varrho)^{p-1}\mu$ or start with the letter combination $(\varrho^0\varrho)^{p-1}\varphi^0$ or $(\varrho^0\varrho)^p$.

The word $(\varrho^0\varrho)^{p-1}\mu$ has degree $2p$, so the shortest monic word in degree 0 modulo $2p$ of degree greater than $2p$, is thus $(\varrho^0\varrho)^{p-1}\varphi^0\varrho^k\varrho\mu_0$ for $k \geq 1$, and it has length $2p+2$.

By a similar argument, we get that the shortest monic word in degree -1 modulo $2p$ of degree greater than $2p$, is $\varrho(\varrho^0\varrho)^{p-2}\varphi^0\varrho^k\varrho\mu_0$ for $k \geq 1$, and it has length $2p+1$. □

Applying the Loday functor $L_*(-)$ to the cofiber sequence

$$S^{n-1} \to D^n \to S^n$$

gives rise to a bar spectral sequence

$$E^2(S^n) = \mathrm{Tor}^{L_*(S^{n-1})}(\mathbb{F}_p, \mathbb{F}_p) \implies L_*(S^n)$$

by Propositions 2.2 and 8.2. The spectral sequence is indexed such that the differentials are of the form $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$. The differentials are only given up to multiplication with a unit.

For an abelian group G and $m > 1$, the bar construction of the Eilenberg–Mac Lane space $K(G,m)$ is equivalent to $K(G,m+1)$, and the spectral sequence $E^2(S^n)$ is analogous to the Eilenberg–Moore spectral sequence calculating $K(G,m+1)$ from $K(G,m)$.

The pinch map ψ induces vertical maps of cofiber sequences

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, "\psi"] & D^n \arrow[r] \arrow[d] & S^n \arrow[d, "\psi"] \\
S^{n-1} \vee S^{n-1} \arrow[r] & D^n \vee D^n \arrow[r] & S^n \vee S^n
\end{tikzcd}$$

and this in combination with the reflection map on S^n gives a map of simplicial spectra

$$B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, H\mathbb{F}_p) \\ \to B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, H\mathbb{F}_p)$$

that endows this spectral sequence with a $\mathbb{F}_p$–Hopf algebra structure as explained in Proposition 8.4. Flatness is no problem since $\mathbb{F}_p$ is a field.

Theorem 3.6 *When $n \le 2p$, there are no differentials in the spectral sequence $E^*(S^n)$, and there is an $\mathbb{F}_p$–Hopf algebra isomorphism*

$$L_*(S^n) \cong B_n.$$

Proof The proof is by induction on n. Bökstedt [8] showed that $L_*(S^1) \cong P(\mu) = B_1$.

Assume we have proved the theorem for $n-1$. The bar spectral sequence then becomes

$$E^2(S^n) = \mathrm{Tor}^{B_{n-1}}(\mathbb{F}_p, \mathbb{F}_p) \cong B_n \implies L_*(S^n).$$

By Proposition 8.6, the shortest differential in lowest total degree goes from an indecomposable element to a primitive element. We have $E^2(S^n)_{0,*} \cong \mathbb{F}_p$, so the indecomposable elements in B_n that can support differentials are generated by $\varrho^k w$ and $\varphi^k w$ with $k \geq 1$, where w is some admissible word of length $n-1$. By Lemma 3.2(3), these elements are all in degrees greater than or equal to $4p$ and equal to 0 modulo $2p$ since $k \geq 1$. Thus if z is an indecomposable element, $d^r(z)$ is in degree -1 modulo $2p$ and greater than or equal to $4p-1$. By Lemma 3.5, there are no primitive elements in these degrees when $n \leq 2p$, so there are no differentials in the spectral sequence. Hence, $E^2(S^n) = E^\infty(S^n)$.

To solve the multiplicative extensions, we must determine $(\varrho^k w)^p$ and $(\varphi^k w)^p$ for all $k \geq 0$ and w an admissible word of length $n-1$.

Assume z is one of the generators $\varrho^k w$ or $\varphi^k w$ of lowest total degree with $z^p \neq 0$. Then since the p^{th} powers of primitives are primitive,

$$\begin{aligned}\psi(z^p) = \psi(z)^p &= (1\otimes z + z\otimes 1 + \textstyle\sum z'\otimes z'')^p \\ &= 1\otimes z^p + z^p\otimes 1 + \textstyle\sum (z')^p\otimes (z'')^p \\ &= 1\otimes z^p + z^p\otimes 1,\end{aligned}$$

so z^p must be a primitive element in degree 0 modulo $2p$. By Lemma 3.5, this is impossible when $n \leq 2p$ and $|z^p| \geq 4p$, so there are no multiplicative extensions.

When $n \geq 2$, the pinch map $\psi\colon S^n \to S^n \vee S^n$ is homotopy cocommutative, ie the following diagram commutes:

$$\begin{tikzcd} & S^n \vee S^n \arrow[dd, "\tau"] \\ S^n \arrow[ur, "\psi"] \arrow[dr, "\psi"'] & \\ & S^n \vee S^n \end{tikzcd}$$

where τ interchanges the two factors. Cocommutativity is shown by suspending a homotopy between the identity and antipodal map on S^1, picking one of the endpoints of the suspension as the basepoint in S^n, and identifying the suspension of two antipodal points on S^1 to a point, to define ψ.

From this, it follows that $L_*(S^n)$ is cocommutative as an $\mathbb{F}_p$–coalgebra when $n \geq 2$. Since $E^2(S^n)$ is a tensor product of exterior algebras and divided power algebras, Proposition 8.7 says that there are no coproduct coextensions. Thus $L_*(S^n) \cong E^\infty(S^n) = E^2(S^n) \cong B_n$ as an $\mathbb{F}_p$–Hopf algebra. $\square$

4 Multifold Hopf algebras

The homotopy groups $L_*(T^n)$ will have several Hopf algebra structures coming from the various circles in T^n. These structures will be interlinked, and in this section, we set up an algebraic framework for this interlinked structure. Our main goal is to state Proposition 5.5, which is a crucial ingredient in the calculation of the multiplicative structure of $L_*(T^n)$. Throughout this section, we will prove statements about the multifold Hopf algebra structure of $L_*(T^n)$ as an illustration of the definitions and propositions.

This exposition of multifold Hopf algebra leaves several unanswered questions. In particular, it would be interesting to have a good description of the module of elements that are primitive in all the Hopf algebra structures simultaneously. In that regard, a generalization of the very special case in Lemma 9.5 would be welcomed.

Let CRings be the category of commutative rings. In this section, we will assume that all our Hopf algebras are connected and commutative.

First we construct a category of Hopf algebras and show that it has all small colimits. Objects in this category are ordinary Hopf algebras, but we use the morphisms to define a multifold Hopf algebra.

Definition 4.1 The category of Hopf algebras has pairs of commutative rings (A, R) as objects, where A is given the structure of a commutative connected R–Hopf algebra. A morphism from (A, R) to (B, S) consists of two maps $f\colon A \to B$ and $g\colon R \to S$ of commutative rings such that f is a map of R–algebras and $f \otimes_g \mathrm{id}\colon A \otimes_R S \to B \otimes_S S \cong B$ is a map of S–coalgebras.

Proposition 4.2 *The category of Hopf algebras has all small colimits, and the colimit* $\mathrm{colim}_J(A_j, R_j)$ *is equal to the pair* $(\mathrm{colim}_J\, A_j, \mathrm{colim}_J\, R_j)$ *of colimits in the category of commutative rings.*

Proof The proof is left to the reader. □

Our multifold Hopf algebras will be functors from the following categories. Let S be a finite set and define $V(S)$ to be the category with objects given by subsets of S and morphisms from U to V given by subsets of $U \cap V$, where composition is intersection. Given an element $s \in S$, we will often write $S \setminus s$ for $S \setminus \{s\}$ to make the formulas more readable.

Equivalently, $V(S)$ is isomorphic to the category of spans in $[2]^S$, where $[2]^S$ is the category with subsets of S as objects and inclusions of sets as morphisms. There is an inclusion $[2]^S \to V(S)$ given by sending a morphism $U \subseteq V$ to the morphism U from U to V.

Example 4.3 Let $S = \{u, v\}$. Then the nonidentity morphisms in the category $V(S)$ are given in the diagram

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] \arrow[dr] & \{v\} \arrow[l, bend right] \arrow[d] \\
\{u\} \arrow[u, bend left] \arrow[r] & \{u,v\} \arrow[l, bend left] \arrow[u, bend right] \arrow[ul, bend right]
\end{tikzcd}$$

and the images of the nonidentity morphisms under the inclusion $[2]^S \to V(S)$ are the straight arrows.

The next definition is only a preliminary step towards the final definition of an S–fold Hopf algebra in Definition 4.18.

Definition 4.4 Let S be a finite set viewed as a discrete category, and let $X \subset V(S) \times S$ denote the full subcategory of pairs (V, v) with $v \in V$. A pre-S–fold Hopf algebra A is a functor $A\colon V(S) \to \mathrm{CRings}$ with the following property:

For every $v \in V \subseteq S$, the pair $(A(V), A(V \setminus v))$ is equipped with the structure of a Hopf algebra with unit and counit induced by the spans $V \setminus v \leftarrow V \setminus v \rightarrow V$ and $V \leftarrow V \setminus v \rightarrow V \setminus v$, respectively, such that with this structure, the composite

$$X \xrightarrow{F} V(S) \times V(S) \xrightarrow{A \times A} \mathrm{CRings} \times \mathrm{CRings},$$

where F is the functor $F(V, v) = (V, V \setminus v)$, becomes a functor to the category of Hopf algebras.

We let ψ_V^v, ϕ_V^v, η_V^v and ϵ_V^v denote the coproduct, product, unit and counit in the Hopf algebra $(A(V), A(V \setminus v))$, respectively.

For a functor $A\colon C \to D$, where the objects of C are finite sets, we will for $V \in C$ write A_V for $A(V)$.

Definition 4.5 A map from a pre-S–fold Hopf algebra A to a pre-S–fold Hopf algebra B is a natural transformation from A to B such that for every $v \in V \subseteq S$, the induced map from $(A_V, A_{V \setminus v})$ to $(B_V, B_{V \setminus v})$ is a map of Hopf algebras.

Fix a basepoint on the circle S^1. Let $\mathcal{I}$ be the category with finite sets of natural numbers as objects, and inclusions as morphisms.

We define the functor $T\colon \mathcal{I} \to \mathrm{Top}$ by $T(\varnothing) = \{\mathrm{pt}\}$ and $T(U) = T^U$, the U–fold torus, for $U \neq \varnothing$. On morphisms, it takes an inclusion $U \subseteq V$ to the inclusion $\mathrm{in}_U^V\colon T^U \to T^V$, where we use the basepoint in the factors not in U. Furthermore, there is the projection map

$$\mathrm{pr}_U^V\colon T^V \to T^U.$$

Give the circle S^1 the minimal CW-structure with one 0–cell and one 1–cell, and give the U–fold torus T^U the product CW-structure. We write T_k^U for the k–skeleton of T^U. If U has cardinality k, the quotient map

$$g^U\colon T^U \to T^U/T_{k-1}^U$$

maps to the U–sphere S^U.

Definition 4.6 Let W be an object in $\mathcal{I}$. Define the functor $\mathcal{L}\colon V(W) \to \mathrm{CRings}$ on an object $V \in V(W)$ by $\mathcal{L}(V) = L_*(T^V)$ and on a map $U\colon V \to X$ by $\mathcal{L}(U) = \mathrm{in}_X^U \circ \mathrm{pr}_U^V$.

Proposition 4.7 *Let W be an object in $\mathcal{I}$.*

(1) *The functor $\mathcal{L}$ is a pre-W–fold Hopf algebra if the pair $(L_*(T^U), L_*(T^{U\setminus v}))$ is equipped with the Hopf algebra structure in Proposition 2.4 for all $v \in U \subseteq W$.*

(2) *The map $g^W\colon T^W \to S^W$ induces a map of Hopf algebras*

$$(L_*(T^W), L_*(T^{W\setminus j})) \to (L_*(S^{|W|}), \mathbb{F}_p).$$

Proof Given $U \subseteq V \subseteq W$ and $v \in U$, we get two homomorphisms of Hopf algebras $(L_*(T^U), L_*(T^{U\setminus v})) \to (L_*(T^V), L_*(T^{V\setminus v})) \to (L_*(T^U), L_*(T^{U\setminus v}))$ induced by the inclusion $U \setminus v \to V \setminus v$. Hence, $\mathcal{L}$ is a pre-W–fold Hopf algebra.

Given subsets $U \subseteq V \subseteq W$, the ring $\mathcal{L}_V^U$, as defined in Definition 4.8, is isomorphic to $L_*(T^{V\setminus U} \times (S^1 \vee S^1)^U)$, since $\mathcal{L}$ commutes with colimits, and the colimit of the composite

$$T(U) \xrightarrow{-\cup-} [2]^U \xrightarrow{-\cup(V\setminus U)} [2]^S \xrightarrow{T^-} \mathrm{Top}$$

is $T^{V\setminus U} \times (S^1 \vee S^1)^U$; see Definition 4.8. □

Let $T(S)$ be the full subcategory of $[2]^S \times [2]^S$ with pairs (U, V) such that $U \cap V = \varnothing$ as objects.

Definition 4.8 Let S be a finite set and A a functor $A\colon [2]^S \to \mathrm{CRings}$. Given finite sets $U \subseteq V \subseteq S$, we define the functor $F^U_{A,V}$ to be the composite

$$T(U) \xrightarrow{-\cup-} [2]^U \xrightarrow{-\cup(V\setminus U)} [2]^S \xrightarrow{A} \mathrm{CRings},$$

and define A^U_V to be the colimit of the functor $F^U_{A,V}$.

Using the inclusion $[2]^S \to V(S)$, this construction applies to any pre-S–fold Hopf algebra A. Note that $A^{\varnothing}_V = A_V$.

Example 4.9 Let $U = \{u,v\} \subseteq V$. The source category $T(U)$ of $F^U_{A,V}$ is the diagram on the left, and the image of $F^U_{A,V}$ in commutative rings is the diagram on the right:

$$\begin{tikzcd}
\{u,v\},\varnothing & \{u\},\varnothing \arrow[l] \arrow[r] & \{u\},\{v\} \\
\{v\},\varnothing \arrow[u] \arrow[d] & \varnothing,\varnothing \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \varnothing,\{v\} \arrow[u] \arrow[d] \\
\{v\},\{u\} & \varnothing,\{u\} \arrow[l] \arrow[r] & \varnothing,\{u,v\}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_V & A_{V\setminus v} \arrow[l] \arrow[r] & A_V \\
A_{V\setminus u} \arrow[u] \arrow[d] & A_{V\setminus\{u,v\}} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & A_{V\setminus u} \arrow[u] \arrow[d] \\
A_V & A_{V\setminus v} \arrow[l] \arrow[r] & A_V
\end{tikzcd}$$

Example 4.10 For $U \subset V \subset W$, we have $\mathcal{L}^U_V \cong L_*(T^{V\setminus U} \times (S^1 \vee S^1)^{\times U})$.

Remark 4.11 We will now describe a helpful way to think about the rings A^U_V. The power set $P(U)$ of U can be thought of as a discrete category, with the subsets of U as objects. There is a functor G from $P(U)$ to $T(U)$ given by mapping $W \subseteq U$ to the pair $(U \setminus W, W)$. The composite $F^U_{A,V} \circ G$ is the constant functor A_V, so this induces a surjective map on colimits from $A_V^{\otimes P(U)}$ to A^U_V.

An element in A^U_V can thus be represented by an element in $A_V^{\otimes P(U)}$. By thinking of the objects in $P(U)$ as the corners of the unit cube in $\mathbb{R}^U$, we can think of the monomials in $A_V^{\otimes P(U)}$ as U–indexed cubes with a monomial from A_V in each corner.

We will write the image of such a representative in $A_V^{\otimes P(U)}$ as a sum of U–indexed cubes as well.

Example 4.12 Let $U = \{u,v\} \subseteq V$. We represent an element of A^U_V by a sum of cubes

$$\begin{bmatrix} x_\varnothing & x_{\{v\}} \\ x_{\{u\}} & x_{\{u,v\}} \end{bmatrix},$$

where the displayed cube is the notation for $\bigotimes_{I\in P(U)} x_I \in A_V^{\otimes P(U)}$. The four entries in the cube correspond to the four corners in the right diagram in Example 4.9, and the subscripts are given by the second set in the four corners in the left diagram. Multiplication is done componentwise, and we have the identifications

$$\begin{bmatrix} a_u x_\varnothing & x_{\{v\}} \\ x_{\{u\}} & x_{\{u,v\}} \end{bmatrix} = \begin{bmatrix} x_\varnothing & a_u x_{\{v\}} \\ x_{\{u\}} & x_{\{u,v\}} \end{bmatrix}, \qquad \begin{bmatrix} x_\varnothing & x_{\{v\}} \\ a_u x_{\{u\}} & x_{\{u,v\}} \end{bmatrix} = \begin{bmatrix} x_\varnothing & x_{\{v\}} \\ x_{\{u\}} & a_u x_{\{u,v\}} \end{bmatrix},$$
$$\begin{bmatrix} a_v x_\varnothing & x_{\{v\}} \\ x_{\{u\}} & x_{\{u,v\}} \end{bmatrix} = \begin{bmatrix} x_\varnothing & x_{\{v\}} \\ a_v x_{\{u\}} & x_{\{u,v\}} \end{bmatrix}, \qquad \begin{bmatrix} x_\varnothing & a_v x_{\{v\}} \\ x_{\{u\}} & x_{\{u,v\}} \end{bmatrix} = \begin{bmatrix} x_\varnothing & x_{\{v\}} \\ x_{\{u\}} & a_v x_{\{u,v\}} \end{bmatrix},$$

where a_u is an element in $A_{V\setminus\{u\}} \subseteq A_V$, and a_v is an element in $A_{V\setminus\{v\}} \subseteq A_V$. Observe that if a is an element in $A_{V\setminus\{u,v\}}$, we can move the element between all four corners of the cube.

Definition 4.13 Observe that the colimits of the columns in the right diagram in Example 4.9 are

$$A_V^{\{u\}} \leftarrow A_{V\setminus\{v\}}^{\{u\}} \rightarrow A_V^{\{u\}}.$$

Given a map of diagrams

$$\begin{tikzcd} A_V \arrow[d, "\psi_V"] & A_{V\setminus v} \arrow[l] \arrow[r] \arrow[d, "\psi_{V\setminus v}"] & A_V \arrow[d, "\psi_V"] \\ A_V^{\{u\}} & A_{V\setminus v}^{\{u\}} \arrow[l] \arrow[r] & A_V^{\{u\}} \end{tikzcd}$$

we will write the map on the colimits of the horizontal direction as

$$\begin{bmatrix} \psi_V & \psi_V \end{bmatrix} \colon A_V^{\{v\}} \to A_V^{\{u,v\}}.$$

Lemma 4.14 *For $U \subseteq V$ and $v \in V \setminus U$, the universal property of colimits induces the commutative-ring isomorphism*

$$A_V^U \otimes_{A_{V\setminus v}^U} A_V^U \cong A_V^{U\cup v}.$$

Proof Both sides are the colimit of the functor $F_{A,V}^{U\cup v}$. On the left-hand side, the colimit is evaluated in two steps, evaluating the v^{th} direction in the diagram $T(U\cup v)$ last.

More explicitly, the middle term $A_{V\setminus v}^U$ is the colimit of the functor $F_{A,V}^{U\cup v}$ precomposed with the inclusion $T(U) \to T(U\cup v)$. The two outer terms A_V^U are the colimit of the functor $F_{A,V}^{U\cup v}$ precomposed with the two maps $T(U) \to T(U\cup v)$ given by adding v to the first and second set in $T(U)$, respectively. □

Given a pre- S–fold Hopf algebra A, there are some related multifold Hopf algebras. Thinking of a pre- S–fold Hopf algebra as a sum of S–cubes with corners indexed by the subset of S, the first part of the next proposition says that every face is a pre-multifold Hopf algebra in a natural way.

Proposition 4.15 *Let A be a pre- S–fold Hopf algebra. If U and W are subsets of S, the composite*

$$V(W) \xrightarrow{-\cup U} V(S) \xrightarrow{A} \mathrm{CRings}$$

is a pre- W–fold Hopf algebra. If U is a subset of S, the functor

$$A^U\colon V(S \setminus U) \to \mathrm{CRings}$$

given by $A^U(V) = A^U_{V\cup U}$ is a pre- $(S\setminus U)$–fold Hopf algebra.

Proof The first case is clear by definition. In the second case, for every $U \subseteq V \subseteq S$ and $v \in V \setminus U$, we need to give a Hopf algebra structure to the pair $(A^U_V, A^U_{V\setminus v})$ satisfying the definition of a pre- $(S\setminus U)$–fold Hopf algebra.

We claim there is a pushout diagram of Hopf algebras:

$$\begin{tikzcd}
(A^U_{V\setminus u}, A^U_{(V\setminus u)\setminus v}) \arrow[r] \arrow[d] & (A^U_V, A^U_{V\setminus v}) \arrow[d] \\
(A^U_V, A^U_{V\setminus v}) \arrow[r] & (A^{U\cup u}_V, A^{U\cup u}_{V\setminus v})
\end{tikzcd}$$

The identification of the pushout follows from Lemma 4.14. The case where $U = \varnothing$ follows from the definition of a pre- S–fold Hopf algebra. The other statements follow by induction on the cardinality of U and Proposition 4.2.

The universal property of pushouts guarantees that these Hopf algebras combine to a functor satisfying the definition of a pre- $(S\setminus U)$–fold Hopf algebra. □

Composing the various coproducts in a pre- S–fold Hopf algebra gives rise to several homomorphisms that we now introduce. An S–fold Hopf algebra is a pre- S–fold Hopf algebra where these various homomorphisms agree.

Definition 4.16 Let A be a pre- S–fold Hopf algebra. Given a pair of sets $U \subseteq V \subseteq S$ with $v \in V \setminus U$, we define

$$\psi^{U,v}_V\colon A^U_V \to A^U_V \otimes_{A^U_{V\setminus v}} A^U_V \cong A^{U\cup v}_V$$

to be the composition of the coproduct in the Hopf algebra $(A_V^U, A_{V\backslash v}^U)$ with the isomorphism from Lemma 4.14

Given a sequence of distinct elements $u_1, u_2, \dots, u_k \in V \subseteq S$, we define

$$\psi_V^{u_1,\dots,u_k}\colon A_V \to A_V^{\{u_1,\dots,u_k\}}$$

by the recursive formula

$$\psi_V^{u_i,\dots,u_k} = \psi_V^{\{u_{i+1},\dots,u_k\},u_i} \circ \psi_V^{u_{i+1},\dots,u_k}.$$

Similarly, we define

$$\widetilde{\psi}_V^{u_1,\dots,u_k}\colon A_V \to A_V^{\{u_1,\dots,u_k\}}$$

using the reduced coproducts $\widetilde{\psi}_V^{\{u_{i+1},\dots,u_k\},u_i} = \psi_V^{\{u_{i+1},\dots,u_k\},u_i} - 1 \otimes \mathrm{id} - \mathrm{id} \otimes 1$.

Definition 4.17 Let S be a finite set, and let A a pre-S–fold Hopf algebra. An *S–fold primitive element* in A is an element which is primitive in $(A_S, A_{S\backslash s})$ for each $s \in S$. When S is clear from context, it will also be called a *simultaneous primitive element*.

Definition 4.18 Let S be a finite set. An S–fold Hopf algebra A is a pre-S–fold Hopf algebra A with the additional requirement that for every sequence $u_1, u_2, \dots, u_k$ of distinct elements in $V \subseteq S$ and all permutations α of k,

$$\psi_V^{u_{\alpha(1)},\dots,u_{\alpha(k)}} = \psi_V^{u_1,\dots,u_k}\colon A_V \to A_V^{\{u_1,\dots,u_k\}}.$$

Letting $U = \{u_1, \dots, u_k\}$, we denote this map by

$$\psi_V^U\colon A_V \to A_V^U.$$

Definition 4.19 A map from an S–fold Hopf algebra A to an S–fold Hopf algebra B is a map of pre-S–fold Hopf algebras.

Proposition 4.20 *Let W be an object in $\mathcal{I}$. The functor $\mathcal{L}\colon V(W) \to \mathrm{CRings}$ is a W–fold Hopf algebra.*

Proof We proved in Proposition 4.7 that $\mathcal{L}$ is a pre-W–fold Hopf algebra. That $\mathcal{L}$ is also a W–fold Hopf algebra follows from the geometric origin of the coproducts ψ_V^i for $i \in V \subseteq W$. Given a sequence $u_1, u_2, \dots, u_k$ of distinct elements in $V \subseteq W$, let $U = \{u_1, u_2, \dots, u_k\}$. The map

$$\psi_V^{u_1,u_2,\dots,u_k}\colon \mathcal{L}(V) \cong L_*(T^V) \to \mathcal{L}_V^U \cong L_*(T^{V\backslash U} \times (S^1 \vee S^1)^U),$$

defined in Definition 4.16, is induced by the pinch map on every circle in $T^U \subseteq T^V$. Hence, it is independent of the order of the elements u_i in $\psi_V^{u_1,u_2,\dots,u_k}$. $\square$

It shouldn't come as a surprise that when the composition of the coproducts agree, the composition of the reduced coproducts agree. More precisely:

Proposition 4.21 *Let A be an S–fold Hopf algebra, and let $u_1, u_2, \dots, u_k$ be a sequence of distinct elements in $V \subseteq S$. Then for all permutations α of k,*

$$\widetilde{\psi}_V^{u_{\alpha(1)},\dots,u_{\alpha(k)}} = \widetilde{\psi}_V^{u_1,\dots,u_k}.$$

Proof By Proposition 4.15, it suffices to check the claim for transpositions since $A^{\{u_i,\dots,u_k\}}$ is an $S \setminus \{u_i, \dots, u_k\}$–fold Hopf algebra for every $i \leq k$. That it holds for transpositions is easily checked by calculating the two sides using the cube notation. □

We end this section by constructing some special S–fold Hopf algebras.

Definition 4.22 Let S be a finite set. We define a subcategory Δ of $V(S)$ to be *saturated* if it has the property that if $W \in \Delta$, then $V(W) \subseteq \Delta$.

Definition 4.23 Let S be a finite set and let Δ be a saturated subcategory of $V(S)$. Define a *partial S–fold Hopf algebra* $A\colon \Delta \to \mathrm{CRings}$ to be a functor A which, for every $W \in \Delta$, is a W–fold Hopf algebra when restricted to W.

Definition 4.24 Let S be a finite set, let Δ be a saturated subcategory of $V(S)$, let $\bar{\Delta}$ be the subcategory $\bigcup_{W\in\Delta} [2]^W \subseteq \Delta$, and let $A\colon \Delta \to \mathrm{CRings}$ be a partial S–fold Hopf algebra. The functor

$$\bar{A}\colon V(S) \to \mathrm{CRings}$$

defined by $\bar{A}(W) = \mathrm{colim}_{U\subseteq W, U\in\bar{\Delta}} A(U)$ has the structure of an S–fold Hopf algebra, and we call it *the extension of A to S*.

Since the category of Hopf algebras has all small colimits, and these are given as pairs of colimits in commutative rings, it is clear that the extension of A is an S–fold Hopf algebra. All the properties of an S–fold Hopf algebra follows from functoriality of the colimit.

Definition 4.25 Let S be a finite set, and let Δ be a saturated subcategory of $V(S)$. Given an S–fold Hopf algebra A, we define *the restriction of A to Δ* to be the S–fold Hopf algebra which is the extension of the functor

$$A|_\Delta\colon \Delta \to \mathrm{CRings}.$$

Example 4.26 Let W be a finite set, let $k > 0$ be an integer, and let Δ be the saturated subcategory of $V(W)$ containing all sets of cardinality at most k. Let $\mathcal{L}_\Delta$ be the restriction of $\mathcal{L}$ to Δ. Then for $U \subseteq W$, and where T^U_k is the k–skeleton of T^U,

$$\mathcal{L}_\Delta(U) \cong L_*(T^U_k).$$

Definition 4.27 Let S be a finite set, and let m be a positive even integer. Let $\Delta \subseteq V(S)$ be the full subcategory containing all sets with at most one element. Let $A\colon \Delta \to \mathrm{CRings}$ be the functor given by $A(\varnothing) = R$ and $A(\{s\}) = P_R(\mu_s)$, with $|\mu_s| = m$.

We define $P_R(\mu_-)$, *the polynomial S–fold Hopf algebra over R in degree m*, to be the extension of the functor A to all of S.

Example 4.28 For the set $S = \{s_1, \dots, s_k\}$, there is an R–algebra isomorphism $P_R(\mu_S) \cong P_R(\mu_{s_1}, \dots, \mu_{s_k})$, and for each $s \in S$, the element μ_s is primitive in the Hopf algebra $(P_R(\mu_S), P_R(\mu_{S\setminus s}))$.

For $U \subseteq V \subseteq S$, there is an R–algebra isomorphism

$$(P_R(\mu_S))^U_V \cong P_R(\mu_V) \otimes_{P_R(\mu_{V\setminus U})} P_R(\mu_V).$$

Since $P_R(\mu_S)$ is generated as an R algebra by $\{\mu_{s_1}, \dots, \mu_{s_k}\}$, and $\psi^{s_j}_S(\mu_{s_i}) = \mu_{s_i}$ when $i \neq j$, the iterated coproduct

$$\psi^U_V\colon P_R(\mu_V) \to (P_R(\mu_S))^U_V \cong P_R(\mu_V) \otimes_{P_R(\mu_{V\setminus U})} P_R(\mu_V)$$

is determined by $\psi^U_V(\mu_{u_i}) = \mu_{u_i} \otimes 1 + 1 \otimes \mu_{u_i}$ for $u_i \in U$ and $\psi^U_V(\mu_{u_i}) = \mu_{u_i} \otimes 1 = 1 \otimes \mu_{u_i}$ for $u_i \in V \setminus U$.

Example 4.29 When $m = 2$, and Δ is the subcategory of $V(W)$ containing all sets with at most one element, the functor $\mathcal{L}_\Delta$ is naturally isomorphic to $P_{\mathbb{F}_p}(\mu_-)$.

5 Coproduct in a multifold Hopf algebra

In this section, we state a proposition that we will need when we calculate the multiplicative structure of $L_*(T^n)$.

Given an integer n divisible by p, we write n/p for the image of n/p under the ring map $\mathbb{Z} \to \mathbb{F}_p$. In the polynomial $\mathbb{F}_p$–Hopf algebra $P_{\mathbb{F}_p}(\mu)$, we write $\widetilde{\psi}(\mu^{p^i})/p$ for the image of $\widetilde{\psi}(\mu^{p^i})/p$ under the ring map $P_{\mathbb{Z}}(\mu_n) \to P_{\mathbb{F}_p}(\mu_n)$ given by mapping μ_n to μ_n. This is well defined since $\binom{p^i}{k}$ is divisible by p for all i and k with $0 < k < p^i$.

Lemma 5.1 *Let M be an $\mathbb{F}_p$–module, and let n be a natural number greater than 2. Let $\{r_{k,n-k}\}_{0\le k\le n}$ be a subset of M which satisfy the relations $\binom{a+b}{b} r_{a+b,c} = \binom{b+c}{b} r_{a,b+c}$ for all $a+b+c=n$ and $0<a,c<n$. Then the following relations hold:*

(1) *If $n=p^{m+1}$ for some $m\ge 0$, then*

$$r_{k,n-k} = \frac{\binom{n}{k}}{p} r_{p^m,(p-1)p^m}$$

for all $0<k<n$.

(2) *If $n=p^{m_1}+p^{m_2}$ with $m_1<m_2$ and $k\ne p^{m_1}, p^{m_2}$, then $r_{k,n-k}=0$.*

(3) *If $n\ne p^{m+1}, p^{m_1}+p^{m_2}$ with $m_1<m_2$, then*

$$r_{k,n-k} = \binom{n}{k} n_m^{-1} r_{p^m,n-p^m}$$

for all $0<k<n$, where $n=n_0+n_1p^1+\dots+n_mp^m$ with $0\le n_i<p$, and $n_m\ne 0$ is the p–adic representation of n.

The only case which is not covered by the lemma is $n=p^{m_1}+p^{m_2}$ with $m_1\ne m_2$, when the relations in the lemma don't give any relation between $r_{p^{m_1},p^{m_2}}$ and $r_{p^{m_2},p^{m_1}}$.

Proof Given a set $\{r_{k,n-k}\}_{0<k<n}$ of elements in an abelian group, let $\sim$ be the equivalence relation generated by $\binom{a+b}{b} r_{a+b,c} \sim \binom{b+c}{b} r_{a,b+c}$. Let $\mathbb{F}_p\{r_{1,n-1},\dots,r_{n-1,1}\}$ be the free $\mathbb{F}_p$–module on the set $\{r_{1,n-1},\dots,r_{n-1,1}\}$. Since M is an $\mathbb{F}_p$–module, there is a homomorphism

$$\mathbb{F}_p\{r_{1,n-1},\dots,r_{n-1,1}\}/\!\sim\ \to M$$

defined by mapping $r_{k,n-k}$ to $r_{k,n-k}$. Hence it suffices to prove the lemma for the module $M=\mathbb{F}_p\{r_{1,n-1},\dots,r_{n-1,1}\}/\!\sim$. Let

$$k=k_0+k_1p^1+\dots+k_jp^j$$

with $0\le k_i<p$ and $k_j\ne 0$ be the p–adic representations of k. Similarly, let

$$n=n_0+n_1p^1+\dots+n_mp^m$$

with $0\le n_i<p$ and $n_m\ne 0$ be the p–adic representation of n, except that when n is a power of p we express it as $n=p^{m+1}$.

The proof consists of two parts. First we prove that unless both k and $n-k$ are powers of p, there is a sequence of equations expressing $r_{k,n-k}$ as a multiple of $r_{p^m,n-p^m}$. The second part is to identify the factor in this equation in terms of $\binom{n}{k}$.

We will now use the relations $\binom{a+b}{b} r_{a+b,c} = \binom{b+c}{b} r_{a,b+c}$ to express $r_{k,n-k}$ as a multiple of $r_{p^m,n-p^m}$. Lucas' theorem says that $\binom{n}{k} = \prod_i \binom{n_i}{k_i} \bmod p$, so $\binom{k}{k-p^j} = \binom{k}{p^j} = k_j$, giving us the equation

$$r_{k,n-k} = \frac{\binom{n-p^j}{k-p^j}}{\binom{k}{k-p^j}} r_{p^j,n-p^j}.$$

If $j = m$, we are done. Otherwise, if $n > p^j + p^m$, the m^{th} coefficient in the p–adic expansion of $n - p^j$ is at least 1. Hence $\binom{n-p^j}{p^m} \neq 0$, and we have two equations

$$r_{p^j,n-p^j} = \frac{\binom{p^j+p^m}{p^m}}{\binom{n-p^j}{p^m}} r_{p^j+p^m,n-p^j-p^m}, \quad r_{p^j+p^m,n-p^j-p^m} = \frac{\binom{n-p^m}{p^j}}{\binom{p^j+p^m}{p^j}} r_{p^m,n-p^m}.$$

If $n < p^j + p^m$, there is an $i < j$ such that $n_i \neq 0$ and the i^{th} coefficient in the p–adic expansion of $n - p^j$ is n_i. Hence $\binom{n-p^j}{p^i} = n_i$ and $\binom{n-p^i}{p^m} = n_m$, and the four equations below move these powers of p back and forth:

$$r_{p^j,n-p^j} = \frac{\binom{p^j+p^i}{p^i}}{\binom{n-p^j}{p^i}} r_{p^j+p^i,n-p^j-p^i}, \qquad r_{p^j+p^i,n-p^j-p^i} = \frac{\binom{n-p^i}{p^j}}{\binom{p^j+p^i}{p^j}} r_{p^i,n-p^i},$$

$$r_{p^i,n-p^i} = \frac{\binom{p^i+p^m}{p^m}}{\binom{n-p^i}{p^m}} r_{p^i+p^m,n-p^m-p^i}, \qquad r_{p^i+p^m,n-p^i-p^m} = \frac{\binom{n-p^m}{p^i}}{\binom{p^i+p^m}{p^i}} r_{p^m,n-p^m}.$$

Combining three or five equations, respectively, when $(k,n) \neq (p^j, p^j + p^m)$ with $j < m$, we get the equation

$$r_{k,n-k} = u r_{p^m,n-p^m}$$

where u is some element in $\mathbb{F}_p$.

To determine the element u, we will take a detour through $\mathbb{Z}_{(p)}$, the integers localized at p. In the $\mathbb{Q}$–module $\mathbb{Q}\{r_{1,n-1},\dots,r_{n-1,1}\}/\sim$, we let $r_{1,n-1} = nr$. The formula $\binom{k}{1} r_{k,n-k} = \binom{n-k+1}{1} r_{k-1,n-k+1}$ and induction give the equality

$$r_{k,n-k} = \frac{n-k+1}{k} r_{k-1,n-k+1} = \frac{n-k+1}{k} \binom{n}{k-1} r = \binom{n}{k} r$$

in $\mathbb{Q}\{r_{1,n-1},\dots,r_{n-1,1}\}/\sim$. Thus if $n = p^{m+1}$, then

$$r_{k,n-k} = \frac{\binom{p^{m+1}}{k}}{\binom{p^{m+1}}{p^m}} r_{p^m,(p-1)p^m} = \frac{\binom{p^{m+1}}{k}/p}{\binom{p^{m+1}-1}{p^m-1}} r_{p^m,(p-1)p^m},$$

and when n is not a power of p,

$$r_{k,n-k} = \frac{\binom{n}{k}}{\binom{n}{p^m}} r_{p^m,n-p^m} = \binom{n}{k} n_m^{-1} r_{p^m,n-p^m}.$$

By Lucas' theorem, $\binom{p^{m+1}}{k}$ is divisible by p for every k, but neither $\binom{p^{m+1}-1}{p^m-1}$ nor n_m are divisible by p. Hence these relations exists in the submodule

$$\mathbb{Z}_{(p)}\{r_{1,n-1}, \dots, r_{n-1,1}\}/\sim \subseteq \mathbb{Q}\{r_{1,n-1}, \dots, r_{n-1,1}\}/\sim.$$

By the universal property of localization, we get a map

$$f\colon \mathbb{Z}_{(p)}\{r_{1,n-1}, \dots, r_{n-1,1}\}/\sim \to \mathbb{F}_p\{r_{1,n-1}, \dots, r_{n-1,1}\}/\sim$$

by mapping $r_{k,n-k}$ to $r_{k,n-k}$.

By Lucas' theorem, $\binom{p^{m+1}-1}{p^m-1} = 1 \mod p$ and $\binom{n}{n_m p^m} = 1 \mod p$. So when n is not a power of p, we have $f\left(\binom{n}{k}/\binom{n}{n_m p^m}\right) = \binom{n}{k} = u$, proving part (3). In particular, when $k \neq p^j$ or p^m, the binomial coefficients $\binom{p^j+p^m}{k}$ are equal to 0, proving part (2) of the lemma.

If $n = p^{m+1}$, then we have

$$f\left(\frac{\binom{p^{m+1}}{k}/p}{\binom{p^{m+1}-1}{p^m-1}}\right) = \binom{p^{m+1}}{k}/p = u,$$

proving part (1). □

Given a graded module M, let $M_{\leq n}$ denote the module $\bigoplus_{i \leq n} M_n$, and similarly for other inequalities $<$, $>$ and $\geq$.

Definition 5.2 Let R be a commutative ring.

(1) Let A and B be graded R–algebras. An R–algebra homomorphism from A to B in degrees less than or equal to q is an R–module homomorphism $f\colon A \to B$ which induces an R–algebra homomorphism on the quotients $A/A_{>q} \to B/B_{>q}$.

(2) Let A and B be graded R–coalgebras. An R–coalgebra homomorphism from A to B in degrees less than or equal to q is an R–module homomorphism $f\colon A \to B$ which induces an R–coalgebra homomorphism $A_{\leq q} \to B_{\leq q}$.

(3) Let A and B be graded R–Hopf algebras. An R–Hopf algebra homomorphism from A to B in degrees less than or equal to q is an R–module homomorphism $f\colon A \to B$ which is both an R–algebra and R–coalgebra homomorphism in degrees less than or equal to q, and which preserves the antipode map in degrees less than or equal to q.

Let $\mathbb{N}$ denote the natural numbers including 0, let $\mathbb{N}_+$ denote the strictly positive natural numbers and let $\mathbb{P}$ denote the set of prime powers $\{p^0, p^1, p^2, \dots\} \subseteq \mathbb{N}$.

Proposition 5.3 *Let R be an $\mathbb{F}_p$–algebra, and let A be an R–Hopf algebra such that:*

(1) *There is an R–Hopf subalgebra $P_R(\mu) \subseteq A$.*

(2) *There is a chosen retraction* pr: $A \to P_R(\mu)$ *of the $\mathbb{F}_p$ module inclusion $P_R(\mu) \subset A$ that is a homomorphism of R–algebras in degrees less than or equal to q.*

(3) *In degree less than or equal to $q-1$, this is a splitting as an R–Hopf algebra; ie the diagram*

$$\begin{tikzcd} A \arrow[r, "\mathrm{pr}"] \arrow[d, "\psi_A"] & P_R(\mu) \arrow[d, "\psi_{P_R(\mu)}"] \\ A \otimes A \arrow[r, "\mathrm{pr} \otimes \mathrm{pr}"] & P_R(\mu) \otimes P_R(\mu) \end{tikzcd}$$

commutes in degree less than or equal to $q-1$.

Let x be an element of degree q in $\ker(\mathrm{pr})$. *Then there exist elements $r_n \in R$ for $n \in \mathbb{N}_+$, and $t_{(n_1<n_2)} \in R$ for pairs $(n_1 < n_2) \in \mathbb{P} \times \mathbb{P}$, such that the coproduct satisfies*

$$(\mathrm{pr}_{P_R(\mu)} \otimes \mathrm{pr}_{P_R(\mu)}) \circ \psi(x) = \sum_{n \in \mathbb{N}_+} r_n \widetilde{\psi}(\mu^n) + \sum_{n \in \mathbb{P}} r_n \frac{\widetilde{\psi}(\mu^n)}{p} + \sum_{(n_1<n_2) \in \mathbb{P}\times\mathbb{P}} t_{(n_1<n_2)} \mu^{n_1} \otimes \mu^{n_2}.$$

Recall that $\widetilde{\psi}(\mu^n) = \sum_{k=1}^{n-1} \binom{n}{k} \mu^k \otimes \mu^{n-k}$ and that if n is a power of p, then $\binom{n}{k}$ is divisible by p for all $0 < k < n$. Hence, $\widetilde{\psi}(\mu^n)/p$ is well defined.

Observe that since $\widetilde{\psi}(\mu^{p^i}) = 0$ for all $i \geq 1$, the first sum is independent of the values of r_{p^i}. An example where this proposition applies is the dual Steenrod algebra A_*

with $P(\xi_1) \subseteq A_*$. Then $\widetilde{\psi}(\xi_2) = \xi_1^p \otimes \xi_1 = \widetilde{\psi}(\xi_1^{p+1}) - \xi_1 \otimes \xi_1^p$, so $t_{1<p} = -1$ and $r_{p+1} = 1$.

Proof In general, $(\mathrm{pr}_{P_R(\mu)} \otimes \mathrm{pr}_{P_R(\mu)}) \circ \psi(x) = \sum_{n\in\mathbb{N}} \sum_{a+b=n} r_{a,b}\mu^a \otimes \mu^b$ for some $r_{a,b} \in R$. Since $(\epsilon \otimes \mathrm{id})\psi = (\mathrm{id} \otimes \epsilon)\psi = \mathrm{id}$ and $x \in \ker(\mathrm{pr})$, we must have that $r_{0,n} = r_{n,0} = 0$ for all n.

By part (3) of the proposition, we have

$$(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr})(\widetilde{\psi} \otimes \mathrm{id})\widetilde{\psi}(x) = \sum_{n\in\mathbb{N}} \sum_{d+c=n} r_{d,c} \sum_{a+b=d} \binom{d}{b} \mu^a \otimes \mu^b \otimes \mu^c$$

and

$$(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr})(\mathrm{id} \otimes \widetilde{\psi})\widetilde{\psi}(x) = \sum_{n\in\mathbb{N}} \sum_{a+d=n} r_{a,d} \sum_{b+c=d} \binom{d}{b} \mu^a \otimes \mu^b \otimes \mu^c.$$

From coassociativity of $\widetilde{\psi}$, we know that the coefficients in front of $\mu^a \otimes \mu^b \otimes \mu^c$ in the two expressions above must be equal. Hence there are relations $\binom{a+b}{b} r_{a+b,c} = \binom{b+c}{b} r_{a,b+c}$ for all $a, c \geq 1$ and $b \geq 0$.

Given such relations, if $n = p^{m+1}$, then $r_{k,n-k} = \big(\binom{n}{k}/p\big) r_{p^m,(p-1)p^m}$ by Lemma 5.1, and we let $r_n = r_{p^m,(p-1)p^m}$. If $n = p^{m_1} + p^{m_2}$ with $m_1 < m_2$, then $r_{k,n-k} = 0$ when $k \neq p^{m_1}, p^{m_2}$. We let $r_n = r_{p^{m_2},p^{m_1}}$ and $t_{(p^{m_1}<p^{m_2})} = r_{p^{m_1},p^{m_2}} - r_{p^{m_2},p^{m_1}}$. Otherwise, let $n = n_0 + n_1 p^1 + \cdots + n_m p^m$, with $0 \leq n_i < p$ and $n_m \neq 0$, be the p–adic representation of n. Then $r_{k,n-k} = \binom{n}{k} n_m^{-1} r_{p^m,n-p^m}$, and we let $r_n = n_m^{-1} r_{p^m,n-p^m}$. □

Definition 5.4 Let $B \xrightarrow{f} A$ be S–fold Hopf algebras with $R = A_\varnothing \cong B_\varnothing$. We say that A is a B split S–Hopf algebra in degree $q \geq 0$ if:

(1) There is a splitting of S–fold Hopf algebras $B \xrightarrow{f} \widetilde{A} \xrightarrow{\mathrm{pr}} B$, where $\widetilde{A}$ is the restriction of A, as in Definition 4.25, to the full subcategory of $V(S)$ not containing S.

(2) In degree less than or equal to q, the map pr can be extended to A_S; ie in degree less than or equal to q, there is an R–algebra homomorphism pr: $A_S \to B_S$ (see Definition 5.2) such that the diagram

$$\begin{tikzcd} \widetilde{A}_S \arrow[r] \arrow[d, "\mathrm{pr}_S"'] & A_S \arrow[dl, "\mathrm{pr}"] \\ B_S & \end{tikzcd}$$

commutes in degree less than or equal to q.

(3) For all $s \in S$, the map $\mathrm{pr}\colon (A_S, A_{S\setminus s}) \to (B_S, B_{S\setminus s})$ is a map of Hopf algebras in degree less than or equal to $q-1$.

The next proposition is similar to the previous one, but involves S–fold Hopf algebras. Although they are similar, the next proposition doesn't specialize to the previous one when S contains exactly one element since, in Definition 5.4(1), $\widetilde{A} = R$ gives us an impossible splitting $P_R(\mu) \to R \to P_R(\mu)$. Given a finite set $U = \{u_1, \dots, u_k\}$, we write $P_R(\mu_U)$ for the polynomial ring $P_R(\mu_{u_1}, \dots, \mu_{u_k})$, and given an element $m \in \mathbb{N}^V$ where $U \subseteq V$, we let μ_U^m in $P_R(\mu_U)$ denote the product $\mu_{u_1}^{m_{u_1}} \cdots \mu_{u_k}^{m_{u_k}}$.

Proposition 5.5 *Let A be a $P_R(\mu_-)$ split S–Hopf algebra in degree q. Let x be an element in $\bigcap_{s\in S} \ker(\epsilon_S^s\colon A_S \to A_{S\setminus s}) \subseteq A_S$ of degree q. If $x \in \ker(\mathrm{pr})$ and $s \in S$, then there exist elements $r_b \in R$ for $b \in \mathbb{N}_+^{\times S}$ such that for every $s \in S$,*

$$(5.6)\quad \begin{bmatrix}\mathrm{pr} & \mathrm{pr}\end{bmatrix} \circ \psi_S^s(x) = \sum_{b\in\mathbb{N}_+^{\times S}} r_b \mu_{S\setminus s}^b \widetilde{\psi}_S^s(\mu_s^{b_s}) + \sum_{b\in\mathbb{P}^{\times S}} r_{b,s} \mu_{S\setminus s}^b \frac{\widetilde{\psi}_S^s(\mu_s^{b_s})}{p} + \sum_{b\in\mathbb{P}^{S\setminus s}} \sum_{c_1<c_2\in\mathbb{P}\times\mathbb{P}} t_{b,c_1<c_2,s} \mu_{S\setminus s}^b \begin{bmatrix}\mu_s^{c_1} & \mu_s^{c_2}\end{bmatrix},$$

where b_s is the s^{th} component of b, and $r_{b,s}$ and $t_{b,c_1<c_2,s}$ are elements in R.

An important observation is that in the first sum, the coefficients r_b are independent of the element s. The $\mathbb{P}^{\times S}$ part in the first sum is zero since $\widetilde{\psi}^s(\mu_s^{p^i}) = 0$ for all $i \geq 0$. The map $\begin{bmatrix}\mathrm{pr} & \mathrm{pr}\end{bmatrix}$ was given in Definition 4.13.

Proof In this proof, we will compare $\widetilde{\psi}_S^{i,k}(x)$ with $\widetilde{\psi}_S^{k,i}(x)$ for all pairs of elements $i \neq k$ in S, where the definition of $\widetilde{\psi}_S^{k,i}$ is found in Definition 4.18.

For every element $i \in S$, the ring A_S is an $A_{S\setminus i}$–Hopf algebra, and $A_{S\setminus i}$ is an $\mathbb{F}_p$–algebra since $R = A_\varnothing = \mathbb{F}_p$. By Definition 5.4(1), the unit $\eta_S^i\colon A_{S\setminus i} \to A_S$ induces an inclusion

$$P_{A_{S\setminus i}}(\mu_i) \cong P_R(\mu_S) \otimes_{P_R(\mu_{S\setminus i})} A_{S\setminus i} \to \widetilde{A}_S \to A_S,$$

so Proposition 5.3(1) is satisfied for the Hopf algebra $(A_S, A_{S\setminus i})$. The splitting in Proposition 5.3(2) comes from the homomorphism

$$A_S \to P_R(\mu_S) \otimes_{P_R(\mu_{S\setminus i})} A_{S\setminus i} \cong P_{A_{S\setminus i}}(\mu_i)$$

induced by ϵ_S^i and the splitting in Definition 5.4(1). From Definition 5.4(3), this splitting induces a map of Hopf algebras

$$(A_S, A_{S\setminus i}) \to \big(P_R(\mu_S) \otimes_{P_R(\mu_{S\setminus i})} A_{S\setminus i}, P_R(\mu_{S\setminus i}) \otimes_{P_R(\mu_{S\setminus i})} A_{S\setminus i}\big) \cong (P_{A_{S\setminus i}}(\mu_i), A_{S\setminus i})$$

satisfying Proposition 5.3(3).

By Proposition 5.3, there exist elements $r_{b,i}$ and $t_{b,c_1<c_2,i}$ in R such that

$$\begin{bmatrix}\mathrm{pr}\\ \mathrm{pr}\end{bmatrix} \circ \psi_S^i(x) = \sum_{b\in\mathbb{N}^S} r_{b,i}\mu_{S\setminus i}^b \widetilde{\psi}(\mu_i^{b_i}) + \sum_{b\in\mathbb{N}^{S\setminus i} | b_i\in\mathbb{P}} r_{b,i}\mu_{S\setminus i}^b \frac{\widetilde{\psi}(\mu_i^{b_i})}{p} + \sum_{b\in\mathbb{N}^S}\ \sum_{c_1<c_2\in\mathbb{P}\times\mathbb{P}} t_{b,c_1<c_2,i}\mu_{S\setminus i}^b \begin{bmatrix}\mu_i^{c_1}\\ \mu_i^{c_2}\end{bmatrix}. \tag{5.7}$$

Observe that if $b_i = 1$, we can choose $r_{b,i}$ arbitrarily.

We will now show that if $b_i \geq 2$ and $b_k = 0$ for some $k \neq i$, then $r_{b,i} = 0$. The counits ϵ_S^k and $\epsilon_{S\setminus i}^k$ induce a map of Hopf algebras $(A_S, A_{S\setminus i}) \to (A_{S\setminus k}, A_{S\setminus\{i,k\}})$. Since x is in $\bigcap_{s\in S}\ker(\epsilon_S^s\colon A_S \to A_{S\setminus s})$, we have $\psi_{S\setminus k}^i \circ \epsilon_S^k(x) = 0$. If $r_{b,i} \neq 0$, then $\epsilon_S^k \otimes \epsilon_S^k(\psi_S^i(x)) \neq 0$, so the commutative diagram

$$\begin{tikzcd} A_S \arrow[r, "\psi_S^i"] \arrow[d, "\epsilon_S^k"] & A_S \otimes_{A_{S\setminus i}} A_S \arrow[d, "\epsilon_S^k\otimes\epsilon_S^k"] \\ A_{S\setminus k} \arrow[r, "\psi_{S\setminus k}^i"] & A_{S\setminus k} \otimes_{A_{S\setminus\{i,k\}}} A_{S\setminus k} \end{tikzcd}$$

gives a contradiction. Thus $r_{b,i} = 0$.

From Definition 5.4(3), we get a commutative diagram

$$\begin{tikzcd} \ker(\epsilon_S^i) \arrow[r, "\widetilde{\psi}_S^i"] \arrow[d, "\widetilde{\psi}_S^i"] & A_S^{\{i\}} \arrow[r, "\widetilde{\psi}_S^{\{i\},k}"] & A_S^{\{i,k\}} \arrow[d, "\mathrm{pr}"] \\ A_S^{\{i\}} \arrow[r, "\mathrm{pr}"] & P_R(\mu_S)^{\{i\}} \arrow[r, "\widetilde{\psi}_S^{\{i\},k}"] & P_R(\mu_S)^{\{i,k\}} \end{tikzcd}$$

in degree less than or equal to q, where the composition of the two morphisms on the top is the definition of $\widetilde{\psi}_S^{k,i}$. The diagram commutes in degree less than or equal to q, and not just $q-1$, since we use the reduced coproduct.

From this diagram, we have the formula

$$\begin{aligned}
\begin{bmatrix} \mathrm{pr} & \mathrm{pr} \\ \mathrm{pr} & \mathrm{pr} \end{bmatrix} \circ \widetilde{\psi}_S^{k,i}(x) \\
= \sum_{b \in \mathbb{N}_+^S} \sum_{0<a_i<b_i} \sum_{0<a_k<b_k} r_{b,i} \binom{b_k}{a_k} \binom{b_i}{a_i} \mu^b_{S\setminus\{i,k\}} \begin{bmatrix} \mu_i^{a_i} \mu_k^{a_k} & \mu_k^{b_k-a_k} \\ \mu_i^{b_i-a_i} & 1 \end{bmatrix} \\
+ \sum_{b \in \mathbb{N}_+^S,\, b_i \in \mathbb{P}} \sum_{0<a_i<b_i} \sum_{0<a_k<b_k} r_{b,i} \binom{b_k}{a_k} \frac{\binom{b_i}{a_i}}{p} \mu^b_{S\setminus\{i,k\}} \begin{bmatrix} \mu_i^{a_i} \mu_k^{a_k} & \mu_k^{b_k-a_k} \\ \mu_i^{b_i-a_i} & 1 \end{bmatrix} \\
+ \sum_{b \in \mathbb{N}_+^{S\setminus i}} \sum_{c_1<c_2 \in \mathbb{P}\times\mathbb{P}} \sum_{0<a_k<b_k} t_{b,c_1<c_2,i} \binom{b_k}{a_k} \mu^b_{S\setminus\{i,k\}} \begin{bmatrix} \mu_i^{c_1} \mu_k^{a_k} & \mu_k^{b_k-a_k} \\ \mu_i^{c_2} & 1 \end{bmatrix}.
\end{aligned}$$

The three lines correspond to the three summands in (5.7).

Since A is an S–fold Hopf algebra, $\widetilde{\psi}_S^{k,i} = \widetilde{\psi}_S^{i,k}$, so

$$\begin{bmatrix} \mathrm{pr} & \mathrm{pr} \\ \mathrm{pr} & \mathrm{pr} \end{bmatrix} \circ \widetilde{\psi}_S^{k,i}(x) = \begin{bmatrix} \mathrm{pr} & \mathrm{pr} \\ \mathrm{pr} & \mathrm{pr} \end{bmatrix} \circ \widetilde{\psi}_S^{i,k}(x).$$

We will now compare the coefficient in front of

$$\mu^b_{S\setminus\{i,k\}} \begin{bmatrix} \mu_i^{a_i} \mu_k^{a_k} & \mu_k^{b_k-a_k} \\ \mu_i^{b_i-a_i} & 1 \end{bmatrix}$$

for $b \in \mathbb{N}_+^S$, with $0 < a_j < b_j$.

We will say that an integer $b_i \geq 2$ is

- type 1 if b_i is equal to a power of the prime p,
- type 2 if b_i is equal to a sum of two distinct powers of p, and
- type 3 otherwise.

There are six possible cases since b_i and b_k may be interchanged.

Case 1 Both b_i and b_k are type 3 We get the equation

$$\binom{b_k}{a_k}\binom{b_i}{a_i} r_{b,i} = \binom{b_i}{a_i}\binom{b_k}{a_k} r_{b,k}.$$

Since neither b_i nor b_k are of type 1, there exist integers $0 < a_i < b_i$ and $0 < a_k < b_k$ such that $\binom{b_i}{a_i} \neq 0$ and $\binom{b_k}{a_k} \neq 0$. Thus $r_{b,i} = r_{b,k}$.

Case 2 $\boldsymbol{b_i}$ **is type 2 and** $\boldsymbol{b_k}$ **is type 3** Let $b_i = p^j + p^l$ with $j < l$. When $a_i = p^j$, we get the equation

$$\binom{b_k}{a_k}\binom{b_i}{p^j} r_{b,i} + \binom{b_k}{a_k} t_{b,p^j<p^l,i} = \binom{b_i}{p^j}\binom{b_k}{a_k} r_{b,k},$$

and when $a_i = p^l$, we get the equation

$$\binom{b_k}{a_k}\binom{b_i}{p^l} r_{b,i} = \binom{b_i}{p^l}\binom{b_k}{a_k} r_{b,k}.$$

By Lucas' theorem, $\binom{b_i}{p^j} = \binom{b_i}{p^l} = 1$. Since b_k is not of type 1, there exists an a_k such that $\binom{b_k}{a_k} \neq 0$. The last equation thus gives $r_{b,i} = r_{b,k}$, and the second equation becomes $r_{b,i} + t_{b,p^j<p^l,i} = r_{b,k}$, so $t_{b,p^j<p^l,i}$ must be equal to 0.

The rest are proven similarly, and we just state the results.

Case 3 $\boldsymbol{b_i}$ **is type 1 and** $\boldsymbol{b_k}$ **is type 3** In this case, $r_{b,i} = 0$.

Case 4 Both $\boldsymbol{b_i}$ **and** $\boldsymbol{b_k}$ **are type 2** In this case, $r_{b,i} = r_{b,k}$.

Case 5 $\boldsymbol{b_i}$ **is type 2 and** $\boldsymbol{b_k}$ **is type 1** In this case, $r_{b,k} = 0$ and $r_{b,i}$ is undetermined.

Case 6 Both $\boldsymbol{b_i}$ **and** $\boldsymbol{b_k}$ **are type 1** In this case, both $r_{b,i}$ and $r_{b,k}$ are undetermined.

From these six cases, we will now deduce (5.6) in the proposition.

Consider an S–tuple $b \in \mathbb{N}_+^S$. These fall in five classes:

(1) All b_i are equal to 1.

(2) All b_i are of type 1 or equal to 1.

(3) Exactly one b_i is of type 2, and the rest are of type 1 or equal to 1.

(4) At least two b_i are of type 2, and the rest are of type 1 or equal to 1.

(5) At least one b_i is of type 3.

We will now consider these cases one by one.

(1) We can choose $r_{b,i}$ arbitrarily since they don't affect the sum, so we let $r_b = 0$.

(2) This corresponds to the middle sum in (5.6).

(3) Assume b_i is of type 2. By case 5, for all b_k of type 1, $r_{b,k} = 0$, but nothing can be said about $r_{b,i}$ nor $t_{b,p^j<p^l,i}$. We choose $r_b = r_{b,i}$, and this corresponds to one summand in the first sum and one summand in the last sum in (5.6).

(4) Assume $b_i = p^{j_i} < p^{l_i}$ and $b_k = p^{j_k} < p^{l_k}$ are of type 2. Then using case 4 twice, we get that $t_{b,p^j<p^l,k} = t_{b,p^j<p^l,k} = 0$ and $r_{b,k} = r_{b,i}$. If b_j is of type 1, case 5 shows that $r_{b,j} = 0$. We choose $r_b = r_{b,i}$, and this corresponds to the first sum in (5.6).

(5) Case 2 shows that for all $b_k = p^j + p^l$ of type 2, $t_{b,p^j<p^l,k} = 0$ and $r_{b,k} = r_{b,i}$. From case 3, $r_{b,k} = 0$ for all k with b_k of type 1. Finally, case 1 says that $r_{b,k} = r_{b,i}$ for all b_k of type 3, so we let $r_b = r_{b,i}$. This also corresponds to the first sum in (5.6). □

6 Calculating the homotopy groups of $\Lambda_{T^n} H\mathbb{F}_p$

In this section, we will calculate the homotopy groups $L_*(T^n)$ for $n \le p$. We will use the bar spectral sequence and the multifold Hopf algebra structure of $L_*(T^n)$ to make the calculation.

The proof of Theorem 6.2 calculating $L_*(T^n)$ is very long and spread over several lemmas, so we will now give a sketch of how the proof is structured.

The outermost layer is a double induction argument on the dimension n of the tori and the degrees of the elements in $L_*(T^n)$, and this induction argument uses several properties of $L_*(T^n)$, all of which must be proven in the induction step, and is thus included in Theorem 6.2.

The main calculation in the induction step is done using the bar spectral sequence $E^*(T^{\boldsymbol{n}})$. We use the Bökstedt spectral sequence to identify the E^2–page of the bar spectral sequence and to show that all d^2–differentials are zero.

The collection $L_*(T^n)$ can be endowed with a multifold Hopf algebra structure, and the degrees, modulo $2p$, of the simultaneously primitive elements in $L_*(T^{\boldsymbol{n}})$ are calculated. Using this, we can show that there are no other nonzero differentials and hence calculate $E^\infty(T^{\boldsymbol{n}})$.

From $E^\infty(T^{\boldsymbol{n}})$, we can choose a set of $\mathbb{F}_p$–algebra generators for $L_*(T^{\boldsymbol{n}})$, and in a couple of steps, we perturb these sets of generators such that they have nicer and nicer properties. Using these nicer properties, we are able to prove all the remaining statements in Theorem 6.2 and thus finish the induction step. In particular, we need the multifold Hopf algebra structure to get hold of the multiplicative structure in $L_*(T^{\boldsymbol{n}})$.

We will now construct a family of bar spectral sequences that will be the backbone in our calculation of $L_*(T^{\boldsymbol{n}})$.

The attaching maps in the CW-structures yield cofiber sequences

$$S^{n-1} \xrightarrow{f^{\boldsymbol{n}}} T^{\boldsymbol{n}}_{n-1} \to T^{\boldsymbol{n}},$$

giving an equivalence of commutative $H\mathbb{F}_p$–algebra spectra

$$B(\Lambda_{D^n} H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, \Lambda_{T^{\boldsymbol{n}}_{n-1}} H\mathbb{F}_p) \simeq \Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p.$$

By Proposition 8.2, there is an $\mathbb{F}_p$–algebra bar spectral sequence

$$E^2(T^{\boldsymbol{n}}) = \mathrm{Tor}^{L_*(S^{n-1})}(L_*(T^{\boldsymbol{n}}_{n-1}), \mathbb{F}_p) \implies L_*(T^{\boldsymbol{n}}).$$

The spectral sequence is indexed such that the differentials are of the form $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$. The differentials are only given up to multiplication with a unit.

For each $i \in \boldsymbol{n}$, the pinch of the i^{th} circle in $T^{\boldsymbol{n}}$ induces a map of cofiber sequences

$$\begin{tikzcd}
S^{n-1} \arrow[r, "f^{\boldsymbol{n}}"] \arrow[d] & T^{\boldsymbol{n}}_{n-1} \arrow[r] \arrow[d] & T^{\boldsymbol{n}} \arrow[d] \\
S^{n-1} \vee S^{n-1} \arrow[r] & T^{\boldsymbol{n}}_{n-1} \amalg_{T^{\boldsymbol{n}\backslash i}} T^{\boldsymbol{n}}_{n-1} \arrow[r] & T^{\boldsymbol{n}} \amalg_{T^{\boldsymbol{n}\backslash i}} T^{\boldsymbol{n}}
\end{tikzcd}$$

inducing a map of simplicial spectra

$$\begin{aligned}
&B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, \Lambda_{T^{\boldsymbol{n}}_{n-1}} H\mathbb{F}_p) \\
&\to B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p \wedge_{H\mathbb{F}_p} \Lambda_{S^{n-1}} H\mathbb{F}_p, \Lambda_{T^{\boldsymbol{n}}_{n-1}} H\mathbb{F}_p \wedge_{\Lambda_{T^{\boldsymbol{n}\backslash i}} H\mathbb{F}_p} \Lambda_{T^{\boldsymbol{n}}_{n-1}} H\mathbb{F}_p) \\
&\simeq B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, \Lambda_{T^{\boldsymbol{n}}_{n-1}} H\mathbb{F}_p) \wedge_{\Lambda_{T^{\boldsymbol{n}\backslash i}} H\mathbb{F}_p} B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, \Lambda_{T^{\boldsymbol{n}}_{n-1}} H\mathbb{F}_p).
\end{aligned}$$

Hence by Proposition 8.4, if $E^r(T^{\boldsymbol{n}})$ is flat as an $L_*(T^{\boldsymbol{n}\backslash i})$–module then $E^*(T^{\boldsymbol{n}})$ is a spectral sequence of $L_*(T^{\boldsymbol{n}\backslash i})$–Hopf algebras, and if $L_*(T^{\boldsymbol{n}})$ is flat as an $L_*(T^{\boldsymbol{n}\backslash i})$–module, then $L_*(T^{\boldsymbol{n}})$ is an $L_*(T^{\boldsymbol{n}\backslash i})$–Hopf algebra, and the spectral sequence converges to $L_*(T^{\boldsymbol{n}})$ as an $L_*(T^{\boldsymbol{n}\backslash i})$–Hopf algebra.

Definition 6.1 Given a finite ordered set $S = \{s_1 < \cdots < s_n\}$, we define an S–labeled admissible word to be an admissible word of length n, where the first letter is labeled with s_n, the second with s_{n-1}, and so forth. We define B_S to be the $\mathbb{F}_p$–Hopf algebra that is a tensor product of exterior algebras on all S–labeled admissible monic words of odd degree and divided power algebras on all S–labeled admissible monic words of even degree. We let $B_\varnothing = \mathbb{F}_p$ be generated by the empty word in degree zero.

The operator $\sigma_{s_n}\colon B_{S\backslash x_n} \to B_S$ is determined by $\sigma_{s_n}(x) = \varrho_{s_n} x$ and $\sigma_{s_n}(x) = \varrho^0_{s_n} x$ when x is an $S\backslash s_n$–labeled admissible word of even and odd degree, respectively.

Forgetting the labels on the letters induces an $\mathbb{F}_p$–Hopf algebra isomorphism between B_S and B_n. An example of an S–labeled word of length 3 is $\varrho^k_{s_3}\varrho_{s_2}\mu_{s_1}$.

Given a finite subcategory $\Delta \subseteq \mathcal{I}$, we define

$$T^\Delta = \operatorname*{colim}_{U\in\Delta} T^U.$$

Theorem 6.2 *Given $1 \le k \le p$ when $p \ge 5$ and $1 \le k \le 2$ when $p = 3$, let Δ be a finite subcategory of $\mathcal{I}$ of dimension at most k, and let V be a nonempty set in $\mathcal{I}$ of cardinality at most k.*

(1) *The map $L(f^{\boldsymbol{k}})\colon L_*(S^{k-1}) \to L_*(T^{\boldsymbol{k}}_{k-1})$ factors through $\mathbb{F}_p$.*

(2) *When $k \ge 2$, the spectral sequence $E^*(T^{\boldsymbol{k}})$ collapses on the E^2–term.*

(3) *There is a natural isomorphism*

$$\alpha_V\colon L_*(T^V) \to \bigotimes_{U\subseteq V} B_U,$$

where B_U is described in Definition 6.1, of functors from $V(\boldsymbol{k})$ to $\mathbb{F}_p$–algebras inducing an isomorphism of $\mathbb{F}_p$–algebras

$$L_*(T^\Delta) \cong \operatorname*{colim}_{U\in\Delta} L_*(T^U) \cong \bigotimes_{U\in\Delta} B_U.$$

In the sequel, we will identify $L_*(T^V)$ and $\bigotimes_{U\subseteq V} B_U$ by means of α_V.

(4) *For every $v \in V$, the projection maps*

$$\mathrm{pr}\colon L_*(T^V) \overset{\alpha}{\cong} \bigotimes_{U\subseteq V} B_U \to \bigotimes_{i\in V} B_{\{i\}}$$

induce maps of Hopf algebras

$$(L_*(T^V), L_*(T^{V\setminus v})) \xrightarrow{\mathrm{pr}} (P(\mu_V), P(\mu_{V\setminus v})).$$

(5) *Assume $|V| \ge 2$ and let v be the greatest integer in V. The operator*

$$\sigma\colon L_*(T^{V\setminus v}) \to L_*(T^V)$$

is determined by the fact that σ is a derivation and that $\sigma(z) = \varrho_v z$ and $\sigma(z) = \varrho^0_v z$ when $\varnothing \ne U \subseteq V \setminus v$ and z is an U–labeled admissible word in $B_U \subseteq L_*(T^{V\setminus v})$ of even and odd degree, respectively. In particular, for any $z \in L_*(T^{V\setminus v})$, we have that $\sigma(z)$ is in the kernel of

$$\mathrm{pr}\colon L_*(T^V) \overset{\alpha}{\cong} \bigotimes_{U\subseteq V} B_U \to \bigotimes_{i\in V} B_{\{i\}}.$$

(6) *There is a commutative diagram*

$$\begin{tikzcd} L_*(T^V) \arrow[r, "\alpha"] \arrow[d, "g^V"] & \bigotimes_{U\subseteq V} B_U \arrow[d, "\mathrm{pr}"] \\ L_*(S^V) \arrow[r, "\cong"] & B_V \end{tikzcd}$$

where the bottom isomorphism is the one from Theorem 3.6 together with the canonical isomorphism $B_{|V|} \cong B_V$ given by labeling the words in $B_{|V|}$.

The range $k \le p$ comes from all the lemmas in Section 9 concerning the degrees of primitive elements. It is possible that this range could be improved by getting better control of the degrees of the primitive elements.

When $k = p = 3$, parts (1) and (2) of the theorem still hold, but we are not able to determine the multiplicative structure of $L_*(T^{\mathbf{3}}) \cong E^\infty(T^{\mathbf{3}}) \cong L_*(T^{\mathbf{3}}_2) \otimes B_3$. This is because the degrees of $\gamma_{p^{k+1}}(\varrho^0\varrho\mu) \in \Gamma(\varrho^0\varrho\mu) = B_3$ are equal to the degrees of $\mu_1^{p^k+p^{k+1}}\mu_2^{p^k}\mu_3^{p^k}$. Thus, we can't use Proposition 5.5 to show that $(\gamma_{p^{k+1}}(\varrho^0\varrho\mu))^p$ is a simultaneously primitive element in the **3**–fold Hopf algebra $L_*(T^{\mathbf{3}})$.

The idea to look at the simultaneously primitive elements to show that the spectral sequence collapses on the E^2–term originated from a note by John Rognes, where he showed that the spectral sequence $E^*(T^{\mathbf{3}})$ collapses on the E^2–term.

Remark 6.3 It should be possible to prove a similar result for $V(0)_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{Z})$. The difference would be the degrees of the elements in the rings, and thus the degrees of the simultaneously primitive elements. The arguments in Section 3 would thus have to be adjusted for these new elements, and it may be preferable to work modulo $2p^2$ instead of modulo $2p$.

We need this corollary to identify the E^2–term $E^2(T^{\mathbf{n}})$ and show that there are no d^2–differentials.

Corollary 6.4 *Given n, assume Theorem 6.2 holds when $1 \le k \le n-1$. Given $m \ge 0$, if $f^{\mathbf{n}}\colon L_*(S^{n-1}) \to L_*(T^{\mathbf{n}}_{n-1})$ factors through $\mathbb{F}_p$ in degrees less than or equal to $2pm-1$, and the spectral sequence $E^*(T^{\mathbf{n}})$ collapses in total degrees less than or equal to $2pm-1$ (that is, $E^2(T^{\mathbf{n}}) = E^\infty(T^{\mathbf{n}})$ in these degrees), then*

(1) *the map $f^{\mathbf{n}}\colon L_*(S^{n-1}) \to L_*(T^{\mathbf{n}}_{n-1})$ factors through $\mathbb{F}_p$ in degrees less than or equal to $2p(m+1)-2$,*

(2) *the spectral sequence $E^*(T^{\mathbf{n}})$ collapses in total degrees less than or equal to $2p(m+1)-2$.*

Proof From Lemma 10.1, we know that as an $\mathbb{F}_p$–module,

$$H_*(\Lambda_{T^n} H\mathbb{F}_p)_{\le 2p(m+1)-2} \cong \Big(A_* \otimes \bigotimes_{U \subseteq \boldsymbol{n}} B'_U\Big)_{\le 2p(m+1)-2}.$$

Since $\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p$ is a generalized Eilenberg–Mac Lane spectrum, the Hurewicz homomorphism induces an isomorphism between the $\mathbb{F}_p$–modules $A_* \otimes L_*(T^{\boldsymbol{n}}) \cong A_* \otimes E^\infty(T^{\boldsymbol{n}})$ and $H_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)$.

The E^1–terms of the bar spectral sequence $E^1(T^{\boldsymbol{n}})$ is the two-sided bar complex

$$E^1_{s,*}(T^{\boldsymbol{n}}) = B_s(L_*(T^{\boldsymbol{n}}_{n-1}), B_{n-1}, \mathbb{F}_p) \cong L_*(T^{\boldsymbol{n}}_{n-1}) \otimes B_{n-1}^{\otimes s} \otimes \mathbb{F}_p,$$

where the $B_{n-1}^{\otimes s}$ module structure on $L_*(T^{\boldsymbol{n}}_{n-1})$ is induced by $f^{\boldsymbol{n}}$, and the differential $d^1\colon E^1_{s,t}(T^{\boldsymbol{n}}) \to E^1_{s-1,t}(T^{\boldsymbol{n}})$ is given by

$$\begin{aligned} &d^1(a \otimes b_1 \otimes \cdots \otimes b_{s+1}) \\ &\qquad = af^{\boldsymbol{n}}(b_1) \otimes b_2 \otimes \cdots \otimes b_{s+1} + \sum (-1)^i a \otimes b_1 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_{s+1}. \end{aligned}$$

If $f^{\boldsymbol{n}}$ factors through $\mathbb{F}_p$ in degrees less than l, then

$$\begin{aligned} E^2(T^{\boldsymbol{n}}) &= \mathrm{Tor}^{L_*(S^{n-1})}(L_*(T^{\boldsymbol{n}}_{n-1}), \mathbb{F}_p) \\ &\cong L_*(T^{\boldsymbol{n}}_{n-1}) \otimes \mathrm{Tor}^{L_*(S^{n-1})}(\mathbb{F}_p, \mathbb{F}_p) = L_*(T^{\boldsymbol{n}}_{n-1}) \otimes B_{\boldsymbol{n}} \end{aligned}$$

in bidegrees (s,t) with $t < l$. Furthermore,

$$E^2_{0,l}(T^{\boldsymbol{n}}) \cong L_*(T^{\boldsymbol{n}}_{n-1}) / \operatorname{im}(f^{\boldsymbol{n}}).$$

If $f^{\boldsymbol{n}}$ doesn't factor through $\mathbb{F}_p$ in degrees $l \le 2p(m+1)-2$, then the dimension of $A_* \otimes E^2(T^{\boldsymbol{n}})$ in total degree l is smaller than the dimension of $H_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)$ in degree l, giving us a contradiction. Thus $f^{\boldsymbol{n}}$ factors through $\mathbb{F}_p$ in degrees less than or equal to $2p(m+1)-2$.

By a similar argument, if there are any nonzero d^r–differentials in $E^r(T^{\boldsymbol{n}})$ starting in total degrees less than or equal to $2p(m+1)-1$, the dimension of $A_* \otimes E^r(T^{\boldsymbol{n}})$ in the degree of the image of this differential will be smaller than the dimension of $H_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)$ in this degree.

Thus the spectral sequence $E^*(T^{\boldsymbol{n}})$ collapses in total degrees less than or equal to $2p(m+1)-2$. □

Proof of Theorem 6.2 The proof is by induction. Given n, with $1 \le n \le p$ when $p \ge 5$ and $1 \le n \le 2$ when $p = 3$, assume the theorem holds for all k with $1 \le k < n$.

In this proof, we call this *the torus-induction hypothesis*. The only place in the proof where there is a difference between $p=3$ and $p\geq 5$ is when we invoke Corollary 9.4 in the proof of part (3).

When $n=2$, the theorem holds since $L_*(T_1^U)\cong P(\mu_U)$.

Parts (1) and (2) We proceed by induction on the degrees of elements in part (1) and total degrees in part (2). Given m, assume that parts (1) and (2) hold in degrees less than or equal to $2pm-1$. This is trivially true when $m=0$.

By Corollary 6.4, parts (1) and (2) hold in (total) degrees at most $2p(m+1)-2$. We must show that they hold in degree $2p(m+1)-1$.

The attaching map $f^{\boldsymbol{n}}\colon L_*(S^{n-1})\to L_*(T^{\boldsymbol{n}}_{n-1})$ is determined by what it does on the set of algebra generators in $L_*(S^{n-1})$ given by the monic words of length $n-1$, and by Lemma 9.2 there are no such element in degrees -1 modulo $2p$. Hence $f^{\boldsymbol{n}}$ factors through $\mathbb{F}_p$ in degrees less than or equal to $2p(m+1)-1$. So in vertical degrees less than or equal to $2p(m+1)-1$, the Künneth isomorphism yields an $L_*(T^{\boldsymbol{n}}_{n-1})$–module isomorphism

$$\begin{aligned} E^2(T^{\boldsymbol{n}}) &= \operatorname{Tor}^{L_*(S^{n-1})}(L_*(T^{\boldsymbol{n}}_{n-1}),\mathbb{F}_p)\\ &\cong L_*(T^{\boldsymbol{n}}_{n-1})\otimes \operatorname{Tor}^{L_*(S^{n-1})}(\mathbb{F}_p,\mathbb{F}_p)\cong L_*(T^{\boldsymbol{n}}_{n-1})\otimes B_n. \end{aligned}$$

It remains to show that there are no d^r–differentials in $E^r(T^{\boldsymbol{n}})$ starting in total degrees $2p(m+1)$. For every i in $\boldsymbol{n}$, we have that $E^2(T^{\boldsymbol{n}})$ is an $L_*(T^{\boldsymbol{n}\setminus i})$–Hopf algebra spectral sequence since $E^2(T^{\boldsymbol{n}})$ is flat over $L_*(T^{\boldsymbol{n}\setminus i})$. The Hopf algebra structure on $E^2(T^{\boldsymbol{n}})$ is the tensor product of the $L_*(T^{\boldsymbol{n}\setminus i})$–Hopf algebra structures on $L_*(T^{\boldsymbol{n}}_{n-1})$ and the $\mathbb{F}_p$ Hopf algebra structure on B_n. Thus by Proposition 8.6, a shortest nonzero differential in lowest total degree must go to a primitive element in the $L_*(T^{\boldsymbol{n}\setminus i})$–Hopf algebra structure. Hence if a shortest nonzero differential starts in total degree $2p(m+1)$, there must be elements in degree $2p(m+1)-1$ that are primitive in the $L_*(T^{\boldsymbol{n}\setminus i})$–Hopf algebra structure for all $i\in\boldsymbol{n}$.

By the graded version of [28, Proposition 3.12], the $L_*(T^{\boldsymbol{n}\setminus i})$–primitive elements in $L_*(T^{\boldsymbol{n}}_{n-1})\otimes B_n$ are linear combinations of primitive elements in $L_*(T^{\boldsymbol{n}}_{n-i})$ and B_n, and the module of $L_*(T^{\boldsymbol{n}\setminus i})$–primitive elements in B_n is $L_*(T^{\boldsymbol{n}\setminus i})\{X_n\}$, where X_n is the set of monic words in B_n. The intersection $\bigcap_{i\in\boldsymbol{n}} L_*(T^{\boldsymbol{n}\setminus i})\{X_n\}$ is equal to $\mathbb{F}_p\{X_n\}$ since $\bigcap_{i\in\boldsymbol{n}} L_*(T^{\boldsymbol{n}\setminus i})=\mathbb{F}_p$. Thus the module of elements in $B_n\subseteq E^2(T^{\boldsymbol{n}})$ that are primitive in the $L_*(T^{\boldsymbol{n}\setminus i})$–Hopf algebra structure for every $i\in\boldsymbol{n}$ is $\mathbb{F}_p\{X_n\}\subseteq B_n$, which is isomorphic to the module of $\mathbb{F}_p$–primitive elements in B_n under the projection map $E^2(T^{\boldsymbol{n}}) \to B_n$. By Lemma 3.5, there are no $\mathbb{F}_p$–primitive elements in B_n in degrees -1 modulo $2p$ when $n \le 2p$. Hence there are no differentials starting in total degree $2p(m+1)$ that have target in filtration 1 or higher.

It remains to show that there are no differentials starting in total degree $2p(m+1)$ that have nonzero target in filtration 0. This is only possible if there are $\boldsymbol{n}$–fold primitive elements in $L_*(T^{\boldsymbol{n}}_{n-1})$ in the target of the differential. If z is an indecomposable element in B_n in degree $2p(m+1)$, Corollary 9.10 says there are no $\boldsymbol{n}$–fold primitive elements in $L_*(T^{\boldsymbol{n}}_{n-1})$ in degree $2p(m+1)-1$ when $n \le p$.

Hence there are no differentials in $E^*(T^{\boldsymbol{n}})$ when $n \le p$, so $E^*(T^{\boldsymbol{n}})$ collapses on the E^2–term. Since $E^2(T^{\boldsymbol{n}})$ is isomorphic to $E^\infty(T^{\boldsymbol{n}})$, it is flat as an $L_*(T^{\boldsymbol{n}\setminus i})$–module, and $L_*(T^{\boldsymbol{n}})$ is flat as an $L_*(T^{\boldsymbol{n}\setminus i})$–module. The spectral sequence converges to $L_*(T^{\boldsymbol{n}})$ as an $L_*(T^{\boldsymbol{n}\setminus i})$–Hopf algebra.

Parts (3) and (4) We will only show the theorem for the set $V = \boldsymbol{n}$.

(1) Let $\mathcal{G}_1^{\mathrm{odd}}$ and $\mathcal{G}_1^{\mathrm{even}}$ and be the sets of all admissible words of length n starting with ϱ or ϱ^0, respectively, and let $\mathcal{G}_1 = \mathcal{G}_1^{\mathrm{odd}} \cup \mathcal{G}_1^{\mathrm{even}}$.

(2) Let $\mathcal{G}_2$ be the set of all admissible words of length n that start with φ^i or ϱ^{i+1} for $i \ge 0$.

The set $\mathcal{G}_2$ only contains elements in degree 0 modulo $2p$, and the sets $\mathcal{G}_1$ and $\mathcal{G}_2$ generate B_n as an $\mathbb{F}_p$–algebra.

We can also think of $\mathcal{G}_1^{\mathrm{odd}}$ and $\mathcal{G}_1^{\mathrm{even}}$ as sets of elements in $E^2_{1,*}(T^{\boldsymbol{n}})$ of odd and even degree, respectively, and $\mathcal{G}_2$ as a set of elements in $E^2_{s,*}(T^{\boldsymbol{n}})$ with $s \ge 2$; together they generate $E^2(T^{\boldsymbol{n}})$ as an $L_*(T^{\boldsymbol{n}}_{n-1})$–algebra.

We will define sets $\overline{\mathcal{G}}_1$ and $\overline{\mathcal{G}}_2$ of elements in $L_*(T^{\boldsymbol{n}})$, with bijections $\overline{\mathcal{G}}_1 \cong \mathcal{G}_1$ and $\overline{\mathcal{G}}_2 \cong \mathcal{G}_2$, which generate $L_*(T^{\boldsymbol{n}})$ as an $L_*(T^{\boldsymbol{n}}_{n-1})$–algebra.

The $\mathbb{F}_p$–isomorphism α is then the composite

$$\alpha\colon L_*(T^{\boldsymbol{n}}) \xrightarrow{\alpha'} L_*(T^{\boldsymbol{n}}_{n-1}) \bigotimes B_n \xrightarrow{\mathrm{id}\otimes\alpha''} L_*(T^{\boldsymbol{n}}_{n-1}) \bigotimes B_{\boldsymbol{n}},$$

where α' is the $\mathbb{F}_p$–isomorphism induced by the bijections $\overline{\mathcal{G}}_i \cong \mathcal{G}_i$, and α'' is the $\mathbb{F}_p$–algebra isomorphism $B_n \cong B_{\boldsymbol{n}}$ given by labeling the words in B_n.

Let $\overline{\mathcal{G}}_1$ be the set of elements $\sigma(z_{\boldsymbol{n-1}})$ in $L_*(T^{\boldsymbol{n}})$, where $z_{\boldsymbol{n-1}}$ runs over all admissible words in $B_{\boldsymbol{n-1}} \subseteq L_*(T^{\boldsymbol{n-1}})$. Define the bijection $\mathcal{G}_1 \cong \overline{\mathcal{G}}_1$ by mapping $\varrho(z)$ or $\varrho^0(z)$, where z is an admissible word in B_{n-1} of odd or even degree, respectively, to $\sigma(z_{\boldsymbol{n-1}})$, where $z_{\boldsymbol{n-1}}$ is the labeled version of z.

We will use induction on the degrees of elements in $L_*(T^{\boldsymbol{n}})$ to define the bijection $\overline{\mathcal{G}}_2 \cong \mathcal{G}_2$ and to prove parts (3) and (4).

Let *the degree-induction hypothesis* be as follows:

(1) In degrees less than $2pl$, we have lifted all elements in $\mathcal{G}_2$ to elements in $\overline{\mathcal{G}}_2$, and in degrees less than $2pl$, this lift induces an isomorphism α satisfying Theorem 6.2(3).

(2) In degrees less than $2pl$, we have defined a homomorphism pr satisfying Theorem 6.2(4).

See Definition 5.2 for the definition of an homomorphism in degrees less than $2pl$. When $l = 0$, there is nothing to prove. Assume we have proved it for m when $m = l$.

First we prove hypothesis (1) of the degree-induction hypothesis for $m+1$. For each $x \in \mathcal{G}_2$, let

$$\hat{x} = \sum_{U \subseteq \boldsymbol{n}} (-1)^{n-|U|} \operatorname{in}_U^{\boldsymbol{n}} \operatorname{pr}_U^{\boldsymbol{n}}(x), \tag{6.5}$$

and let

$$\bar{x} = \hat{x} - \sum_{b \in \mathbb{N}_+^{\boldsymbol{n}}} r_{b,x} \mu_{\boldsymbol{n}}^b, \tag{6.6}$$

where the elements $r_{b,x}$ are given in (6.7). Define $\overline{\mathcal{G}}_2$ to be $\{\bar{x} \mid x \in \mathcal{G}_2,\ |x| \le 2pm\}$.

To prove Theorem 6.2(3) for $m+1$, we will first show that α is well defined in degree $2pm$ on decomposable elements. By the degree-induction hypothesis, it suffices to prove that if $y \in \overline{\mathcal{G}}_1 \cup \overline{\mathcal{G}}_2$ is a nonzero element of degree $2m$, then $y^p = 0$.

By Proposition 4.20, $L_*(T^-)$ is an $\boldsymbol{n}$–fold Hopf algebra. By the degree-induction hypothesis and the definition of α, the only monomials in $L_*(T^{\boldsymbol{n}})$ in degrees less than $2m$ that are nonzero when raised to the power of p are the monomials in the subring $P(\mu_{\boldsymbol{n}}) \subseteq L_*(T^{\boldsymbol{n}})$. Thus, by Frobenius and Theorem 6.2(4), we have

$$\psi_{\boldsymbol{n}}^i(y^p) = \psi_{\boldsymbol{n}}^i(y)^p = \Big(1 \otimes y + y \otimes 1 + \sum y' \otimes y''\Big)^p = 1 \otimes y^p + y^p \otimes 1$$

in all $L_*(T^{\boldsymbol{n}\setminus i})$–Hopf algebra structures. Hence y^p is primitive in the $L_*(T^{\boldsymbol{n}\setminus i})$–Hopf algebra structure for every $i \in \boldsymbol{n}$.

Let y^p be represented by $y_s \in E^{\infty}_{s,*}(T^{\boldsymbol{n}})$ modulo lower filtration. Then, since y^p is primitive in the $L_*(T^{\boldsymbol{n}\setminus i})$–Hopf algebra structure, y_s must be primitive in the $L_*(T^{\boldsymbol{n}\setminus i})$–Hopf algebra $E^{\infty}(T^{\boldsymbol{n}})$ for every $i \in \boldsymbol{n}$. If y_s is not primitive, $\psi^i_{\boldsymbol{n}}(y^p)$ would not be equal to $y_s \otimes 1 + 1 \otimes y_s$ in filtration s.

By the graded version of [28, Proposition 3.12], the $L_*(T^{\boldsymbol{n}\setminus i})$–primitive elements in $L_*(T^{\boldsymbol{n}}_{n-1}) \otimes B_n$ are linear combinations of primitive elements in $L_*(T^{\boldsymbol{n}}_{n-i})$ and B_n, and the module of $L_*(T^{\boldsymbol{n}\setminus i})$–primitive elements in B_n is $L_*(T^{\boldsymbol{n}\setminus i})\{x_j\}$, where x_j runs over the monic words in B_n. The intersection $\bigcap_{i\in\boldsymbol{n}} L_*(T^{\boldsymbol{n}\setminus i})\{x_j\}$ is equal to $\mathbb{F}_p\{x_j\}$ since $\bigcap_{i\in\boldsymbol{n}} L_*(T^{\boldsymbol{n}\setminus i}) = \mathbb{F}_p$. Thus, the module of elements in $B_n \subseteq E^2(T^{\boldsymbol{n}})$ that are primitive in the $L_*(T^{\boldsymbol{n}\setminus i})$–Hopf algebra structure for every $i \in \boldsymbol{n}$ is $\mathbb{F}_p\{x_j\} \subseteq B_n$, which is isomorphic to the module of $\mathbb{F}_p$–primitive elements in B_n, under the projection map $E^2(T^{\boldsymbol{n}}) \to B_n$.

The degree of y^p is at least 4 and is 0 modulo $2p$, so by Lemma 3.5 there are no $\mathbb{F}_p$–primitive elements in B_n in the degree of y^p.

Hence y^p must be equal to an $\boldsymbol{n}$–fold primitive element in $L_*(T^{\boldsymbol{n}}_{n-1})$. By Corollary 9.10, the degree of y^p is not equal to the degree of any $\boldsymbol{n}$–fold primitive element in $L_*(T^{\boldsymbol{n}}_{n-1})$ when $n \le p$. Thus $y^p = 0$.

We will now extend α to degree $2pm$ on all elements. First we show that $L_*(T^-)$ is a $P_{\mathbb{F}_p}(\mu_-)$ split S–fold Hopf algebra in degree q. By the torus-induction hypothesis and Theorem 6.2(4), there is a splitting of $\boldsymbol{n}$–fold Hopf algebras $P_{\mathbb{F}_p}(\mu_-) \to L_*(T^-_{n-1}) \to P_{\mathbb{F}_p}(\mu_-)$. Since $L_*(T^-_{n-1})$ is the restriction (see Definition 4.25) of $L_*(T^-)$ to the full subcategory of $V(\boldsymbol{n})$ not containing $\boldsymbol{n}$, Definition 5.4(1) is thus satisfied for the $\boldsymbol{n}$–fold Hopf algebra $L_*(T^-)$ for $q = 2pm$. By the degree-induction hypothesis and Theorem 6.2(4), this map can be extended to a map

$$\mathrm{pr}\colon L_*(T^{\boldsymbol{n}}) \to P_{\mathbb{F}_p}(\mu_-)$$

in degrees less than $2pm$ such that Definition 5.4(3) is satisfied. We can further extend it to an $\mathbb{F}_p$–algebra map

$$\mathrm{pr}'\colon L_*(T^{\boldsymbol{n}}) \to P_{\mathbb{F}_p}(\mu_-)$$

in degrees less than or equal to $2pm$ by mapping $\hat{x}$, defined in (6.5), to zero for $x \in \mathcal{G}_2$ in degree $2pm$. This is well defined since $\hat{x}$ is indecomposable. The map pr' satisfies Definition 5.4(2).

From (6.5), $\hat{x} \in \bigcap_{i\in\boldsymbol{n}} \ker(\epsilon^i_{\boldsymbol{n}}\colon L_*(T^{\boldsymbol{n}}) \to L_*(T^{\boldsymbol{n}\setminus i}))$, and by the definition of pr', we have $\hat{x} \in \ker(\mathrm{pr}')$.

For $x \in \mathcal{G}_2$, the degree of x is equal to the degree of an admissible word in $B_n \subsetneq E^2(T^{\boldsymbol{n}})$ of even degree. By Corollary 9.4, $\hat{x}$ is thus not in the same degree as any of the elements $\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\cdots\mu_n^{p^{j_n}}$ or $(\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\cdots\mu_n^{p^{j_n}})\mu_s^{p^{j_{n+1}}}$, where $1 \le s \le n$ and $j_i \in \mathbb{N}$ for all $1 \le i \le n+1$.

Hence Proposition 5.5 gives us that for some $r_{b,x}$ in $\mathbb{F}_p$,

$$(\mathrm{pr}'_{P(\mu_{\boldsymbol{n}})} \otimes_{P(\mu_{\boldsymbol{n}\setminus i})} \mathrm{pr}'_{P(\mu_{\boldsymbol{n}})}) \circ \psi^i_{\boldsymbol{n}}(\hat{x}) = \sum_{b \in \mathbb{N}^{\boldsymbol{n}}_+} r_{b,x}\mu^b_{\boldsymbol{n}\setminus i}\tilde{\psi}^i(\mu_i^{b_i}). \tag{6.7}$$

To finish the construction of α in degrees less than $2p(m+1)$, we must show that it is well defined on elements in $\overline{\mathcal{G}}_1$. Note that if $y \in \overline{\mathcal{G}}_1$ is of odd degree, then $y^2 = 0$ since the ring is graded commutative. This shows that α is an $\mathbb{F}_p$–isomorphism in degree less than $2p(m+1)$.

To show that it is a natural isomorphism of functors from $V(\boldsymbol{k})$ to $\mathbb{F}_p$–algebras, we must show that $\mathrm{pr}^{\boldsymbol{n}}_V(y) = 0$ for all $y \in \overline{\mathcal{G}}_1 \cup \overline{\mathcal{G}}_2$. In (6.6), we only sum over sequences of positive integers, so $\mathrm{pr}^{\boldsymbol{n}}_V(\bar{x}) = \mathrm{pr}^{\boldsymbol{n}}_V(\hat{x}) = 0$ for all $V \subsetneq \boldsymbol{n}$, proving naturality for the set $\overline{\mathcal{G}}_2$. Naturality for the elements in $\overline{\mathcal{G}}_1$ follows from Lemma 6.10.

Now we proof hypothesis (2) of the degree-induction hypothesis for $m+1$. To show that $\mathrm{pr}_{P(\mu_{\boldsymbol{n}})}$ is an Hopf algebra morphism for all $i \in \boldsymbol{n}$, we must show that

$$(\mathrm{pr}_{P(\mu_{\boldsymbol{n}})} \otimes_{P(\mu_{\boldsymbol{n}\setminus i})} \mathrm{pr}_{P(\mu_{\boldsymbol{n}})}) \circ \psi^i_{\boldsymbol{n}}(x) = 0 \tag{6.8}$$

for x in $\overline{\mathcal{G}}_1$ or $\overline{\mathcal{G}}_2$. By (6.6) and (6.7), this holds for x in $\overline{\mathcal{G}}_2$.

Each element in $\overline{\mathcal{G}}_1$ is of the form $\sigma(z)$ for some admissible word z in $B_{\boldsymbol{n-1}}$. By Proposition 2.7, $\sigma\colon \pi_*(\Lambda_{T^{\boldsymbol{n-1}}} H\mathbb{F}_p) \to \pi_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)$ is a derivation. From the commutative diagram (2.9), if $\psi^i_{\boldsymbol{n-1}}(z) = 1 \otimes z + z \otimes 1 + \sum z'_i \otimes z''_i$ for $i \in \boldsymbol{n-1}$, then

$$\psi^i_{\boldsymbol{n}}(\sigma(z)) = 1 \otimes \sigma(z) + \sigma(z) \otimes 1 + \sum \sigma(z'_i) \otimes z''_i \pm z'_i \otimes \sigma(z''_i).$$

By part (2) of the degree-induction hypothesis and since σ is a derivation, $\sigma(z'_i)$ and $\sigma(z''_i)$ are in the kernel of $\mathrm{pr}_{P(\mu_{\boldsymbol{n}})}$, and thus (6.8) is satisfied for $i < n$. Proposition 2.8 shows that $\sigma(z)$ is primitive as an element in the $L_*(T^{\boldsymbol{n-1}})$–Hopf algebra $L_*(T^{\boldsymbol{n}})$, proving it for $i = n$.

Part (5) This follows from the definition of $\overline{\mathcal{G}}_1$ and α.

Part (6) It suffices to prove it for the elements in $\overline{\mathcal{G}}_1$ and $\overline{\mathcal{G}}_2$ since they generate $L_*(T^{\boldsymbol{n}})$ as an $L_*(T^{\boldsymbol{n}}_{n-1})$–algebra.

Let X be the ideal generated by the nonunits in $L_*(T^{\boldsymbol{n}}_{n-1})$. By Lemma 6.9, x and $\hat{x}$ in (6.5) are equal modulo X. In (6.6), $\hat{x}$ and $\bar{x}$ are equal modulo X. From this it follows that α maps $\bar{x}$, modulo the ideal X, to the labeled version of x in $B_{\boldsymbol{n}}$, proving part (6) for elements in $\overline{\mathcal{G}}_2$.

By Proposition 8.5 and the commutative diagram

$$\begin{tikzcd}
S^1_+ \wedge \Lambda_{T^{\boldsymbol{n-1}}} H\mathbb{F}_p \arrow[r, "\omega"] \arrow[d, "S^1_+ \wedge g^{\boldsymbol{n-1}}"] & \Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p \arrow[d, "g^{\boldsymbol{n}}"] \\
S^1_+ \wedge \Lambda_{S^{n-1}} H\mathbb{F}_p \arrow[r, "\hat{\omega}"] & \Lambda_{S^n} H\mathbb{F}_p
\end{tikzcd}$$

the element $g^{\boldsymbol{n}}(\sigma(z_{\boldsymbol{n-1}}))$ is equal to $\varrho g^{\boldsymbol{n-1}}(z_{\boldsymbol{n-1}}) = \varrho z$, where z is the unlabeled version of $z_{\boldsymbol{n-1}}$. This proves part (6) for elements in $\overline{\mathcal{G}}_1$. $\square$

Lemma 6.9 *Given $x \in L_*(T^{\boldsymbol{n}})$, let*

$$\hat{x} = \sum_{U \subseteq \boldsymbol{n}} (-1)^{n-|U|} \operatorname{in}^{\boldsymbol{n}}_U \operatorname{pr}^{\boldsymbol{n}}_U(x).$$

Then for every $V \subsetneq \boldsymbol{n}$,

$$\operatorname{pr}^{\boldsymbol{n}}_V(\hat{x}) = 0,$$

and $\hat{x} = x$ in $L_*(T^{\boldsymbol{n}})$ modulo the ideal generated by the nonunits in $L_*(T^{\boldsymbol{n}}_{n-1})$.

Proof For every $U \subsetneq \boldsymbol{n}$, we have $\operatorname{in}^{\boldsymbol{n}}_U \operatorname{pr}^{\boldsymbol{n}}_U(x) \in L_*(T^{\boldsymbol{n}}_{n-1})$, so $\hat{x}$ is equal to x in $L_*(T^{\boldsymbol{n}})$ modulo the ideal generated by the nonunits in $L_*(T^{\boldsymbol{n}}_{n-1})$. The diagram

$$\begin{tikzcd}
T^{\boldsymbol{n}} \arrow[r, "\operatorname{pr}^{\boldsymbol{n}}_U"] \arrow[d, "\operatorname{pr}^{\boldsymbol{n}}_{U\cap V}"] & T^U \arrow[r, "\operatorname{in}^{\boldsymbol{n}}_U"] & T^{\boldsymbol{n}} \arrow[d, "\operatorname{pr}^{\boldsymbol{n}}_V"] \\
T^{U\cap V} \arrow[rr, "\operatorname{in}^V_{U\cap V}"] & & T^V
\end{tikzcd}$$

commutes. Hence if $V \subsetneq \boldsymbol{n}$, then

$$\operatorname{pr}^{\boldsymbol{n}}_V(\hat{x}) = \sum_{U \subseteq \boldsymbol{n}} (-1)^{n-|U|} \operatorname{pr}^{\boldsymbol{n}}_V \operatorname{in}^{\boldsymbol{n}}_U \operatorname{pr}^{\boldsymbol{n}}_U(x) = \sum_{S \subseteq V} \sum_{W \subseteq \boldsymbol{n} \setminus V} (-1)^{n-|S|-|W|} \operatorname{in}^V_S \operatorname{pr}^{\boldsymbol{n}}_S(x) = 0,$$

since

$$\sum_{W \subseteq \boldsymbol{n} \setminus V} (-1)^{n-|S|-|W|} = \sum_{i=0}^{n-|V|} (-1)^{n-|S|-i} \binom{n-|V|}{i} = 0. \qquad \square$$

Lemma 6.10 *Let x be an element in $\bigcap_{i \in \boldsymbol{n-1}} \ker(\epsilon^i_{\boldsymbol{n-1}}\colon L_*(T^{\boldsymbol{n-1}}) \to L_*(T^{\boldsymbol{n-1}\setminus i}))$. Then $\sigma(x)$ is an element in $\bigcap_{i \in \boldsymbol{n}} \ker(\epsilon^i_{\boldsymbol{n}}\colon L_*(T^{\boldsymbol{n}}) \to L_*(T^{\boldsymbol{n}\setminus i}))$.*

Proof Observe that the diagrams

$$\begin{tikzcd} S^1_+ \wedge \Lambda_{T^{\boldsymbol{n-1}}} H\mathbb{F}_p \arrow[r, "\omega"] \arrow[d, "S^1_+ \wedge \mathrm{pr}^{\boldsymbol{n-1}}_{\boldsymbol{n-1}\setminus i}"] & \Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p \arrow[d, "\mathrm{pr}^{\boldsymbol{n}}_{\boldsymbol{n}\setminus i}"] \\ S^1_+ \wedge \Lambda_{T^{\boldsymbol{n-1}\setminus i}} H\mathbb{F}_p \arrow[r, "\omega"] & \Lambda_{T^{\boldsymbol{n}\setminus i}} H\mathbb{F}_p \end{tikzcd} \qquad \begin{tikzcd} S^1_+ \wedge \Lambda_{T^{\boldsymbol{n-1}}} H\mathbb{F}_p \arrow[r, "\omega"] \arrow[d, "\mathrm{pr}_+ \wedge \mathrm{id}"] & \Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p \arrow[d, "\mathrm{pr}^{\boldsymbol{n}}_{\boldsymbol{n-1}}"] \\ S^0 \wedge \Lambda_{T^{\boldsymbol{n-1}}} H\mathbb{F}_p \arrow[r, "\cong"] & \Lambda_{T^{\boldsymbol{n-1}}} H\mathbb{F}_p \end{tikzcd}$$

commute for all $i \in \boldsymbol{n-1}$. From the left diagram, we conclude that $\mathrm{pr}^{\boldsymbol{n}}_{\boldsymbol{n}\setminus i}(\sigma(x))$ is zero when $i \neq n$. From the right, we conclude that $\mathrm{pr}^{\boldsymbol{n}}_{\boldsymbol{n-1}}(\sigma(z))$ is zero since $H_1(S^0) = 0$. $\square$

7 Periodic elements

The connective m^{th} Morava K-theory $k(m)$ is a ring spectrum with coefficient ring $k(m)_* = P_{\mathbb{F}_p}(v_m)$, where $|v_m| = 2p^m - 2$. The unit map of the ring spectrum $\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p$ induces a homomorphism $P_{\mathbb{F}_p}(v_m) \to k(m)_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)$, and we denote the image of v_m with v_m. In this section, we show that the *the Rognes element*

$$t_1\mu_1^{p^{n-1}} + t_2\mu_2^{p^{n-1}} + \cdots + t_n\mu_n^{p^{n-1}},$$

where $\mu_i \in L_2(T^{\boldsymbol{n}})$ is the image of the generator in $L_*(S^1)$ under the inclusion of the i^{th} circle, in the homotopy fixed-point spectral sequence which calculates $k(n-1)_*(F(E_2T^n_+, \Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)^{T^n})$, is not hit by any differential, and that this implies $v_{n-1} \in k(n-1)_*((\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)^{hT^n})$ is nonzero.

Given a prime p, let A_* be the dual Steenrod algebra; see Milnor [27] for details. When p is odd, $A_* = P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_0, \bar{\tau}_1, \dots)$, where $|\bar{\xi}_i| = 2p^i - 2$ and $|\bar{\tau}_i| = 2p^i - 1$, and when $p = 2$, we have $A_* = P(\bar{\xi}_1, \bar{\xi}_2, \dots)$, where $|\bar{\xi}_i| = 2^i - 1$.

See Johnson and Wilson [21] for the following details about Morava K-theory. We have $H_*(k(n)) = A\langle n\rangle_*$, where $A\langle n\rangle_* \cong P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_0, \dots, \bar{\tau}_{n-1}, \bar{\tau}_{n+1}, \dots)$ is isomorphic to the dual Steenrod algebra A_* without the generator $\bar{\tau}_n$. Multiplication by v_n yields a cofiber sequence

$$\Sigma^{2p^n-2} k(n) \to k(n) \to H\mathbb{F}_p,$$

which, in homology, decomposes into short exact sequences

$$0 \to A\langle n\rangle_* \to A_* \to \Sigma^{2p^n-1} A\langle n\rangle_* \to 0.$$

Since $\Lambda_{T^n} H\mathbb{F}_p$ is an $H\mathbb{F}_p$–module spectrum, we have as graded $k(m)_*$–algebras,

$$k(m)_*(\Lambda_{T^n} H\mathbb{F}_p) \cong k(m)_*(H\mathbb{F}_p) \otimes L_*(T^{\boldsymbol{n}}).$$

Let X be a T^n–spectrum. The homotopy fixed-point spectrum of X is defined as the mapping spectrum

$$X^{hT^n} = F(ET^n_+, X)^{T^n}$$

of T^n–equivariant based maps from ET^n_+ to X. Here ET^n_+ should be interpreted as the unreduced suspension spectrum of ET^n, the free contractible T^n–space.

Now we want to construct the first two columns of the homotopy fixed-point spectral sequence for the group T^n. We use the setup of Bruner and Rognes [12]. Let the unit sphere $S(\mathbb{C}^\infty)$ be our model for ES^1 with the S^1–action given by the coordinatewise action. The space $S(\mathbb{C}^\infty)$ is equipped with a free S^1–CW-structure with one free S^1–cell in each even degree. We use the product $S(\mathbb{C}^\infty)^n$ as a model for ET^n with the product T^n–CW-structure.

We now get a T^n–equivariant cofiber sequence

$$E_0T^n \to E_2T^n \to T^n_+ \wedge (\vee S^2),$$

where the wedge sum runs over all 2–cells in ET^n. Here T^n acts trivially on the space $(\vee S^2)$.

Proposition 7.1 *Let X be a bounded-below T^n–spectrum with finite homotopy groups in each degree. Let M be any homology theory. There is a strongly convergent spectral sequence*

$$E^2_{s,t} \cong \mathbb{Z}\{1, t_1, \dots, t_n\} \otimes M_t(X) \implies M_{s+t}(F(E_2T^n_+, X)^{T^n}).$$

Proof Proof is left to the reader. There are no convergence issues since the spectral sequence is concentrated in two columns. □

This spectral sequence is just a long exact sequence in disguise, but since the calculations are relevant for calculations in the homotopy fixed-point spectral sequence, we have chosen to use the language of spectral sequences.

When $X = \Lambda_{T^n} H\mathbb{F}_p$, we will write $E^*(M,n)$ for the spectral sequence

$$E^2(M,n) = M_*(\Lambda_{T^n} H\mathbb{F}_p)\{1, t_1, \dots, t_n\} \implies M_*(F(E_2T^n_+, \Lambda_{T^n} H\mathbb{F}_p)^{T^n})$$

when M is $H\mathbb{F}_p$ or $k(m)$.

Proposition 7.2 *Let $x \in E^2_{-2,2p^{n-1}}(k(n-1),n)$, and suppose that it survives to $E^3_{-2,2p^{n-1}}(k(n-1),n)$. If $d^2(\overline{\tau}_{n-1}) = h(x)$ in $E^2(H\mathbb{F}_p,n)$, where h is the Hurewicz homomorphism, then $[x] = uv_{n-1}$, where $[x] \in k(n-1)_*(F(E_2T^n_+, \Lambda_{T^n} H\mathbb{F}_p)^{T^n})$ is the class of x, for some unit u.*

Proof Smashing the cofiber sequences

$$\Sigma^q k(n-1) \xrightarrow{v_{n-1}} k(n-1) \to H\mathbb{F}_p \to \Sigma^{q+1} k(n-1)$$

and

$$X_2 \to F(E_2T^n_+, \Lambda_{T^n} H\mathbb{F}_p)^{T^n} \to X_0 \xrightarrow{\partial} \Sigma X_2,$$

where $X_i = F(E_iT^n_+/E_{i-2}T^n_+, \Lambda_{T^n} H\mathbb{F}_p)^{T^n}$, and taking homotopy groups, one obtains a diagram with exact rows and columns, where the connecting homomorphism ∂ induces the d^2 differential of the two-column spectral sequences. Since x survives to $E^3_{-2,2p^{n-1}}(k(n-1),n)$, its class $[x]$ is nonzero in $k(n)_{2p^{n-1}-2}(F(E_2T^n_+, \Lambda_{T^n} H\mathbb{F}_p)^{T^n})$. By a diagram chase, it is seen that $[x]$ generates the kernel of

$$k(n)_{2p^{n-1}-2}(F(E_2T^n_+, \Lambda_{T^n} H\mathbb{F}_p)^{T^n}) \to H_{2p^{n-1}-2}(F(E_2T^n_+, \Lambda_{T^n} H\mathbb{F}_p)^{T^n}),$$

and that multiplication by v_{n-1} is zero on $k(n-1)_* X_i$, from which the proposition follows. □

Definition 7.3 Let M be a homology theory. Since T^n_+ splits completely, the homomorphism in Definition 2.6,

$$(\omega_{T^n})_*\colon M_*(T^n_+ \wedge \Lambda_{T^n} H\mathbb{F}_p) \to M_*(\Lambda_{T^n \times T^n} H\mathbb{F}_p),$$

together with the multiplication map in the group T^n, induces a homomorphism

$$H_*(T^n) \otimes M_*(\Lambda_{T^n} H\mathbb{F}_p) \to M_*(\Lambda_{T^n} H\mathbb{F}_p).$$

Given $j \in \boldsymbol{n}$ and $x \in M_*(\Lambda_{T^n} H\mathbb{F}_p)$, we write $\sigma_j(x)$ for the image of $[S^1_j] \otimes x$ under this map, where $[S^1_j] \in H_*(T^n)$ is the image of a fundamental class $[S^1] \in H_*(S^1)$ under the inclusion of the j^{th} circle.

Here, H_* is homology with $\mathbb{F}_p$ coefficients. Since T^n is a pointed space, $\Lambda_{T^n} H\mathbb{F}_p$ is an $H\mathbb{F}_p$–algebra and thus a generalized Eilenberg–Mac Lane spectrum. Let $p \geq 5$ and $1 \leq n \leq p$ or $p = 3$ and $1 \leq n \leq 2$, and define $\mathfrak{p}$ to be the kernel of the projection homomorphism

$$H_*(\Lambda_{T^n} H\mathbb{F}_p) \cong A_* \otimes L_*(T^n) \cong A_* \otimes \bigotimes_{U \subseteq \boldsymbol{n}} B_U \to \bigotimes_{i \in \boldsymbol{n}} B_{\{i\}},$$

where the isomorphism $\alpha\colon L_*(T^n) \cong \bigotimes_{U \subseteq \boldsymbol{n}} B_U$ is the one in Theorem 6.2.

Proposition 7.4 *Let $p \geq 5$ and $1 \leq n \leq p$ or $p = 3$ and $1 \leq n \leq 2$. If x is in the subring $P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes L_*(T^{\boldsymbol{n}}) \subseteq A_* \otimes L_*(T^{\boldsymbol{n}}) \cong H_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)$, then for every $j \in \boldsymbol{n}$, the element $\sigma_j(x)$ is in $\mathfrak{p}$.*

Proof Since σ_j is a derivation by Proposition 2.7, it suffices to check the claim for the set of $\mathbb{F}_p$–algebra generators in $P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes L_*(T^{\boldsymbol{n}}) \cong P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes \bigotimes_{U \subseteq \boldsymbol{n}} B_U$ consisting of $\bar{\xi}_i$ for $i \geq 1$ together with all U–labeled admissible words, where $U \subseteq \boldsymbol{n}$.

By [2, Proposition 4.9], the element $\sigma(\bar{\xi}_i)$ for $i \geq 1$ is represented by $\sigma\bar{\xi}_i$ in filtration 1 in the Bökstedt spectral sequence calculating $H_*(\Lambda_{S^1} H\mathbb{F}_p)$. By the calculation in [8] (see Hunter [20, Theorem 1] for a published account), the element $\sigma\bar{\xi}_i$ is a boundary in the Bökstedt spectral sequence, and hence $\sigma(\bar{\xi}_i) \in A_*$. Thus it must be equal to zero since it is the image of $[S^1] \otimes \bar{\xi}_i$, and $[S^1]$ is mapped to zero on the left-hand side in the following commutative diagram:

$$\begin{tikzcd}
H_*(S^1_+) \otimes H_*(H\mathbb{F}_p) \arrow[r, "\omega_*"] \arrow[d, "\mathrm{pr} \otimes \mathrm{id}"] & H_*(\Lambda_{S^1} H\mathbb{F}_p) \arrow[d, "\mathrm{pr} \otimes \mathrm{id}"] \\
H_*(S^0) \otimes H_*(H\mathbb{F}_p) \arrow[r, "\omega_*"] & H_*(\Lambda_{\{\mathrm{pt}\}} H\mathbb{F}_p)
\end{tikzcd}$$

Hence $\sigma_j(\bar{\xi}_i) = 0$ for all $i \geq 1$ and $1 \leq j \leq n$.

We prove the proposition by induction on the degree m of an element x in $L_*(T^{\boldsymbol{n}}) \cong \bigotimes_{U \subset \boldsymbol{n}} B_U$. When $m = 0$, there is nothing to check since σ_j is trivial on units.

Assume the proposition holds for all elements in degrees less than m. If m is even, the proposition holds because $\sigma_j(x)$ is then of odd degree, and $\bigotimes_{i \in \boldsymbol{n}} B_{\{i\}}$ is concentrated in even degrees. Assume m is odd, and that x is a U–labeled admissible word of degree m for some $U \subseteq \boldsymbol{n}$. By Lemma 3.2(4), x is thus equal to $\varrho_k y$, where k is the largest element in U and y is a $U \setminus k$–labeled admissible word of even degree. By Theorem 6.2(5), x is equal to $\sigma_k(y)$, where we think of y as being an element in $B_{U \setminus k} \subseteq L_*(T^{U \setminus k}) \subseteq L_*(T^{\boldsymbol{k-1}})$.

If $j > k$, the element $\sigma_j(x) = \sigma_j(\sigma_k(y))$ is in $\mathfrak{p}$ by Theorem 6.2(5).

The element $\sigma_j(\sigma_k(y))$ is equal to the image of $[S^1_j] \cdot [S^1_k] \otimes y$, where $[S^1_j] \cdot [S^1_k]$ is the product in $H_*(T^{\boldsymbol{n}})$. When $k = j$, $\sigma_j(\sigma_k(y))$ is thus zero since $[S^1_j]^2 = 0$.

When $j < k$, we have $\sigma_j(\sigma_k(y)) = \pm\sigma_k(\sigma_j(y))$ since the ring $H_*(T^{\boldsymbol{n}})$ is graded commutative. Now, $\sigma_j(y)$ is in $L_*(T^{\boldsymbol{k-1}})$, so by Theorem 6.2(5), the element $\sigma_k(\sigma_j(y))$ is in $\mathfrak{p}$. Hence, $\sigma_j(\sigma_k(y))$ is in $\mathfrak{p}$. □

Proposition 7.5 *The differential in $E^2(H\mathbb{F}_p, n)$ is given, for $x \in E^2_{0,*}(H\mathbb{F}_p, n)$, by*

$$d^2(x) = t_1\sigma_1(x) + \cdots + t_n\sigma_n(x).$$

Thus, if $p \geq 5$ and $1 \leq n \leq p$ or $p = 3$ and $1 \leq n \leq 2$, and x is in the subring $P(\overline{\xi}_1, \overline{\xi}_2, \dots) \otimes L_*(T^{\boldsymbol{n}}) \subseteq H_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p) \cong E^2_{0,*}(H\mathbb{F}_p, n)$, then $d^2(x)$ is in $\mathfrak{p}\{t_1, \dots, t_n\}$.

Proof For each $i \in \boldsymbol{n}$, inclusion of fixed points induces the projection homomorphism from $E^2(H\mathbb{F}_p, n)$ to

$$E^2\langle i\rangle = H_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)\{1, t_i\} \implies H_*(F(E_2S^1_+, \Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)^{S^1}),$$

where S^1 acts on the i^{th} circle in T^n. Now $E^2\langle i\rangle$ maps injectively to the Tate spectral sequence, so by Hesselholt [17, Lemma 1.4.2], the d^2–differential in $E^2\langle i\rangle$ is induced by the operator σ_i.

Since $E^2_{2,*}(H\mathbb{F}_p, n) \cong H_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)\{t_1, t_2, \dots, t_n\}$, and $E^2_{0,*}\langle i\rangle \cong E^2_{0,*}(H\mathbb{F}_p, n)$, the formula for the differential in $E^2_{0,*}(H\mathbb{F}_p, n)$ is thus

$$d^2(x) = t_1\sigma_1(x) + \cdots + t_n\sigma_n(x),$$

and the second claim now follows by Proposition 7.4. □

We will show that the Rognes element $t_1\mu_1^{p^{n-1}} + t_2\mu_2^{p^{n-1}} + \cdots + t_n\mu_n^{p^{n-1}}$ in $E^2(H\mathbb{F}_p, n)$ is not hit by any differential in the homotopy fixed-point spectral sequence. The idea of the proof is that, by the previous propositions, only differentials on $\overline{\tau}_i$ can hit an element in $P(\mu_1, \dots, \mu_n)\{t_1, \dots, t_n\}$. This can only happen when $i \leq n-2$ for dimensional reasons, but since we have one fewer variable $\overline{\tau}_i$ than μ_j, these will not add up correctly.

Definition 7.6 Since the Rognes element $t_1\mu_1^{p^{n-1}} + t_2\mu_2^{p^{n-1}} + \cdots + t_n\mu_n^{p^{n-1}}$ is a cycle, it represents an element in $k(n-1)_*(F(E_2T^n_+, \Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)^{T^n})$, which we call *the Rognes class*.

Proposition 7.7 *Let $p \geq 5$ and $1 \leq n \leq p$ or $p = 3$ and $1 \leq n \leq 2$. The Rognes element $t_1\mu_1^{p^{n-1}} + t_2\mu_2^{p^{n-1}} + \cdots + t_n\mu_n^{p^{n-1}}$ in*

$$\begin{aligned} &E^2(k(n-1), n) \\ &\qquad \cong k(n-1)_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)\{1, t_1, \dots, t_n\} \implies k(n-1)_*(F(E_2T^n_+, \Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)^{T^n}) \end{aligned}$$

is not hit by any differential, and hence the Rognes class is a nonzero element in $k(n-1)_*(F(E_2T^n_+, \Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)^{T^n})$.

Proof The differentials in $E^2(k(n-1),n)$ are determined by the differentials in $E^2(H\mathbb{F}_p,n)$ since $k(n-1)_*(\Lambda_{T^n} H\mathbb{F}_p) \subseteq H_*(\Lambda_{T^n} H\mathbb{F}_p)$. By Hesselholt and Madsen [18, Theorem 5.2], and by [2, Proposition 4.9], $\sigma_i(\overline{\tau}_j) = \mu_i^{p^j}$, so Proposition 7.5 yields $d^2(\overline{\tau}_j) = \sum_{i=1}^n t_i \mu_i^{p^j}$. Assume z is an element in $k(n-1)_*(\Lambda_{T^n} H\mathbb{F}_p)$ with differential $d^2(z) = \sum_{i=1}^n t_i \mu_i^{p^{n-1}}$. It can be written as

$$z = \overline{\tau}_0 z_0 + \cdots + \overline{\tau}_{n-2} z_{n-2} + z',$$

where z' is an element in $P(\overline{\xi}_1, \overline{\xi}_2, \dots) \otimes L_*(T^{\boldsymbol{n}})$ and z_i, for $0 \le i \le n-2$, are elements in $E(\overline{\tau}_{i+1}, \overline{\tau}_{i+2}, \dots) \otimes P(\overline{\xi}_1, \overline{\xi}_2, \dots) \otimes L_*(T^{\boldsymbol{n}})$. By Proposition 7.5, $d^2(z')$ is in $\mathfrak{p}\{t_1, \dots, t_n\}$, and obviously the products $\overline{\tau}_i d^2(z_i)$ for $0 \le i \le n-2$ are in $\mathfrak{p}\{t_1, \dots, t_n\}$, so for some y in $\mathfrak{p}\{t_1, \dots, t_n\}$, we must have

$$d^2(\overline{\tau}_0) z_0 + \cdots + d^2(\overline{\tau}_{n-2}) z_{n-2} = \sum_{j=0}^{n-2} (t_1 \mu_1^{p^j} + \cdots + t_n \mu_n^{p^j}) z_j = \sum_{i=1}^{n} t_i \mu_i^{p^{n-1}} + y. \tag{7.8}$$

Write the elements z_i in the monomial basis in $A_* \otimes L(T^{\boldsymbol{n}}) \cong A_* \otimes \bigotimes_{U \subseteq \boldsymbol{n}} B_U$. For (7.8) to hold, at least one of the z_i must have a nonzero coefficient in front of $\mu_1^{p^{n-1}-p^i}$. We let $k_1 \ge 0$ be the greatest integer i such that this coefficient is nonzero. Let $k_2 < k_1$ be the greatest integer where the coefficient of $\mu_1^{p^{n-1}-p^{k_1}} \mu_2^{p^{k_1}-p^{k_2}}$ in z_{k_2} is nonzero. Such an integer must exist, because the coefficient of $t_2 \mu_2^{p^{k_1}} \mu_1^{p^{n-1}-p^{k_1}}$ on the left-hand side in (7.8) would otherwise be nonzero due to the contribution from $d^2(\overline{\tau}_{k_1}) z_{k_1}$.

Continuing in this way, we get that since there are n variables t_i, there must be a sequence of integers $k_1 > \cdots > k_n$ such that the coefficient in front of the monomial $\mu_1^{p^{n-1}-p^{k_1}} \mu_2^{p^{k_1}-p^{k_2}} \cdots \mu_n^{p^{k_n}-p^{k_n}}$ in z_{k_n} is nonzero. But this is impossible since there are only $n-1$ number of variables z_i.

We thus get a contradiction, so there is no element z in $k(n-1)_*(\Lambda_{T^n} H\mathbb{F}_p)$ with differential $d^2(z) = \sum_{i=1}^n t_i \mu_i^{p^{n-1}}$. □

Theorem 7.9 *Let $p \ge 5$ and $1 \le n \le p$ or $p = 3$ and $1 \le n \le 2$. Then v_{n-1} in $k(n-1)_*((\Lambda_{T^n} H\mathbb{F}_p)^{hT^n})$ is nonzero and is detected by the Rognes class in the homotopy spectral sequence. Equivalently, the homomorphism*

$$k(n-1)_*(\Sigma^{2p^{n-1}-2} F(E_2 T^n_+, \Lambda_{T^n} H\mathbb{F}_p)^{T^n}) \xrightarrow{v_{n-1}} k(n-1)_*(F(E_2 T^n_+, \Lambda_{T^n} H\mathbb{F}_p)^{T^n})$$

maps 1 to a nonzero class.

Whether higher powers of v_{n-1} are nonzero is not known, and similar arguments to those in this article is probably not sufficient to resolve this question as the number of potential differentials in the spectral sequences increases as the power increases.

Proof The unit map $S^0 \to (\Lambda_{T^n} H\mathbb{F}_p)^{hT^n}$ and the inclusion $E_2T^n \to ET^n$ induce the vertical homomorphisms in the following commutative diagram:

$$\begin{array}{ccc} k(n-1)_*(\Sigma^{2p^{n-1}-2}S^0) & \xrightarrow{\quad v_{n-1}\quad} & k(n-1)_*(S^0) \\ \downarrow & & \downarrow \\ k(n-1)_*(\Sigma^{2p^{n-1}-2}(\Lambda_{T^n}H\mathbb{F}_p)^{hT^n}) & \xrightarrow{v_{n-1}} & k(n-1)_*((\Lambda_{T^n}H\mathbb{F}_p)^{hT^n}) \\ \downarrow & & \downarrow \\ k(n-1)_*(\Sigma^{2p^{n-1}-2}F(E_2T^n_+, \Lambda_{T^n}H\mathbb{F}_p)^{T^n}) & \xrightarrow{v_{n-1}} & k(n-1)_*(F(E_2T^n_+, \Lambda_{T^n}H\mathbb{F}_p)^{T^n}) \end{array}$$

By Propositions 7.7 and 7.2, the homomorphism v_{n-1} maps 1 in the lower left-hand corner to the nonzero element represented by the cycle $t_1\mu_1^{p^{n-1}}+t_2\mu_2^{p^{n-1}}+\cdots+t_n\mu_n^{p^{n-1}}$ in the lower right-hand corner. Hence, the image of v_{n-1} must be nonzero in the middle group on the right-hand side of the diagram. □

8 Spectral sequences

In this section, we review the well-known bar spectral sequence, which is the most important tool in our calculations.

Let X_* be a simplicial spectrum, and define the simplicial abelian group $\pi_t(X_*)$ to be $\pi_t(X_q)$ in degree q with face and degeneracy homomorphisms induced by the face and degeneracy maps in X_*. Write $|X_*|$ for the realization of the simplicial spectrum X_*.

The spectral sequence below is well known for spaces and appears for S–modules in Elmendorf, Kriz, Mandell and May [16, Theorem X.2.9].

Proposition 8.1 *Let X_* be a simplicial spectrum, and assume $\mathrm{sk}_s(X_*) \to \mathrm{sk}_{s+1}(X_*)$ is a cofibration for all $s \geq 0$. There is a strongly convergent spectral sequence*

$$E^2_{s,t}(X_*) = H_s(\pi_t(X_*)) \implies \pi_{s+t}(X_*).$$

Let R be a ring spectrum. If X_* is also a simplicial R–algebra, then $E^*_{*,*}(X_*)$ is a $\pi_*(R)$–algebra spectral sequence.

Let R be a commutative ring spectrum, M a cofibrant right R–module, N a left R–module and $B(M,R,N)$ the bar construction. In more detail, $B(M,R,N)$ is the simplicial spectrum which in degree q is equal to $M \wedge R^{\wedge q} \wedge N$, and where the face and degeneracy maps are induced by the same formulas as in the algebra case using the unit map and multiplication map. By Shipley [30, Lemma 4.1.9], there is an equivalence $|B(M,R,N)| \simeq |M \wedge_R N|$.

Proposition 8.2 *Let R be a bounded-below ring spectrum, M a right R–module and N a left R–module. Then there is a strongly convergent spectral sequence*

$$E^2_{s,t} = \mathrm{Tor}^{\pi_* R}_s(\pi_* M, \pi_* N)_t \implies \pi_{s+t}(M \wedge^L_R N).$$

Proof For S–modules, this is a corollary of [16, Theorem X.2.9]. □

Remark 8.3 Let X_* be a cofibrant simplicial R–module. If $\pi_*(X_*)$ is flat as a $\pi_*(R)$–module, this proposition yields an isomorphism

$$\pi_*(X_* \wedge^L_R X_*) \cong \pi_*(X_*) \otimes_{\pi_*(R)} \pi_*(X_*).$$

If X_* is a simplicial R–coalgebra, ie there is a coproduct map $\psi\colon X_* \to X_* \wedge_R X_*$ with a counit map $X_* \to R$ making the obvious diagrams commute up to homotopy, and $\pi_*(X_*)$ is flat as a $\pi_*(R)$–module, then $\pi_*(X_*)$ is a $\pi_*(R)$–coalgebra with coproduct induced by ψ followed by the isomorphism $\pi_*(X_* \wedge^L_R X_*) \cong \pi_*(X_*) \otimes_{\pi_*(R)} \pi_*(X_*)$.

Proposition 8.4 *Assume that X_* is a cofibrant simplicial R–coalgebra, and assume that the map $\mathrm{sk}_s(X_*) \to \mathrm{sk}_{s+1}(X_*)$ is a cofibration for all $s \geq 0$. If each term $E^r(X_*)$ for $r \geq 1$ is flat over $\pi_*(R)$, then $E^*_{*,*}(X_*)$ is a $\pi_*(R)$–coalgebra spectral sequence. If in addition, $\pi_*(X_*)$ is flat as a $\pi_*(R)$–module, then the spectral sequence converges to $\pi_*(X_*)$ as a $\pi_*(R)$–coalgebra.*

Proof A similar statement is proven for $X = \Lambda_{S^1} R$ and mod p homology in [2, Theorem 4.5]. This proof also works for T^n and S^n since we can make the pinch maps simplicial so that they descend to maps of chain complexes. □

In particular, for $B(R, \Lambda_X R, R) \simeq \Lambda_{S^1 \wedge X} R$, we have the following proposition.

Proposition 8.5 *Let R be a commutative ring spectrum, and let X be a simplicial set. The derivation*

$$\hat{\sigma}\colon \pi_*(\Lambda_X R) \to \pi_*(\Lambda_{S^1 \wedge X} R)$$

(see Definition 2.6) *takes $z \in \pi_n(\Lambda_X R)$ to the class of $[z]$ in*

$$E^2_{s,t} = \mathrm{Tor}^{\pi_*(\Lambda_X R)}_s(\pi_*(R), \pi_*(R))_t \implies \pi_{s+t}(\Lambda_{S^1 \wedge X} R),$$

where $[z]$ in the reduced bar complex $B(\pi_*(R), \pi_*(\Lambda_X R), \pi_*(R))$ is represented by z in $E^1_{1,n} \cong B_1(\pi_*(R), \pi_*(\Lambda_X R), \pi_*(R))_n \cong \pi_n(\Lambda_X R)$.

Proof Using the minimal simplicial model for S^1 we get a simplicial spectrum $S^1_+ \wedge \Lambda_X R$, which in simplicial degree q, is equal to $(S^1_q)_+ \wedge \Lambda_X R \cong (\Lambda_X R)^{\vee q}$, the q–fold wedge of $\Lambda_X R$. In the E^2–term of the spectral sequence in Proposition 8.1 associated with this simplicial spectrum, the element $[S^1] \otimes z$ is represented by $1 \oplus z$ in $E^1_{1,*} \cong \pi_*(\Lambda_X R \vee \Lambda_X R) \cong \pi_*(\Lambda_X R) \oplus \pi_*(\Lambda_X R)$, where the second summand corresponds to the nondegenerate simplex in S^1_1.

Similarly, there is a simplicial model for the spectrum $\Lambda_{S^1 \wedge X} R$, which in simplicial degree q, is equal to $\Lambda_{S^1_q \wedge X} R \cong \Lambda_{\vee_q X} R \cong (\Lambda_X R)^{\wedge_R q-1}$, the $(q-1)$–fold smash product over R. The map $\widehat{\omega}$: $S^1_+ \wedge \Lambda_X R \to \Lambda_{S^1 \times X} R$, defined in Definition 2.5, is given on these simplicial models in degree q by the natural map

$$(\Lambda_X R)^{\vee q} \to (\Lambda_X R)^{\wedge q} \to (\Lambda_X R)^{\wedge_R q-1},$$

where the first map is induced by the inclusion into the various smash factors using the unit maps, and the second map is induced by the map $\Lambda_X R \to \Lambda_{\{\mathrm{pt}\}} R$ on the factor indexed by the simplex which is the image of the 0^{th} simplex under a q–fold composition of degeneracy maps. The element $\widehat{\sigma}(z)$ in the spectral sequence from Proposition 8.1 associated with this simplicial spectrum is thus represented by the element z in $E^1_{1,*} \cong \pi_*(\Lambda_X R)$.

Now we have to compare this last spectral sequence with the spectral sequence coming from the bar complex $B(R, \Lambda_X R, R)$. In simplicial degree q, the complex $B(R, \Lambda_X R, R)$ is equal to $R \wedge \Lambda_X R^{\wedge q-1} \wedge R \cong \Lambda_{S^0 \amalg (\coprod_q X)} R$. The equivalence between $B(R, \Lambda_X R, R)$ and the model above is induced by the map $S^0 \amalg \coprod_q X \to \bigvee_q X$ identifying S^0 and the basepoints in X to the base point in $\bigvee_q X$. The element $\widehat{\sigma}(z)$ is thus represented by the class of $[z]$ in

$$E^2_{s,t} = \mathrm{Tor}^{\pi_*(\Lambda_X R)}(\pi_*(R), \pi_*(R)) \implies \pi_{s+t}(\Lambda_{S^1 \wedge X} R),$$

where $[z]$ is in the reduced bar complex $B(\pi_*(R), \pi_*(\Lambda_X R), \pi_*(R))$. □

Proposition 8.6 *Let R be a commutative ring, and let E^* be a first quadrant connected R–Hopf algebra spectral sequence. The shortest nonzero differentials in E^* of lowest total degree, if there are any, are from an indecomposable element in E^* to a primitive element in E^*.*

Proof See [2, Proposition 4.8]. □

The next proposition shows that in certain circumstances, the coalgebra structure of the abutment in a spectral sequence is determined by the algebra structure of the dual spectral sequence. We will use it to calculate the $\mathbb{F}_p$–Hopf algebra structure of $\pi_*(\Lambda_{S^n} H\mathbb{F}_p)$.

Let R be a field with characteristic different than 2, and let

$$\begin{tikzcd} 0 \arrow[r] & A_0 \arrow[r, "i"] \arrow[d, "="'] & A_1 \arrow[r, "i"] \arrow[d, "j"] & A_2 \arrow[r] \arrow[d, "j"] & \cdots \\ & E_0^1 & E_1^1 \arrow[ul, "k"'] & E_2^1 \arrow[ul, "k"'] & \end{tikzcd}$$

be an unrolled exact couple of connected cocommutative R–coalgebras which are of finite type. The unrolled exact couple gives rise to a spectral sequence E^* converging strongly to $\operatorname{colim}_s A_s$ by Boardman [7, Theorem 6.1].

Proposition 8.7 *Assume that in each degree t, the map $A_{s,t} \to A_{s+1,t}$ eventually stabilizes, ie is an isomorphism for all $s \geq u$ for some u depending on t. Assume that the E^∞–term of the spectral sequence is isomorphic, as an R–coalgebra, to a tensor product of exterior algebras and divided power algebras. Then there are no coproduct coextensions in the abutment. Hence $\operatorname{colim}_s A_S \cong E^\infty$ as an R–coalgebra.*

Proof The colimit $\operatorname{colim}_s A_S$ of R–coalgebras is constructed in the underlying category of R–modules. If we apply $D(-) = \hom_R(-, R)$ to the isomorphism $\operatorname{colim}_s A_S \cong E^\infty$, we get an isomorphism $D(\operatorname{colim}_s A_S) \cong D(E^\infty)$, where $D(E^\infty)$ is a free graded commutative algebra. Since $D(E^\infty)$ is the associated graded algebra of the filtered commutative algebra $D(\operatorname{colim}_s A_S)$, this implies that $D(\operatorname{colim}_s A_S)$ is a free graded commutative algebra. Since the maps $A_s \to A_{s+1}$ eventually stabilizes, $D(\lim_s A_s) \cong \operatorname{colim}_s A_s$, so we can dualize again and get that there is an R–coalgebra isomorphism $\operatorname{colim}_s A_s \cong E^\infty$. □

9 Primitive elements

In this section, we prove several technical statements about the degrees of certain admissible words and primitive elements in multifold Hopf algebras. The first two lemmas can obviously be generalized to all n, but we only need them for $n \leq p$, so we keep their formulations as simple as possible.

Lemma 9.1 *Let $n \leq 2p-2$, and let x be an admissible word in B_n of even degree. Let l be the number of occurrences of the letter ϱ in the word x. The sum of the coefficients in the p–adic expansion of the number $\frac{1}{2}|x|$ is equal to $n-l$.*

Proof The proof is by induction on n. It is true for $n=1$ since $l=0$ and $|\mu|=2$. Assume it is true for all $1\le m\le n-1$. An admissible word x in B_n of even degree is, by Lemma 3.2(4), either equal to $\varphi^k y$ or $\varrho^k\varrho z$ for some $k\ge 0$, where y and z are admissible words in B_{n-1} and B_{n-2}, respectively.

First, $\frac{1}{2}|\varphi^k y| = p^k\big(1+\frac{1}{2}p|y|\big)$, so if the sum of the coefficients in the p–adic expansion of $\frac{1}{2}|y|$ is $n-1-l$, where l is the number of occurrences of ϱ in y, the sum of the coefficients in the p–adic expansion of $\frac{1}{2}|\varphi^k y|$ is $n-l$.

Second, $\frac{1}{2}|\varrho^k\varrho z| = p^k\big(1+\frac{1}{2}|z|\big)$, so if the sum of the coefficients in the p–adic expansion of $\frac{1}{2}|z|$ is $n-2-(l-1)=n-1-l$, where $l-1$ is the number of occurrences of ϱ in z, then the sum of the coefficients in the p–adic expansion of $\frac{1}{2}|\varrho^k\varrho z|$ is $n-l$, unless there was carrying involved in the addition $1+\frac{1}{2}|z|$.

There is only carrying involved if the degree of z is equal to -2 modulo $2p$, and by Lemma 3.2(5), this implies that z is equal to $(\varrho^0\varrho)^{p-2}\mu$, or it starts with $(\varrho^0\varrho)^{p-1}$ or $(\varrho^0\varrho)^{p-2}\varphi^0$. In these cases, $\varrho^0\varrho z$ has length at least $2p-1$, so there is no carrying involved when $n\le 2p-2$. □

Lemma 9.2 *Let $Q(B_n)$ be the module of indecomposable elements in B_n. If $2\le n\le 2p$, then $Q(B_n)_{2pi-1}=0$ for all i, and $\bigoplus_{i\ge 1} Q(B_n)_{2pi}$ is equal to the module generated by all nonmonic admissible words of length n.*

Proof The module of indecomposable elements is generated by all admissible words of length n. All nonmonic words are in degree 0 modulo $2p$. All monic words are primitive, so by Lemma 3.5, they are not in degree -1 or 0 modulo $2p$ when $2\le n\le 2p$. □

Lemma 9.3 *The sum of the coefficients in the p–adic expansion of*

$$\tfrac{1}{2}|\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\cdots\mu_n^{p^{j_n}}|,$$

where $j_i\ge 0$ and $|\mu_i|=2$ for $1\le i\le n$, is equal to n when $0<n<p$ and either n or $n-p+1$ when $p\le n<2p$.

Proof If less than p of the numbers j_i are equal, we get the case n, and if at least p of the numbers j_i are equal, we get the case $n-p+1$. □

Corollary 9.4 *Let x be an admissible word in B_n of even degree.*

If $1 \leq n \leq p$, then the degree of x is not equal to the degree of $\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\cdots\mu_n^{p^{j_n}}$, where $j_i \geq 0$ for $1 \leq i \leq n$.

If $p \geq 5$, $1 \leq n \leq p$ and $1 \leq s \leq n$, then the degree of x is not equal to the degree of $(\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\cdots\mu_n^{p^{j_n}})\mu_s^{p^{j_{n+1}}}$, where $j_i \geq 0$ for $1 \leq i \leq n+1$.

Proof By Lemma 9.1, the sum of the coefficients in the p–adic expansion of $\frac{1}{2}|x|$ is equal to $n-l$ where l is the number of occurrences of the letter ϱ in x. Lemma 3.2(2) says that $1 \leq l \leq \frac{1}{2}(n-1)$, so $\frac{1}{2}(n+1) \leq n-l \leq n-1$. By Lemma 9.3, the sum of the coefficients in the p–adic expansion of $\frac{1}{2}|\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\cdots\mu_n^{p^{j_n}}|$ is equal to n when $0 < n < p$ and either n or 1 when $n = p$. Now $n-l \leq n-1 < n < n+1$, and if $n = p$, then $1 < \frac{1}{2}(n+1) = \frac{1}{2}(p+1) \leq n-l$, proving the first claim.

The sum of the coefficients in the p–adic expansion of $\frac{1}{2}|(\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\cdots\mu_n^{p^{j_n}})\mu_i^{p^{j_{n+1}}}|$ is equal to $n+1$ when $0 < n < p-1$ and either $n+1$ or $n-p+2$ when $p-1 \leq n \leq p$. If $n = p-1 \geq 4$, then $1 < \frac{1}{2}(n+1) = \frac{1}{2}p \leq n-l$, and if $n = p \geq 5$, then $2 < \frac{1}{2}(n+1) = \frac{1}{2}(p+1) \leq n-l$, proving the second claim. □

Lemma 9.5 *Let $n \leq p$, and let $P \subseteq \bigotimes_{U \subsetneq \boldsymbol{n}} B_U$ be the $\mathbb{F}_p$–submodule generated by all products $z_{U_1}\cdots z_{U_k}$, where $U_1, \dots, U_k$ is a partition of $\boldsymbol{n}$, and z_{U_i} is a primitive element in B_{U_i} for every i. Then $P_{2pi-1} = 0$ for every $i \geq 2$, and the module $\bigoplus_{i \geq 2} P_{2pi}$ is contained in the module generated by all the elements $\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\cdots\mu_n^{p^{j_n}}$, where $j_i \geq 0$ for $1 \leq i \leq n$.*

Proof In a divided power algebra $\Gamma(x)$, the only primitive elements are nonzero scalar multiples of $\gamma_1(x)$, and in a polynomial algebra $P(x)$, the primitive elements are generated by x^{p^j}. By [28, Proposition 3.12], the primitive elements in B_{U_i} are therefore linear combinations of monic words w_i of length $|U_i|$ when $|U_i| > 1$, and $\mu_{U_i}^{p^{j_i}}$ when $|U_i| = 1$. Assume without loss of generality that z is a product of such elements.

Observe that the degree of a word starting with φ^k, $\varrho^k\varrho$ or μ^{p^k} is 0 modulo $2p$ when $k \geq 1$. Thus multiplication with one of these words will not change the degree of the product modulo $2p$. The degrees of $\varphi^0 x$ and μ are 2 modulo $2p$, and finally the degree of $\varrho^0\varrho x$ is $2+|x|$ modulo $2p$.

Except for the products $\mu_1^{p^{j_1}}\cdots\mu_n^{p^{jn}}$, the smallest n where z has degree 0 modulo $2p$ is thus $n=p+2$, where z may be equal to $\mu_1\cdots\mu_{p-2}\cdot\mu_{p-1}^{p^k}\cdot\varrho_{p+2}^0\varrho_{p+1}\mu_p$. Similarly, the smallest n where the degree of z can be -1 modulo $2p$ is $n=p+1$, where z might be equal to $\mu_1\cdots\mu_{p-2}\cdot\mu_{p-1}^{p^k}\cdot\varrho_{p+1}\mu_p$. □

This lemma is about which elements in $L_*(T^{\boldsymbol{n}}_{n-1})$ are simultaneously primitive in all n Hopf algebra structures. For example, $\mu_1\mu_2^p\mu_3^{p^2}$ is simultaneously primitive in $L_*(T^{\mathbf{3}})$ since it's a product of elements that are primitive in the different circles. We only gain control over the degrees of the elements, but that is sufficient for our needs. It is probably a very special case of a more general statement about simultaneously primitive elements in an S–fold Hopf algebra, but a more general statement has eluded us.

Given a finite subcategory $\Delta\subseteq\mathcal{I}$ and a finite set U, we define $\Delta|_U$, the *restriction* of Δ to U, to be the full subcategory of Δ with objects $\{V\cap U\mid V\in\Delta\}$. The *dimension* of Δ is the maximal cardinality of the sets in Δ.

Lemma 9.6 *Assume Theorem 6.2(3) holds for $1\le k\le n-1$. Let S be an object in $\mathcal{I}$, and let Δ be a saturated subcategory* (see Definition 4.22) *of $V(S)$ with dimension at most $n-1$.*

Let $V\in\mathcal{I}$, and define $\mathbb{N}_V\subseteq\mathbb{N}$ to be the set of degrees of monic words in B_V when $|V|\ge 2$, and the set $\{2p^i\}_{i\ge 0}$ of degrees of $\mu_v^{p^i}$ when $V=\{v\}$. Let $\mathbb{N}_\Delta\subseteq\mathbb{N}$ be the set

$$\mathbb{N}_\Delta=\Big\{\sum_{U_i\in\{U_1,\dots,U_j\}} r_{U_i}\ \Big|\ U_1,\dots,U_j \text{ is a partition of } S \text{ with } U_i\in\Delta \text{ and } r_{U_i}\in\mathbb{N}_{U_i}\Big\}.$$

If $z\in L_*(T^\Delta)$ is S–fold primitive, then $|z|\in\mathbb{N}_\Delta$.

Proof We prove it by induction on the number of sets in Δ. There are no S–fold primitive elements in $L_*(T^\Delta)$ if $S\setminus\big(\bigcup_{U\in\Delta}U\big)=W\neq\varnothing$ since $L_*(T^{\Delta|_{S\setminus j}})=L_*(T^\Delta)$ for any $j\in W$, so $\psi_S^j=\mathrm{id}\colon L_*(T^\Delta)\to L_*(T^\Delta)$. If $S=\{s\}$ and $\Delta=S$, the lemma holds since $L_*(T^\Delta)=B_{\{s\}}=P(\mu_s)$, and the primitive elements are generated by $\mu_s^{p^i}$ for $i\ge 0$.

Let $V\in\Delta$ be a maximal set in Δ; ie if $V\subseteq W\in\Delta$, then $V=W$. Let $\widehat{\Delta}$ be the full subcategory of Δ not containing V.

Let $z_0^V,z_1^V,\dots$ be an ordered monomial basis of $B_V\subseteq L_*(T^V)\cong\bigotimes_{U\subseteq V}B_U$ ordered so that $|z_i^V|\le|z_{i+1}^V|$ for all $i\ge 0$. Note that $z_0^V=1$.

When $z \neq 0$, we can write z uniquely as

$$z = z_l^V x_l^{\widehat{V}} + z_{l-1}^V x_{l-1}^{\widehat{V}} + \cdots + z_0^V x_0^{\widehat{V}}, \tag{9.7}$$

where $x_i^{\widehat{V}}$ are elements in $L_*(T^{\widehat{\Delta}}) \cong \bigotimes_{U \in \Delta, U \neq V} B_U$ and $x_l^{\widehat{V}} \neq 0$. This is possible since $L_*(T^{\Delta}) \cong L_*(T^{\widehat{\Delta}}) \otimes B_V$. If $l = 0$, then $z \in L_*(T^{\widehat{\Delta}})$, and we are done by the induction hypothesis.

Otherwise, given $j \in V$, assume $x_l^{\widehat{V}} \notin L_*(T^{\Delta|_{S \setminus j}}) \subseteq L_*(T^{\Delta})$. Then

$$\begin{aligned}
\psi_S^j(z) &= \psi_S^j(z_l^V)\psi_S^j(x_l^{\widehat{V}}) + \psi_S^j(z_{l-1}^V)\psi_S^j(x_{l-1}^{\widehat{V}}) + \cdots + \psi_S^j(z_0^V)\psi_S^j(x_0^{\widehat{V}}) \\
&= \Big(1 \otimes z_l^V + z_l^V \otimes 1 + \sum (z_l^V)' \otimes (z_l^V)''\Big) \\
&\qquad\qquad \times \Big(1 \otimes x_l^{\widehat{V}} + x_l^{\widehat{V}} \otimes 1 + \sum (x_l^{\widehat{V}})' \otimes (x_l^{\widehat{V}})''\Big) + \cdots \\
&= 1 \otimes z_l^V x_l^{\widehat{V}} + z_l^V x_l^{\widehat{V}} \otimes 1 + z_l^V \otimes x_l^{\widehat{V}} + x_l^{\widehat{V}} \otimes z_l^V + \cdots .
\end{aligned}$$

We have

$$\psi_S^j \colon L_*(T^{\widehat{\Delta}}) \to L_*(T^{\widehat{\Delta}}) \otimes_{L_*(T^{\widehat{\Delta}|_{S\setminus j}})} L_*(T^{\widehat{\Delta}}),$$

so the expression on the last line can not be equal to $z \otimes 1 + 1 \otimes z$ due to the summands $z_l^V \otimes x_l^{\widehat{V}}$ and $x_l^{\widehat{V}} \otimes z_l^V$ and the facts that $z_l^V, \ldots, z_0^V$ is part of a basis and that z_l^V is of highest degree. Hence we get a contradiction, and $x_l^{\widehat{V}} \in L_*(T^{\Delta|_{S\setminus j}}) \subseteq L_*(T^{\Delta})$. Doing this for all j gives us that $x_l^{\widehat{V}} \in L_*(T^{\Delta|_{S\setminus V}}) \subseteq L_*(T^{\Delta})$.

For $U \in \Delta$, the projection maps $\mathrm{pr}^U_{U\setminus V} \colon T^U \to T^{U\setminus V}$ combine into a map

$$\mathrm{pr} \colon T^{\widehat{\Delta}} \to T^{\Delta|_{S\setminus V}}.$$

Since this map collapses $T^V_{|V|-1}$ to a point, the map $g^V \colon T^V \to S^{|V|}$ together with pr induces a map

$$\mathrm{pr} \colon T^{\Delta} \to S^{|V|} \vee T^{\Delta|_{S\setminus V}}.$$

For $j \in V$, the pinch map ψ^j on the j^{th} circle induces a commutative diagram:

$$\begin{tikzcd}
T^{\Delta} \arrow[rr, "\mathrm{pr}"] \arrow[dd, "\psi^j"] & & S^V \vee T^{\Delta|_{S\setminus V}} \arrow[d, "\psi \vee \mathrm{id}"] \\
& & S^V \vee S^V \vee T^{\Delta|_{S\setminus V}} \\
T^{\Delta} \amalg_{T^{\Delta|_{S\setminus j}}} T^{\Delta} \arrow[r, "\mathrm{pr} \amalg \mathrm{pr}"] & (S^V \vee T^{\Delta|_{S\setminus V}}) \amalg_{T^{\Delta|_{S\setminus V}}} (S^V \vee T^{\Delta|_{S\setminus V}}) \arrow[ur, "\cong"] &
\end{tikzcd}$$

Similarly, for $j \in S \setminus V$, the pinch map ψ^j on the j^{th} circle induces a commutative diagram:

$$\begin{tikzcd}
T^\Delta \arrow[rr, "\mathrm{pr}"] \arrow[dd, "\psi^j"] & & S^V \vee T^{\Delta|_{S\setminus V}} \arrow[d, "\mathrm{id} \vee \psi^j"] \\
& & S^V \vee (T^{\Delta|_{S\setminus V}} \amalg_{T^{\Delta|_{(S\setminus V)\setminus j}}} T^{\Delta|_{S\setminus V}}) \\
T^\Delta \amalg_{T^{\Delta|_{S\setminus j}}} T^\Delta \arrow[rr, "\mathrm{pr} \amalg \mathrm{pr}"] & & (S^V \vee T^{\Delta|_{S\setminus V}}) \amalg_{S^V \vee T^{\Delta|_{(S\setminus V)\setminus j}}} (S^V \vee T^{\Delta|_{S\setminus V}}) \arrow[u, "\cong"']
\end{tikzcd}$$

Applying the functor $L_*(-)$ to these two diagrams yields, for $j \in V$, a commutative diagram

$$\begin{tikzcd}
L_*(T^\Delta) \arrow[r, "\mathrm{pr}"] \arrow[d, "\psi^j_S"] & B_V \otimes L_*(T^{\Delta|_{S\setminus V}}) \arrow[d, "\psi_{B_V} \otimes \mathrm{id}"] \\
L_*(T^\Delta) \otimes_{L_*(T^{\Delta|_{S\setminus j}})} L_*(T^\Delta) \arrow[r, "\mathrm{pr} \otimes \mathrm{pr}"] & B_V \otimes B_V \otimes L_*(T^{\Delta|_{S\setminus V}})
\end{tikzcd} \tag{9.8}$$

and for $j \in S \setminus V$, a commutative diagram:

$$\begin{tikzcd}
L_*(T^\Delta) \arrow[r, "\mathrm{pr}"] \arrow[d, "\psi^j_S"] & B_V \otimes L_*(T^{\Delta|_{S\setminus V}}) \arrow[d, "\mathrm{id} \otimes \psi^j_{S\setminus V}"] \\
L_*(T^\Delta) \otimes_{L_*(T^{\Delta|_{S\setminus j}})} L_*(T^\Delta) \arrow[r, "\mathrm{pr} \otimes \mathrm{pr}"] & B_V \otimes \big(L_*(T^{\Delta|_{S\setminus V}}) \otimes_{L_*(T^{\Delta|_{(S\setminus V)\setminus j}})} L_*(T^{\Delta|_{S\setminus V}})\big)
\end{tikzcd} \tag{9.9}$$

We have proved that $x_l^{\widehat{V}} \in L_*(T^{\Delta|_{S\setminus V}})$, so

$$\mathrm{pr}(z) = z_l^V x_l^{\widehat{V}} + z_{l-1}^V \,\mathrm{pr}(x_{l-1}^{\widehat{V}}) + \cdots + z_0^V \,\mathrm{pr}(x_0^{\widehat{V}})$$

is nonzero since $z_l^V, \dots, z_0^V$ is part of a basis. From diagram (9.8) we know that $\mathrm{pr}(z)$ must be primitive in the $L_*(T^{\Delta|_{S\setminus V}})$–Hopf algebra $B_V \otimes L_*(T^{\Delta|_{S\setminus V}})$, where the Hopf algebra structure is induced by the $\mathbb{F}_p$–Hopf algebra structure on $B_V \cong B_{|V|} \cong L_*(S^V)$. By the graded version of [28, Proposition 3.12], this implies that if $\mathrm{pr}(x_i^{\widehat{V}})$ is nonzero, then z_i^V is a V–labeled monic word when $|V| \geq 2$ or an element $\mu_v^{p^m}$ for some m when $V = \{v\}$. It follows from diagram (9.9) that when $\mathrm{pr}(x_i^{\widehat{V}}) \neq 0$, it is $S\setminus V$–fold primitive. By induction, the lemma holds for $\mathrm{pr}(x_i^{\widehat{V}})$, finishing the proof. □

Corollary 9.10 *Given $n \le p$, assume Theorem 6.2(3) holds for $1 \le k < n$. Let y be an $\boldsymbol{n}$–fold primitive element in $L_*(T^{\boldsymbol{n}}_{n-1})$. If x is an admissible word of length n with degree 0 modulo $2p$, then $|x|-1 \neq |y|$. If z is an admissible word of length n with even degree, then $|z^p| \neq |y|$.*

Proof When $2 \le n \le p$, Lemma 9.2 says the admissible words of length n with degree 0 modulo $2p$ are those that start with φ^i or ϱ^i for $i \ge 1$. Hence $x = \varrho^i x'$ or $x = \varphi^i x'$ for x' some admissible word of length $n-1$. The element $\varrho^i x'$ has degree greater than or equal to $4p$. By Lemmas 9.6 and 9.5, there are no $\boldsymbol{n}$–fold primitive elements in $L_*(T^{\boldsymbol{n}}_{n-1})$ in dimension $2pm-1$ for $m \ge 2$, and hence $|x|-1 \neq |y|$.

If z is an admissible word of length n and even degree, then by Lemma 9.2, we have $z = \varrho^i z'$ or $z = \varphi^i z'$ for some admissible word z' of length $n-1$. So $|z^p| = |\varrho^{i+1} z'|$ or $|z^p| = |\varphi^{i+1} z'|$. By Lemmas 9.6 and 9.5, the degrees of the $\boldsymbol{n}$–fold primitive elements in $L_*(T^{\boldsymbol{n}}_{n-1})$ are equal to the degrees of the products $\mu_1^{p^{j_1}} \cdots \mu_n^{p^{j_n}}$. By Corollary 9.4, neither $\varrho^{i+1} z'$ nor $\varphi^{i+1} z'$ is in the same degree as one of the products $\mu_1^{p^{j_1}} \cdots \mu_n^{p^{j_n}}$, and hence $|z^p| \neq |y|$. □

10 Bökstedt spectral sequence argument

To prove that there are no nonzero d^2 differentials when computing $L_*(T^n)$, we look at the Bökstedt spectral sequence.

Lemma 10.1 *Suppose we have $n > 2$ and a prime $p > 2$. Assume that*

$$L_*(T^{\boldsymbol{n-1}}) \cong \bigotimes_{U \subseteq \boldsymbol{n-1}} B_U,$$

where B_U is described in Definition 6.1.

(1) *The Bökstedt spectral sequence calculating $H_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)$ has E^2–page*

$$\bar{E}^2(T^{\boldsymbol{n}}) \cong A_* \otimes L_*(T^{\boldsymbol{n-1}}) \otimes \bigotimes_{\varnothing \neq U \subseteq \boldsymbol{n-1}} B_{U \cup \{n\}} \otimes E(\sigma_n \bar{\xi}_1, \sigma_n \bar{\xi}_2, \dots) \otimes \Gamma(\sigma_n \bar{\tau}_0, \sigma_n \bar{\tau}_1, \dots).$$

(2) *There are no differentials d^r when $r < p-1$, so $\bar{E}^2(T^{\boldsymbol{n}}) = \bar{E}^{p-1}(T^{\boldsymbol{n}})$.*

(3) *There are differentials*

$$d^{p-1}(\gamma_{p^l}(\sigma_n \bar{\tau}_i)) = \sigma_n \bar{\xi}_{i+1} \cdot \gamma_{p^l - l}(\sigma_n \bar{\tau}_i).$$

If, in addition, given $m \geq 0$, the homomorphism $f^{\boldsymbol{n}}\colon L_*(S^{n-1}) \to L_*(T^{\boldsymbol{n}}_{n-1})$ factors through $\mathbb{F}_p$ in degrees less than or equal to $2pm-1$, and the spectral sequence $E^*(T^{\boldsymbol{n}})$ collapses in total degrees less than or equal to $2pm-1$ (that is $E^2(T^{\boldsymbol{n}}) = E^\infty(T^{\boldsymbol{n}})$ in these degrees), then:

(4) *The only other possible nonzero differentials in $\bar{E}^{p-1}(T^{\boldsymbol{n}})$ starting in total degrees less than or equal to $2p(m+1)-1$ are*

$$d^{p-1}(\gamma_{p^l}(\sigma_n x)) = \gamma_{p^l-p}(\sigma_n x)\sum_i r_{x,i}d^{p-1}(\gamma_p(\sigma_n\bar{\tau}_i)),$$

where x is a generator in $L_*(T^{\boldsymbol{n-1}})$ of odd degree and $r_{x,i} \in L_*(T^{\boldsymbol{n-1}}) \subset \bar{E}^{p-1}_{0,*}(T^{\boldsymbol{n}})$.

(5) *Let $B'_U \subsetneq \bar{E}^2(T^{\boldsymbol{n}})$ be the algebra, isomorphic to B_U, that has the same generators as B_U, except that we exchange the generators $\gamma_{p^l}(\sigma_n x)$ in degrees less than $2p(m+1)$ with the infinite cycles*

$$\gamma_{p^l}((\sigma_n x)') = \sum_{j=0}^{p^{l-1}}\Bigl((-1)^j\gamma_{p^l-pj}((\sigma_n x)')\sum_{\alpha\in\mathbb{N}^{\mathbb{N}},|\alpha|=j}\ \prod_{i\in\mathbb{N}} r_i^{\alpha_i}\gamma_{p\alpha_i}(\sigma_n\bar{\tau}_i)\Bigr),$$

where $|\alpha| = \sum_{i\in\mathbb{N}}\alpha_i$, and the convention is that $0^0 = 1$, $\gamma_0(x) = 1$, and $\gamma_i(x) = 0$ when $i < 0$. When $s+t \leq 2p(m+1)-2$, we get an isomorphism

$$\bar{E}^\infty_{s,t}(T^{\boldsymbol{n}}) \cong A_* \otimes L_*(T^{\boldsymbol{n-1}}) \otimes \bigotimes_{\varnothing\neq U\subseteq \boldsymbol{n-1}} B'_{U\cup\{n\}} \otimes P_p(\sigma_n\bar{\tau}_0, \sigma_n\bar{\tau}_1, \dots).$$

Proof By [26, Proposition 2.1] and the Künneth isomorphism, there are isomorphisms of $H_*(\Lambda_{T^{\boldsymbol{n-1}}}H\mathbb{F}_p) \cong A_* \otimes L_*(T^{\boldsymbol{n-1}})$–Hopf algebras

$$\begin{aligned}\bar{E}^2(T^{\boldsymbol{n}}) = \mathrm{HH}_*(H_*(\Lambda_{T^{\boldsymbol{n-1}}}H\mathbb{F}_p)) &\cong H_*(\Lambda_{T^{\boldsymbol{n-1}}}H\mathbb{F}_p)\otimes \mathrm{Tor}^{A_*\otimes L_*(T^{\boldsymbol{n-1}})}(\mathbb{F}_p,\mathbb{F}_p)\\ &\cong A_*\otimes L_*(T^{\boldsymbol{n-1}})\otimes\bigotimes_{U\subseteq\boldsymbol{n-1}}\mathrm{Tor}^{B_U}(\mathbb{F}_p,\mathbb{F}_p)\otimes\mathrm{Tor}^{A_*}(\mathbb{F}_p,\mathbb{F}_p)\\ &\cong A_*\otimes L_*(T^{\boldsymbol{n-1}})\otimes\bigotimes_{\varnothing\neq U\subseteq\boldsymbol{n-1}}B_{U\cup\{n\}}\\ &\qquad\otimes E(\sigma_n\bar{\xi}_1,\sigma_n\bar{\xi}_2,\dots)\otimes\Gamma(\sigma_n\bar{\tau}_0,\sigma_n\bar{\tau}_1,\dots),\end{aligned}$$

where the empty set is left out in the tensor product in the last line since $\mathrm{Tor}^{B_\varnothing}(\mathbb{F}_p,\mathbb{F}_p)$ is isomorphic to $\mathbb{F}_p$. See [2] for more details on the Hopf algebra structure.

Part (2) The Bökstedt spectral sequence $\bar{E}^2(T^{\boldsymbol{n}})$ is an $A_*\otimes L_*(T^{\boldsymbol{n-1}})$–Hopf algebra spectral sequence. By Proposition 8.6, the shortest differential is therefore from an indecomposable element to a primitive element. By the graded version of [28, Proposition 3.12], the primitive elements are linear combinations of the monic words in $\bigotimes_{\varnothing\neq U\subseteq \boldsymbol{n-1}} B_{U\cup\{n\}}$, and the elements $\sigma_n\bar{\xi}_{i+1}$ and $\gamma_1(\sigma_n\bar{\tau}_i)$ for $i\geq 0$. The primitive elements are thus in filtration 1 and 2. The indecomposable elements are linear combinations of the $\mathbb{F}_p$–algebra generators in $\bigotimes_{\varnothing\neq U\subseteq \boldsymbol{n-1}} B_{U\cup\{n\}}\otimes E(\sigma_n\bar{\xi}_1,\sigma_n\bar{\xi}_2,\dots)\otimes\Gamma(\sigma_n\bar{\tau}_0,\sigma_n\bar{\tau}_1,\dots)$, given by the admissible words in $\bigotimes_{\varnothing\neq U\subseteq \boldsymbol{n-1}} B_{U\cup\{n\}}$ together with the elements $\sigma_n\bar{\xi}_j$ and $\gamma_{p^k}(\sigma_n\bar{\tau}_j)$, and they are in filtration $1,2$ and π for $i>0$. The indecomposable elements in filtration p are generated by $\gamma_p(\sigma_n x)$ for a generator x in $A_*\otimes L_*(T^{\boldsymbol{n-1}})$ of odd degree. By [20, Theorem 1], these elements survive to $\bar{E}^{p-1}(T^{\boldsymbol{n}})$, so $\bar{E}^2(T^{\boldsymbol{n}})=\bar{E}^{p-1}(T^{\boldsymbol{n}})$.

Part (3) Theorem 1 of [20] also gives us the differentials

$$d^{p-1}(\gamma_{p+k}(\sigma_n\bar{\tau}_i))=u_i\sigma_n\bar{\xi}_{i+1}\cdot\gamma_k(\sigma_n\bar{\tau}_i), \tag{10.2}$$

where u_i are units in $\mathbb{F}_p$.

Part (4) When $m=0$, there is nothing to prove, since all elements in filtration p and higher are in degrees at least $2p$. Since $\Lambda_{T^{\boldsymbol{n}}}H\mathbb{F}_p$ is an $H\mathbb{F}_p$–module, it is a generalized Eilenberg–Mac Lane spectrum, so the Hurewicz homomorphism induces an isomorphism between the $\mathbb{F}_p$–modules $A_*\otimes L_*(T^{\boldsymbol{n}})$ and $H_*(\Lambda_{T^{\boldsymbol{n}}}H\mathbb{F}_p)$.

From the assumption that $f^{\boldsymbol{n}}$ factors through $\mathbb{F}_p$ in degrees less than or equal to $2pm-1$ and that $E^2(T^{\boldsymbol{n}})_{<2pm}\cong E^\infty(T^{\boldsymbol{n}})_{<2pm}$, we know the dimension of $H_*(\Lambda_{T^{\boldsymbol{n}}}H\mathbb{F}_p)$ as an $\mathbb{F}_p$–module in degrees less than $2pm$. We will show that if there are differentials in the spectral sequence $\bar{E}^2(T^{\boldsymbol{n}})$ starting in degrees less than or equal to $2p(m+1)-1$ other than those in parts (3) and (4) of the lemma, the dimension of $\bar{E}^\infty(T^{\boldsymbol{n}})$ is smaller than the abutment of the spectral sequence, which is equal to $H_*(\Lambda_{T^{\boldsymbol{n}}}H\mathbb{F}_p)$, thus giving us a contradiction.

Assume the only d^{p-1}–differentials in the Bökstedt spectral sequence $\bar{E}^2(T^{\boldsymbol{n}})$ are those generated by (10.2). Lemma 10.3 yields an isomorphism

$$\bar{E}^p(T^{\boldsymbol{n}})\cong A_*\otimes L_*(T^{\boldsymbol{n-1}})\otimes\bigotimes_{\varnothing\neq U\subseteq \boldsymbol{n-1}} B_{U\cup\{n\}}\otimes P_p(\sigma_n\bar{\tau}_0,\sigma_n\bar{\tau}_1,\dots).$$

Proposition 3.4, together with the assumption that $f^{\boldsymbol{n}}$ factors through $\mathbb{F}_p$ in degrees less than $2pm-1$ and that $E^2(T^{\boldsymbol{n}})_{<2pm}=E^\infty(T^{\boldsymbol{n}})_{<2pm}$, gives us an $\mathbb{F}_p$–module isomorphism

$$\begin{aligned}
E^{\infty}(T^{\boldsymbol{n}})_{<2pm} = E^2(T^{\boldsymbol{n}})_{<2pm} &= (L_*(T^{\boldsymbol{n}}_{n-1}) \otimes B_{\boldsymbol{n}})_{<2pm} \\
&\cong \Big(\bigotimes_{U\subseteq \boldsymbol{n}} B_U\Big)_{<2pm} \cong \Big(\bigotimes_{U\subseteq \boldsymbol{n-1}} B_U \otimes \bigotimes_{\varnothing\neq U\subseteq \boldsymbol{n-1}} B_{U\cup\{n\}} \otimes B_{\{n\}}\Big)_{<2pm} \\
&\cong \Big(L_*(T^{\boldsymbol{n-1}}) \otimes \bigotimes_{\varnothing\neq U\subseteq \boldsymbol{n-1}} B_{U\cup\{n\}} \otimes B_{\{n\}}\Big)_{<2pm}.
\end{aligned}$$

By Proposition 3.4, there is an $\mathbb{F}_p$–module isomorphism from $P_p(\sigma_n\overline{\tau}_0, \sigma_n\overline{\tau}_1, \dots)$ to $B_{\{n\}}$ given by mapping $\sigma_n\overline{\tau}_i$ to $\mu_n^{p^i}$, and this isomorphism yields an $\mathbb{F}_p$–module isomorphism $\overline{E}^p(T^{\boldsymbol{n}})_{<2pm} \cong (A_* \otimes L_*(T^{\boldsymbol{n}}))_{<2pm} \cong H_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)_{<2pm}$.

Assume there is a d^{p-1}–differential with image in $\overline{E}^{p-1}(T^{\boldsymbol{n}})_{<2pm}$, which doesn't have image in $(\sigma_n\overline{\xi}_1, \sigma_n\overline{\xi}_2, \dots) \subseteq \overline{E}^{p-1}(T^{\boldsymbol{n}})$, the ideal generated by the images of all the differentials in (10.2). Then in the degree of the target of this differential, the dimension of the $\mathbb{F}_p$–module $\overline{E}^{\infty}(T^{\boldsymbol{n}})_{<2pm}$ would be smaller than the dimension of $H_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)_{<2pm} \cong \overline{E}^p(T^{\boldsymbol{n}})_{<2pm}$, giving us a contradiction.

To find all possible d^{p-1}–differentials with target in $(\sigma_n\overline{\xi}_1, \sigma_n\overline{\xi}_2, \dots)$, it suffices to look at differentials from indecomposable elements. Possible nonzero d^{p-1}–differentials with image in $\overline{E}^{p-1}_{<2pm}$ are thus generated by $d^{p-1}(\gamma_{p^k}(\varrho_n^0 x))$ and $d^{p-1}(\gamma_{p^k}(\varphi_n x))$, where x is a U–admissible word in $B_U \subseteq L_*(T^{\boldsymbol{n-1}})$ for some $\varnothing \neq U \subset \boldsymbol{n-1}$ of odd degree at most $2mp^{1-k}-1$ and even degree at most $(2mp^{1-k}-2)/p$, respectively, and $k \geq 1$. From the calculation

$$\psi(d^{p-1}(\gamma_{p^k}(\varphi_n x))) = d^{p-1}(\psi(\gamma_{p^k}(\varphi_n x))) = d^{p-1}\Big(\sum_{i+j=p^k} \gamma_i(\varphi_n x) \otimes \gamma_j(\varphi_n x)\Big),$$

we see by induction on k that $d^{p-1}(\gamma_{p^k}(\varphi_n x))$ must be primitive. Thus it is zero since when $k \geq 1$, it is in filtration greater than or equal to $p+1$, while the primitive elements are in filtration 1 and 2.

For the elements $\gamma_{p^k}(\varrho_n^0 x)$, [20, Theorem 1] yields the formula

$$d^{p-1}(\gamma_{p+k}(\varrho_n^0 x)) = (\sigma\beta Q^{(|x|+1)/2} x) \cdot \gamma_k(\varrho_n^0 x),$$

so $\gamma_{p+k}(\varrho_n^0 x)$ is a cycle if and only if $\gamma_p(\varrho_n^0 x)$ is a cycle.

In $\overline{E}^{p-1}_{1,*}(T^{\boldsymbol{n}})$, the ideal generated by the elements $\sigma_n\overline{\xi}_1, \sigma_n\overline{\xi}_2, \dots$ is equal to $A_* \otimes L_*(T^{\boldsymbol{n-1}})\{\sigma_n\overline{\xi}_1, \sigma_n\overline{\xi}_2, \dots\}$. Thus, if $d^{p-1}(\gamma_p(\varrho_n^0 x))$ is nonzero, $\sigma_n\beta Q^{(|x|+1)/2} x$

must be an element in $A_* \otimes L_*(T^{\boldsymbol{n-1}})\{\varrho_n\bar{\xi}_1, \varrho_n\bar{\xi}_2, \dots\}$. Since differentials from an A_*–comodule primitive has target an A_*–comodule primitive, $\sigma_n \beta Q^{(|x|+1)/2} x$ must actually be an element in $L_*(T^{\boldsymbol{n-1}})\{\sigma_n\bar{\xi}_1, \sigma_n\bar{\xi}_2, \dots\}$. Hence,

$$\sigma_n \beta Q^{(|x|+1)/2} x = \sum_i r_{x,i} d^{p-1}(\gamma_p(\sigma_n \bar{\tau}_i)),$$

where $r_{x,i}$ are elements in $L_*(T^{\boldsymbol{n-1}})$.

Part (5) By Lemma 10.4, the elements $\gamma_{p^l}((\sigma_n x)')$ in part (5) are cycles, and $\bar{E}^{p-1}$ is isomorphic as an algebra to

$$\bar{E}^{p-1}(T^{\boldsymbol{n}}) \cong A_* \otimes L_*(T^{\boldsymbol{n-1}}) \otimes \bigotimes_{\varnothing \neq U \subseteq \boldsymbol{n-1}} B'_U \otimes E(\sigma_n\bar{\xi}_1, \sigma_n\bar{\xi}_2, \dots) \otimes \Gamma(\sigma_n\bar{\tau}_0, \sigma_n\bar{\tau}_2, \dots).$$

In total degrees less than or equal to $2p(m+1)-1$, all elements in $\bigotimes_{\varnothing \neq U \subseteq \boldsymbol{n-1}} B'_U$ are cycles. Thus, when $s+t \leq 2p(m+1)-2$, the only differentials are those in part (3), so by Lemma 10.3, there is an isomorphism

$$\bar{E}^{p}(T^{\boldsymbol{n}}) \cong A_* \otimes L_*(T^{\boldsymbol{n-1}}) \otimes \bigotimes_{\varnothing \neq U \subseteq \boldsymbol{n-1}} B'_U \otimes P_p(\sigma_n\bar{\tau}_0, \sigma_n\bar{\tau}_1, \dots)$$

in total degrees less than $2p(m+1)-2$.

All the algebra generators in filtration greater than 2 are in total degrees zero modulo $2p$. All generators in total degrees less than or equal to $2pm$ must be cycles, because otherwise, in the degrees of the target of this nonzero differential, the dimension of the $\mathbb{F}_p$–module $\bar{E}^{\infty}(T^{\boldsymbol{n}})_{<2pm}$ will be smaller than the dimension of $H_*(\Lambda_{T^{\boldsymbol{n}}} H\mathbb{F}_p)_{<2pm} \cong \bar{E}^{p}(T^{\boldsymbol{n}})_{<2pm}$. Thus there are no more differentials with source in total degrees less than or equal to $2p(m+1)$, so $\bar{E}^{p}(T^{\boldsymbol{n}})_{\leq 2p(m+1)-2} \cong \bar{E}^{\infty}(T^{\boldsymbol{n}})_{\leq 2p(m+1)-2}$. □

The final two lemmas are one standard homological calculation, and one easy homological calculation that was used in the previous lemma.

Lemma 10.3 *Let R be a field of characteristic p, and let E^* be a connected R–algebra spectral sequence with*

$$E^{p-1} \cong A \otimes_R \Gamma_R(x_0, x_1, \dots) \otimes E_R(y_1, y_2, \dots),$$

where x_i and y_i are in filtration 1. Assume there are nonzero differentials

$$d^{p-1}(\gamma_{p+k}(x_i)) = \gamma_k(x_i) y_{i+1}$$

for all $i, k \geq 0$. Then

$$E^p \cong A \otimes P_R(x_0, x_1, \dots)/(x_0^p, x_1^p, \dots).$$

Proof Consider the R–algebra $\Gamma_R(x_i) \otimes E_R(y_{i+1})$ with differentials given by $d^{p-1}(\gamma_{p+k}(x_i)) = \gamma_k(x_i)y_{i+1}$. The cycles are $\gamma_k(x_i)$ for $k \le p-1$ and $\gamma_k(x_i)y_{i+1}$ for all k, but this last family are also boundaries, so the homology is $P_R(x_i)/(x_i^p)$. Since R is a field, the lemma now follows from the Künneth isomorphism. □

Lemma 10.4 *Let R be a field of characteristic p, and let E^* be a connected R–algebra spectral sequence with*

$$E^{p-1} \cong A \otimes \Gamma_R(x_0, x_1, \dots) \otimes E_R(y_1, y_2, \dots) \otimes \Gamma_R(z).$$

Assume there are differentials

$$d^{p-1}(\gamma_{p+k}(x_i)) = \gamma_k(x_i)y_{i+1}, \quad d^{p-1}(\gamma_{p+k}(z)) = \gamma_k(z) \cdot \sum_{l \in \mathbb{N}} r_l y_{l+1},$$

where r_l are elements in R. For $k > 0$, define $\gamma_{p^k}(z')$ by the formula

$$\gamma_{p^k}(z') = \sum_{j=0}^{p^{k-1}} \left((-1)^j \gamma_{p^k - pj}(z) \sum_{\alpha \in \mathbb{N}^{\mathbb{N}}, |\alpha| = j} \prod_{i \in \mathbb{N}} r_i^{\alpha_i} \gamma_{p\alpha_i}(x_i) \right), \tag{10.5}$$

where $|\alpha| = \sum_{k \in \mathbb{N}} \alpha_i$, and the convention is that $0^0 = 1$, $\gamma_0(x) = 1$, and $\gamma_j(x) = 0$ when $j < 0$. There is an R–algebra isomorphism

$$\begin{aligned} &A \otimes \Gamma_R(x_0, x_1, \dots) \otimes E_R(y_1, y_2, \dots) \otimes \Gamma_R(z') \\ &\qquad \cong A \otimes \Gamma_R(x_0, x_1, \dots) \otimes E_R(y_1, y_2, \dots) \otimes \Gamma_R(z), \end{aligned}$$

induced by (10.5). Furthermore, the elements $\gamma_{p^k}(z')$ are d^{p-1} cycles in E^{p-1}.

Proof First we show that the elements $\gamma_{p^k}(z')$ are cycles. Applying the Leibniz rule several times gives

$$\begin{aligned} &d^{p-1}(\gamma_{p^k}(z')) \\ &= \sum_{j=0}^{p^{k-1}} \left((-1)^j \gamma_{p^k - p(j+1)}(z) \times \left(\sum_{l \in \mathbb{N}} r_l y_{l+1} \right) \sum_{\alpha \in \mathbb{N}^{\mathbb{N}}, |\alpha| = j} \prod_{i \in \mathbb{N}} r_i^{\alpha_i} \gamma_{p\alpha_i}(x_i) \right) \\ &\quad + \sum_{j=0}^{p^{k-1}} \Bigg((-1)^j \gamma_{p^k - pj}(z) \\ &\qquad \times \sum_{\alpha \in \mathbb{N}^{\mathbb{N}}, |\alpha| = j} \sum_{l \in \mathbb{N}} r_l^{\alpha_l} \gamma_{p(\alpha_l - 1)}(x_l) y_{l+1} \prod_{l \neq i \in \mathbb{N}} r_i^{\alpha_i} \gamma_{p\alpha_i}(x_i) \Bigg), \end{aligned} \tag{10.6}$$

and there are no extra signs since all the factors in the expression of $\gamma_{p^k}(z)$ are in even degrees.

In the first sum in (10.6), observe that

$$\Big(\sum_{l\in\mathbb{N}} r_l d^{p-1}(\gamma_p(x_l))\Big) \sum_{\alpha\in\mathbb{N}^{\mathbb{N}},|\alpha|=j} \prod_{i\in\mathbb{N}} r_i^{\alpha_i}\gamma_{p\alpha_i}(x_i) \\ = \sum_{l\in\mathbb{N}} \sum_{\alpha\in\mathbb{N}^{\mathbb{N}},|\alpha|=j+1} r_l^{\alpha_l} y_{l+1}\gamma_{p(\alpha_l-1)}(x_l) \prod_{l\neq i\in\mathbb{N}} r_i^{\alpha_i}\gamma_{p\alpha_i}(x_i).$$

Substituting this expression into (10.6) and increasing the summation index in the first sum by 1, the differential is given by

$$d^{p-1}(\gamma_{p^k}(z')) = \\ \sum_{j=1}^{p^{k-1}+1}\Big((-1)^{j-1}\gamma_{p^k-pj}(z) \sum_{\alpha\in\mathbb{N}^{\mathbb{N}},|\alpha|=j} \sum_{l\in\mathbb{N}} r_l^{\alpha_l}\gamma_{p(\alpha_l-1)}(x_l)y_{l+1} \prod_{l\neq i\in\mathbb{N}} r_i^{\alpha_i}\gamma_{p\alpha_i}(x_i)\Big) \\ +\sum_{j=0}^{p^{k-1}}\Big((-1)^{j}\gamma_{p^k-pj}(z) \sum_{\alpha\in\mathbb{N}^{\mathbb{N}},|\alpha|=j} \sum_{l\in\mathbb{N}} r_l^{\alpha_l}\gamma_{p(\alpha_l-1)}(x_l)y_{l+1} \prod_{l\neq i\in\mathbb{N}} r_i^{\alpha_i}\gamma_{p\alpha_i}(x_i)\Big).$$

The $j=p^{k-1}+1$ summand in the first sum is zero because $\gamma_{p^k-(p^{k-1}+1)p}(z) = \gamma_{-p}(z)=0$. Similarly, the $j=0$ summand in the last sum is zero because $0=j=|\alpha|$ implies that $\alpha_l=0$ for all l, and hence $\gamma_{p(\alpha_l-1)}(x_l)=\gamma_{-p}(x_l)=0$.

The rest of the summands cancel pairwise due to the factors $(-1)^{j-1}$ and $(-1)^j$. Thus $d^{p-1}(\gamma_{p^k}(z'))=0$.

That $(\gamma_{p^k}(z'))^p=0$ is clear by the Frobenius formula since every summand in the expression for $\gamma_{p^k}(z')$ contains a factor in a divided power algebra.

The composite

$$\Gamma_R(z) \xrightarrow{\gamma_{p^k}(z)\mapsto\gamma_{p^k}(z')} \Gamma_R(x_0,x_1,\dots)\otimes E_R(y_1,y_2,\dots)\otimes\Gamma_R(z) \xrightarrow{\mathrm{pr}_{\Gamma_R(z)}} \Gamma_R(z)$$

equals the identity. Hence the map induced by (10.5) induces an R–algebra isomorphism

$$A\otimes\Gamma_R(x_0,x_1,\dots)\otimes E_R(y_1,y_2,\dots)\otimes\Gamma_R(z') \\ \cong A\otimes\Gamma_R(x_0,x_1,\dots)\otimes E_R(y_1,y_2,\dots)\otimes\Gamma_R(z). \qquad \square$$

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