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Goldman algebra, opers and the swapping algebra

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We define a Poisson algebra called the *swapping algebra* using the intersection of curves in the disk. We interpret a subalgebra of the fraction algebra of the swapping algebra, called the *algebra of multifractions*, as an algebra of functions on the space of cross ratios and thus as an algebra of functions on the Hitchin component as well as on the space of $\mathrm{SL}_n(\mathbb{R})$–opers with trivial holonomy. We relate this Poisson algebra to the Atiyah–Bott–Goldman symplectic structure and to the Drinfel'd–Sokolov reduction. We also prove an extension of the Wolpert formula.

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1 Introduction

The purpose of this article is threefold. We first introduce the *swapping algebra*, which is a Poisson algebra generated, as a commutative algebra, by pairs of points on the circle. Then we relate this construction to two well-known Poisson structures:

- The Poisson structure of the character variety of representations of a surface group in $\mathrm{PSL}_n(\mathbb{R})$, discovered by Atiyah, Bott and Goldman [1; 8; 9].
- The Poisson structure of the space of $\mathrm{PSL}_n(\mathbb{R})$–opers introduced by Dickey, Gel'fand and Magri and described in a geometrical way by Drinfel'd and Sokolov [22; 6; 5].

One way to heuristically interpret these relations is to say that the swapping algebra embodies the notion of a "Poisson structure" for the space of all cross ratios, a space that contains both the space of opers and the "universal (in genus) Hitchin component". As a byproduct of the methods of this paper, we also produce a generalization of the Wolpert formula, which computes the brackets of length functions for the Hitchin component.

The results of this article were announced in [19]. The relation, at a topological level, between the character variety and opers was already noted by the author [16], by Fock and Goncharov [7], and foreseen by Witten [30]; see also Govindarajan and Jayaraman [10; 11].

We now explain more precisely the content of this article.

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1.1 The swapping algebra

Our first result is the construction of the *swapping algebra*. To avoid cumbersome expressions, most of the time we shall denote the ordered pair (X, x) of points of the circle by the concatenated symbol Xx. We recall in Section 2.1 the definition and properties of the linking number $[Xx, Yy]$ of the two pairs (X, x) and (Y, y). If P is a subset of the circle, we denote by $\mathcal{L}(\mathsf{P})$ the commutative associative algebra generated by pairs of points of P with the relations $XX = 0$ for all X in P. Our starting result is the following.

Theorem 1 (swapping bracket) *For every complex number α, there exists a unique Poisson bracket on $\mathcal{L}(\mathsf{P})$ such that the bracket of two generators is*

$$\{Xx, Yy\}_\alpha := [Xx, Yy](Xy.Yx + \alpha.Xx.Yy).$$

The *swapping algebra* is the algebra $\mathcal{L}_\alpha(\mathsf{P})$ endowed with the Poisson bracket $\{\cdot,\cdot\}_\alpha$. This theorem is proved in Section 2. The goal of this paper is to relate this swapping algebra to other Poisson algebras.

One should note that this bracket can be used to express very simply some results of Wolpert and in particular, the variation of the length of curve transverse to a shear; see Wolpert [31; 32].

1.2 Cross ratios and the multifraction algebra

We shall concentrate on the interpretation of an offshoot of the swapping algebra. We denote by $\mathcal{Q}_\alpha(\mathsf{P})$ the algebra of fractions of $\mathcal{L}_\alpha(\mathsf{P})$ equipped with the induced Poisson structure. The *multifraction algebra* $\mathcal{B}(\mathsf{P})$ is the vector subspace of $\mathcal{Q}_\alpha(\mathsf{P})$ generated by the *elementary multifractions*

$$[\mathrm{X}, \mathrm{x}; \sigma] := \frac{\prod_{i=1}^n X_i x_{\sigma(i)}}{\prod_{i=1}^n X_i x_i},$$

where $\mathrm{X} = (X_1, \dots, X_n)$ and $\mathrm{x} = (x_1, \dots, x_n)$ are n–tuples of points of P and σ is a permutation of $\{1, \dots, n\}$. Then we have the following easy proposition.

Proposition 2 *The multifraction algebra is a Poisson subalgebra of $\mathcal{Q}_\alpha(\mathsf{P})$. The induced Poisson structure does not depend on α. Finally, $\mathcal{B}(\mathsf{P})$ is generated as a commutative algebra by the* ***cross fractions***

$$[X, Y, x, y] := \frac{Xx.Yy}{Yx.Xy}.$$

In particular, it follows that the multifraction algebra is naturally mapped to the commutative algebra of functions on cross ratios; see Section 3. Thus the existence of a Poisson structure on the multifraction algebra can be interpreted as that of a Poisson structure on the space of cross ratios.

1.3 The multifraction algebra as a "universal" Goldman algebra

We then relate the multifraction algebra to the Goldman algebra. Let Γ be the fundamental group of a surface S, $\partial_\infty\Gamma$ the boundary at infinity of Γ, and P the subset of $\partial_\infty\Gamma$ consisting of fixed points of elements of Γ. The Hitchin component $\mathsf{H}(n,S)$ of the character variety of representations of Γ in $\mathrm{PSL}_n(\mathbb{R})$ was interpreted in Labourie [17] as a space of cross ratios. Thus every multifraction in $\mathcal{B}(\mathsf{P})$ gives a smooth function on the Hitchin component; see Proposition 4.2.4 for details. Thus we have a restriction

$$\mathsf{I}_S\colon \mathcal{B}(\mathsf{P}) \to C^\infty(\mathsf{H}(n,S)).$$

This mapping is not a Poisson morphism, nevertheless it becomes one when we take sequences of well-chosen finite-index subgroups. More precisely, we define and prove, as an immediate consequence of one of the main results of Niblo [24], the existence of *vanishing sequences* of finite-index subgroups $\{\Gamma_n\}_{n\in\mathbb{N}}$ of Γ; these sequences are essentially such that every geodesic becomes eventually simple and for which the intersection of two geodesics becomes eventually minimal; see Section 6.2.1 and the appendix for details.

Then denoting by $\{\cdot,\cdot\}_W$ the swapping bracket, and by $\{\cdot,\cdot\}_{S_n}$ the Goldman bracket for $S_n := \widetilde{S}/\Gamma_n$ coming from the Atiyah–Bott–Goldman symplectic form on the character variety, we prove in Section 9:

Theorem 3 (Goldman bracket for vanishing sequences) *Let $\{\Gamma_m\}_{m\in\mathbb{N}}$ be a vanishing sequence of subgroups of $\pi_1(S)$. Let $\mathsf{P}\subset\partial_\infty\pi_1(S)$ be the set of end points of geodesics. Let b_0 and b_1 be two multifractions in $\mathcal{B}(\mathsf{P})$. Then we have*

$$\lim_{n\to\infty}\{b_0,b_1\}_{S_n} = \{b_0,b_1\}_W. \tag{1}$$

The statement of this theorem actually requires some preliminaries in properly defining the meaning of (1). In a way, this result tells us that the swapping bracket is the Goldman bracket on the universal solenoid.

The proof relies on the description of special multifractions called *elementary functions* (see Section 4.2) as limits of the well-studied functions on the character variety known as *Wilson loops*.

Another result is a precise asymptotic formula, on a fixed surface this time, relating the Goldman and the swapping brackets. With Γ as above, let $\gamma \in \Gamma$, $y \in \mathsf{P}$ and let γ^+ and γ^- be respectively the attractive and repulsive fixed points of γ in $\partial_\infty(\Gamma)$. Define the following formal series of cross fractions, reverting to the notation (X, x) for pairs:

$$\widehat{\ell}_\gamma(y) = \frac{1}{2}\log\left(\frac{(\gamma^+, \gamma(y))\cdot(\gamma^-, \gamma^{-1}(y))}{(\gamma^+, \gamma^{-1}(y))\cdot(\gamma^-, \gamma(y))}\right).$$

In [16] we show that the *period function* $\ell_\gamma := \mathrm{I}_S(\widehat{\ell}_\gamma(y))$, seen as a function on the character variety, is independent of y and is a function of the eigenvalues of the monodromy of γ. These period functions coincide with the length functions for classical Teichmüller theory; that is, $n = 2$.

We now have:

Theorem 4 (bracket of length functions) *Let γ and η be homotopy classes of curves which as simple curves have at most one intersection point. Then we have*

$$\lim_{n\to\infty} \mathrm{I}_S\big(\{\widehat{\ell}_{\gamma^n}(y), \widehat{\ell}_{\eta^n}(y)\}_W\big) = \tfrac{1}{4}\{\ell_\gamma, \ell_\eta\}_S.$$

As a tool of the proof of this result we prove the following extension of the Wolpert formula [32; 31].

Theorem 5 (generalized Wolpert formula) *Let γ and η be two homotopy classes of curves which as simple curves have exactly one intersection point. Then the Goldman bracket of the two length functions ℓ_γ and ℓ_η is*

$$\{\ell_\gamma, \ell_\eta\}_S(\boldsymbol{b}) = \iota(\gamma, \eta) \sum_{\varepsilon, \varepsilon' \in \{-1, 1\}} \varepsilon\varepsilon'.\boldsymbol{b}(\gamma^\varepsilon, \eta^{\varepsilon'}, \gamma^{-\varepsilon}, \eta^{-\varepsilon'}). \tag{2}$$

This formula has recently been extended using different methods by Bridgeman [3].

1.4 The multifraction algebra and $\mathbf{PSL}_n(\mathbb{R})$–opers

We finally relate the multifraction algebra to opers. We recall in Section 10 the definition of real opers and their interpretation as maps to the projective space $\mathbb{P}(\mathbb{R}^n)$ and its dual. In particular, opers with trivial holonomy can be embedded in the space of smooth cross ratios. The Drinfel'd–Sokolov reduction allows us to define the Poisson bracket of pairs of *acceptable observables*, a subclass of functions on the spaces of opers. We then show that this Poisson bracket coincides with the swapping bracket.

Theorem 6 (swapping bracket and opers) *Let $(X_0, x_0, Y_0, y_0, X_1, x_1, Y_1, y_1)$ be pairwise distinct points on the circle $\mathbb{T}$. Then the cross fractions $[X_0, x_0, Y_0, y_0]$ and $[X_1, x_1, Y_1, y_1]$ define a pair of acceptable observables whose Poisson bracket with respect to the Drinfel'd–Sokolov reduction coincides with their Poisson bracket in the multifraction algebra.*

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2 The swapping bracket

In this section, we first recall the definition and properties of the linking number of two ordered pairs of points. We then construct the swapping algebra and prove Theorem 1, which relies on an identity involving the linking numbers of six points.

2.1 Linking number for pairs of points

We recall that if (X, x, Y, y) is a quadruple of points on the real line, the *linking number* of (X, x) and (Y, y) is

(3) $$[Xx, Yy] := \frac{1}{2}\big(\mathrm{Sign}(X-x)\,\mathrm{Sign}(X-y)\,\mathrm{Sign}(y-x) - \mathrm{Sign}(X-x)\,\mathrm{Sign}(X-Y)\,\mathrm{Sign}(Y-x)\big),$$

where $\mathrm{Sign}(x) = -1, 0, 1$ whenever $x < 0$, $x = 0$, $x > 0$ respectively. By definition, the linking number is invariant under orientation-preserving homeomorphisms of the real line. We note that:

(i) When the four points are pairwise distinct, this linking number is also the total linking number of the curve joining X to x with the curve joining Y to y in the upper half-plane.

(ii) The equality cases are as follows:

(a) For all points (X, Y, y) on the circle,

(4) $$[XX, Yy] = 0 = [Xy, Xy].$$

(b) If, up to cyclic permutation, (X, Y, x) are pairwise distinct points and oriented, then

$$[Xx, Yx] = \tfrac{1}{2}. \tag{5}$$

The first observation shows that we can define the linking number of a quadruple of points on the oriented circle S^1 by choosing a point x_0 disjoint from the quadruple and defining the linking number as the linking number of the quadruple in $S^1 \setminus \{x_0\} \sim \mathbb{R}$. The linking number so defined does not depend on the choice of x_0 and is invariant under orientation-preserving homeomorphisms.

2.1.1 Properties of the linking number We summarize the useful properties (for us) of the linking number of pairs of points in the following definition. Let P be any set.

Definition 2.1.1 A *linking number* of pairs of points of P is a map from P^4 to a commutative ring,

$$(X, x, Y, y) \mapsto [Xx, Yy],$$

such that for all points X, x, Y, y, Z, z,

$$[Xx, Yy] + [Yy, Xx] = 0 \qquad \textbf{(first antisymmetry)}, \tag{6}$$

$$[Xx, Yy] + [Xx, yY] = 0 \qquad \textbf{(second antisymmetry)}, \tag{7}$$

$$[zy, XY] + [zy, YZ] + [zy, ZX] = 0 \qquad \textbf{(cocycle identity)}, \tag{8}$$

and moreover, if (X, x, Y, y) are all pairwise distinct, then

$$[Xx, Yy].[Xy, Yx] = 0 \qquad \textbf{(linking number alternative)}. \tag{9}$$

We illustrate the cocycle identity and the alternative for the standard linking number in Figure 1.

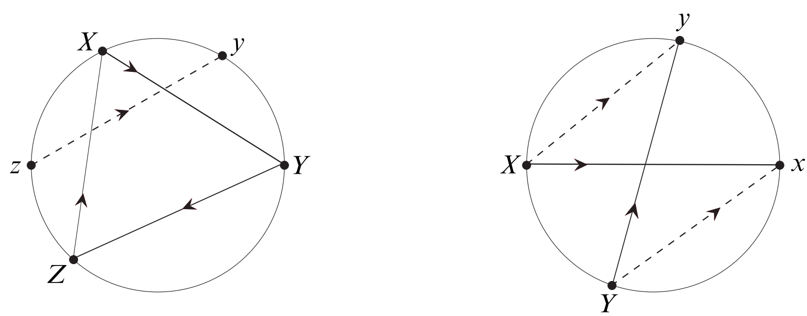

Figure 1: Linking number for pairs of points on the circle: the cocycle identity (left) and linking number alternative (right)

Then we prove:

Proposition 2.1.2 *The canonical linking number for pairs of points of the circle is a linking number in the sense of the previous definition.*

Proof The first two symmetries are checked from the definition. In the case that $\{x, y\} \cap \{X, Y, Z\} = \varnothing$, Equation (8) follows from the geometric definition of the linking number. It remains to check different cases of equality. We can assume that (X, Y, Z) are pairwise distinct; otherwise the equality follows from the two previous ones and (4).

- If $x = y$, the equation is true by (4).

- Assume that $x = X$ and $y \notin \{X, Y, Z\}$ up to cyclic permutations of (X, Y, Z). Then the equality follows from the following remark. Let z and t be points close enough to x so that (z, x, t) is oriented. Then when $A = Y$ or $A = Z$, we have

$$[xy, xA] = \tfrac{1}{2}([zy, xA] + [ty, xA]).$$

- Assume finally that $(x, y) = (X, Y)$. Then the equality reduces to

$$[XY, ZX] + [XY, YZ] = 0,$$

which is true by (5) and the fact that (X, Y, Z) and (Y, X, Z) have opposite orientation.

Equation (9) follows from the geometric definition of linking number. □

A linking number also satisfies more complicated relations.

Proposition 2.1.3 *Let (X, x, Z, z, Y, y) be 6 points on the set* P *equipped with a linking number* $[\cdot,\cdot]$. *Then*

$$[Xy, Zz] + [Yx, Zz] = [Xx, Zz] + [Yy, Zz]. \tag{10}$$

Moreover, if $\{X, x\} \cap \{Y, y\} \cap \{Z, z\} = \varnothing$, then

$$[Xx, Yy][Xy, Zz] + [Zz, Xx][Zx, Yy] + [Yy, Zz][Yz, Xx] = 0, \tag{11}$$

$$[Xx, Yy][Yx, Zz] + [Zz, Xx][Xz, Yy] + [Yy, Zz][Zy, Xx] = 0. \tag{12}$$

Remarks (i) The hypothesis on the configuration of points is necessary: if X, x, Y, Z are pairwise distinct, then for $(X, x, Y, y, Z, z) = (X, x, Y, x, Z, x)$, the left-hand side in (11) is nonzero in the case of the standard linking number of pairs of points on the circle.

(ii) A simple way to prove this proposition is to use mathematical computing software; below we give a mathematical proof.

Proof Formula (10) follows at once from the cocycle identity (8). We now prove formulas (11) and (12). Let us define

$$F(X,x,Y,y,Z,z) := [Xx,Yy][Xy,Zz]+[Zz,Xx][Zx,Yy]+[Yy,Zz][Yz,Xx],$$
$$G(X,x,Y,y,Z,z) := [Xx,Yy][Yx,Zz]+[Zz,Xx][Xz,Yy]+[Yy,Zz][Zy,Xx].$$

We first prove some symmetries of F and G.

Our first observation is that, using the first antisymmetry property (6), we get that

$$F(X,x,Y,y,Z,z) = -G(Y,y,X,x,Z,z). \tag{13}$$

Thus we only need to prove that $F=0$.

Step 1 *The expression F is invariant under all permutations of the pairs (X,x), (Y,y) and (Z,z).*

Using equations (10) and (6), we obtain that

$$\begin{aligned} F(X,x,Y,y,Z,z)+G(X,x,Y,y,Z,z) &= [Xx,Yy][Yy,Zz]+[Xx,Yy][Xx,Zz] \\ &\quad +[Zz,Xx][Zz,Yy]+[Zz,Xx][Xx,Yy] \\ &\quad +[Yy,Zz][Yy,Xx]+[Yy,Zz][Zz,Xx] \\ &= 0. \end{aligned}$$

Hence, by (13),

$$F(X,x,Y,y,Z,z) = F(Y,y,X,x,Z,z). \tag{14}$$

By construction F is invariant by cyclic permutations and thus, from the previous equation, F is invariant by all permutations of the pairs (X,x), (Y,y) and (Z,z).

Step 2 *The expression F satisfies a cocycle equation,*

$$F(X,x,Y,y,Z,z)+F(x,t,Y,y,Z,z) = F(X,t,Y,y,Z,z). \tag{15}$$

We also have the symmetries

$$\begin{aligned} F(X,x,Y,y,Z,z) &= -F(x,X,Y,y,Z,z) \\ &= -F(X,x,y,Y,Z,z) \\ &= -F(X,x,Y,y,z,Z). \end{aligned} \tag{16}$$

The symmetries of equation (16) follow at once from the cocycle equation (15) and the fact that $F(X,X,Y,y,Z,z)=0$.

Let us prove a cocycle equation for F. We shall only use the cocycle identity (8) and the previous symmetries for the linking number. By definition,

$$\begin{aligned}F(X,x,Y,y,Z,z)+F(x,t,Y,y,Z,z)&=[Xx,Yy][Xy,Zz]+[xt,Yy][xy,Zz]\\&\quad+[Zz,Xx][Zx,Yy]+[Zz,xt][Zt,Yy]\\&\quad+[Yy,Zz][Yz,Xx]+[Yy,Zz][Yz,xt].\end{aligned}$$

Using the cocycle identity (8) to expand the first term and regrouping the fifth and sixth terms of the right-hand side, we get

$$\begin{aligned}F(X,x,Y,y,Z,z)+F(x,t,Y,y,Z,z)&=[Xt,Yy][Xy,Zz]+[tx,Yy][Xy,Zz]\\&\quad+[xt,Yy][xy,Zz]+[Zz,Xx][Zx,Yy]\\&\quad+[Zz,xt][Zt,Yy]+[Yy,Zz][Yz,Xt].\end{aligned}$$

Using the cocycle identity (8) for regrouping the second and third term of the right-hand side and rearranging, we get

$$\begin{aligned}F(X,x,Y,y,Z,z)+F(x,t,Y,y,Z,z)&=[Xt,Yy][Xy,Zz]+[Zz,Xx][Zx,Yy]\\&\quad+[Zz,Xx][xt,Yy]+[Zz,xt][Zt,Yy]\\&\quad+[Yy,Zz][Yz,Xt]\\&=F(X,t,Y,y,Z,z).\end{aligned}$$

Using the cocycle identity (8) to regroup the second and third terms, then the fourth, of the right-hand side, we finally get

$$\begin{aligned}F(X,x,Y,y,Z,z)+F(x,t,Y,y,Z,z)&=[Xt,Yy][Xy,Zz]+[Zz,Xt][Zt,Yy]\\&\quad+[Yy,Zz][Yz,Xt]\\&=F(X,t,Y,y,Z,z).\end{aligned}$$

Step 3 *If* (X,x,Y,y) *are pairwise distinct, then*

$$F(X,x,Y,y,Y,x)=0, \tag{17}$$

$$F(X,x,X,x,Y,y)=0. \tag{18}$$

Let us first prove (17). It follows from the linking number alternative (9) and the cocycle identity (8) that

$$\begin{aligned}F(X,x,Y,y,Y,x)&=[Xx,Yy][Xy,Yx]+\big([Yx,Xx][Yx,Yy]+[Yy,Yx][Yx,Xx]\big)\\&=0.\end{aligned}$$

This proves formula (17). Similarly, using the cocycle equation (15) for F for the first equality, symmetries for the second, and our previous formula (17) (for (X,x,y,Y))

for the last, we get

$$\begin{aligned} F(X,x,X,x,Y,y) &= F(X,x,X,y,Y,y) + F(X,x,y,x,Y,y) \\ &= F(X,x,y,x,y,Y) \\ &= F(X,x,y,Y,y,x) \\ &= 0. \end{aligned}$$

Step 4 *If (X,x,Y,y,Z) are pairwise distinct, then*

$$F(X,x,Y,y,Z,x) = 0. \tag{19}$$

Using the cocycle formula (15) for F and the previous step, we get

$$\begin{aligned} F(X,x,Y,y,Z,x) &= F(X,x,Y,Z,Z,x) + F(X,x,Z,y,Z,x) \\ &= -F(X,x,Z,Y,Z,x) \\ &= 0. \end{aligned}$$

Final step *If (X,x,Y,y,Z,z) are pairwise distinct, then*

$$F(X,x,Y,y,Z,z) = 0. \tag{20}$$

Indeed, using the cocycle formula (15) for F for the first equality, symmetries for the second, and the previous step for the last equality, we get

$$\begin{aligned} F(X,x,Y,y,Z,z) &= F(X,x,Y,y,Z,Y) + F(X,x,Y,y,Y,z) \\ &= F(y,Y,x,X,Z,Y) - F(y,Y,x,X,y,z,Y) \\ &= 0. \end{aligned}$$

This concludes the proof. □

2.2 The swapping algebra

Let P be a set and $[\cdot,\cdot]$ be a linking number with values in an integral domain A. We represent a pair (X,x) of points of P by the expression Xx. We consider the free associative commutative algebra $\mathcal{L}(\mathsf{P})$ generated over A by pairs of points on P, together with the relation $XX = 0$ for all $X \in \mathsf{P}$.

Let α be any element in A. We define the *swapping bracket* of two pairs of points as the following element of $\mathcal{L}(\mathsf{P})$:

$$\{Xx, Yy\}_\alpha := [Xx, Yy](\alpha.Xx.Yy + Xy.Yx). \tag{21}$$

We extend the swapping bracket to the whole algebra $\mathcal{L}(\mathsf{P})$ using the Leibniz rule, and call the resulting algebra $\mathcal{L}_\alpha(\mathsf{P})$ the *swapping algebra*.

Theorem 2.2.1 *The swapping bracket satisfies the Jacobi identity. Hence, the swapping algebra $\mathcal{L}_\alpha(\mathsf{P})$ is a Poisson algebra.*

Proof All we need to check is the Jacobi identity

$$\{\{Xx, Yy\}_\alpha, Zz\}_\alpha + \{\{Yy, Zz\}_\alpha, Xx\}_\alpha + \{\{Zz, Xx\}_\alpha, Yy\}_\alpha = 0$$

for the generators of the algebra.

We make preliminary computations, omitting the subscript α in the bracket. The triple bracket $\{\{A, B\}, C\}$ is a polynomial of degree 2 in α and we wish to compute its coefficients. By definition, using the Leibniz rule for the second equality, we have

$$\begin{aligned}\{\{Xx, Yy\}, Zz\} &= [Xx, Yy]\big(\alpha\{Xx.Yy, Zz\} + \{Xy.Yx, Zz\}\big)\\ &= \alpha[Xx, Yy]\big(\{Xx, Zz\}.Yy + \{Yy, Zz\}.Xx\big)\\ &\quad + [Xx, Yy]\big(\{Xy, Zz\}.Yx + \{Yx, Zz\}.Xy\big).\end{aligned} \tag{22}$$

Now we compute two expressions appearing in the right-hand side of the previous equation. We have

$$\begin{aligned}&\{Xx, Zz\}.Yy + \{Yy, Zz\}.Xx\\ &\qquad = \alpha([Xx, Zz] + [Yy, Zz])Xx.Yy.Zz\\ &\qquad\quad + ([Xx, Zz]Xz.Yy.Zx + [Yy, Zz]Xx.Yz.Zy).\end{aligned} \tag{23}$$

Similarly,

$$\begin{aligned}&\{Xy, Zz\}.Yx + \{Yx, Zz\}.Xy\\ &\qquad = \alpha([Xy, Zz] + [Yx, Zz])Xy.Yx.Zz + [Xy, Zz]Xz.Yx.Zy\\ &\qquad\quad + [Yx, Zz]Xy.Yz.Zx.\end{aligned} \tag{24}$$

It follows from (23) and (24) that the coefficient of α^2 in the triple bracket (22) is

$$P_2 := ([Xx, Yy][Xx, Zz] + [Xx, Yy][Yy, Zz])Xx.Yy.Zz. \tag{25}$$

The coefficient of α in the triple bracket (22) is

$$\begin{aligned}P_1 &:= [Xx, Yy][Xx, Zz]Xz.Yy.Zx + [Xx, Yy][Yy, Zz]Xx.Yz.Zy\\ &\quad + ([Xx, Yy][Xy, Zz] + [Xx, Yy][Yx, Zz])Xy.Yx.Zz.\end{aligned} \tag{26}$$

Finally, the constant coefficient is

$$P_0 := [Xx, Yy][Xy, Zz]Xz.Yx.Zy + [Xx, Yy][Yx, Zz]Xy.Yz.Zx, \tag{27}$$

so that

$$\{\{Xx, Yy\}, Zz\} = \alpha^2 P_2 + \alpha.P_1 + P_0. \tag{28}$$

In order to check the Jacobi identity, we have to consider the sums S_2, S_1 and S_0 over cyclic permutations of (Xx, Yy, Zz) of the three terms P_2, P_1 and P_0. We consider successively these three coefficients.

Term of degree 0 We first have

$$S_0 = F(X, x, Y, y, Z, z)(Xz.Yx.Zy - Xy.Yz.Zx). \tag{29}$$

Indeed, we have

$$S_0 = A.Xz.Yx.Zy + B.Xy.Yz.Zx,$$

where

$$\begin{aligned} A &= [Xx, Yy][Xy, Zz] + [Zz, Xx][Zx, Yy] + [Yy, Zz][Yz, Xx] \\ &= F(X, x, Y, y, Z, z), \\ B &= [Xx, Yy][Yx, Zz] + [Zz, Xx][Xz, Yy] + [Yy, Zz][Zy, Xx] \\ &= G(X, x, Y, y, Z, z). \end{aligned}$$

Now (29) follows from (13).

We now prove that $S_0 = 0$. It follows from Proposition 2.1.3 that if

$$\{X, x\} \cap \{Y, y\} \cap \{Z, z\} = \varnothing,$$

then $F = 0$, hence $S_0 = 0$.

Up to cyclic permutations, we just have to consider two cases.

(i) If $x = y = z$ or $X = Y = Z$, then

$$Xz.Yx.Zy - Xy.Yz.Zx = 0,$$

hence $S_0 = 0$.

(ii) If $x = y = Z$ or $X = Y = z$ or the other cases obtained by cyclic permutations, since $aa = 0$, we have

$$Xz.Yx.Zy = Xy.Yz.Zx = 0.$$

Thus $S_0 = 0$.

We have completed the proof that $S_0 = 0$.

Term of degree 1 Next, we write

$$P_1 = A_1(X,x,Y,y,Z,z)Xx.Yz.Zy + A_2(X,x,Y,y,Z,z)Xz.Yy.Zx + A_3(X,x,Y,y,Z,z)Xy.Yx.Zz.$$

Thus

$$S_1 = A_x.Xx.Yz.Zx + A_y.Xz.Yy.Zx. + A_z.Xy.Yx.Zz,$$

where

$$\begin{aligned} A_z &= A_3(X,x,Y,y,Z,z) + A_2(Y,y,Z,z,X,x) + A_1(Z,z,X,x,Y,y) \\ &= [Xx,Yy][Xy,Zz] + [Xx,Yy][Yx,Zz] + [Yy,Zz][Yy,Xx] \\ &\qquad + [Zz,Xx][Xx,Yy] \\ &= [Xx,Yy]([Xy,Zz] + [Yx,Zz] - [Yy,Zz] - [Xx,Zz]). \end{aligned}$$

By (10), $A_z = 0$. Therefore, $A_y = A_z = A_x = 0$ by cyclic permutations. We have completed the proof that $S_1 = 0$.

Term of degree 2 Finally, $S_2 = C.Xx.Yy.Zz$, where

$$C = [Xx,Yy][Xx,Zz] + [Xx,Yy][Yy,Zz] + [Yy,Zz][Yy,Xx] + [Yy,Zz][Zz,Xx] + [Zz,Xx][Zz,Yy] + [Zz,Xx][Xx,Yy].$$

Then $C = 0$ by the antisymmetry of the linking number. Thus $S_2 = 0$.

Now we have

$$\{\{Xx,Yy\}_\alpha, Zz\}_\alpha + \{\{Yy,Zz\}_\alpha, Xx\}_\alpha + \{\{Zz,Xx\}_\alpha, Yy\}_\alpha = \alpha^2 S_2 + \alpha S_1 + S_0 = 0,$$

concluding the proof of the Jacobi identity. □

2.3 The multifraction algebra

The swapping algebra is very easy to define. However, in the sequel we shall need to consider other Poisson algebras built out of the swapping algebra: these algebras will be more precisely subalgebras of the fraction algebra of $\mathcal{L}(\mathsf{P})$. We introduce in this subsection *cross fractions*, *multifractions* and the *multifraction algebra*.

2.3.1 Cross fractions and multifractions Since $\mathcal{L}_\alpha(\mathsf{P})$ is an integral domain (with respect to the commutative product) we can consider its algebra of fractions $\mathcal{Q}_\alpha(\mathsf{P})$.

Let $(X,Y,x,y) =: Q$ be a quadruple of points of P such that $x \neq Y$ and $y \neq X$. The *cross fraction* determined by Q is the element of $\mathcal{Q}_\alpha(\mathsf{P})$ defined by

$$[X;Y;x;y] := \frac{Xx.Yy}{Xy.Yx}.$$

More generally, if $\mathrm{X} := (X_1, \dots, X_n)$ and $\mathrm{x} := (x_1, \dots x_n)$ are two tuples of elements of P such that $x_i \neq X_i$ for all i, and σ is a permutation of $\{1, \dots, n\}$, then the *elementary multifraction — defined over* P — determined by this data is

$$[\mathrm{X}, \mathrm{x}; \sigma] := \frac{\prod_{i=1}^n X_i x_{\sigma(i)}}{\prod_{i=1}^n X_i x_i}.$$

2.3.2 The multifraction algebra Now let $\mathcal{B}(\mathsf{P})$ be the vector space generated by elementary multifractions and let us call any element of $\mathcal{B}(\mathsf{P})$ a *multifraction*. Then:

Proposition 2.3.1 *The vector space $\mathcal{B}(\mathsf{P})$ is a Poisson subalgebra of $\mathcal{Q}_\alpha(\mathsf{P})$. Moreover, it is generated as a Poisson algebra by cross fractions. Finally, the swapping bracket $\{\cdot,\cdot\}_\alpha$ when restricted to $\mathcal{B}(\mathsf{P})$ does not depend on α.*

From now on, we call the Poisson algebra $\mathcal{B}(\mathsf{P})$ the *algebra of multifractions*.

Proof The proposition follows from two immediate observations:

- Every elementary multifraction is a product of cross fractions.
- If A and B are two cross fractions, then $\{A, B\}_\alpha$ is a multifraction and does not depend on α. □

3 Cross ratios and cross fractions

In this section, we interpret cross fractions, and in general multifractions, as functions on the space of cross ratios.

3.1 Cross ratios

Recall that a cross ratio on a set P is a map $\boldsymbol{b}$ from

$$\mathsf{P}^{4*} := \{(X, Y, x, y) \in \mathsf{P} \mid y \neq X,\ x \neq Y\}$$

to a field $\mathbb{K}$, which satisfies some algebraic rules. These rules encode two conditions which constitute a normalization, and two multiplicative cocycle identities which hold for different sets of variables:

- **normalization** $$\begin{cases} \boldsymbol{b}(X, Y, x, y) = 0 \iff x = X \text{ or } Y = y, \\ \boldsymbol{b}(X, Y, x, y) = 1 \iff x = y \text{ or } X = Y, \end{cases}$$
- **cocycle identity** $$\begin{cases} \boldsymbol{b}(X, Y, x, y) = \boldsymbol{b}(X, Y, x, z)\boldsymbol{b}(X, Y, z, y), \\ \boldsymbol{b}(X, Y, x, y) = \boldsymbol{b}(X, Z, x, y)\boldsymbol{b}(Z, Y, x, y). \end{cases}$$

Assume Γ acts on P. We say the cross ratio $\boldsymbol{b}$ is Γ–invariant if it is invariant under the diagonal action.

Remarks • We have changed convention from our previous articles [16; 17] in order to be coherent with the formula for the classical projective cross ratio: if $\boldsymbol{b}$ is a cross ratio with respect to the definition above, and we let $b(X,x,Y,y) := \boldsymbol{b}(X,Y,x,y)$, then b is a cross ratio using our older convention. Observe that the second normalization together with the cocycle identities imply the following symmetries:

$$\boldsymbol{b}(X,Y,x,y) = \boldsymbol{b}(Y,X,x,y)^{-1} = \boldsymbol{b}(Y,X,y,x) = \boldsymbol{b}(X,Y,y,x)^{-1}.$$

• Assume Γ acts on P. Let $\boldsymbol{b}$ be a Γ–invariant cross ratio. Let $\gamma \in \Gamma$, and γ^+ and γ^- be two γ–fixed points in P. Then the quantity

$$\boldsymbol{b}(\gamma^+, \gamma^-, \gamma y, y)$$

does not depend on the choice of y. In particular, let S be a closed connected oriented surface of genus greater than 2, let P be $\partial_\infty \pi_1(S)$ equipped with the action of $\pi_1(S)$. Let γ^+ and γ^- be, respectively, the attractive and repulsive fixed points of a nontrivial element γ of $\pi_1(S)$, and $\boldsymbol{b}$ a $\pi_1(S)$–invariant cross ratio. Then

$$\ell_{\boldsymbol{b}}(\gamma) := \big|\log|\boldsymbol{b}(\gamma^+, \gamma^-, \gamma(y), y)|\big|$$

is called the *period* of γ.

We finally denote by $\mathbb{B}(\mathsf{P})$ the set of cross ratios on P.

These definitions are closely related to those given by Otal [25; 26], discussions from various perspectives by Ledrappier [21], and work of Bourdon [2] in the context of CAT(-1)–spaces.

3.2 Multifractions as functions

To every cross fraction $[X;Y;x;y]$ we associate a function, denoted by $\overline{[X;Y;x;y]}$, on $\mathbb{B}(\mathsf{P})$ by the formula

$$\overline{[X;Y;x;y]}(\boldsymbol{b}) := \boldsymbol{b}(X,Y,x,y).$$

The following proposition follows at once from the definition of cross ratio.

Proposition 3.2.1 *The map $[X;Y;x;y] \to \overline{[X;Y;x;y]}$ extends uniquely to a morphism of commutative associative algebras from $\mathcal{B}(\mathsf{P})$ to the algebra of functions on $\mathbb{B}(\mathsf{P})$.*

In the sequel, we shall use identical notation for a multifraction and its image in the space of functions on $\mathbb{B}(\mathsf{P})$. Also, so far we did not (and will not) consider any topological structure on $\mathbb{B}(\mathsf{P})$ or on P.

3.3 Multifractions and Hitchin components

In [16], we identified the Hitchin component with a space of cross ratios satisfying certain identities. Let us recall some notation and definitions.

3.3.1 Hitchin component Let S be a closed oriented connected surface with genus at least two.

Definition 3.3.1 (Fuchsian and Hitchin homomorphisms) An n–*Fuchsian* homomorphism from $\pi_1(S)$ to $\mathrm{PSL}_n(\mathbb{R})$ is a homomorphism ρ which factorizes as $\rho = \iota \circ \rho_0$, where ρ_0 is a discrete faithful homomorphism with values in $\mathrm{PSL}_2(\mathbb{R})$, and ι is an irreducible homomorphism from $\mathrm{PSL}_2(\mathbb{R})$ to $\mathrm{PSL}_n(\mathbb{R})$.

An n–*Hitchin homomorphism* from $\pi_1(S)$ to $\mathrm{PSL}_n(\mathbb{R})$ is a homomorphism which may be deformed into an n–Fuchsian homomorphism.

The *Hitchin component* $\mathsf{H}(n,S)$ is the space of Hitchin homomorphisms up to conjugacy by an exterior automorphism of $\mathrm{PSL}_n(\mathbb{R})$. All these representations lift to $\mathsf{SL}(n,\mathbb{R})$. By construction $\mathsf{H}(n,S)$ is identified with a connected component of the character variety. It is a result by Hitchin [15] that $\mathsf{H}(n,S)$ is homeomorphic to the interior of a ball of dimension $(2g-2)(n^2-1)$.

As a corollary of the main result of [16], we have:

Theorem 3.3.2 *If ρ is Hitchin, and if γ is a nontrivial element of $\pi_1(S)$, then $\rho(\gamma)$ has n distinct positive real eigenvalues.*

By convention, we write these eigenvalues as $\lambda_1(\rho(\gamma)),\dots,\lambda_n(\rho(\gamma))$ with

$$\lambda_1(\rho(\gamma)) > \cdots > \lambda_n(\rho(\gamma)) > 0.$$

This allows us to introduce the following definition.

Definition 3.3.3 (girth and width) The *width* of a nontrivial element γ of $\pi_1(S)$ with respect to a Hitchin representation ρ is

$$\mathrm{width}_\rho(\gamma) := \log\left(\left|\frac{\lambda_1(\rho(\gamma))}{\lambda_n(\rho(\gamma))}\right|\right).$$

The *girth* of ρ is

$$\mathrm{gh}(\rho) := \sup\left\{\left|\frac{\lambda_2(\rho(\gamma))}{\lambda_1(\rho(\gamma))}\right| \;\middle|\; \gamma \in \pi_1(S)\setminus\{\mathrm{Id}\}\right\}. \tag{30}$$

The following proposition will be used several times.

Proposition 3.3.4 *Let C be a compact subset of $\mathsf{H}(n,S)$. Then:*

(i) *For any positive A, the following subset of $\pi_1(S)$ defined by*

$$S_A = \left\{\gamma \in \pi_1(S) \;\middle|\; \exists \rho \in C \text{ such that } \left|\frac{\lambda_2(\rho(\gamma))}{\lambda_1(\rho(\gamma))}\right| > A\right\}$$

contains only finitely many conjugacy classes.

(ii) *Moreover,*

$$\sup\{\mathrm{gh}(\rho) \mid \rho \in C\} < 1.$$

For the proof of this proposition, we first need:

Lemma 3.3.5 *Let S be a hyperbolic surface with unit tangent bundle $\mathsf{U}S$ equipped with the geodesic flow $\{\phi_t\}_{t\in\mathbb{R}}$. Let ρ_0 be a Hitchin representation in $\mathsf{H}(n,S)$. Then there exists a neighborhood W of ρ_0 in $\mathsf{H}(n,S)$ such that for every ρ in W, there exists a function $f_\rho\colon \mathsf{U}S \to \mathbb{R}$ such that:*

- *For every closed orbit γ and $x \in \gamma$,*

$$\int_0^{\ell(\gamma)} f_\rho \circ \phi_s(x)\,\mathrm{d}s = \log\left|\frac{\lambda_1(\rho(\gamma))}{\lambda_2(\rho(\gamma))}\right|,$$

where $\ell(\gamma)$ is the hyperbolic length of γ.

- *The function $\rho \mapsto f_\rho$ is continuous from W to $C^0(\mathsf{U}S,\mathbb{R})$, and moreover there exists a positive constant ε_0 such that $f_\rho > \varepsilon_0$ for all ρ.*

Proof of Lemma 3.3.5 This follows from the Anosov property of Hitchin representations and results in [4]. One could also use results by Guichard and Wienhard [14] or combine results of Sambarino [28; 27]. Since by [4, Theorem 6.1], the limit maps of a Hitchin representation depend in an analytic way on the representation, we can find

- a neighborhood D of ρ_0 in $\mathsf{H}(n,S)$,
- a vector bundle E over $M := D \times \mathsf{U}S$, smooth along $\mathsf{U}S$,
- a splitting of $E = L_1 \oplus \cdots \oplus L_n$ into line bundles, smooth along the geodesic flow,
- a continuous lift $\{\Phi_t\}_{t\in\mathbb{R}}$ on E of the geodesic flow $\{\phi_t\}_{t\in\mathbb{R}}$ on M preserving this decomposition and smooth along $\mathsf{U}S$,

such that if γ is a closed geodesic of hyperbolic length $\ell(\gamma)$ and $u \in L_i|_{\{\rho\}\times\gamma}$, then

$$\Phi_{\ell(\gamma)}(u) = \lambda_i(\rho(\gamma))\cdot u.$$

In this last equation, we identify the closed geodesic with the corresponding conjugacy class in $\pi_1(S)$. We now construct metrics ω_i on L_i, smooth along the geodesic flow. Let us consider the functions g_i on M such that $g_i\cdot\omega_i = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\Phi_t^*\omega_i$. In particular, we have

$$\Phi_t^*\omega_i(x) = \exp\biggl(\int_0^t g_i\circ\phi_s(x)\,\mathrm{d}s\biggr)\omega_i(x). \tag{31}$$

Then by construction for $x \in \{\rho\}\times\gamma$, we have

$$-\log(\lambda_i(\rho(\gamma))) = \int_0^{\ell(\gamma)} g_i\circ\phi_s(x)\,\mathrm{d}s.$$

Now let $g = g_2 - g_1$; then

$$\log\biggl(\frac{\lambda_1(\rho(\gamma))}{\lambda_2(\rho(\gamma))}\biggr) = \int_0^{\ell(\gamma)} g\circ\phi_s(x)\,\mathrm{d}s.$$

By the Anosov property, there exists some $T > 0$ such that the flow Φ_T contracts uniformly on $\mathrm{Hom}(L_1, L_2)$ along $\{\rho_0\}\times \mathsf{U}S$. In a more precise way, if we denote by ω_1^* the dual metric on L_1^* to ω_1, then there exists some T such that along $\{\rho_0\}\times\mathsf{U}S$ we have

$$\Phi_T^*(\omega_1^*\otimes\omega_2) = H\cdot\omega_1^*\otimes\omega_2,$$

where H is a continuous function on M such that along $\{\rho_0\}\times\mathsf{U}S$,

$$H < \tfrac{1}{2}.$$

By the continuity of H, the previous inequality extends to M after possibly restricting D. As a consequence, we have for $x \in M$,

$$\int_0^T g\circ\phi_s(x)\,\mathrm{d}s = -\log(H(x)) > \log(2).$$

Now let

$$f(x) := \frac{1}{T}\int_0^T g\circ\phi_s(x)\,\mathrm{d}s.$$

Then by construction,

$$f(x) > \frac{1}{T}\log(2) =: \varepsilon_0,$$

and, moreover, for $x \in \{\rho\} \times \mathsf{U}S$,

$$\int_0^{\ell(\gamma)} f \circ \phi_s(x)\,\mathrm{d}s = \int_0^{\ell(\gamma)} g \circ \phi_s(x)\,\mathrm{d}s = \log\left(\frac{\lambda_1(\rho(\gamma))}{\lambda_2(\rho(\gamma))}\right). \qquad \square$$

Proof of Proposition 3.3.4 By compactness, it is enough to prove that every ρ in $\mathsf{H}(n,S)$ possesses a neighborhood W so that the properties of the proposition hold when C is replaced by W. We choose the neighborhood W obtained in the previous lemma. Let then f_ρ be as in the conclusion of this lemma. Since f_ρ is bounded away from zero by a positive constant ε_0, it follows that

$$\log\left(\frac{\lambda_1(\rho(\gamma))}{\lambda_2(\rho(\gamma))}\right) > \varepsilon_0 \cdot \ell(\gamma).$$

The first result immediately follows. Then for the second result, we use the fact that $S_{1/2}$ contains only finitely many conjugacy classes and that given γ, the function

$$\rho \mapsto \frac{\lambda_2(\rho(\gamma))}{\lambda_1(\rho(\gamma))}$$

is continuous, with values less than 1. $\square$

3.3.2 Rank *n* cross ratios For every integer p, let $\partial_\infty\pi_1(S)^p_*$ be the set of pairs

$$(\mathrm{X},\mathrm{x}) = \big((X_0, X_1, \dots, X_p), (x_0, x_1, \dots, x_p)\big)$$

of $(p+1)$–tuples of points in $\partial_\infty\pi_1(S)$ such that $X_j \neq X_i \neq x_0$ and $x_j \neq x_i \neq X_0$ whenever $j > i > 0$. Let $\chi^p(\mathrm{X},\mathrm{x})$ be the multifraction defined by

$$\chi^p(\mathrm{X},\mathrm{x}) := \det_{i,j>0}([X_i; X_0; x_j; x_0]).$$

A cross ratio $\boldsymbol{b}$ has *rank* n if

- $\chi^n(\mathrm{X},\mathrm{x})(\boldsymbol{b}) \neq 0$ for all (X,x) in $\partial_\infty\pi_1(S)^n_*$,
- $\chi^{n+1}(\mathrm{X},\mathrm{x})(\boldsymbol{b}) = 0$ for all (X,x) in $\partial_\infty\pi_1(S)^{n+1}_*$.

The main result of [18], which used a result by Guichard [13], is the following.

Theorem 3.3.6 *There exists a bijection ϕ from the set of n–Hitchin representations to the set of $\pi_1(S)$–invariant rank n cross ratios, such that if $\boldsymbol{b} = \phi(\rho)$ then:*

(i) *For any nontrivial element γ of $\pi_1(S)$,*

$$\ell_{\boldsymbol{b}}(\gamma) = \mathrm{width}_\rho(\gamma),$$

where $\ell_{\boldsymbol{b}}(\gamma)$ is the period of γ given with respect to $\boldsymbol{b} = \phi(\rho)$, and $\mathrm{width}_\rho(\gamma)$ *is the width of γ with respect to ρ.*

(ii) *If γ_1 and γ_2 are two nontrivial elements of $\pi_1(S)$, and if e_i (resp. E_i) is an eigenvector of $\rho(\gamma_i)$ (resp. $\rho^*(\gamma_i)$) of maximal (resp. minimal) eigenvalue, then*

$$\boldsymbol{b}(\gamma_1^+,\gamma_2^+,\gamma_2^-,\gamma_1^-)=\frac{\langle E_2,e_1\rangle\langle E_1,e_2\rangle}{\langle E_1,e_1\rangle\langle E_2,e_2\rangle}. \tag{32}$$

In particular, every multifraction defines a function on the Hitchin component.

4 Wilson loops, multifractions and length functions

In this section, we shall relate Wilson loops, which are regular functions on the character variety, to multifractions. We will also introduce *elementary functions*, which are limits of Wilson loops, prove that they generate the multifraction algebra and that they are smooth functions on the Hitchin component. We finally introduce *length functions* in Section 4.4.

4.1 Wilson loops

Let γ be an element of $\pi_1(S)$ and ρ an element of $\mathsf{H}(n,S)$. The *Wilson loop* associated to γ is the function $\mathsf{W}(\gamma)$ on $\mathsf{H}(n,S)$ defined by

$$\mathsf{W}(\gamma)(\rho):=\operatorname{tr}(\rho(\gamma)).$$

Wilson loops only depend on conjugacy classes. We introduce the following definition.

Definition 4.1.1 (class of an element) Let γ be a nontrivial element of $\pi_1(S)$. Then the *class* $[\gamma]$ of γ is the oriented pair (γ^+,γ^-) of points of $\partial_\infty\pi_1(S)$, where γ^+ and γ^- are the attractive and repulsive fixed points of γ respectively.

Recall that $[\gamma]=[\eta]$ if and only if there exist positive integers m,n such that $\gamma^m=\eta^n$.

4.1.1 Asymptotics of Wilson loops Let ρ be a Hitchin representation. Recall that for any γ in $\pi_1(S)$ we can write

$$\rho(\gamma)=\sum_{1\leqslant i\leqslant n}\lambda_i(\rho(\gamma))\,p_i(\gamma),$$

where $p_i(\gamma)$ is a projector of trace 1, and $\lambda_i(\rho(\gamma))$ are real numbers such that

$$0<|\lambda_n(\rho(\gamma))|<\cdots<|\lambda_1(\rho(\gamma))|.$$

Let us write $\dot\rho(\gamma)=p_1(\gamma)$. We denote by $[\mathsf{A}]$ the set of eigenvectors of a purely loxodromic matrix A, and observe that $[\mathsf{A}^n]=[\mathsf{A}]$. We choose an auxiliary norm, denoted by $\|\cdot\|$, on $\mathbb{R}^n$. Then we have:

Proposition 4.1.2 *For any γ in $\pi_1(S)$ and $p \in \mathbb{N}$, we have*

$$\left\| \frac{\rho(\gamma^p)}{\mathsf{W}(\gamma^p)(\rho)} - \dot\rho(\gamma) \right\| \leqslant \mathrm{gh}(\rho)^p K([\rho(\gamma)]), \tag{33}$$

where K is a continuous function on the set of n lines in general position.

Proof Let $\mathsf{A} = \rho(\gamma)$. Since A is a real diagonalizable matrix,

$$\mathsf{A} = \sum_{i=1}^{n} \lambda_i \mathsf{p}_i,$$

where p_i are projectors and the eigenvalues λ_i satisfy $\lambda_1 > \cdots > \lambda_n > 0$. Thus

$$\begin{aligned} \left\| \frac{\mathsf{A}^p}{\mathrm{tr}(\mathsf{A}^p)} - \mathsf{p}_1 \right\| &\leqslant \frac{1}{\sum_{i=1}^n \lambda_i^p} \left\| \sum_{i=2}^{n} \lambda_i^p \mathsf{p}_i - \Big(\sum_{i=2}^{n} \lambda_i^p\Big) \mathsf{p}_1 \right\| \\ &\leqslant \left(\frac{\lambda_2}{\lambda_1}\right)^p \left(n\|\mathsf{p}_1\| + \sum_{i=2}^{n} \|\mathsf{p}_2\| \right). \end{aligned} \tag{34}$$

Thus the inequality follows by taking

$$K([\mathsf{A}]) = n\|\mathsf{p}_1\| + \sum_{i=2}^{n} \|\mathsf{p}_2\|. \qquad \square$$

As a corollary, we get:

Corollary 4.1.3 *Let $\gamma_1, \gamma_2, \dots, \gamma_q$ be coprime elements of Γ and let $m_1, m_2, \dots, m_q$ be positive numbers. Then*

$$\left\| \frac{\prod_{i=1}^q \rho(\gamma_i^{m_i})}{\mathsf{W}\big(\prod_{i=1}^q \gamma_i^{m_i}\big)(\rho)} - \frac{\dot\rho(\gamma_1)\dot\rho(\gamma_q)}{\mathrm{tr}(\dot\rho(\gamma_1)\dot\rho(\gamma_q))} \right\| \leqslant \mathrm{gh}(\rho)^m K, \tag{35}$$

where $m = \inf(m_i)$, and K depends continuously on the eigenvectors of $\rho(\gamma_i)$ and their relative configurations.

Proof We restate the previous proposition by saying that

$$\rho(\gamma^p) = \mathsf{W}(\gamma^p)(\rho) \cdot (\dot\rho(\gamma) + \mathrm{gh}(\rho)^p \cdot K(\gamma, \rho)), \tag{36}$$

where $K(\gamma, \rho)$ is continuous in ρ and only depends on the eigenvectors of $\rho(\gamma)$. Thus

$$\begin{aligned} \prod_{i=1}^{q} \rho(\gamma_i^{m_i}) &= \prod_{i=1}^{q} \mathsf{W}(\gamma_i^{m_i})(\rho) \cdot \prod_{i=1}^{q} (\dot\rho(\gamma_i) + \mathrm{gh}(\rho)^{m_i} \cdot K(\gamma_i, \rho)) \\ &= \prod_{i=1}^{q} \mathsf{W}(\gamma_i^{m_i})(\rho) \cdot \left(\prod_{i=1}^{q} \dot\rho(\gamma_i) + \mathrm{gh}(\rho)^m \cdot K_0(\gamma_1, \dots, \gamma_p; \rho) \right), \end{aligned} \tag{37}$$

where $K_0(\gamma_1,\dots,\gamma_p;\rho)$ is continuous in ρ and only depends on the eigenvectors of $\rho(\gamma_i)$. Thus

$$\frac{\mathsf{W}\big(\prod_{i=1}^{q}\gamma_i^{m_i}\big)(\rho)}{\prod_{i=1}^{q}\mathsf{W}(\gamma_i^{m_i})(\rho)}=\operatorname{tr}\Big(\prod_{i=1}^{q}\dot\rho(\gamma_i)\Big)+\operatorname{gh}(\rho)^m\cdot K_1(\gamma_1,\dots,\gamma_p;\rho), \tag{38}$$

where $K_1(\gamma_1,\dots,\gamma_p;\rho)$ is continuous in ρ and only depends on the eigenvectors of $\rho(\gamma_i)$. Combining equations (37) and (38), we obtain that

$$\frac{\prod_{i=1}^{q}\rho(\gamma_i^{m_i})}{\mathsf{W}\big(\prod_{i=1}^{q}\gamma_i^{m_i}\big)(\rho)}=\frac{\prod_{i=1}^{q}\dot\rho(\gamma_i)}{\operatorname{tr}\big(\prod_{i=1}^{q}\dot\rho(\gamma_i)\big)}+\operatorname{gh}(\rho)^m\cdot K_2(\gamma_1,\dots,\gamma_p;\rho), \tag{39}$$

where $K_1(\gamma_1,\dots,\gamma_p;\rho)$ is continuous in ρ and only depends on the eigenvectors of $\rho(\gamma_i)$ and their relative positions. To conclude the proof of the corollary, note that if A is an endomorphism and p,q are projectors such that $\operatorname{tr}(\mathrm{p}A\mathrm{q})\neq 0\neq\operatorname{tr}(\mathrm{p}A\mathrm{q})$, then

$$\frac{\mathrm{p}A\mathrm{q}}{\operatorname{tr}(\mathrm{p}A\mathrm{q})}=\frac{\mathrm{pq}}{\operatorname{tr}(\mathrm{pq})}.$$

Using this, we get that

$$\frac{\prod_{i=1}^{q}\dot\rho(\gamma_i)}{\operatorname{tr}\big(\prod_{i=1}^{q}\dot\rho(\gamma_i)\big)}=\frac{\dot\rho(\gamma_1)\dot\rho(\gamma_q)}{\operatorname{tr}(\dot\rho(\gamma_1)\dot\rho(\gamma_q))}.$$

Combining this last equality with (39) yields the statement of the corollary. □

We begin with the following proposition where we consider multifractions as functions on $\mathsf{H}(n,S)$.

Proposition 4.1.4 *Let $\gamma_1,\dots,\gamma_k$ be nontrivial elements of $\pi_1(S)$. Then the sequence*

$$\left\{\frac{\mathsf{W}(\gamma_1^p\cdots\gamma_k^p)}{\mathsf{W}(\gamma_1^p)\cdots\mathsf{W}(\gamma_k^p)}\right\}_{p\in\mathbb{N}}$$

converges uniformly on every compact of $\mathsf{H}(n,S)$ to a multifraction when p goes to infinity. More precisely,

$$\lim_{p\to\infty}\left(\frac{\mathsf{W}(\gamma_1^p\cdots\gamma_k^p)}{\mathsf{W}(\gamma_1^p)\cdots\mathsf{W}(\gamma_k^p)}\right)=\frac{\prod_{i=1}^{k}\gamma_{i+1}^+\gamma_i^-}{\prod_{i=1}^{k}\gamma_i^+\gamma_i^-}=[\mathrm{G}^+,\mathrm{G}^-;\tau],$$

where $\mathrm{G}^\pm=(\gamma_1^\pm,\dots,\gamma_k^\pm)$ and $\tau(i)=i-1$, using the convention that $k+1=1$.

Proof We first observe that if e_i (resp. E_i) is an eigenvector of $\rho(\gamma_i)$ (resp. $\rho^*(\gamma_i)$) of maximal (resp. minimal) eigenvalue, with $\langle E_i,e_i\rangle=1$, then

$$\operatorname{tr}(\dot\rho(\gamma_1)\cdots\dot\rho(\gamma_k))=\prod_i\langle E_i,e_{i+1}\rangle.$$

By Equation (32),

$$\prod_i \langle E_i, e_{i+1}\rangle = \left(\frac{\prod_{i=1}^{p} \gamma_{i+1}^{+}\gamma_i^{-}}{\prod_{i=1}^{k} \gamma_i^{+}\gamma_i^{-}}\right)(\rho).$$

It thus follows that

$$\mathrm{tr}(\dot\rho(\gamma_1)\cdots\dot\rho(\gamma_k)) = [\mathrm{G}^+, \mathrm{G}^-; \tau].$$

Then the result follows at once from Propositions 4.1.2 and 3.3.4. □

4.2 Elementary functions

Proposition 4.1.4 leads us to the following definition.

Definition 4.2.1 The multifraction

$$\mathsf{T}(\gamma_1,\dots,\gamma_p) := \frac{\prod_{i=1}^{p} \gamma_{i+1}^{+}\gamma_i^{-}}{\prod_{i=1}^{p} \gamma_i^{+}\gamma_i^{-}} \tag{40}$$

is an *elementary function of order* p.

By the previous proposition and its proof, we have the equalities

$$\mathsf{T}(\gamma_1,\dots,\gamma_p) = \lim_{n\to\infty} \frac{\mathsf{W}(\gamma_1^n\cdots\gamma_p^n)}{\mathsf{W}(\gamma_1^n)\cdots\mathsf{W}(\gamma_p^n)}, \tag{41}$$

$$\mathsf{T}(\gamma_1,\dots,\gamma_p) = \mathrm{tr}(\dot\rho(\gamma_1)\cdots\dot\rho(\gamma_p)). \tag{42}$$

The following formal properties of elementary functions are then easily checked:

Proposition 4.2.2 (i) **Cyclic invariance** *For every cyclic permutation* σ *of the indexing set* $\{1,\dots,p\}$, *we have*

$$\mathsf{T}(\gamma_1,\dots,\gamma_p) = \mathsf{T}(\gamma_{\sigma(1)},\dots,\gamma_{\sigma(p)}).$$

(ii) **Class invariance** *If* $[\eta_i] = [\gamma_i]$, *then*

$$\mathsf{T}(\gamma_1,\dots\gamma_p) = \mathsf{T}(\eta_1,\dots,\eta_p).$$

(iii) *If* $[\gamma_p] = [\gamma_{p-1}]$, *then*

$$\mathsf{T}(\gamma_1,\dots\gamma_p) = \mathsf{T}(\gamma_1,\dots\gamma_{p-1}).$$

(iv) *If* $[\gamma_p] = [\gamma_{p-1}^{-1}]$, *then*

$$\mathsf{T}(\gamma_1,\dots\gamma_p) = 0.$$

(v) **Relations** *Assume that* $[\gamma_i] \neq [\gamma_{i+1}]$. *Then*

$$\mathsf{T}(\gamma_1,\dots,\gamma_p) = \frac{\mathsf{T}(\gamma_1,\gamma_2)\,\mathsf{T}(\gamma_1,\gamma_p)\,\mathsf{T}(\gamma_2,\gamma_3,\dots,\gamma_p)}{\mathsf{T}(\gamma_p,\gamma_2,\gamma_1)}.$$

From the last statement we deduce the following corollary.

Corollary 4.2.3 *Let* P *be the set of fixed points in* $\partial_\infty\pi_1(S)$ *of nontrivial elements of* $\pi_1(S)$*. Then every restriction of an elementary multifraction over* P *is a quotient of a product of elementary functions of orders 2 and 3.*

Proof Let us consider four nontrivial elements a, b, c, d of $\pi_1(S)$. Then we have

$$\frac{\mathsf{T}(a,b,c).\,\mathsf{T}(c,d)}{\mathsf{T}(a,d,c)\,\mathsf{T}(c,b)} = [b^+; d^+; a^-; c^-]. \tag{43}$$

The result follows. □

Recall that in this section we choose P to be the set of fixed points of nontrivial elements of $\pi_1(S)$. We now prove:

Proposition 4.2.4 *Every multifraction defined over* P *is a smooth function on* $\mathsf{H}(n,S)$*.*

Proof Let $\mathrm{Hom}(n,S)$ be the space of Hitchin homomorphisms, and π the submersion

$$\pi\colon \mathrm{Hom}(n,S) \to \mathsf{H}(n,S) = \mathrm{Hom}(n,S)/\,\mathrm{Aut}(\mathrm{PSL}_n(\mathbb{R})).$$

For every loxodromic element A in $\mathrm{PSL}_n(\mathbb{R})$, let p_A be the projection on the eigenspace of maximal eigenvalue with respect to the other eigenspaces. The map $A \to p_A$ (from the space of loxodromic elements) is smooth. It follows that for any elements $\gamma_1, \dots, \gamma_k$ in $\pi_1(S)$, the map from $\mathrm{Hom}(n,S)$ to $\mathbb{R}$ defined by

$$\Psi\colon \rho \to \mathrm{tr}(p_{\rho(\gamma_1)} \cdots p_{\rho(\gamma_k)})$$

is smooth. We end by observing that Ψ is $\mathrm{Aut}(\mathrm{PSL}_n(\mathbb{R}))$–invariant and that by (42),

$$\Psi = \mathsf{T}(\gamma_1, \dots, \gamma_k) \circ \pi.$$

Thus every elementary function is smooth and by the previous result every multifraction is smooth. □

4.3 The swapping bracket of elementary functions

For the sequel, we shall need to compute the swapping brackets of elementary functions. This is given by the following proposition, whose proof follows by an immediate application of the definition. We first say that two nontrivial elements γ and η in $\pi_1(S)$ are *coprime* if $\gamma^n \neq \eta^m$ for all nonzero integers m and n.

Proposition 4.3.1 *Let $\gamma_0,\dots,\gamma_p$ and $\eta_0,\dots,\eta_q$ be elements of $\pi_1(S)\setminus\{1\}$ such that the pairs (γ_i,γ_{i+1}) and (η_j,η_{j+1}) are coprime. Let*

$$\begin{aligned} \mathrm{a}_{i,j} &:= [\gamma_i^+\gamma_i^-,\eta_j^+\eta_j^-], & \mathrm{b}_{i,j} &:= [\gamma_{i+1}^+\gamma_i^-,\eta_{j+1}^+\eta_j^-],\\ \mathrm{c}_{i,j} &:= [\gamma_i^+\gamma_i^-,\eta_{j+1}^+\eta_j^-], & \mathrm{d}_{i,j} &:= [\gamma_{i+1}^+\gamma_i^-,\eta_j^+\eta_j^-],\\ \mathsf{T}_\gamma &:= \mathsf{T}(\gamma_0,\dots,\gamma_p), & \mathsf{T}_\eta &:= \mathsf{T}(\eta_0,\dots,\eta_q). \end{aligned} \tag{44}$$

Then

$$\begin{aligned} \frac{\{\mathsf{T}_\gamma,\mathsf{T}_\eta\}}{\mathsf{T}_\gamma\cdot\mathsf{T}_\eta} = \sum_{\substack{0\leqslant i\leqslant q\\0\leqslant j\leqslant p}} &\left(\mathrm{a}_{i,j}\,\mathsf{T}(\gamma_i,\eta_j)+\mathrm{b}_{i,j}\frac{\mathsf{T}(\eta_{j+1},\eta_j,\gamma_{i+1},\gamma_i)}{\mathsf{T}(\eta_j,\eta_{j+1})\,\mathsf{T}(\gamma_i,\gamma_{i+1})}\right)\\ &-\sum_{\substack{0\leqslant i\leqslant q\\0\leqslant j\leqslant p}}\left(\mathrm{c}_{i,j}\frac{\mathsf{T}(\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\eta_j,\eta_{j+1})}+\mathrm{d}_{i,j}\frac{\mathsf{T}(\eta_j,\gamma_{i+1},\gamma_i)}{\mathsf{T}(\gamma_i,\gamma_{i+1})}\right). \end{aligned} \tag{45}$$

Proof Using "logarithmic derivatives", we have

$$\begin{aligned} \frac{\{\mathsf{T}_\gamma,\mathsf{T}_\eta\}}{\mathsf{T}_\gamma\cdot\mathsf{T}_\eta} &= \sum_{\substack{0\leqslant i\leqslant p\\0\leqslant j\leqslant q}}\left(\left(\frac{\{\gamma_{i+1}^+\gamma_i^-,\eta_{j+1}^+\eta_j^-\}}{\gamma_{i+1}^+\gamma_i^-.\eta_{j+1}^+\eta_j^-}+\frac{\{\gamma_i^+\gamma_i^-,\eta_j^+\eta_j^-\}}{\gamma_i^+\gamma_i^-.\eta_j^+\eta_j^-}\right)\right.\\ &\qquad\left.-\left(\frac{\{\gamma_i^+\gamma_i^-,\eta_{j+1}^+\eta_j^-\}}{\gamma_i^+\gamma_i^-.\eta_{j+1}^+\eta_j^-}+\frac{\{\gamma_{i+1}^+\gamma_i^-,\eta_j^+\eta_j^-\}}{\gamma_{i+1}^+\gamma_i^-.\eta_j^+\eta_j^-}\right)\right)\\ &= \sum_{\substack{0\leqslant i\leqslant q\\0\leqslant j\leqslant p}}\left(\mathrm{b}_{i,j}\frac{\gamma_{i+1}^+\eta_j^-.\eta_{j+1}^+\gamma_i^-}{\gamma_{i+1}^+\gamma_i^-.\eta_{j+1}^+\eta_j^-}+\mathrm{a}_{i,j}\frac{\gamma_i^+\eta_j^-.\eta_j^+\gamma_i^-}{\gamma_i^+\gamma_i^-.\eta_j^+\eta_j^-}\right.\\ &\qquad\left.-\mathrm{d}_{i,j}\frac{\gamma_{i+1}^+\eta_j^-.\eta_j^+\gamma_i^-}{\gamma_{i+1}^+\gamma_i^-.\eta_j^+\eta_j^-}-\mathrm{c}_{i,j}\frac{\gamma_i^+\eta_j^-.\eta_{j+1}^+\gamma_i^-}{\gamma_i^+\gamma_i^-.\eta_{j+1}^+\eta_j^-}\right). \end{aligned}$$

From the definition of elementary functions (40), we get that

$$\frac{\mathsf{T}(\eta_{j+1},\eta_j,\gamma_{i+1},\gamma_i)}{\mathsf{T}(\eta_j,\eta_{j+1})\,\mathsf{T}(\gamma_i,\gamma_{i+1})}=\frac{\gamma_{i+1}^+\eta_j^-.\eta_{j+1}^+\gamma_i^-}{\gamma_{i+1}^+\gamma_i^-.\eta_{j+1}^+\eta_j^-},\qquad \mathsf{T}(\gamma_i,\eta_j)=\frac{\gamma_i^+\eta_j^-.\eta_j^+\gamma_i^-}{\gamma_i^+\gamma_i^-.\eta_j^+\eta_j^-},$$

$$\frac{\mathsf{T}(\eta_j,\gamma_{i+1},\gamma_i)}{\mathsf{T}(\gamma_i,\gamma_{i+1})}=\frac{\gamma_{i+1}^+\eta_j^-.\eta_j^+\gamma_i^-}{\gamma_{i+1}^+\gamma_i^-.\eta_j^+\eta_j^-},\qquad \frac{\mathsf{T}(\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\eta_j,\eta_{j+1})}=\frac{\gamma_i^+\eta_j^-.\eta_{j+1}^+\gamma_i^-}{\gamma_i^+\gamma_i^-.\eta_{j+1}^+\eta_j^-}.$$

This concludes the proof of the proposition. □

4.4 Length functions

In this section we introduce length functions.

4.4.1 Length functions from the point of view of the multifraction algebra Recall first that $\pi_1(S)$ acts on $\partial_\infty\pi_1(S)$ and thus on $\mathcal{B}(\partial_\infty\pi_1(S))$. For any $y\in\partial_\infty\pi_1(S)$ and β a nontrivial element in $\pi_1(S)$, let us introduce the cross fraction

$$p_\beta(y) = \frac{(\beta^+,\beta(y))\cdot(\beta^-,\beta^{-1}(y))}{(\beta^+,\beta^{-1}(y))\cdot(\beta^-,\beta(y))},$$

where for readability we revert to the classical notation (X,x) for pairs of points, rather than the concatenated notation Xx. We have, for any β in $\pi_1(S)$,

$$\frac{p_\beta(y)}{p_\beta(z)} = \frac{(\beta^2)*F_{y,z}}{F_{y,z}},$$

where

$$F_{y,z} = \frac{(\beta^+,\beta^{-1}(y))\cdot(\beta^-,\beta^{-1}(z))}{(\beta^+,\beta^{-1}(z))\cdot(\beta^-,\beta^{-1}(y))}.$$

In particular, the restriction of $p_\beta(y)$ to the space of $\pi_1(S)$–invariant cross ratios is independent of the choice of y.

For the sake of simplicity, we introduce the following formal series of multifractions and call it a *length function*:

$$\widehat{\ell}_\beta(y) := \tfrac{1}{2}\log(p_\beta(y)),$$

extending the bracket by the "log derivative" formulas

$$\{\widehat{\ell}_\beta(y),q\} := \frac{\{p_\beta(y),q\}}{2\,p_\beta(y)},\qquad \{\widehat{\ell}_\beta(y),\widehat{\ell}_\gamma(z)\} := \frac{\{p_\beta(y),p_\gamma(z)\}}{4\,p_\beta(y).p_\gamma(z)}. \tag{46}$$

Observe that $\mathrm{I}_S(\widehat{\ell}_{\beta^n}(y)) = n.\mathrm{I}_S(\widehat{\ell}_\beta(y))$.

4.4.2 Length functions and the character variety We can further relate these objects with the period and length defined in Section 3.1. Let

$$\mathrm{I}_S\colon \mathcal{B}(\partial_\infty\pi_1(S)) \to C^\infty(\mathsf{H}(n,S))$$

denote the restriction of functions from $\mathbb{B}(\partial_\infty\pi_1(S))$ to $\mathsf{H}(n,S)$.

We have for $\beta\in\pi_1(S)$ that

$$\mathrm{I}_S(\widehat{\ell}_\beta(y)) = \ell_\beta,$$

where

$$\ell_\beta(\rho) := \ell_b(\beta),$$

and ℓ_b is the period of β with respect to the cross ratio associated to ρ; see Section 3.1.

5 The Goldman algebra

In this section, we first recall the construction of the Atiyah–Bott–Goldman symplectic form on the character variety. We then explain the construction of the Goldman algebra, which allows us to compute the bracket of Wilson loops in terms of a Lie bracket on the vector space generated by free homotopy classes of loops.

5.1 The Atiyah–Bott–Goldman symplectic form

In [1], Atiyah and Bott introduced a symplectic structure on the character variety of representations of closed surface groups in compact Lie group, generalizing Poincaré duality. This was later generalized by Goldman for noncompact groups [9; 8] and connected to the Weil–Petersson Kähler form. If we identify the tangent space of $\mathsf{H}(n,S)$ at ρ with $H^1_\rho(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of $\mathrm{PSL}_n(\mathbb{R})$, then the symplectic form is given by

$$\omega_S([A],[B]) = \int_S \mathrm{tr}(A \wedge B), \tag{47}$$

where A and B are de Rham representatives of the cohomology classes $[A]$ and $[B]$. We denote by $\{\cdot,\cdot\}_S$ the associated Poisson bracket, called the *Atiyah–Bott–Goldman (ABG) Poisson bracket* in the sequel, and $\mathcal{A}(S)$ the Poisson algebra of smooth functions on $\mathsf{H}(n,S)$. In the next paragraph, we show how to compute the Atiyah–Bott–Goldman bracket, in the case of $\mathrm{PSL}_n(\mathbb{R})$, for the Wilson loops that we introduced in the previous section.

5.2 Wilson loops and the Goldman algebra

We describe in this subsection the Goldman algebra and how it helps to compute the ABG Poisson bracket. Let C be the set of free homotopy classes of closed curves on an oriented surface S. Let $\mathbb{Q}[\mathsf{C}]$ be the vector space generated by C over $\mathbb{Q}$. We linearly extend Wilson loops so that the map $\gamma \mapsto \mathsf{W}(\gamma)$ is now a linear map from $\mathbb{Q}[\mathsf{C}]$ to $C^\infty(\mathsf{H}(n,S))$.

Goldman [9] introduced a Lie bracket on $\mathbb{Q}[\mathsf{C}]$. We define it for two elements γ_1 and γ_2 of $\mathsf{C} \subset \mathbb{Q}[\mathsf{C}]$ and then extend it to $\mathbb{Q}[\mathsf{C}]$ linearly. We choose two curves representing γ_1 and γ_2, which we denote the same way.

If γ_1 and γ_2 are two curves from S^1 to S, an *intersection point* is a pair (a,b) in $S^1 \times S^1$ such that $\gamma_1(a) = \gamma_2(b)$. By a slight abuse of language, we usually identify an intersection point (a,b) with its image $x = \gamma_1(a) = \gamma_2(b)$. We further assume that γ_1 and γ_2 have transverse intersection points.

For every intersection point x, let ι_x be the local intersection number at x, let $\gamma_1 \circledast_x \gamma_2$ be the free homotopy class of the curve obtained by composing γ_1 and γ_2 in $\pi_1(S,x)$, and finally let

$$\iota(\gamma_1,\gamma_2) := \sum_{x\in\gamma_1\cap\gamma_2} \iota_x$$

be the global intersection number.

Definition 5.2.1 The *Goldman bracket* of γ_1 and γ_2 is the element of $\mathbb{Q}[\mathrm{C}]$ defined by

$$\{\gamma_1,\gamma_2\} := \sum_{x\in\gamma_1\cap\gamma_2} \iota_x \cdot \gamma_1 \circledast_x \gamma_2. \tag{48}$$

We illustrate in Figure 2 the Goldman bracket of two curves.

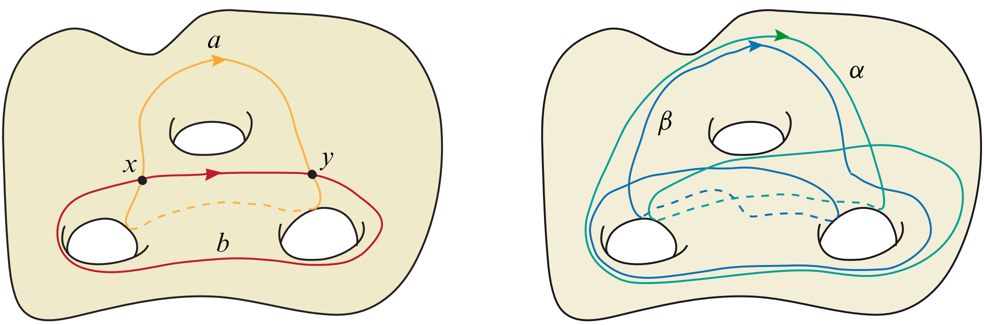

Figure 2: $\{b,a\} = \alpha - \beta$: two curves (left) and their Goldman bracket (right)

Goldman [9] proved that this bracket does not depend on the choice of representatives and is a Lie bracket. This bracket is related to the ABG Poisson bracket as follows.

Theorem 5.2.2 (Goldman) *Let γ_1 and γ_2 be two loops on S. Then the ABG Poisson bracket of the two corresponding Wilson loops in $\mathsf{H}(n,S)$ is*

$$\{\mathsf{W}(\gamma_1),\mathsf{W}(\gamma_2)\}_S = \mathsf{W}(\{\gamma_1,\gamma_2\}) - \frac{\iota(\gamma_1,\gamma_2)}{n}\mathsf{W}(\gamma_1)\cdot\mathsf{W}(\gamma_2). \tag{49}$$

We just stated Goldman's theorem for the case of $\mathsf{H}(n,S)$, but the theorem has a formulation in the general case of character varieties for semisimple groups. A different proof can also be found in [20].

6 Vanishing sequences and the main results

In this section, we first recall the definition of the length functions on the character varieties, then introduce the notion of a vanishing sequence of finite index subgroups of a surface group and state our main results relating the swapping algebra to the Goldman algebra. All these results will be proved in Section 9. As usual, let

$$\mathrm{I}_S\colon \mathcal{B}(\partial_\infty \pi_1(S)) \to C^\infty(\mathsf{H}(n,S))$$

denote the restriction of functions from $\mathbb{B}(\partial_\infty \pi_1(S))$ to $\mathsf{H}(n,S)$.

6.1 Poisson brackets of length functions

We explain in this section our results concerning length functions; see Section 4.4 for notation and definitions. Our first result relates the Goldman and the swapping Poisson brackets.

Theorem 6.1.1 *Let γ and η be two geodesics with at most one intersection point. Then we have*

$$\lim_{n\to\infty} \mathrm{I}_S(\{\widehat{\ell}_{\gamma^n}(y), \widehat{\ell}_{\eta^n}(y)\}) = \tfrac{1}{4}\{\ell_\gamma, \ell_\eta\}_S.$$

In the course of the proof of this result, we prove the following result of independent interest, which is an extension of the Wolpert formula [32; 31].

Theorem 6.1.2 (generalized Wolpert formula) *Let γ and η be two closed geodesics with a unique intersection point. Then the Goldman bracket of the two length functions ℓ_γ and ℓ_η, seen as functions on the Hitchin component, is*

$$\{\ell_\gamma, \ell_\eta\}_S = \iota(\gamma,\eta) \sum_{\varepsilon,\varepsilon' \in \{-1,1\}} \varepsilon\varepsilon'.\,\mathsf{T}(\gamma^\varepsilon.\eta^{\varepsilon'}), \tag{50}$$

where we recall that

$$\mathsf{T}(\xi,\zeta)(\rho) = \boldsymbol{b}_\rho(\xi^+,\zeta^+,\zeta^-,\xi^-).$$

We prove these two results in Section 9.2.

6.2 Poisson brackets of multifractions

We now relate in general the swapping bracket and the Goldman bracket. Our result can be described by saying that the swapping bracket is an inverse limit (with respect to sequences of coverings) of the Goldman bracket, or in other words that the swapping racket is a universal (in genus) Goldman bracket.

6.2.1 Vanishing sequences We now assume that S is equipped with an auxiliary hyperbolic metric. Let $\widetilde{S}$ be the universal cover of S so that $S=\widetilde{S}/\pi_1(S)$. For any γ in $\pi_1(S)$, we denote by $\widetilde{\gamma}$ its axis in $\widetilde{S}$ and $\langle\gamma\rangle$ the cyclic subgroup that it generates. Recall that we say that two elements γ and η of $\pi_1(S)$ are *coprime* if $\langle\gamma\rangle\cap\langle\eta\rangle=\{1\}$.

Let $\{\Gamma_m\}_{m\in\mathbb{N}}$ be a sequence of nested finite index subgroups of $\Gamma_0:=\pi_1(S)$. Then let $S_n:=\widetilde{S}/\Gamma_n$. For any $\gamma\in\Gamma$ let $\langle\gamma\rangle_n:=\langle\gamma\rangle\cap\Gamma_n$. Finally, let π_n be the projection from $\widetilde{S}$ to S_n and let $\widetilde{\gamma}_n:=\pi_n(\widetilde{\gamma})$.

Definition 6.2.1 Let $\{\Gamma_m\}_{m\in\mathbb{N}}$ be a sequence of nested finite index normal subgroups of $\Gamma_0:=\pi_1(S)$. We say that $\{\Gamma_m\}_{m\in\mathbb{N}}$ is a *vanishing sequence* if for all γ and η in $\pi_1(S)$, and for any set H which is invariant by left multiplication by γ and right multiplication by η and whose projection in $\langle\eta\rangle\backslash\pi_1(S)/\langle\gamma\rangle$ is finite, there exists an n_0 such that for all $n>n_0$, $H\cap\Gamma_n\subset\langle\eta\rangle.\langle\gamma\rangle$.

We shall freely use the following immediate consequence.

Proposition 6.2.2 *Let $\{\Gamma_m\}_{m\in\mathbb{N}}$ be a vanishing sequence with $\Gamma_0=\pi_1(S)$. For any η and γ in $\pi_1(S)$, and for any finite subset H_0 of $\pi_1(S)$ such that $H_0\cap(\langle\eta\rangle\cdot\langle\gamma\rangle)=\varnothing$, there exists a p_0 such that for all $p>p_0$,*

$$H_0\cap(\langle\eta\rangle\cdot\Gamma_p\cdot\langle\gamma\rangle)=\varnothing.$$

We prove in the appendix that vanishing sequences exist. This is an immediate consequence of a result by G Niblo [24].

6.2.2 Sequences of subgroups and limits Let P be the subset of $\partial_\infty\pi_1(S)$ given by the end points of periodic geodesics. Let G be the set of pairs of points $\gamma=(\gamma^-,\gamma^+)$ in P which correspond to fixed points of an element of the group $\partial_\infty\pi_1(S)$. Observe that given any finite index subgroup Γ of $\pi_1(S)$, the set G is in bijection with the set of primitive elements of Γ.

In the sequel, we shall freely identify elements of G with primitive elements in $\pi_1(S)$ or any of its finite index subgroups.

We associate to a sequence $\sigma=\{\Gamma_m\}_{m\in\mathbb{N}}$ of finite index subgroups of $\pi_1(S)$ the inverse limit S_σ of $\{S_m:=\widetilde{S}/\Gamma_m\}_{m\in\mathbb{N}}$, where $\widetilde{S}$ is the universal cover of S.

Observe that we have a map I from $\mathcal{B}(\mathsf{P})$ to $\mathcal{A}(S_\sigma)$ which by definition is the projective limit of $\{\mathcal{A}(S_m)\}_{m\in\mathbb{N}}$.

Definition 6.2.3 Let $\{g_m\}_{m\in\mathbb{N}}$ be a sequence of functions such that $g_m \in \mathcal{A}(S_m)$. We say that $\{g_m\}_{m\in\mathbb{N}}$ *converges* to the function h in $\mathcal{A}(S_\sigma)$, and write

$$\lim_{m\to\infty} g_m = h,$$

if for all p,

$$\lim_{n\to\infty} \mathsf{I}_{S_p}(g_n) = \mathsf{I}_{S_p}(h),$$

where I_{S_p} is the restriction with values in $\mathcal{A}(S_p)$.

6.2.3 Poisson brackets of multifractions The following result explains that the algebra of multifractions is an inverse limit of Goldman algebras with respect to vanishing sequences.

Theorem 6.2.4 *Let $\{\Gamma_m\}_{m\in\mathbb{N}}$ be a vanishing sequence of subgroups of $\pi_1(S)$. Let $\mathsf{P}\subset\partial_\infty\pi_1(S)$ be the set of end points of geodesics. Let b_0 and b_1 be two multifractions in $\mathcal{B}(\mathsf{P})$. Then we have*

$$\lim_{n\to\infty}\{\mathrm{I}(b_0),\mathrm{I}(b_1)\}_{S_n} = \mathsf{I}(\{b_0,b_1\}_W).$$

We prove this result in Section 9.1.

7 Product formulas and bouquets in good position

In this section, we wish to describe the Goldman bracket of curves which are compositions of many arcs. We shall call such a description a *product formula* and produce several instances of such formulas. This section is part of the technical core of this article.

The first formula (see Proposition 7.2.1) deals with a rather general situation computing the Goldman bracket of curves which are compositions of many arcs. Then, considering repetition, and using special collections of arcs called *bouquets in good positions* (see Definition 7.3.2), we prove a refinement of the product formula in Proposition 7.3.3. Proposition 7.3.3 is the first key result of this section.

Finally, in Proposition 7.5.2, we explain under which topological conditions we can find bouquets in good position and compute the various intersection numbers involved in Proposition 7.3.3. Proposition 7.5.2 is the second key result of this section.

7.1 An alternative formulation of the Goldman bracket

We first need to give an alternative description of the Goldman bracket.

Let $\overline{\gamma}_1$ and $\overline{\gamma}_2$ be two arcs passing through a basepoint x_0. For any point x in $\overline{\gamma}_i$, let $a_i(x)$ be the path along $\overline{\gamma}_i$ joining x_0 to x.

Definition 7.1.1 (intersection loops) Following this notation, for any $x \in \overline{\gamma}_1 \cap \overline{\gamma}_2$, the homotopy class

$$c_x(\overline{\gamma}_1, \overline{\gamma}_2) := a_1(x).a_2(x)^{-1} \in \pi_1(S, x_0)$$

is called an *intersection loop* at x; see Figure 3.

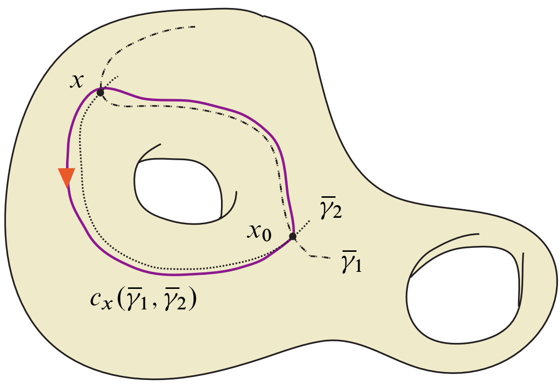

Figure 3: Intersection loop

The goal of this subsection is the following proposition.

Proposition 7.1.2 *Let γ_1 and γ_2 be two free homotopy classes of loops represented by curves $\overline{\gamma}_1$ and $\overline{\gamma}_2$ passing though x_0. Then the Goldman bracket in $\mathbb{Q}[\mathrm{C}]$ of the associated loops is given using intersection loops by*

$$\{\gamma_1, \gamma_2\}_S = \sum_{x \in \gamma_1 \cap \gamma_2} \iota_x \overline{\gamma}_1 \cdot c_x \cdot \overline{\gamma}_2 . c_x^{-1}. \tag{51}$$

This proposition is an immediate consequence of the following.

Proposition 7.1.3 *Let γ_1 and γ_2 be two loops passing though x_0. Then for every $x \in \gamma_1 \cap \gamma_2$, we have*

$$\gamma_1 \circledast_x \gamma_2 = \gamma_1 . c_x(\gamma_1, \gamma_2) \gamma_2 . c_x(\gamma_1, \gamma_2)^{-1},$$

as free homotopy classes of curves.

Proof As before let a_i be the arc along γ_i joining x_0 to x, and $c_x = a_1.a_2^{-1}$. Then

$$\gamma_1 \circledast_x \gamma_2 = a_1^{-1}\gamma_1 a_1 a_2^{-1}\gamma_2 a_2 = a_1^{-1}\gamma_1 c_x \gamma_2 a_2 = a_1^{-1}\gamma_1 c_x \gamma_2 c_x^{-1} a_1.$$

Thus $\gamma_1 \circledast_x \gamma_2$ is freely homotopic to $\gamma_1 c_x \gamma_2 c_x^{-1}$. □

7.2 The product formula

We need to express the Goldman bracket of Wilson loops of curves consisting of many arcs. We work with the following data (see Figure 4 for a partial drawing):

- Two tuples of arcs $\xi_0, \dots, \xi_q$ and $\zeta_0, \dots, \zeta_{q'}$ such that $A = \xi_0 \dots \xi_q$ and $B = \zeta_0 \dots \zeta_q$ are closed curves.
- Assume furthermore that for all pairs (i, j), the arcs ξ_i and ζ_j have transverse intersections and do not intersect at their end points.
- For each i and j, arcs u_i and v_j joining a basepoint x_0 to the origins of ξ_i and ζ_j respectively.

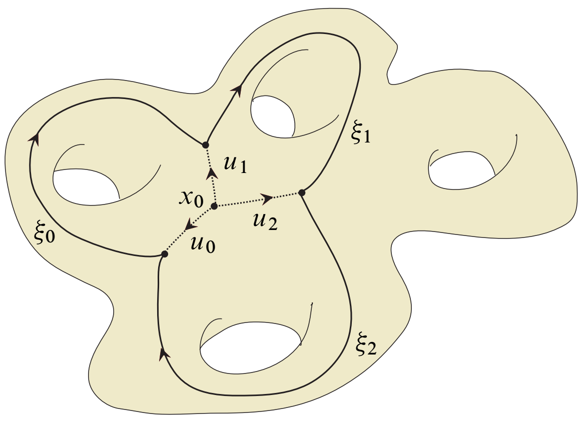

Figure 4: Arcs ξ_i and u_i

Let us introduce the following notation:

- $c_x^{i,j} := c_x(u_i\xi_i, v_j\zeta_j)$ for every $x \in \xi_i \cap \zeta_j$.
- $I_{i,j}(\xi) := \sum_{x \in \xi_i \cap \xi_j \mid \xi = c_x^{i,j}} \iota(x)$ for any $\xi \in \pi_1(S)$.
- $A_i := u_i \xi_i \xi_{i+1} \dots \xi_{i-1} u_i^{-1}$ and $B_j := v_j \zeta_j \zeta_{j+1} \dots \zeta_{j-1} v_j^{-1}$.

Proposition 7.2.1 (product formula) *Using the notation and assumptions described above, we have the following equality in* $\mathbb{Q}[\mathrm{C}]$*:*

$$\{A,B\} = \sum_{\substack{0\leqslant i\leqslant q\\ 0\leqslant j\leqslant q'}} \Bigg(\sum_{x\in\xi_i\cap\zeta_j} \iota_x A_i . c_x^{i,j} . B_j (c_x^{i,j})^{-1} \Bigg) \tag{52}$$

$$= \sum_{\substack{0\leqslant i\leqslant q\\ 0\leqslant j\leqslant q'}} \Bigg(\sum_{\xi\in\pi_1(S)} I_{i,j}(\xi) A_i . \xi . B_j \xi^{-1} \Bigg). \tag{53}$$

We first prove a preliminary proposition and postpone the proof of Proposition 7.2.1 until the next subsection.

7.2.1 A preliminary case We first study the following simple situation:

- Let ξ and η be two closed curves. Assume that $\xi = \xi_1.\xi_2$ and $\zeta = \zeta_1.\zeta_2$. Assume that for all i and j, ξ_i and ζ_j are closed curves with transverse intersections that do not intersect at their origin.
- Let u_i and v_j be arcs from x_0 to ξ_i and η_j respectively.
- Let $\tilde{\xi}_i := u_i \xi_i u_{i+1}^{-1}$, $\tilde{\zeta}_j := v_j \zeta_j v_{j+1}^{-1}$ and $c_x^{i,j} := c_x(\tilde{\xi}_i, \tilde{\zeta}_j) \in \pi_1(S, x_0)$ for $x \in \xi_i \cap \zeta_j$.

Proposition 7.2.2 *We have the following equality in* $\mathbb{Q}[\mathrm{C}]$*:*

$$\sum_{x\in\xi\cap\zeta} \iota_x \cdot \xi \ast_x \zeta = \sum_{\substack{x\in\xi_i\cap\zeta_j\\ 1\leqslant i,j\leqslant 2}} \iota_x \cdot \tilde{\xi}_i . \tilde{\xi}_{i+1} . c_x^{i,j} . \tilde{\zeta}_j . \tilde{\zeta}_{j+1} . (c_x^{i,j})^{-1}. \tag{54}$$

Proof First, we observe that for any two pairs of curves (ξ_1, ξ_2) and (ζ_1, ζ_2), we have

$$(\xi_1.\xi_2) \cap (\zeta_1.\zeta_2) = \bigsqcup_{i,j} (\xi_i \cap \zeta_j).$$

Let us denote

$$c_x := c_x(\tilde{\xi}_1.\tilde{\xi}_2, \tilde{\zeta}_1.\tilde{\zeta}_2).$$

We then have

$$x \in \xi_1 \cap \zeta_1 \Longrightarrow c_x = c_x^{1,1}, \qquad x \in \xi_2 \cap \zeta_1 \Longrightarrow c_x = \tilde{\xi}_1 . c_x^{2,1},$$
$$x \in \xi_1 \cap \zeta_2 \Longrightarrow c_x = c_x^{1,2} . \tilde{\zeta}_1^{-1}, \quad x \in \xi_2 \cap \zeta_2 \Longrightarrow c_x = \tilde{\xi}_1 . c_x^{2,2} . \tilde{\zeta}_1^{-1}.$$

Thus in all cases, if $x \in \xi_i \cap \zeta_j$, we have the equality of free homotopy classes

$$\tilde{\xi}_1 \tilde{\xi}_2 c_x \tilde{\zeta}_1 \tilde{\zeta}_2 = \tilde{\xi}_i \tilde{\xi}_{i+1} . c_x^{i,j} . \tilde{\zeta}_j . \tilde{\zeta}_{j+1} . (c_x^{i,j})^{-1}.$$

Thus we obtain the product formula:

$$\sum_{x\in(\xi_1\xi_2)\cap(\zeta_1\zeta_2)} \iota_x(\tilde{\xi}_1.\tilde{\xi}_2.c_x.\tilde{\zeta}_1.\tilde{\zeta}_2.c_x^{-1}) = \sum_{i,j}\Big(\sum_{x\in\xi_i\cap\zeta_j} \iota_x\big(\tilde{\xi}_i\tilde{\xi}_{i+1}.c_x^{i,j}.\tilde{\zeta}_j.\tilde{\zeta}_{j+1}.(c_x^{i,j})^{-1}\big)\Big). \tag{55}$$

This concludes the proof. □

Proof of Proposition 7.2.1 Obviously formula (53) is an immediate consequence of formula (52), so we concentrate on the latter.

First, we observe that the product formula when ξ_i and ζ_j are closed curves follows by induction from Proposition 7.2.2.

Let us now make the following observation. Let a, ξ and ζ be three arcs, transverse to a curve κ. Assume that $\xi.a.a^{-1}.\zeta$ is a closed curve. Then we have the following equalities in $\mathbb{Q}[\mathrm{C}]$:

$$\xi.a.a^{-1}.\zeta = \xi.\zeta, \tag{56}$$

$$\sum_{x\in(\xi.\zeta)\cap\kappa} \iota_x\xi.\zeta.c_x\kappa.c_x^{-1} = \sum_{x\in(\xi.a.a^{-1}.\zeta)\cap\kappa} \iota_x\xi.a.a^{-1}.\zeta.c_x\kappa.c_x^{-1}.$$

The first equality is obvious. For the second we notice that every intersection point of a with κ appears twice with a different sign.

We can now extend the product formula to arcs. We choose auxiliary arcs α_i joining x_0 to the initial point of ξ_i, similarly auxiliary arcs β_i joining x_0 to the initial point of ζ_i, and replace ξ_i and ζ_i respectively by the closed curves $\hat{\xi}_i = \alpha_i\xi_i\alpha_{i+1}^{-1}$ and $\hat{\zeta}_i = \beta_i\zeta_i\beta_{i+1}^{-1}$. From (56), since the product formula holds for the closed curves $\hat{\zeta}_j$ and $\hat{\xi}_i$, it holds for the arcs ζ_j and ξ_i. □

7.3 Bouquets in good position and the product formula

We shall need a special case of the product formula when we allow some repetitions in the arcs.

7.3.1 Bouquets in good position

Definition 7.3.1 (flowers and bouquets) (i) A *flower* based at $(x_0,\dots,x_q)$ is a collection of arcs

$$\mathcal{S} := ((g_0,\dots,g_q),(\alpha_0,\dots,\alpha_q))$$

such that:

- The g_i are closed curves based at x_i representing primitive elements in the fundamental group,
- The α_i are arcs, called *connecting arcs*, joining x_i to x_{i+1}.

(ii) A *bouquet* is a triple

$$\mathcal{F} = (\mathcal{S}_0, \mathcal{S}_1, V),$$

where $\mathcal{S}_1$ and $\mathcal{S}_0$ are flowers based at $(x_0, \dots, x_q)$ and $(y_0, \dots, y_{q'})$ respectively, and V is an arc joining x_0 and y_0.

(iii) We finally say that the bouquet $\mathcal{F}$ *represents* $((\gamma_0, \dots, \gamma_q), (\eta_0, \dots, \eta_{q'}))$, where γ_i, η_j are elements of $\pi_1(S, x_0)$ defined by $\gamma_i = U_i g_i U_i^{-1}$ and $\eta_j = V_j h_j V_j^{-1}$, for $U_i := \alpha_0 \dots \alpha_{i-1}$ and $V_j := V.\beta_0 \dots \beta_{j-1}$.

We shall also need bouquets which have especially neat configurations. Let

$$\mathcal{F} = \big(((g_0, \dots, g_q), (\alpha_0, \dots, \alpha_q)), ((h_0, \dots, h_{q'}), (\beta_0, \dots, \beta_{q'})), V\big)$$

be a bouquet of flowers based respectively at $(x_0, \dots, x_q)$ and $(y_0, \dots, y_{q'})$.

Definition 7.3.2 (good position) We say that:

(i) $\mathcal{F}$ is in a *good position* if

- the arcs α_i and g_i intersect transversely the arcs β_j and h_j at points different from x_i and y_j for all i, j,
- the closed curves $\alpha_0 \dots \alpha_q$ and $\beta_0 \dots \beta_{q'}$ are homotopic to zero.

(ii) $\mathcal{F}$ is in a *homotopically good position* if it is in a good position and if the following intersection loops are homotopically trivial:

$$\begin{cases} c_x(U_i.\alpha_i, V_j.\beta_j) & \text{for } x \in \alpha_i \cap \beta_j, \\ c_x(U_i.\alpha_i, V_j.h_j) & \text{for } x \in \alpha_i \cap h_j, \\ c_x(U_i.g_i, V_j.\beta_j) & \text{for } x \in g_i \cap \beta_j, \end{cases} \tag{57}$$

where $U_i := \alpha_0 \dots \alpha_{i-1}$ and $V_j := V.\beta_0 \dots \beta_{i-1}$.

In Figure 5, we have represented two flowers, one in blue, the other in red, where the connecting arcs α_i and β_i are dotted. In this figure all intersection loops corresponding to the four yellow transverse intersection points are drawn in the orange contractible region. Thus the bouquet is in a homotopically good position.

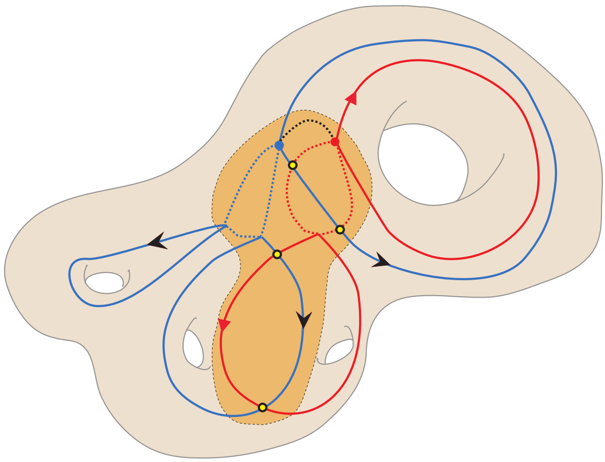

Figure 5: Bouquet in good position

7.3.2 Product formula for bouquets Let $\mathcal{F}$ be a bouquet as above in good position. Let us consider the closed curves

$$\boldsymbol{F}_i^{(p,n)} := U_i\,.\,g_i^n\,.\,(\alpha_i\,.\,g_{i+1}^p\alpha_{i+1}\dots g_{i-1}^p\alpha_{i-1})g_i^{p-n}U_i^{-1},$$
$$\boldsymbol{G}_i^{(p,n)} := V_i\,.\,h_i^n\,.\,(\beta_i\,.\,h_{i+1}^p\beta_{i+1}\dots h_{i-1}^p\beta_{i-1})h_i^{p-n}V_i^{-1}.$$

To simplify notation, let us write $\boldsymbol{F}^{(p)} := \boldsymbol{F}_0^{(p,0)}$ and $\boldsymbol{G}^{(p)} := \boldsymbol{G}_0^{(p,0)}$. Let us denote

$$H_{i,j} := \{c_x(U_i.g_i, V_j.h_j) \mid x \in g_i \cap h_j,\ c_x(U_i.g_i, V_j.h_j) \text{ homotopically trivial}\},$$
$$C_{i,j} := \{c_x(U_i.g_i, V_j.h_j) \mid x \in g_i \cap h_j,\ c_x(U_i.g_i, V_j.h_j) \text{ not homotopically trivial}\}.$$

Finally let

$$\mathrm{f}_{i,j}(\mathcal{F}) := \sum_{\xi\in H_{i,j}} I_{i,j}(\xi), \quad \mathrm{m}_{i,j}(\mathcal{F}) := \iota(g_i, \beta_j),$$
$$\mathrm{n}_{i,j}(\mathcal{F}) := \iota(\alpha_i, h_j), \qquad \mathrm{q}_{i,j}(\mathcal{F}) := \iota(\alpha_i, \beta_j), \tag{58}$$

where we recall that for any $\xi \in \pi_1(S)$, we denote

$$I_{i,j}(\xi) = \sum_{\substack{x\in g_i\cap h_j \\ c_x(U_i.g_i, V_j.h_j)=\xi}} \iota_x.$$

We can rewrite the product formula.

Proposition 7.3.3 (product formula in good position) *Assuming the bouquet $\mathcal{F}$ is in a homotopically good position and using the above notation, we have the following* *equality in* $\mathbb{Q}[\mathrm{C}]$*:*

$$
\begin{aligned}
(59)\quad & \{\boldsymbol{F}^{(p)}, \boldsymbol{G}^{(p)}\} \\
&= \sum_{\substack{0\leqslant i\leqslant q\\ 0\leqslant j\leqslant q'}} \Bigg(\sum_{\substack{1\leqslant m'\leqslant p\\ 1\leqslant m\leqslant p}} \mathrm{f}_{i,j}(\mathcal{F})(\boldsymbol{F}_i^{(p,m')}\mathbf{G}_j^{(p,m)}) + \sum_{1\leqslant m\leqslant p} \mathrm{m}_{i,j}(\mathcal{F})(\boldsymbol{F}_i^{(p,m)}\boldsymbol{G}_j^{(p,0)}) \\
&\qquad + \sum_{1\leqslant m'\leqslant p} \mathrm{n}_{i,j}(\mathcal{F})(\boldsymbol{F}_i^{(p,0)}\boldsymbol{G}_j^{(p,m')}) + \mathrm{q}_{i,j}(\mathcal{F})(\boldsymbol{F}_i^{(p,0)}\boldsymbol{G}_j^{(p,0)})\Bigg) \\
&\qquad + \sum_{\substack{0\leqslant i\leqslant q\\ 0\leqslant j\leqslant q'}} \sum_{\xi\in C_{i,j}} I_{i,j}(\xi)\Bigg(\sum_{\substack{1\leqslant m'\leqslant p\\ 1\leqslant m\leqslant p}} (\boldsymbol{F}_i^{(p,m')}\xi\boldsymbol{G}_j^{(p,m)}\xi^{-1})\Bigg).
\end{aligned}
$$

Proof This will be just another way to write the product formula. We consider the arcs ξ_i defined by

$$\xi_i := \begin{cases} g_j & \text{if } i = j.(p+1)+n \text{ with } 1\leqslant n\leqslant p, \\ \alpha_j & \text{if } i = j.(p+1). \end{cases}$$

Similarly, we consider the arcs ζ_i defined by

$$\zeta_i := \begin{cases} h_j & \text{if } i = j.(p+1)+n \text{ with } 1\leqslant n\leqslant p, \\ \beta_j & \text{if } i = j.(p+1). \end{cases}$$

Let us now finally consider the following arcs:

$$\begin{cases} u_i := U_j = \alpha_0.\dots.\alpha_j & \text{if } i = j.(p+1)+n \text{ with } 1\leqslant n\leqslant p, \\ v_i := V_j = V.\beta_0.\dots.\beta_j & \text{if } i = j.(p+1)+n \text{ with } 1\leqslant n\leqslant p, \end{cases}$$

such that u_i (resp. v_i) goes from x_0 to x_j (resp. x_0 to y_j).

We now apply formulas (52) and (59) to the arcs ξ_i, u_i, ζ_j, v_j. Observe that using the notation of Section 7.2, we have

$$\boldsymbol{F}^{(p)} = A, \quad \boldsymbol{G}^{(p)} = B.$$

We now have to identify the term on the right-hand sides of formulas (52) and (59), and in particular understand the arcs A_i, B_j, $c_x^{i,j}$ that appear in the right-hand side of formula (59). By definition,

$$A_i = u_i\xi_i\xi_{i+1}\dots\xi_{i-1}u_i^{-1}.$$

Thus if $i = j(p+1)+m$ with $0\leqslant m\leqslant p$,

$$A_i = \boldsymbol{F}_j^{(p,m)},$$

and by a similar argument,

$$B_i = \boldsymbol{G}_j^{(p,m)}.$$

By definition if $x \in \xi_i \cap \zeta_j$,

$$c_x^{i,j} = c_x(u_i \xi_i, v_j \zeta_j).$$

We now observe that

(i) if $i = j.(p+1)$, then $u_i \xi_i = U_j.g_j$,

(ii) if $i = j.(p+1) + n$ with $1 \leqslant m \leqslant p$, then $u_i \xi_i = U_j.\alpha_j$,

and similarly

(i) if $i = j.(p+1)$, then $v_i \zeta_i = V_j.h_j$,

(ii) if $i = j.(p+1) + n$ with $1 \leqslant m \leqslant p$, then $v_i \zeta_i = V_j.\beta_j$.

Then the special product formula (59) is a consequence of the product formula (52); indeed, thanks to the "homotopically good position" hypothesis, many of the intersection loops $c_x^{i,j}$ are homotopically trivial. □

7.4 Bouquets and covering

Let $\pi\colon S_1 \to S_0$ be a finite covering. Let

$$\mathcal{F} = \big(((g_0, \dots, g_q), (\alpha_0, \dots, \alpha_q)), ((h_0, \dots, h_{q'}), (\beta_0, \dots, \beta_q)), V\big)$$

be a bouquet of flowers in S_0 based respectively at $(x_0, \dots, x_q)$ and $(y_0, \dots, y_{q'})$. Let $\hat{x}_0$ be a lift of x_0 in S_1.

Definition 7.4.1 The bouquet of flowers in S_1

$$\widehat{\mathcal{F}} = \big(((\hat{g}_0, \dots, \hat{g}_q), (\hat{\alpha}_0, \dots, \hat{\alpha}_q)), ((\hat{h}_0, \dots, \hat{h}_{q'}), (\hat{\beta}_0, \dots, \hat{\beta}_q)), \hat{V}\big)$$

is *the lift of $\mathcal{F}$ through $\hat{x}_0$* if

- all arcs $\hat{V}$, $\hat{\alpha}_i$ and $\hat{\beta}_i$ are lifts of the arcs V, α_i and β_i;
- $\hat{g}_0$ is based at $\hat{x}_0$;
- the closed curves $\hat{g}_i$ and $\hat{h}_j$ are the *primitive lifts* of the curves g_i and h_j, in other words the primitive curves which are lifts of positive powers of the curves g_i and h_j.

Observe that the lift of a bouquet in homotopically good position is itself a bouquet in homotopically good position.

7.5 Finding bouquets in good position

Let S be a closed hyperbolic surface and $\widetilde{S}$ its universal cover. Let $G=(\gamma_0,\dots,\gamma_q)$ and $F=(\eta_0,\dots,\eta_{q'})$ be two tuples of primitive elements of $\pi_1(S)$ such that for all i, $(\gamma_i,\ \gamma_{i+1})$ are pairwise coprime, as are (η_i,η_{i+1}) as well, where the index i lives in $\mathbb{Z}/q\mathbb{Z}$ and $\mathbb{Z}/q'\mathbb{Z}$ respectively. Recall that we denote by $\widetilde{\zeta}$ the axis of the element $\zeta\in\pi_1(S)$.

Definition 7.5.1 We say G and F satisfy the *good position hypothesis* if there exists a metric ball B in $\widetilde{S}$ such that:

(i) For all i and j such that γ_i and η_j are coprime,

$$\widetilde{\gamma}_i\cap\widetilde{\eta}_j\subset B. \tag{60}$$

(ii) For all $\xi\in\pi_1(S)\setminus\{1\}$, we have

$$B\cap\xi(B)=\varnothing. \tag{61}$$

(iii) For all $\zeta\in\{\gamma_0,\dots,\gamma_q,\eta_0,\dots,\eta_{q'}\}$ and for all $\xi\in\pi_1(S)\setminus\langle\zeta\rangle$, we have

$$B\cap\xi(\widetilde{\zeta})=\varnothing. \tag{62}$$

(iv) For all $\zeta\in\{\gamma_0,\dots,\gamma_q,\eta_0,\dots,\eta_{q'}\}$ and for all $\xi\in\pi_1(S)\setminus\langle\zeta\rangle$, we have

$$\widetilde{\zeta}\cap\xi(\widetilde{\zeta})=\varnothing. \tag{63}$$

In other words, the closed geodesic corresponding to ζ is embedded.

Then we have the following result.

Proposition 7.5.2 *With the notation above, assume that G, F and $\pi_1(S)$ satisfy the good position hypothesis. Then there exist two bouquets $\mathcal{F}_L$ and $\mathcal{F}_R$ in S in a homotopically good position, both representing (G,F), such that furthermore,*

$$\tfrac{1}{2}(\mathrm{f}_{i,j}(\mathcal{F}_L)+\mathrm{f}_{i,j}(\mathcal{F}_R))=[\gamma_i^-\gamma_i^+,\eta_j^-\eta_j^+], \tag{64}$$

$$\tfrac{1}{2}(\mathrm{n}_{i,j}(\mathcal{F}_L)+\mathrm{n}_{i,j}(\mathcal{F}_R))=[\gamma_i^-\gamma_{i+1}^-,\eta_j^-\eta_j^+], \tag{65}$$

$$\tfrac{1}{2}(\mathrm{m}_{i,j}(\mathcal{F}_L)+\mathrm{m}_{i,j}(\mathcal{F}_R))=[\gamma_i^-\gamma_i^+,\eta_j^-\eta_{j+1}^-], \tag{66}$$

$$\tfrac{1}{2}(\mathrm{q}_{i,j}(\mathcal{F}_L)+\mathrm{q}_{i,j}(\mathcal{F}_R))=[\gamma_i^-\gamma_{i+1}^-,\eta_j^-\eta_{j+1}^-]. \tag{67}$$

Proof Let G and F be as above and B be a metric ball in $\widetilde{S}$ satisfying the assumptions (60)–(62). We subdivide the proof into several steps. We denote by π the projection from $\widetilde{S}$ to S.

Step 1 (construction of the bouquet in good position) Let $\widetilde{\gamma}_i$ be the axis of γ_i, let ε be some constant that we shall choose later to be very small, and let $\widetilde{\eta}_j^{\varepsilon}$ be a curve (with constant geodesic curvature) at distance ε from the axis $\widetilde{\eta}_j$ of η_j. (Notice that we have two such curves, for the moment we arbitrarily choose one of them.) We choose ε small enough that assertions (60) and (62) still hold when the $\widetilde{\eta}_j$ are replaced by $\widetilde{\eta}_j^{\varepsilon}$.

For every i, choose $x_i \in \widetilde{\gamma}_i \cap B$ so that

$$\widetilde{\gamma}_i \cap B \subset [x_i, \gamma_i^+[,$$

and similarly choose $y_j \in \widetilde{\eta}_j^{\varepsilon}$ so that

$$\widetilde{\eta}_i^{\varepsilon} \cap B \subset [y_i, \eta_i^+[_{\varepsilon},$$

where $[a,b]_{\varepsilon}$ denotes an arc joining a to b along a curve at a distance ε to a geodesic; see Figure 6.

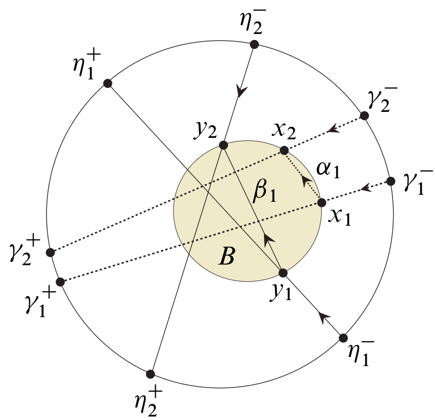

Figure 6: Finding a bouquet in good position

We now consider geodesic arcs $\widetilde{\alpha}_i$, $\widetilde{\beta}_j$ and $\widetilde{V}$ in $\widetilde{S}$ joining, respectively, x_i to x_{i+1}, y_j to y_{j+1} and x_0 to y_0. We furthermore choose B (and ε) so that all the arcs $\widetilde{\alpha}_i$, $\widetilde{\eta}_j^{\varepsilon}$, $\widetilde{\gamma}_j$ and $\widetilde{\beta}_j$ are transverse. In particular, if

$$\begin{aligned} &\alpha_i = \pi(\widetilde{\alpha}_i), && \beta_i = \pi(\widetilde{\beta}_i), && V = \pi(\widetilde{V}), \\ &\widetilde{g}_i = [x_i, \gamma(x_i)], && \widetilde{h}_j = [y_j, \eta_j(y_j)]_{\varepsilon}, \\ &g_i = \pi([x_i, \gamma(x_i)]), && h_j = \pi([y_j, \eta_j(y_j)]_{\varepsilon}), \end{aligned} \tag{68}$$

then

$$\mathcal{F} = \big(((g_0, \dots, g_q), (\alpha_0, \dots, \alpha_q)), ((h_0, \dots, h_{q'}), (\beta_0, \dots, \beta_{q'})), V\big)$$

is in good position. Observe furthermore that $\mathcal{F}$ represents (G, F).

Step 2 (homotopically good position) Let us now prove that $\mathcal{F}$ is in a homotopically good position. Let as usual

$$U_i = \alpha_0 \dots \alpha_{i-1}, \quad V_j = V.\beta_0 \dots \beta_{j-1}, \tag{69}$$

$$\widetilde{U}_i = \widetilde{\alpha}_0 \dots \widetilde{\alpha}_{i-1}, \quad \widetilde{V}_j = \widetilde{V}.\widetilde{\beta}_0 \dots \widetilde{\beta}_{j-1}. \tag{70}$$

Then $\widetilde{U}_i$ and $\widetilde{V}_j$ are the respective lifts of U_i and V_j, starting respectively from x_0 and y_0 and ending respectively in x_i and y_j.

Observe that all the arcs $\widetilde{\alpha}_k$, $\widetilde{\beta}_l$ and $\widetilde{V}$ lie in B. Thus so do the paths $\widetilde{U}_i$ and $\widetilde{V}_j$.

Let W_i be equal to α_i or g_i. Let $\widehat{W}_j$ be equal to β_j or h_j. From now on let us fix $x \in W_i \cap \widehat{W}_j$. Let us introduce some notation.

- Let a (resp. $\hat{a}$) be the path along W_i (resp. $\widehat{W}_j$) from $\pi(x_i)$ (resp. $\pi(y_j)$) to x.
- Let b (resp. $\hat{b}$) be the lift of a (resp. $\hat{a}$) in $\widetilde{S}$, starting from x_i (resp. x_j).
- Let z and $\hat{z}$ be the endpoints of b and $\hat{b}$, and let $\zeta \in \pi_1(S)$ be such that $z = \zeta(\hat{z})$.

By construction, ζ is conjugate to the intersection loop $c_x(U_i W_i, U_j \widehat{W}_j)$.

Let us now consider the various possibilities for the positions of z and $\hat{z}$.

(i) $W_i = \alpha_i$. Then $b \subset \widetilde{\alpha}_i$ and thus z belongs to B.

(ii) $W_i = g_i$. Then $z \in [x_i, \gamma_i(x_i)[\subset [x_i, \gamma_i^+[$.

(iii) $\widehat{W}_j = \beta_j$. Then, symmetrically, $\hat{z} = \zeta(z)$ belongs to B.

(iv) $\widehat{W}_j = h_i$. Then, symmetrically, $\hat{z} \in [y_j, \eta_j^\varepsilon(x_j)[_\varepsilon \subset [y_i, \eta_j^+[_\varepsilon$, where the intervals are subsets of $\widetilde{\eta}_j^\varepsilon$.

Our goal is now to prove that $\zeta = 1$ unless, possibly, $W_i = g_i$ and $\widehat{W}_j = h_j$.

(a) $W_i = \alpha_i$ and $\widehat{W}_j = \beta_j$, so by (i) and (iii) above, both z and $\zeta(z)$ belong to B, and thus by (61), $\zeta = 1$.

(b) $W_i = \alpha_i$ and $\widehat{W}_j = h_j$, so by assertions (i) and (iv), $\zeta(z) \in \widetilde{\eta}_j^\varepsilon$ and $z \in B$. Thus $\zeta^{-1}(\widetilde{\eta}_i^\varepsilon) \cap B \neq \varnothing$. Then by hypothesis (62), $\zeta \in \langle \eta_j \rangle$. In particular $z \in B \cap \widetilde{\eta}_j^\varepsilon$, so

$$z \in [y_j, \eta_j(y_j)[_\varepsilon. \tag{71}$$

Recall that from (iv),

$$\zeta(z) = \hat{z} \in [y_j, \eta_i(y_j)[_\varepsilon. \tag{72}$$

Since η_j is primitive and $\zeta \in \langle \eta_j \rangle$, we obtain from (71) and (72) that $\zeta = 1$.

(c) A symmetric argument proves that when $W_i = g_i$ and $\widehat{W}_j = \beta_j$, then $\zeta = 1$.

This finishes the proof that $\mathcal{F}$ is in a homotopically good position.

Step 3 (computation of the intersection numbers) Recall that for each (oriented) axis $\widetilde{\eta}_j$, we had two choices of curves at distance ε. Let us denote by $\widetilde{\eta}_j^L$ (resp. $\widetilde{\eta}_j^R$) the curve on the left (resp. right) of $\widetilde{\eta}_j$. Then let $\mathcal{F}^L$ and $\mathcal{F}^R$ be the corresponding collections of arcs.

We have proved that both $\mathcal{F}^L$ and $\mathcal{F}^R$ are in homotopically good position. Let us now compute the intersection numbers. We will do that step by step.

We shall repeat the following observation several times. Let g and h be two curves in S which pass through a point x_0 and intersect transversely at a finite number of points $x_1, \dots x_n$. Let $\widetilde{g}$ and $\widetilde{h}$ be the lifts of these curves in $\widetilde{S}$ which pass through a point $\widetilde{x}_0$. Then the projection realizes a bijection between the set of those x_i whose intersection loop is trivial, and the intersection points of $\widetilde{g}$ and $\widetilde{h}$.

In particular,

$$\sum_{\substack{x \in g \cap h \\ c_x(g,h)=1}} \iota(x) = \sum_{z \in \widetilde{g} \cap \widetilde{h}} \iota(z). \tag{73}$$

Proof of (64) If γ_i and η_j are coprime, then by formula (73) and since two geodesics have at most one intersection point, we have that

$$\mathrm{f}_{i,j} = [\gamma_i^- \gamma_i^+, \eta_j^- \eta_j^+].$$

If γ_i and η_j are not coprime, then since g_i is embedded by assumption (63), we have

$$\iota(g_i, h_j) = 0 = [\gamma_i^- \gamma_i^+, \eta_j^- \eta_j^+].$$

Thus in both cases,

$$\mathrm{f}_{\mathrm{i,j}}(\mathcal{F}^L) = \mathrm{f}_{\mathrm{i,j}}(\mathcal{F}^R) = [\gamma_i^- \gamma_i^+, \eta_j^- \eta_j^+].$$

Proof of (65) Since all the corresponding intersection loops are trivial, we see that

$$\iota(g_i, \beta_j) = \iota(\widetilde{\gamma}_i, \widetilde{\beta}_j).$$

We know that $\widetilde{\beta}_j \subset B$. To simplify, let us first consider the case when γ_i and η_k are coprime for $k = j, j+1$. Then $\widetilde{\gamma}_i \cap \widetilde{\eta}_k^\varepsilon \subset B$ and thus

$$\iota\big(\widetilde{\gamma}_i,]\eta_k^-, y_k]_\varepsilon\big) = 0.$$

It follows then that

$$\iota(g_i, \beta_j) = \iota\big(\widetilde{\gamma}_i,]\eta_j^-, y_j] \cup \widetilde{\beta}_j \cup]y_{j+1}, \eta_{j+1}^-[\big) = [\gamma_i^- \gamma_i^+, \eta_j^- \eta_{j+1}^-].$$

We illustrate that situation in Figure 7, on the left.

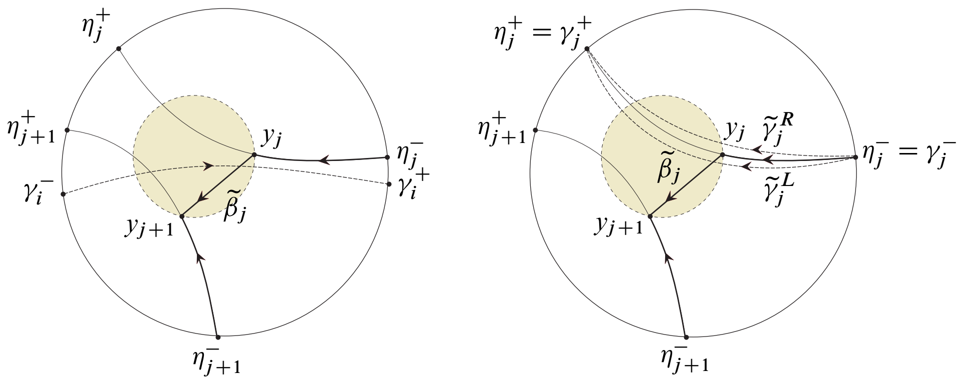

Figure 7: Intersection computations: noncoprime elements (left) and coprime elements (right)

Let us move to the remaining cases. The purpose of taking the "left and right perturbations" of $\widetilde{\eta}_j$ is to take care of the situation when η_j (or η_{j+1}) and γ_i are not coprime. So let us assume now that $\widetilde{\eta}_j = \widetilde{\gamma}_i$ (the case when $\widetilde{\eta}_{j+1} = \widetilde{\gamma}_i$ is symmetric).

Then in this case assume that η_{j+1}^- is on the left of $\widetilde{\eta}_j$ and $\widetilde{\gamma}_i$ has the same orientation as $\widetilde{\eta}_j$ (the other cases being symmetric). It then follows that

$$\mathrm{m}_{i,j}(\mathcal{F}^L) = \iota(\widetilde{\gamma}_i, \widetilde{\beta}_j^L) = 0, \tag{74}$$

$$\mathrm{m}_{i,j}(\mathcal{F}^R) = \iota(\widetilde{\gamma}_i, \widetilde{\beta}_j^R) = 1. \tag{75}$$

It follows that

$$\tfrac{1}{2}(\mathrm{m}_{i,j}(\mathcal{F}^L) + \mathrm{m}_{i,j}(\mathcal{F}^R)) = \tfrac{1}{2} = [\gamma_i^- \gamma_i^+, \eta_j^- \eta_{j+1}^-].$$

We illustrate that case in Figure 7, on the right. This finishes the proof of Equation (65).

Proof of (66) and (67) The proof uses the same ideas as the previous ones. □

8 Asymptotics

This section is the computational core of this article. Our goal is to compute asymptotic product formulas; namely, understand the behavior of the special product formula when the repetition in the arcs becomes infinite. This allows us to describe the limit of certain Wilson loops as elementary functions; see Proposition 8.2.4.

The goal of this section is to obtain Corollary 8.3.2, which is an asymptotic product formula for the Goldman bracket of elementary functions.

We first need some facts about vanishing sequences.

8.1 Properties of vanishing sequences

In this subsection, we shall be given a vanishing sequence $\{\Gamma_p\}_{p\in\mathbb{N}}$ of finite index subgroups of $\pi_1(S)$. We need some notation and definitions.

- Let gh_p be the function defined on $\mathsf{H}(n,S)$ by

$$\mathrm{gh}_p(\rho) := \mathrm{gh}(\rho|_{\Gamma_p}).$$

- For any positive integer p and primitive element ξ in Γ_0, let $\xi(p)$ be the positive integer such that

$$\langle \xi^{\xi(p)}\rangle = \langle\xi\rangle \cap \Gamma_p.$$

We write $\xi_p = \xi^{\xi(p)}$ and we denote the associated closed geodesic by $\widetilde{\xi}_p$.

Definition 8.1.1 (N–nice covering) Let γ and η be primitive coprime elements of $\Gamma_0 = \pi_1(S)$. Let N be a positive integer. We say that Γ_p is *N–nice* with respect to γ and η if the intersection loop $c_x(\widetilde{\gamma}_p,\widetilde{\eta}_p)$ is either trivial or satisfies

$$\pi_p(c_x(\widetilde{\gamma}_p,\widetilde{\eta}_p)) = \gamma^{k_1}.\eta^{-k_2},$$

where k_1 and k_2 satisfy

$$\gamma(p) - N > k_1 > N \quad \text{and} \quad \eta(p) - N > k_2 > N.$$

We need the following properties of vanishing sequences.

Proposition 8.1.2 *Let $\{\Gamma_p\}_{p\in\mathbb{N}}$ be a vanishing sequence of finite index subgroups of $\pi_1(S)$, and let $\{S_p\}_{p\in\mathbb{N}}$ be the corresponding sequence of coverings such that $\pi_1(S_p) = \Gamma_p$. Then:*

(i) *When p goes to infinity, the gh_p converge uniformly to 0 on every compact of $\mathsf{H}(n,S)$.*

(ii) *For any primitive coprime elements γ and η and for all N, there exists a p_0 such that Γ_p is N–nice with respect to γ and η for every $p > p_0$.*

(iii) *Let $G = (\gamma_0,\dots,\gamma_p)$ and $F = (\eta_0,\dots,\eta_q)$ be tuples of primitive elements of $\pi_1(S)\setminus\{1\}$ such that the pairs (γ_i,γ_{i+1}) and (η_j,η_{j+1}) are coprime. Then for q large enough, G and F satisfy the good position hypothesis of Definition 7.5.1 as elements of $\pi_1(S_q)$.*

Proof of Proposition 8.1.2 The proposition will follow from the concatenation of Propositions 8.1.3, 8.1.5 and 8.1.6, proved next. Proposition 8.1.4 is an intermediate step in proving Proposition 8.1.6. □

We now fix a vanishing sequence $\{\Gamma_p\}_{p\in\mathbb{N}}$.

Remember that we identify primitive elements in $\pi_1(S)$ and in any of its finite index subgroups.

Proposition 8.1.3 *When p goes to infinity, the* gh_p *converge uniformly to* 0 *on every compact of* $\mathsf{H}(n,S)$.

Proof For all positive numbers K and compact subsets C in $\mathsf{H}(n,S)$, let us consider the following subset of $\pi_1(S)$:

$$Z_K := \left\{\gamma \in \pi_1(S)\setminus\{\mathrm{Id}\} \,\middle|\, \exists \rho \in C \text{ such that } \left|\frac{\lambda_2(\rho(\gamma))}{\lambda_1(\rho(\gamma))}\right| > K\right\}.$$

By Proposition 3.3.4, the set of conjugacy classes in Z_K is a finite set. Let Z_K^0 be a finite set in $\pi_1(S)$ of representatives of the conjugacy classes of Z_K. From the definition of vanishing sequences, it follows that there exists a p_0 such that for all $p > p_0$, we have

$$Z_K^0 \cap \Gamma_p = \varnothing.$$

Since Γ_p is normal, it follows that

$$Z_K \cap \Gamma_p = \varnothing.$$

Then by definition, the girth of any representation in C restricted to Γ_p is smaller than K. Thus the family of functions gh_p converges uniformly to zero on C when p goes to ∞. □

The following proposition is well known.

Proposition 8.1.4 *Let γ be an element of Γ_0. Then there exists a p_0 such that for all $p > p_0$, the geodesics $\tilde{\gamma}_p$ are simple.*

Proof Let

$$\widehat{A}_\gamma := \{\xi \in \Gamma_0 \mid \xi(\tilde{\gamma}) \cap \tilde{\gamma} \neq \varnothing\} \subset \Gamma_0/\langle\gamma\rangle.$$

Observe that $\widehat{A}_\gamma$ is invariant under right multiplication by γ and that its projection in $\Gamma_0/\langle\gamma\rangle$ is a finite set. Thus there exists a p_0 such that for every $p > p_0$,

$$A_\gamma \cap \Gamma_p \subset \langle\gamma\rangle.$$

This implies that the projection of $\tilde{\gamma}$ in S_p is a simple closed geodesic; indeed the existence of a self-intersection point implies the existence of an element ξ in Γ_p such that $\xi(\tilde{\gamma}) \cap \tilde{\gamma} \neq \varnothing$. □

We finally need:

Proposition 8.1.5 *Let γ and η be two coprime primitive elements of $\Gamma_0 = \pi_1(S)$. Let N be a positive integer. Then there exists a p_0 such that for all $p > p_0$, the group Γ_p is N–nice with respect to γ and η.*

Proof We assume using the previous proposition that $\widetilde{\gamma}_p$ and $\widetilde{\eta}_p$ are simple.

We shall prove the following assertion:

Step 1 *For any $N > 0$, there exists a p_0 such that for any $p > p_0$, for any integers k such that $0 < k \leqslant N$ and for any m,*

$$\gamma^k \eta^m \notin \Gamma_p \quad \text{and} \quad \gamma^m \eta^{-k} \notin \Gamma_p.$$

This is an immediate application of Proposition 6.2.2. Let $H := \{\gamma^k \mid 0 < k \leqslant N\}$. Since γ and η are coprime, $H \cap \langle \eta \rangle = \varnothing$. Using Proposition 6.2.2, we get that there exists a p_0 such that for all $p > p_0$,

$$H \cap (\Gamma_p . \langle \eta \rangle) = \varnothing.$$

In other words, for all n and k such that $0 < k \leqslant N$,

$$\gamma^k . \eta^n \notin \Gamma_p.$$

A symmetric argument concludes the proof.

We now prove:

Step 2 *If $x \in \gamma_p \cap \eta_p$, then there exist positive integers k_1 and k_2 such that the intersection loop $c_p(x) := c_x(\widetilde{\gamma}_p, \widetilde{\eta}_p)$ satisfies*

$$\pi_p(c_p(x)) = \gamma^{k_2} . \eta^{-k_1},$$

where the equality is as homotopy classes in $S_0 = S$.

We may as well assume (using the first step and a shift in p) that the projection of the axis of γ and η are simple geodesics in S_0. Let also

$$A_p := \{\xi \in \Gamma_p \mid \xi(\widetilde{\eta}) \cap \widetilde{\gamma} \neq \varnothing\} \subset \Gamma_p.$$

Observe that A_p is invariant under left multiplication by γ and right multiplication by η. Let $\widehat{A}_p$ be the projection of A_p in $\langle \eta_p \rangle \backslash \Gamma_p / \langle \eta_p \rangle$. Observe also that we have a bijection from $\widehat{A}_p$ to

$$I_p := \pi_p(\widetilde{\gamma}) \cap \pi_p(\widetilde{\eta}) \subset S_p,$$

given by

$$\langle \gamma \rangle . \xi . \langle \eta \rangle \to \pi_p(\xi(\widetilde{\eta}) \cap \widetilde{\gamma}).$$

In particular, $\widehat{A}_p$ is finite since I_p is finite. Moreover, if x in I_p comes in this procedure from an element a in A_p, then a represents the intersection loop of x.

Since $\widehat{A}_0$ is finite, using the double coset separability property, there exists a p_0 such that for all $p > p_0$, we have

$$A_0 \cap \Gamma_p \subset \langle\gamma\rangle\langle\eta\rangle.$$

Since $A_p \subset A_0 \cap \Gamma_p$, it follows that the projection in S_0 of any intersection loop $c_x(\gamma_p, \eta_p)$ is homotopic to $\gamma^n.\eta^{-m}$ with n and m positive integers.

Conclusion of the proof The proposition follows at once from Steps 1 and 2. □

Proposition 8.1.6 *Let $G = (\gamma_0, \dots, \gamma_p)$ and $F = (\eta_0, \dots, \eta_q)$ be tuples of primitive elements of $\pi_1(S)\backslash\{1\}$ such that the pairs (γ_i, γ_{i+1}) and (η_j, η_{j+1}) are coprime. Then for m large enough, G and F satisfy the good position hypothesis of Definition 7.5.1 as elements of $\pi_1(S_m)$.*

Proof Let us check the four conditions of the good position hypothesis. Let $G = (\gamma_0, \dots, \gamma_p)$ and $F = (\eta_0, \dots, \eta_q)$ be primitive elements of $\pi_1(S) \setminus \{1\}$ such that both (γ_i, γ_{i+1}) and (η_j, η_{j+1}) are pairwise coprime.

(i) Let $B \subset \widetilde{S}$ be a ball containing all the intersections $\widetilde{\gamma}_i \cap \widetilde{\eta}_j$ when γ_i and η_j are coprime. Thus condition (i) of the good position hypothesis is satisfied.

(ii) Let

$$F := \{\xi \in \pi_1(S) \mid B \cap \xi(B) \neq \varnothing\}.$$

The set F is finite. Thus, by Proposition 6.2.2 applied to $\gamma = \eta = \mathrm{Id}$, there exists a p_0 such that for all $p > p_0$, we have

$$F \cap \Gamma_p = \{\mathrm{Id}\}.$$

Thus condition (ii) of the good position hypothesis is satisfied.

(iii) Next, for every $\zeta \in \{\gamma_0, \dots, \gamma_p, \eta_0, \dots, \eta_q\}$, the set

$$H_\zeta := \{\xi \in \Gamma/\langle\zeta\rangle \mid \xi(\widetilde{\zeta}) \cap B \neq \varnothing\}$$

is finite. Thus by Proposition 6.2.2 applied to $\gamma = \mathrm{Id}$, $\eta = \zeta$, there exists a p_0 such that for all $p > p_0$, we have

$$H_\zeta\langle\zeta\rangle \cap \Gamma_p = \langle\zeta\rangle.$$

Thus condition (iii) of the good position hypothesis is satisfied.

(iv) Finally, condition (iv) of the good position hypothesis is satisfied for p large enough by Proposition 8.1.4. □

8.2 Asymptotic product formula for Wilson loops

Throughout this subsection, we shall be given a finite index subgroup Γ_k of $\Gamma_0=\pi_1(S)$, corresponding to a covering $S_k\to S_0=S$. Then, if ρ is a Hitchin representation of $\pi_1(S)$ in $\mathrm{PSL}_n(\mathbb{R})$, ρ_k will denote the restriction of ρ to Γ_k.

Let $(\gamma_0,\dots,\gamma_q)$ and $(\eta_0,\dots,\eta_{q'})$ be two tuples of primitive elements of $\pi_1(S)$. We assume that (γ_i,γ_{i+1}) as well as (η_j,η_{j+1}) are all pairwise coprime.

Let then $\widehat{\gamma}_i$ and $\widehat{\eta}_i$ be the representatives of γ_i and η_i in Γ_k, and

(76) $$\boldsymbol{F}^{(p)}=\widehat{\gamma}_1^p\dots\widehat{\gamma}_q^p,\quad \mathbf{G}^{(p)}=\widehat{\eta}_1^p\dots\widehat{\eta}_{q'}^p.$$

We want to understand the asymptotics when p goes to infinity of the function

$$B_p^k(\gamma_0,\dots,\gamma_q;\eta_0,\dots,\eta_{q'})\colon \mathsf{H}(n,S_k)\to\mathbb{R}$$

defined by

(77) $$B_p^k(\gamma_0,\dots,\gamma_q;\eta_0,\dots,\eta_{q'}):=\frac{\mathsf{W}(\{\boldsymbol{G}^{(p)},\boldsymbol{F}^{(p)}\}_{S_k})}{\mathsf{W}(\mathbf{G}^{(p)})\mathsf{W}(\boldsymbol{F}^{(p)})}.$$

Let then

(78) $$\begin{aligned}\mathrm{f}_{i,j}&=[\gamma_i^-\gamma_i^+,\eta_j^-\eta_j^+], & \mathrm{n}_{i,j}&=[\gamma_i^-\gamma_{i+1}^-,\eta_j^-\eta_j^+],\\ \mathrm{m}_{i,j}&=[\gamma_i^-\gamma_i^+,\eta_j^-\eta_{j+1}^-], & \mathrm{q}_{i,j}&=[\gamma_i^-\gamma_{i+1}^-,\eta_j^-\eta_{j+1}^-].\end{aligned}$$

The next subsection is devoted to the proof of the following proposition.

Proposition 8.2.1 (asymptotic product formula) *For every compact set U in $\mathsf{H}(n,S)$, for every positive integer N and for k large enough, we have*

(79) $$\begin{aligned}&B_p^k(\gamma_0,\dots,\gamma_q,\eta_0,\dots,\eta_{q'})(\rho)\\ &=p^2\mathrm{R}_{i,j}\,\mathsf{T}(\gamma_i,\eta_j)+K\cdot(\mathrm{gh}_k(\rho)+\mathrm{gh}_0(\rho)^N)\\ &\quad+\sum_{\substack{0\leqslant i\leqslant q\\0\leqslant j\leqslant q'}}\Bigg((p-1)^2\mathrm{f}_{i,j}\,\mathsf{T}(\gamma_i,\eta_j)+(p-1)\bigg(\frac{\mathsf{T}(\gamma_{i+1},\gamma_i,\eta_j)}{\mathsf{T}(\gamma_i,\gamma_{i+1})}(\mathrm{n}_{i,j}+\mathrm{f}_{i+1,j})\\ &\qquad\qquad+\frac{\mathsf{T}(\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\eta_j,\eta_{j+1})}(\mathrm{m}_{i,j}+\mathrm{f}_{i,j+1})\bigg)\\ &\qquad\qquad+\frac{\mathsf{T}(\gamma_{i+1},\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\gamma_{i+1},\gamma_i)\,\mathsf{T}(\eta_j,\eta_{j+1})}(\mathrm{q}_{i,j}+\mathrm{n}_{i,j+1}+\mathrm{m}_{i+1,j}+\mathrm{f}_{i+1,j+1})\Bigg)\end{aligned}$$

for every ρ in U, where

- *K is bounded on U;*
- *$\mathrm{gh}_k(\rho)=\mathrm{gh}(\rho|_{\Gamma_k})$, where $\mathrm{gh}(\rho)$ is the girth of ρ (see Definition 3.3.3);*
- *the integers $\mathrm{f}_{i,j}$, $\mathrm{m}_{i,j}$, $\mathrm{n}_{i,j}$ and $\mathrm{q}_{i,j}$ are defined as in (78);*
- *$\mathrm{R}_{i,j}$ is an integer that only depends on γ_i and η_j.*

We will use bouquets to express these asymptotics using our product formula for bouquets.

8.2.1 Preliminary asymptotics Let ρ be a representation of $\Gamma_0=\pi_1(S)$. For any k, let $\rho_k := \rho|_{\Gamma_k}$. Let $\gamma_0,\dots,\gamma_q$ and $\eta_0,\dots,\eta_{q'}$ be primitive elements of Γ_0, and let $\widehat{\gamma}_0,\dots,\widehat{\gamma}_q$ and $\widehat{\eta}_0,\dots,\widehat{\eta}_{q'}$ be the corresponding elements in Γ_k given by

$$\widehat{\gamma}_i=\gamma_i^{Q_i} \quad\text{and}\quad \widehat{\eta}_j=\eta_j^{P_j}, \tag{80}$$

where Q_i and P_j are positive integers. In this proof, $K, K_0, K_1,\dots$ will be the generic symbol for a function of ρ bounded by a continuous function that only depends on the relative position of the eigenvectors of $\rho(\gamma_i)$ and $\rho(\eta_i)$ and does not depend on k. Let us define

$$\widehat{\mathsf{g}}_i=\rho_k(\widehat{\gamma}_i),\quad \widehat{\mathsf{h}}_i=\rho_k(\widehat{\eta}_i), \tag{81}$$

$$\mathsf{g}_i=\rho(\gamma_i),\qquad \mathsf{h}_i=\rho(\eta_i), \tag{82}$$

$$\widehat{\mathsf{F}}_i^{(p,m)} := \widehat{\mathsf{g}}_i^m\widehat{\mathsf{g}}_{i+1}^p\dots\widehat{\mathsf{g}}_{i-1}^p\widehat{\mathsf{g}}_i^{p-m}, \tag{83}$$

$$\widehat{\mathsf{G}}_i^{(p,m)} := \widehat{\mathsf{h}}_i^m\widehat{\mathsf{h}}_{i+1}^p\dots\widehat{\mathsf{h}}_{i-1}^p\widehat{\mathsf{h}}_i^{p-m}, \tag{84}$$

$$\mathsf{F}_i^{(p,m)} := \mathsf{g}_i^m\mathsf{g}_{i+1}^p\dots\mathsf{g}_{i-1}^p\mathsf{g}_i^{p-m}, \tag{85}$$

$$\mathsf{G}_i^{(p,m)} := \mathsf{h}_i^m\mathsf{h}_{i+1}^p\dots\mathsf{h}_{i-1}^p\mathsf{h}_i^{p-m}. \tag{86}$$

In this subsection we prove two propositions.

Proposition 8.2.2 *For all positive integers p, for all integers m with $0<m<p$, and for any ρ in a compact set U of $\mathsf{H}(n,S)$, we have*

$$\frac{\widehat{\mathsf{F}}_i^{(p,0)}}{\operatorname{tr}(\widehat{\mathsf{F}}_i^{(p,0)})}=\frac{\dot{\mathsf{g}}_i\dot{\mathsf{g}}_{i-1}}{\operatorname{tr}(\dot{\mathsf{g}}_i\dot{\mathsf{g}}_{i-1})}+K_1\,.\,\mathrm{gh}_k(\rho)^p, \tag{87}$$

$$\frac{\widehat{\mathsf{F}}_i^{(p,p)}}{\operatorname{tr}(\widehat{\mathsf{F}}_i^{(p,0)})}=\frac{\dot{\mathsf{g}}_{i+1}\dot{\mathsf{g}}_i}{\operatorname{tr}(\dot{\mathsf{g}}_i\dot{\mathsf{g}}_{i+1})}+K_2\,.\,\mathrm{gh}_k(\rho)^p, \tag{88}$$

$$\frac{\widehat{\mathsf{F}}_i^{(p,m)}}{\operatorname{tr}(\widehat{\mathsf{F}}_i^{(p,0)})}=\dot{\mathsf{g}}_i+K_3\,.\,\mathrm{gh}_k(\rho)^{\inf(m,p-m)}, \tag{89}$$

where the K_i are locally bounded functions of ρ.

We recall that $\dot{\mathsf{g}}$ is the projector on the eigendirection of the highest eigenvalue of g.

Proof Observe that for all m,

$$\operatorname{tr}(\widehat{\mathsf{F}}_i^{(p,m)})=\operatorname{tr}(\widehat{\mathsf{F}}_i^{(p,0)}).$$

We use Corollary 4.1.3 and get that for all p,

$$\frac{\widehat{\mathsf{F}}_i^{(p,0)}}{\operatorname{tr}(\widehat{\mathsf{F}}_i^{(p,0)})} = \frac{\dot{\mathsf{g}}_i\dot{\mathsf{g}}_{i-1}}{\operatorname{tr}(\dot{\mathsf{g}}_i\dot{\mathsf{g}}_{i-1})} + K_3\,.\,\mathrm{gh}_k(\rho)^p, \tag{90}$$

$$\frac{\widehat{\mathsf{F}}_i^{(p,p)}}{\operatorname{tr}(\widehat{\mathsf{F}}_i^{(p,0)})} = \frac{\dot{\mathsf{g}}_{i+1}\dot{\mathsf{g}}_{i}}{\operatorname{tr}(\dot{\mathsf{g}}_i\dot{\mathsf{g}}_{i+1})} + K_4\,.\,\mathrm{gh}_k(\rho)^p, \tag{91}$$

$$\frac{\widehat{\mathsf{F}}_i^{(p,m)}}{\operatorname{tr}(\widehat{\mathsf{F}}_i^{(p,m)})} = \dot{\mathsf{g}}_i + K_5\,.\,\mathrm{gh}_k(\rho)^{\inf(m,p-m)} \quad \text{for } m \notin \{0,p\}. \tag{92}$$

□

We use the same notation as in the beginning of this subsection.

Proposition 8.2.3 *Let us fix i and j. Let*

- *$\{N_1,\dots,N_r\}$ be a sequence of pairwise distinct integers such that $N_l \geqslant N$ and $Q_j - N_l \geqslant N$,*
- *$\{M_1,\dots,M_r\}$ be a sequence of pairwise distinct integers such that $M_l \geqslant N$ and $P_j - M_l \geqslant N$.*

Then for any ρ in a compact set U in $\mathsf{H}(n,S)$, and for any positive integers p, m and m', we have

$$\sum_{1\leqslant l\leqslant r} \frac{\mathsf{g}_i^{-N_l}\,\widehat{\mathsf{F}}_i^{\,(p,m)}\,\mathsf{g}_i^{N_l}\,.\,\mathsf{h}_j^{-M_l}\,\widehat{\mathsf{G}}_j^{\,(p,m')}\,\mathsf{g}_i^{M_l}}{\operatorname{tr}(\widehat{\mathsf{F}}_i^{(p,0)})\operatorname{tr}(\widehat{\mathsf{G}}_j^{(p,0)})} = r\,.\,\dot{\mathsf{g}}_i\dot{\mathsf{h}}_j + K\cdot \mathrm{gh}_0(\rho)^{M+N} + r\,\mathrm{gh}_0(\rho)^{Np}, \tag{93}$$

where K is a locally bounded function of ρ and

$$M = \inf(Q_i(m-1), P_j(m'-1), Q_i p - Qm', M_j p - m).$$

Proof In this proof, K_i will as usual denote a locally bounded function of ρ. For the purpose of this proof, we define

$$\widetilde{F}_i^{(p)} = \widehat{\mathsf{g}}_{i+1}^p \dots \widehat{\mathsf{g}}_{i-1}^p \quad \text{and} \quad \widetilde{G}_j^{(p)} = \widehat{\mathsf{h}}_{j+1}^p \dots \widehat{\mathsf{h}}_{j-1}^p.$$

By definition, if $m \geqslant 1$, $m' \geqslant 1$, $n < Q_i$ and $r < P_j$,

$$\mathsf{g}_i^{-n}\widehat{\mathsf{F}}_i^{(p,m)}\mathsf{g}_i^{n} = \mathsf{g}_i^{Q_i m - n}\widetilde{\mathsf{F}}_i^{(p)}\mathsf{g}_i^{Q_i(p-m)+n},$$
$$\mathsf{h}_j^{-r}\,\widehat{\mathsf{G}}_j^{(p,m')}\mathsf{h}_j^{r} = \mathsf{h}_j^{P_j m' - r}\,\widetilde{\mathsf{G}}_j^{(p)}\mathsf{h}_i^{P_j(p-m')+r}.$$

Observe also that

$$\operatorname{tr}(\widehat{\mathsf{F}}_i^{(p,0)}) = \operatorname{tr}(\widehat{\mathsf{F}}_i^{(p,m)}), \quad \operatorname{tr}(\widehat{\mathsf{G}}_j^{(p,0)}) = \operatorname{tr}(\widehat{\mathsf{G}}_j^{(p,m')}).$$

Thus, using the asymptotics of Corollary 4.1.3, we get that

$$\frac{\mathsf{g}_i^{-N_l}\,\widehat{\mathsf{F}}_i^{(p,m)}\,\mathsf{g}_i^{N_l}}{\operatorname{tr}(\widehat{\mathsf{F}}_i^{(p,0)})} = \dot{\mathsf{g}}_i + K_3\,.\,\mathrm{gh}_0(\rho)^{R_l},$$

where $A_l = \inf(Q_i m - N_l, Q_i(p-m) + N_l, Np)$, and we have observed that $Q_k \geqslant N$ for all k. Similarly,

$$\frac{\mathsf{h}_j^{-M_l}\,\widehat{\mathsf{G}}_j^{(p,m')}\,\mathsf{h}_i^{M_l}}{\operatorname{tr}(\widehat{\mathsf{G}}_j^{(p,0)})} = \dot{\mathsf{h}}_i + K_4\,.\,\mathrm{gh}_k(\rho)^{M},$$

where $B_l = \inf(P_j m' - N_l, P_j(p-m') + N_l, Np)$. Thus

$$\sum_{1\leqslant l\leqslant r} \frac{\mathsf{g}_i^{-N_l}\,\widehat{\mathsf{F}}_i^{(p,m)}\,\mathsf{g}_i^{N_l}\,.\,\mathsf{h}_j^{-M_l}\,\widehat{\mathsf{G}}_j^{(p,m')}\,\mathsf{h}_i^{M_l}}{\operatorname{tr}(\widehat{\mathsf{F}}_i^{(p,0)})\operatorname{tr}(\widehat{\mathsf{G}}_j^{(p,0)})} = r\,\dot{\mathsf{g}}_i\dot{\mathsf{h}}_j + K_0\cdot\Bigl(\sum_{1\leqslant l\leqslant r} \mathrm{gh}_0(\rho)^{R_l}\Bigr),$$

where

$$R_l = \inf(Q_i m - N_l, Q_i(p-m) + N_l, P_j m' - M_l, P_j(p-m') + M_l, Np)).$$

To conclude the proof, we will show that

$$\sum_{1\leqslant l\leqslant r} \mathrm{gh}_0(\rho)^{R_l} \leqslant \frac{4\,\mathrm{gh}_0(\rho)^{N+M}}{1-\mathrm{gh}_0(\rho)} + r\,\mathrm{gh}_0(\rho)Np. \tag{94}$$

Let

$$\mathcal{A} = \{l \mid R_l = Q_i m - N_l\},\qquad \mathcal{B} = \{l \mid R_l = Q_i p - Qm + N_l\},$$
$$\mathcal{F} = \{l \mid R_l = P_j m' - M_l\},\quad \mathcal{D} = \{l \mid R_l = P_j p - Pm' + M_l\}.$$

By definition,

$$\sum_{l\in\mathcal{A}} \mathrm{gh}_0(\rho)^{R_l} = \sum_{l\in\mathcal{A}} \mathrm{gh}_0(\rho)^{Qm-N_l} \leqslant \sum_{n\geqslant Q_i(m-1)+N} \mathrm{gh}_0(\rho)^{n} \leqslant \frac{\mathrm{gh}_0(\rho)^{N+Q_i(m-1)}}{1-\mathrm{gh}_0(\rho)}.$$

Symmetric arguments show that

$$\sum_{l\in\mathcal{B}} \mathrm{gh}_0(\rho)^{R_l} \leqslant \frac{\mathrm{gh}_0(\rho)^{N+Q_i(p-m)}}{1-\mathrm{gh}_0(\rho)},$$
$$\sum_{l\in\mathcal{F}} \mathrm{gh}_0(\rho)^{R_l} \leqslant \frac{\mathrm{gh}_0(\rho)^{N+P_j(m'-1)}}{1-\mathrm{gh}_0(\rho)},$$
$$\sum_{l\in\mathcal{D}} \mathrm{gh}_0(\rho)^{R_l} \leqslant \frac{\mathrm{gh}_0(\rho)^{N+P_j(p-m')}}{1-\mathrm{gh}_0(\rho)}.$$

Inequality (94), and thus the result, follow. □

8.2.2 Asymptotics and bouquets We use the same notation as in the beginning of this section: let $G=(\gamma_0,\dots,\gamma_q)$ and $F=(\eta_0,\dots,\eta_{q'})$ be two tuples of primitive elements of $\pi_1(S)$. We assume that the pairs (γ_i,γ_{i+1}) and (η_j,η_{j+1}) are all coprime. We shall use the notation of Section 7.3.2.

Proposition 8.2.4 *Assume that G, F and Γ_k satisfy the good position hypothesis. Assume also that Γ_k is N–nice for all pairs (γ_i,η_j). Let C be a bouquet in a good position representing G and F.*

Then for every compact set U in $\mathsf{H}(n,S)$, we have that for every ρ in U,

$$\begin{aligned}B_p^k(\gamma_0,\dots,\gamma_q,\eta_0,\dots,\eta_{q'})(\rho) &=\\ \sum_{\substack{0\leqslant i\leqslant q\\ 0\leqslant j\leqslant q'}}\Bigg((p-1)^2\mathrm{f}_{i,j}(\mathcal{F})\,\mathsf{T}(\gamma_i,\eta_j)+(p-1)\bigg(&\frac{\mathsf{T}(\gamma_{i+1},\gamma_i,\eta_j)}{\mathsf{T}(\gamma_i,\gamma_{i+1})}(\mathrm{n}_{i,j}(\mathcal{F})+\mathrm{f}_{i+1,j}(\mathcal{F}))\\ &+\frac{\mathsf{T}(\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\eta_j,\eta_{j+1})}(\mathrm{m}_{i,j}(\mathcal{F})+\mathrm{f}_{i,j+1}(\mathcal{F}))\bigg)\\ +\frac{\mathsf{T}(\gamma_{i+1},\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\gamma_{i+1},\gamma_i)\,\mathsf{T}(\eta_j,\eta_{j+1})}&(\mathrm{q}_{i,j}(\mathcal{F})+\mathrm{n}_{i,j+1}(\mathcal{F})+\mathrm{m}_{i+1,j}(\mathcal{F})+\mathrm{f}_{i+1,j+1}(\mathcal{F}))\Bigg)\\ &+p^2\bigg(\sum_{i,j}I_{i,j}(1)\,\sharp(C_{i,j})\,\mathsf{T}(\gamma_i,\eta_j)\bigg)+K\cdot(\mathrm{gh}_k(\rho)+\mathrm{gh}_0(\rho)^N),\end{aligned}$$

where

- *K is bounded by a continuous function that only depends on the relative position of the eigenvectors of $\rho(\gamma_i)$ and $\rho(\eta_j)$;*
- *$\mathrm{gh}(\rho)$ is the girth of ρ as defined in Definition 3.3.3, and $\mathrm{gh}_k(\rho)=\mathrm{gh}(\rho|_{\Gamma_k})$;*
- *the integers $\mathrm{f}_{i,j}(\mathcal{F})$, $\mathrm{m}_{i,j}(\mathcal{F})$, $\mathrm{n}_{i,j}(\mathcal{F})$ and $\mathrm{q}_{i,j}(\mathcal{F})$ are as defined in (58).*

Proof of Proposition 8.2.4 We now recall the product formula of (59), which we write using the notation of Section 8.2.1 as

$$B_p=B_p^0+\sum_{\xi\in C_{i,j}}I_{i,j}(\xi)B_p^{\xi}, \tag{95}$$

where

$$\begin{aligned}B_p^0:=&\frac{1}{\mathrm{tr}(\widehat{\mathsf{F}}^{(p,0)})\,.\,\mathrm{tr}(\widehat{\mathsf{G}}^{(p,0)})}\\ &\cdot\sum_{\substack{0\leqslant i\leqslant q\\ 0\leqslant j\leqslant q'}}\Bigg(\sum_{\substack{1\leqslant m'\leqslant p\\ 1\leqslant m\leqslant p}}\mathrm{f}_{i,j}(\mathcal{F})\,\mathrm{tr}(\widehat{\mathsf{F}}_i^{(p,m')}\widehat{\mathsf{G}}_j^{(p,m)})+\sum_{1\leqslant m\leqslant p}\mathrm{m}_{i,j}(\mathcal{F})\,\mathrm{tr}(\widehat{\mathsf{F}}_i^{(p,m)}\widehat{\mathsf{G}}_j^{(p,0)})\\ &\qquad+\sum_{1\leqslant m'\leqslant p}\mathrm{n}_{i,j}(\mathcal{F})\,\mathrm{tr}(\widehat{\mathsf{F}}_i^{(p,0)}\widehat{\mathsf{G}}_j^{(p,m')})+\mathrm{q}_{i,j}(\mathcal{F})\,\mathrm{tr}(\widehat{\mathsf{F}}_i^{(p,0)}\widehat{\mathsf{G}}_j^{(p,0)})\Bigg)\end{aligned}$$

and

$$B_p^\xi := \frac{1}{\mathrm{tr}(\widehat{\mathsf{F}}^{(p,0)})\,.\,\mathrm{tr}(\widehat{\mathsf{G}}^{(p,0)})} \sum_{\substack{1\leqslant m'\leqslant p\\ 1\leqslant m\leqslant p}} \mathrm{tr}\big(\widehat{\mathsf{F}}_i^{(p,m')}\rho(\xi)\widehat{\mathsf{G}}_j^{(p,m)}\rho(\xi)^{-1}\big).$$

Proposition 8.2.4 will follow from the next two propositions, which treat the term B_p^0 and the term involving the B_p^ξ separately. □

Proposition 8.2.5 *We have*

$$\begin{aligned}
&B_p^0(\gamma_0,\dots,\gamma_q,\eta_0,\dots,\eta_{q'}) = \\
&\sum_{\substack{0\leqslant i\leqslant q\\ 0\leqslant j\leqslant q'}}\Bigg((p-1)^2\mathrm{f}_{i,j}(\mathcal{F})\,\mathsf{T}(\gamma_i,\eta_j) + (p-1)\bigg(\frac{\mathsf{T}(\gamma_{i+1},\gamma_i,\eta_j)}{\mathsf{T}(\gamma_i,\gamma_{i+1})}(\mathrm{n}_{i,j}(\mathcal{F})+\mathrm{f}_{i+1,j}(\mathcal{F}))\\
&\qquad\qquad + \frac{\mathsf{T}(\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\eta_j,\eta_{j+1})}(\mathrm{m}_{i,j}(\mathcal{F})+\mathrm{f}_{i,j+1}(\mathcal{F}))\bigg)\\
&\qquad + \frac{\mathsf{T}(\gamma_{i+1},\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\gamma_{i+1},\gamma_i)\,\mathsf{T}(\eta_j,\eta_{j+1})}(\mathrm{q}_{i,j}(\mathcal{F})+\mathrm{n}_{i,j+1}(\mathcal{F})+\mathrm{m}_{i+1,j}(\mathcal{F})+\mathrm{f}_{i+1,j+1}(\mathcal{F}))\Bigg)\\
&\qquad + K\cdot \mathrm{gh}_k(\rho),
\end{aligned}$$

where K only depends on the position of the eigenvectors of $\rho(\gamma_i)$ and $\rho(\eta_j)$.

Proof Using the estimates for $\widehat{\mathsf{F}}_i^{(p,m)}$ and $\widehat{\mathsf{G}}_i^{(p,m)}$ from Proposition 8.2.2 we get

$$\begin{aligned}
&B_p^0 = \\
&\sum_{i,j}\Bigg(\mathrm{f}_{i.j}(\mathcal{F})(p-1)^2\,\mathrm{tr}(\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{h}}_j)\\
&\quad + \mathrm{f}_{i.j}(\mathcal{F})\bigg(\frac{\mathrm{tr}(\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{g}}_{i-1}\,.\,\dot{\mathsf{h}}_j\,.\,\dot{\mathsf{h}}_{j-1})}{\mathrm{tr}(\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{g}}_{i-1})\,\mathrm{tr}(\dot{\mathsf{h}}_j\,.\,\dot{\mathsf{h}}_{j-1})} + (p-1)\bigg(\frac{\mathrm{tr}(\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{h}}_j\,.\,\dot{\mathsf{h}}_{j-1})}{\mathrm{tr}(\dot{\mathsf{h}}_j\,.\,\dot{\mathsf{h}}_{j-1})} + \frac{\mathrm{tr}(\dot{\mathsf{h}}_j\,.\,\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{g}}_{i-1})}{\mathrm{tr}(\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{g}}_{i-1})}\bigg)\bigg)\\
&\quad + \mathrm{m}_{i.j}(\mathcal{F})\bigg((p-1)\frac{\mathrm{tr}(\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{h}}_{j+1}\,.\,\dot{\mathsf{h}}_j)}{\mathrm{tr}(\dot{\mathsf{h}}_j\,.\,\dot{\mathsf{h}}_{j+1})} + \frac{\mathrm{tr}(\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{g}}_{i-1}\,.\,\dot{\mathsf{h}}_{j+1}\,.\,\dot{\mathsf{h}}_j)}{\mathrm{tr}(\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{g}}_{i-1})\,\mathrm{tr}(\dot{\mathsf{h}}_{j+1}\,.\,\dot{\mathsf{h}}_j)}\bigg)\\
&\quad + \mathrm{n}_{i.j}(\mathcal{F})\bigg((p-1)\frac{\mathrm{tr}(\dot{\mathsf{g}}_{i+1}\,.\,\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{h}}_j)}{\mathrm{tr}(\dot{\mathsf{g}}_{i+1}\,.\,\dot{\mathsf{g}}_i)} + \frac{\mathrm{tr}(\dot{\mathsf{g}}_{i+1}\,.\,\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{h}}_j\,.\,\dot{\mathsf{h}}_{j-1})}{\mathrm{tr}(\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{g}}_{i-1})\,\mathrm{tr}(\dot{\mathsf{h}}_{j+1}\,.\,\dot{\mathsf{h}}_j)}\bigg)\\
&\quad + \mathrm{q}_{i.j}(\mathcal{F})\frac{\mathrm{tr}(\dot{\mathsf{g}}_{i+1}\,.\,\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{h}}_{j+1}\,.\,\dot{\mathsf{h}}_j)}{\mathrm{tr}(\dot{\mathsf{g}}_{i+1}\,.\,\dot{\mathsf{g}}_i)\,\mathrm{tr}(\dot{\mathsf{h}}_{j+1}\,.\,\dot{\mathsf{h}}_j)}\Bigg)\\
&\quad + K\cdot \mathrm{gh}_k(\rho),
\end{aligned}$$

where $0\leqslant i\leqslant q$ and $0\leqslant j\leqslant q'$ as before. Using the definition of multifractions, and after reordering terms, we obtain the asymptotics of the proposition. □

Finally we need to understand the last term involving the sum of the terms B_p^ξ.

Proposition 8.2.6 *We have*

$$\sum_{i,j}\sum_{\xi\in C_{i,j}} I_{i,j}(\xi)B_p^\xi = p^2\Bigg(\sum_{i,j} I_{i,j}(1)\,\sharp(C_{i,j})\,\mathrm{tr}(\dot{\mathsf{g}}_i\dot{\mathsf{h}}_j)\Bigg) + K\cdot \mathrm{gh}(\rho_0)^N, \tag{96}$$

where K only depends on the position of the eigenvectors of $\rho(\gamma_i)$ and $\rho(\eta_j)$.

Proof We use again the notation set up in the beginning of Section 8.2.1. By definition of an N–nice covering, any element $\xi\in C_{i,j}$ can be written as

$$\xi = \gamma_i^{N_\xi}\eta_j^{M_\xi},$$

where $N<N_\xi<Q_i-N$ and $N<M_\xi<P_j-N$. Since $\gamma_i^m\notin\Gamma_k$ for $0<m<Q_j$, we obtain that $\xi\to N_\xi$ and $\xi\to M_\xi$ are bijections.

Moreover, since the bouquet C is a lift of a bouquet C^0 in S_0,

$$I_{i,j}(\xi) = I_{i,j}(1).$$

It follows that for any i and j,

$$\sum_{\xi\in C_{i,j}} I_{i,j}(\xi)B_p^\xi = I_{i,j}(1)\sum_{\substack{1\leqslant m'\leqslant p\\ 1\leqslant m\leqslant p}} \frac{\mathrm{tr}(B_p^{m,m',i,j})}{\mathrm{tr}(\widehat{\mathsf{F}}^{(p,0)}).\,\mathrm{tr}(\widehat{\mathsf{G}}^{(p,0)})}, \tag{97}$$

where

$$B_p^{m,m',i,j} = \sum_{\xi\in C_{i,j}} \mathsf{g}_i^{-N_\xi}\widehat{\mathsf{F}}_i^{(p,m)}\mathsf{g}_i^{N_\xi}.\mathsf{h}_j^{-M_\xi}\widehat{\mathsf{G}}_j^{(p,m')}\mathsf{h}_j^{M_\xi}.$$

We now apply Proposition 8.2.3 to get

$$\frac{B_p^{m,m',i,j}}{\mathrm{tr}(\widehat{\mathsf{F}}^{(p,0)}).\,\mathrm{tr}(\widehat{\mathsf{G}}^{(p,0)})} = I_{i,j}(1)\,\sharp(C_{i,j})(\dot{\mathsf{g}}_i\dot{\mathsf{h}}_j + K_0\,\mathrm{gh}_0(\rho)^{N+M(m,m')} + K_0\,\mathrm{gh}_0(\rho)^{Np}), \tag{98}$$

where $M(m,m') = \inf(Q_i(p-m), Q_i(m-1), P_j(p-m'), P_j(m'-1))$. Observe that for any $\lambda<1$,

$$\sum_{\substack{1\leqslant m'\leqslant p\\ 1\leqslant m\leqslant p}} \lambda^{M(m,m')} \leqslant 4\sum_{n\leqslant 0}\lambda^n = \frac{4}{1-\lambda}.$$

Thus (97) and (98) together yield

$$(99)\qquad \sum_{i,j}\sum_{\xi\in C_{i,j}} I_{i,j}(\xi)B_p^{\xi} = p^2\Big(\sum_{i,j} I_{i,j}(1)\,\sharp(C_i,j)\,.\,\mathrm{tr}(\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{h}}_j)\Big) + K\,\mathrm{gh}_0(\rho)^N,$$

where we have used that $p^2\,\mathrm{gh}_0(\rho)^{Np} \leqslant K_5\,\mathrm{gh}_0(\rho)^N$ for some constant K_5 only depending on a compact neighborhood of ρ. The result finally follows from the fact that $\mathrm{tr}(\dot{\mathsf{g}}_i\,.\,\dot{\mathsf{h}}_j) = \mathsf{T}(\gamma_i,\eta_j)$. □

Proof of Proposition 8.2.1 Since G, F and Γ_k satisfy the good position hypothesis, by Proposition 7.5.2 there exist two bouquets $\mathcal{F}_L$ and $\mathcal{F}_R$ in S in a homotopically good position, both representing G and F and such that furthermore,

$$\begin{aligned}
\tfrac{1}{2}(\mathrm{f}_{i,j}(\mathcal{F}_L)+\mathrm{f}_{i,j}(\mathcal{F}_R)) &= \mathrm{f}_{i,j},\\
\tfrac{1}{2}(\mathrm{n}_{i,j}(\mathcal{F}_L)+\mathrm{n}_{i,j}(\mathcal{F}_R)) &= \mathrm{n}_{i,j},\\
\tfrac{1}{2}(\mathrm{m}_{i,j}(\mathcal{F}_L)+\mathrm{m}_{i,j}(\mathcal{F}_R)) &= \mathrm{m}_{i,j},\\
\tfrac{1}{2}(\mathrm{q}_{i,j}(\mathcal{F}_L)+\mathrm{q}_{i,j}(\mathcal{F}_R)) &= \mathrm{q}_{i,j}.
\end{aligned}$$

Thus applying Proposition 8.2.4 twice, once for $\mathcal{F}_L$ and once for $\mathcal{F}_R$, and taking the half sum, we obtain the final result. □

8.3 Asymptotics of brackets of multifractions

The setting of this subsection is the same as the previous one: we shall be given a finite index subgroup Γ_k of $\Gamma_0=\pi_1(S)$, corresponding to a covering $S_k\to S_0=S$. Then, if ρ is a Hitchin representation of $\pi_1(S)$ in $\mathrm{PSL}_n(\mathbb{R})$, ρ_k will denote the restriction of ρ to Γ_k.

Let $G=(\gamma_0,\dots,\gamma_q)$ and $F=(\eta_0,\dots,\eta_{q'})$ be two tuples of primitive elements of $\pi_1(S)$. We assume that the (γ_i,γ_{i+1}) as well as the (η_j,η_{j+1}) are pairwise coprime. Observe that there exists an $M\in\mathbb{N}$ such that for all i and j, both $\widehat{\gamma}:=\gamma_i^M$ and $\widehat{\eta}:=\eta_j^M$ belong to Γ_k.

Then let

$$\overline{\mathsf{W}}_p(\gamma_1,\dots,\gamma_q) := \frac{\mathsf{W}(\widehat{\gamma}_1^p\dots\widehat{\gamma}_q^p)}{\prod_{i=1}^q \mathsf{W}(\widehat{\gamma}_i^p)},$$

so that

$$(100)\qquad \mathsf{T} = \lim_{p\to\infty}\overline{\mathsf{W}}_p.$$

Now let

$$(101)\qquad A_p := \frac{\{\overline{\mathsf{W}}_p(\gamma_0,\dots,\gamma_q),\overline{\mathsf{W}}_p(\eta_0,\dots,\eta_{q'})\}_S}{\overline{\mathsf{W}}_p(\gamma_0,\dots,\gamma_q)\,.\,\overline{\mathsf{W}}_p(\eta_0,\dots,\eta_{q'})}.$$

Let $F=(\gamma_1,\dots,\gamma_q)$ and $G=(\eta_1,\dots,\eta_{q'})$.

Proposition 8.3.1 *We have*

$$A_p = B_p(F,G)-\sum_i B_p(\gamma_i,G)-\sum_j B_p(F,\eta_j)+\sum_{i,j} B_p(\gamma_i,\eta_j). \tag{102}$$

From this proposition and Proposition 8.2.4, we will deduce the following important corollary.

Corollary 8.3.2 *Assume that G and F and Γ_0 satisfy the good position hypothesis. Let k be a positive integer such that Γ_k is N–nice for all pairs (γ_i,η_j). Then*

$$\begin{aligned}\frac{\{\mathsf{T}(\gamma_0,\dots\gamma_q),\mathsf{T}(\eta_0,\dots\eta_{q'})\}_{S_k}}{\mathsf{T}(\gamma_0,\dots\gamma_q).\,\mathsf{T}(\eta_0,\dots\eta_{q'})}&\\ =\sum_{i,j}\Bigg((\mathrm{q}_{i,j}+\mathrm{n}_{i,j+1}+\mathrm{m}_{i+1,j}+\mathrm{f}_{i+1,j+1})&\frac{\mathsf{T}(\gamma_{i+1}\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\gamma_{i+1},\gamma_i)\,\mathsf{T}(\eta_j,\eta_{j+1})}\\ -(\mathrm{n}_{i,j}+\mathrm{f}_{i+1,j})\frac{\mathsf{T}(\gamma_{i+1}\gamma_i,\eta_j)}{\mathsf{T}(\gamma_{i+1},\gamma_i)}&-(\mathrm{m}_{i,j}+\mathrm{f}_{i,j+1})\frac{\mathsf{T}(\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\eta_j,\eta_{j+1})}\\ &+\mathrm{f}_{i,j}.\,\mathsf{T}(\gamma_i,\eta_j)\Bigg)\\ &+K\cdot(\mathrm{gh}_k(\rho)+\mathrm{gh}_0(\rho)^N),\end{aligned}$$

where K is bounded by a continuous function that only depends on the relative position of the eigenvectors of $\rho(\gamma_i)$ and $\rho(\eta_i)$.

We first prove the corollary from the proposition, then prove the proposition.

Proof of Corollary 8.3.2 We study one by one the terms in the right-hand side of the formula of Proposition 8.3.1 using the asymptotics given by Proposition 8.2.4. Let $\varepsilon=\mathrm{gh}_k(\rho)+\mathrm{gh}_0(\rho)^N$. First,

$$\begin{aligned}&B_p(\gamma_0,\dots,\gamma_q,\eta_0,\dots,\eta_{q'})\\ &=\sum_{i,j}\Bigg(\big(p^2\mathrm{R}_{i,j}+(p-1)^2\mathrm{f}_{i,j}\big)\,\mathsf{T}(\gamma_i,\eta_j)\\ &\qquad+(p-1)\bigg(\frac{\mathsf{T}(\gamma_{i+1},\gamma_i,\eta_j)}{\mathsf{T}(\gamma_i,\gamma_{i+1})}(\mathrm{n}_{i,j}+\mathrm{f}_{i+1,j})+\frac{\mathsf{T}(\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\eta_j,\eta_{j+1})}(\mathrm{m}_{i,j}+\mathrm{f}_{i,j+1})\bigg)\\ &\qquad+\frac{\mathsf{T}(\gamma_{i+1},\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\gamma_{i+1},\gamma_i)\,\mathsf{T}(\eta_j,\eta_{j+1})}(\mathrm{q}_{i,j}+\mathrm{n}_{i,j+1}+\mathrm{m}_{i+1,j}+\mathrm{f}_{i+1,j+1})\Bigg)\\ &\qquad+K\varepsilon.\end{aligned}$$

We now consider the term $B_p(\gamma_i,\eta_0,\dots,\eta_{q'})$. Applying the previous formula, using the fact that in this case $\mathrm{q}_{i,j}=\mathrm{n}_{i,j}=0$, we get

$$
\begin{aligned}
&B_p(\gamma_i,\eta_0,\dots,\eta_{q'})\\
&=\sum_{0\leqslant j\leqslant q'}\Biggl(\bigl(p^2\mathrm{R}_{i,j}+(p-1)^2\mathrm{f}_{i,j}\bigr)\,\mathsf{T}(\gamma_i,\eta_j)\\
&\qquad\qquad+(p-1)\biggl(\mathsf{T}(\gamma_i,\eta_j)(\mathrm{f}_{i,j})+\frac{\mathsf{T}(\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\eta_j,\eta_{j+1})}(\mathrm{m}_{i,j}+\mathrm{f}_{i,j+1})\biggr)\\
&\qquad\qquad\qquad\qquad+\frac{\mathsf{T}(\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\eta_j,\eta_{j+1})}(\mathrm{m}_{i,j}+\mathrm{f}_{i,j+1})\Biggr)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+K\varepsilon.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&B_p(\gamma_0,\dots,\gamma_q,\eta_j)\\
&=\sum_{0\leqslant i\leqslant q}\Biggl((p^2\mathrm{R}_{i,j}+(p-1)^2\mathrm{f}_{i,j})\,\mathsf{T}(\gamma_i,\eta_j)\\
&\qquad\qquad+(p-1)\biggl(\frac{\mathsf{T}(\gamma_{i+1},\gamma_i,\eta_j)}{\mathsf{T}(\gamma_i,\gamma_{i+1})}(\mathrm{n}_{i,j}+\mathrm{f}_{i+1,j})+\mathsf{T}(\gamma_i,\eta_j)(\mathrm{f}_{i,j})\biggr)\\
&\qquad\qquad\qquad\qquad+\frac{\mathsf{T}(\gamma_{i+1},\gamma_i,\eta_j)}{\mathsf{T}(\gamma_{i+1},\gamma_i)}(\mathrm{n}_{i,j}+\mathrm{f}_{i+1,j})\Biggr)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+K\varepsilon.
\end{aligned}
$$

Finally,

$$
B_p(\gamma_i,\eta_j)=\mathsf{T}(\gamma_i,\eta_j)\bigl(p^2\mathrm{R}_{i,j}+(p-1)^2\mathrm{f}_{i,j}+2(p-1)+1\bigr)+K\varepsilon. \tag{103}
$$

Thus, using Proposition 8.3.1, regrouping the terms that appear in A_p, we obtain that

- the coefficient of $\mathsf{T}(\gamma_i,\eta_j)$ is $\mathrm{f}_{i,j}$,
- the coefficient of $\mathsf{T}(\gamma_{i+1},\gamma_i,\eta_j)/\mathsf{T}(\gamma_i,\gamma_{i+1})$ is $-(\mathrm{n}_{i,j}+\mathrm{f}_{i+1,j})$,
- the coefficient of $\mathsf{T}(\gamma_i,\eta_{j+1},\eta_j)/\mathsf{T}(\eta_j,\eta_{j+1})$ is $-(\mathrm{m}_{i,j}+\mathrm{f}_{i,j+1})$,
- the coefficient of
$$\frac{\mathsf{T}(\gamma_{i+1},\gamma_i,\eta_{j+1},\eta_j)}{\mathsf{T}(\gamma_{i+1},\gamma_i)\,\mathsf{T}(\eta_j,\eta_{j+1})}$$
is $\mathrm{q}_{i,j}+\mathrm{n}_{i,j+1}+\mathrm{m}_{i+1,j}+\mathrm{f}_{i+1,j+1}$.

Finally, we conclude the proof of the corollary by using (100). □

Proof of Proposition 8.3.1 First we use the "logarithmic derivative formula" for the Poisson bracket,

$$\frac{\{f.g,h\}_S}{fgh} = \frac{\{f,h\}_S}{fh} + \frac{\{g,h\}_S}{gh}.$$

We obtain

$$(104)\quad A_p(F,G) = \frac{\{\mathsf{W}(\gamma_0^p\dots\gamma_q^p),\mathsf{W}(\eta_0^p\dots\eta_{q'}^p)\}_S}{\mathsf{W}(\gamma_0^p\dots\gamma_q^p)\mathsf{W}(\eta_0^p\dots\eta_{q'}^p)} - \sum_i \frac{\{\mathsf{W}(\gamma_i^p),\mathsf{W}(\eta_0^p\dots\eta_{q'}^p)\}_S}{\mathsf{W}(\gamma_i^p)\mathsf{W}(\eta_0^p\dots\eta_{q'}^p)} - \sum_j \frac{\{\mathsf{W}(\gamma_0^p\dots\gamma_q^p),\mathsf{W}(\eta_j)\}_S}{\mathsf{W}(\gamma_0^p\dots\gamma_q^p)\mathsf{W}(\eta_j)} + \sum_{i,j} \frac{\{\mathsf{W}(\gamma_i^p),\mathsf{W}(\eta_j^p)\}_S}{\mathsf{W}(\gamma_i^p)\mathsf{W}(\eta_j^p)}.$$

Then, using the definition of (49) expressing the Goldman Poisson bracket of Wilson loops in terms of the bracket of loops in the Goldman algebra, we get

$$\frac{\{\mathsf{W}(\gamma_0^p\dots\gamma_q^p),\mathsf{W}(\eta_0^p\dots\eta_{q'}^p)\}_S}{\mathsf{W}(\gamma_0^p\dots\gamma_q^p)\mathsf{W}(\eta_0^p\dots\eta_{q'}^p)} = B_p(F,G) - \frac{1}{n}\iota(\gamma_0^p\dots\gamma_q^p,\eta_0^p\dots\eta_{q'}^p).$$

The proposition now follows from the fact that

$$\iota(a.b,c) = \iota(a,c) + \iota(b,c),$$

and thus

$$(105)\quad \iota(\gamma_0^p\dots\gamma_q^p,\eta_0^p\dots\eta_{q'}^p) = \sum_i \iota(\gamma_i,\eta_0^p\dots\eta_{q'}^p) + \sum_j \iota(\gamma_0^p\dots\gamma_q^p,\eta_j) - \sum_{i,j}\iota(\gamma_i^p,\eta_j^p),$$

which completes the proof. □

9 The Goldman and swapping algebras: proofs of the main results

We finally prove the results stated in Section 6. In the course of the proof, we prove the generalized Wolpert formula of Theorem 6.1.2.

9.1 Poisson brackets of elementary functions and proof of Theorem 6.2.4

By Corollary 4.2.3, the algebra $\mathcal{B}(\mathsf{P})$ of multifractions is generated by elementary functions. Thus it is enough to prove the theorem when b_0 and b_1 are elementary functions.

Let $G = (\gamma_0, \dots, \gamma_p)$ and $F = (\eta_0, \dots \eta_{q'})$ be primitive elements of $\pi_1(S)$. We assume that for all i and j, the pairs (γ_i, γ_{i+1}) and (η_j, η_{j+1}) are coprime.

Let $b_0 = \mathsf{T}(\gamma_0, \dots, \gamma_q)$ and $b_1 = \mathsf{T}(\eta_0, \dots, \eta'_q)$.

By Proposition 8.1.6, we can assume that G and F satisfy the good position hypothesis for S_k when $k > k_0$ for some n_0. Let N be a positive integer; we can further assume that $S_k \mapsto S_0$ is N–nice for all pairs (γ_i, η_j) by Proposition 8.1.5 for $k \geqslant k_0$ and k_0 large enough.

Recall also, using the notation of Proposition 4.3.1, that

$$
\begin{aligned}
\mathrm{f}_{i,j} &= [\gamma_i^- \gamma_i^+, \eta_j^- \eta_j^+] &&= \mathrm{a}_{i,j}, \\
\mathrm{q}_{i,j} + \mathrm{n}_{i,j+1} + \mathrm{m}_{i+1,j} + \mathrm{f}_{i+1,j+1} &= [\gamma_i^- \gamma_{i+1}^+, \eta_j^- \eta_{j+1}^+] &&= \mathrm{b}_{i,j}, \\
\mathrm{f}_{i,j+1} + \mathrm{m}_{i,j} &= [\gamma_i^- \gamma_i^+, \eta_j^- \eta_{j+1}^+] &&= \mathrm{c}_{i,j}, \\
\mathrm{f}_{i+1,j} + \mathrm{n}_{i,j} &= [\gamma_i^- \gamma_{i+1}^+, \eta_j^- \eta_j^+] &&= \mathrm{d}_{i,j}.
\end{aligned} \tag{106}
$$

Thus Corollary 8.3.2 and the computation of the swapping bracket in Proposition 4.3.1 yield

$$
\begin{aligned}
&\frac{\{\mathsf{T}(\gamma_0, \dots \gamma_q), \mathsf{T}(\eta_0, \dots \eta_{q'})\}_{S_k}}{\mathsf{T}(\gamma_0, \dots \gamma_q) . \mathsf{T}(\eta_0, \dots \eta_{q'})} \\
&\qquad = \frac{\{\mathsf{T}(\gamma_0, \dots \gamma_q), \mathsf{T}(\eta_0, \dots \eta_{q'})\}_W}{\mathsf{T}(\gamma_0, \dots \gamma_q) . \mathsf{T}(\eta_0, \dots \eta_{q'})} + K \cdot (\mathrm{gh}_k(\rho) + \mathrm{gh}(\rho)^N),
\end{aligned} \tag{107}
$$

where K is a bounded function that only depends on the eigenvectors of $\rho(\gamma_i)$ and $\rho(\eta_j)$. In particular, there exists a real number K_0 and a compact neighborhood C of ρ_0 such that the previous equality holds with $K \leqslant K_0$ and ρ in C.

Let ε be a positive real number. By the last assertion in Proposition 8.1.3, we may furthermore choose k_0 such that if $k > k_0$,

$$\mathrm{gh}_k(\rho) \leqslant \frac{\varepsilon}{2K_0}.$$

Since $\sup\{\mathrm{gh}(\rho) \mid \rho \in C\} < 1$, we may further choose N —and thus k_0 —such that for all ρ in C,

$$\mathrm{gh}(\rho)^N \leqslant \frac{\varepsilon}{2K_0}.$$

It follows that for all ρ in C and all $k \geqslant k_0$, we have

$$\left| \frac{\{\mathsf{T}(\gamma_0, \dots \gamma_q), \mathsf{T}(\eta_0, \dots \eta_{q'})\}_{S_k}}{\mathsf{T}(\gamma_0, \dots \gamma_q) . \mathsf{T}(\eta_0, \dots \eta_{q'})} - \frac{\{\mathsf{T}(\gamma_0, \dots \gamma_q), \mathsf{T}(\eta_0, \dots \eta_{q'})\}_W}{\mathsf{T}(\gamma_0, \dots \gamma_q) . \mathsf{T}(\eta_0, \dots \eta_{q'})} \right| \leqslant \varepsilon. \tag{108}$$

This concludes the proof of Theorem 6.2.4. □

9.2 Poisson brackets of length functions

We shall first prove a result of independent interest, namely the computation of the value of the Goldman bracket of two length functions of geodesics having exactly one intersection point.

Given a Hitchin representation ρ in $\mathrm{PSL}_n(\mathbb{R})$, or alternatively a rank n cross ratio $\boldsymbol{b}_\rho$, the period — or length — of a conjugacy class γ in $\pi_1(S)$ is given by

$$\ell_\gamma(\rho) = \log\left(\frac{\lambda_{\max}(\rho(\gamma))}{\lambda_{\min}(\rho(\gamma))}\right) = \log\big(\boldsymbol{b}_\rho(\gamma^+,\gamma^-,\gamma(y),y)\big) \tag{109}$$

for any $y \in \partial_\infty\pi_1(S)$ different from γ^+ and γ^-, where $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote, respectively, the eigenvalues of greatest and smallest modulus of the endomorphism A.

9.2.1 A generalized Wolpert formula We have the following extension of the Wolpert formula for the bracket of length functions.

Theorem 9.2.1 (generalized Wolpert formula) *Let γ and η be two closed geodesics with a unique intersection point. Then the Goldman bracket of the two length functions ℓ_γ and ℓ_η, seen as functions on the Hitchin component, is*

$$\{\ell_\gamma,\ell_\eta\}_S = \iota(\gamma,\eta)\sum_{\varepsilon,\varepsilon'\in\{-1,1\}} \varepsilon\varepsilon'.\,\mathsf{T}(\gamma^\varepsilon,\eta^{\varepsilon'}), \tag{110}$$

where we recall that

$$\mathsf{T}(\xi,\zeta)(\rho) = \boldsymbol{b}_\rho(\xi^+,\zeta^+,\zeta^-,\xi^-).$$

Proof Let us first remark that

$$\ell_\gamma = \lim_{p\to+\infty}\frac{1}{p}\log\big(\mathrm{tr}(\rho(\gamma^p))\mathrm{tr}(\rho(\gamma^{-p}))\big). \tag{111}$$

Thus, assuming that γ and η have a unique intersection point x whose intersection number is $\iota(\gamma,\eta)$, the product formula (59) gives us, for $\varepsilon_i \in \{-1,1\}$,

$$\{\gamma^{\varepsilon.p},\eta^{\varepsilon'.p}\} = \varepsilon\varepsilon'.\,p^2.\iota(\gamma,\eta)\gamma^{\varepsilon.p}.\eta^{\varepsilon'.p}. \tag{112}$$

It follows that

$$\begin{aligned}
&\big\{\log(\mathsf{W}(\gamma^p)\mathsf{W}(\gamma^{-p})),\log(\mathsf{W}(\eta^p)\mathsf{W}(\eta^{-p}))\big\}_S \\
&\quad= \sum_{\varepsilon,\varepsilon'\in\{-1,1\}}\varepsilon\varepsilon'.\iota(\gamma,\eta)\frac{\{\mathsf{W}(\gamma^{\varepsilon.p}),\mathsf{W}(\eta^{\varepsilon'.p})\}_S}{\mathsf{W}(\gamma^{\varepsilon.p}).\mathsf{W}(\eta^{\varepsilon'.p})} \\
&\quad= \sum_{\varepsilon,\varepsilon'\in\{-1,1\}}p^2.\varepsilon\varepsilon'.\iota(\gamma,\eta)\frac{\mathsf{W}(\gamma^{\varepsilon.p}.\eta^{\varepsilon'.p})}{\mathsf{W}(\gamma^{\varepsilon.p}).\mathsf{W}(\eta^{\varepsilon'.p})} + \frac{1}{n}\iota(\gamma,\eta)\sum_{\varepsilon,\varepsilon'\in\{-1,1\}}\varepsilon\varepsilon'.
\end{aligned} \tag{113}$$

Thus

$$\lim_{p\to\infty}\left\{\frac{1}{p}\log(\mathsf{W}(\gamma^{p})\mathsf{W}(\gamma^{-p})),\frac{1}{p}\log(\mathsf{W}(\eta^{p})\mathsf{W}(\eta^{-p}))\right\}_S = \sum_{\varepsilon,\varepsilon'\in\{-1,1\}} \varepsilon\varepsilon'.\iota(\gamma,\eta)\lim_{p\to\infty}\frac{\mathsf{W}(\gamma^{\varepsilon.p}.\eta^{\varepsilon'.p})}{\mathsf{W}(\gamma^{\varepsilon.p}).\mathsf{W}(\eta^{\varepsilon'.p})} = \sum_{\varepsilon,\varepsilon'\in\{-1,1\}} \varepsilon\varepsilon'.\iota(\gamma,\eta)\,\mathsf{T}(\gamma^{\varepsilon},\eta^{\varepsilon'}). \tag{114}$$

This concludes the proof of the theorem. □

9.2.2 Proof of Theorem 6.1.1 Recall that we want to prove the following result.

Theorem 9.2.2 *Let γ and η be two geodesics with at most one intersection point. Then we have*

$$\lim_{n\to\infty}\mathrm{I}_S(\{\widehat{\ell}_{\gamma^n}(y),\widehat{\ell}_{\eta^n}(y)\})=\tfrac14\{\ell_\gamma,\ell_\eta\}_S.$$

Proof This will be a consequence of the generalized Wolpert formula. By definition,

$$\widehat{\ell}_\gamma(y)=\tfrac12\log\big(\boldsymbol{b}(\gamma^+,\gamma^-,\gamma(y),\gamma^{-1}(y))\big).$$

Thus

$$\{\widehat{\ell}_\alpha(y),\widehat{\ell}_\beta(y)\}=\frac14\sum_{\substack{u,u'\in\{-1,1\}\\ v,v'\in\{-1,1\}}} u.u'\frac{\{(\alpha^v,\alpha^{-uv}(y)),(\beta^{v'},\beta^{-u'v'}(y))\}}{(\alpha^v,\alpha^{-uv}(y)).(\beta^{v'},\beta^{-u'v'}(y))}. \tag{115}$$

But

$$\{(\alpha^v,\alpha^{-uv}(y)),(\beta^{v'},\beta^{-u'v'}(y))\} = [(\alpha^v\alpha^{-uv}(y)),(\beta^{v'}\beta^{-u'v'}(y))]\alpha^v\beta^{-u'v'}(y).\beta^{v'}\alpha^{-uv}(y). \tag{116}$$

We remark that when n is large enough, for all u,v,u',v' we have

$$\begin{aligned}&[(\gamma^v\gamma^{v.n}(y)),(\eta^{v'}\eta^{-u'v'.n}(y))]=0,\\ &[(\gamma^v\gamma^{-uv.n}(y)),(\eta^{v'}\eta^{v'.n}(y))]=0,\\ &[(\gamma^v\gamma^{-v.n}(y)),(\eta^{v'}\eta^{-v'.n}(y))]=vv'[\gamma^+\gamma^-,\eta^+\eta^-].\end{aligned} \tag{117}$$

Combining the remark in Equation (117) with (116) and (115), we have that for n large enough,

$$\{\widehat{\ell}_{\gamma^n}(y),\widehat{\ell}_{\eta^n}(y)\} = \frac{[\gamma^+\gamma^-,\eta^+\eta^-]}{4}\sum_{v,v'\in\{-1,1\}} v.v'\frac{\gamma^v\eta^{-v'.n}(y).\eta^{v'}\gamma^{-v}(y)}{(\gamma^v\gamma^{-v.n}(y)).(\eta^{v'}\eta^{-v'n}(y))}. \tag{118}$$

Thus, taking the limit when n goes to ∞ yields

$$\lim_{n\to\infty} \mathsf{I}_S\{\widehat{\ell}_{\gamma^n}(y), \widehat{\ell}_{\eta^n}(y)\} = \frac{[\gamma^+\gamma^-, \eta^+\eta^-]}{4} \sum_{v,v'\in\{-1,1\}} v\,.\,v' \frac{\gamma^v\eta^{-v'}\,.\,\eta^{v'}\gamma^{-v}}{\gamma^v\gamma^{-v}\,.\,\eta^{v'}\eta^{-v'}} \tag{119}$$

$$= \frac{[\gamma^+\gamma^-, \eta^+\eta^-]}{4} \sum_{v,v'\in\{-1,1\}} v\,.\,v'.\, \mathsf{T}(\gamma^v, \eta^{v'}).$$

The result now follows from this last equation and the generalized Wolpert formula of (110). □

10 Drinfel'd–Sokolov reduction

The purpose of this section is to prove Theorem 10.7.2, which explains the relation of the multifraction algebra with the Poisson structure on $\mathrm{PSL}_n(\mathbb{R})$–opers.

We spend the first three subsections explaining the Poisson structure on $\mathrm{PSL}_n(\mathbb{R})$–opers using the Drinfel'd–Sokolov reduction of the Poisson structure on connections on the circle. Although this is a classical construction (see [5; 23; 12] and the original reference [6]) we take some time explaining the main steps in differential geometric terms, expanding the sketch of the construction given by Graeme Segal [29].

Finally, we relate the swapping algebra and this Poisson structure in Theorem 10.7.2.

10.1 Opers and nonslipping connections

In this subsection, we recall the definition $\mathrm{PSL}_n(\mathbb{R})$–opers and show that they can be interpreted as an equivalence class of "nonslipping" connections on a bundle with a flag structure.

10.1.1 Opers

Definition 10.1.1 (opers) A $\mathrm{PSL}_n(\mathbb{R})$–*oper* is an n^{th}–order linear differential operator on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ of the form

$$D\colon \psi \mapsto \frac{\mathrm{d}^n\psi}{\mathrm{d}t^n} + q_2\frac{\mathrm{d}^{n-2}\psi}{\mathrm{d}t^{n-2}} + \cdots + q_n\psi, \tag{120}$$

where the q_i are functions.

Observe that this definition of an oper requires the choice of a parametrization of the circle. Otherwise the q_i would instead be i^{th}–order differentials.

We denote by $\mathsf{X}_n(\mathbb{T})$ the space of $\mathrm{PSL}_n(\mathbb{R})$–opers on $\mathbb{T}$. Every oper has a natural holonomy, which reflects the fact that the solutions may not be periodic. We consider the space $\mathsf{X}_n(\mathbb{T})^0$ of opers with trivial holonomy; that is, those opers D for which all solutions of $D\psi = 0$ are periodic. A Poisson structure on $\mathsf{X}_n(\mathbb{T})$, whose symplectic leaves are opers with the same holonomy, was discovered in the context of integrable systems and Korteweg–de Vries equations; for a precise account of the history, see Dickey [5]. Later, Drinfel'd and Sokolov [6] interpreted that structure in a more differential geometric way; we shall now retrace the steps of that construction.

10.1.2 Nonslipping connections Let K be the line bundle of $\left(-\frac{1}{2}\right)$–densities over $\mathbb{T}$, so that $\mathsf{T}\mathbb{T} = K^2$, and let $P := J^{n-1}(K^{n-1})$ be the rank n vector bundle of $(n-1)$–jets of sections of the bundles of $\left(-\frac{1}{2}(n-1)\right)$–densities.

Let F_p be the vector subbundle of P defined by

$$F_p := \{j^{n-1}\sigma \mid j^{n-p-1}\sigma = 0\}.$$

The family $\{F_p\}_{1\leqslant p\leqslant n}$ is a filtration of P: we have $F_n = P$, $F_{p-1} \subset F_p$ and $\dim(F_p) = p$. Observe that

$$W_p := F_p/F_{p-1} = (\mathsf{T}^*\mathbb{T})^{n-p} \otimes K^{n-1} = (K^{-2})^{n-p} \otimes K^{n-1} = K^{2p-n-1}.$$

In particular, $W_{n-p-1} = W_p^*$ and it follows that

$$\det(P) = \bigotimes_{p=1}^{n} \det(W_p)$$

is canonically isomorphic to $\mathbb{R}$. Thus P carries a canonical volume form.

We say a family of sections $\{e_1, \dots, e_n\}$ of P is a *basis for the filtration* if for every integer p no greater than n and every $x \in S^1$, $\{e_1(x), \dots, e_p(x)\}$ is a basis of the fiber of F_p at x.

Definition 10.1.2 (nonslipping connections) A connection ∇ on P is *nonslipping* if it satisfies the following conditions:

- $\nabla F_p \subset F_{p+1}$ for all p.
- If α_p is the projection from F_{p+1} to F_{p+1}/F_p, then the map

  $$(X, u) \to \alpha_p(\nabla_X(u)),$$

  considered as a linear map from $K^2 \otimes F_p/F_{p-1} = K^{2p-n+1}$ to $F_{p+1}/F_{p-1} = K^{2p-n+1}$, is the identity.

We denote by D_0 the space of nonslipping connections on P. The first classical proposition is:

Proposition 10.1.3 *Let ∇ be a nonslipping connection. Then there exists a unique basis $\{e_1,\dots,e_n\}$ of determinant 1 for the filtration such that*

$$\begin{cases}\nabla_{\partial_t} e_i = -e_{i+1} & \text{for } i \leqslant n-1,\\ \nabla_{\partial_t} e_n \in F_{n-1},\end{cases} \tag{121}$$

where ∂_t is the canonical vector field on $\mathbb{T}$.

Observe here that the basis depends on the choice of a parametrization of the circle. From this proposition, it follows that we can associate to a nonslipping connection ∇ the differential operator $D = D^{\nabla}$ such that

$$\nabla^*_{\partial_t}\left(\sum_{i=1}^{n} \frac{\mathrm{d}^{i-1}\psi}{\mathrm{d}t^{i-1}}\omega_i\right) = (D\psi)\omega_n,$$

where ∇^* is the dual connection and $\{e_i\}_{1\leqslant i\leqslant n}$ is the dual basis to the basis $\{\omega_i\}_{1\leqslant i\leqslant n}$ associated to ∇ in the previous proposition. One easily checks that

$$D\psi = \frac{\mathrm{d}^n\psi}{\mathrm{d}t^n} + q_2\frac{\mathrm{d}^{n-2}\psi}{\mathrm{d}t^{n-2}} + \cdots + q_n\psi,$$

where the functions q_i are given by $q_i = \omega_{n-j+1}(\nabla_{\partial_t} e_n)$.

We now introduce

(i) the *flag gauge group* as the group N of linear automorphism of the bundle P defined by

$$\mathsf{N} := \{A \in \Omega^0(\mathbb{T}, \mathrm{End}(P)) \mid A(F_p) = F_p,\ A|_{F_p/F_{p-1}} = \mathrm{Id}\},$$

(ii) the *Lie algebra* $\mathfrak{n}$ of the flag gauge group as

$$\mathfrak{n} := \{A \in \Omega^0(\mathbb{T}, \mathrm{End}(P)) \mid A(F_p) \subset F_{p-1}\}.$$

We now have:

Proposition 10.1.4 *The map $\nabla \mapsto D^{\nabla}$ realizes an identification between D_0/N and $\mathsf{X}_n(\mathbb{T})$, and this identification preserves the holonomy.*

It is interesting now to observe that the definition of an oper as an element of D_0/N does not depend on a parametrization.

Proof Let ∇ be a nonslipping connection, $\{e_i\}$ the basis obtained by the previous proposition and $\nabla' = n^*\cdot\nabla$ a connection in the N–orbit of ∇. By definition of N, $\nabla' e_i = \nabla e_i + u_i$ with $u_i \in F_{i-1}$. The result follows. □

10.2 The Poisson structure on the space of connections

The purpose of Drinfel'd–Sokolov reduction is to identify the space $\mathsf{X}_n(\mathbb{T}) = \mathsf{D}_0/\mathsf{N}$ of opers as a symplectic quotient of the space of all connections on $\mathbb{T}$ by the group N.

Again, we shall paraphrase Segal, and define in this section, as a first step of the construction of Drinfel'd–Sokolov reduction, the classical construction of the Poisson structure on the space of connections.

In general, when we deal with a Fréchet space of sections of a bundle, we have to specify functionals that we deem observables and for which we can compute a Poisson bracket. This is done by specifying a subspace of cotangent vectors and describing the Poisson tensor on that subspace. Observables are then functionals whose differentials belong to that specific subset. However, the Poisson bracket can be extended to more general pairs of observables. Rather than describing a general formalism, for which we could refer to [5], we explain the construction in the case of connections.

10.2.1 Connections and central extensions Let G be the gauge group of the vector bundle P. The choice of a trivialization of P gives rise to an isomorphism of G with the loop group of $\mathrm{PSL}_n(\mathbb{R})$. We introduce the following definitions:

- (i) The *Lie algebra* $\mathfrak{g}$ of G is $\Omega^0(\mathbb{T}, \mathrm{End}_0(P))$, where $\mathrm{End}_0(P)$ stands for the vector space of trace free endomorphisms of P. The Lie algebra $\mathfrak{g}$ is equipped naturally with a coadjoint action of G.
- (ii) The *dual Lie algebra* $\mathfrak{g}^\circ$ of G is $\Omega^1(\mathbb{T}, \mathrm{End}_0(P))$.
- (iii) The *duality* is given by the nondegenerate bilinear mapping from $\mathfrak{g}\times\mathfrak{g}^\circ$ defined by

$$\langle \alpha, \beta \rangle = \int_{\mathbb{T}} \mathrm{tr}(\alpha \cdot \beta). \tag{122}$$

Let us choose a connection ∇ on P. Let Ω_∇ be the 2–cocycle on $\mathfrak{g}$ given by

$$\Omega_\nabla(\xi, \eta) = \int_{\mathbb{T}} \mathrm{tr}(\xi \nabla \eta).$$

If ∇ and ∇' are two connections on P, then

$$\Omega_\nabla(\xi, \eta) - \Omega_{\nabla'}(\xi, \eta) = \alpha([\xi, \eta]),$$

where

$$\alpha(\chi) = \int_{\mathbb{T}} \mathrm{tr}\big((\nabla - \nabla').\chi\big).$$

In particular the cohomology class of the cocycle Ω_∇ does not depend on the choice of ∇. Let $\widehat{\mathsf{G}}$, whose Lie algebra is $\hat{\mathfrak{g}}$, be the central extension of G corresponding to this cocycle, so that

$$0 \to \mathbb{R} \to \hat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g}.$$

As we noticed, every connection defines a splitting of this sequence; that is, a way to write $\widehat{\mathfrak{g}}$ as $\mathbb{R}\oplus\mathfrak{g}$.

Dually, we consider the vector space $\widehat{\mathfrak{g}}^\circ$ defined by the exact sequence

$$0\to\mathfrak{g}^\circ\xrightarrow{i}\widehat{\mathfrak{g}}^\circ\to\mathbb{R},$$

with the duality with $\widehat{\mathfrak{g}}$ such that $\langle\gamma, i(\beta)\rangle=\langle\pi(\gamma),\beta\rangle$.

It follows that the space $\mathcal{D}$ of all $\mathrm{PSL}_n(\mathbb{R})$ connections on P can be embedded in the space of such splittings, which is in turn identified with the affine hyperplane D in $\widehat{\mathfrak{g}}^\circ$ defined by

$$\mathsf{D}:=\{\beta\in\widehat{\mathfrak{g}}^\circ\mid\langle Z,\beta\rangle=1\},$$

where $Z\in\widehat{\mathfrak{g}}$ is the generator of the center. The hyperplane D has $\mathfrak{g}^\circ$ as a tangent space. Observe that the embedding $\mathcal{D}\to\mathsf{D}$ is equivariant under the affine action of $\Omega^1(\mathbb{T},\mathrm{End}_0(P))=\mathfrak{g}^\circ\subset\widehat{\mathfrak{g}}^\circ$ as well as the coadjoint action of G itself. In particular, the above embedding is surjective and we now identify D as the space of all $\mathrm{PSL}_n(\mathbb{R})$–connections on P. The coadjoint orbits of G on D are those connections with the same holonomy.

10.2.2 The Poisson structure Since we are working in infinite dimension, we are only going to define the Poisson tensor on certain "cotangent vectors" to D. In our context we consider the set $\mathsf{D}^\circ:=\mathfrak{g}=\Omega^0(\mathbb{T},\mathrm{End}_0(P))$ of cotangent vectors where the duality is given by formula (122). Using this notation, the Poisson structure is described in the following way.

Definition 10.2.1 (Poisson structure for connections) • The *Hamiltonian mapping* from D° to D at a connection ∇ is

$$H\colon\alpha\mapsto\mathrm{d}^\nabla\alpha.$$

- The *Poisson tensor* on D° at a connection ∇ is

$$\Pi_\nabla(\alpha,\beta):=\langle\alpha,H(\beta)\rangle=\int_{\mathbb{T}}\mathrm{tr}(\alpha\cdot\mathrm{d}^\nabla\beta).$$

- We say a functional F is an *observable* if its differential $\mathrm{d}_\nabla F$ belongs to D° for all ∇. The *Poisson bracket* of two observables is

$$\{f,g\}:=\Pi(\mathrm{d}f,\mathrm{d}g)=\langle\mathrm{d}f,H(\mathrm{d}g)\rangle.$$

Remarks (1) The Poisson bracket can be defined for more general pairs of functionals than observables. Observe first that the differential of functionals on a Fréchet space of sections of bundles — for instance connections — are distributions. Thus we can define the Poisson bracket of a general differentiable functional with an observable. For the purpose of this paper, we shall say that two functionals f and g form an *acceptable pair of observables* if their derivatives $\mathrm{d}f$ and $\mathrm{d}g$ are distributions with disjoint singular support, or equivalently if they can be written as

$$\mathrm{d}f = F + f_0, \quad \mathrm{d}g = G + g_0,$$

where F and G have disjoint support and f_0, g_0 are observables in the previous sense. In this case, their Poisson bracket is defined as

$$\{f, g\}(\nabla) = \Pi(f_0, g_0) + \langle F, H(g_0)\rangle - \langle G, H(f_0)\rangle.$$

This Poisson bracket agrees with regularizing procedures.

(2) We further observe that if D_∇ is the space of connections with the same holonomy as ∇ (that is, the coadjoint orbit of ∇), then the tangent space of D_∇ at ∇ is the vector space of exact 1–forms $\mathrm{d}^\nabla(\Omega^0(\mathbb{T}, \mathrm{End}_0(P)))$, and moreover the Poisson tensor on D_∇ is dual to the symplectic form ω defined by

$$\omega(\mathrm{d}^\nabla\alpha, \mathrm{d}^\nabla\beta) := \int_{\mathbb{T}} \mathrm{tr}(\alpha\cdot\mathrm{d}^\nabla\beta).$$

Thus the symplectic leaves of this Poisson structure are connections with the same holonomy. One can furthermore check that this formalism agrees with what we expect from coadjoint orbits.

10.3 Drinfel'd–Sokolov reduction

We now describe the Drinfel'd–Sokolov reduction. We begin by describing more precisely the group that we are going to work with in order to perform the reduction.

10.3.1 Dual Lie algebras Let $\mathfrak{n}$ be the Lie algebra of N as defined above. Let $\mathfrak{u}$ be the subspace of $\mathfrak{g}^\circ$ given by

$$\mathfrak{u} := \{A \in \Omega^1(\mathbb{T}, \mathrm{End}_0(P)) \mid A(F_p) \subset F_p\}.$$

Proposition 10.3.1 *We have* $\mathfrak{u} = \{A \in \widehat{\mathfrak{g}}^\circ \mid \langle\alpha, A\rangle = 0,\ \forall\alpha\in\mathfrak{n}\}$.

Thus if $\mathfrak{n}^\circ := \Omega^1(\mathbb{T}, \mathrm{End}_0(P))/\mathfrak{u}$, we have a duality $\mathfrak{n}^\circ\times\mathfrak{n}\to\mathbb{R}$ given by the map

$$\langle\alpha, \beta\rangle := \int_{\mathbb{T}} \mathrm{tr}(\alpha\beta).$$

We now give another description of $\mathfrak{n}^\circ$ more suitable for our purpose. Let us first consider the natural projections

$$\begin{aligned}\pi_p^+\colon \mathrm{Hom}(F_p, E/F_p) &\to \mathrm{Hom}(F_p, E/F_{p+1}),\\ \pi_p^-\colon \mathrm{Hom}(F_p, E/F_p) &\to \mathrm{Hom}(F_{p-1}, E/F_p).\end{aligned} \tag{123}$$

Let

$$M := \left\{(u_1, \dots, u_{n-1}) \in \bigoplus_{p=1}^{n-1} \mathrm{Hom}(F_p, E/F_p) \,\middle|\, \pi_p^+(u_p) = \pi_{p+1}^-(u_{p+1})\right\}.$$

We leave it to the reader to check the following.

Proposition 10.3.2 *The map from $\mathfrak{n}^\circ$ to $\Omega^1(\mathbb{T}, M)$ defined by*

$$A \to (A|_{F_1}, \dots, A|_{F_{n-1}})$$

is an isomorphism.

10.3.2 Drinfel'd–Sokolov reduction If ∇ is a connection, we define the *slippage* of ∇, denoted by $\sigma(\nabla)$, as the element of $\Omega^1(\mathbb{T}, M) = \mathfrak{n}^\circ$ given by

$$(u_1, \dots, u_p),$$

where $u_p(X, v) = \alpha_p(\nabla_X v)$ and α_p is the projection from E to E/F_p.

We are now going to define a canonical section of $\Omega^1(\mathbb{T}, M)$. We have a natural embedding

$$i_p\colon \mathrm{Hom}(F_p/F_{p-1}, F_{p+1}/F_p) \to \mathrm{Hom}(F_p, F/F_p).$$

Now observe that

$$\Omega^1(\mathbb{T}, \mathrm{Hom}(F_p/F_{p-1}, F_{p+1}/F_p)) = (K^2)^* \otimes (K^{2p-n-1})^* \otimes K^{2p-n+1}.$$

Thus, let

$$\boldsymbol{I}_p := i_p(\mathrm{Id}) \in (K^2)^* \otimes (K^{2p-n-1})^* \otimes K^{2p-n+1}.$$

Finally, we set

$$\boldsymbol{I} := (\boldsymbol{I}_1, \dots, \boldsymbol{I}_{n-1}),$$

and we observe that $\boldsymbol{I}$ is invariant under the coadjoint action of N.

Theorem 10.3.3 (Drinfel'd–Sokolov reduction) *The map σ is a moment map for the action of N. Moreover $\mathsf{D}_0 = \sigma^{-1}(\boldsymbol{I})$ and we thus obtain a Poisson structure on $\mathsf{X}_n(\mathbb{T})$.*

As a particular case of symplectic reduction, we briefly explain the construction of the Poisson bracket in our context of opers and nonslipping connections. If f and g are two functionals on the space of opers, they are observables if their pull-back F and G on the space of nonslipping connections are observables and then their Poisson bracket is $\{f,g\}(D) := \{F,G\}(\nabla)$, where D is the oper associated with ∇.

10.4 Opers and Frenet curves

10.4.1 Curves associated to $\mathsf{PSL}_n(\mathbb{R})$–opers We recall that every oper D gives rise to a curve from $\mathbb{R}$ to $\mathbb{P}(\mathbb{R}^n)$ which is *equivariant* under the holonomy; that is, a curve

$$\xi\colon \mathbb{R} \to \mathbb{P}(\mathbb{R}^n)$$

such that $\xi(t+1) = H(\xi(t))$, where H is the holonomy. The construction runs as follows. The curve ξ is given in projective coordinates by

$$\xi := [v_1, \dots, v_n],$$

where $\{v_1, \dots, v_n\}$ are independent solutions of the equation $D\psi = 0$. The curve ξ is well-defined up to the action of $\mathsf{PSL}_n(\mathbb{R})$. We call ξ the curve *associated* to the oper.

10.4.2 Hitchin opers Let us say an oper is *Hitchin* if it has trivial holonomy and can be deformed through opers with trivial holonomy to the trivial oper $\psi \mapsto \mathrm{d}^n\psi/\mathrm{d}t^n$. Let us denote by $\mathsf{X}_n^0(\mathbb{T})$ the space of Hitchin opers, which by the previous section inherits a Poisson structure.

10.4.3 Frenet curves We say a curve ξ from $\mathbb{T}$ to $\mathbb{P}(\mathbb{R}^n)$ is *Frenet* if there exists a curve $(\xi^1, \xi^2, \dots, \xi^{n-1})$ defined on $\mathbb{T}$, called the *osculating flag curve*, with values in the flag variety such that $\xi(x) = \xi^1(x)$ for every x in $\mathbb{T}$, and moreover:

- For all tuples of pairwise distinct points $(x_1, \dots, x_l)$ in $\mathbb{T}$ and positive integers $(n_1, \dots, n_l)$ such that

  $$\sum_{i=1}^{l} n_i \leqslant n,$$

  the sum

  $$\xi^{n_i}(x_i) + \cdots + \xi^{n_l}(x_l)$$

  is direct.
- For every x in $\mathbb{T}$ and tuple of positive integers $(n_1, \dots, n_l)$ such that

  $$p = \sum_{i=1}^{l} n_i \leqslant n,$$

we have

$$\lim_{\substack{(y_1,\dots,y_l)\to x \\ y_i \text{ all distinct}}} \left(\bigoplus_{i=1}^{l} \xi^{n_i}(y_i)\right) = \xi^p(x).$$

We call $\xi^* := \xi^{n-1}$ the *osculating hyperplane*.

Since the trivial connection is nonslipping with respect to the filtration given by osculating flags, we have the following obvious remark; see also [7, Section 9.12].

Proposition 10.4.1 *Every smooth Frenet curve comes from a* $\mathrm{PSL}_n(\mathbb{R})$*–oper with trivial holonomy.*

Conversely, we now prove:

Proposition 10.4.2 *The curve associated to a Hitchin oper is Frenet.*

Proof Let us first introduce some notation and definitions. A *weighted* p–tuple X is a pair consisting of a p–tuple of pairwise distinct points $(x^1,\dots,x^p)$ in $\mathbb{T}$, called the *support*, and a p–tuple of positive integers $(j_1,\dots,j_p)$ such that

$$\sum_{1\leqslant k\leqslant p} j_k = n.$$

If η is a smooth curve defined on a subinterval I of $\mathbb{T}$ with values in $\mathbb{R}^n\setminus\{0\}$, let

$$\widehat{\eta}^{(p)}(x) := \eta(x)\wedge\dot{\eta}(x)\wedge\cdots\wedge\eta^{(p-1)}(x) \in \Lambda^p(\mathbb{R}^n),$$

where $\dot{\eta}$ and $\eta^{(k)}$ denote the derivative and k^{th} derivatives of η respectively. Moreover, if X is a weighted p–tuple as above with support in I, let

$$\widehat{\eta}(X) := \bigwedge_{1\leqslant k\leqslant p} \widehat{\eta}^{(j^k)}(x^k) \in \Lambda^n(\mathbb{R}^n) = \mathbb{R}.$$

We say that a weighted p–tuple is *degenerate* with respect to η if $\widehat{\eta}(X)=0$. Observe finally that being degenerate only depends on the projection of η as a curve with values in $\mathbb{P}(\mathbb{R}^n)$, and thus makes sense for curves with values in $\mathbb{P}(\mathbb{R}^n)$. By definition, a curve ξ with values in $\mathbb{P}(\mathbb{R}^n)$ is Frenet if it admits no degenerate weighted p–tuple.

Let us work by contradiction and assume that there exists a Hitchin oper whose associated curve is not Frenet. Let m be the smallest integer such that there exists a curve ξ associated to an Hitchin oper which admits a degenerate m–tuple.

Let O_m be the set of Hitchin opers whose associate curve admits a degenerate m–tuple. By our standing assumption, O_m is nonempty, and moreover the trivial oper, which corresponds to the Veronese embedding, does not belong to O_m. We will now prove that O_m is both open and closed, which will yield a contradiction since $\mathsf{X}_n^0(\mathbb{T})$ is connected.

Step 1 *The set O_m is open in $\mathsf{X}_n^0(\mathbb{T})$.*

Let

$$X = ((x^1, \dots, x^m), (i_1, \dots, i_m))$$

be a degenerate m–tuple for the curve ξ associated to the oper D. Without loss of generality we can assume i_1 is the greatest integer j such that $((x^1, \dots, x^m), (j, \dots, i_m))$ is degenerate. Now let η be a lift of ξ (with values in $\mathbb{R}^n \setminus \{0\}$) on an interval containing the support of X. Let us consider the function f_D defined on a neighborhood of x^1 by

$$f_D \colon y \mapsto \widehat{\eta}(X(y)),$$

where $X(y) := ((y, x^2, \dots, x^m), (i_1, \dots, i_m))$. We first prove that $\dot{f}_D(x^1) \neq 0$. A computation yields

$$\dot{f}_D(x^1) = \big(\widehat{\eta}^{(i_1-2)}(x_1) \wedge \eta^{(i_1)}(x_1)\big) \wedge \Big(\bigwedge_{2 \leqslant j \leqslant m} \widehat{\eta}^{i_j}(x_j)\Big).$$

Let us recall the following elementary fact of linear algebra. Let u, v and $e_1 \dots, e_k$ be vectors in $\mathbb{R}^n$ such that

(124) $$u \wedge v \wedge e_1 \wedge \dots \wedge e_{k-1} \neq 0,$$

(125) $$u \wedge e_1 \wedge \dots \wedge e_k = 0.$$

Then

(126) $$v \wedge e_1 \wedge \dots \wedge e_k \neq 0.$$

Indeed, by (125), u belongs to the hyperplane H generated by $(e_1, \dots, e_k)$. If (126) does not hold, then v also belongs to H. Thus the vector space generated by $(u, v, e_1, \dots, e_{k-1})$ also would lie in H, contradicting (124).

By maximality of i_1, we know that $\widehat{\eta}(Y) \neq 0$, where

$$Y = ((x^1, \dots, x^m), (i_1 + 1, i_2 - 1, \dots, i_m)).$$

Since $f_D(x^1) = 0$, the previous remark with $u = \eta^{(i_1-1)}$, $v = \eta^{(i_1)}$ yields $\dot{f}_D(x^1) \neq 0$.

By transversality, it then follows that for D' close to D there exists a z close to x^1 such that $f_{D'}(z) = 0$, and thus $D' \in \mathsf{O}_m$.

Step 2 *The set O_m is closed in $\mathsf{X}^0_n(\mathbb{T})$.*

Let $\{\xi^1_n\}_{n\in\mathbb{N}}$ be a sequence of curves associated to a sequence of opers in O_m converging to an oper D associated to the curve ξ. Let

$$\{X_n = ((x^1_n,\dots,x^m_n),(j^1_n,\dots,j^m_n))\}_{n\in\mathbb{N}}$$

be the corresponding sequence of degenerate m–tuples. We can extract a subsequence such that for every i, the sequence $\{j^i_n\}_{n\in\mathbb{N}}$ is constant and equal to j^i. After permutation of $\{1,\dots,p\}$ and extracting a further subsequence, we can assume that there exists a p–tuple

$$Y = ((y^1,\dots,y^p),(i^1,\dots,i^p)),$$

with $p \leqslant m$, and integers $k_1,\dots,k_p$ such that

(i) $1 = k_1 \leqslant \dots \leqslant k_p = m$,

(ii) for all i with $k_u \leqslant i < k_{u+1}$ we have $\lim_{n\to\infty}(x^i_n) = y^u$,

(iii) for all v with $1 \leqslant v \leqslant p$,

$$i^v = \sum_{k_v \leqslant u < k_{v+1}} j^u.$$

As an application of the Taylor formula, we have

$$\hat{\eta}^{(p)}(x) \wedge \hat{\eta}^{(k)}(y) = (x-y)^{p.k}\hat{\eta}^{p+k}(x) + o((x-y)^k).$$

It follows that for all u,

$$\lim_{n\to\infty}\left(\left(\prod_{v=k_u}^{k_{u+1}-1}\frac{1}{(x^v_n - y^u)^{N_v}}\right)\bigwedge_{v=k_u}^{k_{u+1}-1}\hat{\eta}^{(i^v)}(x^v_n)\right) = \hat{\eta}^{(i_u)}(y^u),$$

where $N_v = i^v\big(\sum_{w=k_u}^{v-1} i^w\big)$. In particular, Y is degenerate for ξ. Thus $p = m$ by minimality, and thus $D \in \mathsf{O}_m$. ☐

Finally, let us say a Frenet curve is *Hitchin* if it can be deformed through Frenet curves to the Veronese embedding. Then we obtain as a consequence of the two previous propositions the following statement, which seems to belong to the folklore but for which we could not find a proper reference.

Theorem 10.4.3 *The map which associates to an $\mathrm{PSL}_n(\mathbb{R})$–oper its associated curve is a homeomorphism from the space of Hitchin opers to the space of Hitchin Frenet curves.*

10.5 Cross ratios and opers

Let ξ be a Frenet curve and ξ^* be its associated osculating hyperplane curve. The *weak cross ratio* associated to this pair of curves is the function on

$$\mathbb{T}^{4*} := \{(x,y,z,t) \in \mathbb{T}^4 \mid z \neq y,\ x \neq t\}$$

defined by

$$\boldsymbol{b}_{\xi,\xi^*}(x,y,z,t) = \frac{\langle \hat{\xi}(x) \mid \hat{\xi}^*(y)\rangle \langle \hat{\xi}(z) \mid \hat{\xi}^*(t)\rangle}{\langle \hat{\xi}(z) \mid \hat{\xi}^*(y)\rangle \langle \hat{\xi}(x) \mid \hat{\xi}^*(t)\rangle}, \tag{127}$$

where for every u, we choose arbitrary nonzero vectors $\hat{\xi}(u)$ and $\hat{\xi}^*(u)$ in $\xi(u)$ and $\xi^*(u)$ respectively. This weak cross ratio only depends on the oper D, and we shall denote it by $\boldsymbol{b}_D$.

10.5.1 Coordinate functions As in Section 10.1.2, let K be the line bundle of $\left(-\frac{1}{2}\right)$–densities over $\mathbb{T}$ and $P := J^{n-1}(K^{n-1})$ be the rank n vector bundle of $(n-1)$–jets of sections of the bundle of $\left(-\frac{1}{2}(n-1)\right)$–densities. We choose once and for all a trivialization of P given by n fiberwise independent sections $\sigma_1,\dots,\sigma_n$ of P, so that F_p is generated by $\sigma_1,\dots,\sigma_p$.

Let ∇ be a connection on P. Let I be an interval in $\mathbb{R}$ with extremities Y and y. We pull back ∇, P and σ_i on $\mathbb{R}$ using the projection

$$\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z} = \mathbb{T}.$$

We denote the pulled back objects by the same symbol. For any $y \in \mathbb{R}$, let σ_y the ∇–parallel section of P on I characterized by $\sigma_y(y) = \sigma_1(y)$. Similarly let σ_Y^* be the ∇^*–parallel section on I of P^* characterized by $\sigma_Y^*(Y) = \sigma_n^*(Y)$, where $(\sigma_1^*,\dots,\sigma_n^*)$ is the dual basis to $(\sigma_1,\dots,\sigma_n)$.

Then the function $t \mapsto \langle \sigma_Y^*(t), \sigma_y(t)\rangle$ is constant on I.

Definition 10.5.1 (coordinate function) The *coordinate function* associated to the points Y and y and the trivialization of P is the function

$$F_{Y,y}\colon \nabla \mapsto F_{Y,y}(\nabla) = \langle \sigma_Y^*(t), \sigma_y(t)\rangle \quad \text{for } t \in I,$$

defined on the space of connections on P.

We shall write $\sigma_Y^* \otimes \sigma_y =: \mathsf{p}^{Y,y} = \mathsf{p}^{Y,y}(\nabla) \in \Omega^0(\mathbb{R}, \mathrm{End}_0(P))$, so that

$$F_{Y,y}(\nabla) = \mathrm{tr}(\mathsf{p}^{Y,y}). \tag{128}$$

We then have:

Proposition 10.5.2 *Assume that ∇ has trivial holonomy. Then the coordinate function $F_{Y,y}$ only depends on the projections of Y and y in $\mathbb{T}$. Moreover there exists a section $\mathsf{p}^{Y_0,y_0} \in \Omega^0(\mathbb{T}, \mathrm{End}_0(P))$ such that $\mathsf{p}^{Y,y}$ is the pullback of p^{Y_0,y_0}.*

Proof Let Y_0 and y_0 be the respective projections of Y and y. Since ∇ has trivial holonomy we may find parallel sections η_{y_0} and $\eta^*_{Y_0}$ such that $\eta_{y_0}(y_0) = \sigma_1(y_0)$ and $\eta^*_{Y_0}(Y_0) = \sigma_1(Y_0)$. Then $\sigma_y = \pi^*(\eta_{y_0})$ and $\sigma^*_Y = \pi^*(\eta_{y_0})$. Thus

$$F_{Y,y}(\nabla) = \langle \eta^*_{Y_0}(t), \eta_{y_0}(t) \rangle.$$

The first part of the result follows. For the second part, we take $\mathsf{p}^{Y_0,y_0} = \eta^*_{Y_0} \otimes \eta_{y_0}$. $\square$

10.5.2 Differential of coordinate functions Our aim in this subsection is to compute the differential of $F_{Y,y}$, where Y and y belong to an interval I.

Proposition 10.5.3 *Let ∇ be a connection, let y_0 be a point in $\mathbb{R} \setminus I$ and let α be an element of $\Omega^1(\mathbb{T}, \mathrm{End}_0(P))$. Then*

$$\langle \mathrm{d}_\nabla F_{Y,y}, \alpha \rangle = \int_{\mathbb{R}} \psi^{Y,y,y_0} \,\mathrm{tr}(\mathsf{p}^{Y,y} \pi^*(\alpha)), \tag{129}$$

where $\psi^{Y,y,y_0}(s) := [y_0 s, Yy]$.

We can observe that the right-hand side of Equation (129) does not depend on the choice of $y_0 \in \mathbb{R} \setminus I$. Indeed, by the cocycle identity, $\psi^{Y,y,x} - \psi^{Y,y,z}$ is constant and equal to $[xz, Yy] = 0$, if $x, z \notin I$.

Proof Let β be a primitive of $\pi^*\alpha$ on I such that $\beta(y) = 0$. Let $t \mapsto \nabla^t$ be a one-parameter smooth family of connections with $\nabla^0 = \nabla$ such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \nabla^t = \alpha.$$

Let G^t be the family of sections of $\mathrm{End}(P)$ such that $G^t(z) = \mathrm{Id}$ and $(G^t)^*\nabla = \nabla^t$. Then by construction

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} G^t = \beta.$$

Moreover,

$$F_{Y,y}(\nabla^t) = \langle \sigma^*_n(Y), G^t(\sigma_y(Y)) \rangle.$$

Thus,

$$\langle \mathrm{d}_\nabla F_{Y,y}, \alpha \rangle = \langle \sigma^*_Y(Y), \beta(Y)\sigma_y(Y) \rangle. \tag{130}$$

Let $c(t)$ be a curve with values in I such that $c(0)=y$ and $c(1)=Y$. Let

$$f(t)=\langle \sigma_Y^*(c(t)), \beta(c(t))\sigma_y^*(c(t))\rangle.$$

Then

$$\langle \mathrm{d}_\nabla F_{Y,y}, \alpha\rangle = f(1)-f(0)=\int_0^1 \dot f(s)\,\mathrm{d}s. \tag{131}$$

Since σ_Y^* and σ_y are parallel,

$$\dot f(s)=\langle \sigma_Y^*(c(s)), \pi^*\alpha(\dot c(s)).\sigma_y(c(s))\rangle,$$

and we have, letting J be the interval whose endpoints are Y and y,

$$\begin{aligned}\langle \mathrm{d}_\nabla F_{Y,y}, \alpha\rangle &= \mathrm{Sign}(Y-y)\int_J \langle \sigma_Y^*, \pi^*(\alpha).\sigma_y\rangle \\ &= \mathrm{Sign}(Y-y)\int_J \mathrm{tr}(\mathrm{p}^{Y,y}.\pi^*(\alpha)).\end{aligned} \tag{132}$$

We finally deduce the result from (132) and the fact that for any $y_0 \notin I$ we have

$$\mathrm{Sign}(Y-y)\int_J \gamma = \int_{\mathbb{R}} \psi^{Y,y,y_0}\gamma.$$

□

10.6 Poisson brackets on the space of connections

Since $F_{X,x}$ is not an observable in the sense of Section 10.2.2, we first need to regularize these functions.

10.6.1 Regularization Let μ and ν be two C^∞ measures compactly supported in a bounded interval $]a,b[$ of $\mathbb{R}$. Let us consider the function

$$F_{\mu,\nu} := \int_{\mathbb{R}^2} F_{X,x}\,\mathrm{d}\mu\cdot\mathrm{d}\nu(X,x).$$

We consider this function as defined on the space of connections over the bundle $P\to\mathbb{T}$. We obviously have:

Proposition 10.6.1 *Let $\{(\mu_n,\nu_n)\}_{n\in\mathbb{N}}$ be two sequences of measures weakly converging to (μ,ν). Then $\{F_{\mu_n,\nu_n}\}_{n\in\mathbb{N}}$ converges uniformly on every compact to $F_{\mu,\nu}$.*

We say the sequence $\{(\mu_n,\nu_n)\}_{n\in\mathbb{N}}$ is *regularizing* for the pair (X,x) if μ_n,ν_n are smooth measures weakly converging to the Dirac measures supported at X and x respectively.

10.6.2 Poisson brackets of regularization We now have:

Proposition 10.6.2 *For any pair of smooth measures (μ,ν) with compact support, $F_{\mu,\nu}$ is an observable. Let (μ,ν) and $(\overline{\mu},\overline{\nu})$ be two pairs of C^∞ measures on $\mathbb{R}$. Then the Poisson bracket $\{F_{\mu,\nu},F_{\overline{\mu},\overline{\nu}}\}$ is equal to*

$$\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}^4}[m(Y)m(y),Xx]\Big(F_{X,y}F_{Y,x}-\frac{1}{n^2}F_{X,x}F_{Y,y}\Big)\,\mathrm{d}\Lambda(X,x,Y,y), \tag{133}$$

where $m(u)=u+m$ and $\Lambda=\mu\otimes\nu\otimes\overline{\mu}\otimes\overline{\nu}$. In particular, if all measures are supported on $[0,1]$, then the bracket $\{F_{\mu,\nu},F_{\overline{\mu},\overline{\nu}}\}$ is equal to

$$\int_{[0,1]^4}[Yy,Xx]\Big(F_{X,y}F_{Y,x}-\frac{1}{n^2}F_{X,x}F_{Y,y}\Big)\,\mathrm{d}\Lambda(X,x,Y,y). \tag{134}$$

Proof By Proposition 10.5.3, we have that if a does not belong to the union K of the supports of μ and ν,

$$\langle \mathrm{d}F_{\mu,\nu},\alpha\rangle=\int_{\mathbb{R}^2}\psi^{X,x,a}\,\mathrm{tr}(\mathsf{p}^{X,x}\pi^*\alpha)\,\mathrm{d}\mu\cdot\mathrm{d}\nu(X,x).$$

Let us denote by $\mathsf{C}_0:=\mathsf{C}-\frac{1}{n}\,\mathrm{tr}(\mathsf{C})\,\mathrm{Id}$ the trace-free part of the endomorphism C. For any s in $\mathbb{R}$, let

$$\Lambda_{\mu,\nu}(s):=\int_{\mathbb{R}^2}\psi^{X,x,a}(s)\,\mathsf{p}_0^{X,x}(s)\,\mathrm{d}\mu\cdot\mathrm{d}\nu(X,x). \tag{135}$$

Observe that $\Lambda_{\mu,\nu}\in\Omega^0(\mathbb{R},\mathrm{End}_0(P))$ and the support of $\Lambda_{\mu,\nu}$ is included in K. Let us trivialize P using the connection ∇. Then let

$$G_{\mu,\nu}(s):=\sum_{m\in\mathbb{Z}}\Lambda_{\mu,\nu}(s+m).$$

Then $G_{\mu,\nu}(s)$ is periodic and thus of the form $\pi^*\beta$, with $\beta\in\Omega^0(\mathbb{T},P)$. Moreover,

$$\int_{\mathbb{T}}\mathrm{tr}(\beta.\alpha)=\int_0^1\mathrm{tr}(\pi^*\beta.\pi^*\alpha)=\int_{\mathbb{R}}\mathrm{tr}(\Lambda_{\mu,\nu}.\pi^*\alpha)=\langle\mathrm{d}F_{\mu,\nu},\alpha\rangle. \tag{136}$$

It follows by (136) that

$$\mathrm{d}F_{\mu,\nu}(s)=\beta\in\mathfrak{g}=\mathsf{D}^0. \tag{137}$$

In particular, according to Definition 10.2.1, $F_{\mu,\nu}$ is an observable. From (135) we have

$$\Lambda_{\mu,\nu}(s)=-\int_{-\infty}^{s}\int_{s}^{\infty}\mathsf{p}_0^{X,x}(s)\,\mathrm{d}\mu\cdot\mathrm{d}\nu(X,x)+\int_{s}^{\infty}\int_{-\infty}^{s}\mathsf{p}_0^{X,x}(s)\,\mathrm{d}\mu\cdot\mathrm{d}\nu(X,x).$$

For any smooth probability measure ξ let us write $\mathrm{d}\xi = \dot{\xi}\,\mathrm{d}\lambda$, where λ is the Lebesgue measure. Then, since $\mathsf{p}^{X,x}$ is parallel, we have

$$\begin{aligned}\nabla_{\partial_t}\Lambda_{\mu,\nu}(s) = -\dot{\mu}(s)\int_s^\infty \mathsf{p}_0^{s,x}(s)\,\mathrm{d}\nu(x) + \dot{\nu}(s)\int_{-\infty}^s \mathsf{p}_0^{X,s}(s)\,\mathrm{d}\mu(X)\\ -\dot{\mu}(s)\int_{-\infty}^s \mathsf{p}_0^{s,x}(s)\,\mathrm{d}\nu(x) + \dot{\nu}(s)\int_s^\infty \mathsf{p}_0^{X,s}(s)\,\mathrm{d}\mu(X)\\ = \dot{\nu}(s)\int_{\mathbb{R}} \mathsf{p}_0^{X,s}\,\mathrm{d}\mu(X) - \dot{\mu}(s)\int_{\mathbb{R}} \mathsf{p}_0^{s,x}\,\mathrm{d}\nu(x).\end{aligned}$$

It follows that

$$\begin{aligned}&\operatorname{tr}(\Lambda_{\mu,\nu}(s+m)\nabla_{\partial_t}\Lambda_{\overline{\mu},\overline{\nu}}(s))\\ &= \dot{\overline{\nu}}(s)\int_{\mathbb{R}^3}\psi^{X,x,a}(s+m)\operatorname{tr}(\mathsf{p}_0^{Y,s}\mathsf{p}_0^{X,x})\,\mathrm{d}\mu\cdot\mathrm{d}\nu\cdot\mathrm{d}\overline{\mu}(X,x,Y)\\ &\quad - \dot{\overline{\mu}}(s)\int_{\mathbb{R}^3}\psi^{X,x,a}(s+m)\operatorname{tr}(\mathsf{p}_0^{s,y}\mathsf{p}_0^{X,x})\,\mathrm{d}\mu\cdot\mathrm{d}\nu\cdot\mathrm{d}\overline{\nu}(X,x,y).\end{aligned}\tag{138}$$

We can now compute the Poisson bracket as defined in Definition 10.2.1:

$$\begin{aligned}\{F_{\mu,\nu},F_{\overline{\mu},\overline{\nu}}\} &= \int_{\mathbb{R}}\operatorname{tr}\big(\Lambda_{\mu,\nu}(s)\pi^*(\nabla_{\partial_t}\mathrm{d}F_{\overline{\mu},\overline{\nu}}(s))\big)\,\mathrm{d}\lambda(s)\\ &= \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}\operatorname{tr}\big(\Lambda_{\mu,\nu}(s)\nabla_{\partial_t}\Lambda_{\overline{\mu},\overline{\nu}}(s+m)\big)\,\mathrm{d}\lambda(s)\\ &= \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}\operatorname{tr}\big(\Lambda_{\mu,\nu}(s+m)\nabla_{\partial_t}\Lambda_{\overline{\mu},\overline{\nu}}(s)\big)\,\mathrm{d}\lambda(s).\end{aligned}\tag{139}$$

Using (138), we get that

$$\begin{aligned}\{F_{\mu,\nu},F_{\overline{\mu},\overline{\nu}}\} &= \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}^4}\psi^{X,x,a}(s+m)\operatorname{tr}(\mathsf{p}_0^{Y,s}\mathsf{p}_0^{X,x})\,\mathrm{d}\Lambda(X,x,Y,s)\\ &\quad - \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}^4}\psi^{X,x,a}(s+m)\operatorname{tr}(\mathsf{p}_0^{s,y}\mathsf{p}_0^{X,x})\,\mathrm{d}\Lambda(X,x,s,y).\end{aligned}\tag{140}$$

Using the dummy changes of variable $s=y$ on line one and $s=Y$ on line two of (140), we finally get

$$\begin{aligned}&\{F_{\mu,\nu},F_{\overline{\mu},\overline{\nu}}\}\\ &= \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}^4}(\psi^{X,x,a}(y+m)-\psi^{X,x,a}(Y+m))\operatorname{tr}(\mathsf{p}_0^{Y,y}\mathsf{p}_0^{X,x})\,\mathrm{d}\lambda(X,x,Y,y)\\ &= \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}^4}[(Y+m)(y+m),Xx]\operatorname{tr}(\mathsf{p}_0^{Y,y}\mathsf{p}_0^{X,x})\,\mathrm{d}\Lambda(X,x,Y,y).\end{aligned}\tag{141}$$

We conclude the proof of the proposition by remarking that

$$\operatorname{tr}(\mathsf{p}^{X,x}\mathsf{p}^{Y,y}) = \operatorname{tr}(\mathsf{p}^{X,y})\operatorname{tr}(\mathsf{p}^{Y,x}),$$

and thus

$$\operatorname{tr}(\mathsf{p}_0^{X,x}\mathsf{p}_0^{Y,y}) = \operatorname{tr}(\mathsf{p}^{X,y})\operatorname{tr}(\mathsf{p}^{Y,x}) - \frac{1}{n^2}\operatorname{tr}(\mathsf{p}^{X,x})\operatorname{tr}(\mathsf{p}^{Y,y}). \tag{142}$$

Combining equations (141) and (142) yields the result. □

As corollaries, we obtain:

Corollary 10.6.3 *Let (μ_n,ν_n) and $(\overline{\mu}_n,\overline{\nu}_n)$ be regularizing sequences for (X,x) and (Y,y) respectively. Assume that $\{X,x,Y,y\}\subset\,]0,1[$. Then*

$$\lim_{n\to\infty}\{F_{\mu_n,\nu_n},F_{\overline{\mu}_n,\overline{\nu}_n}\} = [Yy,Xx]\big(F_{X,y}F_{Y,x} - \tfrac{1}{n^2}F_{X,x}F_{Y,y}\big).$$

Corollary 10.6.4 *Let (X,x,Y,y) be a quadruple of pairwise distinct points. Then $(F_{X,x},F_{Y,y})$ is a pair of acceptable observables. Moreover,*

$$\{F_{X,x},F_{Y,y}\} = [Yy,Xx]\big(F_{X,y}F_{Y,x} - \tfrac{1}{n^2}F_{X,x}F_{Y,y}\big).$$

This last corollary interprets the swapping algebra as an algebra of "observables" on the space of connections.

10.7 Drinfel'd–Sokolov reduction and the multifraction algebra

We introduced in Section 10.5.1 functions of connections depending on the choice of a trivialization of P. We now introduce functions that only depend on the associated oper and do not rely on the choice of trivialization of P.

We first relate cross ratios to our previously defined coordinate functions.

10.7.1 Cross ratios The following proposition follows at once from the definitions.

Proposition 10.7.1 *Let D be a Hitchin oper associated to the connection ∇ with trivial holonomy. Let X,x,Y,y be a quadruple of pairwise distinct points of $\mathbb{T}$. Let $\widetilde{X},\widetilde{x},\widetilde{Y},\widetilde{y}$ be lifts of X,x,Y,y in $\mathbb{R}$. Then*

$$\boldsymbol{b}_D(X,x,Y,y) = \frac{F_{\widetilde{X},\widetilde{y}}(\nabla)\,.\,F_{\widetilde{Y},\widetilde{x}}(\nabla)}{F_{\widetilde{X},\widetilde{x}}(\nabla)\,.\,F_{\widetilde{Y},\widetilde{y}}(\nabla)}.$$

10.7.2 The main theorem We can now prove our main theorem, which relates the Poisson structure on the space of opers and the multifraction algebra.

Theorem 10.7.2 *Let $(X_0, x_0, Y_0, y_0, X_1, x_1, Y_1, y_1)$ be pairwise distinct points. Then the cross fractions $[X_0; x_0; Y_0; y_0]$ and $[X_1; x_1; Y_1; y_1]$ define a pair of acceptable observables whose Poisson bracket with respect to the Drinfel'd–Sokolov reduction coincides with their Poisson bracket in the multifraction algebra.*

Proof This is an immediate consequence of Proposition 10.7.1 and Corollary 10.6.4, as well as the definition of the Poisson structure coming from the symplectic reduction in Theorem 10.3.3. □

Appendix: Existence of vanishing sequences

We prove the existence of vanishing sequences.

Definition A.1.1 (separability in groups) Let G be a group.

- We say G is *subgroup separable* if given any finitely generated subgroup H in G, any $g \in G$ and any $h \notin Hg$, there exists a finite index subgroup G_0 in G such that if π is the projection of G onto G/G_0, then $\pi(h) \notin \pi(Hg)$.
- We say G is *double coset separable* if given any finitely generated subgroups H and K in G, any $g \in G$ and any $h \notin HgK$, there exists a finite index subgroup G_0 in G such that if π is the projection of G onto G/G_0, then $\pi(h) \notin \pi(HgK)$.

Observe that a double coset separable group is then subgroup separable and residually finite. G Niblo [24] proved:

Theorem A.1.2 *A surface group is double coset separable.*

As an immediate consequence, since $\pi_1(S)$ is countable, we have:

Corollary A.1.3 *Vanishing sequences exist.*

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