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Deforming convex projective manifolds

DARYL COOPER
DARREN LONG
STEPHAN TILLMANN

We study a properly convex real projective manifold with (possibly empty) compact, strictly convex boundary, and which consists of a compact part plus finitely many convex ends. We extend a theorem of Koszul, which asserts that for a compact manifold without boundary the holonomies of properly convex structures form an open subset of the representation variety. We also give a relative version for noncompact (G, X) manifolds of the openness of their holonomies.

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Given a subset $\Omega \subset \mathbb{RP}^n$, the *frontier* is $\mathrm{Fr}(\Omega) = \mathrm{cl}(\Omega) \setminus \mathrm{int}(\Omega)$ and the *boundary* is $\partial\Omega = \Omega \cap \mathrm{Fr}(\Omega)$. A *properly convex projective manifold* is $M = \Omega/\Gamma$, where $\Omega \subset \mathbb{RP}^n$ is a convex set with nonempty interior, and $\mathrm{cl}(\Omega)$ does not contain any $\mathbb{RP}^1$, and $\Gamma \subset \mathrm{PGL}(n+1, \mathbb{R})$ acts freely and properly discontinuously on Ω. If, in addition, $\mathrm{Fr}(\Omega)$ contains no line segment then M and Ω are *strictly convex*. The *boundary of M is strictly convex* if $\partial\Omega$ contains no line segment.

If M is a *compact* (G, X) manifold then a sufficiently small deformation of the holonomy gives another (G, X)–structure on M. Koszul [24; 25] proved a similar result holds for closed, properly convex, projective manifolds. In particular, nearby holonomies continue to be discrete and faithful representations of the fundamental group.

Koszul's theorem cannot be generalized to the case of noncompact manifolds without some qualification — for example, a sequence of hyperbolic surfaces whose completions have cone singularities can converge to a hyperbolic surface with a cusp. The holonomy of a cone surface in general is neither discrete nor faithful. Therefore we must impose conditions on the holonomy of each end.

If M is a geometrically finite hyperbolic manifold with a convex core that has compact boundary, then every end of M is topologically a product, and is foliated by strictly convex hypersurfaces. These hypersurfaces are either convex towards M, so that cutting

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along one gives a submanifold of M with convex boundary, and the holonomy of the end contains only hyperbolics; or else convex *away* from M, in which case the end is a cusp and the holonomy of the end contains only parabolics.

This paper studies properly convex manifolds whose ends are either convex towards or away from M. An end that is convex towards M may be compactified by adding a convex boundary. *Generalized cusps* are those that are convex away from M with virtually nilpotent fundamental group. The holonomy of a generalized cusp may contain both hyperbolic and parabolic elements.

Definition 0.1 A *generalized cusp* is a properly convex manifold C homeomorphic to $\partial C \times [0,\infty)$ with compact, strictly convex boundary and with $\pi_1 C$ virtually nilpotent.

For instance, all ends of a finite-volume hyperbolic manifold are generalized cusps. For an n–manifold M, possibly with boundary, define

$$\mathrm{Rep}(\pi_1 M) = \mathrm{Hom}(\pi_1 M, \mathrm{GL}(n+1,\mathbb{R}))$$

and $\mathrm{Rep}_{\mathrm{ce}}(M)$ to be the subset of $\mathrm{Rep}(\pi_1 M)$ consisting of holonomies of properly convex structures on M with ∂M strictly convex and such that each end is a generalized cusp. A group $\Gamma \subset \mathrm{GL}(n+1,\mathbb{R})$ is a *virtual flag group* if it contains a subgroup of finite index that is conjugate into the upper-triangular group. The set of virtual flag groups is written VFG.

Theorem 0.2 *Suppose N is a compact, connected n–manifold and $\mathcal{V} = \bigcup_i V_i \subseteq \partial N$ is the union of some of the boundary components of N. Let $M = N \setminus \mathcal{V}$. Assume $\pi_1 V_i$ is virtually nilpotent for each i. Let $B_i \cong V_i \times [0,1)$ be the end of M corresponding to V_i. Then $\mathrm{Rep}_{\mathrm{ce}}(M)$ is an open subset of*

$$\mathrm{VFG}(M) := \{\rho \in \mathrm{Rep}(\pi_1 M) : \rho(\pi_1 B_i) \in \mathrm{VFG} \textit{ for all } i\}.$$

A similar statement holds for orbifolds since a properly convex orbifold has a finite cover which is a manifold, and the property of being properly convex is unchanged by coverings. This theorem is a consequence of our main theorem, Theorem 6.29, that a certain map is open. By Corollary 6.11, $\rho(\pi_1 B_i) \in \mathrm{VFG}$ if and only if there is a finite-index subgroup $\Gamma < \rho(\pi_1 B_i)$ such that every eigenvalue of every element of Γ is real.

There is a Margulis lemma for *properly* convex manifolds that says the local fundamental group is virtually nilpotent; see Cooper, Long and Tillmann [13, 0.1], also Crampon and Marquis [14]. There is a *thick-thin* decomposition for *strictly* convex manifolds — see [13, 0.2], also Crampon and Marquis [15] — but not for *properly* convex manifolds. Each component of the thin part of a *strictly* convex manifold is a Margulis tube or a *cusp* and has virtually nilpotent fundamental group consisting of parabolics. This motivates the definition of *generalized cusp* above. There is a discussion of cusps in properly convex manifolds in Section 5 of [13].

Here is some intuition for the proof of Theorem 0.2. Many of the ideas are already present for surfaces. Suppose M is a projective compact surface without boundary. There is a developing map dev: $\widetilde{M} \to \mathbb{RP}^2$. If M is properly convex then this map is injective and the image is a domain $\Omega \subset \mathbb{R}^2$ bounded by a convex closed curve $C \subset \mathbb{R}^2$. There is a compact polygonal fundamental domain $D \subset \Omega$ and the images of D under the holonomy $\rho\colon \pi_1 M \to \Gamma \subset \mathrm{PGL}(3,\mathbb{R})$ tessellate Ω. Small images of D accumulate on C. Suppose ρ' is a nearby homomorphism. Our aim is to show there is a convex closed curve C' close to C that bounds a domain Ω' that is preserved by $\Gamma' = \rho'(\pi_1 M)$.

The convexity of C is a phenomenon that *takes place at infinity* with regard to Ω. A priori, there is no reason to expect this phenomenon to be stable with respect to small deformations. A finite generating set for $\pi_1 M$ determines a word metric on $\pi_1 M$. If $g \in \pi_1 M$ is not too far from the identity in the word metric then $\rho'(g)$ and $\rho(g)$ are close and send D to almost the same set. However, for large elements g, one *loses control,* and there is no obvious reason why images of D should accumulate on some convex curve C'. Thus *convexity* of $C = \partial\Omega$ is a *limiting feature* of large group elements in $\pi_1 M$, and this might be destroyed by arbitrarily small deformations ρ'. In fact this is what happens with the example of surfaces with cone singularities discussed above. Convexity is stable for *closed* surfaces for reasons that we now outline.

Let $U \subset \mathbb{R}^3$ be the half space $x_3 \geq 0$ and $P = \partial U$ the plane $x_3 = 0$. Suppose $D \subset U$ is a disc, bounded by the simple closed curve $C = D \cap P$, and $S = \mathrm{int}(D)$. If S is convex down then C is convex and bounds a convex domain $\Omega \subset P$, and S is a graph over Ω. The condition that S is convex down is a condition over (finite) points in Ω rather than a condition at infinity. One might imagine that S is the surface of a mountain. If this surface is convex then the boundary of the base of the mountain is also convex.

The *tautological line bundle* ξM over M is an affine manifold. For a properly convex projective structure on M, there is a section $\sigma(M)$ of this bundle that is a surface in ξM which is *strictly* convex in the sense that the Hessian is strictly positive. The image of the universal cover of $\sigma(M)$ under the developing map is a surface $S \subset \mathbb{R}^3$ that limits on the sphere at infinity. By viewing $\mathbb{R}^3 \subset \mathbb{RP}^3$, and choosing a suitable new affine patch where the sphere at infinity becomes P, we can instead view S as a surface in U as above, and then $M = \Omega/\rho(\pi_1 M)$. If M is compact then $\sigma(M)$ is compact. A small deformation M' of M gives a nearby strictly convex surface $\sigma(M')$ in $\xi M'$, which develops to another convex surface S' in $\mathbb{R}^3$ and gives a convex domain $\Omega' \subset P$. This is the intuition for Koszul's theorem when M is closed.

Now suppose that M is a union of a compact manifold N and a (generalized) cusp C. One first shows, for the deformed cusp C', that $\xi C'$ contains a nearby strictly convex surface $\sigma(C')$. Since N is compact, $\xi N'$ contains a nearby convex surface $\sigma(N')$. One now deforms and joins $\sigma(N')$ and $\sigma(C')$, maintaining convexity, to obtain a strictly convex hypersurface in $\xi M'$. This implies M' is properly convex.

Section 1 describes the (G, X)*–extension theorem*, Theorem 1.7. This generalizes a well-known result for compact manifolds (the holonomies of (G, X)–structures form an open subset of the representation variety) by providing a *relative* version. Section 2 recalls the definition and properties of the tautological bundle. Section 3 reviews Hessian metrics and gives a characterization of properly convex manifolds in terms of the existence of a certain kind of Hessian metric on the tautological line bundle. Section 4 shows that various functions on properly convex projective manifolds are uniformly bounded, including a proof of the folklore result that they admit Riemannian metrics with all sectional curvatures bounded in terms of dimension.

The *convex extension theorem*, Theorem 5.8, is a version of Theorem 1.7 for properly convex manifolds with strictly convex boundary. A consequence is Theorem 0.3 below. Roughly this says that if you can convexly deform the ends of a properly convex manifold then you can convexly deform the manifold. There is no assumption that the fundamental group of an end is virtually nilpotent for this result.

Theorem 0.3 *Suppose $M = A \cup \mathcal{B}$ is a properly convex manifold with* (possibly empty) *compact, strictly convex boundary, and A is a compact, connected submanifold of M with $\partial A = \partial M \sqcup \partial\mathcal{B}$ and $\mathcal{B} \cong \partial\mathcal{B} \times [0,\infty) = B_1 \sqcup \cdots \sqcup B_k$, and each B_i is connected and π_1–injective in M.*

Suppose $\rho\colon (-1,1) \to \mathrm{Rep}(\pi_1 M)$ is continuous and $\rho_t := \rho(t)$ and ρ_0 is the holonomy of M. Let $\mathfrak{C}$ denote the space of closed subsets of $\mathbb{RP}^n$ with the Hausdorff topology. Suppose for all $1 \le i \le k$ and all $t \in (-1,1)$ that

(1) *there is a properly convex set* $\Omega_i(t) \subset \mathbb{RP}^n$ *that is preserved by* $\rho_t(\pi_1 B_i)$,

(2) $P_i(t) = \Omega_i(t)/\rho_t(\pi_1 B_i)$ *is a properly convex manifold and* $\partial P_i(t)$ *is strictly convex,*

(3) *there is a projective diffeomorphism from* $P_i(0)$ *to* B_i,

(4) $P_i(t)$ *is diffeomorphic to* B_i,

(5) *the two maps* $t \mapsto \mathrm{cl}(\Omega_i(t))$ *and* $t \mapsto \mathrm{cl}(\partial\Omega_i(t))$ *into* $\mathfrak{C}$ *are continuous.*

Then there is $\epsilon > 0$ *such that for all* $t \in (-\epsilon, \epsilon)$ *there is a properly convex projective structure on* M *with holonomy* $\rho(t)$ *such that* ∂M *is strictly convex and* B_i *is projectively diffeomorphic to* $P_i(t)$.

If $\Omega \subset \mathbb{RP}^n$, we write $\mathrm{PGL}(\Omega)$ for the subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$ that preserves Ω. Section 6 proves that generalized cusps contain *homogeneous* cusps; see Theorem 6.3:

Theorem 0.4 *Suppose* $C = \Omega/\Gamma$ *is a generalized cusp. Then* C *contains a generalized cusp* $C' = \Omega'/\Gamma$ *such that* $\mathrm{PGL}(\Omega')$ *acts transitively on* $\partial\Omega'$.

An algebraic argument (Theorem 6.18) uses that Γ is virtually nilpotent to show that if $C = \Omega/\Gamma$ is a generalized cusp then Γ has a finite-index subgroup that is a lattice in a connected Lie group $T = T(\Gamma)$ that is conjugate into the upper-triangular group.

Next, Proposition 6.24 shows that the T–orbit of some point $p \in \Omega$ is a strictly convex hypersurface $S = T \cdot p$. The convex hull of S is a domain Ω_T that is preserved by all of Γ, and we may shrink C to be Ω_T/Γ, giving Theorem 0.4.

From Theorem 0.4 it follows that generalized cusps are *stable* (see Theorem 6.28): if Γ is deformed to a nearby virtual flag group Γ', then $T' = T(\Gamma')$ is a nearby Lie group, so $S' = T(\Gamma') \cdot p$ is a nearby strictly convex hypersurface which gives a nearby domain $\Omega_{T'}$ and a nearby generalized cusp $C' = \Omega_{T'}/\Gamma'$.

The convex extension theorem and the stability of generalized cusps imply the main theorem, Theorem 0.2. Ballas, Cooper and Leitner [3] classify generalized cusps, and their properties are studied. This classification for 3–manifolds is given without proof in Section 7.

A function is *Hessian-convex* if it is smooth and has positive definite Hessian. This property is preserved by composition with diffeomorphisms that are close to affine. Section 8 reviews various types of convexity and contains a theorem about approximating strictly convex functions on affine manifolds by Hessian-convex ones. As a result, for our purposes *strictly* and *Hessian* convex are more or less equivalent. Section 9 is a short proof of Benzécri's theorem. We have put these results at the end of the paper with the intention of not breaking the narrative.

There is an entirely PL approach to Theorem 0.2 which, however, we do not develop in this paper. It is based on using the convex hull of the orbit of one point instead of a characteristic surface.

Theorem 0.2 does not always remain true if ∂M is convex but not strictly convex. However, in some cases, the theorem can still be applied. For instance, a hyperbolic manifold M with compact, totally geodesic boundary is a submanifold of a finite-volume hyperbolic manifold with strictly convex smooth boundary obtained by fattening. In particular, any small deformation in $\mathrm{PGL}(4,\mathbb{R})$ of the holonomy in $\mathrm{PO}(3,1)$ of a compact Fuchsian manifold is the holonomy of a strictly convex projective structure on (surface) $\times\,[0,1]$.

The reader only interested in the proof of Theorem 0.2 when M is compact need only read Section 1 up to Proposition 1.2, and then Sections 2–4, stopping before Definition 4.2. Those interested only in the proof of Theorem 0.3 can omit Section 6.

Most of Sections 1-4 is not new, and there is considerable overlap in the first five sections with the results and methods of Choi [8]. Marquis [29; 28] determined the holonomies of properly convex surfaces with cusps. In [12], Cooper and Long give a method of constructing fundamental domains for some strictly convex manifolds with cusps. Using the main result of this paper, Ballas [2] found new properly convex structures on the figure eight knot obtained by deforming the complete hyperbolic structure. The type of geometry in a generalized cusp can change during a deformation. For example a generalized cusp with diagonal holonomy can *transition* to one with parabolic holonomy. This is related to the study of geometric transition; see Cooper, Danciger and Wienhard [11].

In Section 5 of [13] there is a discussion of properly convex n–manifolds with parabolic holonomy. Such a manifold is diffeomorphic to a product $C\times\mathbb{R}$ and $\Gamma=\pi_1 C$ is virtually nilpotent. The manifold C is compact if and only if the Hirsch rank of Γ is maximal, namely $n-1$. In [13], these cusps are called *maximal*, and it is shown in this case that Γ is conjugate into $\mathrm{PO}(n,1)$. In general, parabolics are not conjugate into $\mathrm{PO}(n,1)$. In this paper, frequent use is made of the fact that the compactness of ∂C is equivalent to $H_{n-1}(C;\mathbb{Z}_2)\cong\mathbb{Z}_2$. For consistency with [13] one might call such generalized cusps *maximal*. However, to keep this paper from becoming even longer, these are the only type of generalized cusp we consider, therefore we do not use the term *maximal*.

1 (G, X) structures and extending deformations

The goal of this section is a relative version of the well-known fact (Proposition 1.2) that for *compact* manifolds the set of holonomies of (G, X)–structures is an open subset of the representation variety. The *extension theorem*, Theorem 1.7, implies that if $\mathcal{B}$ is a codimension-0 submanifold of a connected manifold M with $M \setminus \mathcal{B}$ compact and connected, then given a (G, X)–structure on M with holonomy ρ, together with a nearby representation σ, and given a nearby (G, X)–structure on each component of $\mathcal{B}$ with holonomy the restriction of σ, there is a nearby (G, X)–structure on M with holonomy σ that extends the structure on $\mathcal{B}$.

A *geometry* is a pair (G, X) where G is a Lie group which acts transitively and real-analytically on a manifold X. A (G, X)*–structure* on a manifold M (possibly with boundary) is a maximal atlas of charts which takes values in X such that transitions maps are locally the restriction of elements of G. A map between (G, X) manifolds is a (G, X) *map* if locally it is conjugate via (G, X)–charts to an element of G.

Let $\pi\colon \widetilde{M} \to M$ be (a fixed choice for) the universal cover of M. We regard $\pi_1 M$ as being *defined* as the group of covering transformations of this covering. A local diffeomorphism $f\colon \widetilde{M} \to X$ determines a (G, X)–structure on $\widetilde{M}$. If all the covering transformations are (G, X)–maps then there is a unique (G, X)–structure on M such that the covering space projection is a (G, X)–map. In this case f is called a *developing map* for this structure and determines a homomorphism $\mathrm{hol} = \mathcal{H}ol(f)\colon \pi_1 M \to G$ called *holonomy*.

For smooth manifolds M^m and N^n, the set of smooth maps $C^\infty_w(M, N)$ has the *weak topology*; see Hirsch [21, page 35]. The space of diffeomorphisms $\mathrm{Diff}(M)$ is a subspace of $C^\infty_w(M, M)$. If $N = \mathbb{R}$ then $C^\infty_w(M) := C^\infty_w(M, \mathbb{R})$.

The set of all developing maps is denoted by $\mathcal{D}ev(M, (G, X))$ or just $\mathcal{D}ev(M)$. The (G, X)–structure on M given by $\mathrm{dev} \in \mathcal{D}ev(M)$ is written (M, dev). There is a natural map of $\mathcal{D}ev(M)$ into $C^\infty_w(\operatorname{int} \widetilde{M}, X)$ given by restricting the developing map to $\operatorname{int} \widetilde{M}$. This map is injective because $\operatorname{int} \widetilde{M}$ is dense in $\widetilde{M}$.

Definition 1.1 The *geometric topology* on $\mathcal{D}ev(M)$ is the subspace topology from $C^\infty_w(\operatorname{int} \widetilde{M}, X)$.

Thus two developing maps for M are close if they are close on a large compact set in $\widetilde{M}$ that is disjoint from $\partial \widetilde{M}$. The following is due to Thurston [37]; see also Goldman [17] and Choi [7]. The topology on $\mathrm{Hom}(\pi_1 M, G)$ is the compact–open topology.

Proposition 1.2 (holonomy is open) *Suppose M is a compact, connected smooth manifold possibly with boundary. Then $\mathcal{H}ol\colon \mathcal{D}ev(M,(G,X)) \to \mathrm{Hom}(\pi_1 M, G)$ is continuous and open.*

Given $\mathrm{dev}_M \in \mathcal{D}ev(M)$ and $\mathrm{dev}_N \in \mathcal{D}ev(N)$, a smooth map $f\colon M \to N$ is *close* to a (G,X) map if it is covered by $F\colon \widetilde{M} \to \widetilde{N}$ and there is $g \in G$ such that $g \circ \mathrm{dev}_N \circ F$ is close to dev_M in $C^\infty_w(\widetilde{M}, X)$. This means there is a large compact set $K \subset \operatorname{int} \widetilde{M}$ and some $g \in G$ such that for each $x \in K$ there is an open neighborhood $U \subset \widetilde{M}$ with $V = \mathrm{dev}_M(U \cap K)$ and the map $g \circ \mathrm{dev}_N \circ F \circ (\mathrm{dev}_M|_{U\cap K})^{-1}$ is close to the inclusion map $V \hookrightarrow X$ in $C^\infty_w(V, X)$. This notion of *close* depends on dev_M but not on the choice of developing map dev_N for a given (G,X)–structure on N.

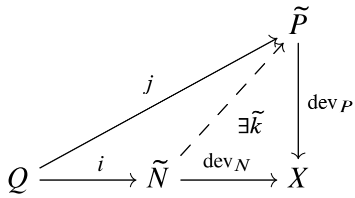
There is a nice description of what it means for two developing maps in $\mathcal{D}ev(M)$ to be close when one of them is *injective*. Suppose $\mathrm{dev} \in \mathcal{D}ev(M)$ is injective and $\Omega = \mathrm{dev}(\widetilde{M})$ and $\rho = \mathcal{H}ol(\mathrm{dev})$ and $\Gamma = \rho(\pi_1 M)$. Then $N = \Omega/\Gamma$ is a (G,X) manifold that is (G,X)–diffeomorphic to M. We choose the universal cover $\widetilde{N} = \Omega$; then $\pi_1 N = \Gamma$ by our definition as the group of covering transformations. There is a homeomorphism between spaces of developing maps $\mathcal{D}ev(M) \to \mathcal{D}ev(N)$.

Definition 1.3 Replacing $\mathcal{D}ev(M)$ by $\mathcal{D}ev(N)$ is called *choosing* dev *as the basepoint for the space of developing maps.*

The developing map $\mathrm{dev}_* \in \mathcal{D}ev(N)$ for N is the inclusion map $i\colon \widetilde{N} \hookrightarrow X$ and $\mathcal{H}ol(\mathrm{dev}_*)\colon \Gamma \hookrightarrow G$ is also the inclusion map. If N has no boundary then $\mathcal{D}ev(N)$ is a subspace of $C^\infty_w(\widetilde{N}, X)$ so $\mathrm{dev}' \in \mathcal{D}ev(N)$ is close to dev_* if dev' is close to i in $C^\infty_w(\widetilde{N}, X)$.

The idea for the extension theorem is the following. Suppose ($M = A_1 \cup A_2$, dev) is a (G,X) manifold with holonomy ρ, and $C = A_1 \cap A_2 \cong \partial A_i \times [0,1]$ is a connected collar neighborhood of ∂A_i and the inclusion map $C \hookrightarrow M$ is π_1–injective. Suppose $\rho'\colon \pi_1 M \to G$ is a nearby homomorphism and (A_i, dev'_i) is a nearby (G,X)–structure to $(A_i, \mathrm{dev}|_{A_i})$ with holonomy $\rho'|_{\pi_1 A_i}$. If C is compact we show there is a *nearby* (G,X)–structure on M obtained by gluing (A_1, dev'_1) to (A_2, dev'_2) by a (G,X)–diffeomorphism. This is done in Lemma 1.4, which uses analytic continuation to map the submanifold $C \subset (A_1, \mathrm{dev}'_1)$ into (A_2, dev'_2) by a (G,X)–diffeomorphism.

Lemma 1.4 (lifting developing maps) *In this statement all manifolds and maps are (G,X). Suppose N and P are connected manifolds and $\theta\colon \pi_1 N \to \pi_1 P$ is a* *homomorphism such that* $\mathrm{hol}_N = \mathrm{hol}_P \circ \theta$. *Suppose* $\pi_P\colon \widetilde{P} \to P$ *and* $\pi_N\colon \widetilde{N} \to N$ *are universal covers and* $i\colon Q \hookrightarrow \widetilde{N}$ *is the inclusion map of a path-connected set* Q *with* $\pi_N(Q) = N$. *Suppose* $\mathrm{dev}_N \circ i\colon Q \to X$ *lifts to a map* $j\colon Q \to \widetilde{P}$ *such that* $\mathrm{dev}_P \circ j = \mathrm{dev}_N \circ i$. *Then there is* $k\colon N \to P$ *covered by* $\widetilde{k}\colon \widetilde{N} \to \widetilde{P}$ *that extends* j:



Proof Because the covering translates of Q cover $\widetilde{N}$ and the manifolds N and P have (via θ) the same holonomy, j can be extended by analytic continuation to an equivariant (G, X)–map $\widetilde{k}\colon \widetilde{N} \to \widetilde{P}$. Equivariance implies $\widetilde{k}$ covers a (G, X)–map $k\colon N \to P$. □

For a *closed* manifold M, two nearby developing maps with the same holonomy differ by composition with a diffeomorphism of M close to the identity. If M has boundary then the boundary can move, so this result only holds outside a small neighborhood of the boundary; see the discussion in I.1.5 and I.1.6 leading to the proof of I.1.7.1 in Canary, Epstein and Green [6].

If P is a smooth manifold then $\mathrm{Diff}_0(\widetilde{P}, P) \subset \mathrm{Diff}(\widetilde{P})$ is defined to be the identity component of the subgroup of diffeomorphisms that cover an element of $\mathrm{Diff}(P)$. The next result says that if two developing maps are close, and have the same holonomy then, after changing one by a small isotopy, the developing maps are equal on a compact submanifold in the interior.

Corollary 1.5 *Suppose* P *is a smooth manifold. Let* $\rho \in \mathrm{Hom}(\pi_1 P, G)$ *be the holonomy of* $\mathrm{dev} \in \mathcal{D}ev(P, (G, X))$ *and* $\mathcal{D}ev_\rho(P) \subset \mathcal{D}ev(P)$ *be the subspace of developing maps with holonomy* ρ. *Then the map* $\mathrm{Diff}_0(\widetilde{P}, P) \to \mathcal{D}ev_\rho(P)$ *given by* $f \mapsto \mathrm{dev} \circ f$ *is an open map.*

It follows that, if N *is a compact codimension-*0 *manifold in the interior of* P, *and* $\mathrm{dev}' \in \mathcal{D}ev_\rho(P)$ *is close enough to* dev, *then there is* $k \in \mathrm{Diff}(P)$ *covered by* $\widetilde{k} \in \mathrm{Diff}_0(\widetilde{P}, P)$ *such that* $\mathrm{dev} = \mathrm{dev}' \circ \widetilde{k}$ *on* N, *and* k *is isotopic to the identity by a small isotopy supported in a small neighborhood of* N.

Proof Let $\pi_P\colon \widetilde{P} \to P$ and $\pi_N\colon \widetilde{N} \to N$ be universal covers. Let $Q \subset \widetilde{N}$ be a compact, connected manifold such that $\pi_N(Q) = N$. Since $\mathrm{dev}|_Q\colon Q \to X$ factors through the inclusion $j\colon Q \hookrightarrow \widetilde{P}$ and $(\pi_p \circ j)(Q) \subset \operatorname{int}(P)$, it follows that if dev' is close enough to dev then $\operatorname{dev}'|_Q\colon Q \to X$ has a nearby lift $j'\colon Q \to \widetilde{P}$. By Lemma 1.4, there is a (G,X)–map $k\colon (N, \operatorname{dev}'|_N) \to (P, \operatorname{dev}|_N)$ that lifts to a map that extends j'. If dev' is sufficiently close to dev, then k is close to the inclusion, and the result now follows from the fact that a diffeomorphism (namely k) close to an inclusion is ambient isotopic to the inclusion by a small ambient isotopy; see Lima [27]. □

Given a connected π_1–injective submanifold $B \subset M$, we fix a choice of some component $\widetilde{B} \subset \widetilde{M}$ of the preimage B in the universal cover of M and identify $\pi_1 B$ with those covering transformations of $\widetilde{M}$ that preserve $\widetilde{B}$. If dev_M is a developing map for a (G,X)–structure on M, the *restriction to B is* $\operatorname{dev}_{M|B} := \operatorname{dev}_M|_{\widetilde{B}}$.

Suppose M is a smooth manifold with (possibly empty) boundary and $\mathcal{B} \subset M$ is a codimension-0 submanifold that is a closed subset such that $A = \operatorname{cl}(M \setminus \mathcal{B})$ is a compact manifold. Suppose $\mathcal{B} = B_1 \sqcup \cdots \sqcup B_k$ has $k < \infty$ connected components and each component is π_1–injective. Define the *relative holonomy space*

$$\mathcal{RelHol}(M, \mathcal{B}, (G,X)) \subset \operatorname{Hom}(\pi_1 M, G) \times \prod_{i=1}^{k} \mathcal{Dev}(B_i, (G,X))$$

to be the subset of all $(\rho, \operatorname{dev}_1, \ldots, \operatorname{dev}_k)$ such that $\mathcal{Hol}(\operatorname{dev}_i) = \rho|_{\pi_1 B_i}$. This space has the subspace topology of the product topology.

Definition 1.6 A developing map for M restricts to give developing maps on each component of $\mathcal{B}$ and this defines the *relative holonomy map* $\mathcal{E}\colon \mathcal{Dev}(M,(G,X)) \to \mathcal{RelHol}(M, \mathcal{B}, (G,X))$,

$$\mathcal{E}(\operatorname{dev}_M) = (\mathcal{Hol}(\operatorname{dev}_M), \operatorname{dev}_{M|B_1}, \ldots, \operatorname{dev}_{M|B_k}).$$

This map depends on a fixed choice of one component $\widetilde{B}_i \subset \widetilde{M}$ for each i. In the special case that $\mathcal{B}$ is empty, $\mathcal{E} = \mathcal{Hol}$. We will apply this when $\mathcal{B}$ consists of the ends of M, which is why the symbol $\mathcal{E}$ is used. However, the result is of interest even when everything is compact.

Theorem 1.7 (extension theorem) *Suppose M is a smooth manifold with* (possibly empty) *boundary and $\mathcal{B} \subset M$ is a π_1–injective codimension-*0 *submanifold that is a closed subset such that $A = \operatorname{cl}(M \setminus \mathcal{B})$ is a compact, connected manifold. Then $\mathcal{E}\colon \mathcal{Dev}(M,(G,X)) \to \mathcal{RelHol}(M, \mathcal{B}, (G,X))$ is continuous and open.*

Proof Continuity is easy. We prove openness. For simplicity we will assume that $\mathcal{B} = B$ is connected; the multiend case merely requires more notation. Suppose $\mathcal{E}(\mathrm{dev}_{\rho,M}) = (\rho, \mathrm{dev}_{\rho,M|B})$ and $(\sigma, \mathrm{dev}_{\sigma,B})$ are nearby in $\mathcal{R}el\mathcal{H}ol(M, \mathcal{B}, (G, X))$.

Let $E \subset B$ be a compact collar of ∂B and $C = A \cup E$. By Proposition 1.2, there is $\mathrm{dev}_{\sigma,C}\colon \widetilde{C} \to X$ close to $\mathrm{dev}_{\rho,M|C}$ with holonomy (the restriction of) σ. Using Corollary 1.5 to change $\mathrm{dev}_{\sigma,E}$ by a small isotopy, we may assume $\mathrm{dev}_{\sigma,C}$ and $\mathrm{dev}_{\sigma,B}$ are equal on a smaller collar $E^- \subset E$. This gives a developing map $\mathrm{dev}_{\sigma,M}\colon \widetilde{M} \to X$ close to $\mathrm{dev}_{\rho,M}$ that is given by $\mathrm{dev}_{\sigma,C}$ on $\widetilde{A}$ and $\mathrm{dev}_{\sigma,B}$ on $\widetilde{B}$. □

2 Tautological bundles

There is a bundle $\xi M \to M$ over a real projective manifold M called the tautological line bundle. In the next section we show that M is properly convex if and only if ξM admits a certain kind of metric.

Radiant affine geometry is $\mathbb{L} = (\mathrm{GL}(n+1, \mathbb{R}), \mathbb{R}^{n+1} \setminus 0)$. A manifold with this structure is called a *radiant affine manifold*. It *ought* to be called a *linear manifold*, since transition functions are linear maps.

Projective geometry over a real vector space V is $\mathbb{P} = (\mathrm{PGL}(V), \mathbb{P}(V))$, where $\mathbb{P}(V) = (V \setminus 0)/\mathbb{R}^*$. *Positive projective space* is $\mathbb{P}_+(V) = (V \setminus 0)/\mathbb{R}_+$ and the action of $\mathrm{GL}(V)$ on V induces an effective action of $\mathrm{P}_+\mathrm{GL}(V) = \mathrm{GL}(V)/\mathbb{R}_+$ on $\mathbb{P}_+(V)$ which gives *positive projective geometry* $\mathbb{P}_+ = (\mathrm{P}_+\mathrm{GL}(V), \mathbb{P}_+(V))$. If $X \subset V$, we write $\mathbb{P}(X)$ for its image in $\mathbb{P}(V)$ and similarly $\mathbb{P}_+(X) \subset \mathbb{P}_+(V)$.

We identify $\mathbb{P}_+(\mathbb{R}^{n+1})$ with the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, and the *radial projection* $\pi_\xi\colon \mathbb{R}^{n+1} \setminus 0 \to \mathbb{S}^n$ is $\pi_\xi(x) = x/\|x\|$. An action of $A \in \mathrm{SL}_\pm(n+1, \mathbb{R})$ on $\mathbb{S}^n$ is given by $A(\pi_\xi x) = \pi_\xi(Ax)$. Clearly $\mathbb{P}_+ \cong \mathbb{S} := (\mathrm{SL}_\pm(n+1, \mathbb{R}), \mathbb{S}^n)$.

For each of the geometries $\mathbb{G}$ above there is a space of developing maps $\mathcal{D}ev(M, \mathbb{G})$ with the geometric topology. By lifting developing maps one obtains:

Proposition 2.1 *The natural map $\mathcal{D}ev(M, \mathbb{S}) \to \mathcal{D}ev(M, \mathbb{P})$ is two-to-one.*

Thus every projective structure on M lifts to a positive projective structure. If M is a real projective n–manifold, then the holonomy $\rho\colon \pi_1 M \to \mathrm{PGL}(n+1, \mathbb{R})$ lifts to $\widetilde{\rho}\colon \pi_1 M \to \mathrm{SL}_\pm(n+1, \mathbb{R})$ and $\mathrm{dev}\colon \widetilde{M} \to \mathbb{RP}^n$ lifts to $\widetilde{\mathrm{dev}}\colon \widetilde{M} \to \mathbb{S}^n$.

We will pass back and forth between projective geometry and positive projective geometry without mention. The *tautological bundle* over $\mathbb{S}^n$ is $\pi_\xi\colon \mathbb{R}^{n+1} \setminus 0 \to \mathbb{S}^n$.

The total space is a radiant affine manifold. There is an action of $(\mathbb{R},+)$ on the total space called the *radial flow* given by $\Phi_t(x)=\exp(-t)x$. This bundle is a principal $(\mathbb{R},+)$–bundle. All this structure is preserved by the action of $\mathrm{GL}(n+1,\mathbb{R})$ on $\mathbb{R}^{n+1}\setminus 0$ covering the action of $\mathrm{SL}_{\pm}(n+1,\mathbb{R})$ on $\mathbb{S}^n$.

Suppose M is a projective n–manifold defined by a developing map $\mathrm{dev}_M\colon \widetilde{M}\to\mathbb{S}^n$ with holonomy $\rho\colon \pi_1 M\to \mathrm{SL}_{\pm}(n+1,\mathbb{R})$ and with universal cover $\pi_M\colon \widetilde{M}\to M$. Then pullback gives a line bundle $\pi_\xi\colon \xi\widetilde{M}\to\widetilde{M}$, where

$$\xi\widetilde{M}=\{(\tilde{m},x)\in\widetilde{M}\times(\mathbb{R}^{n+1}\setminus 0):\mathrm{dev}(\tilde{m})=\pi_\xi(x)\},\quad \pi_\xi(\tilde{m},x)=\tilde{m}.$$

Recall that we *defined* $\pi_1 M$ as the group of covering transformations of $\widetilde{M}$. There is an action of $\tau\in\pi_1 M$ on $\xi\widetilde{M}$ given by $\tau\cdot(\tilde{m},x)=(\tau(\tilde{m}),(\rho(\tau))(x))$. The quotient of $\xi\widetilde{M}$ by $\pi_1 M$ is called the *tautological bundle* ξM. There is a natural bundle map $\xi_M\colon \xi M\to M$ given by $\xi_M[\tilde{m},x]=\pi_M(\tilde{m})$. There is also a natural radiant affine manifold structure on ξM with developing map $\mathrm{dev}_\xi\colon \xi\widetilde{M}\to\mathbb{R}^{n+1}\setminus 0$ given by $\mathrm{dev}_\xi(\tilde{m},x)=x$, and with holonomy $\rho\circ(\xi_M)_*$.

There is a radial flow on ξM given by $\Phi_t[m,x]=[m,\exp(-t)\cdot x]$, so ξM is a principal $(\mathbb{R},+)$ bundle over M. Orbits are called *flow-lines*. The *tautological circle bundle* is $\xi_1 M=\xi M/\Phi_1$. It is sometimes called an *affine suspension*. Observe that the developing maps of ξM and $\xi_1 M$ are the same.

Definition 2.2 We make use of the following *covering space trick*. If M is a compact projective manifold (possibly with boundary) then $\xi_1 M$ is a compact affine manifold. Since $\pi_1(\xi_1 M)=\pi_1 M\oplus\mathbb{Z}$, small deformations of the holonomy of M give small deformations of the holonomy of $\xi_1 M$. The latter is compact, so Proposition 1.2 implies there is a nearby affine structure on $\xi_1 M$ with the deformed holonomy, and hence also a nearby affine structure on the noncompact manifold ξM.

Definition 2.3 A *flow function* is a function $c\colon \xi M\to\mathbb{R}$ that is *flow equivariant*, which means that $c(\Phi_t(p))=t+c(p)$ for all p and t.

A flow function determines a section $\sigma\colon M\to\xi M$ of the bundle $\xi_M\colon \xi M\to M$ defined by $c(\sigma(x))=0$. Conversely, a section σ determines a flow function c via $c(x)=-t$ if $\Phi_t(x)=\sigma(\pi x)$. So the negative of the flow function is the amount of time it takes a point to flow onto this section.

We will mostly be concerned with the situation where $\Omega\subset\mathbb{RP}^n$ is properly convex and $\mathrm{dev}_M\colon \widetilde{M}\to\Omega$ is injective. In this case dev_ξ is a diffeomorphism onto the cone $\mathcal{C}\Omega\subset\mathbb{R}^{n+1}\setminus 0$ defined in Section 4. This identifies ξM with $\mathcal{C}\Omega/\Gamma$, where

$\Gamma = \mathrm{hol}(\pi_1 M)$. Moreover, dev_M identifies $\widetilde{M}$ with a subset of $\mathbb{S}^n$. Using these identifications, $\xi_M\colon \xi M \to M$ is covered by π_ξ.

3 Hessian metrics and convexity

The ideas in this section go back to Koszul [24; 25], and we have followed the exposition in Shima and Yagi [34]. However our notation and terminology are somewhat different.

Suppose M is a simply connected affine manifold and dev: $M \to \mathbb{R}^n$ is some developing map. Given $a, b \in M$, a *segment in M from a to b* is a map $\gamma\colon [u, v] \to M$ such that $\gamma(u) = a$ and $\gamma(v) = b$ and $\mathrm{dev} \circ \gamma$ is affine. We often denote such a segment by $[a, b]$. It is a *unit segment* if $[u, v]$ is the unit interval $I := [0, 1]$. A *ray* in M is a nonconstant affine map $\gamma\colon [0, s) \to M$ with $s \in (0, \infty]$ which does not extend to a segment. A *unit triangle* in M is a map $\tau\colon \Delta \to M$ such that $\mathrm{dev} \circ \tau$ is affine, where $\Delta \subset \mathbb{R}^2$ is the triangle with vertices 0, e_1, e_2. The sides of a triangle are segments.

A C^2 function $c\colon M \to \mathbb{R}$ is *Hessian* convex if, for every (nondegenerate) segment $\gamma\colon [-1, 1] \to M$, the function $F = c \circ \gamma$ satisfies $F'' > 0$. Then c defines a Riemannian metric on M via $\|\gamma'(0)\|^2 = F''(0)$ called a *Hessian metric*. See Shima [33] for a discussion.

An affine manifold M has *convex boundary* if, for each $p \in \partial M$, there is an affine coordinate chart (U, ϕ) with $p \in U$ and a closed half-space $H \subset \mathbb{R}^n$ such that $\phi(U) \subset H$ and $\phi(p) \in \partial H$.

Theorem 3.1 *Suppose M is a simply connected affine n–manifold with convex boundary and M has a Hessian metric that makes M into a complete metric space. Then* dev: $M \to \mathbb{R}^n$ *is an affine isomorphism onto a convex subset of* $\mathbb{R}^n$.

Proof It suffices to show that for every pair of segments $[p, a]$ and $[p, b]$ in M there is a segment $[a, b]$ in M. This is because every pair of points in M can be connected by a polygonal path composed of finitely many segments. One may replace a pair of adjacent segments in this path by one segment. It follows that a and b are contained in a single segment. Since dev: $M \to \mathbb{R}^n$ sends segments to segments, if $\mathrm{dev}(a) = \mathrm{dev}(b)$ then the segment in M from a to b maps to a segment in $\mathbb{R}^n$ with both endpoints the same. Hence this segment is a single point, so $a = b$, and dev is injective. Since every pair of points in M are contained in a segment, the same is true of $\mathrm{dev}(M)$, therefore $\mathrm{dev}(M)$ is convex. Thus dev is an affine isomorphism onto a convex set.

Given unit segments $\alpha\colon I \to [p,a]$ and $\beta\colon I \to [p,b]$, let $\mathcal{I} \subset I$ be the set of $t \in I$ such there is a unit triangle $\tau\colon \Delta \to M$ with vertices $p = \tau(0)$ and $\alpha(t) = \tau(e_1)$ and $\beta(t) = \tau(e_2)$. Then $\mathcal{I}$ is connected and contains 0. It suffices to show $\mathcal{I} = I$, since then $\gamma(t) = \tau(te_1 + (1-t)e_2)$ is a segment containing a and b.

Since ∂M is convex, it easily follows from the standard argument about sets with convex boundary that $\mathcal{I}$ is open. To show $\mathcal{I}$ is closed we may assume $\mathcal{I} = [0,1)$ by reparametrizing. After this reparametrization, τ is defined on the interior of Δ and also on the two sides given by α and β since $a, b \in M$. However, τ might not be defined on part of the side connecting e_1 to e_2.

The Hessian metric is given by some function $c\colon M \to \mathbb{R}$. Given any segment γ in M, define $\ell(\gamma)$ to be its length. If γ is a unit segment and $F = c \circ \gamma$, then

$$\ell(\gamma) = \int_0^1 \sqrt{F''(t)}\, dt.$$

By the Cauchy–Schwarz inequality for L^2,

$$\ell(\gamma) \le \left(\int_0^1 F''(t)\, dt\right)^{1/2} \left(\int_0^1 dt\right)^{1/2} \le \sqrt{|F'(1)| + |F'(0)|}.$$

For $s \in [0,1)$ there is a unit segment γ_s given by $\gamma_s(t) = \tau\big(s(te_1 + (1-t)e_2)\big)$ with endpoints $\alpha(s)$ and $\beta(s)$. By the triangle inequality,

$$d(p, \gamma_s(t)) \le d(p, \gamma_s(0)) + d(\gamma_s(0), \gamma_s(t)) \le \ell(\alpha) + \ell(\gamma_s).$$

The function $G(s,t) = \tau\big(s(te_1 + (1-t)e_2)\big)$ is defined and smooth for all (s,t) in the domain $[0,1) \times [0,1] \cup \{1\} \times \{0,1\}$. By compactness, there is $K > 0$ such that $|\partial G/\partial t| \le K$ for all $s \in [0,1]$ and $t \in \{0,1\}$. It follows that for all $s \in [0,1)$ and $t \in [0,1]$ we have

$$d(p, \gamma_s(t)) \le \ell(\alpha) + \sqrt{2K} =: R.$$

Since the metric on M is complete, the ball $P \subset M$ with center p and radius R is compact and contains all the segments γ_s with $s \in [0,1)$. It follows that γ_s converges to a segment $\gamma_1 \subset P$ as $s \to 1$, so $1 \in \mathcal{I}$. □

Definition 3.2 If M is a projective n–manifold a *convexity function for M* is a Hessian-convex flow function $c\colon \xi M \to \mathbb{R}$. It is *complete* if the Hessian metric given by c is complete.

The flow-equivariance of c implies the radial flow acts by isometries of the Hessian metric on ξM given by c. The 1–form dc is preserved by the flow and therefore is the pullback of a 1–form α on $\xi_1 M$. Koszul works with α but we work with c.

The *Hilbert metric* on a properly convex subset $\Omega \subset \mathbb{RP}^n$ is a Finsler metric given by the *Hilbert–Finsler norm* on $T_x\Omega$; see Papadopoulos and Troyanov [31] and Marquis [30]. For the definition of *Hessian-convex hypersurface* see the start of Section 8. The next result is that the flow function c is Hessian-convex if and only if the level set $c^{-1}(0)$ is a Hessian-convex hypersurface that is convex in the *backward* direction of the flow, ie $c^{-1}(-\infty, 0]$ is convex.

Lemma 3.3 *Suppose M is properly convex and $N = \xi M$ and $\|\cdot\|$ is the Hilbert–Finsler norm on $T_x N$. Suppose $c\colon N \to \mathbb{R}$ is a flow function and $S = c^{-1}(0)$. Then at $x \in S$ there is a splitting $T_x N = V \oplus E$ which is orthogonal with respect to $Q := D^2_x c$, where $V = \ker d_x c \subset T_x N$ is the tangent hyperplane to the hypersurface S and $E = \langle e \rangle$, where $e = \Phi_0'(x)$ is a tangent vector to the flow.*

Moreover, $Q(e,e) = \|e\|^2 = 1$. Thus, if $\kappa \in [0,1]$ then $Q \geq \kappa \|\cdot\|^2$ if and only if $Q|_V \geq \kappa\big(\|\cdot\||_V\big)^2$.

In particular, c is Hessian-convex if and only if S is a Hessian-convex hypersurface that is convex in the backward *direction of the radial flow.*

Proof This is a local question so it suffices to assume ξM is a properly convex cone in $\mathbb{R}^{n+1} \setminus 0$, and S is a hypersurface, and the radial flow is $\Phi_t(x) = \exp(-t)\cdot x$. Since c is a flow function, $c(\Phi_t(x)) = c(x) + t$. This implies $c(s\cdot x) = c(x) - \log s$. From this it follows that $D^2_x c(e, v) = 0$ for all $v \in V$, which proves the Q–orthogonality of the direct sum.

The Hilbert–Finsler norm on $(0,\infty)$ is ds/s. The radial flow on $(0,\infty)$ is $\Phi_t(s) = \exp(-t)s$, so $e = \Phi_0'(s) = s\cdot \partial/\partial s$ and $\|e\| = 1$. Moreover,

$$Q(e,e) = s^2 Q\Big(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\Big) = s^2 \frac{d^2(-\log s)}{ds^2} = 1.$$

Observe that $Q|_V$ is positive definite if and only if S is Hessian-convex in the backward direction of the flow. □

Theorem 3.4 *Suppose M is a projective manifold with* (possibly empty) *convex boundary and $c\colon \xi M \to \mathbb{R}$ is a complete convexity function. Then M is properly convex.*

Proof By Theorem 3.1, dev: $\xi\widetilde{M} \to \mathbb{R}^{n+1} \setminus 0$ is injective and the image is a convex cone $\mathcal{C} \subset \mathbb{R}^{n+1}$. It suffices to show that $\Omega := \pi_\xi(\mathcal{C})$ is properly convex. Let $\pi\colon \xi\widetilde{M} \to \xi M$ be the universal cover and $\widetilde{c} = c \circ \pi$. The function $f = \widetilde{c} \circ \mathrm{dev}^{-1}\colon \mathcal{C} \to \mathbb{R}$ is a complete convexity function: it is strictly convex, and the hypersurfaces $S_t = f^{-1}(t)$ are connected, and strictly convex, and foliate $\mathcal{C}$. The radial flow on $\xi\widetilde{M}$ is conjugate to the radial flow $\Phi_t(x) = \exp(-t) \cdot x$ on $\mathbb{R}^{n+1}$, so $\Phi_s(S_t) = S_{t+s}$. Define $S := S_0$.

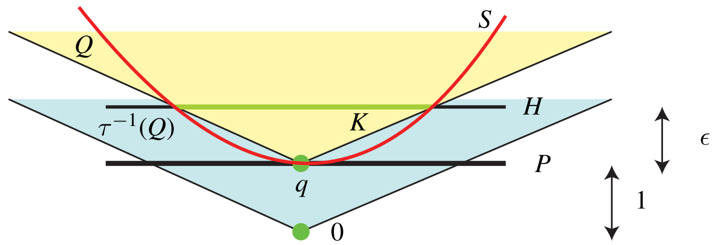


Figure 1: Flowing S backward

Let q be a point in the interior of S. We can choose coordinates in $\mathbb{R}^{n+1}$ so that S is tangent at $q = (1, 0, \dots, 0) = e_1$ to the hyperplane P given by $x_1 = 1$ and S lies on the opposite side of P to 0.

The sublevel set $W = f^{-1}(-\infty, 0] = \bigcup_{t \le 0} \Phi_t(S) \subset \mathcal{C}$ is obtained by flowing S backward. Let H be the hyperplane $x_1 = 1 + \epsilon$. Refer to Figure 1. We do not know that S is *properly* embedded in $\mathbb{R}^{n+1}$. However, if $\epsilon > 0$ is small enough, we can work in a chart for a small neighborhood of $\mathrm{dev}^{-1}(q)$ in $\xi\widetilde{M}$ and see that $K = H \cap W$ is a compact convex set and $\partial K = H \cap S$.

Let Q be the convex cone consisting of the set of rays starting at q and intersecting K. Since $q \in \partial W = S$ and W is convex it follows that Q contains the subset of W above H. Unit vertical translation upwards $\tau\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is given by $\tau(x) = x + e_1$. Note that $\tau(Q) \subset Q$. Since we can assume $\epsilon < 1$ it follows that $\tau(S)$ is above H, therefore Q contains $\tau(S)$. Hence $\tau^{-1}(Q)$ contains S. Since $\tau^{-1}(Q)$ is the cone from 0 of $\tau^{-1}(K)$, it is preserved by Φ, so it contains the entire orbit $\Phi \cdot S = \mathcal{C}$. It follows that $\Omega = \pi_\xi(\mathcal{C}) \subset \mathbb{RP}^n$ is contained in $\pi_\xi(\tau^{-1}(K))$. Since $\tau^{-1}(K)$ is a compact convex set in $x_n = \epsilon$, it follows that Ω is properly convex. □

4 The characteristic convexity function

In this section, $V = \mathbb{R}^{n+1}$ and $\Omega \subset \mathbb{S}(V) = \mathbb{S}^n$ is an open properly convex set. The open convex cone $\mathcal{C}\Omega \subset V$ consists of all $t \cdot v$ with $v \in \Omega$ and $t > 0$. The *dual* *cone* $\mathcal{C}\Omega^* \subset V^*$ is the set of all $\phi \in V^*$ with $\phi(x) > 0$ for all $x \in \mathcal{C}\overline{\Omega}$. The *dual domain* is $\Omega^* = \mathbb{P}(\mathcal{C}\Omega^*) \subset \mathbb{P}(V^*)$. The *characteristic function* $\chi = \chi_\Omega \colon \mathcal{C}\Omega \to \mathbb{R}^+$ of Koecher [23] and Vinberg [38] is defined by

$$\chi(x) = \int_{\mathcal{C}\Omega^*} e^{-\psi(x)}\, d\psi,$$

where $d\psi$ is a fixed choice of Euclidean volume form on V^*. This function is real analytic and nonnegative, and $\chi(tx) = t^{-(n+1)}\chi(x)$ for $t > 0$. More generally, if A is in the subgroup $\mathrm{GL}(\mathcal{C}\Omega) \subset \mathrm{GL}(V)$ that preserves $\mathcal{C}\Omega$, then $\chi(Ax) = (\det A)^{-1}\chi(x)$. The level sets of χ, called *characteristic hypersurfaces*, are smooth, convex, and meet each ray in $\mathcal{C}\Omega$ once transversely. The *characteristic section* is the map $\sigma_\Omega \colon \Omega \to \mathcal{C}\Omega$ given by

$$\sigma_\Omega(x) = x \cdot (\chi(x))^{1/(n+1)}.$$

It has image *the characteristic hypersurface* $S_\Omega = \chi^{-1}(1)$. The radial flow $\Phi_t(x) = e^{-t} \cdot x$ on V preserves $\mathcal{C}\Omega$ and there is a flow function on $\mathcal{C}\Omega$ given by

$$c = c_\Omega = (n+1)^{-1} \log \chi.$$

The Hessian $D^2 c$ is a positive definite quadratic form at each point of $\mathcal{C}\Omega$ and gives a complete metric on $\mathcal{C}\Omega$. Thus $c_\Omega \colon \mathcal{C}\Omega \to \mathbb{R}$ is a complete convexity function called the *characteristic convexity function*. A reference for the above is Goldman [18].

If $\Gamma \subset \mathrm{SL}_\pm(\mathcal{C}\Omega)$ is the holonomy of a properly convex manifold $M = \Omega/\Gamma$ with developing map dev, then ξM is identified with $\mathcal{C}\Omega/\Gamma$. Since c_Ω is preserved by Γ, it covers a map $c_{\mathrm{dev}} = c_M \colon \xi M \to \mathbb{R}$. This is a convexity function for M called the *characteristic convexity function* for M.

Definition 4.1 The subspace $\mathcal{D}ev_c(M) \equiv \mathcal{D}ev_c(M, \mathbb{P}_+) \subset \mathcal{D}ev(M, \mathbb{P}_+)$ consists of the developing maps of properly convex structures for which ∂M is strictly convex.

Proof of Theorem 0.2 when M is closed Suppose M is properly convex with holonomy ρ and $c_M \colon \xi M \to \mathbb{R}$ is a characteristic convexity function. If ρ' is close to ρ then, by Proposition 1.2, there is a radiant affine manifold N_1 with holonomy ρ' and a diffeomorphism $f \colon \xi_1 M \to N_1$ that is close to an affine map. Taking infinite cyclic covers gives a map $F \colon \xi M \to N$ that is close to affine. The hypersurface $S = c^{-1}(0) \subset \xi M$ maps to a hypersurface in N. Since S is compact, Hessian-convex and transverse to the radial flow, if F is close enough in $C^2(\xi M, N)$ to affine, then $F(S) \subset N$ is Hessian-convex, and transverse to the radial flow Φ_N on N. This section of the radial flow defines a convexity function on N by Lemma 3.3. This convexity function is complete because N_1 is compact and every Riemannian metric on a compact manifold is complete. It follows from Theorem 3.4 that N/Φ_N is properly convex. □

From here until Definition 4.2 we allow Ω to have boundary $\partial\Omega$ that is an open subset of $\mathrm{Fr}(\Omega)$. Let $\mathfrak{C}$ be the set of closed subsets of $\mathbb{S}^n$ equipped with the Hausdorff topology. Let $\mathcal{P}$ be the set of properly convex n–manifolds $\Omega \subset \mathbb{S}^n$ with (possibly empty) strictly convex boundary. There is an injective map $\iota\colon \mathcal{P} \to \mathfrak{C} \times \mathfrak{C}$ defined by $f(\Omega) = (\mathrm{cl}(\Omega), \mathrm{cl}(\partial\Omega))$. The *Hausdorff boundary topology* on $\mathcal{P}$ is the subspace topology given by this embedding. Thus a neighborhood of Ω consists of all $\Omega' \in \mathcal{P}$ close to Ω such that $\partial\Omega'$ is also close to $\partial\Omega$. This topology is given by a metric.

Definition 4.2 The *strong geometric topology* on $\mathcal{D}ev_c(M)$ is the smallest refinement of the geometric topology such that the map $\mathcal{D}ev_c(M) \to \mathcal{P}$ given by $\mathrm{dev} \mapsto \mathrm{Im}(\mathrm{dev})$ is continuous.

If M is closed, the strong geometric topology equals the geometric topology because fixed points of elements of the holonomy are dense in $\partial\,\mathrm{Im}(\mathrm{dev})$. In general, two developing maps are close in this topology if they are close in the C^∞ topology on a large compact set in the universal cover of the interior and, in addition, their images are close in the above sense. This can be expressed more simply using basepoints in the space of developing maps as in Definition 1.3:

Suppose $\mathrm{dev}_\rho \in \mathcal{D}ev_c(M)$ and $\rho = \mathcal{H}ol(\mathrm{dev}_\rho)$ and $\Gamma = \rho(\pi_1 M) \subset \mathrm{SL}_\pm(n+1,\mathbb{R})$ and $\Omega_\rho = \mathrm{Im}(\mathrm{dev}_\rho) \subset \mathbb{S}^n$. *Choosing* dev_ρ *as a basepoint* means to replace M by Ω_ρ/Γ. Thus $\mathrm{dev}_\rho = i\colon \widetilde{M} \hookrightarrow \mathbb{S}^n$ is now the inclusion. Then $\mathrm{dev}_\sigma \in \mathcal{D}ev_c(M)$ *is close to* dev_ρ *in the strong geometric topology* if the restrictions of dev_σ and i are close in $C^\infty_w(\mathrm{int}(\widetilde{M}), \mathbb{S}^n)$ and $\Omega_\sigma = \mathrm{Im}(\mathrm{dev}_\sigma)$ is close to Ω_ρ in $\mathcal{P}$.

There is a similar notion for the radiant affine manifolds. The radiant affine manifold $N = \mathcal{C}\Omega_\rho/\Gamma$ is $\mathbb{L}$–equivalent to ξM_ρ. The developing map for N is the inclusion $i = \mathrm{dev}^\xi_\rho\colon \mathcal{C}_\rho\Omega \hookrightarrow \mathbb{R}^{n+1}$ and $\mathrm{dev}^\xi_\rho \in \mathcal{D}ev(N,\mathbb{L})$. *A nearby developing map* $\mathrm{dev}^\xi_\sigma \in \mathcal{D}ev(N,\mathbb{L})$ *in the strong geometric topology* is one such that the restrictions of dev^ξ_σ and i are close in $C^\infty_w(\mathrm{int}(\mathcal{C}\Omega_\rho), \mathbb{R}^{n+1})$ and in addition $\mathcal{C}\Omega_\sigma$ is close to $\mathcal{C}\Omega_\rho$ in $\mathcal{P}$.

Let $\mathcal{P}' \subset \mathcal{P}$ be the subspace of open properly convex sets. For $K \subset V$ define $\mathcal{P}'(K) = \{\Omega \in \mathcal{P}' : K \subset \mathcal{C}\Omega\}$. The map $\mathcal{P}' \to \mathcal{P}'$ given by $\Omega \mapsto \Omega^*$ is continuous.

Lemma 4.3 *If* $K \subset \mathbb{R}^{n+1}\backslash 0$ *is compact, then the function* $\overline{\chi}\colon \mathcal{P}'(K) \to C^\infty(K)$ *defined by* $\overline{\chi}(\Omega) = \chi_\Omega|_K$ *is continuous.*

Proof Since both topologies are metrizable, it suffices to show that the image of a convergent sequence converges. Suppose the sequence $\Omega_k \in \mathcal{P}'(K)$ converges to $\Omega_\infty \in \mathcal{P}'(K)$, and denote the respective characteristic functions by χ_k and χ_∞. Define the smooth function $h\colon V \times V^* \to \mathbb{R}$ by $h(x,\phi) = \exp(-\phi x)$. Then for $x \in K$, if ∂^α is an n^{th} order mixed partial derivative on V, then $\partial^\alpha h(x,\phi) = p(\phi)h(x,\phi)$, where $p(\phi)$ is a monomial of degree n in the coordinates of ϕ. Let $U = \Omega_\infty^* \triangle \Omega_k^*$ be the symmetric difference; then

$$|\partial^\alpha \chi_\infty(x) - \partial^\alpha \chi_k(x)| \le \int_{\mathcal{C}U} |p(\phi)h(x,\phi)|\, d\phi.$$

Since $K \subset \mathcal{C}(\Omega_k \cap \Omega_\infty)$, it follows that $\phi(x) > 0$ for all $x \in K$ and $\phi \in \mathcal{C}U$. Now $p(\phi)$ is polynomial in ϕ and $h(x,\phi)$ is exponential in ϕ, so $p(\phi)h(x,\phi) \to 0$ exponentially fast as $\phi \to \infty$ in $\mathcal{C}U$. It follows that if U is small enough, then $|\partial^\alpha \chi_\infty - \partial^\alpha \chi_k| < \epsilon$ on K. See Faraut and Korányi [16, I.3.1] for more details. □

It follows that nearby properly convex manifolds (without boundary) have nearby characteristic convexity functions:

Lemma 4.4 *Suppose $\partial M = \varnothing$. The map $\mathcal{D}ev_c(M) \to C_w^\infty(\xi M)$ given by* $\mathrm{dev} \mapsto c_{\mathrm{dev}}$ *is continuous.* (Here, the strong geometric topology is used on $\mathcal{D}ev_c(M)$.)

Proof If $\mathrm{dev}, \mathrm{dev}' \in \mathcal{D}ev_c(M)$ are close in the strong geometric topology then $\Omega' = \mathrm{Im}(\mathrm{dev}')$ is close to $\Omega = \mathrm{Im}(\mathrm{dev})$ in $\mathcal{P}$. By Lemma 4.3, the restrictions to K of χ_Ω and $\chi_{\Omega'}$ are close. Composing with log shows that c_Ω and $c_{\Omega'}$ are close on K. Thus the characteristic convexity functions c_{dev} and $c_{\mathrm{dev}'}$ are close. □

We wish to give universal bounds on the derivatives of certain real-valued functions defined on radiant affine manifolds of the form $N = \mathcal{C}\Omega/\Gamma$. If M is a smooth manifold and $f \in C^\infty(M)$ is a smooth function, then the k^{th} derivative $D^k f_x$ at $x \in M$ is a symmetric k–linear map on the vector space $V = T_x M$ (an element of $\mathrm{Hom}(\mathrm{Sym}^k(V), \mathbb{R})$). Given a norm on V we get an operator norm $\|D^k f_x\|$ defined as the infimum of K for which $|D^k f_x(v_1, \dots, v_k)| \le K\|v_1\| \cdots \|v_k\|$. In our case, $M = \mathcal{C}\Omega$ is properly convex, and hence a Finsler manifold using the Hilbert metric on $\mathcal{C}\Omega$. This gives a norm $\|\cdot\|_{\mathcal{C}\Omega}$ called the *Hilbert–Finsler norm* on the tangent space to $\mathcal{C}\Omega$. There is a corresponding operator norm. The group $\mathrm{GL}(\mathcal{C}\Omega)$ acts by isometries of this norm, which therefore pushes down to a norm on the tangent space of $N = \mathcal{C}\Omega/\Gamma$.

Given a point $x \in \mathcal{C}\Omega$, there is a *Benzécri chart* τ for $\mathcal{C}\Omega$ (see Theorem 9.1) centered on x. This chart determines a Euclidean metric d_E on $\mathcal{C}\Omega$, and there is also the Hilbert metric $d_H = d_{\mathcal{C}\Omega}$. There is a constant $K > 0$ depending only on dimension such that in the ball of d_H–radius 1 around x we have $K^{-1} \cdot d_E \le d_H \le K \cdot d_E$.

It follows that universal bounds on operator norms using the Hilbert metric give bounds in the Euclidean metric for Benzécri coordinates, and vice-versa. Thus we may regard these universal bounds as bounds on ordinary partial derivatives of functions defined in a small neighborhood of the origin in $\mathbb{R}^n$ by means of Benzécri coordinates. We now use Benzécri's compactness theorem (Corollary 9.2) with Lemma 4.3 to provide uniform bounds on various properties of characteristic functions.

The restriction of the Hessian metric D^2c to the characteristic hypersurface $S = S_\Omega$ is a Riemannian metric that is preserved by $\mathrm{SL}_\pm(\mathcal{C}\Omega)$. If $M = \Omega/\Gamma$ is a properly convex manifold, then radial projection gives a natural identification $\widetilde{M} \equiv S$ and this puts a Riemannian metric on M called the *induced metric*. The following seems to be folklore:

Corollary 4.5 (bounded curvature) *For each dimension $n > 0$ there is $k_n > 0$ such that if M is a properly convex projective manifold of dimension n, then all sectional curvatures κ of the induced metric on M satisfy $|\kappa| < k_n$. Moreover, the induced metric is k_n–bilipschitz equivalent to the Hilbert metric, and is therefore complete.*

Proof If the first assertion is false, there is a sequence $M_k = \Omega_k/\Gamma_k$, a point $x_k \in M_k$ and a sectional curvature $\kappa > k$ at x_k. By Benzécri compactness (Corollary 9.2), we may assume these domains are in Benzécri position (Theorem 9.1) with $x_k = 0$ and $\Omega_k \to \Omega_\infty$. The sectional curvature is given by a function that is a formula involving various partial derivatives of the flow function c. By Lemma 4.3, these functions converge to some (finite) sectional curvature for M_∞, a contradiction. This also proves the bilipschitz result. □

Lemma 4.6 (uniform Hessian-convexity) *For each dimension n there is $\kappa = \kappa(n) > 0$ with the following property: Suppose $\Omega \subset \mathbb{RP}^n$ is open and properly convex and $c\colon \mathcal{C}\Omega \to \mathbb{R}$ is the characteristic convexity function. Then $D^2c \ge \kappa \|\cdot\|^2_{\mathcal{C}\Omega}$ everywhere.*

Proof Since c is preserved by each element of $\mathrm{GL}(\mathcal{C}\Omega)$ up to adding a constant, it suffices to show there is κ such that the result holds at the center of every Benzécri domain $\Omega = \mathbb{S}^n \cap \mathcal{C}\Omega$. The set of all such domains is compact (Corollary 9.2) and, by

Lemma 4.3, the characteristic function varies smoothly with the Benzécri domain, so the result follows. □

If $f\colon (-\epsilon,\epsilon) \to \mathcal{C}\Omega$ is an arc parametrized by arc length then $(c \circ f)''$ is a *second directional derivative*. The conclusion can be rephrased as $(c \circ f)'' \ge \kappa$ for every second directional derivative. We will abuse notation and write this as $c'' \ge \kappa$.

Suppose B is a properly convex submanifold of a properly convex manifold M, both without boundary; then $\xi B \subset \xi M$. The next result says that far inside B the characteristic convexity functions for B and M are almost equal.

Lemma 4.7 (convexity functions on submanifolds) *Given $\epsilon > 0$ and a dimension n, there is $R = R(\epsilon, n) > 0$ with the following property: Suppose $B \subset M$ are properly convex n–manifolds with characteristic convexity functions c_B and c_M. Let $U \subset B$ be the subset of all x with $d_M(x, M \setminus B) > R$ and define $g = c_M - c_B\colon \xi U \to \mathbb{R}$. Then $\|D^k g\| < \epsilon$ for $0 \le k \le 2$.*

Proof Let $\Omega_U \subset \Omega_B \subset \Omega_M \subset \mathbb{S}^n$ be images of the developing maps of $U \subset B \subset M$, respectively. Since g is constant along rays from the origin in $\mathcal{C}\Omega_U$, it suffices to show the bounds hold for $x \in \Omega_U := \mathbb{S}^n \cap \mathcal{C}\Omega_U$. Choose a Benzécri chart for $\mathcal{C}\Omega_M$ centered on x. In this chart the Euclidean distance between $\partial(\mathcal{C}\Omega_M)$ and $\partial(\mathcal{C}\Omega_B)$ is bounded above by a function $f(R)$ independent of Ω_M and Ω_B and $f(R) \to 0$ as $R \to \infty$. The result now follows using Corollary 9.2 and Lemma 4.3. □

5 Deforming properly convex manifolds rel ends

In this section we prove a version of Theorem 1.7 for convex manifolds. We show that the only obstruction to deforming a properly convex manifold is whether the ends have such a deformation. Suppose M_ρ is a properly convex manifold with holonomy ρ. The main result of this section, Theorem 5.8, is that for representations σ sufficiently close to ρ, if the ends of M_ρ can be deformed to properly convex manifolds with holonomy the restriction of σ, then these deformations can be extended to all of M_ρ to give a properly convex structure M_σ.

Definition 5.1 A Finsler manifold $M = \Omega/\Gamma$ has *controlled ends* if there is a smooth proper function $f\colon M \to [0,\infty)$, called an *exhaustion function*, and $K > 0$ such that $\|Df\|, \|D^2 f\| < K$ in the Finsler norm.

For example, every finite-volume complete hyperbolic manifold has controlled ends. If $C \cong \partial C \times [0,\infty)$ is a horocusp in a hyperbolic manifold M then the horofunction $f(x) = d_M(x, \partial C)$ is an exhaustion function. A similar construction works on a *generalized cusp* (Lemma 6.30). There are complete Riemannian manifolds with no exhaustion function. However:

Proposition 5.2 *Every properly convex manifold has controlled ends.*

Proof By Corollary 4.5, every properly convex manifold admits a complete Riemannian metric that is bilipschitz equivalent to the Hilbert metric and which has bounded sectional curvature. It is a result of Schoen and Yau [32] (see also Tam [36] and Proposition 26.49 in Chow et al [10]) that a complete Riemannian manifold of bounded sectional curvature has a proper function with bounded gradient and Hessian. □

Definition 5.3 A *localization function* on a Finsler manifold M is a smooth function $\lambda\colon M \to [0,1]$ with compact support and $\|D\lambda\|, \|D^2\lambda\| \le 1$.

Corollary 5.4 *If M is a properly convex manifold and $X \subset M$ is compact, then there is a localization function λ on M with $\lambda(X) = 1$.*

Proof By Proposition 5.2, there is an exhaustion function $f\colon M \to [0,\infty)$. By multiplying f by a small positive scalar, we may assume $\|Df\|, \|D^2 f\| < 1$ and that $f(X) \subset [0,1]$. Let $\beta\colon [0,\infty) \to [0,1]$ be a smooth function with compact support and $\beta(t) = 1$ for all $t \le 1$, and $\|\beta'(t)\|, \|\beta''(t)\| \le \frac{1}{10}$ for all t. Then $\lambda = \beta \circ f$ has compact support and $\lambda(X) = 1$. By the chain rule, $\|D\lambda\|, \|D^2\lambda\| \le 1$. □

Suppose $M = A \cup \mathcal{B}$ is a connected n–manifold and A is a compact submanifold with $\partial A = \partial M \sqcup \partial\mathcal{B}$ and $\mathcal{B}$ has k components B_i with $1 \le i \le k$ such that $B_i = \partial B_i \times [0,\infty)$. By Definition 1.6, there is a relative holonomy map

$$\mathcal{E}_{\mathbb{P}}\colon \mathcal{D}ev(M,\mathbb{P}) \to \mathcal{R}el\mathcal{H}ol(M,\mathcal{B},\mathbb{P}).$$

The subspace $\mathcal{D}ev_c(M,\mathbb{P}) \subset \mathcal{D}ev(M,\mathbb{P})$ consists of the developing maps of properly convex structures for which ∂M is strictly convex. The subspace $\mathcal{R}el\mathcal{H}ol_e(M,\mathcal{B},\mathbb{P}) \subset \mathcal{R}el\mathcal{H}ol(M,\mathcal{B},\mathbb{P})$ consists of the data for which each B_i is properly convex and ∂B_i is strictly convex. Then $\mathcal{D}ev_e(M,\mathbb{P}) = \mathcal{E}_{\mathbb{P}}^{-1}\, \mathcal{R}el\mathcal{H}ol_e(M,\mathcal{B},\mathbb{P})$ consists of developing maps for which these ends are properly convex with strictly convex boundary. Finally, $\mathcal{D}ev_{\mathrm{ce}}(M,\mathbb{P}) = \mathcal{D}ev_c(M,\mathbb{P}) \cap \mathcal{D}ev_e(M,\mathbb{P})$ is the subspace of developing maps for properly convex structures on M, with ∂M strictly convex and for which each B_i is properly convex and ∂B_i is strictly convex. The following is well known for manifolds of negative sectional curvature; see Proposition 2.3 in Baker and Cooper [1].

Lemma 5.5 *Suppose M is a properly convex real projective manifold, possibly with boundary, and $B \subset M$ is a properly convex submanifold. Then B is π_1–injective in M.*

Proof The holonomy for B is injective because B is properly convex. The holonomy for B factors through the holonomy for M, therefore the map induced by inclusion $\pi_1 B \to \pi_1 M$ is injective. □

Theorem 5.6 *$\mathcal{E}_{\mathbb{P}}\colon \mathcal{D}ev_{\mathrm{ce}}(M,\mathbb{P}) \to \mathcal{R}el\mathcal{H}ol_e(M,\mathcal{B},\mathbb{P})$ is open using the geometric topology on the domain and the strong geometric topology on the codomain.*

Proof Initially assume M has no boundary. Given a developing map in $\mathcal{D}ev_{\mathrm{ce}}(M,\mathbb{P})$, the first step is to show a nearby relative holonomy is given by a nearby (possibly not convex) projective structure that has the given end data. Then this structure is shown to be properly convex.

By Lemma 5.5, each component of $\mathcal{B}$ is π_1–injective in M. It follows from Theorem 1.7 that $\mathcal{E}_{\mathbb{P}}\colon \mathcal{D}ev(M,\mathbb{P}) \to \mathcal{R}el\mathcal{H}ol(M,\mathcal{B},\mathbb{P})$ is open using the geometric topologies in domain and codomain. Hence the restriction $\mathcal{E}_{\mathbb{P}}\colon \mathcal{D}ev_e(M,\mathbb{P}) \to \mathcal{R}el\mathcal{H}ol_e(M,\mathcal{B},\mathbb{P})$ is also open with these topologies. Thus it is open using the strong geometric topology (which is finer than the geometric topology) on the codomain and the geometric topology on the domain. The *end geometric topology* on $\mathcal{D}ev_e(M,\mathbb{P})$ is defined to be the smallest refinement of the geometric topology such that $\mathcal{E}_{\mathbb{P}}$ is continuous. Then $\mathcal{E}_{\mathbb{P}}$ is open and continuous with the end geometric topology on the domain and the strong geometric topology on the codomain. This completes the first step.

As usual we will assume that $\mathcal{B} = B$ is connected. It suffices to show that $\mathcal{D}ev_{\mathrm{ce}}(M,\mathbb{P})$ is open in $\mathcal{D}ev_e(M,\mathbb{P})$ with respect to the end geometric topology. A neighborhood $\mathcal{U} \subset \mathcal{D}ev_e(M,\mathbb{P})$ of dev_ρ in this topology consists of all developing maps dev_σ that are nearby in $C^\infty_w(\widetilde{M},\mathbb{RP}^n)$ and in addition have the property that $\mathrm{dev}_\sigma(\widetilde{B})$ is close to $\mathrm{dev}_\rho(\widetilde{B})$ in $\mathcal{P}$.

Suppose $\mathrm{dev}_\rho \in \mathcal{D}ev_{\mathrm{ce}}(M,\mathbb{P})$ has holonomy ρ and $\mathrm{dev}_\sigma \in \mathcal{U}$ has holonomy σ. The corresponding projective structures on M are denoted by M_ρ and M_σ. We must show

$\text{dev}_\sigma \in \mathcal{D}ev_c(M, \mathbb{P})$. To do this we construct a complete convexity function on the tautological bundle ξM_σ. It then follows that M_σ is properly convex by Theorem 3.4.

In this sketch various manifolds should be replaced by the corresponding tautological line bundles, but for ease of exposition we do not do this. There are convexity functions for M_ρ and B_σ. If ρ is close to σ then there is a diffeomorphism $M_\rho \to M_\sigma$ that is close to the identity over a large compact set K whose complement is far out in the cusp B. The convexity function on M_σ is obtained by using the one for M_ρ over most of K, and the one for B_σ outside K. We slowly transition from one function to the other over $\partial K \times [0,1]$ using a localization function to give a convex combination that changes in the $[0,1]$ direction. This ends the sketch.

We will use M_ρ as a basepoint for $\mathcal{D}ev_c(M)$ as in Definition 1.3; see also Definition 4.2. Thus we replace M by M_ρ and will usually omit the subscript ρ. Then $\widetilde{M} = \Omega_\rho \subset \mathbb{S}^n$ and $\text{dev}_\rho\colon \Omega_\rho \hookrightarrow \mathbb{S}^n$ is the inclusion map. Similarly, we use $\xi M := \mathcal{C}\Omega_\rho/\Gamma$ as a basepoint for $\mathcal{D}ev(\xi M_\rho, \mathbb{L})$ and write this as $\mathcal{D}ev(\xi M, \mathbb{L})$. Then $\xi \widetilde{M}_\rho = \mathcal{C}\Omega_\rho \subset \mathbb{R}^{n+1}$.

We use the Hilbert–Finsler norm $\|\cdot\|$ on ξM to calculate operator norms. Recall $\xi_1 M = \xi M/\Phi_1$ is the tautological circle bundle and has an infinite-cyclic cover ξM. Let $\kappa = \kappa(\dim(M)) > 0$ be the lower bound on the Hessian of characteristic functions given by Lemma 4.6 and $\epsilon = \frac{1}{10}\kappa$. Let $R = R(\epsilon, \dim(\xi M))$ be the constant given by Lemma 4.7 and $K \subset M$ a compact, connected submanifold such that $\xi_1 K$ contains the R–neighborhood of $\xi_1 A$ in $\xi_1 M$. Hence the characteristic functions $c_{\rho,B}$ and $c_{\rho,M}$ are ϵ–close in $C^2(\xi(M \setminus K))$.

By Corollary 5.4, there is a localization function $\lambda\colon \xi_1 M \to [0,1]$ with $\lambda(\xi_1 K) = 1$ that has support inside a compact, connected submanifold $\xi_1 L$. Define $J = \text{cl}(L \setminus K)$. Then every point in $\xi_1 J$ has distance at least R from $\partial(\xi_1 B)$. All these submanifolds depend on the choice of ϵ. Let $\widetilde{\lambda}\colon \xi M \to [0,1]$ be the function that covers λ. We abuse notation by writing $\widetilde{\lambda}$ as λ. Observe that $\lambda^{-1}(0,1) \subset \xi J$.

Claim 1 *There is a convexity function $c\colon \xi M \to \mathbb{R}$ which equals $c_{\rho,M}$ on ξK and equals $c_{\rho,B}$ on $\xi(M \setminus L)$ and $D^2 c \geq \left(\frac{1}{2}\kappa\right)\|\cdot\|^2$.*

Proof of Claim 1 First blend $c_{\rho,M}$ and $c_{\rho,B}$ inside ξJ using λ to get $f\colon \xi M \to \mathbb{R}$ given by

$$f = \lambda \cdot c_{\rho,M} + (1-\lambda) \cdot c_{\rho,B} = c_{\rho,M} + (1-\lambda) \cdot g,$$

where $g = c_{\rho,B} - c_{\rho,M}$. The map f is well defined even though $c_{\rho,B}$ is only defined on ξB because $1 - \lambda = 0$ outside ξB.

Subclaim $D^2 f \geq \left(\frac{1}{2}\kappa\right)\|\cdot\|^2$.

Outside ξJ this follows from Lemma 4.6 since $f = c_{\rho,M}$ on ξK and $f = c_{\rho,B}$ on $\xi(M \setminus L)$. On ξJ we show this using directional derivatives. By the product rule,

$$f'' = c''_{\rho,M} + g'' - (\lambda'' g + 2\lambda' g' + \lambda g'').$$

Since M_ρ is properly convex, $c''_{\rho,M} \geq \kappa$ by Lemma 4.6. Also $|\lambda|, |\lambda'|, |\lambda''| \leq 1$ because λ is a localization function and $|g|, |g'|, |g''| < \epsilon = \frac{1}{10}\kappa$ on ξJ by definition of R and K, so

$$|g'' - (\lambda'' g + 2\lambda' g' + \lambda g'')| \leq 5\epsilon = \tfrac{1}{2}\kappa.$$

Thus $f'' \geq \frac{1}{2}\kappa$, which proves the subclaim.

The level set $S = f^{-1}(0)$ is Hessian-convex in the backward direction of the flow and is the 0–set of a unique flow function c which coincides with $c_{\rho,B}$ outside ξL. It follows from Lemma 3.3 that $c'' \geq \frac{1}{2}\kappa$ also. This proves Claim 1. □

To avoid a proliferation of notation, and because what we are about to do is similar to what we just did, we reuse notation as follows. We define the new K to be the old L, and the new λ is a localization function on $\xi_1 M$ with $\lambda(\xi_1 K) = 1$, and the new $L \subset M$ is a compact, connected manifold such that $\xi_1 L$ contains the support of λ. Then redefine $J = \mathrm{cl}(L \setminus K)$. Let $E := \mathrm{cl}(M \setminus K) \subset B$. Again we write the lift as $\lambda\colon \xi M \to \mathbb{R}$. There are characteristic convexity functions $c_{\rho,B}\colon \xi B_\rho \to \mathbb{R}$ and $c_{\sigma,B}\colon \xi B_\sigma \to \mathbb{R}$.

Since $\xi_1 L_\rho$ is compact, if $\mathcal{U}$ is small then there is a diffeomorphism $H\colon \xi_1 M_\rho \to \xi_1 M_\sigma$ such that $H|_{\xi_1 L_\rho}$ is close in C^∞ to the identity in the following sense. The map H is covered by $\widetilde{H}\colon \xi \widetilde{M}_\rho \to \xi \widetilde{M}_\sigma$ and the restriction of $\widetilde{H}$ is close to the inclusion $\xi \widetilde{L}_\rho \hookrightarrow \mathbb{R}^{n+1}$ in $C^\infty_w(\xi \widetilde{L}_\rho, \mathbb{R}^{n+1})$. The map $\widetilde{H}$ also covers a map $h\colon \xi M_\rho \to \xi M_\sigma$.

Set $g = (c_{\sigma,B}) \circ h - c_{\rho,B}\colon \xi E_\rho \to \mathbb{R}$. If $\mathcal{U}$ is small enough then, by Lemma 4.4, $c_{\sigma,B}$ and $c_{\rho,B}$ are close on ξJ_ρ. Since $\widetilde{H}$ is close to the identity map and covers h, it follows that $\|D^k g\| < \epsilon$ for $k \in \{0,1,2\}$ everywhere on $\xi_1 J_\rho$. Define $f\colon \xi M_\rho \to \mathbb{R}$ by

$$f = \lambda \cdot c + (1-\lambda) \cdot (c_{\sigma,B}) \circ h.$$

As before, this is well defined.

Claim 2 $f'' \geq \frac{1}{2}\kappa$ *on* ξL_ρ.

Proof of Claim 2 If $\lambda = 1$ then $f'' = c'' \geq \frac{1}{2}\kappa$ by Claim 1. The set where $\lambda < 1$ is contained in ξJ_ρ. On ξJ_ρ we have $c = c_{\rho,B}$, so

$$f = \lambda \cdot c_{\rho,B} + (1-\lambda)\cdot(c_{\sigma,B})\circ h = c_{\rho,B} + (1-\lambda)\cdot g$$

and

$$f'' = c''_{\rho,B} + g'' - (\lambda''\cdot g + 2\lambda' g' + g'').$$

Then $c''_{\rho,B} \geq \frac{1}{2}\kappa$ by Lemma 4.6. As before, $|\lambda|, |\lambda'|, |\lambda''| \leq 1$ and, by the above, $|g|, |g'|, |g''| < \epsilon$. Since $\epsilon < \frac{1}{10}\kappa$, this proves Claim 2. □

Since $\widetilde{H}$ is close to the inclusion in $C^\infty_w(\xi\widetilde{L}_\rho, \mathbb{R}^{n+1})$ it follows that $f\circ h^{-1}$ is Hessian-convex on ξL_σ. Outside this set, $f\circ h^{-1} = c_{\sigma,B}$, which is Hessian-convex. This proves $f\colon \xi M_\sigma \to \mathbb{R}$ is Hessian-convex everywhere.

Again, it follows from Lemma 3.3 that there is a Hessian-convex flow function $c_\sigma\colon \xi M_\sigma \to \mathbb{R}$ defined by $f\circ h^{-1}$. The corresponding Hessian metric on $\xi_1 M_\sigma$ is complete because $\xi_1 L_\sigma$ is compact, so the metric is complete on $\xi_1 L_\sigma$, and outside $\xi_1 L_\sigma$ it is the complete metric given by the properly convex end $\xi_1 B_\sigma$. It follows that the Hessian metric on ξM_σ is also complete. This completes the proof when M has no boundary.

Now suppose M has (compact) boundary and set $P = \operatorname{int}(M)$. Then P is properly convex with a characteristic convexity function $c\colon \xi P_\rho \to \mathbb{R}$. The idea is to shrink M a bit to obtain a submanifold $N \subset P$ with Hessian-convex boundary. The restriction to N of the convexity function for P can be used in the above arguments. It is a complete metric with ∂N at a finite distance.

By Proposition 8.3, there is a submanifold $N \subset M$ with Hessian-convex, compact boundary such that $\operatorname{cl}(M\setminus N)$ is a collar of ∂M. The restriction of c to ξN is a complete convexity function. There is a diffeomorphism $F\colon \xi M \to \xi N$ close to the identity in C^2 that is the identity outside a small collar of $\partial(\xi M)$. Then $c_{\rho,M} := (c|_{\xi N})\circ F\colon \xi M \to \mathbb{R}$ is a complete convexity function. The pullback of the restriction to ξN of the Hilbert metric on ξP is a complete metric on ξM. The proof now proceeds as above to construct a complete convexity function on ξM_σ. □

To apply Theorem 5.6 involves finding deformations of the cusps that are nearby in the strong geometric topology. This involves finding a diffeomorphism from the original cusp to the deformed cusp that is close to projective. To make this task easier we show such a map exists for a small deformation of the holonomy if the deformed domain is close to the original domain.

The *projective Kleinian group space* for a smooth manifold M is

$$\mathcal{K}(M) = \{(\Omega, \rho) \in \mathcal{P} \times \mathrm{Rep}(M) : M \text{ diffeomorphic to } \Omega/\rho(\pi_1 M)\}$$

with topology given by the subspace topology of the product topology on $\mathcal{P} \times \mathrm{Rep}(M)$. This topology is given by a metric. If $(\Omega, \rho) \in \mathcal{K}(M)$ then $\Omega/\rho(\pi_1 M)$ is a properly convex manifold with strictly convex boundary. If M is closed then ρ determines Ω. There is a natural map

$$\mathcal{K}\colon \mathcal{D}ev_c(M, \mathbb{P}) \to \mathcal{K}(M)$$

given by $\mathcal{K}(\mathrm{dev}) = (\mathrm{dev}(\widetilde{M}), \mathrm{hol}(\mathrm{dev}))$.

Proposition 5.7 *Suppose $M \cong \partial M \times [0, \infty)$ is a connected smooth manifold and ∂M is compact. Then $\mathcal{K}$ is a continuous open map for the* strong *geometric topology on $\mathcal{D}ev_c(M, \mathbb{P})$.*

Proof Continuity is obvious. Suppose $\mathrm{dev}_\rho \in \mathcal{D}ev_c(M, \mathbb{P})$ and $\mathcal{K}(\mathrm{dev}_\rho) = (\Omega_\rho, \rho)$ and that $(\Omega_\sigma, \sigma) \in \mathcal{K}(M)$ is close. Then $Q = \Omega_\sigma/\sigma(\pi_1 M)$ is a properly convex manifold. We identify $M \equiv \Omega_\rho/\rho(\pi_1 M)$. It suffices to show there is a diffeomorphism $M \to Q$ which is almost a projective map between large compact sets in the interiors.

By Proposition 8.3, there is a diffeomorphism $M \cong \partial M \times [0, \infty)$ such that $\partial M \times t$ is $\mathrm{dev}_{\rho,M}$–Hessian-convex for all $0 < t \le 1$. For $k > 1$ define $N = \partial M \times [1/k, k]$ and $W = \partial M \times [0, k+1]$ and $E = \partial M \times 1/k$. These are all compact. By Proposition 1.2, there is a $\mathrm{dev}_{\sigma,W} \in \mathcal{D}ev(W, \mathbb{P})$ with holonomy σ that is close to $\mathrm{dev}_{\rho,M}|_W$ over a compact set in $\widetilde{W}$ that covers N.

By Corollary 1.5, we may change $\mathrm{dev}_{\sigma,W}$ by a small isotopy so that there is a projective embedding $f\colon N \to Q$. If σ is close enough to ρ then, since the hypersurface E is Hessian-convex for $\mathrm{dev}_{\rho,M}$, it follows that E is also Hessian-convex for $\mathrm{dev}_{\sigma,W}$. Hence $f(E)$ is Hessian-convex in Q.

Let P be the closure of the component of $Q \setminus f(N)$ that contains ∂Q. Since ∂M is compact, for homology reasons $f(E)$ separates ∂Q from the end of Q, thus P is compact and $\partial P = \partial Q \sqcup f(E)$.

Claim *P is diffeomorphic to $E \times I$.*

Since E is $\mathrm{dev}_{\sigma,N}$–Hessian-convex, there is a nearest-point retraction (using the Hilbert metric on Q) $r\colon P \to E$ with fibers that are lines, and this gives a homeomorphism $P \to E \times I$. By [39], smooth manifolds are PL. The $M \times I$ *theorem* (Hirsch and

Mazur [22]) says that if M is a PL manifold, then every smoothing of $M \times I$ is diffeomorphic to a product. Thus P is diffeomorphic to $E \times I$, which proves the claim.

It follows that P is a collar of ∂Q, so $R = P \cup f(N) \cong E \times [0,k]$ is also a collar of ∂Q. Thus $Q' = \mathrm{cl}(Q \setminus R)$ is diffeomorphic to $E \times [k,\infty)$. Clearly P lies in a small neighborhood of ∂Q. We can now extend f to a diffeomorphism $f\colon M \to Q$ by sending $\partial M \times [0,1/k]$ to P and $\partial M \times [k,\infty)$ to Q'. This is close to a projective map on N. Define $\mathrm{dev}_{\sigma,M}\colon \widetilde{M} \to \mathbb{RP}^n$ by $\mathrm{dev}_{\sigma,M} = \mathrm{dev}_{\sigma,Q} \circ \tilde{f}$. Since f is close to projective over N, it follows that $\mathrm{dev}_{\sigma,M}$ is close to $\mathrm{dev}_{\rho,M}$. □

Suppose $M = A \cup \mathcal{B}$ is a smooth manifold with (possibly empty) boundary and A is a compact submanifold of M with $\partial A = \partial M \sqcup \partial\mathcal{B}$. Suppose $\mathcal{B} = B_1 \sqcup \cdots \sqcup B_k$ has k connected components, and $B_i \cong \partial B_i \times [0,\infty)$. Define the *Kleinian relative holonomy space*

$$\mathcal{R}el\mathcal{H}ol(M,\mathcal{B},\mathcal{K}) \subset \mathrm{Rep}(\pi_1 M) \times \prod_{i=1}^{k} \mathcal{K}(B_i) \tag{1}$$

to be the subset of all $(\rho,(\Omega_1,\rho_1),\dots,(\Omega_k,\rho_k))$ such that $\rho_i = \rho|_{\pi_1 B_i}$. This space has the subspace topology of the product topology.

For each $B_i \subset M$ we fix a choice of some component $\widetilde{B}_i \subset \widetilde{M}$ of the preimage B_i in the universal cover of M. Then $\Omega_i = \mathrm{dev}(\widetilde{B}_i)$ and $\Gamma_i = \mathrm{hol}(\pi_1 B_i)$ gives a point in $\mathcal{K}(B_i)$. This defines the *Kleinian relative holonomy map*

$$\mathcal{E}_{\mathcal{K}}\colon \mathcal{D}ev_{\mathrm{ce}}(M,\mathbb{P}) \to \mathcal{R}el\mathcal{H}ol(M,\mathcal{B},\mathcal{K}).$$

Theorem 5.8 (convex extension theorem) $\mathcal{E}_{\mathcal{K}}\colon \mathcal{D}ev_{\mathrm{ce}}(M,\mathbb{P}) \to \mathcal{R}el\mathcal{H}ol(M,\mathcal{B},\mathcal{K})$ *is open using the geometric topology on the domain.*

Proof This follows immediately from Theorem 5.6 and Proposition 5.7. □

Proof of Theorem 0.3 The map $\gamma\colon (-1,1) \to \mathcal{R}el\mathcal{H}ol(M,\mathcal{B},\mathcal{K})$ defined by

$$\gamma(t) = (\rho_t, (\Omega_1(t), \rho_t|_{\pi_1 B_1}), \dots, (\Omega_k(t), \rho_t|_{\pi_1 B_k}))$$

is continuous by hypothesis (5) of Theorem 0.3. By Theorem 5.8, $\mathcal{E}_{\mathcal{K}}$ is open, and $\gamma(0) \in \mathrm{Im}(\mathcal{E}_{\mathcal{K}})$, thus $\gamma(-\epsilon,\epsilon) \subset \mathrm{Im}(\mathcal{E}_{\mathcal{K}})$ for some $\epsilon > 0$. So for $|t| < \epsilon$ there is $\mathrm{dev}_t \in \mathcal{D}ev_{\mathrm{ce}}(M,\mathbb{P})$ with $\mathcal{E}_{\mathcal{K}}(\mathrm{dev}_t) = \gamma(t)$. Define M_t to be the projective structure on M defined by dev_t. Then M_t is properly convex, with holonomy ρ_t, and ∂M_t is strictly convex. Moreover, the projective structure on M_t restricted to B_i is diffeomorphic to $P_i(t)$ by definition of $\mathcal{E}_{\mathcal{K}}$. □

6 Generalized cusps

A generalized cusp is a certain kind of properly convex projective manifold. The main result of this section is that holonomies of generalized cusps with fixed topology form an open subset in a certain semialgebraic set (Theorem 6.28). This follows from the fact that *a generalized cusp contains a homogeneous cusp* (Theorem 6.3). We then prove the main theorem, Theorem 6.29.

A cusp in a *hyperbolic manifold* viewed as a projective manifold is characterized by being projectively equivalent to an affine manifold that has a foliation by strictly convex hypersurfaces that are images of horospheres, together with a transverse foliation by parallel lines. This characterization does not work in general. Consider the affine manifold $M = U/\Gamma \cong T^2 \times [0,\infty)$, where

$$U = \{(x_1, x_2, x_3) : x_3 \geq x_1^2 + x_2^2 > 0\}$$

and Γ is the cyclic group generated by $(x_1, x_2, x_3) \mapsto (2x_1, 2x_2, 4x_3)$. It has a foliation by tori that are the images of the strictly convex hypersurfaces $z = K(x^2 + y^2)$ for $K \geq 1$, and it has a transverse foliation by vertical lines. However M is *not convex*.

Definition 6.1 A *generalized cusp* is a properly convex manifold $C = \Omega/\Gamma$ homeomorphic to $\partial C \times [0,\infty)$ with ∂C a closed manifold and $\pi_1 C$ virtually nilpotent such that $\partial\Omega$ contains no line segment, ie ∂C is strictly convex. The group Γ is called a *generalized cusp group*.

A *quasicusp* is a properly convex manifold with interior homeomorphic to $Q \times \mathbb{R}$, where Q is a closed manifold and $\pi_1 Q$ is virtually nilpotent.

If Γ contains no hyperbolics, then C is called a *cusp* and Γ is conjugate to a subgroup of $\mathrm{PO}(n,1)$ by Theorem 0.5 in [13]. An example of a quasicusp is Δ/Γ for any discrete subgroup $\Gamma \cong \mathbb{Z}^{n-1}$ of the diagonal group in $\mathrm{SL}(n+1,\mathbb{R})$, where $\Delta \subset \mathbb{RP}^n$ is the interior of an n–simplex that is preserved by Γ.

Definition 6.2 A generalized cusp Ω/Γ is *homogeneous* if $\mathrm{PGL}(\Omega)$ acts transitively on $\partial\Omega$. The group $\mathrm{PGL}(\Omega)$ is called a *(generalized) cusp Lie group*.

For example a cusp in a hyperbolic manifold is homogeneous if and only if it is the quotient of a horoball $\Omega \subset \mathbb{H}^n$. In this case $\mathrm{PGL}(\Omega)$ is conjugate to the subgroup of $\mathrm{PO}(n,1) \cong \mathrm{Isom}(\mathbb{H}^n)$ that fixes one point at infinity. Cusp Lie groups for 3–manifolds are listed in Section 7.

Theorem 6.3 *Every generalized cusp contains a homogeneous generalized cusp.*

There is an equivalence relation on generalized cusps generated by the property that one cusp can be projectively embedded in another. Equivalent cusps have conjugate holonomy. One can shrink a cusp by removing a collar from the boundary. However, sometimes one can remove a submanifold at the other end. For example, there might be a totally geodesic, codimension-1 compact submanifold in the interior of the cusp, which one could cut along. It simplifies matters to do this ahead of time.

Definition 6.4 A generalized cusp C is *minimal* if, for every cusp $C' \subset C$ with $\partial C' = \partial C$, it follows that $C = C'$.

If M is a convex manifold and $X \subset M$ then the *convex hull*, $\mathrm{CH}(X)$, of X is the intersection of all convex submanifolds of M that contain X. Suppose $f\colon S^1 \times (-1,1] \to C$ is a diffeomorphism, and C has a hyperbolic metric such that $\gamma = f(S^1 \times 0)$ is a geodesic and the distance satisfies $d(f(e^{i\theta}, t), \gamma) = |t|$. Thus C is a hyperbolic annulus with one convex boundary component and the other boundary component deleted. Moreover, C is a generalized cusp, and $C' = \mathrm{CH}(\partial C) = f(S^1 \times (0,1])$ is a minimal cusp.

Lemma 6.5 *Every generalized cusp contains a unique minimal cusp. A finite cover of a minimal cusp is minimal.*

Proof Suppose $C = \Omega/\Gamma$ is a generalized cusp. Let Ω' be the convex hull of $\partial\Omega$. Then $\Omega' \subset \Omega$ is properly convex and Γ–invariant and $\partial\Omega' = \partial\Omega$. The cusp $C' = \Omega'/\Gamma$ is the unique minimal cusp contained in C. If M is a finite cover of C' then $M = \mathrm{CH}(\partial M)$, so M is also minimal. □

The following will be used frequently:

Lemma 6.6 *Suppose M is a quasicusp of dimension n, and $P \subset M$ is a convex submanifold, and $\pi_1 P \to \pi_1 M$ is an isomorphism; then:*

(1) *The universal cover $\widetilde{M}$ is contractible.*

(2) *M is an Eilenberg–Mac Lane space: a $K(\pi_1(M), 1)$.*

(3) $H_{n-1}(\pi_1 M; \mathbb{Z}_2) \cong H_{n-1}(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

(4) *If $\dim P < n$ then $\dim(P) = n-1$ and P is a closed manifold.*

(5) *If $P \subset \mathrm{int}(M)$ and $\dim(P) < \dim(M)$ then P separates.*

Proof (1)–(3) follow immediately from the definition. Since P and M are convex, they are aspherical. Hence the inclusion $P \hookrightarrow M$ is a homotopy equivalence. By (3), $H_{n-1}(P;\mathbb{Z}_2) \cong \mathbb{Z}_2$, from which (4) follows.

For (5), it follows from (4) that $\dim P = n-1$. By definition of quasicusp, the interior of M is homeomorphic to $Q\times\mathbb{R}$ for some closed manifold Q. Since P and Q are both closed, for sufficiently large $t\in\mathbb{R}$ it follows that P separates $Q\times t$ from $Q\times(-t)$. □

By Proposition 2.1, the holonomy of a projective structure lifts to $\mathrm{GL}(n+1,\mathbb{R})$, and we will use this lift in what follows.

If V is a finite-dimensional vector space of dimension n then a *(complete) flag for* V is a sequence of subspaces $0 = V_0 < V_1 < \cdots < V_n = V$ with $\dim(V_i) = i$. The subgroup $\mathrm{UT}(n) < \mathrm{GL}(n,\mathbb{R})$ consists of all upper-triangular matrices with positive diagonal entries. A group $\Gamma \subset \mathrm{GL}(n,\mathbb{R})$ is conjugate into $\mathrm{UT}(n)$ if and only if Γ preserves a flag and every weight of Γ is positive.

A *connected* nilpotent subgroup Γ of $\mathrm{GL}(n,\mathbb{C})$ preserves a flag for $\mathbb{C}^n$. However, if Γ is not connected, this need not be true. For example, the quaternionic group of order 8 in $\mathrm{GL}(2,\mathbb{C})$ does not preserve a flag. First we show (Corollary 6.11) that there is a finite-index subgroup of Γ that preserves a flag. The index of a subgroup $H < G$ is written $|G:H|$. A subgroup $H \le G$ is *characteristic* if every automorphism of G preserves H. It is routine to show:

Lemma 6.7 *There exists $h(n,k) > 0$ such that if the group G is generated by k elements, then there is a characteristic subgroup $C \le G$ with $|G:C| \le h(n,k)$ such that if $H \le G$ is any subgroup with index $|G:H| \le n$ then $C \le H$.*

Suppose V is a vector space over $\mathbb{C}$. A *weight* of a subgroup $\Gamma \subset \mathrm{GL}(V)$ is a homomorphism (*character*) $\lambda\colon \Gamma \to \mathbb{C}^*$ such that the *weight space* $E(\lambda)$ and *generalized weight space* $V(\lambda)$ are both nontrivial. Here,

$$E(\lambda) = \bigcap_{\gamma\in\Gamma} \ker(\gamma - \lambda(\gamma)) \quad\text{and}\quad V(\lambda) = \bigcup_{n>0}\bigcap_{\gamma\in\Gamma} \ker(\gamma-\lambda(\gamma))^n.$$

A (generalized) weight space is Γ–invariant. A one-dimensional weight space is the same thing as a one-dimensional Γ–invariant subspace. The vector space V has a *generalized weight decomposition* if $V = \bigoplus V(\lambda)$, where the sum is over all weights.

The group Γ is *polycyclic of (Hirsch) length (at most) k* if there is a subnormal series $\Gamma = \Gamma_k \rhd \Gamma_{k-1} \rhd \cdots \rhd \Gamma_1 \rhd \Gamma_0 = 1$ with Γ_{i+1}/Γ_i cyclic for every i. A subgroup of a polycyclic group of length k is polycyclic of length at most k. Every finitely generated nilpotent group is polycyclic.

Lemma 6.8 *There exists $c = c(n,k)$ such that if $\Gamma < \mathrm{GL}(\mathbb{C}^n)$ is polycyclic of length at most k, then there is a characteristic subgroup $C \le \Gamma$ with $|\Gamma : C| \le c$ and C preserves a one-dimensional subspace of $\mathbb{C}^n$.*

Proof We use induction on k. For $k = 1$ the result follows from Jordan normal form with $c = 1$. Assume the result true for k. Suppose Γ is polycyclic of length $k+1$. Then Γ contains a normal polycyclic group Γ_k of length k with Γ/Γ_k cyclic. There is a characteristic subgroup $C_k \le \Gamma_k$ of index at most $c(n,k)$ that preserves a one-dimensional subspace W.

There is some weight $\lambda\colon C_k \to \mathbb{C}^*$ with W contained in the weight space $E = E(\lambda)$. There are at most n weights for C_k. If θ is an automorphism of C_k then $\lambda \circ \theta$ is a weight for C_k. Since C_k is a characteristic subgroup of Γ_k and Γ_k is normal in Γ, it follows that C_k is preserved by all inner automorphisms of Γ. Thus an inner automorphism of Γ permutes these weights, so an element $\gamma \in \Gamma$ induces a permutation of the weights with order $m \le n!$. Choose $\gamma \in \Gamma$ which generates Γ/Γ_k. Then γ^m induces the identity permutation. Hence the subgroup $\Gamma' = \langle C_k, \gamma^m \rangle$ preserves E. Applying Jordan normal form to $\gamma^m|_E$ gives a one-dimensional subspace of E that is preserved by γ^m. This subspace is also preserved by C_k. Then $|\Gamma : \Gamma'| \le m|\Gamma_k : C_k| \le n! \cdot c(n,k)$ since $m \le n!$. By Lemma 6.7, there is a characteristic subgroup $C \le \Gamma' \le \Gamma$ with $|\Gamma : C| \le c(n,k+1) = h(n! \cdot c(n,k), k+1)$. □

Proposition 6.9 *There exists $d(n,k)$ such that for all polycyclic groups G of length at most k there is a characteristic subgroup $C \le G$ with $|G : C| \le d(n,k)$ such that if $\rho\colon G \to \mathrm{GL}(n,\mathbb{C})$, then $\rho(C)$ preserves a flag in $\mathbb{C}^n$.*

Proof Below we show by induction on n that, for a fixed ρ, there is a subgroup of index at most $e(n,k) = \prod_{i=1}^n c(i,k)$ that preserves a flag. The result follows from Lemma 6.7 with $d(n,k) = h(e(n,k),k)$.

For $n = 1$, the result is clear. By Lemma 6.8, there is a subgroup $\Gamma' < \Gamma = \rho(G)$ of index at most $c(n,k)$ that preserves a one-dimensional subspace $E \subset V = \mathbb{C}^n$. Then Γ' acts on $V/E \cong \mathbb{C}^{n-1}$. By induction there is $\Gamma'' < \Gamma'$ with $|\Gamma' : \Gamma''| \le e(n-1,k)$ that preserves a flag $\mathcal{F}$ in V/E. The preimage of $\mathcal{F}$ in V, together with E, forms a flag for V which is preserved by Γ''. Moreover, $|\Gamma : \Gamma''| = |\Gamma : \Gamma'| \cdot |\Gamma' : \Gamma''| \le c(n,k)e(n-1,k) = e(n,k)$. □

Definition 6.10 Suppose G is a finitely generated, virtually nilpotent group. Let k be the smallest integer such that G is polycyclic of length k. Given $n > 0$, the *n–core of G* is the subgroup $\mathrm{core}(G,n)$ of G that is the intersection of all subgroups of G of index at most $2^n \cdot d(n,k)$.

Clearly, $\mathrm{core}(n,G)$ is a characteristic subgroup of finite index in G that is contained in every subgroup of index at most 2^n in the subgroup $C \le G$ from Proposition 6.9.

Corollary 6.11 *Suppose G is a finitely generated, virtually nilpotent group and $H = \mathrm{core}(G,n)$. Then, for every homomorphism $\rho\colon G \to \mathrm{GL}(n,\mathbb{F})$:*

1. *If $\mathbb{F} = \mathbb{C}$, then $\rho(H)$ preserve a flag in $\mathbb{C}^n$.*
2. *If $\mathbb{F} = \mathbb{R}$ and every weight of $\rho(H)$ is real, then $\rho(H)$ is conjugate into $\mathrm{UT}(n)$.*
3. *If $\mathbb{F} = \mathbb{R}$, then $\rho(G) \in \mathrm{VFG}$ if and only if every weight of $\rho(H)$ is real.*
4. $\mathrm{VFG}(G,n) = \{\rho \in \mathrm{Hom}(G,\mathrm{GL}(n,\mathbb{R})) : \rho(G) \in \mathrm{VFG}\}$ *is semialgebraic.*

Proof Let C be the characteristic subgroup of G given by Proposition 6.9. Then $H \le C$, so (1) follows. For (2) set $U = \mathbb{R}^n$ and $V = U \otimes \mathbb{C}$, so $G \subset \mathrm{GL}(U) \subset \mathrm{GL}(V)$. By (1), $V = \bigoplus V(\lambda)$, where $V(\lambda) = \bigcap_{c\in C} \ker(\rho(c) - \lambda(c))^n$. Observe that $V(\lambda) \subset \mathbb{C}^n$ is given by linear equations that are defined over $\mathbb{R}$ because $\lambda(C) \subset \mathbb{R}$ and $\rho(C) \subset \mathrm{GL}(n,\mathbb{R})$. Thus $V(\lambda)$ is the complexification of $U(\lambda) = \bigcap_{c\in C} \ker(\rho(c) - \lambda(c))^n \subset \mathbb{R}^n$, so $U = \bigoplus U(\lambda)$. Hence $\rho(C)$ preserves a flag in $\mathbb{R}^n$. By replacing C by a certain subgroup, C_0, of index at most 2^n, we may ensure that all real weights are positive. Since $H \le C_0$ it follows that $\rho(H)$ is conjugate into $\mathrm{UT}(n)$, which proves (2). Clearly (2) implies (3). (4) follows from (2) and the observation that the condition that every weight is real is defined by the semialgebraic equations that say every eigenvalue of every element of $\rho(H)$ is real. □

Suppose U is a real vector space and $\Gamma < \mathrm{GL}(U)$ preserves a flag in $V = U \otimes \mathbb{C}$. Then combining each weight λ for V with the complex-conjugate weight $\bar\lambda$ gives a real invariant subspace $U(\lambda,\bar\lambda) = (V(\lambda) + V(\bar\lambda)) \cap U \subset U$ and $U = \bigoplus U(\lambda,\bar\lambda)$. We call $U(\lambda,\bar\lambda)$ a *conjugate generalized weights space*. For each $\gamma \in \Gamma$, the eigenvalues of $\gamma|_{U(\lambda,\bar\lambda)}$ are $\lambda(\gamma)$ and $\bar\lambda(\gamma)$.

Proposition 6.12 *If $P = \Omega/\Gamma$ is a quasicusp of dimension n, then* $\mathrm{core}(\Gamma, n+1)$ *is conjugate into* $\mathrm{UT}(n+1)$. *In particular,* $\Gamma \in \mathrm{VFG}$.

Proof Write $V = \mathbb{R}^{n+1}$, so $\Gamma \subset \mathrm{PGL}(V)$. By Proposition 2.1, we may lift to get $\Gamma \subset \mathrm{GL}(V)$. By Corollary 6.11(1), we can conjugate so that $H = \mathrm{core}(\Gamma, n+1)$ is contained in the upper-triangular subgroup in $\mathrm{GL}(n+1,\mathbb{C})$. We replace Γ by H. Then $V = A \oplus B$, where A is the sum of the generalized weight spaces for real weights and $B = \bigoplus B_i$ is the sum of the remaining conjugate generalized weights spaces. It suffices to show $B = 0$, since then, by Corollary 6.11(2), Γ is conjugate into $\mathrm{UT}(n+1)$.

Each vector $x \in V$ is uniquely expressed as a linear combination $a + b_1 + \cdots + b_k$ with $a \in A$ and $b_i \in B_i$. Define $n(x)$ to be the number of distinct i with $b_i \neq 0$. Choose $x \neq 0$ with $[x] \in \Omega$, so that $n(x)$ is minimal.

Claim $n(x) = 0$.

Proof of the claim If $n(x) \neq 0$, then some $b_j \neq 0$. There is $\gamma \in \Gamma$ which has eigenvalues $\lambda_j(\gamma), \overline{\lambda}_j(\gamma)$ that are not real. Let $\langle\gamma\rangle$ be the cyclic group generated by γ. Let $C \subset B_j$ be the convex hull of the orbit $\langle\gamma\rangle \cdot b_j$.

Suppose $0 \notin C$. Then $K = \mathrm{cl}(\mathbb{P}_+(C))$ is a closed convex cell in $\mathbb{P}_+(B_j)$ that is preserved by γ. By the Brouwer fixed point theorem, γ fixes a point $[v] \in K$, so $v \in B_j$ is an eigenvector of $\gamma|_{B_j}$ with a positive eigenvalue. However every eigenvector for γ in B_j has eigenvalue $\lambda_j(\gamma)$ or $\overline{\lambda}_j(\gamma)$, which are both not real. This contradiction shows that $0 \in C$.

The convex cone $\mathcal{C}\Omega \subset V$ is preserved by Γ. Since $0 \in C$, there is a finite convex combination $\sum t_i \gamma^i b_j = 0$ with $t_i \geq 0$ and $\sum t_i = 1$. Since $x \in \mathcal{C}\Omega$ and this cone is Γ–invariant, it follows that $\gamma^i x \in \mathcal{C}\Omega$. Since $\mathcal{C}\Omega$ is convex, the convex combination $x' = \sum t_i \gamma^i x$ is also in $\mathcal{C}\Omega$. In particular, $x' \neq 0$ and $[x'] \in \Omega$. The component of x' in B_j is $\sum t_i \gamma^i b_j = 0$. Since the conjugate weights spaces are Γ–invariant, the property that a point has a zero component in some B_i is preserved by Γ, so $n(x') < n(x)$, contradicting minimality. Hence no such b_j exists, and this proves the claim. □

Since $x \neq 0$, it follows that $A \neq 0$ and $[x] \in W := \Omega \cap \mathbb{P}(A)$ is a nonempty properly convex set that is preserved by Γ. The submanifold $M = W/\Gamma$ of P is convex and $\pi_1 M \to \pi_1 P$ is an isomorphism, so $\dim(M) \geq n-1$ by Lemma 6.6. Now B_i has real dimension at least 2, so $\dim A \leq \dim V - \dim B_i \leq n-1$. But $\dim M = \dim \mathbb{P}(A) = \dim A - 1 \leq n-2$, which is a contradiction. □

Suppose H is a Lie group. A *virtual syndetic hull* of a discrete subgroup $\Gamma < H$ is a connected Lie subgroup $G < H$ such that $|\Gamma : G \cap \Gamma| < \infty$ and $(G \cap \Gamma)\backslash G$ is compact. In other words Γ is virtually a (cocompact) lattice in H. When syndetic hulls exist, they are not always unique because the exponential map on $\mathfrak{gl}(n)$ is not injective for $n \geq 2$. It is useful to have a unique version of a syndetic hull. For more about syndetic hulls, see Witte [40]. Some of the arguments that follow are inspired by Section 9 of [13], which derives the classification of cusps in *strictly* convex projective manifolds. In particular, this applies to the role of the syndetic hull.

Let $\mathfrak{r} \subset \mathfrak{gl}$ be the subset of all matrices M such that all the eigenvalues of M are real. The set $R = \exp(\mathfrak{r})$ consists of all matrices A such that every eigenvalue of A is positive. Then $\exp\colon \mathfrak{r} \to R$ is a diffeomorphism with inverse $\log$. An element of R is called an *e–matrix* and a group $G \subset R$ is called an *e–group*. For example, $\mathrm{UT}(n)$ is an e–group. The property of being an e–group is preserved by conjugation. If G is a connected e–group, then $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism. If $S \subset R$ define $\langle \log S\rangle$ to be the vector subspace of $\mathfrak{gl}$ spanned by $\log S$.

Definition 6.13 Given a discrete subgroup $\Gamma \subset \mathrm{GL}(n,\mathbb{R})$, a *virtual e–hull* for Γ is a connected Lie group G that is an e–group such that $|\Gamma : G \cap \Gamma| < \infty$ and $(G\cap\Gamma)\backslash G$ is compact. There might not be such G.

Proposition 6.14 [13, 9.3] *Suppose that Γ is a finitely generated, discrete nilpotent subgroup of $\mathrm{GL}(n,\mathbb{R})$. Then Γ contains a subgroup Γ_0 of finite index which has a syndetic hull $G \leq \mathrm{GL}(n,\mathbb{R})$ that is nilpotent, simply connected, and a subgroup of the Zariski closure of Γ_0.*

Lemma 6.15 *If $\Gamma \subset \mathrm{UT}(n)$ is a finitely generated discrete nilpotent subgroup, then it has an e–hull $G \subset \mathrm{UT}(n)$.*

Proof By Proposition 6.14, there is a finite-index subgroup $\Gamma_0 \subset \Gamma$ which has a syndetic hull G. The Zariski closure of $\mathrm{UT}(n)$ is the Borel subgroup B of all upper-triangular matrices in $\mathrm{GL}(n,\mathbb{R})$. It follows that the Zariski closure of Γ is in B, so $G \subset B$. Moreover, G is connected, so $G \subset \mathrm{UT}(n)$. Since $|\Gamma:\Gamma_0| < \infty$ and $\Gamma_0 \subset G$, it follows that $\langle \log\Gamma\rangle = \langle\log\Gamma_0\rangle \subset \mathfrak{g}$, thus $\Gamma \subset G$. Since $G \subset \mathrm{UT}(n)$, it is an e–group. Moreover, $\Gamma\backslash G$ is a quotient of $\Gamma_0\backslash G$ and so is compact. Hence G is a syndetic hull of Γ. □

Lemma 6.16 *If G_0 and G_1 are virtual e–hulls of a discrete subgroup $\Gamma \subset \mathrm{GL}(n,\mathbb{R})$, then $G_0 = G_1$.*

Proof The group $H = G_0 \cap G_1$ is connected because if $h \in H$, then the one-parameter group $\exp\langle \log h\rangle$ is contained in both G_0 and G_1. With R as defined above, set $\Gamma' = \Gamma \cap R$. If $\gamma \in \Gamma'$ then $\gamma^m \in G_i$ for some $m > 0$. Thus $\log\gamma^m = m\log\gamma \in \log G_i$, so $\gamma \in G_i$. Thus $\Gamma' = \Gamma \cap G_i \subset G_i$, so $\Gamma' \subset H$. Since Γ' is a lattice in G_i and H is a closed subgroup of G_i, it follows that Γ' is also a lattice in H. Since H and G_i are diffeomorphic to their Lie algebras, if $H \neq G_i$ then $\dim H < \dim G_i$, which contradicts that Γ' is a lattice in G_i. □

Definition 6.17 If $\Gamma \subset \mathrm{GL}(n,\mathbb{R})$ is finitely generated and $\Gamma \in \mathrm{VFG}$, then the *translation group* of Γ is $T(\Gamma) = \exp\langle \log(\mathrm{core}(\Gamma,n))\rangle$.

Theorem 6.18 *If Ω/Γ is a quasicusp then $T(\Gamma)$ is the unique virtual e–hull of Γ.*

Proof Set $n = 1 + \dim\Omega$. Recall that the definition of quasicusp implies Γ is virtually nilpotent. By Proposition 6.12, $\mathrm{core}(\Gamma,n)$ is conjugate into $\mathrm{UT}(n)$ and is therefore an e–group. By Lemma 6.15, $\mathrm{core}(\Gamma,n)$ has an e–hull, T, that is conjugate into $\mathrm{UT}(n)$. Thus T is a virtual e–hull of Γ. Uniqueness of T follows from Lemma 6.16. It is now clear that $T = T(\Gamma)$. □

The next thing to do is show that, if Ω/Γ is a generalized cusp of dimension n, the orbit under $T(\Gamma)$ of a point $x \in \partial\Omega$ is a strictly convex hypersurface. The key to doing this is to show that, if the cusp is minimal, then Ω is a closed convex subset of $\mathbb{R}^n$ bounded by $\partial\Omega$; see Lemma 6.22.

A *projective flow* Φ on $\mathbb{RP}^n$ is a continuous monomorphism $\Phi\colon \mathbb{R} \to \mathrm{PGL}(n+1,\mathbb{R})$. There is an infinitesimal generator $A \in \mathfrak{gl}_{n+1}$ with $\Phi_t := \Phi(t) = \exp(tA)$. If $p \in \mathbb{RP}^n$ and $\Phi_t(p) = p$ for all t, then p is a *stationary point* of Φ. A *radial flow* is a projective flow that is stationary on a hyperplane $H \cong \mathbb{RP}^{n-1}$ and that is parametrized so that $\Phi_t(p) \to r \in H$ as $t \to -\infty$ whenever p is not stationary. It follows that $\Phi_t = \exp(tA)$, where $A \in \mathfrak{gl}_{n+1}$ is a rank-one matrix and H is the projectivization of $\ker A$. The projectivization of the image of A is a point $p \in \mathbb{RP}^n$, called the *center of the flow*, that is also fixed by Φ. Every orbit is contained in a line containing the center. This property characterizes radial flows.

A radial flow is *parabolic* if $p \in H$ and *hyperbolic* otherwise. Every radial flow is conjugate to one generated by an elementary matrix $E_{i,j}$. A parabolic flow is conjugate to $(I + t\cdot E_{1,n+1})$ and a hyperbolic flow is conjugate to the diagonal group $(\exp(t), 1, \dots, 1)$. The *backward orbit* of $X \subset \mathbb{RP}^n$ is $\Phi_{(-\infty,0]}(X)$. A set $X \subset \mathbb{RP}^n$ is *backward-invariant* if X contains its backward orbit, and it is *backward-vanishing* if $\bigcap_{t<0} \Phi_t(X) = \varnothing$.

A *displacing hyperplane* for a radial flow Φ is a hyperplane P such that P and $\Phi_t(P)$ are disjoint in $\mathbb{RP}^n \setminus H$ for all $t \neq 0$. A hyperplane P is displacing if and only if $P \neq H$ and P does not contain the center of Φ.

Proposition 6.19 *Suppose Ω/Γ is a quasicusp and $\Gamma \subset \mathrm{UT}(n+1)$. If $\lambda\colon \Gamma \to \mathbb{R}^*$ is a weight with generalized weight space $V = V(\lambda)$ then there is a radial flow $\Phi = \Phi^\lambda$ that is centralized by Γ, and Φ acts trivially on each generalized weight space other than V.*

If $\dim V \geq 2$, then Φ is parabolic, and if $\dim V = 1$, then Φ is hyperbolic. The center of Φ is contained in $\mathbb{P}(E(\lambda))$. The group $G(\Gamma) := T(\Gamma) \times \Phi(\mathbb{R})$ generated by $T(\Gamma)$ and $\Phi(\mathbb{R})$ is their internal direct product. If the orbit of $x \in \mathbb{RP}^n$ under $T(\Gamma)$ is a strictly convex hypersurface then $G(\Gamma) \cdot x \subset \mathbb{RP}^n$ is open.

Proof We may assume Γ is upper-triangular and block diagonal, with one block for each generalized weight space. We may assume V is the first block and set $m = \dim V$. As above, let $E_{i,j} \in \mathfrak{gl}(n+1)$ be the elementary matrix with 1 in row i and column j. Define $\Phi(t) = \exp(tE_{1,m})$. Then $\mathbb{R}^{n+1} = V \oplus W$, where W is the sum of the other generalized weight spaces and the action of Φ on W is trivial. If $m = 1$ then $\Phi(t) = \mathrm{diag}(\exp(t), 1, \dots, 1)$ is a hyperbolic flow. If $m \geq 2$ then $\Phi(t)$ is a parabolic flow given by the unipotent subgroup with t in the top right corner of the block for V. The center is $\mathbb{P}(e_1)$ and the stationary hyperplane is $H = \mathbb{P}(\langle e_1, \dots, e_{m-1}\rangle \oplus W)$. It is easy to check that Γ centralizes Φ.

Since $T(\Gamma) = \exp(\mathfrak{t})$ and $\Phi(\mathbb{R}) = \exp(\mathfrak{f})$ are e–groups, if they have a nontrivial intersection, then $\Phi(\mathbb{R}) \subset T(\Gamma)$. The orbits of Φ are lines. If $S = T(\Gamma) \cdot x$ is a strictly convex hypersurface, then it does not contain a line, so $\Phi(\mathbb{R}) \cap T(\Gamma)$ is trivial, and $\Phi(\mathbb{R}) \cdot S \subset \mathbb{RP}^n$ is open. Since $\Phi(\mathbb{R})$ and $T(\Gamma)$ commute, they generate $G = T(\Gamma) \times \Phi(\mathbb{R})$. □

Definition 6.20 A radial flow Φ_t is *compatible* with a properly convex manifold $M = \Omega/\Gamma$ if $\Phi(\mathbb{R})$ commutes with Γ, and Ω is disjoint from the stationary hyperplane of Φ, and Ω is backward-invariant and backward-vanishing.

A *radial flow end* is a properly convex manifold $M = \Omega/\Gamma$ with compact, strictly convex boundary and for which there is a compatible radial flow. A *radial flow cusp* is a radial flow end that is also a generalized cusp.

The hypersurfaces $\widetilde{S}_t := \Phi_{-t}(\partial\Omega)$ are strictly convex and Γ–invariant. Those with $t \geq 0$ foliate Ω. They are all disjoint from H. Their images under the projection

$\pi\colon \Omega \to M$ give a product foliation of M by compact, strictly convex hypersurfaces $S_t = \pi(\widetilde{S}_t)$. There is a transverse foliation of Ω by flow-lines that limit on the center of Φ. These project to a transverse foliation of M by rays.

The *flow time function* is $\widetilde{T}\colon \Omega \to [0,\infty)$ defined by $\widetilde{T}(x) = t$ if $\Phi_t(x) \in \partial\Omega$. Thus $\widetilde{T}(x)$ is the amount of time for x to flow into $\partial\Omega$ and $\widetilde{T}(\widetilde{S}_t) = t$. Let $\pi\colon \Omega \to M$ be the projection. Then there is a map $T\colon M \to [0,\infty)$ defined by $T(\pi x) = \widetilde{T}(x)$. The level sets of T are the hypersurfaces S_t.

Lemma 6.21 *Suppose Φ is a radial flow with center p and stationary hyperplane H. Suppose $\Omega \subset \mathbb{RP}^n \setminus H$ is properly convex. If Φ is hyperbolic and $p \notin \operatorname{cl}(\Omega)$, then Ω is backward-vanishing. If Φ is parabolic, then Ω is backward-vanishing for either $\Phi(t)$ or $\Phi'(t) := \Phi(-t)$.*

Proof If Φ is hyperbolic and $p \notin \operatorname{cl}(\Omega)$, then by the Hahn–Banach separation theorem there is a hyperplane P that separates Ω from p. If Φ is parabolic, then choose any hyperplane P disjoint from Ω that does not contain p. In either case, P is a displacing hyperplane. After possibly reversing Φ in the parabolic case, the component of $\mathbb{R}^n \setminus P$ that contains Ω is a half-space that is backward-vanishing, and hence so is Ω. □

The reason for introducing radial flow ends is:

Lemma 6.22 *Suppose $\Omega \subset \mathbb{RP}^n$ and $M = \Omega/\Gamma$ is a radial flow end with radial flow Φ. Let $H \subset \mathbb{RP}^n$ be the stationary hyperplane for Φ, and $\overline{\Omega} = \operatorname{cl}(\Omega) \subset \mathbb{RP}^n$. Then $\partial\overline{\Omega} = \partial\Omega \sqcup (H \cap \overline{\Omega})$. In particular, Ω is a closed convex subset of $\mathbb{R}^n = \mathbb{RP}^n \setminus H$ bounded by the properly embedded, strictly convex hypersurface $\partial\Omega$.*

Proof Let $\mathbb{R}^n = \mathbb{RP}^n \setminus H$, so that $\Omega \subset \mathbb{R}^n$. It suffices to show that $\partial\Omega$ is properly embedded in $\mathbb{R}^n$ and therefore Ω is a closed convex set in $\mathbb{R}^n$ bounded by $\partial\Omega$.

Let p be the center of Φ. Choose a displacing hyperplane $P \subset \mathbb{RP}^n$ that is disjoint from Ω such that if Φ is hyperbolic then P separates p from Ω in $\mathbb{R}^n$.

Let U be the closure of the component of $\mathbb{R}^n \setminus P$ that is the half-space containing Ω. Then U is backward-invariant. Thus U is the backward orbit of P. Define the function $\tau\colon U \to [0,\infty)$ by $\tau(x) = t$ if $\Phi_t(x) \in P$. This is the amount of time it takes x to flow into P. Observe that if $x, y \in \Omega$, then $\widetilde{T}(x) - \widetilde{T}(y) = \tau(x) - \tau(y)$.

Because $\widetilde{S}_t := \Phi_{-t}(\partial\Omega)$ is strictly convex, the only critical points of the restriction of $\widetilde{T}$ to a segment are maxima, and therefore there is at most one critical point on a segment. Thus $T\colon M \to [0,\infty)$ has the same critical point behavior along segments.

Choose a metrically complete Riemannian metric on M and use the lifted metric on Ω. Suppose $\partial\Omega$ is not properly embedded in $\mathbb{R}^n$. Then there is a sequence $\widetilde{p}_k \in \partial\Omega$ which converges in $\mathbb{R}^n$ to a point $\widetilde{p}_\infty \notin \partial\Omega$.

Let α_k be the length of $[\widetilde{p}_0, \widetilde{p}_k] \subset \Omega$. Then $\alpha_k \to \infty$ because $\widetilde{p}_\infty \notin \Omega$ and the metric on Ω is complete. Let $\widetilde{\ell}_k\colon [0,1] \to [\widetilde{p}_0, \widetilde{p}_k]$ be the unit segment. Then $\widetilde{\ell}_k$ converges to $\widetilde{\ell}_\infty\colon [0,1] \to [\widetilde{p}_0, \widetilde{p}_\infty]$. The restriction of $\widetilde{\ell}_\infty$ to $[0,1)$ is a ray, $\widetilde{\ell}\colon [0,1) \to \Omega$, of infinite length in Ω. Since $\widetilde{p}_k \to \widetilde{p}_\infty$, there is $\beta > 0$ such that $\widetilde{T} \circ \widetilde{\ell}_k \le \beta$ for all $k \in [0,\infty]$.

The projection $\ell_k = \pi \circ \widetilde{\ell}_k\colon [0,1] \to M$ is an immersed affine segment and $T \circ \ell_k \le \beta$. Thus ℓ_k is contained in the compact set $M_\beta := \bigcup_{0 \le t \le \beta} S_t$. These segments converge to the ray $\ell = \pi \circ \widetilde{\ell}$ of infinite length that is also contained M_β. Now $T \circ \ell\colon [0,1) \to [0,\beta]$ is eventually monotonic. Thus there is a segment, $\ell^*\colon [0,1] \to M_\beta$, of length 1, that is a limit of subsegments of ℓ of length 1, and $T \circ \ell^*$ is some constant α. Thus ℓ^* is contained in S_α. But this contradicts the fact that S_α is strictly convex. It follows that $\partial\Omega$ is properly embedded in $\mathbb{R}^n$. Hence Ω is a closed convex set in $\mathbb{R}^n$. □

To apply this we need:

Proposition 6.23 *Every minimal generalized cusp $C = \Omega/\Gamma$ with $\Gamma \subset \mathrm{UT}(n+1)$ is a radial flow cusp.*

Proof It suffices to show there is a radial flow that is compatible with C.

Claim 1 *Ω is disjoint from every Γ–invariant hyperplane H.*

Proof of Claim 1 If $H \cap \Omega \neq \varnothing$, then $H \cap \partial\Omega \neq \varnothing$ since C is minimal. Observe that $H \cap \Omega$ is properly convex and preserved by Γ. Thus $R = (H \cap \Omega)/\Gamma$ is a convex codimension-1 submanifold of C with nonempty boundary, which contradicts Lemma 6.6. □

There are now two cases:

Parabolic case There is a generalized weight space W for Γ with $\dim W \ge 2$. Let Φ be the parabolic radial flow that centralizes Γ given by Proposition 6.19. Let H be the stationary hyperplane and $p \in H$ the center of Φ. Let P be a displacing hyperplane that is tangent to Ω at $q \in \partial\Omega$.

Hyperbolic case Every generalized weight space has dimension 1, so Γ is diagonalizable. The weight spaces projectivize to give points $p_0, \dots, p_n \in \mathbb{RP}^n$ that are in general position. The hyperplane P_i contains all these points except p_i. These hyperplanes divide $\mathbb{RP}^n$ into 2^n open n–simplices. These hyperplanes are Γ–invariant, so Ω is contained in one of these simplices, say Δ, by minimality of the cusp. There is a vertex p of Δ with $p \notin \overline{\Omega}$ because $\partial\Omega$ is a strictly convex hypersurface in Δ. After relabelling $p = p_0$, let $H = P_0$ and let Φ be the radial flow with center p and stationary hyperplane H. Then Φ centralizes Γ and p is disjoint from $\mathrm{cl}(\Omega)$. By the proof of Lemma 6.21, there is a displacing hyperplane P that separates p from Ω.

In each case, Ω is disjoint from H by Claim 1. Set $\mathbb{R}^n = \mathbb{RP}^n \setminus H$, so $\Omega \subset \mathbb{R}^n$. Let U be the closure of the component of $\mathbb{R}^n \setminus P$ that contains Ω. Choose linear coordinates on $\mathbb{R}^n$ such that $q = e_1 = (1, 0, \dots, 0)$ and U is the half-space $x_1 \geq 1$, and, moreover, $p = 0$ in the hyperbolic case and p is the limit of the positive x_1–axis in the parabolic case. Then $P = \partial U$ is the horizontal hyperplane $x_1 = 1$.

We may assume U is backward-invariant after possibly reversing the flow in the parabolic case. We reparametrize Φ so that, in these coordinates, $\Phi_t(x) = x - t \cdot e_1$ in the parabolic case and $\Phi_t(x) = \exp(-t) \cdot x$ in the hyperbolic case.

Let $p_1\colon U \to P$ be the projection along flow-lines, ie $p_1(x_1, \dots, x_n) = (1, x_2, \dots, x_n)$ in the parabolic case and $p_1(x_1, \dots, x_n) = (1, x_2/x_1, \dots, x_n/x_1)$ in the hyperbolic case. Let Ω_1 be the backward orbit of $\mathrm{int}\,\Omega$.

Claim 2 Ω_1 *is open in* $\mathbb{R}^n$*, properly convex, backward-invariant and contains* $\mathrm{int}(\Omega)$.

Proof of Claim 2 Clearly Ω_1 is open, backward-invariant and contains $\mathrm{int}(\Omega)$. Suppose $a, b \in \Omega_1$. Then $a = \Phi_\alpha(a')$ and $b = \Phi_\beta(b')$ for some $\alpha, \beta \leq 0$ and $a', b' \in \mathrm{int}\,\Omega$. Let $\ell = [a', b']$ be the line segment with endpoints a' and b'. Since Ω is convex, $\ell \subset \Omega$. Then $\bigcup_{t \leq 0} \Phi_t(\ell)$ is a planar convex set in Ω_1 that contains a and b. Hence Ω_1 is convex.

Let C be the cone of Ω from 0. Since Ω is properly convex and $0 \notin \overline{\Omega}$, it follows that C is properly convex. Moreover, C contains Ω_1, so Ω_1 is properly convex, proving Claim 2. □

We want to add a boundary to Ω_1 and show this gives Ω. Define Ω_M to be the *flow closure* of Ω_1, ie the set of all points x such that $\Phi_t(x) \in \Omega_1$ for all $t < 0$. Clearly $\Omega_M \subset \mathrm{cl}(\Omega_1) \subset \mathbb{R}^n$. There is a homeomorphism $\widetilde{F}\colon \partial\Omega_M \times [0, \infty) \to \Omega_M$ given by

$F(x,t) = \Phi_{-t}(x)$. Since Ω_1 is open, Ω_M is a manifold with boundary $\partial\Omega_M$. It is clear that Ω_M is disjoint from H, backward-invariant and backward-vanishing.

Claim 3 *Γ acts freely and properly discontinuously on Ω_M.*

Proof of Claim 3 Since Γ commutes with Φ, it follows that Ω_1 is preserved by Γ. Also Γ acts freely on Ω, it contains no elliptics, and therefore acts freely on Ω_1. By 1.3 of [13], Γ is discrete and therefore acts properly discontinuously on Ω_1. The map Φ_{-1} embeds Ω_M into Ω_1 and, since Φ_{-1} commutes with Γ, it follows that Γ acts freely and properly discontinuously on Ω_M. □

Thus $M = \Omega_M/\Gamma$ is a properly convex manifold and there is a homeomorphism $F\colon \partial M \times [0,\infty) \to M$ covered by $\widetilde{F}$.

Claim 4 $M = C$.

First we show $C \subset M$. Since $\operatorname{int}(\Omega) \subset \Omega_1$, it follows that $\operatorname{int}(C) \subset \operatorname{int}(M)$. By Proposition 8.3, there is a collar neighborhood $P \subset C$ of ∂C with $\partial P = \partial C \sqcup Q$ and Q strictly convex. Let $R = \operatorname{cl}(C \setminus P)$, so $\partial R = Q$ and R is a generalized cusp contained in M. Thus $X = \operatorname{cl}(M \setminus R) \cong \partial X \times I$ is compact, and $P \setminus \partial C \subset X$, so it follows that $\partial C \subset X \subset M$, thus $C \subset M$.

The intersection of an orbit under the flow Φ with Ω_M, and also its image in M, is called a *flow-line*. Every flow-line λ in M ends on ∂M and is properly embedded. Moreover, $C \cap \lambda$ is convex. Since ∂C separates ∂M from the end of M, it follows that $\lambda \cap C$ is all of λ except, possibly, a bounded interval at the end of λ. It follows that for all $t < 0$ that $\Phi_t(C) \subset C$, so C is backward-invariant. It now follows that $\Omega_1 = \Omega$ and $M = C$, and this implies Φ is compatible with C. □

Now we know that Ω is a closed subset of $\mathbb{R}^n$ we are ready to show the orbits of the translation group are properly embedded convex hypersurfaces in $\mathbb{R}^n$.

Proposition 6.24 *Suppose $C = \Omega/\Gamma$ is a minimal generalized cusp and $\Gamma \subset \mathrm{UT}(n+1)$. Let $T = T(\Gamma)$ be the translation group. Then C contains a minimal homogeneous cusp $C_T = \Omega_T/\Gamma$ and T acts transitively on $\partial\Omega_T$.*

Proof By Proposition 6.23, C is a radial flow cusp for some flow Φ. Let H be the stationary hyperplane of Φ and set $\mathbb{R}^n = \mathbb{RP}^n \setminus H$. Since T centralizes Φ, it preserves H and acts affinely on $\mathbb{R}^n$.

Claim *There is $x \in \Omega$ such that $T \cdot x \subset \Omega$.*

Proof of the claim Let $\pi\colon \Omega \to C$ be the covering space projection. There is a continuous map $F\colon T \times \partial\Omega \to \mathbb{R}^n/\Gamma$ given by $F(t,x) = \pi(t \cdot x)$. Since $\partial C = \partial\Omega/\Gamma$ is compact, there is a compact subset $D \subset \partial\Omega$ such that $\Gamma \cdot D = \partial\Omega$. So $T \cdot \partial\Omega = T \cdot D$ because $\Gamma \subset T$. There is compact $X \subset T$ such that $\Gamma X = T$. So $T \cdot D = (\Gamma X) \cdot D$. Hence $\mathrm{Im}(F) = \pi(X \cdot D)$ because $\pi(\Gamma \cdot x) = \pi(x)$. Thus $K = \mathrm{Im}(F) \subset \mathbb{R}^n/\Gamma$ is compact, because it equals $F(X \times D)$, and $X \times D$ is compact. Choose $x \in \mathrm{int}(\Omega)$ such that $\pi(x) \notin K$. Then $T \cdot x$ is connected, and disjoint from $\partial\Omega$. By Lemma 6.22, $\mathrm{int}(\Omega) \subset \mathbb{R}^n$ is bounded by $\partial\Omega$. It follows that $T \cdot x \subset \Omega$. This proves the claim. □

The set $\Omega_T = \mathrm{cl}(\mathrm{CH}(T \cdot x)) \subset \Omega$ is properly convex and T–invariant.

Claim *$\partial\Omega_T$ is strictly convex.*

Since Ω_T is a closed properly convex set in $\mathbb{R}^n$, there is an extreme point $y \in \partial\Omega_T$ at which $\partial\Omega_T$ is strictly convex. Thus $\partial\Omega_T$ is strictly convex at every point in the orbit $S = T \cdot y \subset \partial\Omega_T$. Then $\Omega_S := \mathrm{CH}(S) \subset \Omega_T$ is properly convex. Since $\Gamma \subset T$ it follows that Ω_S is Γ–invariant. Thus $C_S = \Omega_S/\Gamma$ is a generalized cusp and a submanifold of C. It follows from Lemma 6.6 that $\dim \Omega_S = \dim \Omega$. Moreover, $S \subset \partial\Omega_T$, so $S \subset \partial\Omega_S$. Since $\Omega_S = CH(S)$, it follows that $S = \partial\Omega_S = \partial\Omega_T$ and $\Omega_S = \Omega_T$, which proves the claim, and thus Proposition 6.24. □

If Ω/Γ is a generalized cusp, by Proposition 6.24 there is a homogeneous domain $\Omega_T \subset \Omega$ that is preserved by the finite-index subgroup of $\Gamma \cap T(\Gamma)$. Next we show that Ω_T is preserved by all of Γ.

Lemma 6.25 *Suppose $C = \Omega/\Gamma$ is a minimal generalized cusp and $T = T(\Gamma) \subset \mathrm{UT}(n+1)$ and $\Gamma_0 = T \cap \Gamma$. Suppose Ω/Γ_0 contains a homogeneous cusp Ω_T/Γ_0 and Ω_T is preserved by T. Then Γ preserves Ω_T, so C contains the homogeneous generalized cusp Ω_T/Γ.*

Proof By Proposition 6.23, $C^* = \Omega_T/\Gamma_0$ is a radial flow cusp and, by Lemma 6.22, $\Omega_T \subset \mathbb{R}^n$ is bounded by the strictly convex properly embedded hypersurface $\partial\Omega_T$. By Theorem 6.18, $T = T(\Gamma)$ is the unique translation group that contains Γ.

Since Γ normalizes itself, it follows that Γ normalizes T and therefore Γ permutes the decomposition of $\mathbb{RP}^n$ into T–orbits. The domain Ω_T is foliated by T–orbits and $\Omega_T/T \cong [0,1)$. Since $\Gamma \cap T$ preserves Ω_T and $|\Gamma : \Gamma \cap T| < \infty$, it follows the

Γ–orbit of Ω_T is a finite number of pairwise disjoint convex sets all contained in Ω. Thus $\Gamma \cap T$ permutes these domains. There is a finite-index subgroup $\Gamma_1 \subset \Gamma \cap T$ that preserves each domain. We may assume Γ_1 is normal in Γ. Thus $M = \Omega/\Gamma_1$ is a regular cover of C that contains one copy of $P = \Omega_T/\Gamma_1$ for each domain. However, M and each copy of P is a generalized cusp. Each copy of ∂P separates ∂M from the end of M. Since the copies of P are disjoint, there is only one copy of P and Γ preserves Ω_T. □

Proof of Theorem 6.3 Suppose $C = \Omega/\Gamma$ is a generalized cusp of dimension n. We may assume C is minimal by Lemma 6.5. Since Γ is virtually nilpotent, it follows from Proposition 6.12 that there is a finite-index subgroup $\Gamma' < \Gamma$ that is conjugate into $\mathrm{UT}(n+1)$. We will assume this conjugacy has been done. Then $\widetilde{C} = \Omega/\Gamma'$ is a generalized cusp that is a finite cover of C and is minimal by Lemma 6.5. It follows from Theorem 6.18 that Γ' is a lattice in a connected upper-triangular Lie group $T = T(\Gamma)$. By Proposition 6.23, it follows that $\widetilde{C}$ is a radial flow cusp for a radial flow Φ with stationary hyperplane H. Let $\mathbb{R}^n = \mathbb{RP}^n \setminus H$. By Lemma 6.22, $\Omega \subset \mathbb{R}^n$ is a closed strictly convex set bounded by the strictly convex hypersurface $\partial\Omega$. By Proposition 6.24, there is a properly convex $\Omega_T \subset \Omega$ that is T–invariant and thus Γ'–invariant. By Lemma 6.25, Ω_T is preserved by all of Γ, hence Ω_T/Γ is a homogeneous cusp in C and $\Gamma < \mathrm{PGL}(\Omega_T)$. □

Lemma 6.26 *Suppose G is a connected group with* $\dim G = n-1$. *For $x \in \mathbb{RP}^n$ the subset of* $\mathrm{Hom}(G, \mathrm{GL}(n+1,\mathbb{R}))$ *consisting of all ρ with $\rho(G)\cdot x$ a Hessian convex hypersurface is open.*

Proof Suppose the map $f\colon G \to \mathbb{RP}^n$ given by $f(g) = (\rho(g))\cdot x$ has image a strictly convex hypersurface S. Because G acts transitively on S by projective maps, it follows that S is strictly convex everywhere if and only if it is strictly convex at the single point x. Let ν be the normal to S at x and $e \in G$ the identity. Hessian convexity of S at x is equivalent to the quadratic form $Q = \nu \cdot D_e^2 f$ being positive or negative definite. This form $Q = Q(\rho)$ is a smooth function of ρ and the set of definite quadratic forms is open in the set of all quadratic forms. □

Definition 6.27 If C is a generalized cusp of dimension n, then $\mathcal{GC}(C)$ is the subspace of $\mathrm{Hom}(\pi_1 C, \mathrm{GL}(n+1,\mathbb{R}))$ consisting of holonomies of all generalized cusp structures on C and $\mathrm{VFG}(C)$ is the subspace of all ρ with $\rho(\pi_1 C) \in \mathrm{VFG}$.

Theorem 6.28 (stability of generalized cusps) *Suppose C is a generalized cusp of dimension n. Then:*

(1) $\mathrm{VFG}(C)$ *is semialgebraic.*

(2) $\mathcal{GC}(C) \subset \mathrm{VFG}(C)$.

(3) $\theta\colon \mathcal{K}(C) \to \mathrm{VFG}(C)$ *given by $\theta(\Omega, \rho) = \rho$ is a continuous open map.*

(4) $\mathcal{GC}(C) = \mathrm{Im}(\theta)$.

Proof By Corollary 6.11, $\mathrm{VFG}(C)$ is semialgebraic. By definition $(\Omega, \rho) \in \mathcal{K}(C)$ if and only if $\Omega/\mathrm{Im}(\rho)$ is a generalized cusp diffeomorphic to C, thus $\mathcal{GC}(C) = \mathrm{Im}(\theta)$. Every generalized cusp is homogeneous by Theorem 6.3, and Proposition 6.12 implies the holonomy contains a finite-index subgroup that is conjugate into an upper-triangular group, so $\mathcal{GC}(C) \subset \mathrm{VFG}(C)$. It is obvious that θ is continuous; it only remains to show it is open.

Set $H = \mathrm{core}(\pi_1 C, n+1)$. By Definition 6.17, given $(\Omega, \rho) \in \mathcal{K}(C)$, the translation group is

$$T(\rho) := T(\rho(\pi_1 C)) = \exp\langle \log(\rho H)\rangle.$$

This is clearly a continuous function of ρ.

Choose $x \in \partial\Omega$. By Lemma 6.26 for $\sigma \in \mathrm{VFG}(C)$ close enough to ρ, the hypersurface $S = T(\sigma)\cdot x$ is Hessian-convex and close to $\partial\Omega$. By Proposition 6.19, there is a radial flow Φ that is centralized by $T(\sigma)$ and the group $G = T(\sigma) \oplus \Phi(\mathbb{R})$ has an open orbit W in $\mathbb{RP}^n$. Moreover, W is foliated by the strictly convex hypersurfaces $S_t = \Phi_t(S)$.

After replacing t by $-t$ we may assume for $t < 0$ and close to 0 that S_t is on the convex side of $S = S_0$. Let $\Omega^+ = \bigcup_{t \le 0} S_t$. Then $\partial\Omega^+ = S_0$. This set is preserved by $T(\sigma)$ and therefore by $\sigma(H)$. It is contained in a properly convex cone by the argument of Theorem 3.4 using Figure 1. Hence $\Omega(\sigma) := \mathrm{CH}(\Omega^+)$ is properly convex and $T(\sigma)$–invariant. The argument of Claim 3 in Proposition 6.23 shows that $\sigma(H)$ acts freely and properly discontinuously on $\Omega(\sigma)$. Since $\Omega(\sigma)$ is $T(\sigma)$–invariant, $\partial\Omega(\sigma)$ is Hessian-convex. Thus $\Omega(\sigma)/\sigma(H)$ is a homogeneous generalized cusp.

It only remains to show that $\Omega(\sigma)$ is preserved by all of $\sigma(\pi_1 C)$. The argument is very similar to the proof of Lemma 6.25. The $\sigma(\pi_1 C)$–orbit of $\Omega(\sigma)$ is finite because $|\pi_1 CM : H| < \infty$. By Lemma 6.16, $T(\sigma)$ is the unique virtual e–hull of $\sigma(\pi_1 C)$. Thus $\sigma(\pi_1 C)$ preserves the decomposition of $\mathbb{RP}^n$ into $T(\sigma)$–orbits. Moreover,

$\Omega(\sigma)$ is a union of such orbits. Thus if $g \in \pi_1 C$ then $(\sigma g)(\Omega(\sigma))$ is either $\Omega(\sigma)$ or disjoint from $\Omega(\sigma)$. We need only look at finitely many such g. Observe that $\Omega(\sigma)$ is close to $\Omega(\rho)$ and $\rho(g)$ is close to $\sigma(g)$ and $\rho(\pi_1 C)$ preserves $\Omega(\rho)$. Thus $\rho(g)$ preserves $\Omega(\rho)$, so $(\sigma g)(\Omega(\sigma))$ intersects $\Omega(\sigma)$. It follows that $(\sigma g)(\Omega(\sigma)) = \Omega(\sigma)$. □

We remark that when $C \cong T^2 \times [0,\infty)$, the subset of diagonal representations in $\mathrm{GL}(3,\mathbb{R})$ given by holonomies of generalized cusps is not closed. The boundary consists of properly convex structures on C with ∂C flat, thus not strictly convex.

Theorem 6.29 (main theorem) *Suppose N is a compact, connected n–manifold and $\mathcal{V} = \bigcup_{i=1}^k V_i \subseteq \partial N$ is the union of some of the boundary components of N. Assume $\pi_1 V_i$ is virtually nilpotent for each i. Let $M = N \setminus \mathcal{V}$. Then the holonomy map $\mathcal{H}ol\colon \mathcal{D}ev_{\mathrm{ce}}(M) \to \mathrm{VFG}(M)$ is continuous and open, and $\mathrm{VFG}(M)$ is a semialgebraic subset of $\mathrm{Hom}(\pi_1 M, \mathrm{GL}(n+1,\mathbb{R}))$.*

Proof Continuity is obvious. That $\mathrm{VFG}(M)$ is semialgebraic follows from Corollary 6.11. Let B_i be the end of M corresponding to V_i and $\mathcal{B} = \bigcup B_i$. Given a developing map $\mathrm{dev} \in \mathcal{D}ev_{\mathrm{ce}}(M)$, the holonomy $\rho = \mathcal{H}ol(\mathrm{dev})$ is in $\mathrm{VFG}(M)$ by Proposition 6.12. The restriction $\mathrm{dev}|_{B_i}$ determines a generalized cusp $B_i \cong \Omega_i/\Gamma_i$ with holonomy $\Gamma_i = \rho(\pi_1 B_i)$. If $\rho' \in \mathrm{VFG}(M)$ is sufficiently close to ρ, then, by Theorem 6.28, there are nearby generalized cusps $B_i \cong \Omega_i'/\Gamma_i'$ with holonomy $\Gamma_i' = \rho'(\pi_1 B_i)$. Thus $x = (\rho, (\Omega_1, \Gamma_1), \dots, (\Omega_k, \Gamma_k))$ and $x' = (\rho', (\Omega_1', \Gamma_1'), \dots, (\Omega_k', \Gamma_k'))$ are close in $\mathcal{R}el\mathcal{H}ol(M, \mathcal{B}, \mathcal{K})$. By Theorem 5.8, the map

$$\mathcal{E}_{\mathcal{K}}\colon \mathcal{D}ev_{\mathrm{ce}}(M, \mathbb{P}) \to \mathcal{R}el\mathcal{H}ol(M, \mathcal{B}, \mathcal{K})$$

is open, so there is dev' close to dev with $\mathcal{E}_{\mathcal{K}}(dev') = x'$ and $\mathcal{H}ol(x') = \rho'$. Hence $\mathcal{H}ol$ is open. □

Proof of Theorem 0.2 By definition, $\mathrm{Rep}_{\mathrm{ce}}(M) = \mathrm{Im}(\mathcal{H}ol)$, so the result follows from Theorem 6.29. □

We may avoid appealing to the theorem of Schoen and Yau used in the proof of Proposition 5.2 for the manifolds appearing in the main theorem using:

Lemma 6.30 *Every homogeneous cusp has an exhaustion function.*

Proof Suppose $C = \Omega/\Gamma$ is a homogeneous cusp and $T = T(\Gamma)$ is the translation subgroup. Then Ω has a codimension-1 foliation by T–orbits that covers a smooth foliation of C. Pick y in the interior of Ω and define $F\colon \Omega \to \mathbb{R}$ by $F(x) = d_\Omega(x, T\cdot y)$ if $T \cdot y$ separates $\partial\Omega$ from x, and define $F(x) = 0$ otherwise. Then F covers a map $f\colon C \to \mathbb{R}$ and $Y = f^{-1}(0)$ is a compact collar neighborhood of ∂C and $f(x) = d(y, Y)$ and $f^{-1}(t)$ is a leaf of the foliation of C for $t > 0$.

It is clear that $\|df\| \le 1$ when $f > 0$. Thus it suffices to show that $\|D^2 f\|$ is bounded. Suppose there is a sequence (C_k, f_k, x_k) such that $\|D^2 f_k\|_{x_k} > k$. Then $C_k = \Omega_k/\Gamma_k$ and Γ_k is a lattice in $G_k = \mathrm{PGL}(\Omega_k)$. We may assume all the Ω_k are in Benzécri position and 0 covers x_k. We may also assume $\Omega_k \to \Omega$ in the Hausdorff topology. Then $G_k \to G \subset \mathrm{PGL}(\Omega)$. The T–orbits form a smooth foliation of Ω and we define a smooth function $F\colon \Omega \to \mathbb{R}$ using $y = 0$ as above. Then F_k converges to F in C^∞ on compact sets. But $\|D^2 f_k\| = \|D^2 F_k\| \to \infty$ contradicts $\|D^2 F\| < \infty$ because F is smooth. □

7 Three-dimensional generalized cusps

An orientable three-dimensional generalized cusp is diffeomorphic to $T^2 \times [0, \infty)$, and Leitner [26] shows that the holonomy is conjugate into a unique group C_n of the form below with $\beta \ge \alpha > 0$ fixed, where n is the number of nontrivial weights:

$$C_0 = \begin{pmatrix} 1 & s & t & \frac{1}{2}(s^2+t^2) \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad C_1 = \begin{pmatrix} e^s & 0 & 0 & 0 \\ 0 & 1 & t & \frac{1}{2}t^2 - s \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C_2(\alpha) = \begin{pmatrix} e^s & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & 1 & -t-\alpha s \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_3(\alpha,\beta) = \begin{pmatrix} e^s & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^{-\alpha s - \beta t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A related statement, due to Benoist, is in 2.7 of [4]. There is a *compact*, properly convex domain $\Omega_n = \Omega_n(\alpha,\beta)$ preserved by $C_n = C_n(\alpha,\beta)$ and $\partial\Omega_n = A \sqcup B$ where $A = C_n \cdot x$ is an orbit and B is a simplex contained in a projective hyperplane. If $\Gamma_n \subset C_n$ is a lattice then $\overline{M} = \Omega_n/\Gamma_n$ is a compactification of the generalized cusp $M = (\Omega_n \setminus B)/\Gamma_n$ obtained by adding $\partial_\infty \overline{M} = B/\Gamma$, which is a point for C_0, a circle for C_1 and a torus for C_2 or C_3. The group C_n is a *translation group* and is contained in the cusp Lie group $\mathrm{PGL}(\Omega_n)$.

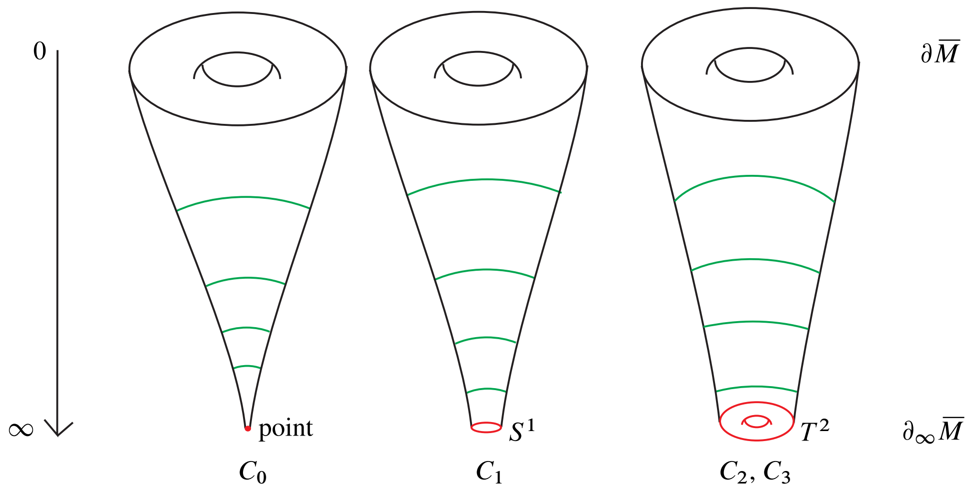


Figure 2: Generalized cusps in dimension 3

The group C_0 is conjugate into $\mathrm{PO}(3,1)$ and contains the holonomy of a cusp of a hyperbolic 3–manifold. Ballas [2] found a lattice in C_1 that is the holonomy of a generalized cusp for a properly convex structure on the figure eight knot complement. The groups $C_3(\alpha,\beta)$ are diagonal affine groups that satisfy the *uniform middle eigenvalue condition* of Choi [9]. Gye-Seon Lee found lattices in some of these groups that are holonomies of generalized cusps for a properly convex structure on the figure eight knot complement. At the time of writing it is not known if there is a 3–manifold that admits a finite-volume hyperbolic structure and also a properly convex structure that is a lattice in some $C_2(\alpha)$. The classification of generalized cusps in all dimensions is given in Ballas, Cooper and Leitner [3].

8 Convex smoothing

We are concerned with various kinds of convexity. A function $f\colon (a,b)\to\mathbb{R}$ is *convex* if

$$\forall (c,d)\subset(a,b)\ \ \forall t\in(0,1)\qquad f(tc+(1-t)d)\le tf(c)+(1-t)f(d)$$

and *strictly convex* if the above inequality is always *strict*. It is *Hessian-convex* if f is smooth and $f''>0$ everywhere. A convex function is *strictly convex* at $p\in(a,b)$ if the graph of $y=f(x)$ intersects the tangent line at $x=p$ at the single point $(p,f(p))$. Each of these definitions extends to a function $f\colon M\to\mathbb{R}$ on an affine manifold M by requiring its restriction to each line segment in M to have the corresponding property.

In the case of Hessian-convex we also require f is smooth. It follows that if a convex function $f\colon M \to \mathbb{R}$ is strictly convex at the point $p \in M$, then there is hyperplane H and a neighborhood $U \subset M$ of p such the graph of $f|_U$ intersects H only at p. For affine manifolds, we show how to approximate a convex function *which is strictly convex somewhere* by a smooth, Hessian-convex one.

The main application is that given a projective manifold which has a convex boundary that is strictly convex at some point, we can shrink the manifold slightly to produce a submanifold with Hessian-convex boundary, ie locally the graph of a Hessian-convex function. One might imagine using sandpaper to smooth the boundary and produce a submanifold with smooth, strictly convex boundary.

The idea is to improve a convex function which is already Hessian-convex on some open subset, by changing it in a small convex set C, and leaving it unchanged outside C. This is done so that it is Hessian-convex inside a slightly smaller convex set $C^- \subset C$, and also Hessian-convex at any point where it was previously Hessian-convex. In this way the problem is reduced to a local one in Euclidean space.

Greene and Wu [20, Theorem 2], and also [19], showed that on a Riemannian manifold, any function f with the property that locally there is a function g with positive definite Hessian such that $f-g$ is convex along geodesics (*they* call f *strictly convex*) can be uniformly approximated by smooth, Hessian-convex functions. Smith [35] gives an example, for each $k \geq 0$, of a C^k convex function on a noncompact Euclidean surface which is not approximated by a C^{k+1} convex function.

A function f is *convex down* if $-f$ is convex. This means secant lines lie *below* the graph: $tf(a)+(1-t)f(b) \leq f(ta+(1-t)b)$ for all a,b and $0 \leq t \leq 1$. Equivalently, the set of points *below* the graph of f is convex.

If f and g are smooth convex-down functions, then $\min(f,g)$ is convex down, but need not be smooth at points where $f=g$. We construct a smooth approximation m^κ on $\mathbb{R}^2_+$ which agrees with min outside a certain neighborhood of the diagonal and has good convexity properties.

Lemma 8.1 (smoothing min) *Given $\kappa \in (0,1)$, there is a smooth function $m^\kappa\colon \mathbb{R}^2_+ \to \mathbb{R}_+$ which is convex down and nondecreasing in each variable ($m^\kappa_x \geq 0, m^\kappa_y \geq 0$) such that if $x \leq \kappa y$ or $y \leq \kappa x$, then $m^\kappa(x,y) = \min(x,y)$. Moreover, m^κ is linear along rays: $m^\kappa(tx,ty) = t \cdot m^\kappa(x,y)$ for $t \geq 0$. It follows that if $f,g\colon C \to \mathbb{R}_+$ are convex down, then so is $h(x) = m^\kappa(f(x),g(x))$.*

Proof On $\mathbb{R}^2_+$,

$$\min(x,y) = (x+y)\cdot k\left(\frac{x}{x+y}\right), \quad \text{where } k(t) = \min(t, 1-t).$$

Choose δ so that $\kappa = \delta/(1-\delta)$. Then $\delta \in \left(0, \frac{1}{2}\right)$. Let $K\colon [0,1] \to [0,1]$ be a convex-down smooth function that agrees with k outside $(\delta, 1-\delta)$. Define $m\colon \mathbb{R}^2_+ \to \mathbb{R}$ by

$$m(x,y) = (x+y)\cdot K\left(\frac{x}{x+y}\right).$$

Clearly m is linear along rays. If $x/(x+y) \le \delta$, then $m(x,y) = x$. This happens when $x \le \kappa y$. Similarly, $m(x,y) = y$ when $y \le \kappa x$, thus

$$m(x,y) = \min(x,y) \quad \text{if } x \le \kappa y \text{ or } y \le \kappa x.$$

The subset of $\mathbb{R}^2_+$ where neither $x \le \kappa y$ nor $y \le \kappa x$ is called the *transition region*. Outside the transition region, $m = \min$.

The graph of m is a convex-down surface above $\mathbb{R}^2_+$ that is a union of rays starting at the origin. One can picture the graph of m: it is the cone from the origin of the convex-down arch that is the part of the graph lying above $x+y=1$. This arch is given by $K(x)$. Since $K(x)$ is convex down, the graph of m is convex down, though in the radial direction it is, of course, linear.

This surface is comprised of three parts. The central part is curved down. The other two parts are sectors of flat planes, one containing the x–axis and the other containing the y–axis.

We claim $m(x,y)$ is a nondecreasing function of each variable. This is clear on the two parts of the graph of m that are flat, since they are planes containing either the x–axis or the y–axis. Now

$$m_x(a,b) = \frac{\partial m}{\partial x} = K\left(\frac{a}{a+b}\right) + (a+b)\cdot b(a+b)^{-2}K'\left(\frac{a}{a+b}\right).$$

Since $m_x(ta,tb) = m_x(a,b)$, we may assume $a+b=1$. Then

$$m_x(a,b) = K(a) + (1-a)K'(a) =: c.$$

The point $(x,y) = (1,c)$ lies on the tangent line to the graph of $y = K(x)$ at $x = a$. Since this graph is convex down, and underneath the graph of $y = k(x)$, it follows that $c \ge 0$. This is best seen by staring at a diagram. Similar calculations work for m_y. This proves the claim.

Next we deduce that h is convex down using these two properties of m, where

$$h(ta+(1-t)b)=m\big(f(ta+(1-t)b),g(ta+(1-t)b)\big).$$

Since $m_x \geq 0$ and f is convex down,

$$m(f(ta+(1-t)b),g(ta+(1-t)b)) \geq m\big(tf(a)+(1-t)f(b),g(ta+(1-t)b)\big).$$

Similarly, since $m_y \geq 0$ and g is convex down,

$$\begin{aligned}m\big(tf(a)+(1-t)f(b),g(ta+(1-t)b)\big) &\geq m(tf(a)+(1-t)f(b),tg(a)+(1-t)g(b))\\ &= m\big(t(f(a),g(a))+(1-t)(f(b),g(b))\big).\end{aligned}$$

Finally, since m is convex down,

$$\begin{aligned}m\big(t(f(a),g(a))+(1-t)(f(b),g(b))\big) &\geq t\cdot m(f(a),g(a))+(1-t)m(f(b),g(b))\\ &= t\cdot h(a)+(1-t)h(b).\end{aligned}$$

□

Corollary 8.2 (relative convex smoothing) *Suppose $C\subset\mathbb{R}^n$ is a compact convex set with nonempty interior and C^- is a compact convex set in the interior of C. Suppose $f\colon C\to\mathbb{R}$ is a nonconstant, convex function, which is Hessian-convex on a* (possibly empty) *subset $S\subset C$. Assume $f|_{\partial C}=0$. Then there is a convex function $F\colon C\to\mathbb{R}$ such that F is Hessian-convex on $S\cup C^-$ and $f=F$ on some neighborhood of ∂C.*

Proof Observe $f<0$ on the interior of C. Let g be a Hessian-convex function on $\mathbb{R}^n$ which is negative everywhere on C with $g\geq\frac{1}{2}f$ everywhere on C^-. Since f is not identically zero, this can be done with, for example, $g(x)=\alpha\|x\|^2+\beta$ with suitable constants.

For $\kappa\in\big(0,\frac{1}{2}\big)$, define $F(x)=-m^{\kappa}(-f(x),-g(x))$; then $F(x)=\max(f(x),g(x))$ except when $f(x)$ is close enough to $g(x)$, depending on κ. Since $g<f=0$ on ∂C, it follows that $F=f$ on some neighborhood of ∂C. Moreover, $F=g$ on C^- and therefore F is Hessian-convex on C^-.

By Lemma 8.1, F is convex. Since $m=m^{\kappa}$ and g are smooth and the composition of smooth functions is smooth, it follows F is smooth on S. It only remains to show D^2F is positive definite on S. It suffices to show, for every $a\in S$ and every unit vector $u\in\mathbb{R}^n$, the function $p(t)=-F(a+t\cdot u)$ satisfies $p''(0)<0$. Computing,

$$p'=-m_xf_u-m_yg_u,$$

where

$$m_x = \frac{\partial m}{\partial x}, \quad m_y = \frac{\partial m}{\partial y}$$

and f_u and g_u are the derivatives in direction u at $a \in C$,

$$f_u = df(u), \quad g_u = dg(u).$$

Then

$$p'' = [m_{xx}(f_u)^2 + 2m_{xy} f_u g_u + m_{yy}(g_u)^2] - [m_x f_{uu} + m_y g_{uu}].$$

Since m is smooth and convex down, it follows that $D^2 m$ is negative semidefinite, so the first term is ≤ 0. By Lemma 8.1, we have $m_x \geq 0$ and $m_y \geq 0$. Also $g_{uu} > 0$ everywhere and $f_{uu} > 0$ on S. Since m is linear along rays and $m(x, y) > 0$ on the positive quadrant ($x > 0$ and $y > 0$), it follows that $m_x + m_y > 0$ on the positive quadrant, so the $m_x f_{uu} + m_y g_{uu} > 0$ everywhere on S. Hence $p'' < 0$ everywhere on S, as required. □

A component N of the boundary of a projective manifold M is *Hessian-convex* if N is locally the graph over the tangent hyperplane of a smooth function with positive definite Hessian in some chart.

Proposition 8.3 (smoothing convex boundary) *Suppose M is a projective manifold and ∂M is everywhere locally convex, and also strictly convex at one point on each component of ∂M. Then there is a submanifold $N \subset M$ such that $M \setminus \mathrm{int}(N) \cong [0, 1] \times \partial N$ and ∂N is Hessian-convex.*

Proof Suppose ∂M is strictly convex at $x \in \partial M$. Choose a (subset of a) hyperplane $H \subset M$ close to x so that the component, C, of $M \setminus H$ containing x is a small convex set V. Using local affine coordinates, $S = C \cap \partial M$ is the graph over H of a convex function f which is 0 on $H \cap \partial M$. Apply Corollary 8.2 to produce a smooth function g with positive definite Hessian and satisfying $0 \leq g \leq f$. The graph of g is a smooth hypersurface between H and S. Replace S by this graph. This smoothes out part of ∂M. Repeating this procedure smoothes the entire boundary. □

In a similar way one can prove:

Corollary 8.4 (smoothing convex functions) *Suppose M is a connected affine manifold and $f\colon M \to \mathbb{R}$ is a convex function which is strictly convex at some point. Given $\epsilon > 0$, there is $g\colon M \to \mathbb{R}$ which is smooth, Hessian-convex and satisfies $|f - g| < \epsilon$.*

9 Benzécri's theorem

Theorem 9.1 (Benzécri [5]) *For each* $n > 1$ *there is a* Benzécri constant $R = R_{\mathcal{B}}(n) \leq 5^{n-1}$ *with the following property: Suppose* Ω *is a properly convex open subset of* $\mathbb{RP}^n$ *and* $p \in \Omega$. *Then there is a projective transformation* $\tau \in \mathrm{PGL}(n+1, \mathbb{R})$ *such that* $\tau(p) = 0$ *and* $B(1) \subset \tau(\Omega) \subset B(R)$, *where* $B(t)$ *is the closed ball of radius* t *in* $\mathbb{R}^n$ *centered at* 0.

The projective transformation τ is called a *Benzécri chart for* Ω *centered at* p and the image $\tau(\Omega, p)$ is called *Benzécri position*. The following proof is shorter and more elementary than the traditional proof using John ellipsoids, and also provides an algorithm to find a Benzécri chart. The set of Benzécri charts for (Ω, p) is a compact subset of $\mathrm{PGL}(n+1, \mathbb{R})$.

Proof The proof is by induction on n. If $n = 1$, then Ω is an open interval in $\mathbb{RP}^1$ with closure a closed interval. There is a projective transformation taking Ω to $(-1, 1)$ and p to 0, so $R_{\mathcal{B}}(1) = 1$.

For the inductive step, choose a projective hyperplane $H^{n-1} \subset \mathbb{RP}^n$ containing p. Then $\Omega' = \Omega \cap H$ is an open convex set in $H \cong \mathbb{RP}^{n-1}$ and $p \in \Omega'$. Since Ω is properly convex, $\overline{\Omega}$ is disjoint from some projective hyperplane K^{n-1}. Thus $\overline{\Omega}' = \overline{\Omega} \cap H$ is disjoint from $H \cap K$, which is a hyperplane in H. It follows that Ω' is properly convex in H. By induction, and after choosing appropriate coordinates on an affine patch in H (or using a fixed coordinate system and applying a Benzécri transformation to Ω'), we may assume that $\Omega' \subset \mathbb{R}^{n-1} \times 0 \subset H$ with $p = 0$ and $B^{n-1}(1) \subset \Omega' \subset B^{n-1}(r)$, where $r = R_{\mathcal{B}}(n-1)$.

There are affine coordinates on $\mathbb{RP}^n \setminus K = \mathbb{R}^n$ such that the affine part of H is $\mathbb{R}^{n-1} \times 0$. In what follows we will apply projective transformations in $\mathrm{PGL}(n+1, \mathbb{R})$ which are the identity on H. This moves Ω while keeping Ω' fixed. The first step is to arrange that

$$\Omega \subset \mathbb{R}^{n-1} \times [-1, 1]$$

and $\partial\Omega$ contains a point $z \in \mathbb{R}^{n-1} \times 1$. Then we may shear so that $z = (0, \ldots, 0, 1)$.

Next consider the one-parameter group $A(t) \in \mathrm{PGL}(n+1, \mathbb{R})$ fixing z and H. As t varies, the points that are not fixed move between z and H. This group preserves the family of affine planes $\{x_n = \text{const}\}$ in $\mathbb{R}^n$. Since it fixes z, the affine plane $\mathbb{R}^{n-1} \times 1$

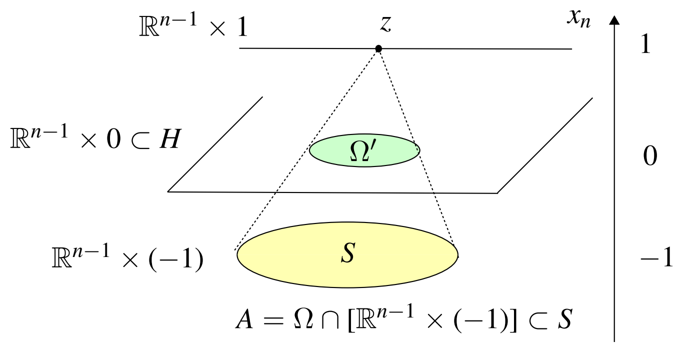


Figure 3: Shadows

is preserved (though not fixed) by this group. Thus we may move Ω by an element of this group so that it is still contained in $\mathbb{R}^{n-1}\times[-1,1]$, still contains z, and

$$A=\Omega\cap(\mathbb{R}^{n-1}\times(-1))$$

is not empty. Let $C\subset\mathbb{R}^{n-1}\times[-1,1]$ be the set of points on all lines ℓ passing through z and some point in Ω'. Then $S=C\cap[\mathbb{R}^{n-1}\times(-1)]$ is the shadow from the point z of Ω' on $\mathbb{R}^{n-1}\times(-1)$. Since Ω is convex, it follows that $A\subset S$. Since $\Omega'\subset B_r(0)$, it follows that S is contained in the shadow of $B_r(0)$, which is the ball $D\subset\mathbb{R}^{n-1}\times(-1)$ of radius $2r$ and center $(0,\dots,0,-1)$. Finally, let X be the union of all line segments in $\mathbb{R}^{n-1}\times[-1,1]$ containing a point of S and B_r. This is contained in the union of the shadows on $\mathbb{R}^{n-1}\times 1$ of $B_r(0)\subset H$ from points in D. This is a ball in $\mathbb{R}^{n-1}\times 1$ of radius $4r$ and center z. It lies within distance $1+r\le 5r$ of 0. □

Let $\mathcal{S}$ be the set of all $\Omega\subset\mathbb{RP}^n$ which are disjoint from some hyperplane and compact, convex and with nonempty interior, equipped with the Hausdorff topology. Let $\mathcal{S}_*\subset\mathcal{S}\times\mathbb{RP}^n$ be the set of all pairs (Ω,p) with p in the interior of Ω with the subspace topology of the product topology. There is an action of $\tau\in\mathrm{PGL}(n+1,\mathbb{R})$ on $\mathcal{S}_*$ given by $\tau(\Omega,p)=(\tau\Omega,\tau p)$. The quotient of $\mathcal{S}_*$ by this action is given the quotient topology and denoted by $\mathcal{B}$.

Corollary 9.2 (Benzécri's compactness theorem [5]) *$\mathcal{B}$ is compact.*

It follows that there is a compact set of preferred charts centered on a point in a properly convex manifold M. Different preferred charts give Euclidean coordinates around p which vary in a compact family independent of M, depending only on dimension.

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DC, DL: *Department of Mathematics, University of California*
Santa Barbara, CA, United States

ST: *School of Mathematics and Statistics, The University of Sydney*
Sydney, NSW, Australia

`cooper@math.ucsb.edu`, `long@math.ucsb.edu`,
`stephan.tillmann@sydney.edu.au`

`http://web.math.ucsb.edu/~cooper/`, `http://web.math.ucsb.edu/~long/`,
`http://www.maths.usyd.edu.au/u/tillmann/`

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