Orderability and Dehn filling

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Motivated by conjectures relating group orderability, Floer homology and taut foliations, we discuss a systematic and broadly applicable technique for constructing left-orders on the fundamental groups of rational homology 3–spheres. Specifically, for a compact 3–manifold M with torus boundary, we give several criteria which imply that whole intervals of Dehn fillings of M have left-orderable fundamental groups. Our technique uses certain representations from $\pi_1(M)$ into $\widetilde{\mathrm{PSL}_2\mathbb{R}}$, which we organize into an infinite graph in $H^1(\partial M; \mathbb{R})$ called the translation extension locus. We include many plots of such loci which inform the proofs of our main results and suggest interesting avenues for future research.

1 Introduction

A group is called *left-orderable* when it admits a total ordering that is invariant under left multiplication (see Clay and Rolfsen [21] for an introduction to the role of orderable groups in topology). We will say that a closed 3–manifold Y is *orderable* when $\pi_1(Y)$ is left-orderable. (Technical aside: by convention, the trivial group is not left-orderable, and so S^3 is not orderable.) Our focus here is on the following question: given a compact orientable 3–manifold M with torus boundary, which Dehn fillings of M are orderable? We care about this question because of its relationship with the following conjecture:

1.1 Conjecture *For an irreducible $\mathbb{Q}$–homology 3–sphere Y, the following are equivalent:*

(1) *Y is orderable.*

(2) *Y is not an L–space.*

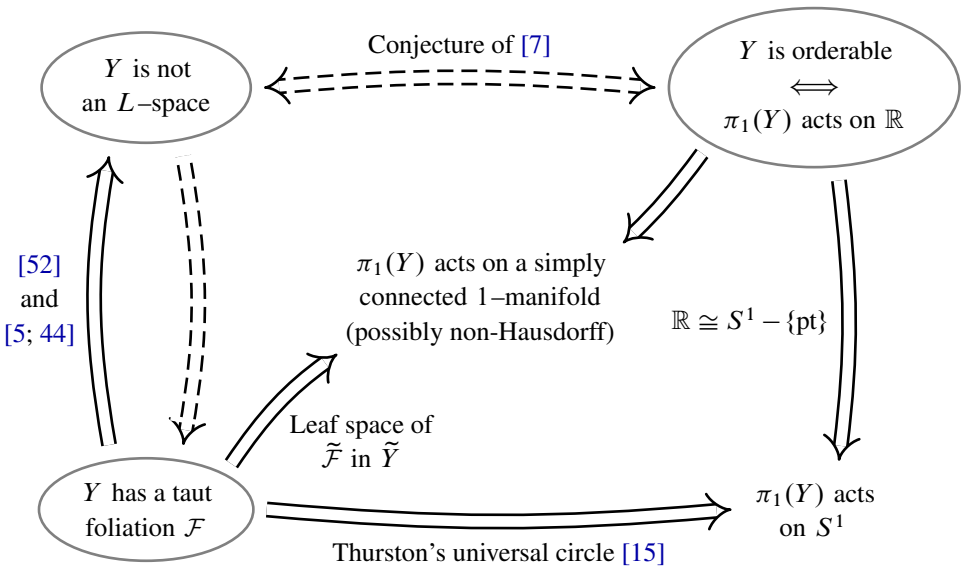(3) *Y admits a coorientable taut foliation.*

Figure 1: Some results related to Conjecture 1.1, which asserts the equivalence of the three circled conditions. Here Y is an irreducible $\mathbb{Q}$–homology 3–sphere, all foliations are coorientable, and all actions are nontrivial, faithful, and orientation-preserving; the solid arrows are theorems and dotted ones conjectures. See Boyer, Gordon and Watson [7] for a complete discussion.

Recall from Ozsváth and Szabó [53] that an L–*space* is a $\mathbb{Q}$–homology 3–sphere with minimal Heegaard Floer homology, specifically one where $\dim \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|$. The equivalence of (1) and (2) was boldly postulated by Boyer, Gordon and Watson [7], which includes a detailed discussion of this conjecture. The equivalence of (2) and (3) was formulated as a question by Ozsváth and Szabó after they proved that (3) implies (2); see Ozsváth and Szabó [52], Kazez and Roberts [44] and Bowden [5]. On its face, Conjecture 1.1 is quite surprising given the disparate nature of these three conditions, but there are actually a number of interconnections between them summarized in Figure 1. Despite much initial skepticism, substantial evidence has accumulated in favor of Conjecture 1.1. For example, it holds for *all* graph manifolds — see Boyer and Clay [6] and Hanselman, Rasmussen, Rasmussen and Watson [38] — many branched covers of knots in the 3–sphere — see Gordon and Lidman [35] — as well as more than 100 000 small-volume hyperbolic 3–manifolds; see Dunfield [28].

Here, we provide further evidence for Conjecture 1.1 by giving tools that order whole families of $\mathbb{Q}$–homology 3–spheres arising by Dehn filling a fixed manifold with torus

boundary. To formulate our first main result, we introduce a new concept: a knot exterior is called *lean* when the longitudinal Dehn filling $M(0)$ is prime and every closed essential surface in $M(0)$ is a fiber in a fibration over S^1. (See Section 2 for precise definitions of standard terminology and conventions used in this introduction.)

1.2 Theorem *Suppose M is the exterior of a knot in a $\mathbb{Z}$–homology 3–sphere. If M is lean and its Alexander polynomial Δ_M has a simple root on the unit circle, then there exists $a > 0$ such that for every rational $r \in (-a, a)$ the Dehn filling $M(r)$ is orderable.*

In fact, with slightly more technical hypotheses, we extend this result to $\mathbb{Q}$–homology 3–spheres in Theorem 7.1 below. The latter result also weakens the requirement that M be lean, replacing it by a condition involving $\mathrm{PSL}_2\mathbb{C}$–character varieties. Combining Theorem 1.2 with Roberts' construction [56] of foliations on Dehn fillings of fibered manifolds immediately gives:

1.3 Corollary *Suppose M is the exterior of a knot in a $\mathbb{Z}$–homology 3–sphere. Suppose that M is lean and fibers over the circle. If Δ_M has a simple root on the unit circle, then Conjecture 1.1 holds for all $M(r)$ with $r \in (-a, a)$ for some $a > 0$. In particular, these $M(r)$ are orderable and have a coorientable taut foliation.*

Our second main result is the following, and applies to branched covers as well as Dehn fillings; see Section 8 for the definition of the trace field of a hyperbolic 3–manifold.

1.4 Theorem *Let K be a hyperbolic knot in a $\mathbb{Z}$–homology 3–sphere Y. If the trace field of the knot exterior M has a real embedding then:*

(1) *For all sufficiently large n, the n–fold cyclic cover of Y branched over K is orderable.*

(2) *There is an interval I of the form $(-\infty, a)$ or (a, ∞) such that the Dehn filling $M(r)$ is orderable for all rational $r \in I$.*

(3) *There exists $b > 0$ such that for every rational $r \in (-b, 0) \cup (0, b)$ the Dehn filling $M(r)$ is orderable.*

The reason the slope 0 is excluded from the conclusion in (3) is that $M(0)$ might have a lens space connect-summand and hence not be orderable. Part (1) of Theorem 1.4 was also proven independently by Steven Boyer (personal communication); the lemma behind part (3) was pointed out to us by Ian Agol and David Futer.
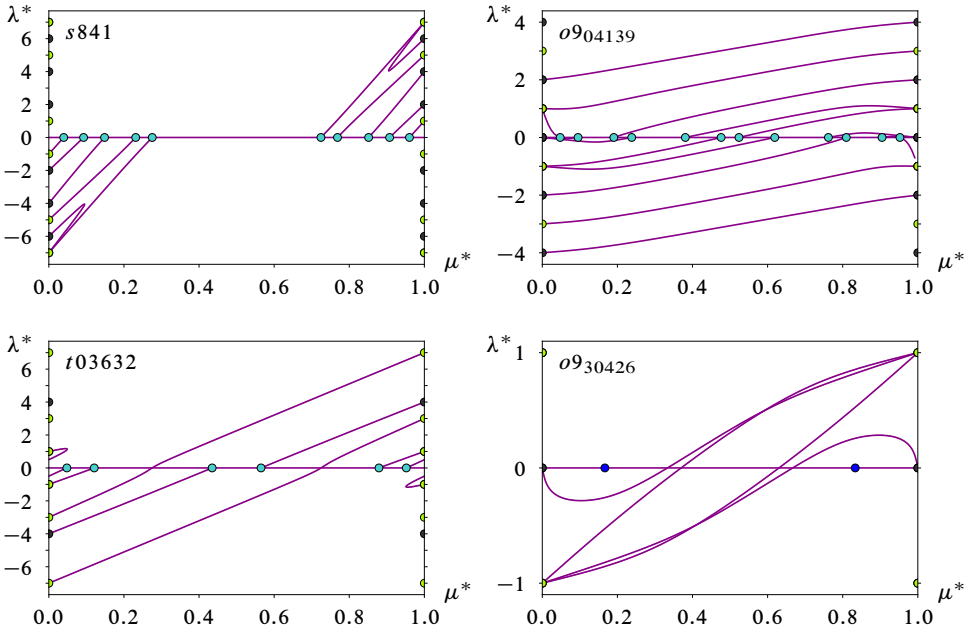
Figure 2: Some translation extension loci that are discussed in detail in Section 5

1.5 Translation extension loci

We prove Theorems 1.2 and 1.4 by studying representations of 3–manifold groups into the nonlinear Lie group $\widetilde{G} = \widetilde{\mathrm{PSL}_2\mathbb{R}}$. Using such representations to order 3–manifold groups goes back at least to Eisenbud, Hirsch and Neumann [30], and has been exploited repeatedly of late to provide evidence for Conjecture 1.1. Closest to our results here, representations to $\widetilde{G}$ were used to obtain ordering results for Dehn surgeries on two-bridge knots in Hakamata and Teragaito [37] and Tran [62], as well as branched covers of two-bridge knots in Hu [42] and Tran [61]. Indeed, some of the results on branched covers in Gordon [34], Hu [42] and Tran [61] can be viewed as special cases of both the statement and the proof of Theorem 1.4(a).

Our main contribution here is to provide a framework for systematically studying representations to $\widetilde{G}$ in a way tailored for applications such as Theorems 1.2 and 1.4. Specifically, given the exterior M of a knot in a $\mathbb{Q}$–homology sphere, we organize the representations of $\pi(M) \to \widetilde{G}$ that are elliptic on $\pi_1(\partial M)$ into a graph in $H^1(\partial M; \mathbb{R})$ called the *translation extension locus* and denoted by $EL_{\widetilde{G}}(M)$. Very roughly, the locus $EL_{\widetilde{G}}(M)$ is the image of the "character variety of $\widetilde{G}$ representations" of $\pi_1(M)$ in the corresponding object for $\pi_1(\partial M)$ under the map induced by $\partial M \hookrightarrow M$; as such,

it parallels the A–polynomial story of Cooper, Culler, Gillet, Long and Shalen [22]. This locus was first studied by Khoi in his computations of Seifert volumes of certain hyperbolic 3–manifolds; see Khoi [45]. While the graph $EL_{\widetilde{G}}(M)$ is infinite, it is actually compact modulo a discrete group of symmetries, and so it is possible to draw a picture of it: see Figure 2 for some examples, and Section 5 for many more.

To prove Theorems 1.2 and 1.4, we give a simple criterion (Lemma 4.4) which says roughly that if the line in $H^1(\partial M; \mathbb{R})$ of slope $-r$ meets $EL_{\widetilde{G}}(M)$ away from the origin, then the Dehn surgery $M(r)$ is orderable. (Lemma 4.5 is our analogous result for branched covers.) In Section 5, we use Lemma 4.4 to order large intervals of Dehn fillings in some specific examples; indeed, the conclusions of Theorems 1.2 and 1.4 are much weaker than what we typically observe in practice.

Once we establish the basic properties of these loci in Theorem 4.3, our main results are proved by using the given hypotheses to produce at least a small arc of $EL_{\widetilde{G}}(M)$ in a certain location in $H^1(\partial M; \mathbb{R})$, and then applying Lemma 4.4 at many points along the arc. For Theorem 1.2, we build the arc by using Heusener and Porti [40] to deform reducible representations corresponding to a root of Δ_M on the unit circle to more interesting representations in $\mathrm{PSL}_2\mathbb{R}$. In Theorem 1.4, we first use a combination of hyperbolic geometry and algebraic geometry to produce an arc which contains (the image of) the representation associated with the real embedding of the trace field. The key issue of the arc's location in $H^1(\partial M; \mathbb{R})$ hinges on the result of Calegari [13, Corollary 2.4] that, for a lift of the holonomy representation of the hyperbolic structure of M to $\mathrm{SL}_2\mathbb{C}$, the trace of the longitude is -2 rather than 2.

1.6 Applicability

While there are many cases where neither Theorem 1.2 nor Theorem 1.4 applies, we next argue that some of their hypotheses are quite generic and therefore our results should be interpreted as providing a profusion of orderable 3–manifolds.

For example, the Alexander polynomial hypothesis of Theorem 1.2 holds for the vast majority of the simpler 3–manifolds that one can tabulate: specifically, it occurs for 81.2% of the 1.7 million prime knots with at most 16 crossings — see Hoste, Thistlethwaite and Weeks [41] — and 96.2% of the 59 068 hyperbolic $\mathbb{Q}$–homology solid tori that can be triangulated with at most 9 ideal tetrahedra; see Burton [11]. We also looked at more complicated knots by taking braid closures of random 10–strand braids with 100–1 000 crossings (conditioned on the closure being a knot rather

than a link); of the 100 000 such knots we examined, some 99.87% had Alexander polynomial with a simple root on the unit circle. Finally, of particular interest in light of Conjecture 1.1 are the L–space knots in S^3, that is, those with a nontrivial Dehn surgery producing an L–space. The Alexander polynomials of such knots have a very special form — see Ozsváth and Szabó [53, Corollary 1.3] — and it follows from Konvalina and Matache [46] that L–space knots have Δ_M with a root on the unit circle; experimentally, there is always a simple root, but we are unable to prove this.

Turning to Theorem 1.4, it is also very common for a hyperbolic 3–manifold to have a trace field with a real embedding. For example, Goerner [33] has calculated the trace fields of all 61 911 cusped manifolds that can be triangulated with at most 9 ideal tetrahedra; see Burton [11]. Among these, some 95.5% had trace fields with a real embedding. Indeed, about 36.3% of the roots of the polynomials defining these fields were real. We conjecture that, for any reasonable model of a random hyperbolic 3–manifold, the trace field has a real embedding with probability 1.

In contrast, the leanness condition of Theorem 1.2, whose use in the proof is more technical, is hardly ubiquitous. While it can easily be arranged by, for example, taking the exterior of a knot in $S^2 \times S^1$ which generates $H_1(S^2 \times S^1; \mathbb{Z})$, it seems that a generic knot in S^3 is not lean: work of Gabai [31] implies that a lean knot must be fibered, and the latter condition is experimentally of probability 0 in the above models of random knots (see also Dunfield and Thurston [29]). That said, the strengthened version of Theorem 1.2, namely Theorem 7.1, requires the weaker hypothesis that M is longitudinally rigid (see Section 7). This condition might well be generic — we know of only a few cases where it fails — but unfortunately it is hard to study in bulk.

1.7 Outline of the rest of the paper

Sections 2 and 3 review some definitions and background results; Section 2 discusses topology, character varieties, and real algebraic geometry, whereas Section 3 is focused on the group $\widetilde{G}$. Section 4 defines the translation extension locus and states its basic properties. Section 5 is the longest and arguably the heart of the paper; it gives twelve examples of translation extension loci and discusses their properties as they relate to our results here. Section 6 proves the basic structure result for these loci (Theorem 4.3), as well as Lemmas 4.4 and 4.5. Sections 7 and 8 then prove Theorems 1.2 and 1.4, respectively. Section 8 includes Remark 8.7, which answers Question 6 of Lidman and Watson [48] by giving an example of a hyperbolic $\mathbb{Q}$–homology solid torus that is not

fibered and where every nonlongitudinal Dehn filling is an L–space. Finally, Section 9 lists ten related open problems.

2 Background

2.1 Topological terminology and conventions

We first review some basic concepts that will be used throughout this paper, and in the process set some standing conventions. First, all 3–manifolds will be assumed connected and orientable unless noted otherwise. A *knot K* in a 3–manifold Y is a smoothly embedded S^1 inside of Y. The *exterior* of K is Y with an open tubular neighborhood about K removed; this is a compact 3–manifold with boundary a torus. A $\mathbb{Q}$–homology 3–sphere is a closed 3–manifold whose rational homology is the same as that of S^3; a $\mathbb{Z}$–homology 3–sphere is defined analogously. A $\mathbb{Q}$–homology solid torus is a compact 3–manifold M with boundary a torus where $H_*(M; \mathbb{Q}) \cong H_*(D^2 \times S^1; \mathbb{Q})$; this is equivalent to M being the exterior of a knot in some $\mathbb{Q}$–homology 3–sphere. Analogously, a $\mathbb{Z}$–homology solid torus is a compact 3–manifold M with boundary a torus where $H_*(M; \mathbb{Z}) \cong H_*(D^2 \times S^1; \mathbb{Z})$; again, this is equivalent to M being the exterior of a knot in a $\mathbb{Z}$–homology 3–sphere.

We call a compact orientable surface F in a 3–manifold M *essential* when it is properly embedded, incompressible, and not boundary parallel; here, incompressible means that $\pi_1(F) \to \pi_1(M)$ is injective and that F is not a 2–sphere bounding a 3–ball.

2.2 Framings and slopes

For a $\mathbb{Q}$–homology solid torus M, we denote the inclusion map of its boundary by $\iota\colon \partial M \to M$. We define a *homologically natural framing* to be a generating set (μ, λ)

for $H_1(\partial M; \mathbb{Z})$, where $\iota_*(\lambda) = 0$ in the rational homology $H_1(M; \mathbb{Q})$. While the *homological longitude* λ is defined up to sign, there are infinitely many choices for μ.

An isotopy class of unoriented essential simple closed curves in ∂M is called a *slope*. A slope can be recorded by a primitive element in $H_1(\partial M; \mathbb{Z})$, which is well-defined up to sign. Once we fix a framing (μ, λ) for ∂M, we shall identify slopes with elements of $\mathbb{Q} \cup \{\infty\}$ via $p/q \leftrightarrow \pm(p\mu + q\lambda)$.

2.3 Representation and character varieties

Throughout this paper, we will use $G_{\mathbb{C}}$ to denote the Lie group $\mathrm{PSL}_2\mathbb{C} \cong \mathrm{PGL}_2\mathbb{C}$. We now review some basic facts about representation and character varieties with target group $G_{\mathbb{C}}$; for details, see eg [39]. For a compact manifold M, the representation space $R(M) = \mathrm{Hom}(\pi_1(M), G_{\mathbb{C}})$ is an affine algebraic set in some $\mathbb{C}^n$. However, we are usually only interested in representations up to conjugacy by elements of $G_{\mathbb{C}}$, so we consider the minimal Hausdorff quotient $X(M)$ of $R(M)$ generated by the orbits of the conjugation action of $G_{\mathbb{C}}$. Equivalently, $X(M)$ is the invariant theory quotient $R(M) /\!\!/ G_{\mathbb{C}}$; hence $X(M)$ is again an affine algebraic set, which is referred to as the $G_{\mathbb{C}}$–*character variety* of M. (These algebraic sets are not always irreducible, but we still refer to them as "varieties" for historical reasons.) For each group element $\gamma \in \pi_1(M)$, there is a regular function $\mathrm{tr}_\gamma^2 \colon X(M) \to \mathbb{C}$ given by $\mathrm{tr}_\gamma^2([\rho]) = \mathrm{tr}(\rho(\gamma))^2$; we must take the square here because the trace of a matrix in $G_{\mathbb{C}}$ is only defined up to sign. One can always choose a finite set of elements in $\pi_1(M)$ so that the corresponding tr^2 functions give a complete system of coordinates for the affine algebraic set $X(M)$ [39, Corollary 2.3].

We will frequently regard $G_{\mathbb{C}}$ as the group of orientation-preserving isometries of hyperbolic 3–space $\mathbb{H}^3$. The group $G_{\mathbb{C}}$ acts on $\mathbf{P}^1(\mathbb{C})$ by Möbius transformations, in a way that extends the action on $\mathbb{H}^3$ to the sphere at infinity $\partial\mathbb{H}^3 = S^2_\infty \cong \mathbf{P}^1(\mathbb{C})$. A representation $\rho \in R(M)$ is called *reducible* when $\rho(\pi_1(M))$ fixes a point in $\mathbf{P}^1(\mathbb{C})$ under the Möbius action of $G_{\mathbb{C}}$; otherwise ρ is called *irreducible*. A character $\chi \in X(M)$ is called *reducible* if any (equivalently all) representations ρ mapping to χ are reducible, and analogously for *irreducible*. While nonconjugate representations can have the same character, this can only happen in the reducible case [39, Lemma 3.15].

Now suppose M is a compact 3–manifold with torus boundary, and let $\iota \colon \partial M \to M$ be the inclusion map. By restricting representations, we get an induced regular map $\iota^* \colon X(M) \to X(\partial M)$. We will need the following fact in the proof of Theorem 4.3:

2.4 Lemma *The image of $\iota^*\colon X(M) \to X(\partial M)$ has complex dimension at most 1.*

For $SL_2\mathbb{C}$–character varieties, rather than the $G_\mathbb{C}$ ones we work with here, the corresponding result is [23, Corollary 10.1]. As not every representation $\pi_1(M) \to G_\mathbb{C}$ lifts to $SL_2\mathbb{C}$ by [39, Theorem 1.4], we must prove Lemma 2.4 directly. However, the argument is essentially identical to the $SL_2\mathbb{C}$ case, and the proof may be safely skipped at first reading.

Proof of Lemma 2.4 We identify $\pi_1(\partial M)$ with $\mathbb{Z} \oplus \mathbb{Z}$ via a fixed framing (μ, λ). We will view the character variety $X(\partial M)$ as the minimal Hausdorff quotient of $R(\partial M) = \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}, G_\mathbb{C})$ by the conjugation action. It is shown in [40, Lemma 7.4] that $R(\partial M)$ has exactly two irreducible components. The first consists exactly of the conjugacy class of a representation onto a Klein 4–group whose nontrivial elements are rotations about three mutually orthogonal lines in $\mathbb{H}^3$. The other component consists of representations that either send both μ and λ to nonparabolic elements with a common axis, or to parabolic elements with a common fixed point. By [40, Lemma 7.4], this component is 4–dimensional and smooth away from the trivial representation. We will denote its image in $X(\partial M)$ by D. The conjugacy class of a nonparabolic representation is closed and isomorphic to the coset space $G_\mathbb{C}/S$, where S is the stabilizer of the axis in $G_\mathbb{C}$, and so the conjugacy class has dimension 2. The conjugacy class of a parabolic representation, on the other hand, is not closed and contains the trivial representation in its closure. Thus the conjugacy class of any parabolic representation maps to the same point in $X(\partial M)$ as the trivial representation. These two facts imply that the complex variety D is 2–dimensional.

Since D is the only irreducible component of $X(\partial M)$ with dimension larger than 1, to prove the lemma it suffices to show that if Z is an irreducible component of $X(M)$ such that $\iota^*(Z) \subset D$ then $\iota^*(Z)$ has dimension at most 1. For this it is convenient to pass to a 2–fold cover of $X(M)$.

Following [27], we define the *augmented representation variety* $\hat{R}(M)$ to be the subalgebraic set of $R(M) \times \mathbf{P}^1$ consisting of all pairs (ρ, x) where x is a point of $\mathbf{P}^1(\mathbb{C})$ that is fixed by the image of $\pi_1(\partial M)$ under $\rho \circ \iota$. On a typical irreducible component of $R(M)$, there are generically two points fixed by the group $\rho(\pi_1(\partial M))$, and so the projection $(\rho, x) \mapsto \rho$ gives a regular map of degree 2 onto an irreducible component of $R(M)$. There is a natural diagonal action of $G_\mathbb{C}$ on $\hat{R}(M)$ which acts by conjugation on the representation ρ and by the induced Möbius transformation on $\mathbf{P}^1$. The quotient $\hat{X}(M) = \hat{R}(M) /\!\!/ G_\mathbb{C}$ is the *augmented character variety* of M.

The augmented representation and character varieties of the boundary torus, $\widehat{R}(\partial M)$ and $\widehat{X}(\partial M)$, are defined analogously. These augmented varieties for ∂M are in fact irreducible, since the pesky Klein 4–group representations have no fixed points on $\boldsymbol{P}^1(\mathbb{C})$ and hence are missing from $\widehat{R}(\partial M)$. Our choice of framing (μ, λ) determines an identification of $\widehat{X}(\partial M)$ with $\mathbb{C}^\times \times \mathbb{C}^\times$ as follows. If ρ is given by

$$\rho(\mu) = \pm \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \quad \text{and} \quad \rho(\lambda) = \pm \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix},$$

and ∞ denotes the point of $\boldsymbol{P}^1(\mathbb{C})$ with homogeneous coordinates $[1 : 0]$, then the $G_\mathbb{C}$–orbit of the pair (ρ, ∞) is identified with the point (z^2, w^2), that is, with the pair consisting of the holonomies of $\rho(\mu)$ and $\rho(\lambda)$.

Since the conjugacy class of any parabolic representation of $\pi_1(\partial M)$ has the same image in $X(\partial M)$ as the trivial representation, when $\widehat{X}(\partial M)$ is identified with $\mathbb{C}^\times \times \mathbb{C}^\times$ in this way, any pair $(\rho, x) \in \widehat{R}(\partial M)$ where ρ is parabolic will be mapped to $(1, 1)$ by the quotient map $\widehat{R}(\partial M) \to \widehat{X}(\partial M)$. Also, the deck transformation for the branched covering $\widehat{X}(\partial M) \to D$ is given by $(z, w) \mapsto (1/z, 1/w)$.

The augmented and unaugmented character varieties fit into the commutative diagram

$$\begin{array}{ccc} \widehat{X}(M) & \xrightarrow{\hat{\iota}^*} & \widehat{X}(\partial M) \\ \downarrow & & \downarrow \\ X(M) & \xrightarrow{\iota^*} & X(\partial M) \end{array}$$

in which the vertical maps are induced by the projection $(\rho, x) \mapsto \rho$. Since the vertical maps are finite, it suffices to show that for each irreducible component $\widehat{Z}$ of $\widehat{X}(M)$, its image $\hat{\iota}^*(\widehat{Z})$ is at most 1–dimensional.

To prove this we apply the same argument used in [23, Corollary 10.1] in the case of $SL_2\mathbb{C}$. Namely, we consider the real 1–form

$$\omega = \log|z|\, d\arg w - \log|w|\, d\arg z$$

defined on $\mathbb{C}^\times \times \mathbb{C}^\times$, viewed as a real manifold. The form ω is not closed since

$$d\omega = d\log|z| \wedge d\arg w - d\log|w| \wedge d\arg z.$$

However, since $d\omega$ is the imaginary part of the complex 2–form $d\log z \wedge d\log w$, it does restrict to a closed form on any complex curve in $\mathbb{C}^\times \times \mathbb{C}^\times$. Moreover, it follows from a result in Craig Hodgson's thesis (see [22, Section 4.4]) that ω pulls back under $\hat{\iota}^*$ to an exact 1–form on $\widehat{Z}$. In fact, the pull-back of ω is equal to $-2\, d\mathrm{vol}$, where

vol is the real analytic function on $\widehat{X}(M)$ that assigns to $([\rho], z)$ the volume of the representation ρ, as defined in [26, Section 2.1]. (In particular, there is a mysterious cohomology class which obstructs a given complex curve in $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ from arising as a component of the image of $\hat{\imath}^{*}$.) To complete the argument, we just observe as in [23] that, since ω is not exact on $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ but pulls back to an exact 1–form on $\widehat{Z}$, we would obtain a contradiction if $\hat{\imath}^{*}(\widehat{Z})$ were dense in $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$. Thus $\hat{\imath}^{*}(\widehat{Z})$ must be at most 1–dimensional, as required. $\qquad\square$

2.5 Real points

We will need a few basic facts from real algebraic geometry; for a general reference, see [1]. If X is an affine algebraic set in $\mathbb{C}^n$, we denote the real points $X \cap \mathbb{R}^n$ by $X_{\mathbb{R}}$. When X can be cut out by polynomials with real coefficients, we say that X is *defined over* $\mathbb{R}$; in this case, the set X is invariant under coordinatewise complex conjugation $\tau \colon \mathbb{C}^n \to \mathbb{C}^n$, and $X_{\mathbb{R}}$ is precisely the set of fixed points of τ. If X is a quasiprojective variety in $\boldsymbol{P}^n(\mathbb{C})$ that can be defined by real polynomials, then the real points $X_{\mathbb{R}}$ are again the fixed points of the involution τ on $\boldsymbol{P}^n(\mathbb{C})$ which acts by complex conjugation of the projective coordinates; in any affine chart whose hyperplane at infinity is defined by a real linear form, the points of $X_{\mathbb{R}}$ are precisely the points of X whose coordinates are real.

In real algebraic geometry, the projective space $\boldsymbol{P}^n(\mathbb{R})$ is isomorphic to an *affine* algebraic variety, and hence any quasiprojective variety is isomorphic to an affine one. When working with real algebraic varieties, it is often natural to consider the larger collection of semialgebraic sets, that is, those defined by polynomial inequalities, and to consider properties such as irreducibility of a real algebraic set in that enlarged category. The dimension of a real semialgebraic set is equal to its topological dimension. Here, we will need only the following three results:

2.6 Proposition [1, Theorem 5.43] *Suppose X is an affine real semialgebraic set which is closed and bounded. If the dimension of X is at most 1, then X is homeomorphic to a finite graph, where graphs are allowed to have isolated vertices.*

2.7 Proposition [1, Theorem 5.48] *A real semialgebraic set is locally path connected.*

2.8 Proposition *Suppose X is a complex affine algebraic curve defined over $\mathbb{R}$. If x_0 is a smooth point of X that lies in $X_{\mathbb{R}}$, then x_0 has a classical neighborhood in $X_{\mathbb{R}}$ which is a smooth arc.*

Proof The curve X has finitely many singular points which are permuted by τ. Let X' be the complementary set of smooth points. Now X' is a smooth surface and the restriction of τ to X' is an orientation-reversing involution. Using a Riemannian metric on X' which is invariant under τ, it is easy to see that $X'_{\mathbb{R}}$ is a smooth 1–manifold, proving the proposition. $\square$

2.9 Real representations

Throughout this paper, we will set $G = \mathrm{PSL}_2\mathbb{R}$ and $K = \mathrm{PSU}_2$, where both groups are viewed as subgroups of $G_{\mathbb{C}} = \mathrm{PSL}_2\mathbb{C}$. We will also occasionally consider the subgroup $\mathrm{PGL}_2\mathbb{R}$ in $G_{\mathbb{C}}$, which makes sense via the identification of $\mathrm{PSL}_2\mathbb{C}$ with $\mathrm{PGL}_2\mathbb{C}$; geometrically, the subgroup $\mathrm{PGL}_2\mathbb{R}$ is the full stabilizer in $G_{\mathbb{C}}$ of the copy of $\mathbb{H}^2$ fixed by G (in particular, $\mathrm{PGL}_2\mathbb{R}$ includes orientation-reversing isometries of $\mathbb{H}^2$). We will view $R_G(M) = \mathrm{Hom}(\pi_1(M), G)$ as a subset of $R(M)$, and we will denote by $X_G(M)$ the image of $R_G(M)$ under the quotient map $t: R(M) \to X(M)$. Thus $X_G(M) \subset X_{\mathbb{R}}(M)$. By [39, Lemma 10.1], in fact the set $X_{\mathbb{R}}(M)$ is the image of $R_{\mathrm{PGL}_2\mathbb{R}}(M) \cup R_K(M)$ under t. Since $X_G(M)$ is the image of a real algebraic set under a polynomial map, it is a real semialgebraic subset of $X_{\mathbb{R}}(M)$. Note that $X_G(M)$ is *not* the quotient of $R_G(M)$ under the action of G by conjugation, even neglecting the issue of nonclosed orbits; rather, it is essentially the quotient of $R_G(M)$ under conjugation by the larger group $\mathrm{PGL}_2\mathbb{R}$. Geometrically, the point is that $\mathrm{PGL}_2\mathbb{R}$, not G, is the full stabilizer in $G_{\mathbb{C}} = \mathrm{Isom}^+(\mathbb{H}^3)$ of the standard $\mathbb{H}^2$ in $\mathbb{H}^3$. Since G can be characterized as the subgroup of $\mathrm{PGL}_2\mathbb{R}$ which preserves the orientation of $\mathbb{H}^2$, considering representations into G up to conjugacy in $G_{\mathbb{C}}$ amounts to forgetting the orientation on $\mathbb{H}^2$. We also use $X_K(M)$ to denote the image of $R_K(M)$ in $X_{\mathbb{R}}(M)$; in this case, the set $X_K(M)$ is the ordinary quotient $R_K(M)/K$. Let S be the subgroup $\mathrm{PSO}_2 = G \cap K \cong S^1$, which is the stabilizer of a point in $\mathbb{H}^2$ under the action of G. As usual, we use $X_S(M)$ to denote the image of $R_S(M)$ in $X_{\mathbb{R}}(M)$; note here that any representation in $R_S(M)$ factors through $H_1(M; \mathbb{Z})$ since S itself is abelian. The next two lemmas will be used in the proof of Theorem 1.4.

2.10 Lemma *The intersection $X_K(M) \cap X_G(M)$ is exactly $X_S(M)$. In particular, if $[\rho]$ is in $X_K(M) \cap X_G(M)$ then $[\rho]$ is reducible over $G_{\mathbb{C}}$.*

Proof Consider $[\rho] \in X_K(M) \cap X_G(M)$. If $[\rho]$ is irreducible over $G_{\mathbb{C}}$, then any two representatives of $[\rho]$ are conjugate in $G_{\mathbb{C}}$, so we can assume that $\rho \in R_G(M)$ and that every $\rho(\gamma)$ is elliptic or trivial as ρ is also conjugate into $R_K(M)$. However,

every subgroup of G consisting solely of elliptic elements has a global fixed point x_0 in $\mathbb{H}^2$ (see eg [2, Theorem 4.3.7]). The representation ρ then fixes pointwise the geodesic in $\mathbb{H}^3$ that is perpendicular to $\mathbb{H}^2$ and contains x_0; in particular, it is reducible over $G_{\mathbb{C}}$, contradicting our initial assumption. So we have reduced to the case where $[\rho]$ is reducible over $G_{\mathbb{C}}$, and there we can choose the representative ρ to be diagonal. As $[\rho]$ is in $X_K(M)$, we have $\mathrm{tr}_\gamma^2(\rho)$ in $[0, 4]$ for all $\gamma \in \pi_1(M)$. A diagonal matrix A in $G_{\mathbb{C}}$ with $\mathrm{tr}^2(A)$ in $[0, 4]$ has nonzero entries on the unit circle, and so ρ comes from a homomorphism $\pi_1(M) \to S^1/\{\pm 1\}$. In particular, the representation ρ is conjugate into $R_S(M)$, as desired. $\qquad\qquad\square$

2.11 Lemma *The map $t\colon R(M) \to X(M)$ has the weak path lifting property, that is, given a path $c\colon I \to X(M)$ there is a $\tilde{c}\colon I \to R(M)$ with $c = t \circ \tilde{c}$. The same is true for its restrictions $R_K(M) \to X_K(M)$, $R_G(M) \to X_G(M)$ and $R_{\mathrm{PGL}_2\mathbb{R}}(M) \to X_{\mathrm{PGL}_2\mathbb{R}}(M)$. Moreover, if $c(0)$ is an irreducible character, then we can require $\tilde{c}(0)$ to be any specified representation in $t^{-1}(c(0))$.*

Proof With regards to the main claim, for the group K this is [10, Section II.6], for the group $G_{\mathbb{C}}$ this is [47, Corollary 3.3], and the other two cases follow from [4, Lemma 2.1]. We may specify $\tilde{c}(0)$ arbitrarily because in this case all of the representations in $t^{-1}(c(0))$ are conjugate. $\qquad\qquad\square$

The next lemma is a comforting fact, but it is not needed for any of the main results in this paper; indeed, we use it only in Lemma 6.8, which is just a remark to justify a claim about the examples in Section 5; you should therefore skip the proof at first reading.

2.12 Lemma *The subsets $X_K(M)$ and $X_G(M)$ are closed in $X(M)$ in the classical topology.*

Proof For ease of notation, we set $G^\star = \mathrm{PGL}_2\mathbb{R}$ and $\Gamma = \pi_1(M)$, and also suppress the manifold M from our representation and character varieties. The subset X_K is compact since it is the image of the compact set R_K under a continuous map, and so we turn immediately to X_G. Given $\chi \in \overline{X}_G$, which must be in $X_{\mathbb{R}}$, we need to show that χ is in X_G. We give separate arguments depending on whether χ is reducible over $G_{\mathbb{C}}$.

First, suppose χ is reducible over $G_{\mathbb{C}}$. Let ρ be a diagonal representation into $G_{\mathbb{C}}$ with character χ. The top-left entry of ρ gives a homomorphism $\psi\colon \Gamma \to \mathbb{C}^\times/\{\pm 1\}$.

Since $\chi \in \overline{X}_G$, we have that $\operatorname{tr}_\gamma^2(\rho) \in [0, \infty)$ for all $\gamma \in \Gamma$. Consequently, all $\psi(\gamma)$ are in $S^1 \cup \mathbb{R}^\times$. In fact, the image $\psi(\Gamma)$ must be contained entirely in one of S^1 or $\mathbb{R}^\times$, as otherwise we can easily find a $\psi(\gamma)$ that is not in $S^1 \cup \mathbb{R}^\times$. If $\psi(\Gamma)$ is in S^1, then ρ is conjugate into $S \le G$, and if instead $\psi(\Gamma)$ is in $\mathbb{R}^\times$ then ρ is already in R_G. Thus, when χ is reducible we have shown that $\chi \in X_G$, as desired.

Suppose instead that χ is irreducible over $G_\mathbb{C}$. By [39, Lemma 10.1], we need to consider two cases, depending on whether χ is in the image of R_K or $R_{G^\star}$. To start, suppose χ can be realized by a $\rho \in R_K$; in particular, $\rho(\Gamma)$ fixes a point $x_0 \in \mathbb{H}^3$. As ρ is irreducible, there must be γ_1 and γ_2 in Γ where the $\rho(\gamma_i)$ are elliptic elements with rotation axes L_i and $L_1 \cap L_2 = \{x_0\}$. By Proposition 2.7 and Lemma 2.11, we can approximate ρ by an irreducible ρ' in R whose character χ' is in X_G and where the $\rho'(\gamma_i)$ are still elliptic with axes L_i' very close to the L_i. As ρ' is conjugate into R_G, there is a totally geodesic plane P' preserved by $\rho'(\Gamma)$, and the axes L_i' must meet P' in right angles; in particular, the angle between L_1' and L_2', as measured along their perpendicular bisector (which is contained in P'), is 0. For ρ' close enough to ρ, this is impossible as $L_1 \cap L_2 = \{x_0\}$. So we cannot have χ in X_K.

Thus our final case is when χ is irreducible and in $X_{G^\star}$. As $X_{G^\star}$ is locally path connected by Proposition 2.7 and χ is a limit point of $X_G \subset X_{G^\star}$, we can find a path χ_t from χ_0 in X_G to χ. Applying Lemma 2.11 to $R_{G^\star} \to X_{G^\star}$, we lift χ_t to a path ρ_t starting at $\rho_0 \in R_G$. As G is a connected component of $G^\star$ and each $\rho_0(\gamma) \in G$, it follows by continuity that $\rho_1(\gamma)$ is also in G. Thus ρ_1 is in R_G and so χ is in X_G, as desired, proving the lemma. $\qquad\square$

3 Basic facts about $\widetilde{\mathrm{PSL}_2\mathbb{R}}$

For the group $G = \mathrm{PSL}_2\mathbb{R}$, consider its universal covering Lie group $\widetilde{G} = \widetilde{\mathrm{PSL}_2\mathbb{R}}$, which is also its universal central extension (see [32, Section 5] or [14, Section 2.3.3]):

$$0 \to \mathbb{Z} \to \widetilde{G} \xrightarrow{p} G \to 1$$

Concretely, we realize $\widetilde{G}$ as follows. We identify $S^1 = \boldsymbol{P}^1(\mathbb{R})$ with $\mathbb{R}/\mathbb{Z}$ and view the quotient map as the universal covering map $\mathbb{R} \to \boldsymbol{P}^1(\mathbb{R})$. The projective action of G on $\boldsymbol{P}^1(\mathbb{R})$ is faithful, so we identify G with its image subgroup in $\mathrm{Homeo}^+(S^1)$. Every homeomorphism of $\boldsymbol{P}^1(\mathbb{R})$ in G lifts to countably many homeomorphisms of $\mathbb{R}$. We define $\widetilde{G}$ to be the subgroup of $\mathrm{Homeo}^+(\mathbb{R})$ consisting of all lifts of elements of G. The kernel of $p\colon \widetilde{G} \to G$, which is also the center of $\widetilde{G}$, is the deck group of

$\mathbb{R} \to \boldsymbol{P}^1(\mathbb{R})$, namely the group of integer translations. We let s be the element of the center which acts by $x \mapsto x+1$, and write elements of the center multiplicatively as s^k. An element of $\widetilde{G}$ is called elliptic, parabolic, or hyperbolic when its image in G is of that type. The disjoint partition of G into elliptic, parabolic, hyperbolic, and trivial elements means that $\widetilde{G}$ is similarly partitioned into elliptic, parabolic, hyperbolic, and central elements.

3.1 Translation number

An important concept for us is the *translation number* of an element $\widetilde{g} \in \widetilde{G}$, which is defined as

$$(3.2) \qquad \mathrm{trans}(\widetilde{g}) = \lim_{n \to \infty} \frac{\widetilde{g}^n(x) - x}{n} \quad \text{for some } x \in \mathbb{R}.$$

This is well-defined since the value of the limit is independent of the choice of x.

Here are some key properties of the translation number; see [32, Section 5] or [14, Section 2.3.3] for extensive background and details. First, the map $\mathrm{trans}\colon \widetilde{G} \to \mathbb{R}$ is continuous and is constant on conjugacy classes in $\widetilde{G}$. Also, it is a homogenous quasimorphism for $\widetilde{G}$ in the sense discussed in Section 6.3 below. Considering the center $Z(\widetilde{G}) = \langle s \rangle$ as above, we have $\mathrm{trans}(s^k) = k$, and moreover $\mathrm{trans}(\widetilde{g} \cdot s^k) = \mathrm{trans}(\widetilde{g}) + k$ for any $\widetilde{g}$ in $\widetilde{G}$.

Since they map to elements in G that have a fixed point in $\boldsymbol{P}^1(\mathbb{R})$, all parabolic and hyperbolic elements of $\widetilde{G}$ have integral translation numbers. In contrast, any real number arises as the translation number of an elliptic element. Moreover, if $\widetilde{g}$ is an elliptic element of $\widetilde{G}$, then $2\pi \mathrm{trans}(\widetilde{g})$ is equal, modulo 2π, to the rotation angle of $p(\widetilde{g})$ at its unique fixed point in $\mathbb{H}^2$.

3.3 The Euler class

Given a group Γ and a representation $\rho\colon \Gamma \to G$, the Euler class $Euler(\rho) \in H^2(\Gamma; \mathbb{Z})$ is a complete obstruction to lifting ρ to a representation $\widetilde{\rho}\colon \Gamma \to \widetilde{G}$ such that $p \circ \widetilde{\rho} = \rho$. Here is a review of its definition; see eg [32, Section 6.2] for details. Choose an arbitrary section $\sigma\colon \Gamma \to \widetilde{G}$, that is, a function satisfying $p \circ \widetilde{\rho} = \rho$. Define a function $\phi_\sigma\colon \Gamma \times \Gamma \to \mathbb{Z}$ by

$$s^{\phi_\sigma(\gamma_1, \gamma_2)} = \sigma(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_1\gamma_2)^{-1}, \quad \text{where } Z(\widetilde{G}) = \langle s \rangle.$$

Associativity of group multiplication implies that ϕ_σ satisfies the 2–cocycle relation

$$\phi_\sigma(\gamma_2, \gamma_3) - \phi_\sigma(\gamma_1\gamma_2, \gamma_3) + \phi_\sigma(\gamma_1, \gamma_2\gamma_3) - \phi_\sigma(\gamma_1, \gamma_2) = 0.$$

We define *Euler*(ρ) to be the class in $H^2(\Gamma; \mathbb{Z})$ represented by ϕ_σ. To see that this is well-defined, note that if σ' is another section, then $\phi_\sigma - \phi_{\sigma'}$ is the coboundary of the 1–cochain $\tau \colon \Gamma \to \mathbb{Z}$ determined by $s^{\tau(\gamma)} = \sigma(\gamma)\sigma'(\gamma)^{-1}$. Now a section σ is actually a lift of the representation ρ when the 2–cocycle ϕ_σ is identically zero; if ϕ_σ is merely a coboundary, say $\phi_\sigma = \delta(\tau)$, then the section σ' determined by $\sigma'(\gamma) = \sigma(\gamma)s^{-\tau(\gamma)}$ has $\phi_{\sigma'} = 0$ on the nose. Thus a lift of ρ exists precisely when the cohomology class *Euler*(ρ) vanishes.

Now suppose that ρ_t is a continuous path of representations $\Gamma \to G$. We may choose a continuous family σ_t of sections, for example by choosing generators for Γ and defining $\sigma_t(\gamma)$ in terms of a fixed representation of γ as a word in the generators. This gives a continuous 1–parameter family of cocycles. In the general setting, since the coboundaries are a closed subspace of the cocycles, this implies that the map $\rho \mapsto Euler(\rho)$ is continuous. In our setting, this means that, for any 3–manifold M, the Euler class is constant on connected components of $R_G(M)$.

3.4 Parametrizing lifts

When $\rho \colon \Gamma \to G$ lifts to $\widetilde{G}$, there are many lifts. Specifically, when ρ lifts, the set of all lifts is a 1–dimensional affine space over $H^1(\Gamma; \mathbb{Z})$. Concretely, given some lift $\widetilde{\rho} \colon \Gamma \to \widetilde{G}$ and a $\phi \in H^1(\Gamma; \mathbb{Z})$, then, taking $Z(\widetilde{G}) = \langle s \rangle$, we can construct another lift $\phi \cdot \widetilde{\rho}$ via $\gamma \mapsto \widetilde{\rho}(\gamma)s^{\phi(\gamma)}$, where we are viewing $\phi \in H^1(\Gamma; \mathbb{Z})$ as a homomorphism $\Gamma \to \mathbb{Z}$. Conversely, if $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ are two lifts of ρ, then we claim that they differ by some $\phi \in H^1(\Gamma; \mathbb{Z})$. Since $p \circ \widetilde{\rho}_1(\gamma) = p \circ \widetilde{\rho}_2(\gamma)$ for all $\gamma \in \Gamma$, we have $\widetilde{\rho}_1(\gamma) = \widetilde{\rho}_2(\gamma)s^{\phi(\gamma)}$ for some well-defined function $\phi \colon \Gamma \to \mathbb{Z}$. To see that ϕ is a homomorphism, note that

$$\widetilde{\rho}_1(\gamma_1\gamma_2)s^{\phi(\gamma_1\gamma_2)} = \widetilde{\rho}_2(\gamma_1\gamma_2) = \widetilde{\rho}_1(\gamma_1)s^{\phi(\gamma_1)}\widetilde{\rho}_1(\gamma_2)s^{\phi(\gamma_2)}$$
$$= \widetilde{\rho}_1(\gamma_1)\widetilde{\rho}_1(\gamma_2)s^{\phi(\gamma_1)+\phi(\gamma_2)},$$

which implies that $\phi(\gamma_1\gamma_2) = \phi(\gamma_1) + \phi(\gamma_2)$.

3.5 Representations of $\mathbb{Z}^2$

For $\Lambda = \mathbb{Z}^2$, consider the set of representations $R_{\widetilde{G}}(\Lambda) = \mathrm{Hom}(\Lambda, \widetilde{G})$. A representation $\widetilde{\rho} \in R_{\widetilde{G}}(\Lambda)$ is called elliptic, parabolic, or hyperbolic if the image group $\widetilde{\rho}(\Lambda)$ contains

an element of the corresponding type. Since Λ is abelian, every nontrivial element of $\tilde{\rho}(\Lambda)$ must be of the same type. Thus these categories are disjoint; the remaining representations which are not in any of these categories are called central since $\tilde{\rho}(\Lambda)$ lies there. For a fixed $\tilde{\rho} \in R_{\tilde{G}}(\Lambda)$, we get a map $(\text{trans} \circ \tilde{\rho}) \colon \Lambda \to \mathbb{R}$. The map $\text{trans} \circ \tilde{\rho}$ is actually a homomorphism; this is because a homogenous quasimorphism is actually a homomorphism on any abelian subgroup (see [14, Proposition 2.65] or [32, Theorem 6.16]).

Identifying $\mathrm{Hom}(\Lambda, \mathbb{R})$ with $H^1(\Lambda; \mathbb{R})$, we get a map

$$\text{trans} \colon R_{\tilde{G}}(\Lambda) \to H^1(\Lambda; \mathbb{R}) \quad \text{defined by } \tilde{\rho} \mapsto \text{trans} \circ \tilde{\rho}.$$

This map is far from injective: any parabolic or hyperbolic element of $\tilde{G}$ has an integral translation number, and it follows easily that the preimage of any class in $H^1(\Lambda; \mathbb{Z})$ contains many nonconjugate parabolic and hyperbolic representations. However, for elliptic and central representations, the homomorphism $\text{trans}(\tilde{\rho})$ is a complete conjugacy invariant. In particular, it is easy to see that:

3.6 Lemma *Suppose $\tilde{\rho} \in R_{\tilde{G}}(\Lambda)$ is elliptic or central. If $\text{trans}(\tilde{\rho})(\nu) = 0$ for some $\nu \in \Lambda$, then $\tilde{\rho}(\nu) = 1$.*

4 Translation extension loci

We will now define the translation extension locus, which is the central object in this paper. Let M be an irreducible $\mathbb{Q}$–homology solid torus, and let $\iota \colon \partial M \to M$ be the inclusion map. Inside $R_{\tilde{G}}(M) = \mathrm{Hom}(\pi_1(M), \tilde{G})$, let $PE_{\tilde{G}}(M)$ be the subset of representations whose restriction to $\pi_1(\partial M)$ is either elliptic, parabolic, or central in the sense of Section 3.5. Consider the composition

$$R_{\tilde{G}}(M) \xrightarrow{\iota^*} R_{\tilde{G}}(\partial M) \xrightarrow{\text{trans}} H^1(\partial M; \mathbb{R}).$$

The closure in $H^1(\partial M; \mathbb{R})$ (with respect to the vector space topology) of the image of $PE_{\tilde{G}}(M)$ under $\text{trans} \circ \iota^*$ is called the *translation extension locus* and denoted by $EL_{\tilde{G}}(M)$. We distinguish two special kinds of points of $EL_{\tilde{G}}(M)$. First, those which are not in the image of $PE_{\tilde{G}}(M)$, but only its closure, are called *ideal points*. Second, those coming from elements of $PE_{\tilde{G}}(M)$ which restrict to parabolic representations in $R_{\tilde{G}}(\partial M)$ are called *parabolic*; such points necessarily lie on the integer lattice $H^1(\partial M; \mathbb{Z})$. The translation extension locus was first considered by Khoi [45] in his work on computing Seifert volumes of hyperbolic 3–manifolds.

Let $T = \iota^*(H^1(M; \mathbb{Z})) \subset H^1(\partial M; \mathbb{R})$. Consider the group of affine isomorphisms of $H^1(\partial M; \mathbb{R})$ generated by the map $x \mapsto -x$ together with all translations by elements of T. As T is isomorphic to $\mathbb{Z}$, this affine group is isomorphic to an infinite dihedral group whose action on $H^1(\partial M; \mathbb{R})$ preserves the line containing T; we will denote this dihedral group by $D_\infty(M)$.

4.1 Coordinates and lines

It will be helpful to have concrete coordinates for the translation extension locus. To this end, fix a homologically natural framing (μ, λ) for $H_1(\partial M; \mathbb{Z})$ as discussed in Section 2.2. We now identify $H^1(\partial M; \mathbb{R})$ with $\mathbb{R}^2$ by using the basis (μ^*, λ^*) that is algebraically dual to the basis (μ, λ) of $H_1(\partial M; \mathbb{R})$, that is, $\mu^*(\mu) = \lambda^*(\lambda) = 1$ and $\mu^*(\lambda) = \lambda^*(\mu) = 0$. Note that while λ is unique up to sign and μ depends on our choice of framing, it is μ^* that is unique (up to sign) and λ^* that depends on the framing; geometrically, the point is that μ^* is the Poincaré dual of $\pm\lambda$. Let $k \in \mathbb{N}$ be the order of $\iota_*(\lambda)$ in $H_1(M; \mathbb{Z})$; by Poincaré duality, the number k is also the index of $\langle \iota_*(\mu) \rangle$ in $H_1(M; \mathbb{Z})_{\text{free}} = H_1(M; \mathbb{Z})/_{\text{(torsion)}} \cong \mathbb{Z}$. Hence, in our coordinates, the subgroup $T = \iota^*(H^1(M; \mathbb{Z}))$ is the points $(kn, 0)$ for $n \in \mathbb{Z}$. Moreover, the group $D_\infty(M)$ consists of horizontal translations by shifts in $k\mathbb{Z}$ and π–rotations about every point of the form $\left(\frac{1}{2}kn, 0\right)$ for $n \in \mathbb{Z}$.

To state our tool for constructing orders, we need the following concept. Given a slope r on ∂M, which we can specify by a primitive element $\gamma \in H_1(\partial M; \mathbb{Z})$, define the line $L_r = L_\gamma$ to be the subspace of $H^1(\partial M; \mathbb{R})$ consisting of linear functionals that vanish on the 1–dimensional subspace of $H_1(\partial M; \mathbb{R})$ determined by γ. Thus the line $L_\infty = L_\mu$ is the span of λ^*, which is the vertical axis in our coordinates, and the line $L_0 = L_\lambda$ is the span of μ^*, which is the horizontal axis. In general, L_r is a line through the origin in $\mathbb{R}^2$ of slope $-r$.

4.2 Key results

Here is the basic structural result about $EL_{\widetilde{G}}(M)$, which is roughly that it is a family of immersed arcs invariant under $D_\infty(M)$ such that the quotient is a finite graph.

4.3 Theorem *The extension locus $EL_{\widetilde{G}}(M)$ is a locally finite union of analytic arcs and isolated points. It is invariant under $D_\infty(M)$ with quotient homeomorphic to a finite graph. The quotient contains finitely many points which are ideal or parabolic in the sense defined above. The locus $EL_{\widetilde{G}}(M)$ contains the horizontal axis L_λ, which comes from representations to $\widetilde{G}$ with abelian image.*

Moreover, here are our key tools for constructing orders.

4.4 Lemma *Suppose M is a compact orientable irreducible 3–manifold with ∂M a torus, and assume the Dehn filling $M(r)$ is irreducible. If L_r meets $EL_{\widetilde{G}}(M)$ at a nonzero point which is not parabolic or ideal, then $M(r)$ is orderable.*

4.5 Lemma *Suppose K is a knot in a $\mathbb{Z}$–homology 3–sphere Y whose exterior M is irreducible. Let (μ, λ) be a homologically natural framing with $M(\mu) = Y$. Assume also that the n–fold cyclic cover $\widetilde{Y}$ of Y branched over K is irreducible. If the vertical line $\mu^* = 1/n$ meets $EL_{\widetilde{G}}(M)$ at a point which is not ideal, then $\widetilde{Y}$ is orderable.*

The proofs of Theorem 4.3, Lemma 4.4, and Lemma 4.5 occupy Section 6. Before tackling them, we show pictures of various $EL_{\widetilde{G}}(M)$ to get a feel for these objects.

5 A menagerie of translation extension loci

We now give twelve examples of translation extension loci which will motivate the various results in this paper. Indeed, for us these examples form the intellectual core of this paper, directly inspiring all of the theorems here. The reader should peruse these examples carefully before continuing, as they illustrate both the ideas and the potential pitfalls in the proofs of the main theorems. The first pictures of this type appeared in Figure 8 of [45].
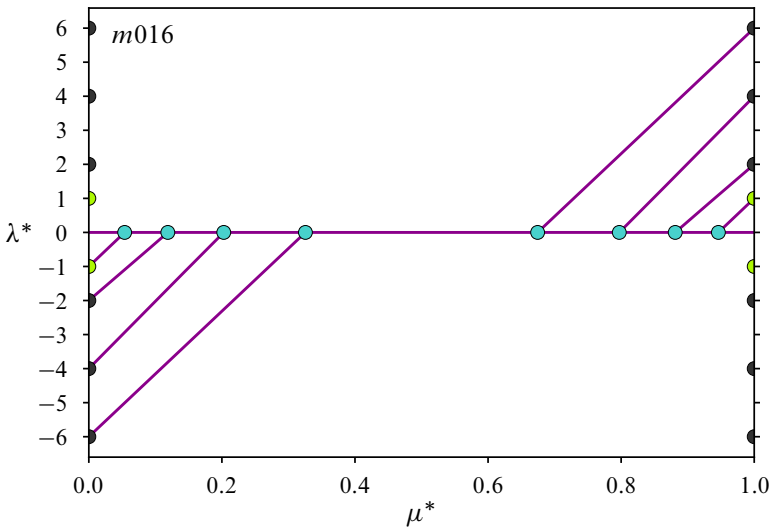

Figure 3: The translation extension locus for $m016$. See Section 5 for how to read this and subsequent plots.
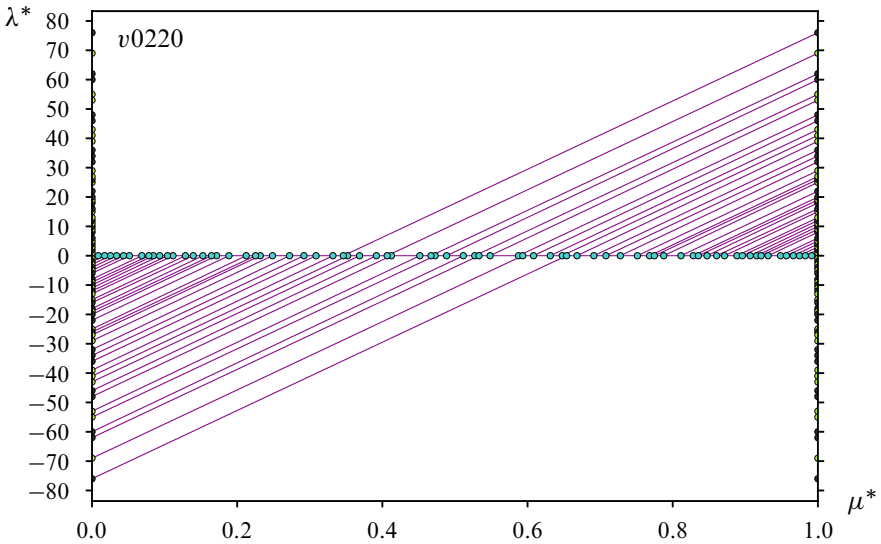
Figure 4: This extension locus follows the same basic pattern as Figure 3, but there are some 72 diagonal arcs each joining a parabolic point to an Alexander point. The manifold here is $M = v0220$, which is the exterior of the knot $k7_6 = T(7, -17, 2, 1)$ in S^3 found in [19]. The framing is such that $M(\mu) = S^3$ and $M(-117)$ is the lens space $L(117, 43)$; in SnapPy's default framing, $\mu = (1, 0)$ and $\lambda = (-116, 1)$. The manifold M fibers over the circle with fiber of genus 47. Here, we can use Lemma 4.4 to order $M(r)$ for all $r \in (-75, \infty)$; in contrast, the interval of non-L–space slopes is $(-93, \infty)$. It is remarkable how complicated $EL_{\widetilde{G}}(M)$ is given that M has an ideal triangulation with only seven tetrahedra!

The twelve examples come from hyperbolic 3–manifolds that have ideal triangulations with at most 9 tetrahedra, and the nomenclature follows [17; 11; 24]. We selected them from a sample of about 600 translation extension loci of such manifolds to illustrate a range of behaviors.

We start with $M = m016$, which is homeomorphic to the exterior of the $(-2, 3, 7)$ pretzel knot in S^3. Its translation extension locus is shown in Figure 3, and we discuss it in detail to explain how to read the plots here. We use a homological framing (μ, λ) where $M(\mu) = S^3$ and $M(-18)$ and $M(-19)$ are lens spaces. (In SnapPy's default framing, $\mu = (1, 0)$ and $\lambda = (18, 1)$.) The figure shows the intersection of $EL_{\widetilde{G}}(M)$ with the strip $0 \leq x \leq 1$ in our usual (μ^*, λ^*)–coordinates on $H^1(\partial M; \mathbb{R})$. This strip is a fundamental domain for the action of $T \leq D_\infty(M)$ which is generated by translation by μ^*. The symmetry of $EL_{\widetilde{G}}(M)$ under the element of $D_\infty(M)$ which is π–rotation about $\left(\frac{1}{2}, 0\right)$ is visually clear.
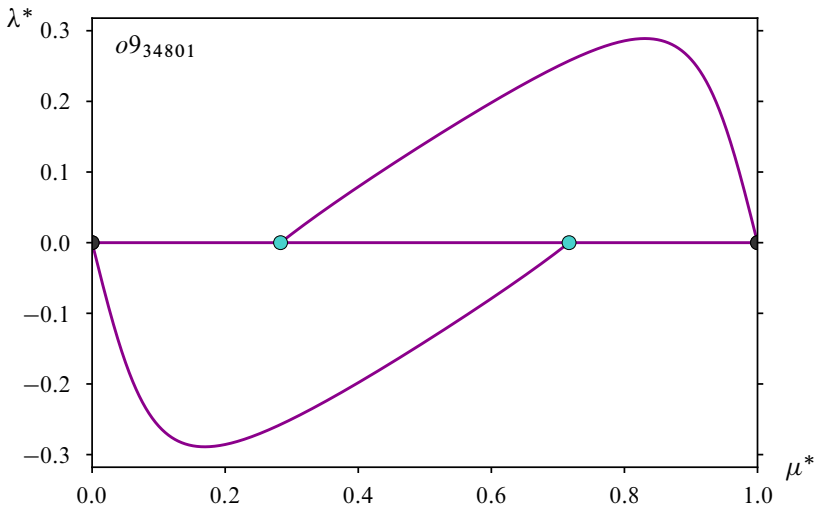
Figure 5: Like those in Figures 3 and 4, this locus consists of arcs that run between parabolic and Alexander points, but a key difference is that the parabolic points lie on the horizontal axis. The manifold $M = o9_{34801}$ here is the exterior of a genus 2 fibered knot in S^3, and as usual $M(\mu) = S^3$. (In SnapPy's default framing, $\mu = (1, 0)$ and $\lambda = (-1, 1)$.) Using Lemma 4.4, we can order $M(r)$ for $r \in [-0.36, 3.6)$, where the endpoints of the interval are approximate. In contrast, the interval of non-L–space slopes is $(-\infty, \infty)$ since the Alexander polynomial $t^4 - 2t^3 + t^2 - 2t + 1$ does not satisfy the condition of [53, Corollary 1.3]. This example illustrates the difficulty of strengthening the proof of Theorem 1.2 to give a lower bound on the size of the interval $(-a, a)$ in the conclusion.

There are 16 parabolic points of $EL_{\widetilde{G}}(M)$ in this picture, which are marked by the dark and light half disks on the vertical sides of the strip. (As mentioned, parabolic points are necessarily integer lattice points.) When the sides of the strip are glued by T, these 16 half disks are paired up to form 8 full discs; down in the full quotient $BL_{\widetilde{G}}(M) = EL_{\widetilde{G}}(M)/D_\infty(M)$, there are only 4 parabolic points.

The color of the half disks indicates when the corresponding representation to G is Galois conjugate to the holonomy representation of the complete hyperbolic structure on M (see Section 8.1 for the definition), with the light green being "geometric" in this limited sense and black indicating other "random" parabolic G–representations.

There are no ideal points in this $EL_{\widetilde{G}}(M)$ or in any of our example translation loci; all of the manifolds involved are small, and Lemma 6.8 below rules out any ideal points in this situation. (The smallness of these manifolds was checked using Regina [12].)
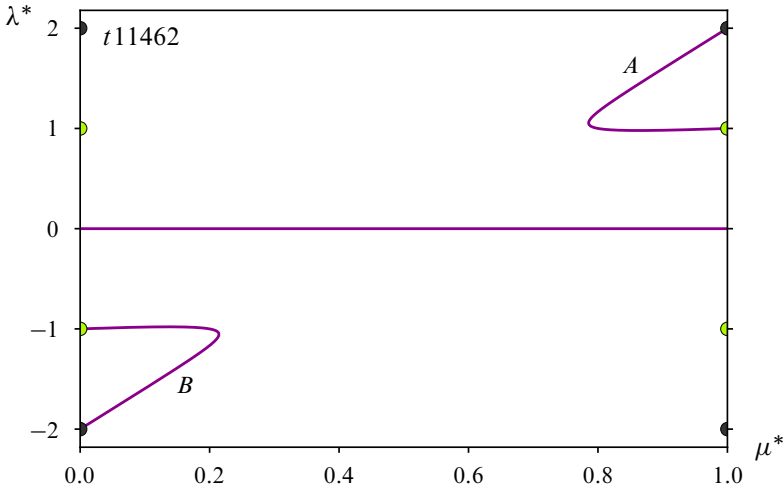
Figure 6: This locus has arcs that run between two parabolic points, rather than from parabolic to Alexander points. The manifold $M = t11462$ is the exterior of a genus 3 fibered knot in S^3, namely $k8_{249}$ from [18]. (In SnapPy's default framing, $\mu = (1, 0)$ and $\lambda = (3, 1)$, and as usual $M(\mu) = S^3$.) Using Lemma 4.4, we claim that we can order $M(r)$ for r in $(-2, -1) \cup (-1, 2) \cup [a, \infty)$, where $a \approx 4.84$. For example, the arc labeled A in the figure gives orderings for $r \in (-2, -1)$, and the arc labeled B shown gives orderings for $r \in [a, \infty)$. The translates of A by positive shifts contribute the intervals $(-2/k, -1/k)$ for $k \geq 1$, as do all the translates of B by negative shifts; the union of these intervals is $(-2, -1) \cup (-1, 0)$. The other translates of A and B contribute half-open intervals that contain, but are slightly larger than, $[1/k, 2/k]$ for $k \geq 1$; the union of these is $(0, 2)$. The interval of non-L–space slopes is $(-\infty, \infty)$ since the Alexander polynomial $t^6 - 2t^5 + 3t^4 - 5t^3 + 3t^2 - 2t + 1$ does not satisfy the condition of [53, Corollary 1.3]. This example illustrates the difficulty of strengthening the proof of Theorem 1.4(2) to give an interval (a, ∞) where a is bounded above.

The disks on the μ^*–axis L_λ correspond to the roots of the Alexander polynomial that lie on the unit circle. Specifically, for each such root ξ, we plot $\left(\frac{1}{2\pi} \arg(\xi), 0\right)$ and call this an *Alexander point*. Simple roots, such as all the ones for this manifold, are shown as light turquoise disks; in later examples, multiple roots will be shown in dark blue. Notice that there is a nonhorizontal arc of $EL_{\widetilde{G}}(M)$ leaving each Alexander point. Such arcs are used to prove Theorem 1.2 and come from deforming an abelian representation to irreducible representations, which is only possible at Alexander points (see Section 7 for a complete discussion).

Since the line L_r has slope $-r$ in our picture, and M has no reducible Dehn fillings, we see that Lemma 4.4 applies to show $M(r)$ is orderable for all $r \in (-6, \infty)$. To
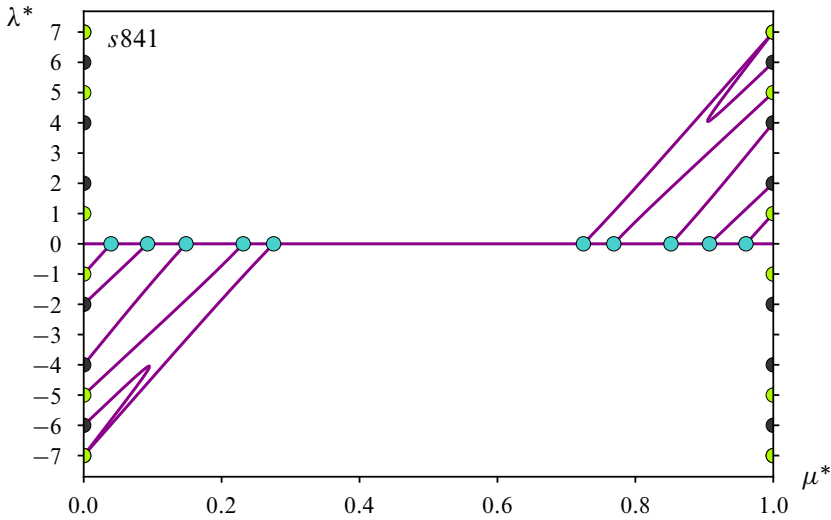
Figure 7: This locus has a mix of the behaviors shown in the previous figures. The manifold $M = s841$ is the exterior of a genus 7 fibered knot in S^3, namely $k6_{38}$ from [18]. (In SnapPy's default framing, $\mu = (1, 0)$ and $\lambda = (22, 1)$, and as usual $M(\mu) = S^3$.) Using Lemma 4.4, we can order $M(r)$ for r in $(-7, \infty)$; the interval of non-L–space slopes is $(-\infty, \infty)$ since the Alexander polynomial does not satisfy the condition of [53, Corollary 1.3]. There are actually two distinct Galois conjugates of the holonomy representation that give rise to each of the points $(0, \pm 7)$ and $(1, \pm 7)$. This is why there are two separate arcs of $EL_{\widetilde{G}}(M)$ emerging from these parabolic points instead of the one you might expect from the proof of Theorem 1.4.

compare with Conjecture 1.1, the interval of non-L–space fillings for M is precisely $(-9, \infty)$ for the following reason. As M has two lens space fillings, it is *Floer simple* in the sense of [55], and hence the interval of L–space fillings is $[-\infty, -(2g - 1)]$, where g is the Seifert genus; the latter is 5 as that is the genus of the fiber in the fibration of M over the circle. In fact, both Theorems 1.2 and 1.4 apply to M, though we got much better results by applying Lemma 4.4 directly.

A summary of the overall structure of this $EL_{\widetilde{G}}(M)$ is that, besides the horizontal line of abelian representations, it consists of diagonal arcs with a parabolic point at one end and an Alexander point at the other. Moreover, none of the arcs overlap. This pattern was quite common in our sample, and a much more complicated instance is shown in Figure 4. Overall, there are many different behaviors that are relevant to us here; please see Figures 5–12 and their captions for details.
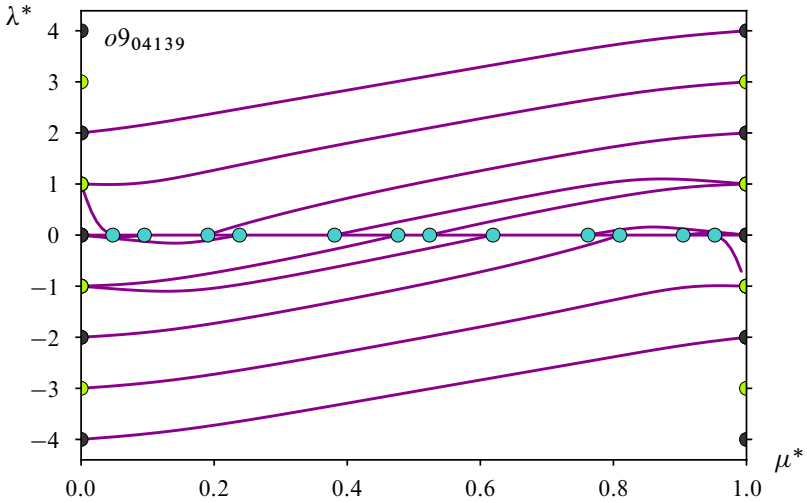
Figure 8: This locus has arcs between parabolics that are on opposite sides of the strip. It is one of the few instances we found where Lemma 4.4 allows us to order $M(r)$ for all r in $(-\infty, \infty)$. (Another such example is $v1971$ in Figure 10.) The manifold $M = o9_{04139}$ is the exterior of a genus 6 fibered knot in S^3. As usual $M(\mu) = S^3$; in SnapPy's default framing, $\mu = (1, 0)$ and $\lambda = (-1, 1)$. While the Alexander polynomial does satisfy [53, Corollary 1.3], it turns out that the set of non-L–space slopes is $(-\infty, \infty)$; using the criterion of [56] and the program flipper [3], Mark Bell and the second author were able to show that every nontrivial Dehn filling on M has a coorientable taut foliation. There are actually four distinct Galois conjugates of the holonomy representation that give rise to each of the points $(0, \pm 1)$ and $(1, \pm 1)$, explaining the arcs that emerge from them.

5.1   Numerical methods and caveats

To compute points in $X_{\mathbb{R}}(M)$ corresponding to representations which send μ to an elliptic isometry, we worked with the *gluing variety* $\mathcal{G}(\mathcal{T})$, where $\mathcal{T}$ is an ideal triangulation of M. Each $\mathcal{G}(\mathcal{T})$ is an affine algebraic set described in coordinates which are the shape parameters for the tetrahedra in $\mathcal{T}$. There is one equation for each edge, specifying that the tetrahedra match around that edge, and the variety determined by these has dimension 1 in our examples. The holonomy H_μ is the square of an eigenvalue of the image of μ, and can be expressed in these coordinates to give a polynomial map $H_\mu \colon \mathcal{G}(\mathcal{T}) \to \mathbb{C}$. We randomly chose a complex number z_0 near the unit circle and used homotopy continuation with a start system given by the mixed volume method to find the 0–dimensional algebraic set $H_\mu^{-1}(z_0)$. This computation
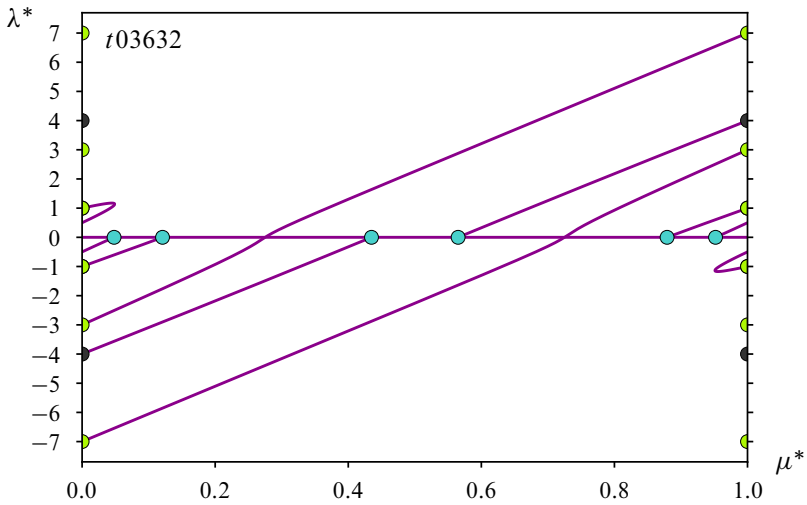
Figure 9: The manifold $M = t03632$ is our first example that is not the exterior of a knot in S^3, with $M(\mu)$ being the small Seifert fibered space $S^2((2, 1), (3, 1), (7, -6))$ which is a $\mathbb{Z}$–homology 3–sphere. (In SnapPy's default framing, $\mu = (1, 0)$ and $\lambda = (-8, 1)$.) One new phenomenon is that $EL_{\widetilde{G}}(M)$ meets the sides of the strip at a point which is not an integer lattice point, namely the intersections at approximately $\left(0, \pm\frac{1}{2}\right)$ and $\left(1, \pm\frac{1}{2}\right)$. Such nonintegral points of $EL_{\widetilde{G}}(M)$ come from representations to $\widetilde{G}$ which factor through $M(\mu)$, which is why they could not appear in the earlier examples, where $M(\mu) = S^3$. Another new phenomenon is that some arcs of $EL_{\widetilde{G}}(M)$ cross the horizontal axis away from the Alexander points; the crossing points correspond to representations to $\widetilde{G}$ which factor through $M(\lambda)$ and have nonabelian image. Such crossings also happen for certain exteriors of knots in S^3, for example with $o9_{21236}$, though not in any of the examples we show here.

was done with PHCpack [63; 64]. Once the fiber over z_0 had been computed, we used the Newton–Raphson method to do path-lifting to our branched cover of $\mathbb{C}$ by $\mathcal{G}(\mathcal{T})$. With some care to avoid singularities, this allowed us to compute the fiber over all N^{th} roots of unity, where N was typically 128 to start with, but sometimes needed to be increased. Each point of one of these fibers determined a character in $X(M)$ corresponding to a representation sending μ to an elliptic with rotation angle $2k\pi/N$ for some k. These representations were computed to standard floating point accuracy (53 bits) and it was numerically decided which of them gave points of $X_G(M)$.

Once we had constructed a representation $\rho\colon \pi_1(M) \to G$, we used the Newton–Raphson method to polish it to very high precision (typically 1 000 bits). The Euler
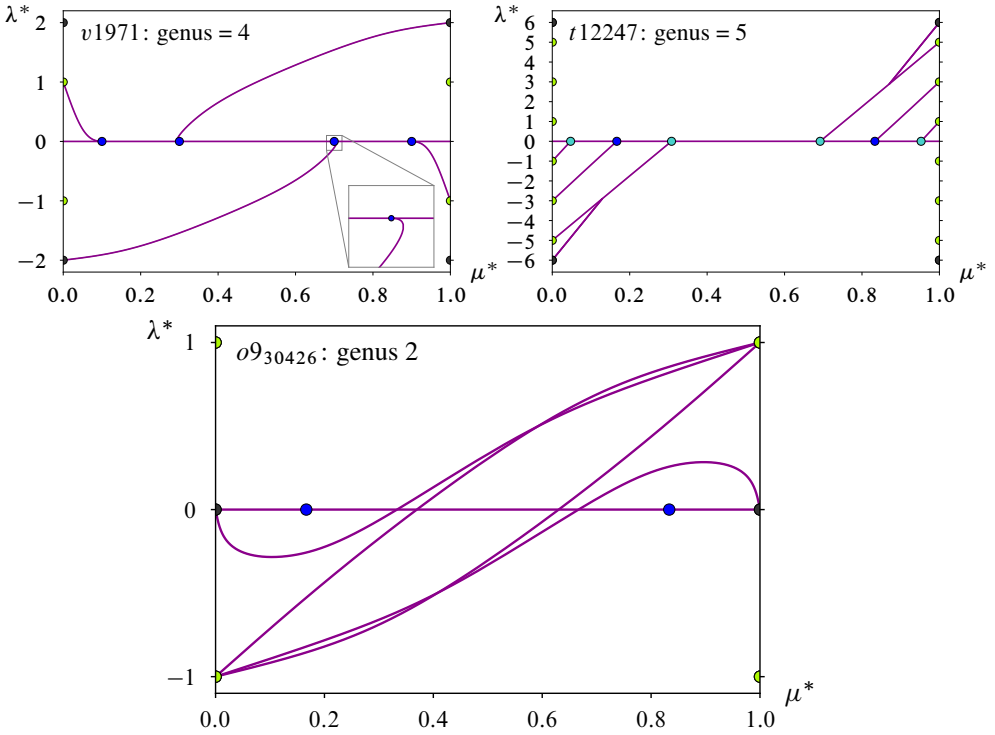
Figure 10: These three loci show some possible behaviors when the Alexander polynomial has a multiple root (for such a root, the corresponding Alexander point is dark blue rather than light blue). The top left example is the knot exterior $v1971 = k7_{74}$ from [19]. There, the arcs leaving the Alexander points are tangent to the horizontal axis, which is a common pattern for multiple roots. However, such tangencies are not required as the top right example of the knot exterior $t12247 = k8_{279} = 12n574$ from [18] shows. The last example of $M = o9_{30426}$ is perhaps the most interesting: there are no nonhorizontal arcs of $EL_{\widetilde{G}}(M)$ leaving the two Alexander points at all! In fact, the corresponding reducible representations to $G_{\mathbb{C}}$ are deformable to irreducible representations, but only into PSU_2, not G. Here, $M(\mu)$ is the Seifert fibered space $S^2((2, 1), (3, 1), (11, -9))$, which is a $\mathbb{Z}$–homology 3–sphere, and there are three separate Galois conjugates of the holonomy representation at the points $(0, \pm 1)$ and $(1, \pm 1)$ in $EL_{\widetilde{G}}(M)$. The bottom example shows why we need the hypothesis that Δ_M has a simple root in the proof of Theorem 1.2, since the picture near the Alexander points does not match Figure 14.

cocycle of Section 3.3 was then computed and used to lift ρ to $\widetilde{\rho}\colon \pi_1(M) \to \widetilde{G}$. The peripheral translations of $\widetilde{\rho}$ were computed and then normalized under the action of $D_{\infty}(M)$ to be plotted in the figure. (For the examples in Figures 11 and 12, frequently there was no lift $\widetilde{\rho}$ as $Euler(\rho)$ was nonzero in $H^2(M; \mathbb{Z})$.) For each figure, we
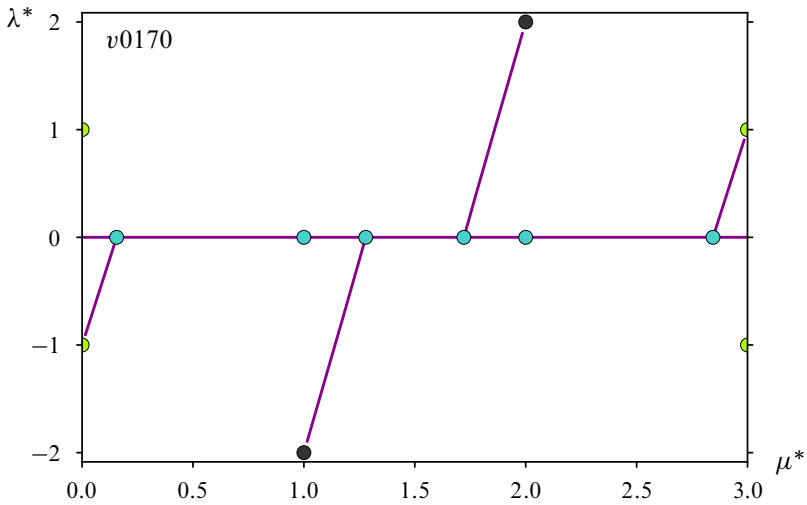
Figure 11: The manifold $M = v0170$ is our first example of something that is not a $\mathbb{Z}$–homology solid torus. In particular, the homological longitude λ has order $k = 3$ in $H_1(M;\mathbb{Z})$, which is why the shown fundamental domain for the action of $T \le D_\infty(M)$ has width 3. The filling $M(\mu)$ is the lens space $L(9,2)$ with the core of the added solid torus representing three times a generator of $H_1(L(9,2);\mathbb{Z}) \cong \mathbb{Z}/9\mathbb{Z}$. (In SnapPy's default framing, $\mu = (1,0)$ and $\lambda = (-5,1)$.) The manifold M fibers over the circle with fiber a genus 4 surface with 3 boundary components. For a root ξ of Δ_M, the corresponding Alexander point is plotted as $\frac{3}{2\pi}\arg(\xi)$ to account for the fact that μ maps to three times a generator in $H_1(M;\mathbb{Z})_{\text{free}}$. The two Alexander points at $(1,0)$ and $(2,0)$ demonstrate the necessity of the hypothesis that $\xi^k \ne 1$ for the proof of Theorem 1.2, since the local picture there does not match Figure 14. The trace field of M has 6 real embeddings, but above there is only one parabolic point modulo $D_\infty(M)$; this is because most of the Galois conjugates into G do not lift to $\widetilde{G}$.

sampled as many as 2 000 different holonomy values for μ in order to get the smooth curves you see.

While we believe our plots of these loci are accurate, they were not rigorously computed. Moreover, there are reasons beyond numerical accuracy that sometimes cause computations using gluing varieties to produce incomplete results, with some arcs missing from the diagram. (On the other hand, using gluing varieties rather than character varieties hugely simplifies the computation, making it feasible to handle larger examples.) The key issue is that the natural map $\mathcal{G}(\mathcal{T}) \to X(M)$ is not always onto; while each irreducible component of $\mathcal{G}(\mathcal{T})$ corresponds to some irreducible
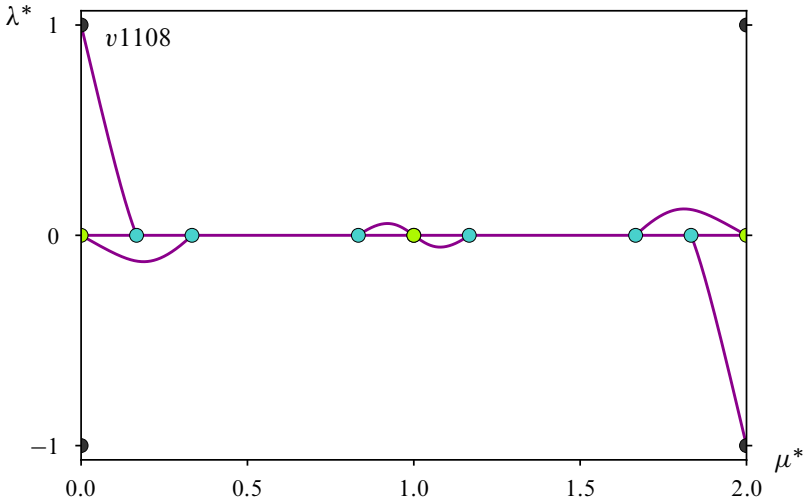
Figure 12: The manifold $M = v1108$ is another example that is not a $\mathbb{Z}$–homology solid torus. In particular, the homological longitude λ has order $k = 2$ in $H_1(M; \mathbb{Z})$ and the filling $M(\mu)$ is the lens space $L(4, 1)$. (For once, the (μ, λ) framing is the same as SnapPy's default.) The manifold M fibers over the circle with fiber a genus 3 surface with 2 boundary components. The parabolic points $(0, 0)$, $(1, 0)$, and $(2, 0)$ are all double, that is, come from two distinct Galois conjugates of the holonomy representation. In addition to being a parabolic point, the point $(1, 0)$ is also a simple Alexander point. However, this Alexander point doesn't contribute an arc to $EL_{\widetilde{G}}(M)$ because it corresponds to the root $\xi = -1$ and $\xi^k = 1$.

component of $X(M)$, there can be components of $X(M)$ that are not seen in $\mathcal{G}(\mathcal{T})$ [27, Section 10.3]. As $\mathcal{G}(\mathcal{T})$ depends fundamentally on the triangulation $\mathcal{T}$, such "missing components" can sometimes be dealt with by changing the triangulation. In other cases, especially when there are components corresponding to representations that factor through a proper quotient of $\pi_1(M)$, changing the triangulation did not help. (In [57], Segerman constructs an "extended" version of $\mathcal{G}(\mathcal{T})$ and shows that there always exists a triangulation such that all components of the character variety can be parametrized in terms of the associated extended gluing variety. However, his technique has not been implemented in software.)

In some cases we were able to detect missing components from inconsistencies in our picture of $EL_{\widetilde{G}}(M)$. In the case of $M = m389$, we obtained a plot of $EL_{\widetilde{G}}(M)$ with a simple Alexander point from which no arcs emerged, violating the proof of Theorem 1.2. It turns out that for the Dehn filling $Y = m389(\mu + \lambda)$ there is a surjection from $\pi_1(Y)$

onto $\mathrm{PSL}_2\mathbb{Z} \cong C_2 * C_3$, giving a component of $X(M)$ that could not be seen by our $\mathcal{G}(\mathcal{T})$. Another fairly common situation that leads to missing components is when a Dehn filling Y contains an essential torus: if $X(Y)$ is nonempty, then $\dim_{\mathbb{C}} X(Y) \geq 1$ because it is possible to "bend" representations using the structure of $\pi_1(Y)$ as a free product with amalgamation along the $\mathbb{Z}^2$–subgroup corresponding to the essential torus. It seems to be common that components of $X(M)$ obtained by bending do not appear in the image of $\mathcal{G}(\mathcal{T})$.

Another issue with gluing varieties is that points at infinity of $\mathcal{G}(\mathcal{T})$ can correspond to nonideal points of the character variety $X(M)$. Geometrically, this means that the shapes of some tetrahedra degenerate even though the associated characters converge. We call these *Tillmann points* after [60]. These points cause numerical difficulties and complicate determining which points of $EL_{\widetilde{G}}(M)$ are ideal. Such Tillmann points occur reasonably frequently in our examples. Specifically, we used Goerner's database [33] of boundary parabolic representations to $G_{\mathbb{C}}$ to identify which of the parabolic points correspond to Galois conjugates of the holonomy representation of the hyperbolic structure, and as a check to our own computations. While Goerner used Ptolemy equations rather than gluing equations, his method still depends on a choice of triangulation, and parabolic representations can go missing for the same reason. In our examples, there were five cases where our plot of $EL_{\widetilde{G}}(M)$ indicated a parabolic or ideal point on the vertical sides of the diagram that were not present in [33]. For example, this occurred with the point $(0, -2)$ in Figure 6. Using Lemma 6.8, we were able to conclude that these are all Tillmann points missed by our preferred triangulation, rather than ideal points.

6 Proof of the structure theorem

This section is devoted to the proofs of Theorem 4.3, Lemma 4.4, and Lemma 4.5. An impatient and trusting reader can skip ahead as the rest of the paper only relies on the statements of these three results. We begin attacking Theorem 4.3 by proving the following two lemmas:

6.1 Lemma *The extension locus $EL_{\widetilde{G}}(M)$ is invariant under $D_{\infty}(M)$.*

6.2 Lemma *The quotient space $BL_{\widetilde{G}}(M) = EL_{\widetilde{G}}(M)/D_{\infty}(M)$ has finitely many connected components.*

Proof of Lemma 6.1   Since invariance is preserved under taking closures, it suffices to show that the image I of $PE_{\widetilde{G}}(M)$ under trans $\circ\, \iota^*$ is invariant under $D_\infty(M)$. Consider any $\widetilde{\rho} \in PE_{\widetilde{G}}(M)$ and let $t = \mathrm{trans}(\widetilde{\rho} \circ \iota)$ be the corresponding point in I. If $\phi \in H^1(M;\mathbb{Z})$, then, as described in Section 3.4, one has $\phi \cdot \widetilde{\rho}$ in $PE_{\widetilde{G}}(M)$ which is also a lift of $p \circ \widetilde{\rho}$. The image of $\phi \cdot \widetilde{\rho}$ in I differs from t via translation by $\iota^*(\phi)$ in $H^1(\partial M;\mathbb{Z})$ since

$$\mathrm{trans}(\phi \cdot \widetilde{\rho}(\gamma)) = \mathrm{trans}(\widetilde{\rho}(\gamma)s^{\phi(\gamma)}) = \mathrm{trans}(\widetilde{\rho}(\gamma)) + \phi(\gamma) \quad \text{for all } \gamma \in \pi_1(M).$$

In particular, this shows that I is invariant under translation by elements of $T = \iota^*(H^1(M;\mathbb{Z})) \subset H^1(\partial M;\mathbb{R})$.

To complete the proof, it remains to show I is invariant under $x \mapsto -x$. To this end, we will exhibit an automorphism $\nu\colon \widetilde{G} \to \widetilde{G}$ where $\mathrm{trans}(\nu(\widetilde{g})) = -\mathrm{trans}(\widetilde{g})$ for all $\widetilde{g} \in \widetilde{G}$. Given such a ν, the image of $\nu \circ \widetilde{\rho}$ in I will be $-t$, proving invariance. To start, consider the element $r \in \mathrm{Homeo}(\mathbb{R})$ which sends $y \mapsto -y$. Conjugation by r preserves the subgroup $\widetilde{G}$ because r descends to the map of $\boldsymbol{P}^1(\mathbb{R})$ induced by $C = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{PGL}_2\mathbb{R}$, and conjugation by C normalizes $G \leq \mathrm{PGL}_2\mathbb{R}$. Let ν be conjugation of $\widetilde{G}$ by r. Taking $x = 0$ in the definition (3.2) of translation number we get

$$\mathrm{trans}(\nu(\widetilde{g})) = \lim_{n\to\infty} \frac{(r \circ \widetilde{g} \circ r)^n(0)}{n} = \lim_{n\to\infty} \frac{-\widetilde{g}^n(-0)}{n} = -\mathrm{trans}(\widetilde{g})$$

as required. $\square$

Proof of Lemma 6.2   Consider the map $P\colon R_{\widetilde{G}}(M) \to R_G(M)$ induced by $p\colon \widetilde{G} \to G$. Let $PE_G(M)$ be the subset of $R_G(M)$ consisting of representations whose restrictions to $\pi_1(\partial M)$ consist only of elliptic, parabolic, and trivial elements. Note that $PE_G(M)$ is a real semialgebraic set. Let $PE_G^{lift}(M) \subset PE_G(M)$ be the image of $PE_{\widetilde{G}}(M)$ under P. By continuity of the Euler class (see Section 3.3), the subset $PE_G^{lift}(M)$ is a union of connected components of $PE_G(M)$, and hence also a real semialgebraic set. As described in Section 3.4, the cohomology $H^1(M;\mathbb{Z})$ acts freely on $PE_{\widetilde{G}}(M)$ with quotient $PE_G^{lift}(M)$; consequently, $P\colon PE_{\widetilde{G}}(M) \to PE_G^{lift}(M)$ is a (regular) covering map. Because the action of $H^1(M;\mathbb{Z})$ on $PE_{\widetilde{G}}(M)$ induces the action of $T \leq D_\infty(M)$ on $EL_{\widetilde{G}}(M)$, the map trans $\circ\, \iota^*$ below factors through ψ as shown:

$$
\begin{array}{ccc}
PE_{\widetilde{G}}(M) & \xrightarrow{\mathrm{trans}\circ\iota^*} & BL_{\widetilde{G}}(M) \\[4pt]
{\scriptstyle P}\Big\downarrow & \nearrow & \\[4pt]
& {\scriptstyle \psi} & \\[2pt]
PE_G^{lift}(M) & &
\end{array}
$$

The map ψ must be continuous as the vertical arrow P is a covering map. As the set $PE_G^{lift}(M)$ has finitely many connected components, it follows that

$$\overline{\psi(PE_G^{lift}(M))} = BL_{\widetilde{G}}(M)$$

has finitely many components, proving the lemma. □

6.3 Milnor–Wood bounds

The remaining tool we need to prove Theorem 4.3 is:

6.4 Lemma *The space $BL_{\widetilde{G}}(M)$ is compact.*

The proof of Lemma 6.4 hinges on knowing that $EL_{\widetilde{G}}(M)$ is contained in a horizontal strip of bounded height; to show this, we use the following result, which is closely related to the Milnor–Wood inequality.

6.5 Proposition *Suppose S is a compact orientable surface with one boundary component. For all $\widetilde{\rho}\colon \pi_1(S) \to \widetilde{G}$ one has*

$$|\mathrm{trans}(\widetilde{\rho}(\delta))| \leq \max(-\chi(S), 0), \quad \text{where } \delta \text{ is a generator of } \pi_1(\partial S).$$

Before discussing Proposition 6.5, let us derive Lemma 6.4 from it.

Proof of Lemma 6.4   Recall that M is a $\mathbb{Q}$–homology solid torus and let k be the order of the homological longitude $\lambda \in \pi_1(\partial M)$ in $H_1(M; \mathbb{Z})$. There is a proper map of an oriented surface $f\colon S \to M$ where S has one boundary component and where $f_*(\delta) = \lambda^k$ in $\pi_1(M)$ for δ a generator of $\pi_1(\partial S)$. Because trans is a homomorphism on cyclic subgroups of $\widetilde{G}$, we have

$$k \cdot |\mathrm{trans}(\widetilde{\rho}(\lambda))| = |\mathrm{trans}(\widetilde{\rho}(\lambda^k))| = |\mathrm{trans}((\widetilde{\rho} \circ f_*)(\delta))|$$

Applying Proposition 6.5 to $\widetilde{\rho} \circ f_*$ bounds the rightmost term in the previous equation, giving

$$|\mathrm{trans}(\widetilde{\rho}(\lambda))| \leq \frac{\max(-\chi(S), 0)}{k}$$

In particular, in our usual (μ^*, λ^*)–coordinates on $H^1(\partial M; \mathbb{R})$, the locus $EL_{\widetilde{G}}(M)$ lies in a horizontal strip whose height is bounded by something that only depends on topological information about M. Thus, since $D_\infty(M)$ contains horizontal translations of $\mathbb{R}^2$ by shifts in $k\mathbb{Z}$, the quotient $BL_{\widetilde{G}}(M)$ is compact. □

We now discuss Proposition 6.5 in detail. Recall that a real-valued function ϕ on a group Γ is called a *quasimorphism* if there exists a number D such that

$$|\phi(xy) - \phi(x) - \phi(y)| \leq D \quad \text{for all } x, y \in \Gamma,$$

and that the infimum of all such D is called the *defect* of ϕ. The standard references [32, Section 5] and [14, Section 2.3.3] contain proofs that for any representation $\tilde{\rho}\colon \Gamma \to \tilde{G}$, the function given by $\phi = \text{trans} \circ \rho$ is a quasimorphism. It is also well-known that this quasimorphism has defect at most 1, although it is harder to extract this fact from the literature. It is stated in [59, Proposition 3.7], with a sketch of a proof that uses a construction of a connection on a circle bundle over a surface in terms of a harmonic measure on a foliation transverse to the fibers. It is also a consequence of the "ab theorem" of [16, Theorem 3.9], which was conjectured and almost proved by Jankins and Neumann [43], the proof having been completed by Naimi [51]. The proof of Calegari and Walker is simpler and effective (see also [50]). With these facts in hand, we turn to the proof of the proposition.

Proof of Proposition 6.5 Let g be the genus of S. The case of $g = 0$ is immediate as then $\tilde{\rho}$ must be trivial and so $\text{trans}(\tilde{\rho}(\delta)) = 0$; thus we will assume $g > 0$. Choose standard generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ for $\pi_1(S)$, where

$$\delta = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g].$$

Because trans: $\tilde{G} \to \mathbb{R}$ is a quasimorphism of defect at most 1, we have

$$|\text{trans}(xy)| \leq |\text{trans}(x)| + |\text{trans}(y)| + 1$$

for all $x, y \in \tilde{G}$. It follows by induction that

$$|\text{trans}(x_1 \cdots x_n)| \leq |\text{trans}(x_1)| + \cdots + |\text{trans}(x_n)| + (n-1)$$

for all $x_1, \ldots, x_n \in \tilde{G}$. As trans is constant on conjugacy classes and satisfies $\text{trans}(x^{-1}) = -\text{trans}(x)$, we have

$$|\text{trans}([x, y])| = |\text{trans}([x, y]) - \text{trans}(xyx^{-1}) - \text{trans}(y^{-1})| \leq 1$$

for all $x, y \in \tilde{G}$. Combining these properties, we have

$$|\text{trans}(\delta)| = |\text{trans}([\tilde{\rho}(\alpha_1), \tilde{\rho}(\beta_1)] \cdots [\tilde{\rho}(\alpha_g), \tilde{\rho}(\beta_g)])| \leq g + (g-1) = -\chi(S),$$

as required.   □

Proof of Theorem 4.3 Define $c: H^1(\partial M; \mathbb{R}) \to X_G(\partial M)$ by sending $\phi: \pi_1(\partial M) \to \mathbb{R}$ to the character of the elliptic representation ρ given by

$$\rho(\mu) = \pm \begin{pmatrix} e^{2\pi i\phi(\mu)} & 0 \\ 0 & e^{-2\pi i\phi(\mu)} \end{pmatrix} \quad \text{and} \quad \rho(\lambda) = \pm \begin{pmatrix} e^{2\pi i\phi(\lambda)} & 0 \\ 0 & e^{-2\pi i\phi(\lambda)} \end{pmatrix}$$

We may use the dual basis to (μ, λ) and the trace-squared coordinates on $X_G(\partial M)$ to express the map c in coordinates as

$$c(x, y) = 4\big(\cos^2(2\pi x), \cos^2(2\pi y), \cos^2(2\pi(x + y))\big)$$

For integers m and n we have $c(x + m, y + n) = c(x, y)$, and also $c(\pm x, \pm y) = c(x, y)$. Thus the map c is topologically an orbifold covering map from $\mathbb{R}^2$ onto a pillowcase, ie a Euclidean orbifold with underlying manifold S^2 and four cone points of angle π. Moreover, the following commutes:

$$
\begin{CD}
PE_{\widetilde{G}}(M) @>{\text{trans} \circ \iota^*}>> H^1(\partial M; \mathbb{R}) \\
@VVV @VV{c}V \\
X_G(M) @>{\iota^*}>> X_G(\partial M)
\end{CD}
$$

Note that c maps $BL_{\widetilde{G}}(M)$ into $\overline{\iota^*(X_G(M))}$. Now, by Lemma 2.4, the complex algebraic set $\iota^*(X(M)) \subset X(\partial M)$ has complex dimension at most 1; hence the real semialgebraic set $\iota^*(X_G(M))$ has real dimension at most 1. Moreover, the set $\overline{\iota^*(X_G(M))}$ is compact since the subset of $X(\partial M)$ corresponding to representations that are parabolic, elliptic, or trivial is compact. Hence by Proposition 2.6, the set $\overline{\iota^*(X_G(M))}$ is a finite graph. Thus, its preimage under c is a locally finite graph with analytic edges that is invariant under $D_\infty(M)$ by Lemma 6.1. As $BL_{\widetilde{G}}(M)$ is compact by Lemma 6.4, we can conclude that it lives in some finite graph in $H^1(\partial M; \mathbb{R})/D_\infty(M)$ with analytic edges. Now, since $BL_{\widetilde{G}}(M)$ has finitely many connected components by Lemma 6.2, it follows that it too must be a finite graph in $H^1(\partial M; \mathbb{R})/D_\infty(M)$ with analytic edges. This proves the hardest part of the theorem.

To see that there are only finitely many parabolic points, note that these only occur at images of lattice points in $H^1(\partial M; \mathbb{Z})$, and there can only be finitely many such points in the compact set $BL_{\widetilde{G}}(M)$. Also, the space $BL_{\widetilde{G}}(M)$ is the closure in a finite graph of a set with finitely many components, and thus there are only finitely many ideal points. Finally, consider the copy of $\mathbb{R}$ in $\widetilde{G}$ sitting above $PSO_2 \leq G$. As $H_1(M; \mathbb{Z})_{\text{free}} = H_1(M; \mathbb{Z})/_{\text{(torsion)}} \cong \mathbb{Z}$, we get a 1–parameter family of abelian representations $\pi_1(M) \to \widetilde{G}$ by sending the generator of $H_1(M; \mathbb{Z})_{\text{free}}$ to any chosen

element of $\mathbb{R}$. Since λ is zero in $H_1(M;\mathbb{Z})_{\text{free}}$ whereas μ is nonzero, we see that these abelian representations give rise to the line L_λ inside of $EL_{\widetilde{G}}(M)$, finishing the proof of the structure theorem. $\square$

6.6 Constructing orderings

We now turn to the proofs of the lemmas that we use to construct orderings of 3–manifold groups.

Proof of Lemma 4.4 Let ϕ be such a point in $L_r \cap EL_{\widetilde{G}}(M)$. As it is neither parabolic nor ideal, there is a $\widetilde{\rho} \in R_{\widetilde{G}}(M)$ which maps to ϕ where the restriction of $\widetilde{\rho}$ to $\pi_1(\partial M)$ is either elliptic or central. Let γ be an element of $\pi_1(\partial M)$ realizing the slope r. By the definition of L_r, we have $\phi(\gamma) = (\text{trans} \circ \widetilde{\rho})(\gamma) = 0$. It follows from Lemma 3.6 that γ is in the kernel of $\widetilde{\rho}$, and hence we get an induced representation $\overline{\rho} \colon \pi_1(M(r)) \to \widetilde{G}$. As ϕ is not the origin in $H^1(M;\mathbb{R})$, the new representation $\overline{\rho}$ is nontrivial since some element of $\pi_1(\partial M)$ is mapped to an element of $\widetilde{G}$ with nonzero translation number. Thus we have found a nontrivial homomorphism $\pi_1(M(r)) \to \widetilde{G}$. Regarding $\widetilde{G}$ as a subgroup of $\text{Homeo}^+(\mathbb{R})$ and using that $M(r)$ is irreducible, Theorem 1.1 of [8] applies to promote this nontrivial homomorphism $\pi_1(M(r)) \to \text{Homeo}^+(\mathbb{R})$ to a faithful one; equivalently, the group $\pi_1(M(r))$ is left-orderable, as claimed. $\square$

Proof of Lemma 4.5 Let $\pi \colon \widetilde{M} \to M$ be the covering map corresponding to $\widetilde{Y} \to Y$. Restricting representations from $\pi_1(M)$ to $\pi_1(\widetilde{M})$, we get a natural subset of $EL_{\widetilde{G}}(\widetilde{M})$ from $EL_{\widetilde{G}}(M)$. Specifically, the locus $EL_{\widetilde{G}}(\widetilde{M})$ contains the image of $EL_{\widetilde{G}}(M)$ under $\pi^* \colon H^1(\partial M;\mathbb{R}) \to H^1(\partial \widetilde{M};\mathbb{R})$. We use $(\widetilde{\mu}, \widetilde{\lambda})$ as a basis for $H_1(\partial \widetilde{M};\mathbb{Z})$, where $\widetilde{\mu}$ maps to $n\mu$ in $H_1(\partial M;\mathbb{Z})$ and $\widetilde{\lambda}$ maps to λ. In the dual bases, we thus have that $\pi^* \colon H^1(\partial M;\mathbb{R}) \to H^1(\partial \widetilde{M};\mathbb{R})$ is given by $\mu \mapsto n\widetilde{\mu}$ and $\lambda \mapsto \widetilde{\lambda}$. Hence $\pi^*(EL_{\widetilde{G}}(M))$ is basically $EL_{\widetilde{G}}(M)$ stretched horizontally by a factor of n. If we act on $\pi^*(EL_{\widetilde{G}}(M))$ by $D_\infty(\widetilde{M})$, we get additional copies of $\pi^*(EL_{\widetilde{G}}(M))$ as shown in Figure 13. (These additional translates still come from representations $\pi_1(M) \to G$, but correspond to lifts $\pi_1(\widetilde{M}) \to \widetilde{G}$ that do not extend to all of $\pi_1(M)$; the point is that we can adjust a lift by any element in $H^1(\widetilde{M};\mathbb{Z})$ and the image of $H^1(M;\mathbb{Z})$ has index n.) The key observation is that as $EL_{\widetilde{G}}(M)$ meets the line $\mu^* = 1/n$, the locus $EL_{\widetilde{G}}(\widetilde{M})$ meets the line $\widetilde{\mu}^* = 1$, and hence by the action of $D_\infty(\widetilde{M})$ the locus $EL_{\widetilde{G}}(\widetilde{M})$ meets L_μ at a point $t = (0, y)$. The desired conclusion now follows from Lemma 4.4 provided we can show that t is neither ideal nor parabolic. The former is
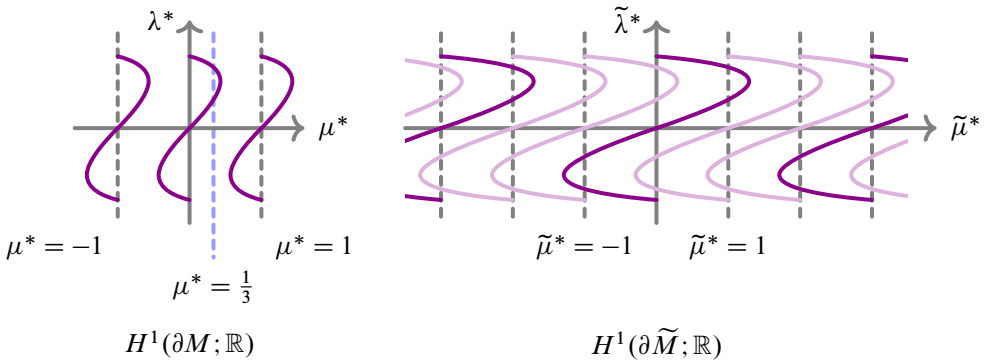
Figure 13: This picture illustrates the proof of Lemma 4.5 in a case where the covering map $\pi\colon \widetilde{M} \to M$ has degree 3. At left is $EL_{\widetilde{G}}(M)$, where its intersections with the vertical axis are parabolic points and there are no ideal points. Note that $EL_{\widetilde{G}}(M)$ meets the vertical line $\mu^* = \frac{1}{3}$ in two points. Its image under $\pi^*\colon H^1(\partial M; \mathbb{R}) \to H^1(\partial \widetilde{M}; \mathbb{R})$ is shown at right as the darker curves; the image is just a copy of $EL_{\widetilde{G}}(M)$ stretched horizontally by a factor of 3. In addition, $EL_{\widetilde{G}}(\widetilde{M})$ contains the lighter curves shown, which are other translates of $\pi^*(EL_{\widetilde{G}}(M))$ under $D_{\infty}(\widetilde{M})$. It is the lighter curves that contribute nonparabolic intersections of $EL_{\widetilde{G}}(\widetilde{M})$ with the vertical axis $L_{\widetilde{\mu}}$, corresponding to the original intersections of $EL_{\widetilde{G}}(M)$ with $\mu^* = \frac{1}{3}$, and so allow us to order $\widetilde{Y} = \widetilde{M}(\widetilde{\mu})$ via Lemma 4.4.

ruled out by the hypothesis that the initial intersection of $EL_{\widetilde{G}}(M)$ with $\mu^* = 1/n$ was not an ideal point. The latter is impossible since, when restricting a representation $\mathbb{Z}^2 \to \widetilde{G}$ to a finite-index subgroup, the only possible change of type (as defined in Section 3.5) is from elliptic to trivial, and the initial intersection of $EL_{\widetilde{G}}(M)$ with $\mu^* = 1/n$ is not parabolic as it is not in the lattice $H^1(M; \mathbb{Z})$. Thus we can apply Lemma 4.4 to order $\widetilde{Y}$, as required. $\qquad\square$

6.7 Ideal points

The following result was used in Section 5, but is not central to this paper and the proof can be safely skipped.

6.8 Lemma *Suppose M is a $\mathbb{Q}$–homology solid torus which is small, that is, contains no closed essential surfaces. Then $EL_{\widetilde{G}}(M)$ has no ideal points.*

Proof Suppose t_0 is an ideal point of $EL_{\widetilde{G}}(M)$. Pick a sequence $\widetilde{\rho}_i \in PE_{\widetilde{G}}(M)$ whose images in $EL_{\widetilde{G}}(M)$ converge to t. Consider the representations $\rho_i = p \circ \widetilde{\rho}_i$ in

$R_G(M)$ and the corresponding characters $[\rho_i]$ in $X_G(M)$. Passing to a subsequence, we arrange that the $[\rho_i]$ lie in a single irreducible component X' of $X(M)$. As M is small, the variety X' must be a complex affine curve by [22, Section 2.4]. As $X_G(M)$ is closed in $X(M)$ by Lemma 2.12, we have that $X'_G = X' \cap X_G(M)$ is closed in X'. Passing to a subsequence, either the $[\rho_i]$ limit to a character in $X_G(M)$ or the $[\rho_i]$ march off to infinity in the noncompact curve X'. In the latter case, since we have $\{\mathrm{tr}^2_\gamma \rho_i\} \in [0, 4]$ for all $\gamma \in \pi_1(\partial M)$, the argument of [22, Section 2.4] produces a closed essential surface associated to a certain ideal point of X', contradicting our hypothesis that M is small.

Now consider the case when the $[\rho_i]$ limit to χ in $X_G(M)$. By Proposition 2.6, we pass to a subsequence where there is an arc $\overline{c}$ in $X_G(M)$ starting at $[\rho_0]$, ending at χ and containing all the $[\rho_i]$. Using Lemma 2.11, lift $\overline{c}$ to a path c in $R_G(M)$ starting at ρ_0 and ending at some ρ whose character is χ. In the notation of the proof of Lemma 6.2, we have that the ρ_i are in $PE_G(M)$. Note that ρ is also in $PE_G(M)$ as it is in $R_G(M)$ and $\mathrm{tr}^2_\gamma \rho$ must be in $[0, 4]$ by continuity for all $\gamma \in \pi_1(\partial M)$. As in the proof of Lemma 6.2, we have that c is in $PE_G^{lift}(M)$ and so we can lift c to a path $\widetilde{c}$ in $PE_{\widetilde{G}}(M)$ starting at $\widetilde{\rho}_0$. After possibly changing $\widetilde{c}$ by a deck transformation of $PE_{\widetilde{G}}(M) \to PE_G^{lift}(M)$, we can assume that the image of $\widetilde{c}(1)$ in $EL_{\widetilde{G}}(M)$ is exactly t_0. Thus t_0 is not actually an ideal point, proving the lemma. $\qquad\square$

7 Alexander polynomials and orderability

In this section we prove our first main result, Theorem 7.1, which implies Theorem 1.2 from the introduction. To state the more general result, we need a pair of definitions. First, we say a compact 3–manifold Y has *few characters* if each positive-dimensional component of $X(Y)$ consists entirely of characters of reducible representations. An irreducible $\mathbb{Q}$–homology solid torus M is called *longitudinally rigid* when its Dehn filling along the homological longitude $M(0)$ has few characters. Here is the statement of Theorem 7.1, where the manifold M has a fixed homologically natural framing (μ, λ).

7.1 Theorem *Suppose that M is a longitudinally rigid irreducible $\mathbb{Q}$–homology solid torus and that the Alexander polynomial of M has a simple root ξ on the unit circle. When M is not a $\mathbb{Z}$–homology solid torus, further suppose that $\xi^k \neq 1$, where $k > 0$ is the order of the homological longitude λ in $H_1(M; \mathbb{Z})$. Then there exists $a > 0$ such that for every rational $r \in (-a, 0) \cup (0, a)$ the Dehn filling $M(r)$ is orderable.*

Boyer told us in a private communication that there is an analog of Theorem 1.2 when the simple root ξ is on the positive real axis. Here is the argument that this implies Theorem 1.2.

Proof of Theorem 1.2 Comparing the statements, there are two things to do: show that M being lean implies that M is longitudinally rigid, and establish that $M(0)$ is orderable. The latter is immediate from Theorem 1.1 of [8] since $H^1(M(0); \mathbb{Z}) \cong \mathbb{Z}$ and $M(0)$ is either irreducible or $S^2 \times S^1$. The former is an immediate consequence of:

7.2 Claim *Suppose Y is an irreducible closed 3–manifold. If the only essential surfaces in Y are fibers in fibrations over the circle, then Y has few characters.*

Here is the proof of the claim. Suppose instead that $X(Y)$ has a positive-dimensional component Z containing an irreducible character χ_0. Recall from Section 2.3 that the functions tr_α^2 for a finite set of $\alpha \in \pi_1(Y)$ give coordinates on the complex affine algebraic set $X(Y)$. Pick an irreducible curve $X_0 \subset Z$ that contains χ_0, which we can do by eg Corollary 1.9 of [20]. As affine algebraic curves over $\mathbb{C}$ are noncompact, there is at least one ideal point of X_0 in the sense of [9, Section 4]. This gives an action of $\pi_1(Y)$ on a simplicial tree, which in turn has an essential dual surface. Let F be a connected component of this dual surface. By hypothesis, the surface F must be a fiber in a fibration of Y over the circle. In fact, since *every* essential surface in Y is a fiber, it follows from [58, pages 113–115] that $b_1(Y) = 1$ and that F is the unique connected essential surface in Y up to isotopy. Therefore, the surface associated to any other ideal point of X_0 must also consist of parallel copies of F. Hence, for every $\gamma \in \pi_1(F)$, the function tr_γ^2 takes a finite value at every ideal point of the curve X_0, which forces the function tr_γ^2 to actually be constant on X_0. Thus every character in X_0 has the same restriction to $\pi_1(F)$, which we denote by $\eta \in X(F)$. There are two cases depending on whether or not η is reducible.

Suppose η is irreducible. As per Section 2.3, all representations $\pi_1(F) \to G_\mathbb{C}$ with character η are irreducible and conjugate, and let us fix one such representation ρ. If $f \colon F \to F$ is the monodromy of the fibration, we have the usual presentation

$$\pi_1(M) = \langle \tau, \pi_1(F) \mid \tau \gamma \tau^{-1} = f_*(\gamma) \text{ for all } \gamma \in \pi_1(F) \rangle$$

Thus, a representation $\hat{\rho} \colon \pi_1(M) \to G_\mathbb{C}$ that restricts to ρ on $\pi_1(F)$ is determined by the element $T = \hat{\rho}(\tau) \in G_\mathbb{C}$; moreover, T must conjugate ρ to $\rho \circ f_*$. As ρ is irreducible, its stabilizer under conjugation is finite [39, Proposition 3.16(i)], and hence there are only finitely many possibilities for T. But then X_0 is finite, a contradiction.

Suppose instead that η is reducible. Let $\psi\colon \pi_1(M) \to G_{\mathbb{C}}$ be an irreducible representation with character in X_0. Note that $\psi|_{\pi_1(F)}$ is nontrivial as otherwise ψ factors through $\pi_1(M)/\pi_1(F) \cong \mathbb{Z}$ making ψ itself reducible. As $\psi|_{\pi_1(F)}$ has character η, it is reducible and has either exactly one or exactly two fixed points on $\boldsymbol{P}^1(\mathbb{C})$. If $\psi|_{\pi_1(F)}$ had a unique fixed point $p_0 \in \boldsymbol{P}^1(\mathbb{C})$, then, since $\pi_1(F)$ is normal, it follows that ψ itself fixes p_0, making ψ reducible. So $\psi|_{\pi_1(F)}$ has exactly two fixed points on $\boldsymbol{P}^1(\mathbb{C})$, and we conjugate ψ so that these are $0 = [0:1]$ and $\infty = [1:0]$. After this conjugation, the image of $\psi|_{\pi_1(F)}$ consists of diagonal matrices and its nontrivial elements are hyperbolic or elliptic with axis the geodesic L in $\mathbb{H}^3$ that joins 0 to ∞. Now consider how $\psi(\tau)$ acts on the points 0 and ∞. It must not fix them individually, as then ψ would be reducible. Hence $\psi(\tau)$ is an elliptic element of order two whose axis is orthogonal to L. We can conjugate ψ by a diagonal matrix, which does not change $\psi|_{\pi_1(F)}$, so that $\psi(\tau) = \pm\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. In particular, up to conjugacy, ψ is completely determined by $\psi|_{\pi_1(F)}$. As a diagonal representation such as $\psi|_{\pi_1(F)}$ is determined up to conjugacy by its character, we have shown that X_0 contains a unique irreducible character. But this contradicts the fact that the irreducible characters in X_0 are Zariski open [39, Corollary 3.6]. This completes the proof of Claim 7.2 and shows that Theorem 1.2 follows from Theorem 7.1. $\qquad\square$

We now sketch the proof of Theorem 7.1, which we also illustrate in Figure 14. Recall that to order the Dehn filling $M(r)$ by applying Lemma 4.4, we need an intersection of the translation locus $EL_{\widetilde{G}}(M)$ with the line L_r, which is the line through the origin of slope $-r$. So to prove the theorem, we construct a cone $\mathcal{C}$ of lines through the origin that contains the horizontal axis L_0 and where every line in $\mathcal{C}$ meets $EL_{\widetilde{G}}(M)$. To this end, we use a result of Heusener and Porti [40] to build an arc A in $EL_{\widetilde{G}}(M)$ which starts at a point in L_0 but is otherwise disjoint from it. The symmetries of $EL_{\widetilde{G}}(M)$ guarantee that if we have such an arc on one side L_0 then we will have one on the other side as well, giving us a big enough chunk of $EL_{\widetilde{G}}(M)$ to have the desired cone $\mathcal{C}$; see Figure 14 for more. A key technical point is that we must take care to ensure that the arc A is not completely contained in L_0, and this is where the hypothesis of longitudinally rigid comes in.

A key component of the proof is the following result, derived from [40].

7.3 Lemma *Suppose M is an irreducible $\mathbb{Q}$–homology solid torus. If ξ is a simple root of the Alexander polynomial that lies on the unit circle, then there exists an analytic path $\rho_t\colon [0,1] \to R_G(M)$, where:*
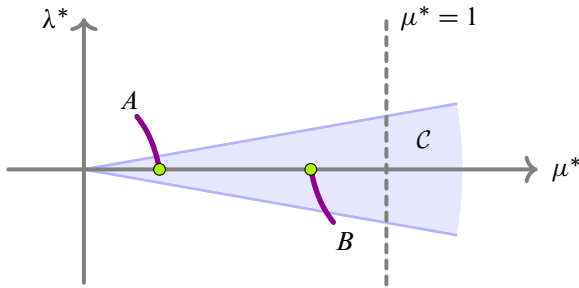
Figure 14: Here is an outline of the proof of Theorem 7.1. From the simple root ξ of Δ_M, we use [40] to produce an arc A in $EL_{\widetilde{G}}(M)$ leaving the horizontal axis at a corresponding Alexander point. Using the action of $D_\infty(M)$, we can assume the arc A lies in the strip $0 \le x \le 1$ as shown. The element of $D_\infty(M)$ which is π–rotation about $\left(\frac{1}{2}, 0\right)$ means there will be a second arc B in this strip on the opposite side of the horizontal axis from A. This allows us to find a cone $\mathcal{C}$ whose lines through the origin meet $EL_{\widetilde{G}}(M)$ in a point which is neither parabolic or ideal. The theorem will then follow from Lemma 4.4.

(1) *The representation ρ_0 acts by rotations about a unique fixed point in $\mathbb{H}^2$, and factors through $H = H_1(M;\mathbb{Z})_{\mathrm{free}} = H_1(M;\mathbb{Z})/_{(\mathrm{torsion})} \cong \mathbb{Z}$. A generator of H acts via rotation by angle $\arg(\xi)$.*

(2) *The representations ρ_t are irreducible over $G_\mathbb{C}$ for $t > 0$.*

(3) *The corresponding path $[\rho_t]$ of characters in $X_G(M)$ is also a nonconstant analytic path.*

(4) *There exists $\gamma \in \pi_1(\partial M)$, where $\mathrm{tr}_\gamma^2(\rho_t)$ is nonconstant in t.*

Proof Except for part (4), the lemma follows straightforwardly from the statement of Proposition 10.3 of [40] and Lemma 2.11 of our paper. However, it is even easier to derive claims (1)–(3) directly from the discussion in Section 10 of [40] and we take that approach. Throughout, we will follow the notation of [40] closely. Fix a generator h of H, and let $\alpha: \pi_1(M) \to \mathbb{C}^\times$ be the homomorphism which factors through the homomorphism $H \to \mathbb{C}^\times$ that sends h to ξ. Consider the associated diagonal representation $\rho_\alpha: \pi_1(M) \to G_\mathbb{C}$ given by

$$\rho_\alpha(\gamma) = \pm \begin{pmatrix} \alpha^{1/2}(\gamma) & 0 \\ 0 & (\alpha^{1/2}(\gamma))^{-1} \end{pmatrix},$$

where $\alpha^{1/2}(\gamma)$ is either square root of $\alpha(\gamma)$. Now the image of ρ_α is contained in the following subgroup of $G_\mathbb{C}$:

$$\mathrm{PSU}(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C} \text{ with } |a|^2 - |b|^2 = 1 \right\},$$

which is a conjugate of G in $G_{\mathbb{C}}$ as it corresponds to the Möbius transformations that stabilize the unit disc D in $\mathbb{C} \subset \mathbf{P}^1(\mathbb{C})$.

The proof of Proposition 10.3 in [40] shows that the cocycle defined there,

$$d_+ + d_- \in H^1(\pi_1(M); \mathfrak{su}(1,1)_{\rho_\alpha}),$$

can be integrated to an analytic path $\rho_t: [0,1] \to R_{\mathrm{PSU}(1,1)}(M)$ with $\rho_0 = \rho_\alpha$ and ρ_t irreducible over $G_{\mathbb{C}}$ for $t > 0$, which gives (2). Note that ρ_α stabilizes the center of D and acts on the tangent space there via α, which gives (1). Next, claim (3) that $[\rho_t]$ is nonconstant follows from (2), since, over $G_{\mathbb{C}}$, a reducible representation cannot have the same character as an irreducible representation.

Finally, we tackle claim (4), whose proof is more involved; please note that claim (4) is not actually used in this paper and so you can safely skip it. By Theorem 1.3 of [40], the character $\chi_\alpha = [\rho_\alpha]$ is contained in precisely two irreducible components of $X(M)$, both of which are (complex) curves: one consisting solely of characters of abelian representations and the other, which we will call X, whose characters generically come from representations that are irreducible over $G_{\mathbb{C}}$. Of course, our path $[\rho_t]$ lies in X. To study X near χ_α, we move away from ρ_α to the representation $\rho^+ \in R(M)$ constructed in [40, Section 5]. The representation ρ^+ is also reducible with character χ_α but has nonabelian image. Proposition 7.6 of [40] gives that ρ^+ is a smooth point of $R(M)$ of local dimension 4. Let $\mathfrak{sl}_2(\mathbb{C})_{\rho^+}$ denote the Lie algebra of $G_{\mathbb{C}}$ as a $\pi_1(M)$–module via the action $\mathrm{Ad} \circ \rho^+$. The proof of Proposition 7.6 of [40] shows that the Zariski tangent space of $R(M)$ at ρ^+ can be identified with the space of cocycles $Z^1(M; \mathfrak{sl}_2(\mathbb{C})_{\rho^+})$. (Unlike [40], we are assuming that M is irreducible and consequently aspherical, and so do not distinguish between cohomology of M and of $\pi_1(M)$.) As the tangent space to the orbit of ρ^+ is the space of coboundaries $B^1(M; \mathfrak{sl}_2(\mathbb{C})_{\rho^+})$, we can identify the Zariski tangent space of X at χ_α with $H^1(M; \mathfrak{sl}_2(\mathbb{C})_{\rho^+})$, which is $\mathbb{C}$ by Corollary 5.4 of [40]. As the restriction $\rho^+ \circ \iota$ in $R(\partial M)$ is nontrivial, the proof of Lemma 7.4 of [40] gives that $\rho^+ \circ \iota$ is a smooth point of $R(\partial M)$, and so again we can identify the Zariski tangent space of $X(\partial M)$ at $[\rho^+ \circ \iota]$ with $H^1(\partial M; \mathfrak{sl}_2(\mathbb{C})_{\rho^+}) \cong \mathbb{C}^2$. The claim (4) boils down to showing that

$$(7.4) \qquad \iota^*: H^1(M; \mathfrak{sl}_2(\mathbb{C})_{\rho^+}) \to H^1(\partial M; \mathfrak{sl}_2(\mathbb{C})_{\rho^+})$$

is injective, since coordinates on $X(\partial M)$ are precisely the functions tr^2_γ for $\gamma \in \pi_1(M)$.

To understand the map in (7.4), start by calculating that the 0–cohomologies, or equivalently the $\pi_1(M)$–invariant subspaces of $\mathfrak{sl}_2(\mathbb{C})_{\rho+}$, are

$$H^0(\partial M; \mathfrak{sl}_2(\mathbb{C})_{\rho+}) \cong \mathbb{C} \quad \text{and} \quad H^0(M; \mathfrak{sl}_2(\mathbb{C})_{\rho+}) \cong 0.$$

As M has Euler characteristic 0, this forces $H^2(M; \mathfrak{sl}_2(\mathbb{C})_{\rho+}) \cong \mathbb{C}$, and so by duality we have $H^1(M, \partial M; \mathfrak{sl}_2(\mathbb{C})_{\rho+}) \cong \mathbb{C}$ as well. Suppressing the coefficients, the long exact sequence of the pair includes

$$H^1(\partial M) \xleftarrow{\iota^*} H^1(M) \longleftarrow H^1(M, \partial M) \xleftarrow{\delta} H^0(\partial M) \longleftarrow H^0(M)$$
$$\mathbb{C}^2 \qquad\qquad \mathbb{C} \qquad\qquad \mathbb{C} \qquad\qquad\qquad \mathbb{C} \qquad\qquad 0$$

which forces ι^* at left to be injective, as claimed. This establishes (4) and hence the lemma. $\qquad\square$

Proof of Theorem 7.1 We will use the coordinate system described in Section 4.1 to identify $H^1(\partial M; \mathbb{R})$ with $\mathbb{R}^2$. We will show that there exists a cone $\mathcal{C}$ in $\mathbb{R}^2$ containing the positive part of the horizontal axis in its interior such that every line contained in $\mathcal{C}$ meets the subset $EL_{\widetilde{G}}(M)$ in a point which is neither ideal nor parabolic. The theorem then follows directly from Lemma 4.4 once we invoke [36, Theorem 1.2] to know that all but at most three Dehn fillings on M are irreducible.

We claim it suffices to produce a path A in $EL_{\widetilde{G}}(M)$ which begins at a point on the horizontal axis, and not at the origin, such that the image of A is not completely contained in the horizontal axis. If the image of A contains points of either the upper or lower open half-plane, then the symmetries imply that there also exists a path whose image contains points of the other half-plane; compare Figure 14. Thus the images of the two paths will meet every line in some cone $\mathcal{C}$. By Theorem 4.3, after shrinking these paths if necessary, we may assume that they contain no ideal points. Since parabolic points occur only at integer lattice points, we may also assume that these paths contain no parabolic points in their interior.

Let ξ be a simple root of Δ_M that lies on the unit circle. Note that ξ is different from 1 since, as M is a $\mathbb{Q}$–homology solid torus, the value $|\Delta_M(1)|$ is the order of the torsion subgroup in $H_1(M; \mathbb{Z})$ and hence positive. Let ρ_t be the associated path in $R_G(M)$ given by Lemma 7.3. Now ρ_0 factors through the free abelianization H of $\pi_1(M)$, which is just $\mathbb{Z}$, and so it is trivial to lift ρ_0 to $\widetilde{\rho}_0 \colon \pi_1(M) \to \widetilde{G}$ that still factors through H. As λ is 0 in H, we have $\text{trans}(\widetilde{\rho}_0(\lambda)) = 0$. As ξ is not 1, we have $\text{trans}(\widetilde{\rho}_0(\mu)) \neq 0$. As noted in Section 4.1, the index of $\langle \iota_*(\mu) \rangle$ in H is

the order k of $\iota_*(\lambda)$ in $H_1(M;\mathbb{Z})$. Thus, using Section 3.4, we adjust $\widetilde{\rho}_0$ so that $\mathrm{trans}(\widetilde{\rho}_0(\mu))$ is in $(0, k]$. In particular, $\widetilde{\rho}_0$ gives a point $(x, 0) \in EL_{\widetilde{G}}(M)$ with $x > 0$ in our coordinates on $H^1(\partial M; \mathbb{R})$.

As discussed in Section 3.3, the Euler class is the complete obstruction to lifting a representation to $\widetilde{G}$ and is constant on connected components of $R_G(M)$. Hence, as ρ_0 lifts to $\widetilde{\rho}_0$, we can extend this to a continuous path $\widetilde{\rho}_t \colon [0, 1] \to R_{\widetilde{G}}(M)$ lifting the original ρ_t. Because $\xi^k \neq 1$, we have $\mathrm{tr}_\mu^2(\widetilde{\rho}_0) = \xi^k + 2 + \xi^{-k} < 4$, so there exists $\epsilon > 0$ such that $\mathrm{tr}_\mu^2(\rho_t) < 4$ for $t \in [0, \epsilon]$. This means that the representation ρ_t sends μ to an elliptic element and, since λ commutes with μ, it must also send λ to an elliptic or trivial element. By replacing ρ_t by its restriction to a subinterval of positive length, we have that ρ_t is a path in $PE_G(M)$ and that $\widetilde{\rho}_t$ is a path in $PE_{\widetilde{G}}(M)$.

We now build our path A by composing $\widetilde{\rho}_t$ with $\mathrm{trans} \circ \iota^* \colon PE_{\widetilde{G}}(M) \to EL_{\widetilde{G}}(M)$. By Lemma 7.3(3), we know $[\rho_t]$ is a nonconstant path in $X(M)$ and hence $\widetilde{\rho}_t$ is a nonconstant path in $PE_{\widetilde{G}}(M)$. However, we must still prove that A is not contained in the horizontal axis, ie that $\mathrm{trans}(\widetilde{\rho}_t(\lambda))$ is not the zero function in t. If it were, then since $\rho_t(\lambda)$ is always elliptic or trivial, we would have that $\rho_t(\lambda) = 1$ for all t; in particular, all the ρ_t factor through $\pi_1(M(0))$ and so the path $[\rho_t]$ lies in $X(M(0)) \subset X(M)$. Thus the $[\rho_t]$ are in an irreducible component Z of $X(M(0))$ of complex dimension at least 1. By Lemma 7.3(2), the ρ_t are irreducible for $t > 0$, and thus Z is a component of $X(M(0))$ of positive dimension which contains an irreducible character. This contradicts our hypothesis that M is longitudinally rigid, and completes the proof of the theorem. $\qquad \square$

7.5 Remark For general $\mathbb{Q}$–homology solid tori, there can be reducible representations that deform to irreducible representations but *do not* come from roots of the Alexander polynomial; rather, they correspond to roots of certain *twisted* Alexander invariants as described in [40]. However, it would not help to consider such representations in the context of Theorem 7.1: as we now explain, the additional representations never lift to $\widetilde{G}$ and hence are of no interest to us here. Specifically, consider a representation $\rho \colon \pi_1(M) \to S \leq G$, where $S = \mathrm{PSO}_2 \cong S^1$; in the proof of Theorem 7.1, we considered such ρ that factor through $H_1(M;\mathbb{Z})_{\mathrm{free}}$ and deform to irreducible representations in $R_G(M)$. More generally, we could consider any deformable $\rho \colon \pi_1(M) \to S \leq G$. However, the preimage of S in $\widetilde{G}$ is $\mathbb{R}$, which is abelian and torsion-free; thus if ρ lifts to $\widetilde{\rho} \colon \pi_1(M) \to \widetilde{G}$, the lift $\widetilde{\rho}$ must factor through $H_1(M;\mathbb{Z})_{\mathrm{free}}$, and so we are back in the case considered in Theorem 7.1.

8 Real embeddings of trace fields and orderability

This section gives the proof of Theorem 1.4, whose statement we recall below after giving some needed background.

8.1 Trace fields and Galois conjugate representations

Let M be a compact orientable 3–manifold whose boundary is a torus. The trace field of a representation $\rho\colon \pi_1(M) \to G_{\mathbb{C}}$ is the subfield of $\mathbb{C}$ generated over $\mathbb{Q}$ by the traces of all $\rho(\gamma)$ for $\gamma \in \pi_1(M)$; this is well-defined even though the trace of each $\rho(\gamma)$ only makes sense up to sign. Of course, the trace field depends only on the conjugacy class of ρ. If ρ_{hyp} is a holonomy representation of a finite-volume hyperbolic structure on the interior of M, by local rigidity its trace field F is a number field, that is, a finite extension of $\mathbb{Q}$ [49, Theorem 3.1.2]. In particular, F is contained in the subfield $\overline{\mathbb{Q}} \subset \mathbb{C}$ of all algebraic numbers.

As the hyperbolic structure has a cusp, we can conjugate ρ_{hyp} so that its image lies in $\mathrm{PSL}_2\, F$ [49, Theorem 3.3.8]. Given an embedding $\sigma\colon F \to \mathbb{C}$, which must have image contained in $\overline{\mathbb{Q}}$, we get a *Galois conjugate representation* $\rho\colon \pi_1(M) \to G_{\mathbb{C}}$ by composing ρ_{hyp} with the induced map $\mathrm{PSL}_2\, F \to \mathrm{PSL}_2(\sigma(F))$. As irreducible representations into $G_{\mathbb{C}}$ are determined by their characters, up to conjugacy in $G_{\mathbb{C}}$ this ρ depends only on σ and not on how we conjugated ρ_{hyp} to lie in $\mathrm{PSL}_2\, F$.

Here is the statement that this section is devoted to proving:

Theorem 1.4 *Let K be a hyperbolic knot in a $\mathbb{Z}$–homology 3–sphere Y. If the trace field of the knot exterior M has a real embedding then:*

(1) *For all sufficiently large n, the n–fold cyclic cover of Y branched over K is orderable.*

(2) *There is an interval I of the form $(-\infty, a)$ or (a, ∞) such that the Dehn filling $M(r)$ is orderable for all rational $r \in I$.*

(3) *There exists $b > 0$ such that for every rational $r \in (-b, 0) \cup (0, b)$ the Dehn filling $M(r)$ is orderable.*

The proof relies on the following three lemmas, the third of which was suggested to us by Agol and Futer.

8.2 Lemma *Suppose M is a hyperbolic $\mathbb{Z}$–homology solid torus, with homological longitude $\lambda \in \pi_1(\partial M)$. Suppose the trace field F of M has a real embedding $\sigma \colon F \to \mathbb{R}$ and let $\rho \colon \pi_1(M) \to G$ be the corresponding Galois conjugate of the holonomy representation. If $\widetilde{\rho} \colon \pi_1(M) \to \widetilde{G}$ is any lift of ρ, then $\mathrm{trans}(\widetilde{\rho}(\lambda))$ is an odd integer.*

8.3 Lemma *Suppose M is an orientable 1–cusped hyperbolic 3–manifold whose trace field has a real embedding. Then there exists an arc c in $R_G(M)$ and a representation ρ in its interior such that:*

(1) *The representation ρ is a Galois conjugate of a holonomy representation of the hyperbolic structure on M.*

(2) *For any slope $\gamma \in \pi_1(\partial M)$, the arc c is parametrized near ρ by tr_γ^2.*

8.4 Lemma (Agol and Futer) *Suppose $EL_{\widetilde{G}}(M)$ contains an arc A that is not horizontal, ie that has points with different vertical coordinates. Then there exists an $a > 0$ so that the line L_r meets $EL_{\widetilde{G}}(M)$ for all r in $(-a, a)$.*

Proof of Lemma 8.2 Let $\rho_{hyp} \colon \pi_1(M) \to \mathrm{PSL}_2 F$ be a holonomy representation for the hyperbolic structure on M. Let $\rho'_{hyp} \colon \pi_1(M) \to \mathrm{SL}_2 F$ be any lifted representation, which exists by [25, Proposition 3.1.1]. By Corollary 2.4 of [13], we know $\mathrm{tr}(\rho'_{hyp}(\lambda)) = -2$ since λ is the boundary of an orientable spanning surface. The Galois conjugate $\rho' = \sigma \circ \rho'_{hyp}$ also has $\mathrm{tr}(\rho'(\lambda)) = -2$, and note that ρ' is a lift of ρ to $\mathrm{SL}_2\mathbb{R}$. Consider the successive quotients

$$\widetilde{G} \xrightarrow[q]{\overset{p}{}} \mathrm{SL}_2\mathbb{R} \xrightarrow{} G$$

The lemma will follow immediately from the fact that $\mathrm{tr}(\rho'(\lambda)) = -2$ once we show:

8.5 Claim *Suppose $\widetilde{g}$ is a parabolic or central element of $\widetilde{G}$ and $\overline{g}$ is its image in $\mathrm{SL}_2\mathbb{R}$. Then the parity of $\mathrm{trans}(\widetilde{g})$ is odd precisely when $\mathrm{tr}(\overline{g}) = -2$ rather than $+2$.*

To see this, consider the subset P of all parabolic or central elements of $\widetilde{G}$. (Figure 1 of [45] has a detailed picture of P as well as the subsets of elliptic and hyperbolic elements; this picture informs our approach here but is not directly used.) Note that every path component of P contains a central element; this is because downstairs in G any parabolic element can be connected to the trivial element by a path all of whose interior points are parabolic, and paths lift to covering spaces. The functions trans and

tr $\circ q$ are both continuous on $\widetilde{G}$ and are integer-valued on P. Hence they are constant on each path component of P, and it suffices to prove the claim for central elements. There, note that the center $\{s^k\}$ of $\widetilde{G}$ maps to the center $\{\pm I\}$ of $\mathrm{SL}_2\mathbb{R}$ via the unique epimorphism $\mathbb{Z} \to \mathbb{Z}/2$; hence the s^k which map to $-I$ are exactly those with k odd. This proves the claim and hence the lemma. $\qquad\square$

Proof of Lemma 8.3 Let F be the trace field of M and $\rho_{hyp}\colon \pi_1(M) \to \mathrm{PSL}_2 F$ be a holonomy representation. As F has a real embedding, choose $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(F) \subset \mathbb{R}$, and define $\rho \in R_G(M)$ as $\sigma \circ \rho_{hyp}$.

Now both $R(M)$ and $X(M)$ are defined over $\mathbb{Q}$, that is, they can be cut out by polynomials with rational coefficients. Hence $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts coordinatewise on their $\overline{\mathbb{Q}}$–points. Since $[\rho_{hyp}]$ comes from the complete hyperbolic structure on M, it is a smooth point of $X(M)$ where the local dimension is 1 — see [54, Corollaire 3.28] — in particular, the Zariski tangent space to $X(M)$ at $[\rho_{hyp}]$ is 1–dimensional. Let X be the unique $\mathbb{Q}$–irreducible component X of $X(M)$ that contains $[\rho_{hyp}]$. (You can construct X by taking the $\mathbb{C}$–irreducible component X_0 of $X(M)$ containing $[\rho_{hyp}]$, which must be defined over some number field, and then taking the union of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$–orbit of X_0.) Since X is invariant under the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$–action, it contains $[\rho]$ as well as $[\rho_{hyp}]$. Finally, the dimension of X (thought of as an algebraic set over either $\mathbb{Q}$ or $\mathbb{C}$) is 1.

Again by [54, Corollaire 3.28], for any slope $\gamma \in \pi_1(\partial M)$, the trace function tr_γ^2 is a local parameter for X on a small classical neighborhood of $[\rho_{hyp}]$ (the reference [54] works with $\mathrm{SL}_2\mathbb{C}$ rather than $G_{\mathbb{C}}$ character varieties, but this makes no difference since near both 2 and -2 in $\mathbb{C}$ the map $z \mapsto z^2$ is injective). Since σ acts on the $\overline{\mathbb{Q}}$–points of X taking $[\rho_{hyp}]$ to $[\rho]$, it follow that $[\rho]$ is also a smooth point of X, where again any tr_γ^2 is a local parameter for the nearby $\mathbb{C}$ points; this is because whether a regular function is a local parameter at a smooth point on the curve X can be expressed purely algebraically and hence is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$–invariant.

Let τ denote the action of complex conjugation on $X(M)$ as in Section 2.5. As $[\rho]$ is a smooth point of a 1–dimensional irreducible component of $X(M)$, by Proposition 2.8 there is a smooth arc $\overline{c}$ of real points in $X_{\mathbb{R}}(M)$ containing $[\rho]$ in its interior. Since tr_γ^2 gives a local parameter for X near ρ, the arc $\overline{c}$ must be locally defined simply by the requirement that tr_γ^2 is real. Thus $\overline{c}$ is parametrized near $[\rho]$ by the value of tr_γ^2 in the interval $[4-\epsilon, 4+\epsilon]$. Moreover, by restricting ϵ we can assume every character in $\overline{c}$ comes from a $G_{\mathbb{C}}$–irreducible representation; by Lemma 2.10, this means $C \subset X_G(M)$
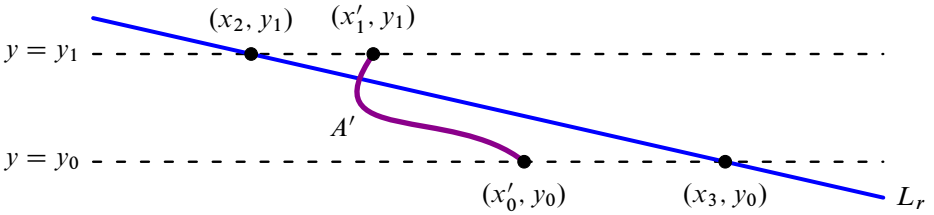
Figure 15: When the slope of L_r is small enough, it must meet a translate A' of A.

since $[\rho] \in X_G(M)$. By Lemma 2.11, we can lift $\bar{c}$ to an arc in c in $R_G(M)$. As the function tr_γ^2 must also be a local parameter for c, we have proved the lemma. □

Proof of Lemma 8.4 By shortening the arc if necessary, we first arrange that A lies to one side of the horizontal axis. As $D_\infty(M)$ preserves $EL_{\widetilde{G}}(M)$ and contains π–rotation about the origin, we may assume that A lies below this axis. We will show that there exists an $a_1 > 0$ such that L_r meets $EL_{\widetilde{G}}(M)$ for all r in $(0, a_1)$. Applying the symmetric argument to the π–rotation of A about the origin will give an $a_2 > 0$ such that L_r meets $EL_{\widetilde{G}}(M)$ for all r in $(-a_2, 0)$; taking $a = \min(a_1, a_2)$ will then give the promised interval, since the horizontal axis itself is always part of $EL_{\widetilde{G}}(M)$.

As usual, let k be the order of $\iota_*(\lambda)$ in $H_1(M; \mathbb{Z})$, so that $D_\infty(M)$ contains the subgroup of horizontal translations by multiples of k. By shortening A if necessary, we can label its endpoints as (x_0, y_0) and (x_1, y_1), where $y_0 < y_1 < 0$ and $|x_1 - x_0| < k$. We claim that L_r meets $EL_{\widetilde{G}}(M)$ for all r where

$$(8.6) \qquad\qquad 0 < r < \frac{y_1 - y_0}{2k}$$

To see this, let (x_2, y_1) be the point where L_r meets the horizontal line $y = y_1$, and let (x_3, y_0) be the point where L_r meets $y = y_0$. Let n be the largest integer such that $x_0 + nk \le x_3$, and let $A' \subset EL_{\widetilde{G}}(M)$ be A translated to the right by nk, so the endpoints of A' are $(x_0 + nk, y_0)$ and $(x_1 + nk, y_1)$. Set $x_0' = x_0 + nk$ and $x_1' = x_1 + nk$.

We now argue that L_r meets A', using Figure 15 as a guide. Since the slope of L_r is $-r$, and since $(y_1 - y_0)/r > 2k$ by (8.6), we have

$$x_3 - x_2 = \frac{y_1 - y_0}{r} > 2k.$$

Our choice of n guarantees that $x_0' < x_3$ and $|x_3 - x_0'| < k$. We also have $|x_1' - x_0'| = |x_1 - x_0| < k$. Thus

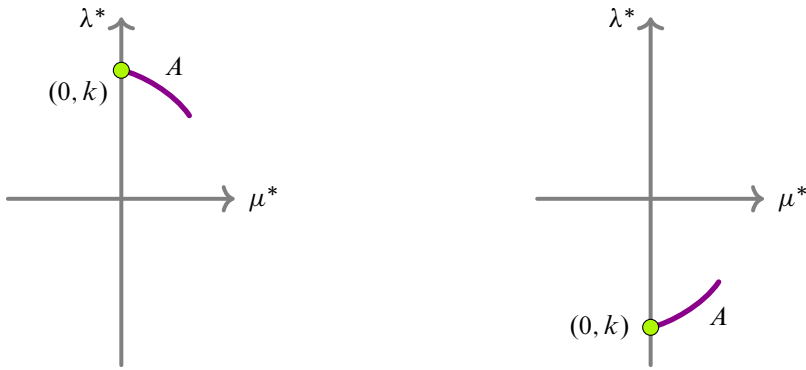$$|x_3 - x_1'| \le |x_3 - x_0'| + |x_0' - x_1'| < 2k.$$

Figure 16: The two possibilities for the arc A in the locus $EL_{\widetilde{G}}(M)$ originating from the image $(0,k)$ of a Galois conjugate of the holonomy representation: $k > 0$ (left) and $k < 0$ (right).

Combining, we conclude that $x_2 < x_1'$. We also have $x_0' < x_3$, so we have shown that the endpoints of A' lie on opposite sides of L_r, as in Figure 15. This implies that L_r must meet A', completing the proof of the lemma. ☐

Proof of Theorem 1.4 Let c be the arc in $R_G(M)$ given by Lemma 8.3, and ρ the Galois conjugate of the holonomy representation which is in c. As $H^2(M;\mathbb{Z}) = 0$, the Euler class of any representation in c vanishes, and hence we can lift c to an arc $\widetilde{c}$ in $R_{\widetilde{G}}(M)$. We fix a particular lift by requiring that ρ lifts to $\widetilde{\rho}$ with $\mathrm{trans}(\widetilde{\rho}(\mu)) = 0$. By Lemma 8.2, we have that $\mathrm{trans}(\widetilde{\rho}(\lambda)) = k$ is an odd integer, and so $\widetilde{\rho}$ gives rise to the point $(0,k)$ in $EL_{\widetilde{G}}(M)$.

Since this is true downstairs for c, the function tr_μ^2 is a local parameter for $\widetilde{c}$ where the parameter takes values in $[4-\epsilon, 4+\epsilon]$. For the subinterval $[4-\epsilon, 4]$, the representations on $\widetilde{c}$ must lie in $PE_{\widetilde{G}}(M)$ since they each send μ to a parabolic or elliptic element of $\widetilde{G}$. In particular, the translation number of μ is a local parameter for this portion of $\widetilde{c}$.

Thus we get an arc A in $EL_{\widetilde{G}}(M)$ which starts from $(0,k)$, where k is the aforementioned odd integer, and is locally parametrized by the μ^*–coordinate on some small interval $[0, \delta]$. Moreover, by construction no point on A is an ideal point, and the only parabolic point on A is $(0,k)$ itself. Depending on the sign of k, we get one of the two pictures in Figure 16.

To prove conclusion (1), consider an n–fold cyclic cover $\widetilde{Y}$ of Y branched over K. First, by the hyperbolic Dehn surgery theorem, the manifold $\widetilde{Y}$ is hyperbolic and hence

irreducible for all large n. Moreover, from Figure 16 it is clear that for all large n the arc A meets the line $\mu^* = 1/n$, so we now get (1) directly from Lemma 4.5.

For (2), for concreteness let us focus on the possibility in Figure 16, right. Since there are at most three Dehn fillings on M that are reducible [36, Theorem 1.2], we can construct an interval $I = (a, \infty)$ where $M(r)$ is irreducible and L_r meets A for all $r \in I$. The claim now follows immediately from Lemma 4.4.

Finally, for part (3), by Lemma 8.3 the arc c in $R_G(M)$ is parametrized near ρ by tr_λ^2; thus the corresponding arc A in $EL_{\widetilde{G}}(M)$ is not horizontal. Hence, by Lemma 8.4, the line L_r meets $EL_{\widetilde{G}}(M)$ for all r in some open interval $(-b, b)$. Shrinking b, we can ensure that $M(r)$ is irreducible for all r in $(-b, 0) \cup (0, b)$. Again, claim (3) now follows immediately from Lemma 4.4, completing the proof of the theorem.   □

8.7 Remark The hypothesis that Y is a $\mathbb{Z}$–homology 3–sphere is certainly necessary for the proof of Theorem 1.4 to work, and it is likely that the conclusion of Theorem 1.4 does not hold in general if one drops this hypothesis. Specifically, consider the 1–cusped hyperbolic 3–manifold $M = v2503$, which has $H_1(M; \mathbb{Z}) = \mathbb{Z} + \mathbb{Z}/10$ and $H^2(M; \mathbb{Z}) = \mathbb{Z}/10$. The trace field here is $\mathbb{Q}(\xi)$, where ξ is a root of

$$x^{10} - 4x^8 + 9x^6 - 15x^4 + 12x^2 - 2,$$

which has six real embeddings. However, none of the resulting representations $\pi_1(M) \to G$ lift to $\widetilde{G}$, completely stymying our technique for constructing orders.

This M is interesting from the point of view of Floer theory; specifically, Lidman and Watson recently gave infinitely many $\mathbb{Q}$–homology solid tori which were not fibered and where every nonlongitudinal Dehn filling is an L–space [48]. As their examples all have essential annuli, they asked [48, Question 6] whether there are hyperbolic examples with these same properties; the manifold $v2503$ answers that question affirmatively, as we now explain. We will use the homological framing (μ, λ) which corresponds to $(0, 1)$ and $(-1, 0)$ in SnapPy's default conventions. Then $M(\mu)$ is the lens space $L(50, 19)$ and $M(\lambda)$ is $S^2 \times S^1 \# \mathbb{R}P^3$. Using [55], it is possible to show that *every* nonlongitudinal Dehn filling on M is an L–space, even though it is not a fibered 3–manifold as $\Delta_M = 2(t^4 + t^3 + t^2 + t + 1)$.

Of course, if Conjecture 1.1 is true, then every Dehn filling on M is *not* orderable (the filling $M(\lambda)$ is not orderable as its fundamental group has torsion). We checked the 16 examples where the Dehn filling coefficients are at most 3, and in each case we

were able to show that the corresponding Dehn filling was not orderable. It would be interesting to show that this is the case for all Dehn fillings.

9 Open questions

Our results in this paper and especially the examples in Section 5 suggest many interesting questions and possible avenues for future research; here are some of them:

(1) Find topological hypotheses on a $\mathbb{Z}$–homology solid torus which imply that all Dehn surgeries in $(-1, 1)$ are orderable.

(2) Find topological hypotheses which give rise to the behavior, shown in Figure 8, where one can use $EL_{\widetilde{G}}(M)$ to order all but one Dehn filling on M.

(3) Do all Berge knots have $EL_{\widetilde{G}}(M)$ of the simple form shown in Figures 3 and 4? What about twisted torus knots? In the latter case, perhaps one can view $EL_{\widetilde{G}}(M)$ as some kind of "perturbation" of the very simple $EL_{\widetilde{G}}(M)$ of the underlying torus knot.

(4) In Lemma 6.4 we show that $EL_{\widetilde{G}}(M)$ lives in a horizontal strip whose size is bounded. When M is a $\mathbb{Z}$–homology solid torus, our proof shows that the maximum y–coordinate of a point in $EL_{\widetilde{G}}(M)$ is $2g - 1$, where g is the Seifert genus of M. In our examples, this bound is never sharp. Is this always the case, and regardless, is there some way to understand this gap?

(5) Does every polynomial satisfying the conclusion of [53, Corollary 1.3] have a *simple* root on the unit circle? Note that by [46] such a polynomial always has a root on the unit circle. Experimental evidence says yes.

(6) Can the longitudinally rigid hypothesis in Theorem 7.1 be eliminated by placing additional conditions on Δ_M? In the known examples where longitudinal rigidity comes into play, the "bad" roots of Δ_M are all roots of unity.

(7) Also motivated by Theorem 7.1, are there closed atoroidal 3–manifolds with $\dim H_1(M; \mathbb{Q}) \leq 1$ which do not have few characters? What if one restricts to 0–surgery on a knot in S^3?

(8) There is a Chern–Simons invariant/Seifert volume/Godbillon–Vey invariant associated to each representation in $R_G(M)$, see [45]. In our usual coordinates on $EL_{\widetilde{G}}(M)$, the derivative is really simple, basically $x\,dy - y\,dx$. Can this invariant be used to prove something interesting about $EL_{\widetilde{G}}(M)$?

(9) How can one explore the space of actions of $\pi_1(M)$ on $\mathbb{R}$ so as to include some which do not arise from $\widetilde{G}$ representations? It is natural to try to use some analog of

the character variety to do this. What is the appropriate setting for this? Is it possible to draw pictures like those in Section 5 that are built from some larger class of maps to

$$\widetilde{\mathrm{Homeo}^+(\mathbb{R})}?$$

(10) Motivated by Remark 8.7, prove that every Dehn filling on $v2503$ is not orderable.

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