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Ricci flow on asymptotically Euclidean manifolds

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In this paper, we prove that if an asymptotically Euclidean manifold with nonnegative scalar curvature has long-time existence of Ricci flow, the ADM mass is nonnegative. We also give an independent proof of the positive mass theorem in dimension three.

53C44; 83C99

1 Introduction

A smooth orientable Riemannian manifold (M^n, g) for $n \geq 3$ is called an *asymptotically Euclidean* (AE) manifold if for some compact $K \subset M^n$, the set $M^n \backslash K$ consists of a finite number of components $E_1, \dots, E_k$ such that for each E_i there exists a C^∞ diffeomorphism $\Phi_i \colon E_i \to \mathbb{R}^n \backslash B(0, A_i)$ whereby under this identification,

$$g_{ij} = \delta_{ij} + O(r^{-\sigma_i}) \quad \text{and} \quad \partial^{|k|} g_{ij} = O(r^{-\sigma_i - k}) \tag{1}$$

for any partial derivative of order k as $r \to \infty$, where r is the Euclidean distance function. We call the positive number σ_i the order of end E_i.

The ADM mass (see Arnowitt, Deser and Misner [2]) from general relativity of an AE manifold (M, g) is defined as

$$m(g) = \lim_{r \to \infty} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) \, dA^j,$$

where $dA^j = \partial_j \lrcorner dV_{g_E}$ and g_E is the canonical Euclidean metric on $\mathbb{R}^n$.

The definition of mass involves a choice of asymptotic coordinates. But it follows from Bartnik's result [6] that if the order satisfies $\sigma > (n-2)/2$ and the scalar curvature is integrable, then the mass is finite and independent of AE coordinates. In other words, $m(g)$ depends only on the metric g.

The general positive mass conjecture is the following; see [26, Theorem 10.1].

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Conjecture 1.1 (positive mass conjecture) *Let (M^n, g) be an AE manifold of dimension $n \geq 3$, with order $\sigma > (n-2)/2$ and nonnegative integrable scalar curvature. Then $m(g) \geq 0$, with equality if and only if $(M, g) = (\mathbb{R}^n, g_E)$.*

In dimension three, the positive mass conjecture was first proved by Schoen and Yau [36] in 1979 by constructing a stable minimal surface and considering its stability inequality. In addition, Schoen and Yau [35; 37] showed that their method could be extended to the case when the dimension was less than eight. In 1981, Witten [43] proved the positive mass conjecture for spin manifolds of any dimension. In 2001, Huisken and Ilmanen [23] proved the stronger Riemannian Penrose inequality in dimension three by using the inverse mean curvature flow. In 2015, Hein and LeBrun gave a proof of the positive mass conjecture for Kähler AE manifolds; see [22]. To the author's knowledge, there is no proof of the positive mass conjecture in general dimension.

A natural question arises: can we prove the positive mass conjecture by using other geometric flows? Since the Ricci flow is one of the most powerful geometric flows, using which Perelman completely solved Thurston's geometrization conjecture [31; 33; 32], it is of interest to know how Ricci flow interacts with AE manifolds and the ADM mass.

Recall that Ricci flow is a geometric flow such that a family of metrics $g(t)$ on a smooth manifold M evolves under the PDE

$$\partial_t g(t) = -2\,\mathrm{Rc}(g(t)). \tag{2}$$

We will focus on the case when $(M, g(0))$ is an AE manifold.

It has been proved by Dai and Ma in [16] that Ricci flow preserves the ALE condition, nonnegative integrable scalar curvature and the ADM mass. Hence, it is important to understand the change of mass at possible singular times and infinity, if long-time existence of Ricci flow is assumed.

One of the main theorems in this paper shows that if we have long-time existence of Ricci flow, an AE manifold will converge to the Euclidean space in some strong sense. The proof is partially motivated by considering possible steady solitons on ALE manifolds; see the appendix. The convergence at time infinity will indicate that the mass is nonnegative along the flow.

We assume throughout this paper that the scalar curvature R is nonnegative and integrable, the manifold has only one end E [1] and the order σ of the end is greater

[1] In fact, all the arguments below apply to the multi-end case with slight modifications.

than $(n-2)/2$. Moreover, we fix a positive smooth function $r(x)$ on M such that $r(x) = |\Phi(x)|$ when $x \in E$, where Φ is the diffeomorphism in the definition of AE manifolds. We also identify $x \in E$ with $\Phi(x) \in \mathbb{R}^n$ without mentioning Φ explicitly.

Moreover, we assume that the order $\sigma \le n-2$, since if an AE manifold is of order greater than $n-2$, then it is also of order $n-2$.

Theorem 1.2 *Let (M^n, g) be an AE manifold satisfying the above assumptions. If there exists a solution $g(t)$, $0 \le t < \infty$, of the Ricci flow with $g(0) = g$, then $m(g) \ge 0$, with equality if and only if $(M^n, g) = (\mathbb{R}^n, g_E)$.*

Under Ricci flow, it is possible that the metric becomes singular at some finite time. In dimension three, we can continue Ricci flow by performing surgeries. We prove that the mass and other related conditions are preserved under Ricci flow with surgery. Moreover, if we choose the surgery parameter function $\delta(t)$ small enough, there are only finitely many surgeries. The finiteness of surgeries is proved by carefully examining the change of Perelman's μ–functional over surgery times. By choosing one appropriate Ricci flow with surgery, we have the long-time existence of Ricci flow after the last surgery time, and Theorem 1.2 applies.

Theorem 1.3 *When $n = 3$, the mass $m(g) \ge 0$, with equality if and only if $(M^3, g) = (\mathbb{R}^3, g_E)$.*

For the remainder of the paper, C may vary from line to line. Moreover $\Delta = \Delta_{g(t)}$, $\nabla = \nabla_{g(t)}$ and $dV = dV_{g(t)}$ unless otherwise specified.

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2 Mass under Ricci flow

We prove in this section that Ricci flow preserves the AE condition and the mass is unchanged under Ricci flow. Our argument differs from that of Dai and Ma in [16] in that we fix an AE coordinate system along the flow. The main tool we use is the following maximum principle on the noncompact manifold with evolving metrics; see [11, Theorem 12.14].

Theorem 2.1 *Suppose that $g(t)$, $t \in [0,T]$, is a complete solution to the Ricci flow on a noncompact manifold M with $|\mathrm{Rm}(g(t))| \le k_0$ for some $k_0 > 0$. Let*

$$Lu = u_t - \Delta u - \langle X(t), \nabla u\rangle - G(u,t),$$

where $X(t)$ is a smooth family of bounded vector fields and $G\colon \mathbb{R}\times[0,T]\to\mathbb{R}$ is locally Lipschitz in the $\mathbb{R}$ factor and continuous in the $[0,T]$ factor. Suppose that u is a smooth function such that

$$Lu \le 0 \quad \text{and} \quad |u(x,t)| \le \exp(b(d_{g(t)}(O,x)+1))$$

for some constant b. For any $c\in\mathbb{R}$, let $U(t)$ be the solution to the corresponding ordinary differential equation

$$\frac{dU}{dt} = G(U,t), \quad U(0)=c.$$

If $u(x,0)\le c$ for all $x\in M$, then we have

$$u(x,t)\le U(t)$$

for all $x\in M$ and $t\in[0,T]$ as long as the ODE exists.

Theorem 2.2 *Suppose that $(M,g(t))$ is a Ricci flow solution with bounded curvature on $M\times[0,T]$ and $(M,g(0))$ is an AE manifold of order $\sigma>0$. Then:*

(i) *The AE condition is preserved, with the same AE coordinates and order.*

(ii) *If $\sigma > (n-2)/2$ and R is integrable, the mass is unchanged.*

Proof (i) Since $(M,g(0))$ is an AE manifold, there exist an end E and C^∞ diffeomorphism $\Phi\colon E\to\mathbb{R}^n\backslash B(0,A)$ such that under this coordinate system,

(3) $$g_{ij} = \delta_{ij} + O(r^{-\sigma}), \quad \partial^{|k|} g_{ij} = O(r^{-\sigma-k}),$$

for all $k=1,2,\dots$. From this it is easy to conclude that $|\nabla^k \mathrm{Rm}(0)| = O(r^{-\sigma-k-2})$.

Since the Riemannian curvature is uniformly bounded on $[0,T]$, there exists an $S>0$ such that $|\mathrm{Rm}|\le S$ on $M\times[0,T]$. Now we consider the evolution equation of $|\mathrm{Rm}|^2$ (see [13, Equations (2.57) and (6.1)]),

$$\partial_t |\mathrm{Rm}|^2 \le \Delta|\mathrm{Rm}|^2 + 16|\mathrm{Rm}|^3 \le \Delta|\mathrm{Rm}|^2 + 16S|\mathrm{Rm}|^2.$$

Let $u = |\mathrm{Rm}|^2 e^{-16St}$; then $\partial_t u \le \Delta u$ on $M\times[0,T]$.

Next we prove that u has the same spatial decaying condition as $u(0)$, see also [16].

Let $h(x) = r^{4+2\sigma}$ on M. We set $w = hu$ and it satisfies

$$(\partial_t - \Delta)w \le Bw - 2\nabla \log h \nabla w$$

on $M \times [0, T]$, where $B = (2|\nabla h|^2 - h\Delta h)/h^2$.

We first show that $|\mathrm{Rm}| \le S$ implies B is uniformly bounded on $M \times [0, T]$.

Since $|\mathrm{Rm}| \le S$, the metrics $g(t)$ are uniformly comparable to $g(0)$. That is,

$$C^{-1} g(0) \le g(t) \le C g(0) \tag{4}$$

on $M \times [0, T]$.

We have the following evolution equations for $|\nabla h|^2 = |\nabla_{g(t)} h|^2_{g(t)}$ and $\Delta h = \Delta_{g(t)} h$:

$$\partial_t |\nabla h|^2 = 2\,\mathrm{Rc}(\nabla h, \nabla h), \tag{5}$$

$$\partial_t (\Delta h) = 2\langle \mathrm{Rc}, \nabla^2 h\rangle. \tag{6}$$

The proof of (5) is straightforward and the proof of (6) can be found in [13, Lemma 2.30].

Therefore, from the curvature bound and (4), we have

$$|\partial_t |\nabla h|^2| \le C|\nabla h|^2, \tag{7}$$

$$|\partial_t (\Delta h)| \le C|\nabla^2 h| \le C|\nabla^2_{g(0)} h|_{g(0)}, \tag{8}$$

and by integration,

$$|\nabla_{g(t)} h|^2_{g(t)} \le C|\nabla_{g(0)} h|^2_{g(0)}, \tag{9}$$

$$|\Delta_{g(t)} h| \le C|\nabla^2_{g(0)} h|_{g(0)}. \tag{10}$$

To estimate $|\nabla_{g(0)} h|^2_{g(0)}$ and $|\nabla^2_{g(0)} h|_{g(0)}$ we use the given coordinate system of $g(0)$ at infinity. From the definition of h and direct computations, it is easy to show that

$$|\nabla_{g(0)} h|^2_{g(0)} \le C r^{6+4\sigma}, \tag{11}$$

$$|\nabla^2_{g(0)} h|_{g(0)} \le C r^{2+2\sigma}. \tag{12}$$

Therefore we have

$$|B| = \left|\frac{2|\nabla h|^2 - h\Delta h}{h^2}\right| \le C\left|\frac{|\nabla_{g(0)} h|^2_{g(0)}}{h^2}\right| + C\left|\frac{|\nabla^2_{g(0)} h|_{g(0)}}{h}\right| \le C r^{-2} \le C, \tag{13}$$

where the last inequality is true since r has a positive minimum.

From Theorem 2.1 we conclude that $|w| \le C$, hence $|\mathrm{Rm}| \le C r^{-2-\sigma}$ on $M \times [0, T]$.

Claim For each $k \geq 0$, we have

$$|\nabla^k \operatorname{Rm}| \leq C r^{-2-k-\sigma}. \tag{14}$$

Proof of claim Assume that the claim holds for all $0 \leq l < k$. Let $h_k = r^{4+2\sigma+2k}$ and $w_k = h_k |\nabla^k \operatorname{Rm}|^2$. The evolution equation of $|\nabla^k \operatorname{Rm}|^2$ (see [13, Equation (6.24)]) is given by

$$\begin{aligned} \partial_t |\nabla^k \operatorname{Rm}|^2 &= \Delta |\nabla^k \operatorname{Rm}|^2 - 2|\nabla^{k+1} \operatorname{Rm}|^2 + \sum_{l=0}^{k} \nabla^l \operatorname{Rm} * \nabla^{k-l} \operatorname{Rm} * \nabla^k \operatorname{Rm} \\ &\leq \Delta |\nabla^k \operatorname{Rm}|^2 + C \sum_{l=0}^{k} |\nabla^l \operatorname{Rm}| |\nabla^{k-l} \operatorname{Rm}| |\nabla^k \operatorname{Rm}|, \end{aligned} \tag{15}$$

which implies

$$(\partial_t - \Delta) w_k \leq B_k w_k - 2\nabla \log h_k \nabla w_k + C \sum_{l=0}^{k} h_k |\nabla^l \operatorname{Rm}| |\nabla^{k-l} \operatorname{Rm}| |\nabla^k \operatorname{Rm}|, \tag{16}$$

where $B_k = (2|\nabla h_k|^2 - h_k \Delta h_k)/h_k^2$ is uniformly bounded as before. Moreover, by induction we have

$$h_k |\nabla^l \operatorname{Rm}| |\nabla^{k-l} \operatorname{Rm}| |\nabla^k \operatorname{Rm}| = h_k |\operatorname{Rm}| |\nabla^k \operatorname{Rm}|^2 \leq C w_k$$

for $l = 0$ or $l = k$, and

$$h_k |\nabla^l \operatorname{Rm}| |\nabla^{k-l} \operatorname{Rm}| |\nabla^k \operatorname{Rm}| \leq h_k r^{-4-2\sigma-k} |\nabla^k \operatorname{Rm}| = r^k |\nabla^k \operatorname{Rm}| \leq C w_k^{1/2}$$

for $0 < l < k$.

From (16) we have

$$(\partial_t - \Delta) w_k \leq -2\nabla \log h_k \nabla w_k + C(w_k + w_k^{1/2}).$$

From Theorem 2.1 we conclude that w_k is uniformly bounded on $M \times [0, T]$, since the solution of the ODE

$$\frac{d\phi}{dt} = C(\phi + \phi^{\frac{1}{2}}), \quad \phi(0) = c, \tag{17}$$

is bounded on $[0, T]$. Therefore $|\nabla^k \operatorname{Rm}| \leq C r^{-2-k-\sigma}$.

For any vector field U on M, we have

$$\begin{aligned}(18)\qquad |\log g(x,t)(U,U)-\log g(x,0)(U,U)| &= \left|\int_0^t \frac{-2\,\mathrm{Rc}(x,s)(U,U)}{g(x,s)(U,U)}\,ds\right| \\ &\le C\int_0^t |\mathrm{Rm}|\,ds \le Cr^{-\sigma-2}.\end{aligned}$$

Therefore

$$(19)\qquad g(t)(U,U)=g(0)(U,U)(1+O(r^{-2-\sigma})),$$

and in particular,

$$\begin{aligned}(20)\qquad g_{ii}(t) &= g_{ii}(0)(1+O(r^{-2-\sigma})) \\ &= (1+O(r^{-\sigma}))(1+O(r^{-2-\sigma})) \\ &= 1+O(r^{-\sigma}).\end{aligned}$$

By the polarization identity and (20), we conclude that $g_{ij}(t)=O(r^{-\sigma})$ when $i\neq j$.

The evolution equation of the Christoffel symbol (see [13, Equation (2.25)])

$$\partial_t\Gamma_{ij}^k=-g^{kl}(\nabla_i R_{jl}+\nabla_j R_{il}-\nabla_l R_{ij})$$

and (14) now imply that $\Gamma_{ij}^k=O(r^{-\sigma-3})$, and hence $\partial_i R_{jk}=O(r^{-\sigma-3})$ follows from the relation $\nabla_i R_{jk}=\partial_i R_{jk}-\Gamma_{ij}^l R_{lk}-\Gamma_{ik}^l R_{jl}$.

Since $\partial_t(\partial_i g_{jk})=-2\partial_i R_{jk}$, it follows that $\partial_i g_{jk}(t)=O(r^{-\sigma-1})$. Now by induction, $\partial^{|k|}g_{ij}=O(r^{-\sigma-k})$ for all k, and hence $(E,g_{ij}(t))$ is an AE coordinate system with the same order σ.

(ii) From the definition of the mass,

$$m(g(t))=\lim_{r\to\infty}\int_{S_r}(\partial_i g_{ij}(t)-\partial_j g_{ii}(t))\,dA^j.$$

Since we have a common coordinate system at infinity,

$$\begin{aligned}m'(g(t)) &= \lim_{r\to\infty}\int_{S_r}(\partial_i g'_{ij}(t)-\partial_j g'_{ii}(t))\,dA^j \\ &= \lim_{r\to\infty}-2\int_{S_r}(\partial_i R_{ij}(t)-\partial_j R_{ii}(t))\,dA^j \\ &= \lim_{r\to\infty}-2\int_{S_r}(\nabla_i R_{ij}(t)-\nabla_j R(t))\,dA^j \\ &= \lim_{r\to\infty}\int_{S_r}\nabla_j R(t)\,dA^j.\end{aligned}$$

Now from [28, Lemma 11],

$$\lim_{r\to\infty}\int_{S_r}|\nabla R(t)|\,d\sigma=0$$

for $t>0$, so $m'(g(t))=0$ for $t>0$.

On the other hand, it is easy to show that $m(g(t))$ is continuous at 0 (see [28, Corollary 12]), hence the mass is unchanged. □

Remark 2.3 The proof of Theorem 2.2 actually shows that if $g_{ij}(0)-\delta_{ij}\in C^k_{-\sigma}$, then $g_{ij}(t)-\delta_{ij}\in C^{k-2}_{-\sigma}$ for any integer $k\geq 4$ and $t>0$. In addition, using the argument in [16] we can prove that if $g_{ij}(0)-\delta_{ij}\in C^2_{-\sigma}$, then $g_{ij}(t)-\delta_{ij}\in C^{1,\alpha}_{-\sigma}$ for $t>0$. The definition of the weighted space can be found in Section 5.

Let $(M,g(t)), 0\leq t\leq T$ be a Ricci flow solution with bounded curvature on $M\times[0,T]$ such that $(M,g(0))$ is an AE manifold. By our assumption, the scalar curvature satisfies $R(x,0)\geq 0$. The evolution equation of R is $\partial_t R=\Delta R+2|\mathrm{Rc}|^2\geq\Delta R$, which together with Theorem 2.1 implies $R(x,t)\geq 0$ on $M\times[0,T]$.

Now from the strong maximum principle under Ricci flow [13, Lemma 6.57], either $R(x,t)>0$ for $(x,t)\in M\times(0,T]$, or $R(x,t)=0$ on $M\times[0,T]$.

In the first case, we redefine the Ricci flow as $g_1(t)=g(t+\epsilon_1)$, where $\epsilon_1\in(0,T)$ is fixed such that the corresponding scalar curvature satisfies $R_1(x,0)>0$ for all $x\in M$.

In the second case, the evolution equation of R implies that $\mathrm{Rc}(0)=0$, that is, $(M,g(0))$ is Ricci-flat. Now we have:

Theorem 2.4 *If (M,g) is a Ricci-flat AE manifold, then (M,g) is isometric to $(\mathbb{R}^n,g_E)$.*

We fix a point p on M and let $d(x)=d_g(x,p)$ be the distance function to p. We first prove the following two lemmas.

Lemma 2.5 *We have*

$$\lim_{r\to\infty}\frac{r(x)}{d(x)}=1, \tag{21}$$

where $r=r(x)$.

Proof From the definition of AE manifolds, there exists a large positive number r_0 such that

$$(1+Cr^{-\sigma})^{-1}g_E(x) \le g(x) \le (1+Cr^{-\sigma})g_E(x) \tag{22}$$

for all $r(x) \ge r_0$.

Given $r_1 \ge r_0$ and large $r(x)$, let $\{\gamma(t),\ t \in [0, d(x)]\}$ be the minimizing geodesic from p to x. Then there exists an $r_x \in [0, d(x)]$ such that $r(\gamma(r_x)) = r_1$ and $r(\gamma(t)) \ge r_1$ for $t \in [r_x, d(x)]$. We assume that $r_x \in [C_1^{-1}r_1, C_1 r_1]$, where C_1 depends on r_1.

Now we estimate the distance between $\gamma(r_x)$ and x under g_E. We have

$$\begin{aligned} r(x) - r_1 \le \int_{r_x}^{d(x)} |\gamma'(t)|_{g_E}\,dt &\le (1+Cr_1^{-\sigma})\left(\int_{r_x}^{d(x)} |\gamma'(t)|_g\,dt\right) \\ &\le (1+Cr_1^{-\sigma})\,d(x), \end{aligned} \tag{23}$$

where we have used (22) to estimate $|\gamma'(t)|_{g_E}$. Then we obtain from (23) that

$$r(x) \le (1+Cr_1^{-\sigma})d(x) + r_1. \tag{24}$$

On the other hand, let $\{\gamma_1(t),\ t \in [0, a]\}$ be the minimizing geodesic from $\gamma(r_x)$ to x under g_E. Then similarly we have

$$\begin{aligned} d(x) - r_x \le \int_0^a |\gamma_1'(t)|_g\,dt &\le (1+Cr_1^{-\sigma})\left(\int_0^a |\gamma_1'(t)|_{g_E}\,dt\right) \\ &\le (1+Cr_1^{-\sigma})(r(x)+r_1), \end{aligned} \tag{25}$$

and hence

$$d(x) \le (1+Cr_1^{-\sigma})(r(x)+r_1) + r_x \le (1+Cr_1^{-\sigma})r(x) + (1+Cr_1^{-\sigma}+C_1)r_1. \tag{26}$$

Combining (24) and (26), we have

$$(1+Cr_1^{-\sigma})^{-1} \le \liminf_{r\to\infty} \frac{r(x)}{d(x)} \le \limsup_{r\to\infty} \frac{r(x)}{d(x)} \le 1 + Cr_1^{-\sigma}. \tag{27}$$

Since r_1 can be chosen as large as we want,

$$\lim_{r\to\infty} \frac{r(x)}{d(x)} = 1 \tag{28}$$

and the proof of the lemma is complete. □

Lemma 2.6 *We have*

$$\lim_{r\to\infty} \frac{\mathrm{Vol}_g B(p, d(x))}{w_n r^n(x)} = 1, \tag{29}$$

where w_n is the volume of the unit ball in $\mathbb{R}^n$.

Proof For the AE manifold, there exists an $r_0 > 0$ sufficiently large such that

$$(1 + Cr^{-\sigma})^{-1} g_E(x) \le g(x) \le (1 + Cr^{-\sigma}) g_E(x), \tag{30}$$

and hence, for any $r(x) \ge r_0$,

$$(1 + Cr^{-\sigma})^{-1} \mathrm{Vol}_{g_E}(x) \le \mathrm{Vol}_g(x) \le (1 + Cr^{-\sigma}) \mathrm{Vol}_{g_E}(x). \tag{31}$$

For any $r(x) \ge r_0$, from Lemma 2.5 there exists a function $\epsilon(r) > 0$ with $\epsilon(r) \to 0$ as $r \to \infty$ such that

$$e^{-\epsilon(r)} \le \frac{r(x)}{d(x)} \le e^{\epsilon(r)}. \tag{32}$$

Now we fix an $r_1 \ge r_0$. Then for any $r(x) > r_1$, we have

$$\begin{aligned} w_n((e^{-\epsilon(r)} r)^n - r_1^n) &= \mathrm{Vol}_{g_E}(B(0, e^{-\epsilon(r)} r) \backslash B(0, r_1)) \\ &\le (1 + Cr_1^{-\sigma}) \mathrm{Vol}_g(B(0, e^{-\epsilon(r)} r) \backslash B(0, r_1)) \\ &\le (1 + Cr_1^{-\sigma}) \mathrm{Vol}_g(B(p, d)), \end{aligned} \tag{33}$$

where the last inequality holds since $B(0, e^{-\epsilon(r)} r) \backslash B(0, r_1) \subset B(p, d)$ by (32). Hence

$$w_n(e^{-\epsilon(r)} r)^n \le (1 + Cr_1^{-\sigma}) \mathrm{Vol}_g(B(p, d)) + w_n r_1^n. \tag{34}$$

On the other hand,

$$\begin{aligned} \mathrm{Vol}_g(B(p, d) \backslash B(p, e^{\epsilon(r_1)} r_1)) &\le \mathrm{Vol}_g(B(0, e^{\epsilon(r)} r) \backslash B(0, r_1)) \\ &\le (1 + Cr_1^{-\sigma}) \mathrm{Vol}_{g_E}(B(0, e^{\epsilon(r)} r) \backslash B(0, r_1)) \\ &= (1 + Cr_1^{-\sigma}) w_n((e^{\epsilon(r)} r)^n - r_1^n), \end{aligned} \tag{35}$$

and hence

$$\mathrm{Vol}_g(B(p, d)) \le (1 + Cr_1^{-\sigma}) w_n(e^{\epsilon(r)} r)^n + \mathrm{Vol}_g(B(p, e^{\epsilon(r_1)} r_1)). \tag{36}$$

Combining (34) and (36), we have

$$\begin{aligned} (1 + Cr_1^{-\sigma})^{-1} &\le \liminf_{r\to\infty} \frac{\mathrm{Vol}_g B(p, d(x))}{w_n r^n} \le \limsup_{r\to\infty} \frac{\mathrm{Vol}_g B(p, d(x))}{w_n r^n} \\ &\le 1 + Cr_1^{-\sigma}. \end{aligned} \tag{37}$$

By taking r_1 to ∞, we conclude that

$$\lim_{r\to\infty} \frac{\mathrm{Vol}_g B(p, d(x))}{w_n r^n} = 1. \tag{38}$$ □

Proof of Theorem 2.4 From Lemmas 2.5 and 2.6, we have

$$\lim_{d(x)\to\infty} \frac{\mathrm{Vol}_g B(p, d(x))}{w_n d^n} = \lim_{r(x)\to\infty} \frac{\mathrm{Vol}_{g_E} B(p, r(x))}{w_n r^n} = 1. \tag{39}$$

Then from a corollary of the Bishop–Gromov volume comparison theorem (see [13, Corollary 1.134]), we conclude that (M, g) is isometric to $(\mathbb{R}^n, g_E)$.

3 Perelman's μ–functional

Recall that Perelman's $\mathcal{W}$ entropy [31] is defined as

$$\mathcal{W}(g, f, \tau) = \int \bigl(\tau(|\nabla f|^2 + R) + f - n\bigr) \frac{e^{-f}}{(4\pi\tau)^{n/2}}\, dV \tag{40}$$

for a smooth function f and a positive parameter τ. Let $u = e^{-f/2}$. Then (40) becomes

$$\overline{\mathcal{W}}(g, u, \tau) = \int \bigl(\tau(4|\nabla u|^2 + Ru^2) - u^2 \log u^2 - nu^2\bigr)(4\pi\tau)^{-n/2}\, dV. \tag{41}$$

Moreover, for a general (possibly incomplete) Riemannian manifold (M, g), the μ–functional is defined as

$$\mu(g, \tau) = \inf\left\{ \overline{\mathcal{W}}(g, u, \tau) \,\middle|\, u \in W_0^{1,2}(M),\ \int_M u^2 (4\pi\tau)^{-n/2}\, dV = 1 \right\}. \tag{42}$$

Note that when M is complete, $W^{1,2}(M) = W_0^{1,2}(M)$. Moreover, from the definition we have $\mu_U(g, \tau) \ge \mu_M(g, \tau)$ for any open set $U \subset M$.

We have the following monotonicity result under Ricci flow for the complete noncompact manifold:

$$\mu(g(t_2), \tau(t_2)) \ge \mu(g(t_1), \tau(t_1)) \quad \text{for all } 0 \le t_1 \le t_2 < \overline{\tau},$$

where $\tau(t) = \overline{\tau} - t$ for $0 < \overline{\tau} < T$. Here we assume that Ricci flow exists for $[0, T]$ and $|\mathrm{Rm}|$ is uniformly bounded in spacetime. The proof of the monotonicity formula can be found in [9, Theorem 7.1(ii)]. Although in [9] they have only proved the case for the conjugate heat kernel, the same proof works for all f satisfying [31, Equations (3.3) and (3.4)].

It is proved in [39] that $\mu(g,\tau)$ is finite if g has bounded geometry, that is, the curvature is bounded and the injective radius is positive. In particular, for any AE manifold the μ–functional is finite.

Moreover, it is shown in [45] that for a manifold with bounded geometry, $\overline{\mathcal{W}}(g,u,1)$ has a smooth positive minimizer if $\mu(g,1)$ is less than the corresponding value at infinity. Note that by our definition of $\overline{\mathcal{W}}$,

$$\overline{\mathcal{W}}(g,u,1) = L(g,v) - \frac{n}{2}\log 4\pi - n,$$

where the functional $L(g,v)$ is defined in [45, Equation (1.1)] and $v = u(4\pi)^{-n/4}$. Therefore,

$$\mu(g,1) = \lambda(M) - \frac{n}{2}\log 4\pi - n, \tag{43}$$

where (see [45, Definition 1.1])

$$\lambda(M) = \inf\left\{ L(v,g) \,\middle|\, \int_M v^2\, dV_g = 1 \right\}.$$

To be more precise, if for any sequence $p_n \to \infty$ on the manifold M such that (M,g,p_n) converges smoothly in the Cheeger–Gromov sense to $(M_\infty, g_\infty, p_\infty)$ one has that $\mu_M(g,1) < \mu_{M_\infty}(g_\infty,1)$, then $\mu_M(g,1)$ has a smooth positive minimizer.

In the case of Euclidean space, the log-Sobolev inequality of Gross [19] implies:

Theorem 3.1 *For any smooth f such that $\int_{\mathbb{R}^n} e^{-f}(4\pi\tau)^{-n/2}\, dV_{g_E} = 1$, we have*

$$\mathcal{W}(g_E, f, \tau) \ge 0. \tag{44}$$

The proof can be found in [42, Lemma 8.17].

It is immediate from (44) that $\overline{\mathcal{W}}(g_E,u,\tau) \ge 0$, where equality holds if $u^2 = e^{-|x|^2/(4\tau)}$. Therefore, $\mu_{\mathbb{R}^n}(g_E,\tau) = 0$. For an AE manifold M^n we have

$$(M,g,p_n) \xrightarrow{C^\infty} (\mathbb{R}^n, g_E, p_\infty)$$

for any sequence $p_n \to \infty$, by the Cheeger–Gromov compactness theorem. Therefore $\overline{\mathcal{W}}(g,u,\tau)$ has a smooth positive minimizer if $\mu(g,\tau) = \mu(\tau^{-1}g,1) < 0$ from the above result. Note that $\tau^{-1}g$ is still an AE metric.

We have the following lemma.

Lemma 3.2 *Assume that (M_i, g_i) converges to (M_∞, g_∞) smoothly in the Cheeger–Gromov sense and $\mu(g_\infty, \tau)$ is finite. Then*

$$\mu(g_\infty, \tau) \geq \limsup_{i\to\infty} \mu(g_i, \tau).$$

Proof For any $\epsilon > 0$ we can find a $u \in W_0^{1,2}(M_\infty)$ such that $\overline{\mathcal{W}}(g_\infty, u, \tau) \leq \mu(g_\infty, \tau)+\epsilon$. For large i, we can find $u_i \in W_0^{1,2}(M_i)$ which are the pull-back functions of u, and $\lim_{i\to\infty} \overline{\mathcal{W}}(g_i, u_i, \tau) = \overline{\mathcal{W}}(g_\infty, u, \tau)$ by the convergence. Therefore

$$\limsup_{i\to\infty} \mu(g_i, \tau) \leq \lim_{i\to\infty} \overline{\mathcal{W}}(g_i, u_i, \tau) \leq \mu(g_\infty, \tau) + \epsilon.$$

Since the above holds for any $\epsilon > 0$, we have $\limsup_{i\to\infty} \mu(g_i, \tau) \leq \mu(g_\infty, \tau)$. □

It follows immediately from the above lemma that $\mu(g, \tau) \leq 0$ for any AE manifold, since $(M, g, p_n) \xrightarrow{C^\infty} (\mathbb{R}^n, g_E, p_\infty)$ for any $p_n \to \infty$.

The Euler–Lagrange equation for the minimizer of $\mu(g, \tau)$ is

$$\tau(-4\Delta u + Ru) - u \log u^2 - nu = \mu(g, \tau)u. \tag{45}$$

For the general Ricci flow on the noncompact manifold we have the following result, whose proof is almost identical to that of the compact case; see [31, Section 3.1].

Theorem 3.3 *If (M^n, g) is a manifold with bounded geometry such that a solution $g(t)$ of bounded curvature to the Ricci flow with $g(0) = g$ exists for $t \in [0, T)$, then $\mu(g, \overline{\tau}) < 0$ for any $\overline{\tau} \in (0, T)$, unless (M^n, g) is isometric to $(\mathbb{R}^n, g_E)$.*

Proof Let $\tau(t) = \overline{\tau} - t$, $y \in M$ and consider the corresponding fundamental solution

$$v(x, t) = (4\pi\tau(t))^{-n/2} e^{-f(x,t)} \quad \text{for } t \in [0, \overline{\tau}) \tag{46}$$

to the adjoint heat equation

$$\frac{\partial v}{\partial t} = -\Delta v + Rv$$

with $\lim_{t\nearrow\overline{\tau}} v(\cdot, t) = \delta_y$.

The existence of the fundamental solutions to the adjoint heat equation on noncompact manifolds and their basic properties can be found in [12, Chapters 24–25].

Then by the monotonicity of the entropy,

$$\begin{aligned} \mu(g, \overline{\tau}) = \mu(g, \tau(0)) &\leq \mathcal{W}(g(0), f(0), \tau(0)) \\ &\leq \limsup_{t\nearrow\overline{\tau}} \mathcal{W}(g(t), f(t), \tau(t)) \leq 0, \end{aligned} \tag{47}$$

where the proof of the last limit in (47) can be found in [9, Theorem 7.1]. If $\mu(g,\overline{\tau})=0$, then $\mathcal{W}(g(t), f(t), \tau(t)) = 0$ since it is monotone. Therefore from the formula

$$\frac{d\mathcal{W}(g(t), f(t), \tau(t))}{dt} = 2\tau \int_M \left|\mathrm{Rc} + \nabla^2 f - \frac{g}{2\tau}\right|^2 \frac{e^{-f}}{(4\pi\tau)^{n/2}}\, dV \tag{48}$$

we have

$$\mathrm{Rc} + \nabla^2 f - \frac{g}{2\tau} \equiv 0 \tag{49}$$

for $t \in [0, \overline{\tau}]$, so $g(t)$ is a shrinking soliton with singular time $\overline{\tau}$. From

$$\tau(t) \max_M |\mathrm{Rm}(g(t))| \equiv \text{constant}$$

for $t \in [0, \overline{\tau}]$, we conclude that $|\mathrm{Rm}(g(t))| \equiv 0$. In particular, g is Ricci-flat and we have from (49) that

$$\nabla^2 f - \frac{g}{2\overline{\tau}} \equiv 0. \tag{50}$$

Set $\overline{f} = 4\overline{\tau} f$. Then $\nabla^2 \overline{f} = 2g$ and hence $\overline{f}$ is a convex function.

Let O be a fixed point. Then for any point $x \in M$ we have a minimizing geodesic $s(t)$, $0 \le t \le d(x, O)$, such that $|\dot{s}(t)| = 1$. Then we have

$$\frac{d^2 \overline{f}(s(t))}{dt^2} = \nabla^2 \overline{f}(\nabla d, \nabla d) = 2g(\nabla d, \nabla d) = 2. \tag{51}$$

Therefore,

$$\frac{d\overline{f}(s(t))}{dt} = \langle \nabla \overline{f}, \nabla d\rangle = 2t + \langle \nabla \overline{f}, \nabla d\rangle_{t=0}. \tag{52}$$

From (52) we have $\overline{f}(s(t)) = \overline{f}(O) + t\langle \nabla \overline{f}, \nabla d\rangle_{t=0} + t^2$. In other words, $\overline{f}$ is quadratically increasing and therefore it has a minimal point O_1. By choosing $O = O_1$, we have $\overline{f}(x) = \overline{f}(O_1) + d^2(x, O_1)$. In particular, by taking the trace of (50) we have

$$\Delta d^2 = 2n.$$

Therefore (M^n, g) is isometric to $(\mathbb{R}^n, g_E)$ by the Bishop–Gromov comparison theorem [13, Theorems 1.128 and 1.132] since g is Ricci-flat. □

Now we have the following crucial result.

Theorem 3.4 *If (M^n, g) is an AE manifold with scalar curvature $R > 0$, then* $\lim_{\tau\to\infty} \mu(g, \tau) = 0$.

Proof If the conclusion does not hold, we can find a sequence $\tau_k \to \infty$ such that $\lim_{k\to\infty} \mu(g,\tau_k) = \mu_\infty$, where μ_∞ is either a finite negative number or $\mu_\infty = -\infty$.

We have previously shown that $\mu(g,\tau_k)$ has a positive minimizer u_k and it satisfies

$$\tau_k(-4\Delta u_k + Ru_k) - u_k \log u_k^2 - nu_k = \mu(g,\tau_k)u_k, \tag{53}$$

$$\int_M u_k^2 (4\pi\tau_k)^{-n/2}\, dV = 1. \tag{54}$$

Claim 1 *The u_k are uniformly bounded.*

We first prove a lemma.

Lemma 3.5 *For $u \in W^{1,2}(M)$, the Sobolev inequality*

$$\left(\int_M u^{2n/(n-2)}\, dV\right)^{\frac{n-2}{n}} \le C\int_M (4|\nabla u|^2 + Ru^2)\, dV \tag{55}$$

holds, where the constant C depends on the dimension, curvature bound, injective radius lower bound, AE coordinate system and infimum of R on a compact set.

Proof Let $M^n = K \sqcup E$ be the disjoint union of a compact set K and AE end E, and K_1 a compact set such that $K \Subset K_1$. We choose a cutoff function ϕ_0 supported on K_1 and with $\phi_0 = 1$ on K. Let $\phi_1 = 1 - \phi_0$.

For any $u \in W^{1,2}(M)$, we have

$$\|u\|_{2n/(n-2)} = \|\phi_0 u + \phi_1 u\|_{2n/(n-2)} \le \|\phi_0 u\|_{2n/(n-2)} + \|\phi_1 u\|_{2n/(n-2)}.$$

By the L^2 Sobolev inequality on manifolds with bounded geometry [3, Theorem 2.21],

$$\begin{aligned}\left(\int_M (\phi_0 u)^{2n/(n-2)}\, dV\right)^{\frac{n-2}{n}} &\le C\int_M (|\nabla(\phi_0 u)|^2 + \phi_0^2 u^2)\, dV \\ &\le C\int_{K_1} (|\nabla\phi_0 u|^2 + |\phi_0 \nabla u|^2 + \phi_0^2 u^2)\, dV \\ &\le C\int_{K_1} (|\nabla u|^2 + u^2)\, dV \\ &\le C\int_{K_1} (4|\nabla u|^2 + Ru^2)\, dV.\end{aligned} \tag{56}$$

The last inequality holds since we assume $R > 0$.

On the AE end E, by enlarging K and K_1 if necessary we can assume the L^2 Sobolev inequality of the Euclidean type holds. To be precise, on $\mathbb{R}^n$ we have the L^2 Sobolev inequality [1]

$$\left(\int_{\mathbb{R}^n} u^{2n/(n-2)}\,dV_{g_E}\right)^{\frac{n-2}{n}} \le C\int_{\mathbb{R}^n} |\nabla_{g_E} u|^2\,dV_{g_E} \tag{57}$$

for any $u \in C_0^1(\mathbb{R}^n)$ and some constant $C > 0$ depending only on dimension.

Since E is the AE end, by shrinking it if necessary we can assume that there exists a $C > 0$ such that

$$C^{-1}dV_{g_E} \le dV \le C dV_{g_E}, \quad C^{-1}|\nabla_{g_E} u|^2 \le |\nabla u|^2 \le C|\nabla_{g_E} u|^2.$$

Hence, for any $u \in C_0^1(E)$,

$$\begin{aligned}\left(\int_E u^{2n/(n-2)}\,dV\right)^{\frac{n-2}{n}} &\le \left(C\int_{\mathbb{R}^n} u^{2n/(n-2)}\,dV_{g_E}\right)^{\frac{n-2}{n}} \\ &\le C\int_{\mathbb{R}^n} |\nabla_{g_E} u|^2\,dV_{g_E} \le C\int_{\mathbb{R}^n} |\nabla u|^2\,dV_{g_E} \\ &\le C\int_{\mathbb{R}^n} |\nabla u|^2\,dV \le C\int_E |\nabla u|^2\,dV.\end{aligned} \tag{58}$$

So we have

$$\begin{aligned}\left(\int_M (\phi_1 u)^{2n/(n-2)}\,dV\right)^{\frac{n-2}{n}} &\le C\int_M |\nabla(\phi_1 u)|^2\,dV \\ &\le C\int_M (|\nabla\phi_1 u|^2 + |\phi_1 \nabla u|^2)\,dV \\ &\le C\int_M |\nabla u|^2\,dV + C\int_{K_1} u^2\,dV \\ &\le C\int_M (4|\nabla u|^2 + Ru^2)\,dV.\end{aligned} \tag{59}$$

Combining (56) and (59), we get (55). □

We can now prove the claim by using Moser iteration. This is known to experts but we write it down for the convenience of readers. For the sake of simplicity, we will not write the subscript k explicitly throughout, setting $\mu = \mu(g, \tau_k)$.

Proof of Claim 1 (See also [45, Lemma 2.1].) From (53) we have

$$4\Delta u - Ru + \frac{2}{\tau} u \log u + \frac{n+\mu}{\tau} u = 0.$$

Since $\mu \le 0$, we have

$$4\Delta u - Ru + \frac{2}{\tau} u \log u + \frac{n}{\tau} u \ge 0. \tag{60}$$

By direct computation, for $p \ge 1$,

$$\begin{aligned} 4\Delta u^p &= 4p(p-1)u^{p-2}|\nabla u|^2 + 4pu^{p-1}\Delta u \\ &\ge 4pu^{p-1}\Delta u \\ &\ge -\frac{2p}{\tau} u^p \log u - \frac{np}{\tau} u^p + pRu^p. \end{aligned} \tag{61}$$

We set $w = u^p$ and ϕ to be a test function. From (61) we have

$$4\int \langle \nabla(w\phi^2), \nabla w\rangle \, dV \le \frac{2p}{\tau}\int w^2\phi^2 \log u \, dV + \frac{np}{\tau}\int w^2\phi^2 \, dV - \int pRw^2\phi^2 \, dV.$$

On the other hand, since

$$\langle \nabla(w\phi^2), \nabla w\rangle = |\nabla(w\phi)|^2 - |\nabla\phi|^2 w^2,$$

we have

$$\begin{aligned} 4\int |\nabla(w\phi)|^2 \, dV \le 4\int |\nabla\phi|^2 w^2 \, dV &+ \frac{2p}{\tau}\int w^2\phi^2 \log u \, dV \\ &+ \frac{np}{\tau}\int w^2\phi^2 \, dV - \int pRw^2\phi^2 \, dV. \end{aligned} \tag{62}$$

There is a constant $c_1 > 0$ such that

$$\log u \le u^{2/n} + c_1.$$

Hence, since (54) holds,

$$\begin{aligned} &\frac{2p}{\tau}\int w^2\phi^2 \log u \, dV \\ &\quad \le \frac{2p}{\tau}\int w^2\phi^2 u^{\frac{2}{n}} \, dV + \frac{2c_1 p}{\tau}\int w^2\phi^2 \, dV \\ &\quad \le \frac{2p}{\tau}\left(\int (w\phi)^{2n/(n-1)} \, dV\right)^{\frac{n-1}{n}} \left(\int u^2 \, dV\right)^{\frac{1}{n}} + \frac{2c_1 p}{\tau}\int w^2\phi^2 \, dV \\ &\quad = \frac{\sqrt{4\pi}\, 2p}{\sqrt{\tau}}\left(\int (w\phi)^{2n/(n-1)} \, dV\right)^{\frac{n-1}{n}} + \frac{2c_1 p}{\tau}\int w^2\phi^2 \, dV. \end{aligned} \tag{63}$$

From Hölder's inequality $\|fh\|_1 \le \|f\|_p \|h\|_q$ and by choosing $f = h = (w\phi)^{n/(n-1)}$, $p = 2(n-1)/(n-2)$ and $q = 2(n-1)/n$, we have

$$\left(\int (w\phi)^{2n/(n-1)}\,dV\right)^{\frac{n-1}{n}} \le \left(\int (w\phi)^{2n/(n-2)}\,dV\right)^{\frac{n-2}{2n}} \left(\int w^2\phi^2\,dV\right)^{\frac{1}{2}} \tag{64}$$
$$\le \lambda\left(\int (w\phi)^{2n/(n-2)}\,dV\right)^{\frac{n-2}{n}} + \frac{1}{4\lambda}\int w^2\phi^2\,dV,$$

where the last line is from Young's inequality for a positive λ to be determined below.

So from (63),

$$\frac{2p}{\tau}\int w^2\phi^2 \log u\,dV \le \frac{c_2\lambda p}{\sqrt{\tau}}\left(\int (w\phi)^{2n/(n-2)}\,dV\right)^{\frac{n-2}{n}} \tag{65}$$
$$+ \frac{c_2 p}{4\lambda\sqrt{\tau}}\int w^2\phi^2\,dV + \frac{2c_1 p}{\tau}\int w^2\phi^2\,dV,$$

where $c_2 = 2\sqrt{4\pi}$.

From Lemma 3.5 and Equations (62) and (65), we have

$$\frac{1}{C}\left(\int (w\phi)^{2n/(n-2)}\,dV\right)^{\frac{n-2}{n}} \tag{66}$$
$$\le \int (4|\nabla(w\phi)|^2 + R(w\phi)^2)\,dV$$
$$\le 4\int |\nabla\phi|^2 w^2\,dV + \frac{2p}{\tau}\int w^2\phi^2\log u\,dV + \frac{np}{\tau}\int w^2\phi^2\,dV$$
$$\le 4\int |\nabla\phi|^2 w^2\,dV + \frac{c_2\lambda p}{\sqrt{\tau}}\left(\int (w\phi)^{2n/(n-2)}\,dV\right)^{\frac{n-2}{n}}$$
$$+ \frac{c_2 p}{4\lambda\sqrt{\tau}}\int w^2\phi^2\,dV + \frac{2c_1 p}{\tau}\int w^2\phi^2\,dV + \frac{np}{\tau}\int w^2\phi^2\,dV.$$

If we choose λ such that $c_2\lambda p/\sqrt{\tau} = 1/(2C)$, that is, $\lambda = \sqrt{\tau}/(2Cc_2 p)$, then from (66) there exists a $C_0 > 0$ such that

$$\left(\int (w\phi)^{2n/(n-2)}\,dV\right)^{\frac{n-2}{n}} \le C_0\int |\nabla\phi|^2 w^2\,dV + \frac{C_0 p^2}{\tau}\int w^2\phi^2\,dV. \tag{67}$$

For any point x on M, we choose ϕ_k so that it is supported on $B(x, \sqrt{\tau}(1 + 1/2^k))$ and $\phi_k = 1$ on $B(x, \sqrt{\tau}(1 + 1/2^{k+1}))$ with $|\nabla\phi_k| \le C2^k/\sqrt{\tau}$.

From (67) we have

$$\begin{aligned}
(68)\quad \left(\int_{B(x,\sqrt{\tau}(1+1/2^{k+1}))} w^{2n/(n-2)}\,dV\right)^{\frac{n-2}{n}}
&\le \left(\int (w\phi_k)^{2n/(n-2)}\,dV\right)^{\frac{n-2}{n}} \\
&\le C_0\int |\nabla\phi_k|^2 w^2\,dV + \frac{C_0 p^2}{\tau}\int w^2\phi_k^2\,dV \\
&\le \frac{C_1 2^{2k} p^2}{\tau}\int_{B(x,\sqrt{\tau}(1+1/2^k))} w^2\,dV.
\end{aligned}$$

If we set $p_0 = n/(n-2)$ and choose $p = p_0^k$, from (68) we have

$$(69)\quad \left(\int_{B(x,\sqrt{\tau}(1+1/2^{k+1}))} u^{2p_0^{k+1}}\,dV\right)^{\frac{n-2}{n}} \le \frac{C_1(2p_0)^{2k}}{\tau}\int_{B(x,\sqrt{\tau}(1+1/2^k))} u^{2p_0^k}\,dV,$$

or equivalently,

$$\begin{aligned}
(70)\quad \left(\int_{B(x,\sqrt{\tau}(1+1/2^{k+1}))} u^{2p_0^{k+1}}\,dV\right)^{1/p_0^{k+1}}
&\le \frac{C_1^{1/p_0^k}(2p_0)^{2k/p_0^k}}{\tau^{1/p_0^k}}\left(\int_{B(x,\sqrt{\tau}(1+1/2^k))} u^{2p_0^k}\,dV\right)^{1/p_0^k}.
\end{aligned}$$

Let $k = 0, 1, \ldots$. Then by iteration,

$$\begin{aligned}
(71)\quad \max_{B(x,\sqrt{\tau})} u^2 &\le \frac{C_1^{\sum_{k\ge 0}(1/p_0^k)}\, p_0^{\sum_{k\ge 0}(2k/p_0^k)}}{\tau^{\sum_{k\ge 0}(1/p_0^k)}}\left(\int_{B(x,2\sqrt{\tau})} u^2\,dV\right) \\
&\le \frac{C_2}{\tau^{n/2}}\left(\int_{B(x,2\sqrt{\tau})} u^2\,dV\right),
\end{aligned}$$

since $\sum_{k\ge 0}(1/p_0^k) = n/2$ and $\sum_{k\ge 0}(2k/p_0^k)$ converges. As

$$\int_{B(x,2\sqrt{\tau})} u^2\,dV \le \int_M u^2\,dV = (4\pi\tau)^{n/2},$$

we conclude from (71) that for some constant $C_3 > 0$,

$$\max_M u^2 \le C_3.$$

Hence all u_k are uniformly bounded.

Since every minimizer is exponentially decaying (see [45, Lemma 2.3]), there is a maximum point p_k for u_k. Since $\Delta u_k(p_k) \le 0$, at p_k we have in (53) that

$$\tau_k R u_k - u_k \log u_k^2 - n u_k - \mu_k u_k \le 0.$$

As $u_k > 0$, we have

$$u_k(p_k) \ge \exp\left(\frac{R(p_k)\tau_k - n - \mu_k}{2}\right) \ge \exp\left(\frac{-n-\mu_k}{2}\right).$$

As we have proved that u_k is uniformly bounded, μ_k cannot tend to $-\infty$. In other words, μ_∞ is finite.

From (54) we have

$$\int_K u_k^2 \, dV + \int_E u_k^2 \, dV = (4\pi\tau_k)^{n/2}.$$

Since u_k are uniformly bounded and K has finite volume, the first integral is uniformly bounded. Hence there is a $c_0 \in (0,1]$ satisfying

$$\int_E u_k^2 \, dV \ge c_0 (4\pi\tau_k)^{n/2}. \tag{72}$$

We define functions $\tilde{u}_k(x) = u_k(\sqrt{\tau_k}x)$, a new metric on E as $\tilde{g}_{ij}(x) = g_{ij}(\sqrt{\tau_k}x)$, the corresponding Laplace operator $\tilde{\Delta}_k = (1/\sqrt{\det \tilde{g}})\partial_i \sqrt{\det \tilde{g}}\tilde{g}^{ij}\partial_j$ and scalar curvature $\tilde{R}(x) = (1/\tau_k)R(\sqrt{\tau_k}x)$.

The metric $\tilde{g}$ on E, after a diffeomorphism, is nothing but $\tau_k^{-1}g$. So by the AE condition, $(E,\tilde{g})$ converges to $(\mathbb{R}^n \backslash \{0\}, g_E)$ in the Cheeger–Gromov sense, and the convergence is smooth away from the origin.

Now (53) becomes

$$-4\tilde{\Delta}_k \tilde{u}_k + \tilde{R}\tilde{u}_k - \tilde{u}_k \log \tilde{u}_k^2 - n\tilde{u}_k = \mu_k \tilde{u}_k. \tag{73}$$

All $\tilde{u}_k$ can be regarded as functions defined on all of $\mathbb{R}^n$ except for a ball with center 0. We next prove that there is a limit in $W^{1,2}(\mathbb{R}^n)$ for the sequence $\{\tilde{u}_k\}$.

Since μ_k are bounded, from (53) and (54) we have (see [39, Equation (29)])

$$\tau_k \int_M |\nabla u_k|^2 (4\pi\tau_k)^{-n/2} \, dV \le C, \tag{74}$$

where the bound C is independent of k.

Therefore, for any annulus $C_{a,A} = \{x \in \mathbb{R}^n \mid a < |x| < A\}$, we have a uniform constant $C_1 > 0$ such that for k sufficiently large,

$$\int_{C_{a,A}} \tilde{u}_k^2 \, d\tilde{V} \le C_1 \quad \text{and} \quad \int_{C_{a,A}} |\tilde{\nabla}\tilde{u}_k|^2 \, d\tilde{V} \le C_1.$$

In other words, the $\tilde{u}_k$ are bounded in $W^{1,2}(C_{a,A})$, hence a subsequence of $\{\tilde{u}_k\}$ converges weakly to a function u_∞ in $W^{1,2}(C_{a,A})$, and by Sobolev embedding it converges strongly to u_∞ in $L^p(C_{a,A})$ if $1 \le p < 2n/(n-2)$. Choosing two sequences $a_m \to 0$ and $A_m \to \infty$ for $m = 1, 2, \dots$, by the diagonal argument, replacing $\{\tilde{u}_k\}$ by a subsequence if necessary, we have a function u_∞ defined on $\mathbb{R}^n \backslash \{0\}$ such that for every compact set C in $\mathbb{R}^n \backslash \{0\}$, there is an $N > 0$ such that $\{\tilde{u}_k,\ k \ge N\}$ converges weakly to u_∞ in $W^{1,2}(\mathbb{R}^n \backslash \{0\})$, and strongly in $L^p(\mathbb{R}^n \backslash \{0\})$ if $1 \le p < 2n/(n-2)$.

By the standard L^p–regularity property of the elliptic equation (73) (see [18, Theorem 9.11]), the convergence is in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$ for some $\alpha > 0$. Therefore if $k \to \infty$ in (73), we have

$$-4\Delta_{g_E} u_\infty - u_\infty \log u_\infty^2 - n u_\infty = \mu_\infty u_\infty. \tag{75}$$

By the standard regularity property of elliptic operators and bootstrapping (see [18, Theorem 6.17]), we know that $u_\infty \in C^\infty(\mathbb{R}^n \backslash \{0\})$ and either $u_\infty \equiv 0$ or $u_\infty > 0$ by the strong maximum principle [34].

Moreover we have

$$\int_{\mathbb{R}^n \backslash \{0\}} u_\infty^2 \, dV_{g_E} \le (4\pi)^{n/2}, \tag{76}$$

and there exists a $C > 0$ such that

$$\int_{\mathbb{R}^n \backslash \{0\}} |\nabla u_\infty|^2 \, dV_{g_E} \le C. \tag{77}$$

Claim 2 $$u_\infty \in W^{1,2}(\mathbb{R}^n).$$

Proof of Claim 2 We first prove a lemma.

Lemma 3.6 *For a function $f \in C^1(\mathbb{R}^n \backslash \{0\})$, if $|f(x)| \le C|x|^{-\alpha}$ for some $\alpha < n-1$ and small x and $|\nabla f|$ is integrable on the punctured ball $B(0,1)\backslash\{0\}$, then the function*

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \ne 0, \\ 0 & \text{for } x = 0, \end{cases}$$

has the weak derivative

$$g_i(x) = \begin{cases} \partial_i f(x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

for $i = 1, 2, \ldots, n$.

Proof For any $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned} \int_{\mathbb{R}^n} \tilde{f} \partial_i \phi \, dV_{g_E} &= \lim_{r \to 0} \int_{\mathbb{R}^n \setminus B(0,r)} f \partial_i \phi \, dV_{g_E} \\ &= - \lim_{r \to 0} \int_{\mathbb{R}^n \setminus B(0,r)} \partial_i f \phi \, dV_{g_E} + \lim_{r \to 0} \int_{S(0,r)} f \phi \upsilon^i \, d\sigma \\ &= - \int_{\mathbb{R}^n} g_i \phi \, dV_{g_E} + \lim_{r \to 0} \int_{S(0,r)} f \phi \upsilon^i \, d\sigma, \end{aligned}$$

where υ^i is the i^{th} component of the inner normal vector of $S(0,r)$. The first integral in the last line is finite since g_i is integrable by our assumption.

From the condition,

$$\left| \int_{S(0,r)} f \phi \upsilon^i \, d\sigma \right| \leq C' r^{n-1} \max_{x \in S(0,r)} |f| \leq C' C r^{n-1-\alpha}.$$

Since $\alpha < n-1$, we conclude that

$$\lim_{r \to 0} \int_{S(0,r)} f \phi \upsilon^i \, d\sigma = 0$$

and the lemma follows. □

Applying Moser's iteration to (75) as in the proof of Claim 1, we have for any $0 < r \leq 1$ and $|p| = r$,

$$\max_{B(p,r/4)} u_\infty^2 \leq \frac{C}{r^n} \int_{B(p,r/2)} u_\infty^2 \, dV_{g_E} \leq \frac{C'}{r^n}.$$

Hence we have

$$u_\infty(x) \leq \frac{C}{|x|^{n/2}}$$

for $|x| \leq 1$. Therefore, by combining with (77) we can apply Lemma 3.6 to conclude that u_∞ can be extended to $\mathbb{R}^n$. Moreover, from (76) and (77), $u_\infty \in W^{1,2}(\mathbb{R}^n)$.

Case 1 ($u_\infty > 0$) From (76) we have

$$0 < \int_{\mathbb{R}^n} u_\infty^2 (4\pi)^{-n/2} \, dV_{g_E} = c_1^2 \leq 1.$$

So if we set $\widetilde{u}_\infty = u_\infty/c_1$, from (75) we have

$$\begin{aligned}(78)\quad &\int_{\mathbb{R}^n} (4|\nabla \widetilde{u}_\infty|^2 - \widetilde{u}_\infty^2 \log \widetilde{u}_\infty^2 - n\widetilde{u}_\infty^2)(4\pi)^{-n/2}\, dV_{g_E} \\ &\qquad = \frac{1}{c_1^2}\int_{\mathbb{R}^n} (4|\nabla u_\infty|^2 - u_\infty^2 \log u_\infty^2 - nu_\infty^2)(4\pi)^{-n/2}\, dV_{g_E} + \log c_1^2 \\ &\qquad = \mu_\infty + \log c_1^2 < 0,\end{aligned}$$

since $\mu_\infty < 0$ and $c_1^2 < 1$. But it contradicts the fact that $\mu_{\mathbb{R}^n}(g_E, 1) = 0$.

Case 2 $(u_\infty \equiv 0)$ In this case it means that $\widetilde{u}_k(x) = u_k(\sqrt{\tau_k}x)$ converges uniformly to 0 on any compact set of E.

We can assume that

$$\limsup_{k\to\infty} \max_{x\in\mathbb{R}^n\setminus B(0,1)} \widetilde{u}_k(x) = 0.$$

For otherwise, if there existed a sequence $\{p_k\}_{k\in\mathbb{N}}$ such that $\widetilde{u}_k(p_k) \ge c > 0$, by assumption $p_k \to \infty$. On the other hand, $(M, \widetilde{g}_k, p_k)$ converges smoothly to $(\mathbb{R}^n, g_E, p_\infty)$ and hence $\widetilde{u}_k(x)$ converges to u'_∞, which is not identically zero. Then, like in Case 1, we would have a contradiction.

Choose a small constant $a > 0$ such that

$$(79)\qquad \int_{E\setminus B(0,2a\sqrt{\tau_k})} u_k^2\, dV \ge \frac{c_0}{2}(4\pi\tau_k)^{n/2}.$$

This is possible since (72) holds and the u_k are uniformly bounded.

Choose a function ϕ such that $\phi \in C_0^\infty(\mathbb{R}^n\setminus B(0,a))$ and $\phi = 1$ on $\mathbb{R}^n\setminus B(0,2a)$. Then we have, like (62),

$$\begin{aligned}(80)\quad &\int (4|\widetilde{\nabla}(\phi\widetilde{u}_k)|^2 + (\widetilde{R} - n)(\phi\widetilde{u}_k)^2 - (\phi\widetilde{u}_k)^2 \log \widetilde{u}_k^2)(4\pi)^{-n/2}\, d\widetilde{V} \\ &\qquad = \int 4|\widetilde{\nabla}\phi|^2 \widetilde{u}_k^2 (4\pi)^{-n/2}\, d\widetilde{V} + \mu_k \int (\phi\widetilde{u}_k)^2 (4\pi)^{-n/2}\, d\widetilde{V} \\ &\qquad \le C\int_{C_{a,2a}} \widetilde{u}_k^2 (4\pi)^{-n/2}\, d\widetilde{V} + \mu_k \int (\phi\widetilde{u}_k)^2 (4\pi)^{-n/2}\, d\widetilde{V}.\end{aligned}$$

But from our assumption that $\{\widetilde{u}_k\}$ converges to 0 uniformly on $C_{a,2a}$, there exists a sequence $\{\epsilon_k\} \searrow 0$ such that if k is sufficiently large,

$$\begin{aligned}(81)\quad &\int (4|\widetilde{\nabla}(\phi\widetilde{u}_k)|^2 + (\widetilde{R} - n)(\phi\widetilde{u}_k)^2 - (\phi\widetilde{u}_k)^2 \log((\phi\widetilde{u}_k)^2))(4\pi)^{-n/2}\, d\widetilde{V} \\ &\qquad \le \epsilon_k + \mu_k \int (\phi\widetilde{u}_k)^2 (4\pi)^{-n/2}\, d\widetilde{V}.\end{aligned}$$

On the other hand,

$$(4\pi)^{n/2} \geq \int \widetilde{u}_k^2 \, d\widetilde{V} \geq \int (\phi \widetilde{u}_k)^2 \, d\widetilde{V} \geq \int_{\mathbb{R}^n \setminus B(0,2a)} \widetilde{u}_k^2 \, d\widetilde{V} \geq \frac{c_0}{2} (4\pi)^{n/2}.$$

So if we set

$$\int (\phi \widetilde{u}_k)^2 \, d\widetilde{V} = \eta_k^2 (4\pi)^{n/2}$$

and $\psi_k = \phi \widetilde{u}_k / \eta_k$, then $\eta_k \in [c_0/2, 1]$ and

$$\int \psi_k^2 (4\pi)^{-n/2} \, d\widetilde{V} = 1.$$

From (81) we have

$$\begin{aligned} \text{(82)} \quad \int (4|\widetilde{\nabla} \psi_k|^2 + (\widetilde{R} - n)\psi_k^2 - \psi_k^2 \log \psi_k^2)(4\pi)^{-n/2} \, d\widetilde{V} &\leq \eta_k^{-2} \epsilon_k + \mu_k + \log \eta_k^2 \\ &\leq \eta_k^{-2} \epsilon_k + \mu_k \\ &\leq 4 c_0^{-2} \epsilon_k + \mu_k. \end{aligned}$$

When k is sufficiently large, $4c_0^{-2}\epsilon_k + \mu_k$ is negative. Since ψ_k converges to 0 uniformly on $\mathbb{R}^n$, it is easy to check that $4|\widetilde{\nabla}\psi_k|^2 + (\widetilde{R} - n)\psi_k^2 - \psi_k^2 \log \psi_k^2$ is positive when k is large.

Thus we have derived a contradiction, and the proof of Theorem 3.4 is complete. □

With the same proof as Theorem 3.4, we have the following uniform version which will be used in Section 7.

Theorem 3.7 *Let (M_i^n, g_i) be a family of AE manifolds of the same order $\sigma > 0$, with positive scalar curvature. For some compact sets $K_i \subset M_i^n$, we have a family of diffeomorphisms $\Phi_i \colon M_i^n \setminus K_i \to \mathbb{R}^n \setminus B(0, A)$ such that under these identifications, for $1 \leq u, v \leq n$ we have*

$$\text{(83)} \qquad |(g_i)_{uv} - \delta_{uv}| \leq C_0 r^{-\sigma_i}, \quad |\partial^{|k|} (g_i)_{uv}| \leq C_k r^{-\sigma - k},$$

for some constants C_k, $k = 0, 1, \ldots$, which are independent of i. Moreover, there exist compact sets K_i' containing K_i such that $\mathrm{dis}_{g_E}(K_i, K_i') \geq d_0$ and, for some $d_0 > 0$, $C > 0$ and any $u \in C_0^1(M_i - K_i')$,

$$\left(\int_{M_i - K_i'} u^{2n/(n-2)} \, dV \right)^{\frac{n-2}{n}} \leq C \int_{M_i - K_i'} |\nabla u|^2 \, dV.$$

In addition, if $|\mathrm{Rm}|_{g_i} \le R_0$, $\mathrm{inj}_{g_i} \ge i_0$, $\mathrm{Vol}_{g_i}(K_i') \le V_0$ *and* $\inf_{p\in K_i'} R_{g_i}(p) \ge r_0$ *for some positive constants* R_0, r_0, i_0 *and* V_0, *we have*

$$\lim_{\tau\to\infty} \mu_{M_i}(g_i,\tau) = 0$$

for all g_i *uniformly.*

Remark 3.8 We can get a uniform constant for Lemma 3.5 since the Sobolev constant only depends on the bounds of curvature and injective radius. The volume control of K_i' is used to prove (72).

Next, we use Theorem 3.4 to prove the no local collapsing theorem in the case of an AE manifold. Recall that a Riemannian manifold is κ–noncollapsed on all scales if for any metric ball $B(x,r)$ satisfying $|\mathrm{Rm}| \le r^{-2}$ for all $y \in B(x,r)$, we have

$$\frac{\mathrm{Vol}\,B(x,r)}{r^n} \ge \kappa.$$

Following the celebrated work of Perelman, we have:

Theorem 3.9 *Let* $g(t)$, $t\in[0,\infty)$, *be the Ricci flow solution on an AE manifold* M^n *with* $R>0$. *Then there exists a* $\kappa>0$ *such that* $g(t)$ *is* κ*–noncollapsed on all scales.*

Proof Since Ricci flow preserves the AE condition, there exists a $\kappa_1>0$ such that

$$\frac{\mathrm{Vol}\,B_{g(t)}(x,r)}{r^n} \ge \kappa_1 \tag{84}$$

for any $t\in[0,1]$ and $r>0$, where $B_{g(t)}(x,r)$ is a metric ball in $(M^n, g(t))$.

For $t\in[1,\infty)$, $r>0$ and $p\in M$ such that $|\mathrm{Rm}| \le r^{-2}$ in $B_{g(t)}(x,r)$, we have the following inequality, whose proof can be found in [13, Proposition 5.37]:

$$\mu(g(t),r^2) \le \log \frac{\mathrm{Vol}\,B_{g(t)}(x,r)}{r^n} + C(n). \tag{85}$$

Then by (85), Theorem 3.4 and the continuity and monotonicity of $\mu(g,\tau)$, there exists a constant C depending on $g(0)$ such that

$$C \le \mu(g(0), r^2+t) \le \mu(g(t),r^2) \le \log \frac{\mathrm{Vol}\,B_{g(t)}(x,r)}{r^n} + C(n).$$

We conclude that there exists a $\kappa_2>0$ such that

$$\frac{\mathrm{Vol}\,B_{g(t)}(x,r)}{r^n} \ge \kappa_2. \tag{86}$$

Combining (84) and (86), we can find a $\kappa = \min(\kappa_1,\kappa_2) > 0$ such that $g(t)$ is κ–noncollapsed on all scales. □

4 Analysis of singularity at time infinity

For the Ricci flow $(M, g(t))$, $t \in [0, \infty)$, there are two different types of singularity at infinity classified by Hamilton; see [20].

Case 1 (Type IIb: $\sup_{M\times[0,\infty)} t|\mathrm{Rm}| = \infty$) In this case, we take any sequence of times $T_i \to \infty$ and then choose $p_i = (x_i, t_i) \in M^n \times [0, T_i]$ such that

(87) $$t_i(T_i - t_i)|\mathrm{Rm}|(x_i, t_i) = \sup_{M^n\times(0,T_i]} t(T_i - t)|\mathrm{Rm}|(x, t).$$

It can be seen from the above choice that $t_i \to \infty$. Indeed, from the definition of Type IIb, we can find two sequences $L_i \to \infty$ and $y_i \in M$ such that $\lim_{i\to\infty} L_i|\mathrm{Rm}|(y_i, L_i) = \infty$ and $L_i \le T_i/2$. Then we have

(88) $$\begin{aligned}\sup_{M^n\times(0,T_i]} t(T_i - t)|\mathrm{Rm}|(x, t) &\ge L_i(T_i - L_i)|\mathrm{Rm}|(y_i, L_i)\\ &\ge \tfrac{1}{2} T_i L_i|\mathrm{Rm}|(y_i, L_i).\end{aligned}$$

Then it is clear from (87) and (88) that $t_i \to \infty$.

If we set $Q_i = |\mathrm{Rm}|(x_i, t_i)$, it can be proved that $(M, g_i(t), p_i)$ converges smoothly in the Cheeger–Gromov sense to a complete eternal Ricci flow solution $(M_\infty, g_\infty(t), p_\infty)$, $t \in (-\infty, \infty)$, where $g_i(t) = Q_i g(t_i + Q_i^{-1} t)$.

Then for any $\tau > 0$,

(89) $$\begin{aligned}\mu(g_\infty(0), \tau) \ge \limsup_{i\to\infty} \mu(Q_i g(t_i), \tau) &\ge \limsup_{i\to\infty} \mu(g(t_i), \frac{\tau}{Q_i})\\ &\ge \limsup_{i\to\infty} \mu(g(0), \frac{\tau}{Q_i} + t_i) = 0,\end{aligned}$$

where the first inequality follows from Lemma 3.2, the last from the monotonicity of μ, and the equality is from Theorem 3.4.

From Theorem 3.3, it must be the case that M^n is isometric to $\mathbb{R}^n$. But this is impossible since $|\mathrm{Rm}|_{g_\infty(0)}(x_\infty) = \lim_{i\to\infty}|\mathrm{Rm}|_{g_i(0)}(x_i) = 1$.

Case 2 (Type III: $\sup_{M\times[0,\infty)} t|\mathrm{Rm}| < \infty$) In this case, suppose that $p_i = (x_i, t_i)$ is a sequence of points and times with $t_i \to \infty$ and that, for some $c > 0$,

$$t_i|\mathrm{Rm}|(x_i, t_i) = t_i \sup_{x\in M} |\mathrm{Rm}|(x, t_i) \ge c.$$

Then as in the first case, $(M, g_i(t) = Q_i g(t_i + Q_i^{-1} t), p_i)$ with $t \in [-t_i Q_i, \infty)$ converges to $(M_\infty, g_\infty(t), x_\infty)$ with $t \in (-c, \infty)$, where $g_i(t) = Q_i g(t_i + Q_i^{-1} t)$. Again we derive a contradiction.

Therefore, we have proved that the singularity at infinity is of Type III, and

$$\lim_{t\to\infty} t \sup_M |\mathrm{Rm}(t)| = 0. \tag{90}$$

Fix an $\epsilon \in (0,1)$ to be determined later. From (90) we assume that for t large enough,

$$\sup_M |\mathrm{Rm}| \le \frac{\epsilon}{1+t}. \tag{91}$$

So by a translation of time, we assume (91) holds for any $t \ge 0$.

Next, we prove a gradient estimate and Harnack inequality for the solution of the heat equation under the condition (91). The proof is a long-time version of the Li–Yau estimates; see [27].

Set $u_0 = r^{-2-\sigma}$, where r is the function defined in the introduction. We consider the positive solution u of the heat equation

$$u_t = \Delta u \tag{92}$$

with the initial condition $u(0) = u_0$.

It can be proved by using the maximum principle, as in the proof of Theorem 2.2, that for any $T > 0$ and $t \in [0,T]$, the functions $u(t)$ and $|\nabla u|(t)$ have the same decay rates as $u(0)$ and $|\nabla_{g(0)} u|(0)$ respectively. To be precise, there exist $c_1(T) > 0$ and $c_2(T) > 0$ such that

$$\begin{aligned} c_1(T) r^{-2-\sigma} \le{} & \quad u(t) \quad \le c_2(T) r^{-2-\sigma}, \\ c_1(T) r^{-3-\sigma} \le{} & |\nabla u|(t) \le c_2(T) r^{-3-\sigma}. \end{aligned} \tag{93}$$

Let $f = \log u$. Then f satisfies

$$f_t = \Delta f + |\nabla f|^2.$$

If we set $H(x,t) = t(|\nabla f|^2 - 2f_t)$, then we have the following lemma.

Lemma 4.1 *Under the condition* $\sup_M |\mathrm{Rm}|(x,t) \le \epsilon/(1+t)$, *we have*

$$\Delta H - H_t \ge -2\nabla f \cdot \nabla H + \frac{t}{n}(|\nabla f|^2 - f_t)^2 - (|\nabla f|^2 - 2f_t) - 3|\nabla f|^2 - \frac{4\epsilon^2}{1+t}. \tag{94}$$

Proof We have

$$\Delta H = t\Delta(|\nabla f|^2 - 2f_t). \tag{95}$$

By Bochner's formula,

$$
\begin{aligned}
(96) \qquad \Delta|\nabla f|^2 &= 2|\nabla^2 f|^2 + 2\operatorname{Rc}(\nabla f, \nabla f) + 2\langle \nabla\Delta f, \nabla f\rangle \\
&= 2|\nabla^2 f|^2 + 2\operatorname{Rc}(\nabla f, \nabla f) - 2\langle \nabla(|\nabla f|^2 - f_t), \nabla f\rangle \\
&\geq 2|\nabla^2 f|^2 - \frac{2}{1+t}|\nabla f|^2 - 2\langle \nabla(|\nabla f|^2 - f_t), \nabla f\rangle,
\end{aligned}
$$

where the last inequality follows from our curvature estimate.

On the other hand,

$$\Delta f_t = (\Delta f)_t - 2R_{ij} f_{ij} \leq (\Delta f)_t + 2|\operatorname{Rc}|^2 + \tfrac{1}{2}|\nabla^2 f|^2.$$

So we get

$$
\begin{aligned}
(97) \quad \Delta H &\geq t\Big(|\nabla^2 f|^2 - 2\langle \nabla(|\nabla f|^2 - f_t), \nabla f\rangle - 2(\Delta f)_t - \frac{2}{1+t}|\nabla f|^2 - 4|\operatorname{Rc}|^2\Big) \\
&\geq \frac{t}{n}(|\nabla f|^2 - f_t)^2 - 2t\langle \nabla(|\nabla f|^2 - f_t), \nabla f\rangle + 2t(|\nabla f|^2 - f_t)_t \\
&\qquad - 2|\nabla f|^2 - \frac{4\epsilon^2}{1+t}.
\end{aligned}
$$

Then we have

$$H_t = |\nabla f|^2 - 2f_t + t(|\nabla f|^2 - 2f_t)_t.$$

Therefore,

$$
\begin{aligned}
(98) \quad \Delta H - H_t &\geq \frac{t}{n}(|\nabla f|^2 - f_t)^2 - 2t\langle \nabla(|\nabla f|^2 - f_t), \nabla f\rangle + 2t(|\nabla f|^2 - f_t)_t \\
&\qquad - t(|\nabla f|^2 - 2f_t)_t - (|\nabla f|^2 - 2f_t) - 2|\nabla f|^2 - \frac{4\epsilon^2}{1+t} \\
&= \frac{t}{n}(|\nabla f|^2 - f_t)^2 - 2t\langle \nabla(|\nabla f|^2 - f_t), \nabla f\rangle + t|\nabla f|^2_t \\
&\qquad - (|\nabla f|^2 - 2f_t) - 2|\nabla f|^2 - \frac{4\epsilon^2}{1+t} \\
&= \frac{t}{n}(|\nabla f|^2 - f_t)^2 - 2t\langle \nabla(|\nabla f|^2 - f_t), \nabla f\rangle + 2t\langle \nabla f_t, \nabla f\rangle \\
&\qquad + 2t\operatorname{Rc}(\nabla f, \nabla f) - (|\nabla f|^2 - 2f_t) - 2|\nabla f|^2 - \frac{4\epsilon^2}{1+t} \\
&\geq \frac{t}{n}(|\nabla f|^2 - f_t)^2 - 2\langle \nabla H, \nabla f\rangle - (|\nabla f|^2 - 2f_t) - 3|\nabla f|^2 - \frac{4\epsilon^2}{1+t}.
\end{aligned}
$$

This proves the lemma. □

Now we can use the above equation to derive the Li–Yau inequality by following the same method as in [38, Theorem 4.2] to conclude that for some $c_1 > 0$,

$$\frac{|\nabla u|^2}{u^2} - 2\frac{u_t}{u} \leq \frac{c_1}{t}. \tag{99}$$

Note that in [38, Equation (1.10)] the extra term $2nk$ when $\alpha = 2$ can be bounded by $C/(1+t)$ in our case.

With the gradient estimate (99), we prove the following Harnack inequality for u.

Theorem 4.2 *For any $x, y \in M^n$ and $0 < t_1 < t_2$,*

$$\frac{u(y,t_2)}{u(x,t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-c_1/2} \exp\left(-\frac{d_{g(t_1)}(x,y)^2}{2(t_2-t_1)}(1+t_2-t_1)^{2\epsilon}\right).$$

Proof Suppose $\gamma(t)\colon [t_1,t_2] \to M$ is a geodesic with respect to the metric $g(t_1)$ such that for $t_1 \leq t \leq t_2$,

$$|\dot{\gamma}(t)| = \frac{d_{g(t_1)}(x,y)}{t_2-t_1}, \quad \gamma(t_1) = x, \quad \gamma(t_2) = y.$$

Then we have

$$\begin{aligned}
\text{(100)} \qquad \log\frac{u(y,t_2)}{u(x,t_1)} &= \int_{t_1}^{t_2} \frac{d}{dt}(\log u(\gamma(t),t))\,dt \\
&= \int_{t_1}^{t_2}\left(\frac{\partial}{\partial t}\log u + \nabla \log u \cdot \frac{\partial \gamma}{\partial t}\right)dt \\
&\geq \int_{t_1}^{t_2}\left(\frac{|\nabla \log u|^2}{2} - \frac{c_1}{2t} + \nabla \log u \cdot \frac{\partial \gamma}{\partial t}\right)dt \qquad \text{(using (99))} \\
&\geq -\frac{c_1}{2}\log\frac{t_2}{t_1} - \frac{1}{2}\int_{t_1}^{t_2}\left|\frac{\partial \gamma}{\partial t}\right|^2_{g(t)}\,dt.
\end{aligned}$$

The evolution equation of the metric along the Ricci flow and inequality (91) imply

$$\int_{t_1}^{t_2}\left|\frac{\partial \gamma}{\partial t}\right|^2_{g(t)} dt \leq (1+t_2-t_1)^{2\epsilon}\int_{t_1}^{t_2}\left|\frac{\partial \gamma}{\partial t}\right|^2_{g(t_1)} dt = (1+t_2-t_1)^{2\epsilon}\,\frac{d_{g(t_1)}(x,y)^2}{t_2-t_1}.$$

Therefore (100) completes the proof. □

Remark 4.3 We note that the proof of the above estimates does not depend on the order of decay for the initial condition u_0.

Theorem 4.4 *We have the following estimate. There exist $\delta > 0$ and $C > 0$ such that*

$$u(x,t) \leq \frac{C}{(1+t)^{1+\delta}}.$$

Proof We fix a constant $p \in (n/(2+\sigma), n/2)$. Then from the decay property (93), u^p is integrable and

$$\frac{d}{dt}\left(\int u^p\,dV\right) = \int (pu^{p-1}u_t - Ru^p)\,dV \le \int pu^{p-1}\Delta u\,dV \tag{101}$$
$$= \lim_{r\to\infty}\int_{r(x)=r} pu^{p-1}\langle\nabla u,\nabla r\rangle\,d\sigma - \lim_{r\to\infty}\int_{r(x)\le r} p(p-1)u^{p-2}|\nabla u|^2\,dV$$
$$= -\int p(p-1)u^{p-2}|\nabla u|^2\,dV \le 0,$$

where the boundary term from the integration by parts vanishes since, by our definitions of r and AE manifolds,

$$|\nabla u|u^{p-1} \le Cr^{-3-\sigma+(p-1)(-2-\sigma)} \le Cr^{-1-p(2+\sigma)} < Cr^{-1-n}, \tag{102}$$

$$\lim_{r\to\infty}\frac{\mathrm{Vol}(\{r(x)=r\})}{nw_n r^{n-1}} = 1. \tag{103}$$

Moreover, (101) is true since $p > n/(2+\sigma) \ge 1$ by our assumption $\sigma \le n-2$.

So from (101) there exists a $c_2 > 0$ such that on any time slice,

$$\int u^p\,dV \le c_2. \tag{104}$$

For a fixed $x \in M^n$ and any $t \ge 1$, the Harnack inequality of Theorem 4.2 implies

$$u^p(y,2t) \ge 2^{-c_1p/2}\exp\left(\frac{-p(1+t)}{2t}\right)u^p(x,t) \tag{105}$$

for any $y \in B_{g(t)}(x,(1+t)^{\frac{1}{2}-\epsilon})$. Therefore,

$$c_2 \ge \int_M u^p(y,2t)\,dV_{g(2t)}(y) \ge \int_{B_{g(t)}(x,(1+t)^{\frac{1}{2}-\epsilon})} u^p(y,2t)\,dV_{g(2t)}(y) \tag{106}$$
$$\ge 2^{-c_1p/2}\exp\left(\frac{-p(1+t)}{2t}\right)\mathrm{Vol}_{g(2t)}\big(B_{g(t)}(x,(1+t)^{\frac{1}{2}-\epsilon})\big)u^p(x,t)$$
$$\ge c_3\mathrm{Vol}_{g(2t)}\big(B_{g(t)}(x,(1+t)^{\frac{1}{2}-\epsilon})\big)u^p(x,t)$$

for some constant $c_3 = 2^{-c_1p/2}e^{-p} \le 2^{-c_1p/2}\exp(-p(1+t)/2t)$ for any $t \ge 1$.

The evolution equation for the volume of any compact set $K \subset M^n$ is

$$\frac{d}{dt}\left(\int_K dV\right) = \int_K -R\,dV \ge \frac{-\epsilon}{1+t}\int_K dV.$$

So we have

$$\operatorname{Vol}_{g(t)}(K) \geq (1+t)^{-\epsilon}\operatorname{Vol}_{g(0)}(K). \tag{107}$$

On the other hand, by the same reason, for any $x, y \in M^n$ we have

$$d_{g(t)}(x,y) \leq (1+t)^{\epsilon} d_{g(0)}(x,y). \tag{108}$$

So from Equations (106)–(108) we have

$$\begin{aligned} c_2 &\geq c_3 \operatorname{Vol}_{g(2t)}(B_{g(t)}(x,(1+t)^{\frac{1}{2}-\epsilon}))u^p(x,t) \\ &\geq c_3(1+2t)^{-\epsilon}\operatorname{Vol}_{g(0)}(B_{g(t)}(x,(1+t)^{\frac{1}{2}-\epsilon}))u^p(x,t) \\ &\geq c_3(1+2t)^{-\epsilon}\operatorname{Vol}_{g(0)}(B_{g(0)}(x,(1+t)^{\frac{1}{2}-2\epsilon}))u^p(x,t) \\ &\geq c_4(1+2t)^{-\epsilon}(1+t)^{(\frac{1}{2}-2\epsilon)n}u^p(x,t) \end{aligned} \tag{109}$$

for some $c_4 > 0$, by the AE condition of $g(0)$.

Hence we have

$$u(x,t) \leq C(1+t)^{(\epsilon-(1/2-2\epsilon)n)/p}. \tag{110}$$

If ϵ is sufficiently small, which depends on p and n, then $\big(\epsilon - \big(\frac{1}{2} - 2\epsilon\big)n\big)/p < -1$ and we can choose $\delta = -1 - \big(\epsilon - \big(\frac{1}{2} - 2\epsilon\big)n\big)/p > 0$.

On the other hand if $t \leq 1$ the conclusion is obvious since u is uniformly bounded on a compact time interval. □

With Theorem 4.4, we prove the following estimate for the curvature operator.

Theorem 4.5 *For some constants $C_0, \delta_0 > 0$,*

$$|\mathrm{Rm}| \leq \frac{C_0}{(1+t)^{1+\delta_0}}.$$

Proof Under Ricci flow, we have the following lemma by direct computation.

Lemma 4.6 *Let T be a time-dependent tensor on M, and u a positive solution of $\partial_t u = \Delta u$. Then*

$$(\partial_t - \Delta)\frac{|T|^2}{u^2} = \frac{2}{u}\nabla u \cdot \nabla \frac{|T|^2}{u^2} - 2\frac{|u\nabla T - \nabla u T|^2}{u^4} + \frac{(\partial_t - \Delta)|T|^2}{u^2}.$$

Let $W = |\mathrm{Rm}|^2/u^2$. Then from Lemma 4.6 we have

$$\begin{aligned}\partial_t W &= \Delta W + \frac{2}{u}\nabla u \cdot \nabla W - 2\frac{|u\nabla \mathrm{Rm} - \nabla u\,\mathrm{Rm}|^2}{u^4} + P \\ &\le \Delta W + \frac{2}{u}\nabla u \cdot \nabla W + P,\end{aligned} \tag{111}$$

where

$$P = \frac{8(B_{ijkl} + B_{ikjl})R_{ijkl}}{u^2} \quad \text{and} \quad B_{ijkl} = -R_{pijq}R_{qlkp}.$$

We have the following estimate for P:

$$P \le \frac{16|\mathrm{Rm}|^3}{u^2} \le \frac{16\epsilon}{1+t}W, \tag{112}$$

where the last inequality is from (91).

As in the proof of Theorem 2.2, $2\nabla u/u$ is bounded on $M^n \times [0,T]$ for any $T > 0$. From Theorem 2.1 we conclude that

$$W = \frac{|\mathrm{Rm}|^2}{u^2} \le C(1+t)^{16\epsilon} \tag{113}$$

for some constant $C > 0$.

Therefore, from Theorem 4.4 we know that there exists a $C_0 > 0$ such that

$$|\mathrm{Rm}| \le C_0 u(1+t)^{8\epsilon} \le \frac{C_0}{(1+t)^{1+\delta-8\epsilon}}, \tag{114}$$

where we can take $\delta_0 = \delta - 8\epsilon > 0$ by choosing ϵ to be small enough. □

Now from the proof of Theorem 4.4, we know that for any σ_0 slightly smaller than σ,

$$u(x,t) \le Ct^{-1-\sigma_0/2}.$$

Therefore, $|\mathrm{Rm}| \le Ct^{-1-\sigma_0/2}$. In other words, we have shown δ_0 can be chosen to be any number less than $\sigma/2$.

We have the following version of Shi's estimate; see also [40].

Theorem 4.7 *For any $k = 0, 1, \ldots,$*

$$|\nabla^k \mathrm{Rm}| \le C_k t^{-1-\delta_0-k/2}.$$

Proof By Theorem 4.5 the conclusion is true for $k=0$. We assume by induction that it holds for any $0 \le l < k$.

For any fixed $s \ge 1$, we let

$$F(x,t) = (t-s)^k |\nabla^k \operatorname{Rm}|^2 + C_1 (t-s)^{k-1} |\nabla^{k-1} \operatorname{Rm}|^2 + \cdots + C_k |\operatorname{Rm}|^2$$

on $M \times [s,\infty)$. From the evolution equation of $|\nabla^k \operatorname{Rm}|^2$,

$$(115)\quad \begin{aligned} \partial_t |\nabla^k \operatorname{Rm}|^2 &= \Delta |\nabla^k \operatorname{Rm}|^2 - 2|\nabla^{k+1} \operatorname{Rm}|^2 + \sum_{l=0}^{k} \nabla^l \operatorname{Rm} * \nabla^{k-l} \operatorname{Rm} * \nabla^k \operatorname{Rm} \\ &\le \Delta |\nabla^k \operatorname{Rm}|^2 - 2|\nabla^{k+1} \operatorname{Rm}|^2 \\ &\qquad + C \sum_{l=0}^{k} |\nabla^l \operatorname{Rm}||\nabla^{k-l} \operatorname{Rm}||\nabla^k \operatorname{Rm}|, \end{aligned}$$

we have by induction that

$$(t-s)^k |\nabla^l \operatorname{Rm}||\nabla^{k-l} \operatorname{Rm}||\nabla^k \operatorname{Rm}| \le C t^{-2-2\delta_0} (t-s)^{k/2} |\nabla^k \operatorname{Rm}| \le C t^{-2-2\delta_0} F^{\frac{1}{2}}$$

for $0 < l < k$, and

$$(t-s)^k |\nabla^l \operatorname{Rm}||\nabla^{k-l} \operatorname{Rm}||\nabla^k \operatorname{Rm}| = (t-s)^k |\operatorname{Rm}||\nabla^k \operatorname{Rm}|^2 \le C t^{-1-\delta_0} F$$

for $l=0$ or $l=k$.

Therefore we can find nonnegative constants $C_1, C_2, \dots, C_k$ such that F satisfies

$$(116)\qquad \partial_t F \le \Delta F + C t^{-2-2\delta_0} (F^{\frac{1}{2}} + t^{1+\delta_0} F).$$

We consider the ODE

$$(117)\qquad \frac{d\phi}{dt} = C t^{-2-2\delta_0} (\phi^{\frac{1}{2}} + t^{1+\delta_0} \phi), \quad \phi(s) = \bar{C} s^{-2-2\delta_0},$$

where $\bar{C} = C_k C_0^2$. Now $F(x,s) \le \phi(s)$, since $F(s) = C_k |\operatorname{Rm}|^2 \le \bar{C} s^{-2-2\delta_0}$.

Since $\phi(t)$ is increasing, $\phi(t) \ge \bar{C} s^{-2-2\delta_0} \ge \bar{C} t^{-2-2\delta_0}$ for $t \ge s$ and hence

$$(118)\qquad \frac{d\phi}{dt} = C t^{-2-2\delta_0} \phi^{\frac{1}{2}} + C t^{-1-\delta_0} \phi \le C t^{-1-\delta_0} \phi.$$

Then it is easy to show $\phi(t) \le C s^{-2-2\delta_0} e^{C t^{-\delta_0}} \le C s^{-2-2\delta_0}$ for $t \ge s \ge 1$.

Now from Theorem 2.1, we conclude that

$$F(2s) \le C s^{-2-2\delta_0}.$$

In other words,

$$s^k |\nabla^k \mathrm{Rm}|^2(2s) \le C_k s^{-2-2\delta_0}.$$

Since s is an arbitrary positive number, we have

$$|\nabla^k \mathrm{Rm}|(t) \le C t^{-1-\delta_0-k/2},$$

which completes the induction process. □

Thus there exists a metric g_∞ such that $g(t)$ converges to g_∞ smoothly as $t \to \infty$. Moreover, arguing as before, we have for any $\tau > 0$,

$$\mu(g_\infty, \tau) \ge \limsup_{t\to\infty} \mu(g(t), \tau) \ge \limsup_{t\to\infty} \mu(g(0), \tau + t) = 0.$$

Then by Theorem 3.3, $(M^n, g_\infty) = (\mathbb{R}^n, g_E)$. In particular, M^n is diffeomorphic to $\mathbb{R}^n$.

5 Proof of Theorem 1.2

In this section, we prove our first main theorem.

We first recall the definition of weighted function space; see for example [26]. Let (M, g) be an AE manifold with AE end E. The weighted space $C^k_\beta(E)$ consists of C^k functions u for which the norm

$$\|u\|_{C^k_\beta} = \sum_{i=0}^{k} \sup_M r^{-\beta+i} |\nabla^i u|$$

is finite. The weighted Hölder space $C^{k,\alpha}_\beta(E)$ is defined for $0 < \alpha < 1$ as the set of $u \in C^k_\beta(E)$ for which the norm

$$\|u\|_{C^{k,\alpha}_\beta} = \|u\|_{C^k_\beta} + \sup_{x,y} (\min\{r(x), r(y)\})^{-\beta+k+\alpha} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha}$$

is finite.

Then we have the following convergence result in the weighted space.

Theorem 5.1 *For any $\sigma' \in ((n-2)/2, \sigma)$, we have that $g_{ij}(t)$ converges to $g_{ij}(\infty)$ in $C^\infty_{-\sigma'}$ as $t \to \infty$. In particular, $(g_{ij}(\infty), E)$ is an AE coordinate system on M^n.*

Proof We first prove a lemma.

Lemma 5.2 *For $k=0,1,\dots$ there exist $C_k,\eta_k>0$ such that*

$$|\nabla^k \mathrm{Rm}|(x,t) \le C_k t^{-1-\eta_k} r^{-k-\sigma'}$$

for all $(x,t)\in M\times[0,\infty)$.

Proof of the lemma We choose σ_1,σ_0 such that $\sigma'<\sigma_1<\sigma_0<\sigma$ and $\delta_0=\sigma_0/2$ in Theorem 4.7.

We consider a domain $D_k=\{(x,t)\in M\times[0,\infty) \mid r(x)\ge t^{a_k}\}$ in the spacetime where $a_k>\frac{1}{2}$, to be determined later.

For $(x,t)\notin D_k$, from Theorem 4.7 we have

$$|\nabla^k \mathrm{Rm}| \le C_k t^{-1-\sigma_0/2-k/2} \le C_k t^{-1-\eta_k} r^{-k-\sigma'} \tag{119}$$

for some $\eta_k>0$, when a_k is sufficiently close to $\frac{1}{2}$.

Claim *We have the estimate $|\nabla^k \mathrm{Rm}|^2 \le C r^{-4-2\sigma_1-2k}$ for $(x,t)\in D_k$.*

Proof of the claim Let $h_k=r^{4+2\sigma_1+2k}$ and $w_k=h_k|\nabla^k \mathrm{Rm}|^2$. From (15) we have

$$(\partial_t-\Delta)w_k \le B_k w_k - 2\nabla \log h_k \nabla w_k + C\sum_{l=0}^{k} h_k |\nabla^l \mathrm{Rm}||\nabla^{k-l} \mathrm{Rm}||\nabla^k \mathrm{Rm}|, \tag{120}$$

where $B_k=(2|\nabla h_k|^2-h_k\Delta h_k)/h_k^2$ is uniformly bounded by $r^{-2}\le t^{-2a_k}$.

For $k=0$, we have

$$(\partial_t-\Delta)w_0 \le -2\nabla\log h_0 \nabla w_0 + C t^{-1-\delta_0'} w_0$$

for some $\delta_0'=\min\{2a_0-1,\frac{1}{2}\sigma_0\}>0$.

Moreover, on ∂D_0 we have

$$|\mathrm{Rm}| \le C t^{-1-\sigma_0/2} = C r^{-(1+\sigma_0/2)/a_0} \le C r^{-2-\sigma_1} \tag{121}$$

for a_0 sufficiently close to $\frac{1}{2}$.

Now we apply Theorem 2.1 on D_k to conclude that the claim holds for $k=0$. Note that even though in Theorem 2.1 there is no boundary in spacetime for $t>0$, if we go through the proof (see [11, Theorem 12.14]), the contradiction is derived at an interior point as long as the conclusion holds also on the boundary.

Now we assume that the claim holds for all $0 \le l < k$. Then by induction on D_k,

$$h_k |\nabla^l \mathrm{Rm}||\nabla^{k-l} \mathrm{Rm}||\nabla^k \mathrm{Rm}| = h_k |\mathrm{Rm}||\nabla^k \mathrm{Rm}|^2 \le t^{-1-\sigma_0/2} w_k$$

for $l = 0$ or $l = k$, and

$$h_k |\nabla^l \mathrm{Rm}||\nabla^{k-l} \mathrm{Rm}||\nabla^k \mathrm{Rm}| \le C r^k |\nabla^k \mathrm{Rm}| = C r^{-\sigma_1 - 2} w_k^{1/2} \le C t^{-a_k \sigma_1 - 2a_k} w_k^{1/2}$$

for $0 < l < k$.

Therefore from (120) we have

$$(\partial_t - \Delta) w_k \le -2 \nabla \log h_k \nabla w_k + C t^{-1-\delta_k'} (w_k + w_k^{1/2})$$

for some $\delta_k' > 0$.

On the other hand, on ∂D_k we have by Shi's estimate

$$|\nabla^k \mathrm{Rm}| \le C_k t^{-k/2} t^{-1-\sigma_0/2} = C_k r^{-(1+k/2+\sigma_0/2)/a_k} \le C_k r^{-2-k-\sigma_1}, \tag{122}$$

when a_k is chosen to be sufficiently close to $\frac{1}{2}$.

So from the maximum principle, we conclude that w_k is uniformly bounded on D_k and the claim holds for k as well. □

Therefore, on D_k we have

$$|\nabla^k \mathrm{Rm}| \le C_k r^{-2-k-\sigma_1} \le C_k t^{-1-\eta_k} r^{-k-\sigma'}$$

for some $\eta_k > 0$ and a_k close to $\frac{1}{2}$.

Thus the proof of the lemma is complete. □

With the same argument as for Theorem 2.2, we conclude that $g_{ij}(t)$ converges to $g_{ij}(\infty)$ in $C^\infty_{-\sigma'}$ because the term $t^{-1-\eta_k}$ guarantees that $|\nabla^k \mathrm{Rm}|$ is integrable with respect to time at infinity. In other words, $g_{ij}(\infty)$ is an AE coordinate system with a smaller order σ' for the Euclidean space. □

Now we continue to prove Theorem 1.2. We choose a smooth function η such that $\eta = 0$ outside of the AE end E, and $\eta = 1$ when r is large.

Let $\chi(t) = (\partial_i g_{ij}(t) - \partial_j g_{ii}(t))\partial_j$ be a vector field on the AE end. By the definition of mass,

$$
\begin{aligned}
m(g(t)) &= \lim_{r\to\infty} \int_{S^r} \chi(t) \lrcorner \, dV_{g_E} \\
&= \lim_{r\to\infty} \int_{S^r} \eta\chi(t) \lrcorner \, dV_{g_E} \\
&= \int \eta \operatorname{div}(\chi(t)) + \langle \chi(t), \nabla\eta\rangle \, dV.
\end{aligned} \tag{123}
$$

On the other hand, we have (see [26, Equation (9.2)])

$$
\begin{aligned}
R &= g^{jk}(\partial_i \Gamma^i_{jk} - \partial_k \Gamma^i_{ij} + \Gamma^i_{il}\Gamma^l_{jk} - \Gamma^i_{kl}\Gamma^l_{ij}) \\
&= \partial_j(\partial_i g_{ij} - \partial_j g_{ii}) + E(g),
\end{aligned} \tag{124}
$$

where $E(g)$ is some universal analytic expression that is polynomial in g, ∂g and $\partial^2 g$ such that $E = O(r^{-2\sigma'-2})$. Moreover,

$$
|E(g(t)) - E(g(\infty))| \le C \|g(t) - g(\infty)\|_{C^2_{-\sigma'}} r^{-2\sigma'-2}.
$$

By taking the difference of equations of $R(t)$ and $R(\infty) = 0$, we have

$$
\begin{aligned}
R(t) &= \partial_j(\partial_i g_{ij}(t) - \partial_j g_{ii}(t)) - \partial_j(\partial_i g_{ij}(\infty) - \partial_j g_{ii}(\infty)) \\
&\qquad + E(g(t)) - E(g(\infty)) \\
&= \operatorname{div}\chi(t) - \operatorname{div}\chi(\infty) + E(g(t)) - E(g(\infty)),
\end{aligned} \tag{125}
$$

and hence

$$
|\operatorname{div}\chi(t) - \operatorname{div}\chi(\infty) - R(t)| \le C\|g(t) - g(\infty)\|_{C^2_{-\sigma'}} r^{-2\sigma'-2}. \tag{126}
$$

From (123) and (126) we have

$$
\begin{aligned}
m(g(0)) &= \lim_{t\to\infty} m(g(t)) = \lim_{t\to\infty} m(g(t)) - m(g(\infty)) \\
&= \lim_{t\to\infty} \int \eta(\operatorname{div}\chi(t) - \operatorname{div}\chi(\infty)) + \langle \chi(t) - \chi(\infty), \nabla\eta\rangle \, dV_{g_E} \\
&\ge \lim_{t\to\infty} \int \big(\eta R(t) - C\eta \|g(t) - g(\infty)\|_{C^2_{-\sigma'}} r^{-2\sigma'-2} \\
&\qquad + \langle \chi(t) - \chi(\infty), \nabla\eta\rangle\big) \, dV_{g_E}.
\end{aligned} \tag{127}
$$

Now $\eta r^{-2\sigma'-2}$ is integrable since $\sigma' > (n-2)/2$. In addition, $\chi(t) - \chi(\infty)$ converges to 0 on the support of $\nabla\eta$ and $\|g(t) - g(\infty)\|_{C^2_{-\sigma'}}$ tends to 0, so we have from (127),

$$
m(g(0)) \ge \lim_{t\to\infty} \int \eta R(t) \, dV_{g_E} \ge 0. \tag{128}
$$

Remark 5.3 From the above proof, we have shown that

$$m(g(0)) = \lim_{t\to\infty} \int R(t)\, dV_t, \tag{129}$$

since $g(t)$ converges to g_E uniformly on any compact set.

If the equality holds, we have by (129) that $\lim_{t\to\infty} \int R(t)\, dV_t = 0$.

On the other hand,

$$\begin{aligned} \frac{d}{dt}\left(\int R\, dV\right) &= \int \Delta R + 2|\mathrm{Rc}|^2 - R^2\, dV \\ &= \int 2|\mathrm{Rc}|^2 - R^2\, dV \\ &\geq -\frac{n-2}{n}\int R^2\, dV \qquad\qquad (\text{from } |\mathrm{Rc}|^2 \geq R^2/n) \\ &\geq -\frac{C}{(1+t)^{1+\delta}}\int R\, dV, \end{aligned} \tag{130}$$

where the second inequality holds since $\lim_{r\to\infty} \int_{S_r} |\nabla R(t)|\, d\sigma = 0$ and therefore $\int \Delta R\, dV = 0$. The last inequality follows from Theorem 4.5.

Integrating both sides, $\lim_{t\to\infty} \int R(t)\, dV_t$ cannot be 0 unless $R(t) \equiv 0$, which is a contradiction by our original assumptions. In other words, the only possibility for $m(g(0)) = 0$ is when $(M^n, g) = (\mathbb{R}^n, g_E)$.

Thus, we have completed the proof of Theorem 1.2.

6 Ricci flow with surgery on AE manifolds

In this section we define the Ricci flow with surgery on an AE manifold. Most definitions and notation are from [33; 29; 7; 24], with slight modifications. We assume from now on that M is an orientable Riemannian AE 3–manifold with $R > 0$ unless otherwise specified.

First of all we fix a surgery model; see [33, Section 2] and [29, Chapter 12].

Definition 6.1 (surgery model) Consider $M_{\text{stan}} = \mathbb{R}^3$ with its natural SO(3)–action. Then there is a complete metric g_{stan} on M_{stan} such that:

(i) g_{stan} is SO(3)–invariant.

(ii) g_{stan} has nonnegative sectional curvature.

(iii) There is a compact ball $B \subset M_{\text{stan}}$ such that the restriction of the metric g_{stan} to the complement of this ball is isometric to the product $(S^2, h) \times (\mathbb{R}^+, ds^2)$, where h is the round metric of scalar curvature 1 on S^2.

(iv) There is a standard Ricci flow $(M_{\text{stan}}, g_{\text{stan}}(t))$, $0 \le t < 1$, such that 1 is the singular time.

For an AE manifold M^3, under Ricci flow we either have long-time existence or the metric goes singular at some finite time. In the latter case, we modify the resulting limit by surgery, which cuts off high curvature parts, and add standard capped tubes, so as to produce a new manifold with an AE end which serves as a new initial condition for Ricci flow. Now we clarify the process of surgery at the first singular time, as an example.

Let $(M, g(t))$, $0 \le t < T$, be the Ricci flow solution where T is the first singular time. Let $\Omega \subset M$ be a subset defined by

$$\Omega = \Big\{ x \in M \;\Big|\; \limsup_{t \to T} R_g(x,t) < \infty \Big\}.$$

Then we have the following properties.

Theorem 6.2 (i) *As $t \to T$ the metric $g(t)|_\Omega$ converges to $g(T)$ uniformly in the C^∞–topology on every compact subset of Ω.*

(ii) *Every end of a connected component of Ω is contained in a strong ϵ–tube.*

(iii) *There exists an $r > 0$ such that any $x \in \Omega \times \{T\}$ with $R(x) \ge r^{-2}$ has a strong (C, ϵ)–canonical neighborhood in $\widehat{\mathcal{M}} = M \times [0,T) \cup_{\Omega \times [0,T)} (\Omega \times [0,T])$.*

(iv) *There exists a compact set $K \subset M$ such that $|\mathrm{Rm}|$ is bounded on $K^c \times [0,T)$. In particular, $K^c \subset \Omega$.*

(v) *The scalar curvature $R(g(T))$ is a proper function from $\Omega \to (0,\infty)$.*

Proof The proof of (i)–(iii) can be found in [29, Theorem 11.19]. Property (iv) is proved by pseudolocality; see [9, Theorem 1.1]. To prove (v), we need the following lemma.

Lemma 6.3 *There exists a compact set K such that $g(T)$ has an AE coordinate system on $K^c = M - K$.*

Proof of the lemma From [9, Theorem 1.1], there exist a compact set K and $S>0$ such that $|\mathrm{Rm}(x,t)| \le S$ on $K^c \times [0,T)$. Enlarging K if necessary, we can assume $g_{ij}(0)$ is an AE coordinate system on K^c and ∂K is smooth. Then we can use the same argument as in Theorem 2.2 on the parabolic cylinder $K^c \times [0,T)$ to conclude that $g(T)$ has an AE coordinate system on K^c. □

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in Ω such that $0<c\le R(x_n,T)\le C$ for some constants $0<c<C$. Since by Lemma 6.3 $g(T)$ has curvature bounded by $Cr^{-2-\sigma}$, all x_n are contained in a compact set of M. Then we assume, by taking a subsequence if necessary, that the x_n converge to a point x_∞ in M. If x_∞ is not in Ω, then by Lemma 7.2 in the next section, we have that $R(x_n,T)$ goes to infinity, which is a contradiction.

Thus, the proof of Theorem 6.2 is complete. □

Remark 6.4 We call K^c in Lemma 6.3 the AE end of Ω.

We fix $0<\rho<r$, where r is the constant from Theorem 6.2(iii), and define $\Omega_\rho \subset \Omega$ to be the closed subset of all $x\in\Omega$ for which $R(x,T)\le \rho^{-2}$. For a component Ω_1 of Ω which contains no point of Ω_ρ, by the canonical neighborhood theorem one of the following holds, see [29, Lemma 11.28]:

(i) Ω_1 is a strong double ϵ–horn and is diffeomorphic to $S^2\times\mathbb{R}$.

(ii) Ω_1 is a C–capped ϵ–horn and is diffeomorphic to $\mathbb{R}^3$ or a punctured $\mathbb{R}\mathrm{P}^3$.

(iii) Ω_1 is a compact component and is diffeomorphic to S^3/Γ, $S^1\times S^2$ or $\mathbb{R}\mathrm{P}^3\,\#\,\mathbb{R}\mathrm{P}^3$.

Those are all possibilities if M is orientable.

Let $\Omega^0(\rho)$ be the union of all components of Ω containing points of Ω_ρ. Then $\Omega^0(\rho)$ has finitely many components and is a union of the AE end and finitely many strong ϵ–horns, each of which is disjoint from Ω_ρ. The finiteness of horns can be derived from the properness of $R(T)\to(0,\infty)$, and the remaining arguments can be found in [29, Lemma 11.30].

Next we have the following lemma, which asserts the existence of a strong δ–neck on which we will do surgeries.

Lemma 6.5 [7, Theorem 5.1] *For any $\delta > 0$, there exist $h \in (0, \delta\rho)$ and a constant $D = D(\delta, \rho)$ such that the following holds. Let $x, y, z \in \Omega$ be such that $R(x,t) \le \rho^{-2}$, $R(y,t) = h^{-2}$ and $R(z,t) \ge Dh^{-2}$. Assume that there is a curve γ in Ω_ρ^c connecting x to z via y. Then (y,t) is the center of a strong δ–neck.*

For the surgery parameters $r, \delta < 1$ we set $\rho = r\delta$. Then the scale $h = h(\rho, r) = h(\delta, r)$ and $D = D(\rho, r) = D(\delta, r)$ are determined. Moreover, we require that

$$\lim_{\delta \to 0} \frac{D(\delta, r) h(\delta, r)}{\rho^3} = 0, \tag{131}$$

since the proof of Lemma 6.5 argues by contradiction by choosing two independent sequences $h_i \to 0$ and $D_i \to \infty$.

We say that (M_+, g_+) is obtained from $(\Omega, g(T))$ by (r, δ)–surgery at time T if:

(i) M_+ is obtained from Ω by removing components disjoint from Ω_ρ and cutting along a locally finite collection of disjoint 2–spheres, capping off 3–balls.

(ii) All $x \in M_+ \backslash M(T)$ are contained in a surgery cap and the cutting and capping are done on a strong δ–neck centered at a point y with $R(t, T) = h^{-2}$.

(iii) (M_+, g_+) is pinched toward positive curvature.

Now we show that (r, δ)–surgery must exist; see [7, Lemma 7.6].

By Zorn's lemma, on Ω there exists a maximal collection $\{N_i\}$ of pairwise disjoint δ–necks centered at y_i with $R(y_i, T) = h^{-2}$. Then from Lemma 6.5, every component of $\Omega \backslash \bigcup_i N_i$ has scalar curvature either less than Dh^{-2} or greater than ρ^{-2}. Then we remove all the components of the second kind and do surgeries on those δ–necks N_i.

Now we let M_+ be the resulting manifold and $R(g_+) \in (0, Dh^{-2}]$. From the construction we know that each component of M_+ contains at least one point p at which $R(p, T) \le \rho^{-2}$, hence there are at most finitely many components, by the properness of R. Moreover, one of the components M_+^0 containing the AE end of M_+ is an AE manifold with the same order σ as M. In addition, the mass of $(M_+^0, g(T))$ is well defined and is equal to that of M, by the same argument as in [16].

In general, we can construct three weakly decreasing parameter functions $r(t)$, $\delta(t)$ and $\kappa(t)$, with $t \in [0, \infty)$, to regulate the surgery process so that $r(t)$ is a canonical neighborhood scale function. The following existence theorem is proved in [29, Theorem 15.9]; see also [7, Theorem 1.2].

Theorem 6.6 *There exists a Ricci flow with surgery* $(\mathcal{M}, g_{\mathcal{M}})$ *on* $[0,\infty)$ *with the initial condition* (M, g) *and decreasing functions* $\delta(t)$, $r(t)$, $\kappa(t)$: $[0,\infty) \to \mathbb{R}^+$ *such that the following hold:*

(i) $(\mathcal{M}, g_{\mathcal{M}})$ *has curvature pinched toward positive.*

(ii) *The flow satisfies the strong* (C,ϵ)*–canonical neighborhood theorem with parameter* $r(t)$ *on* $[0,\infty)$.

(iii) *The flow is* $k(t)$*–noncollapsed on* $[0,\infty)$ *on scales* $\le \epsilon$.

(iv) *For any singular time* t *the surgery is performed with control* $\delta(t)$ *at scale* $h(t) = h(\rho(t), \delta(t)) = h(r(t)\delta(t), \delta(t))$.

Next we show that surgery times do not accumulate.

Theorem 6.7 *Let* $(\mathcal{M}, G)$ *be a Ricci flow with surgery on* $[0,\infty)$ *with the initial condition* (M, g) *with parameter functions* $\delta(t)$, $r(t)$, $\kappa(t)$. *On each compact interval* I *of* $[0,\infty)$, *we have at most finitely many surgeries.*

Proof Since all the parameter functions are decreasing, we can choose uniform parameters δ, r and κ on I. Therefore functions h and D are uniformly determined as well. At each singular time t, by our construction $R(x,t) \le Dh^{-2}$. Since curvature is pinched toward positive curvature, we can assume $|\mathrm{Rm}| \le CDh^{-2}$. Now from the evolution equation of $|\mathrm{Rm}|^2$,

$$\partial_t |\mathrm{Rm}|^2 \le \Delta |\mathrm{Rm}|^2 + 16|\mathrm{Rm}|^3,$$

the regular Ricci flow exists at least for time $h^2/16CD$ from t. Since all constants are uniformly chosen, there are at most finitely many surgeries performed on I. □

Remark 6.8 Theorem 6.7 holds for all Ricci flows with surgery with normalized initial condition, which is satisfied after a scaling, if necessary, for our original manifold M.

From the construction of Ricci flow with surgery, each time slice $(\mathcal{M}(t), g(t))$ consists of an AE manifold and a finite number of compact components. Moreover, we can recover the topology of $\mathcal{M}(0) = M$ by performing connected-sum operations among $\mathcal{M}(t)$ and finitely many S^3/Γ and $S^1 \times S^2$, for any $t > 0$.

7 Proof of Theorem 1.3

We first introduce the following definition.

Definition 7.1 For a Ricci flow with surgery $\mathcal{M}$, a connected open subset $\mathcal{X} \subset \mathcal{M}$ is called a path of components if for every time t, the intersection $\mathcal{X}(t)$ of $\mathcal{X}$ with each time-slice $\mathcal{M}(t)$ is a connected component of $\mathcal{M}(t)$.

We set $\mathcal{M}_0$ to be the path of components of $\mathcal{M}$ such that $\mathcal{M}_0(t)$ is an AE manifold for any $t \geq 0$.

Next we quote a local regularity lemma.

Lemma 7.2 [25, Lemma 3.1] *Let $\mathcal{M}$ be a Ricci flow with surgery, with normalized initial condition. Given $T > \frac{1}{100}$, there are numbers $\mu = \mu(T)$ in $(0,1)$, $\sigma = \sigma(T)$ in $(0,1)$, $i_0 = i_0(T) > 0$ and $A_k = A_k(T) < \infty$ for $k \geq 0$, with the following property. If $t \in \left(\frac{1}{100}, T\right]$ and $|R(x,t)| < \mu\rho(0)^{-2} - r(T)^{-2}$ and we put $Q = |R(x,t)| + r(t)^{-2}$, then:*

(i) *The forward/backward parabolic ball $P_{\pm}(x,t,\sigma Q^{-\frac{1}{2}})$ is unscathed, that is, does not intersect the surgery cap.*

(ii) *We have that $|\mathrm{Rm}| \leq A_0 Q$, $\mathrm{inj} \geq i_0 Q^{-\frac{1}{2}}$ and $|\nabla^k \mathrm{Rm}| \leq A_k Q^{1+k/2}$ on the union $P_+(x,t,\sigma Q^{-\frac{1}{2}}) \cup P_-(x,t,\sigma Q^{-\frac{1}{2}})$ of the forward and backward parabolic balls.*

Now we consider a sequence $\{\mathcal{M}^i\}$ of Ricci flows with surgery, where we let $\delta_i(0) \to 0$, hence ρ_i and h_i also go to 0. We first prove a stability result, which shows that on the finite time interval, all surgeries are done in a compact set.

Theorem 7.3 *Let $\{\mathcal{M}^i\}$ be a sequence of Ricci flows with surgery with $\mathcal{M}^i(0) = M$ and $\lim_{i\to\infty} \delta_i(0) = 0$. For any $S > 0$ and $T > 0$, there exists a compact set $K \subset M$ such that for sufficiently large i, the cylinder $K^c \times [0,T]$ exists in $\mathcal{M}^i$ and $|\mathrm{Rm}_i| \leq S$.*

Proof We prove it by contradiction.

Assume there is a sequence $\{x_j\}_{j\in\mathbb{N}}$ on M with $d_g(x_j,\star) = 2r_i$, where $\star$ is a fixed point on M and $r_i \to \infty$ so that $|\mathrm{Rm}_j|(x_j,t_j) > S$ for some $t_j \in [0,T]$.

By the AE condition, the balls $(B_g(x_j,r_j), g, x_j)$ converge smoothly to $(\mathbb{R}^n, g_E, 0)$. Then there exists a $\theta > 0$ sufficiently small that $B_g(x_j,r_j) \times [0,\theta]$ exists in $\mathcal{M}^j$ and for any $A > 0$, restriction of g_j on $B_g(x_j, A) \times [0,\theta]$ converges smoothly to the Euclidean metric on $B_{g_E}(0, A) \times [0,\theta]$.

Therefore for any $A > 0$, we assume $|\mathrm{Rm}| \le S/2$ on $B_g(x_j, A) \times [0,\theta]$ for j sufficiently large. By Lemma 7.2, there exist Q, σ, A_k and $\theta' = \sigma Q^{-\frac{1}{2}}$, all of which depend on S, T, r, κ and (M, g), such that the forward parabolic ball $P_+(x_j, \theta, \theta')$ and the backward parabolic ball $P_-(x_j, \theta, \theta')$ are unscathed and $|\nabla^k \mathrm{Rm}| \le A_k Q^{1+k/2}$ with $\mathrm{inj} \ge i_0 Q^{-\frac{1}{2}}$ on $B_g(x_j, A) \times [\theta - \theta', \theta + \theta']$ for j sufficiently large. By taking a diagonal subsequence, we have that $B_g(x_j, r_j) \times [0, \theta + \theta']$ converges smoothly to the Euclidean metric on $\mathbb{R}^n \times [0, \theta + \theta']$.

Now we can continue this process, since θ' does not depend on δ^j, to conclude that $B_g(x_j, r_i) \times [0, T]$ converges smoothly to the Euclidean metric on $\mathbb{R}^n \times [0, T]$ and $|\mathrm{Rm}_j| \le S/2$ on $B_g(x_j, 1) \times [0, T]$. This is a contradiction. □

By Theorem 3.3, we can find a constant $\epsilon_0 > 0$ such that $\mu_{S^2 \times \mathbb{R}}(g_c, 1) \le -2\epsilon_0$, where g_c is the standard metric on the cylinder with scalar curvature $R = 1$. Therefore, we choose the parameter ϵ for the surgery so that for any ϵ–neck with metric g and center p, we have $\mu_{S^2 \times (-\epsilon^{-1}, \epsilon^{-1})}(R(p)g, 1) \le -\epsilon_0$.

Let $\mathcal{M}$ be a Ricci flow with surgery such that r, ρ, h and δ are uniform surgery parameters. If T is a surgery time, we consider the change of the μ–functional from $(\mathcal{M}(T), g(T))$ to $(\mathcal{M}(T^-), g(T^-))$. Henceforth we assume that $(\mathcal{M}(T^-), g(T^-))$ and $(\mathcal{M}(T), g(T))$ are pre-surgery and post-surgery Riemannian manifolds, respectively.

Now for a Riemannian manifold (M, g) we have the following definition.

Definition 7.4 [44, Equation (2-11)] Set

$$\lambda_{\sigma^2}(g) = \inf\left\{ \int (\sigma^2(4|\nabla v|^2 + Rv^2) - v^2 \log v^2)\, dV - n \log\sigma \;\middle|\; v \in C^\infty(M),\ \|v\|_2 = 1 \right\}.$$

By our definition of $\overline{\mathcal{W}}(g, u, \tau)$ in (41), it is straightforward to compute, by setting $u = v(4\pi\sigma^2)^{n/4}$, that

$$\mu(g, \sigma^2) = \lambda_{\sigma^2}(g) - n - \frac{n}{2}\log 4\pi. \tag{132}$$

In other words, $\mu(g, \sigma^2)$ and $\lambda_{\sigma^2}(g)$ differ by a constant.

If we set $g_1 = \sigma^{-2} g$ and let u_1 be a minimizer of $\lambda_1(g_1)$, then we have (see [44, Equation (2-12)])

$$4\Delta_1 u_1 - R_1 u_1 + 2u_1 \log u_1 + \Lambda u_1 = 0, \tag{133}$$

where $\Lambda = \lambda_1(g_1)$.

Now from [44, Equation (2-13)] we have

$$\lambda_{\sigma^2}(g(T^-)) \le \Lambda + ck\Big(1 + \frac{4c\sigma^2}{h^2}\Big)\frac{\int_U u_1^2\, dV_{g_1}}{1 - \int_U u_1^2\, dV_{g_1}}, \tag{134}$$

where k is the number of surgery caps with scale h, and U is any surgery cap.

To estimate the term $\int_U u_1^2\, dV_{g_1}$ we have the following two lemmas; see [44, Lemmas 2.2 and 2.3].

Lemma 7.5
$$\sup_{\Omega_\rho^c} u_1^2 \le c \max\Big\{\Big(\frac{\rho}{\sigma}\Big)^{-3}, 1\Big\}.$$

Lemma 7.6 *Let u be a positive solution to the inequality*

$$4\Delta u - Ru + 2u \log u + \Lambda u \ge 0.$$

Given a nonnegative function $\phi \in C^\infty(M)$ with $\phi \le 1$, suppose there is a smooth function f that, when $R \ge 0$ in the support of ϕ, satisfies

$$4|\nabla f|^2 \le R - 2\log^+ u - \tfrac{3}{2}|\Lambda|$$

in the support of ϕ. Then

$$\tfrac{1}{2}|\Lambda| \|e^f \phi u\|_2^2 \le 8 \sup_{x \in \operatorname{supp}\nabla\phi} \big(e^{2f}\big(R - 2\log^+ u - \tfrac{3}{2}|\Lambda|\big) + \|e^f \nabla\phi\|_\infty^2\big)\|u\|_2^2.$$

Note that our Lemma 7.6 is slightly different than Lemma 2.3 in [44] as we do not assume $\Lambda \le 0$. Since we impose a stronger restriction on $4|\nabla f|^2$, the proof is identical.

Now we fix a constant $\Lambda_0 = n + \frac{1}{2} n \log 4\pi - \frac{1}{2}\epsilon_0$. By Lemma 7.5, Lemma 7.6 and the proof of [44, Theorem 1.6], there exists a small constant $\epsilon_1 > 0$ such that if $\rho/\sigma \le \epsilon_1$, then either $\lambda_{\sigma^2}(g(T^-)) \ge \Lambda_0$ or

$$\lambda_{\sigma^2}(g(T^-)) \le \lambda_{\sigma^2}(g(T)) + ck(\sigma+1)^3 h^3. \tag{135}$$

Here the condition $\rho/\sigma \le \epsilon_1$ guarantees (see [44, Equation (2-14)]) that on Ω_ρ^c,

$$\tfrac{1}{2} R_1(x) \le R_1(x) - 2\log^+ u_1(x) - \tfrac{3}{2}\Lambda_0 \le R_1(x). \tag{136}$$

In μ–functional terms, it shows that if $\rho/\sigma \le \epsilon_1$, then either $\mu(g(T),\sigma^2) \ge -\frac{1}{2}\epsilon_0$ or

$$\mu(g(T^-),\sigma^2) \le \mu(g(T),\sigma^2) + ck(\sigma+1)^3h^3. \tag{137}$$

Now we take a sequence of Ricci flows with surgery $\{\mathcal{M}^i\}$, with a fixed AE manifold (M,g) as the initial condition, subject to a uniform $r(t)>0$ and surgery parameter function $\delta_i(0)\to 0$.

By Theorem 3.4 there exists a constant $T>0$ such that

$$\mu_M(g,\tau) \ge -\tfrac{1}{2}\epsilon_0 \tag{138}$$

for any $\tau \ge T$.

Then from Theorem 7.3, there exists a compact set $K\subset M$ such that $|\mathrm{Rm}_i| \le 1$ on $(M\setminus K)\times[0,T]$ and we can find a common AE coordinate system for all $g_i(T)$. On the other hand, by the maximum principle it is easy to show that the $\mathcal{M}^i_0(T)\setminus(M-K)$ have a uniform positive lower bound of scalar curvatures. Hence by Theorem 3.7 there exists a $T'>T$ such that

$$\mu_M(g_i(T),\tau) \ge -\tfrac{1}{2}\epsilon_0 \tag{139}$$

for any $\tau \ge T'-T$ and i.

Now since all $r(t)$ and $\delta_i(t)$ are decreasing, we can choose $r>0$ and $\delta_i\to 0$ as constant parameters on the time interval $[0,T']$.

With all these preparations, Theorem 1.3 follows immediately from Theorem 1.2 and the following theorem.

Theorem 7.7 *For i sufficiently large, there are finitely many surgeries for $\mathcal{M}^i_0$.*

Proof Suppose the conclusion is false. Then we can assume that $\mathcal{M}^i_0$ has infinitely many surgeries for all i. In particular, we denote the first surgery time past T by $T^i_{k_i}$ for $\mathcal{M}^i_0$, and all previous surgery times by $\{T^i_1,T^i_2,\dots,T^i_{k_i-1}\}$. We also set $(\sigma^i_j)^2 = T^i_{k_i}-T^i_{k_i-j}$ for $1\le j\le k_i$ and $T^i_0=0$.

If $T^i_{k_i}\ge T'$, then as $T^i_{k_i}$ is a singular time we can find a sequence $\{p^i_v=(x^i_v,t^i_v)\}_{v\in\mathbb{N}}$ of points in $\mathcal{M}^i_0$ such that $t^i_v\to T^i_{k_i}$ and if $Q^i_j=R(x^i_v,t^i_v)$, then $(\mathcal{M}^i_0(t^i_v),Q^i_v g(t^i_v),x^i_v)$

converges smoothly as $v \to \infty$ to a standard cylinder $(S^2 \times \mathbb{R}, g_c)$. Then we have

$$
\begin{aligned}
-2\epsilon_0 \geq \mu_{S^2\times\mathbb{R}}(g_c, 1) &\geq \lim_{v\to\infty} \mu(Q_v^i g_i(t_v^i), 1) \\
&= \lim_{v\to\infty} \mu\left(g_i(t_v^i), \frac{1}{Q_v^i} \right) \\
&\geq \lim_{v\to\infty} \mu\left(g_i(T), \frac{1}{Q_v^i} + t_v^i - T \right) \\
&= \mu(g_i(T), T_{k_i}^i - T),
\end{aligned} \tag{140}
$$

which contradicts (139) since $T_{k_i}^i - T \geq T' - T$.

Therefore, we can assume all $T_{k_i}^i \leq T'$.

By the same point-picking method as above, we have

$$
\mu(g(T_{k_i-1}^i), (\sigma_1^i)^2) = \mu(g(T_{k_i-1}^i), T_{k_i}^i - T_{k_i-1}^i) \leq -2\epsilon_0. \tag{141}
$$

We assume that s is the largest integer between 1 and k_i such that $(\sigma_s^i)^2 < r^2$, where r is the canonical neighborhood scale. As T is a large number and r is small, the time $T_{k_i-s}^i$ is singular. Now we can find a point p which is the center of an ϵ–neck such that $R(p) = (\sigma_s^i)^{-2}$. By our choice of ϵ, we have $\mu(g(T_{k_i-s}^{i-}), (\sigma_s^i)^2) \leq -\epsilon_0$. By using the monotonicity formula,

$$
\mu(g(T_{k_i-(s+1)}^i), (\sigma_{s+1}^i)^2) \leq -\epsilon_0. \tag{142}
$$

Now let l be the largest integer from $s+1$ to k_i such that

$$
\mu(g(T_{k_i-j}^i), (\sigma_j^i)^2) \leq -\tfrac{2}{3}\epsilon_0 \tag{143}
$$

for any $s+1 \leq j \leq l$.

If $l \neq k_i$, then $T_{k_i-j}^i$ is a surgery time for any $j \in [s+1, l]$. Recall that by assumption, $(\sigma_j^i)^2 \geq r^2$. In this case, from (137), (141) and (143) we have

$$
\begin{aligned}
\mu(g(T_{k_i-j}^{i-}), (\sigma_j^i)^2) &\leq \mu(g(T_{k_i-j}^i), (\sigma_j^i)^2) + ck(\sigma_j^i + 1)^3 h_i^3 \\
&\leq \mu(g(T_{k_i-j}^i), (\sigma_j^i)^2) + Ckh_i^3,
\end{aligned} \tag{144}
$$

since in this case $\rho_i/\sigma_j^i \leq \rho_i/r = \delta_i \leq \epsilon_1$ if i is sufficiently large and $(\sigma_j^i)^2 \leq T'$.

Now we estimate k. On $\mathcal{M}_0^i(T_{k_i-j}^{i-})$ we can find k disjoint ϵ–tubes and each contains an ϵ–neck with center p and $R(p) = \rho_i^{-2}$. The total volume of all k tubes is at least $ck\rho_i^3$. Since all surgeries are done in a compact set K whose volume is decreasing along the flow, we have

$$k \le C\rho_i^{-3}. \tag{145}$$

Combining (144) and (145), we have

$$\mu(g(T^{i-}_{k_i-j}, (\sigma^i_j)^2) \le \mu(g(T^i_{k_i-j}), (\sigma^i_j)^2) + C\frac{h_i^3}{\rho_i^3}. \tag{146}$$

Now we take the sum from $s+1$ to l, so

$$\mu(g(T^{i-}_{k_i-l}, (\sigma^i_l)^2) \le -\epsilon_0 + Ck_i\frac{h_i^3}{\rho_i^3}. \tag{147}$$

By Theorem 6.7 the gap between two consecutive surgeries is at least $CD_i^{-1}h_i^2$, so

$$k_i \le CD_iT'h_i^{-2}. \tag{148}$$

Hence from (147),

$$\mu(g(T^{i-}_{k_i-l}), (\sigma^i_l)^2) \le -\epsilon_0 + CT'\frac{D_ih_i}{\rho_i^3}. \tag{149}$$

By our choice of parameters, ie (131), $\lim_{i\to\infty} D_ih_i/\rho_i^3 = 0$, so for i sufficiently large, $CT'D_ih_i/\rho_i^3 \le \epsilon_0/3$.

Therefore we have $\mu(g(T^{i-}_{k_i-l}), (\sigma^i_l)^2) \le -\frac{2}{3}\epsilon_0$. Again by the monotonicity formula,

$$\mu(g(T^i_{k_i-(l+1)}), (\sigma^i_{l+1})^2) \le -\tfrac{2}{3}\epsilon_0. \tag{150}$$

But this contradicts the maximality of l.

Hence l must be k_i and in this case

$$\mu(g(0), (\sigma^i_k)^2) \le -\tfrac{2}{3}\epsilon_0. \tag{151}$$

But this contradicts (138) since $(\sigma^i_{k_i})^2 \ge T$.

Thus, the proof of Theorem 7.7 is complete. □

Proof of Theorem 1.3 By Theorem 7.7 there exists a Ricci flow with surgery from (M, g) such that there are only finitely many surgeries. Since the mass is preserved along Ricci flow and surgery times, $m(g)$ is nonnegative by Theorem 1.2. If the equality holds, then by Theorem 1.2 there is no surgery and $(M, g) = (\mathbb{R}^n, g_E)$.

Corollary 7.8 [21, Corollary 6] *Any orientable AE* 3*–manifold* M *with scalar curvature* $R \geq 0$ *has diffeomorphism type*

$$M \cong \mathbb{R}^3 \# S^3/\Gamma_1 \# \cdots \# S^3/\Gamma_k \# (S^2 \times S^1) \# \cdots \# (S^2 \times S^1),$$

where there are finitely many connected sums.

Proof By Theorem 7.7, we have a Ricci flow with surgery $\mathcal{M}$ such that there are only finitely many surgeries on $\mathcal{M}_0$. After a large time T, the Ricci flow on $\mathcal{M}_0(T)$ has long-time existence, each of whose time-slice by Theorem 1.2 is diffeomorphic to $\mathbb{R}^3$. Moreover, at time T, all other finitely many components of $\mathcal{M}(T)$ are compact manifolds with $R > 0$. Therefore they must become extinct after finite time. Therefore we can recover the diffeomorphism type of M by performing the connected sum of $\mathbb{R}^3$ with finitely many S^3/Γ and $S^2 \times S^1$. □

Remark 7.9 Robert Haslhofer obtained the same result by using the min-max argument of Colding and Minicozzi [15]; see details in [21, Corollary 6].

A natural question is whether we have the same result if we only assume $g_{ij} - \delta_{ij} \in C^2_{-\sigma}$.

Appendix: Gradient Ricci solitons on ALE manifolds

In this section we prove some results about Ricci gradient solitons on ALE manifolds.

Definition A.1 A smooth Riemannian manifold (M^n, g) is called an asymptotically locally Euclidean (ALE) end of order $\sigma > 0$ if there exist a finite subgroup $\Gamma \subset O(n)$ acting freely on $\mathbb{R}^n \backslash B(0, R)$, a compact set $K \subset M^n$ and a C^∞ diffeomorphism $\Phi\colon M^n \backslash K \to (\mathbb{R}^n \backslash B(0, A))/\Gamma$ such that, under this identification,

$$g_{ij} = \delta_{ij} + O(r^{-\sigma}), \tag{152}$$

$$\partial^{|k|} g_{ij} = O(r^{-\sigma-k}), \tag{153}$$

for any partial derivatives of order k as $r \to \infty$, where r is the Euclidean distance. A complete, noncompact manifold (M^n, g) is called ALE if M^n can be written as the disjoint union of a compact set and finitely many ALE ends [5; 41]. For an ALE end, if the group Γ in the definition is trivial, we call it a *trivial end* or AE end, otherwise we call it a *nontrivial end*. As before, we assume that r is a positive function defined on the entire manifold M^n.

Definition A.2 [13, Equation (4.1)] A metric g for a manifold M^n is called a gradient Ricci soliton if there is a smooth function $f\colon M^n \to \mathbb{R}$ such that

$$\mathrm{Rc} + \mathrm{Hess}(f) + \frac{\lambda}{2} g = 0. \tag{154}$$

It is called steady when $\lambda = 0$, shrinking when $\lambda = -1$ and expanding when $\lambda = 1$.

In [20], Hamilton proved the following identity for gradient steady Ricci solitons:

$$R + |\nabla f|^2 = \Lambda, \tag{155}$$

where Λ is a constant. Since on an ALE manifold the scalar curvature satisfies $R = O(r^{-2-\sigma})$, it follows from (155) that $|\nabla f|$ is bounded. It can be proved (see for example in [13, Theorem 4.1]) that there exists an eternal solution $g(t)$, $-\infty < t < \infty$, of the Ricci flow with $g(0) = g$ such that $g(t) = \phi(t)^* g$, where $\phi(t)$ is the 1–parameter family of diffeomorphisms generated by ∇f.

Since the solution $g(t)$ is selfsimilar, its curvature operator $|\mathrm{Rm}(x,t)|$ is uniformly bounded, as $|\mathrm{Rm}(x,0)|$ is bounded for an ALE manifold. Moreover, $R \geq 0$ for every ancient complete solution of Ricci flow; see [10, Corollary 2.5]. By the strong maximum principle either $R > 0$ or M is Ricci-flat. In the first case, it implies in particular that the constant Λ in (155) is positive.

In addition, if the steady gradient Ricci soliton is nontrivial, the manifold has to be one-ended; see [30, Corollary 1.1].

Now we have:

Theorem A.3 *If (M^n, g) is an ALE manifold such that g is a gradient steady Ricci soliton, then g is Ricci-flat.*

Proof for nontrivial end If M^n is not Ricci-flat, we assume that (155) holds for a positive Λ. Moreover we assume that $|\Gamma| > 1$.

From (155), we have $|\nabla f| \leq \Lambda^{\frac{1}{2}}$ and hence f increases at most linearly. We can assume

$$|f(x)| \leq C(1 + r(x)), \tag{156}$$

where r is the function in the definition of ALE manifold.

Now if we take any sequence $r_i \to \infty$, then $(M, r_i^{-2} g)$ converges to $(\mathbb{R}^n/\Gamma, g_E)$ in the Gromov–Hausdorff sense. Moreover, the convergence is smooth away from 0 by Definition A.1. If we set $f_i = r_i^{-1} f$, it is straightforward to see from (156) that the f_i are locally uniformly bounded on $\mathbb{R}^n/\Gamma$.

Now by taking the trace of (154), we have

$$R + \Delta f = 0. \tag{157}$$

Rewriting (157) in terms of $r_i^{-2} g$ and f_i, we have

$$\Delta_{g_i} f_i = r_i^2 \Delta_g f_i = r_i \Delta_g f = -r_i R. \tag{158}$$

From the elliptic equation (158) and the fact that R decays more than quadratically, f_i converges to a function f_E in $C^\infty_{\mathrm{loc}}(\mathbb{R}^n/\Gamma - \{0\})$. Moreover,

$$\Delta_{g_E} f_E = 0. \tag{159}$$

By lifting everything from $\mathbb{R}^n/\Gamma$ to $\mathbb{R}^n$, we know that since f_E is a bounded harmonic function near 0, it must be smooth on all of $\mathbb{R}^n$; see [4, Theorem 3.9].

In addition,

$$|\nabla_{g_i} f_i|^2_{g_i} = r_i^2 |\nabla_g f_i|^2_g = |\nabla_g f|^2_g = \Lambda - R. \tag{160}$$

Hence by taking the limit we obtain

$$|\nabla_{g_E} f_E|^2 = \Lambda. \tag{161}$$

Now by (154),

$$|\mathrm{Hess}_{g_i}\, f_i|_{g_i} = r_i^2 |\mathrm{Hess}_g\, f_i|_g = r_i |\mathrm{Rc}|_g. \tag{162}$$

Therefore, by taking the limit,

$$|\mathrm{Hess}_{g_E}\, f_E|_{g_E} = 0. \tag{163}$$

By considering (161) and (163), we know that f_E must be a nontrivial linear function. But this is not possible as f_E is also defined on $\mathbb{R}^n/\Gamma$.

Proof for trivial end Assume that the ALE end E of M^n is trivial. By Theorem 3.3, we can assume for all $\tau > 0$ that $\mu(g,\tau) < 0$, since the Ricci flow solution of the steady soliton is eternal and M^n is not Ricci-flat.

For any $\overline{\tau} > 0$, by the monotonicity formula $\mu(g(t), \overline{\tau} - t)$ is increasing for all $0 \le t < \overline{\tau}$. Therefore the function

$$\mu(g(t), \overline{\tau} - t) = \mu(\phi(t)^* g, \overline{\tau} - t) = \mu(g, \overline{\tau} - t)$$

is increasing for all $0 \le t < \overline{\tau}$. As $\overline{\tau}$ can be any positive number, $\mu(g,\tau)$ is decreasing for all $\tau > 0$, contradicting Theorem 3.4. The proof of Theorem A.3 is complete. □

For a complete Ricci shrinking soliton, we have:

Theorem A.4 *If (M^n, g) is an ALE manifold such that g is a gradient-shrinking Ricci soliton, then $(M^n, g) = (\mathbb{R}^n, g_E)$.*

The proof of Theorem A.4 follows immediately from the next theorem since by the ALE condition, $|\mathrm{Rm}| \le Cr^{-2-\sigma}$.

Theorem A.5 *Let $(M^n, g(t))$, $t \in (-\infty, 0]$, be a non-flat κ–noncollapsed type I ancient solution; that is, $|\mathrm{Rm}|(x,t) \le D/(1+|t|)$ for all $t \le 0$. Then we have*

$$\limsup_{d_0(x,O)\to\infty} |\mathrm{Rm}|(x,0)d_0^2(x,O) > 0$$

for a fixed point O.

Proof Assume the contrary, and let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence of points going to infinity such that $|\mathrm{Rm}|(x_i,0)\lambda_i \to 0$, where $\lambda_i = d_0^2(x_i,O)$. Then by [8, Theorem 4.1], it follows that $(M, (1/\lambda_i)g(\lambda_i t), x_i)$ converges smoothly to a nonflat shrinking soliton $(M_\infty, g_\infty(t), x_\infty)$ for $t < 0$.

By our hypothesis and the κ–noncollapsed condition, $\big(B_{\lambda_i^{-1}g(0)}\big(x_i, \frac{1}{2}\big), \lambda_i^{-1}g(\lambda_i t), x_i\big)$ converges to $\big(B_{g_\infty(0)}\big(x_\infty, \frac{1}{2}\big), g_\infty(t), x_\infty\big)$ for $t \le 0$ by the Cheeger–Gromov compactness theorem. Then we have $\mathrm{Rm}_\infty(0) = 0$ on the metric ball $B_{g_\infty(0)}\big(x_\infty, \frac{1}{2}\big)$. Since any shrinking soliton has nonnegative scalar curvature, by the strong maximum principle $R_\infty(t) = 0$ for any $t < 0$ and hence is Ricci-flat since $\partial_t R = \Delta R + 2|\mathrm{Rc}|^2$. Then since $\mathrm{Rc}_\infty + \mathrm{Hess}(f_\infty) - \frac{1}{2}g_\infty = 0$, we have $\mathrm{Hess}(f_\infty) = \frac{1}{2}g_\infty$. Therefore $(M_\infty, g_\infty) = (\mathbb{R}^n, g_E)$ by the same argument as in Theorem 3.3.

Therefore we have a contradiction. □

Remark A.6 It was proved in [14] that $\liminf_{d(x,O)\to\infty} R(x)d^2(x,O) > 0$ for any nonflat shrinking soliton.

There are nontrivial examples of expanding solitons on ALE manifolds; see the constructions in [17].

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