From operator categories to higher operads

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We introduce the notion of an *operator category* and two different models for homotopy theory of ∞–operads over an operator category — one of which extends Lurie's theory of ∞–operads, the other of which is completely new, even in the commutative setting. We define perfect operator categories, and we describe a category $\Lambda(\Phi)$ attached to a perfect operator category Φ that provides Segal maps. We define a wreath product of operator categories and a form of the Boardman–Vogt tensor product that lies over it. We then give examples of operator categories that provide universal properties for the operads A_n and E_n ($1 \le n \le +\infty$) and also a collection of new examples.

0 Introduction

A monoid structure on a set X is the data of a product $\prod_{j \in J} x_j \in X$ of a collection of elements $\{x_j\}_{j \in J}$ indexed on a totally ordered finite set J. These multiplications are compatible with each other in the following sense: if $\phi \colon J \to I$ is an order-preserving map, then one has

$$\prod_{j \in J} x_j = \prod_{i \in I} \prod_{j \in J_i} x_j.$$

(Here J_i is the fiber of ϕ over $i \in I$.) When ϕ is the inclusion $\{1\} \hookrightarrow \{1, 2\}$ or the inclusion $\{2\} \hookrightarrow \{1, 2\}$, this expresses the existence of the (right and left) unit in X. One extracts the associativity from the consideration of the two surjective, order-preserving maps $\{1, 2, 3\} \to \{1, 2\}$. If one drops the ordering on these sets, then one has access to an involution on $\{1, 2\}$ that expresses the commutativity of the monoid structure. The suggestion, therefore, is that the category $\mathbf{O}$ of totally ordered finite sets and order-preserving maps "controls" the theory of monoids, and the category $\mathbf{F}$ of finite sets "controls" the theory of commutative monoids.

If X is instead a space (or a category, or a higher category), one may turn the identities in the description above into specified homotopies to obtain the structures observed on loopspaces and infinite loop spaces, respectively. There are (at least) two ways to make this idea precise:

- One selects a suitably cofibrant model E_1 for the unit nonsymmetric operad and a suitably cofibrant model E_∞ of the unit symmetric operad. Peter May showed that a grouplike E_1 (resp. E_∞) algebra structure on X is essentially equivalent to a single delooping (resp. an infinite delooping) structure thereupon.

- Alternatively, one considers the category Δ of nonempty totally ordered finite sets and the category Γ opposite to that of pointed finite sets. Graeme Segal observed that an E_1 algebra structure on a space X is equivalent to a functor $\Delta^{\mathrm{op}} \to \mathbf{Top}$ that carries 1 to X and satisfies the so-called Segal condition, and an E_∞ algebra structure on a space X is equivalent to a functor $\Gamma^{\mathrm{op}} \to \mathbf{Top}$ that carries 1 to X and satisfies the Segal condition.

Our motivating — but inchoate — observation is that the category $\mathbf{O}$ completely determines both the operad E_1 and Segal's theory of special Δ–spaces, while the category $\mathbf{F}$ completely determines both the operad E_∞ and Segal's theory of special Γ–spaces.

To make precise what one means by "determines", we introduce here the notion of an *operator category*. An operator category is a locally finite category that admits

a terminal object 1 and, for any object I, any morphism $i: 1 \to I$, and any map $\phi: J \to I$, a *fiber* J_i (ie a pullback of ϕ along i). Roughly speaking, an object I of an operator category can be regarded as a finite set $|I| = \mathrm{Mor}(1, I)$ equipped with a suitable additional structure, and morphisms are structure-preserving maps. Of course **O** and **F** are operator categories. Attached to any operator category Φ is a theory of Φ–*operads*. In effect, a Φ–operad P consists of the following: for each object $I \in \Phi$, a space $P(I)$, a point $e \in P(1)$ and for each morphism $\phi: J \to I$, a composition map

$$P(I) \times \prod_{i \in I} P(J_i) \to P(J).$$

These data are required to satisfy an associativity condition for any composable pair of maps $K \to J \to I$ and a condition exhibiting e as the unit. (In praxis, it is far simpler for us to work directly with a suitable theory of ∞–operads.) When $\Phi = \mathbf{O}$, one obtains the theory of nonsymmetric operads, and when $\Phi = \mathbf{F}$, one obtains the theory of symmetric operads. Of course, E_1 is a suitably cofibrant replacement of the unit nonsymmetric operad, and E_∞ is a suitably cofibrant replacement of the unit symmetric operad.

At the same time, the categories **O** and **F** themselves support canonically defined monads, which we shall denote by $T_{\mathbf{O}}$ and $T_{\mathbf{F}}$. (For this, **O** and **F** are required to enjoy an additional technical property, which we call *perfection*, but we emphasize that under this hypothesis the monad is *not* additional structure.) In effect, these monads add points to any object in all the ways one can do so functorially. So $T_{\mathbf{O}}$ adds both a new minimal point and a new maximal point to any totally ordered finite set, whereas $T_{\mathbf{F}}$ simply adds a point to a finite set. An old observation of Joyal then shows that the *Kleisli category* of the monad (ie the full subcategory of algebras spanned by the free algebras) on **O** is naturally equivalent to Δ^{op}, and it is obvious that the Kleisli category of the monad on **F** is Γ^{op}. The *Segal condition* for a functor from one of these categories to spaces ensures that $X(I)$ is equivalent to $\prod_{i \in |I|} X(\{i\})$.

For each operator category Φ, one obtains an associated theory of Φ–operads, and one can form a suitably cofibrant replacement of the unit Φ–operad, whose algebras we will just call Φ–*algebras*. (Again, we will actually work with models of ∞–operads, which permit us to dodge the delicate cofibrancy issues for strict operads.) If Φ happens to be perfect, then it supports a monad T, and the Kleisli category $\Lambda(T)$ of T contains a class of Segal maps χ_i, and we obtain an equivalence of homotopy theories between Φ–algebras in spaces and functors $\Lambda(\Phi) \to \mathbf{Top}$ that satisfy the Segal condition as above.

All of this is determined the instant one has a tiny quantity of combinatorial data — the operator category Φ. Moreover, the axioms for an operator category are so invitingly uncomplicated that one cannot help constructing other examples. The surprise — the biggest of this paper — is that many operads arise in this manner.

For example, there is a wreath product of operator categories, which we describe in Section 3: if Φ and Ψ are operator categories, then $\Psi \wr \Phi$ is the category whose objects are pairs (M_I, I) consisting of an object $I \in \Phi$ and an object $M_I \in \Psi^{\times |I|}$. (Morphisms are defined in the obvious way.) If Φ and Ψ are perfect, then so is $\Psi \wr \Phi$. We also construct an *external* Boardman–Vogt [8] tensor product, which takes a Ψ–operad Q and a Φ–operad P and constructs a $(\Psi \wr \Phi)$–operad $Q \otimes P$. In effect, $(Q \otimes P)$–algebras are Q–algebras in P–algebras. At the same time, we show that if P is the unit Φ–operad and Q is the unit Ψ–operad, then $Q \otimes P$ is the unit $(\Psi \wr \Phi)$–operad.

So if we form the iterated wreath product

$$\mathbf{O}^{(n)} := \mathbf{O} \wr \mathbf{O} \wr \cdots \wr \mathbf{O},$$

then we get a perfect operator category. One sees readily that the theory of (strict) $\mathbf{O}^{(n)}$–operads coincides with Michael Batanin's theory of $(n-1)$–terminal n–operads [5]. Using our theory of ∞–operads, we prove the following.

Theorem *The homotopy theory of $\mathbf{O}^{(n)}$–algebras is equivalent to that of E_n–algebras, and the Kleisli category of the monad on $\mathbf{O}^{(n)}$ is precisely Joyal's disk category Θ_n^{op}.*

This can be regarded as an ∞–categorical analogue of Batanin's main result [5]; the flexibility of the present context allows for a dramatically simpler and more conceptual proof. Moreover, since the tensor product of unit operads is again a unit operad, this yields a description of $E_{k_1 + \cdots + k_m}$ as an iterated Boardman–Vogt tensor product $E_{k_1} \otimes E_{k_2} \otimes \cdots \otimes E_{k_m}$. This is not in itself new. In 1988, Gerald Dunn proved a very strict version of this fact using ordinary operads, and Michael Brinkmeier extended this result in 2000. Later, Zbigniew Fiedorowicz and Rainer Vogt proved a more general additivity theorem, stating that any cofibrant E_k–operad tensored with any cofibrant E_ℓ–operad is $E_{k+\ell}$. However, because the Boardman–Vogt tensor product of strict operads does not preserve weak equivalences, these results are homotopically delicate (and some early attempts to prove a result of this kind appear to be incorrect). For this reason, Jacob Lurie's approach via ∞–operads is far better adapted to our work, and our expression of additivity using $\mathbf{O}^{(n)}$–operads is completely combinatorial.

Another, even simpler, construction of operator categories takes an operator category Φ

and forms the full subcategory $\Phi_{\leq m}$ spanned by those objects I such that $|I|$ is of cardinality $\leq m$. This is again an operator category, but it is not perfect, so only the operad story is accessible. A $\Phi_{\leq m}$–algebra is then much like a Φ–algebra, except that one has only operations of arity $\leq m$. Consequently, one sees that $\mathbf{O}_{\leq m}$–algebras are precisely A_m–algebras. We do not know a standard name for $\mathbf{F}_{\leq m}$–algebras; we write F_m for the operad freely generated by the operations of the E_∞–operad of arity $\leq m$, so that $\mathbf{F}_{\leq m}$–algebras are F_m–algebras. More generally still, one has an orbital category $\mathbf{O}^{(n)}_{\leq m}$, providing a bifiltration of the E_∞–operad (Example 11.5).

Here is a table summarizing the situation:

Φ	Φ–algebras in V	perfect?	$\Lambda(\Phi)$
$\{1\}$	$\mathbf{Alg}_{E_0}(V)$	yes	$\{1\}$
$\mathbf{O}$	$\mathbf{Alg}_{E_1}(V)$	yes	Δ^{op}
$\mathbf{O}^{(n)}$	$\mathbf{Alg}_{E_n}(V)$	yes	Θ_n^{op}
$\mathbf{F}$	$\mathbf{Alg}_{E_\infty}(V)$	yes	Γ^{op}
$\mathbf{O}_{\leq n}$	$\mathbf{Alg}_{A_n}(V)$	no	
$\mathbf{F}_{\leq n}$	$\mathbf{Alg}_{F_n}(V)$	no	

We introduce the 2–category of operator categories in Section 1. Given an operator category Φ, we develop the homotopy theory of weak operads in two ways. The first and simplest of these models (Section 2) is as suitable families of spaces over the nerve of a tree-like category $\Delta_\Phi^{\mathrm{op}}$ of finite sequences of objects of Φ that is related to a version of the dendroidal category with level structure when $\Phi = \mathbf{F}$; this homotopy theory — that of *complete Segal Φ–operads* — can be described for any operator category and is easy to describe without many additional combinatorial complications. The second, a generalization of Lurie's theory of ∞–operads, requires more technology, and will come later (Section 7).

To prove this, we pass to a different way of describing weak operads and their algebras, which will only work for a special class of operator categories that we call *perfect* (Section 4). In effect, a perfect operator category Φ admits a universal way of adding elements to any object. This defines a monad on Φ (Section 5), and the free algebras for this monad form a supplementary category $\Lambda(\Phi)$, called the *Leinster category* (Section 6), which provides an alternative way of parametrizing the operations of an operad over Φ.

Our second model of the homotopy theory of weak operads over a perfect operator category — that of Φ–*quasioperads* — is a natural generalization of Lurie's theory of

∞–operads (Section 7). In effect, a Φ–quasioperad is an inner fibration $X^{\otimes} \to N\Lambda(\Phi)$ enjoying certain properties analogous to the ones developed by Lurie. We show that this and our original model of the homotopy theory of weak operads over Φ are equivalent in a manner compatible with changes of operator category and the formation of ∞–categories of algebras (Section 10).

There is a monoidal structure on the ∞–category of all operator categories called the *wreath product*, which we describe in Section 3. An object of the wreath product $\Psi \wr \Phi$ is a pair consisting of an object $I \in \Phi$ and a collection of objects $\{J_i \mid i \in I\}$ indexed by the elements of I. The Boardman–Vogt tensor product of operads is externalized relative to this wreath product in Section 8, and we deduce our identification of $\mathbf{O}^{(n)}$–algebras with E_n–algebras.

An apology The results of this paper largely date from 2005–2006. Since that time, there has been a series of dramatic advances in the study of homotopy coherent algebraic structures, spearheaded by Moerdijk–Weiss and Lurie. As soon as I managed to record some of these results, the techniques I employed had, discouragingly, become outmoded. It was some time before it became clear to me how the work here interacts with some of these new advances — particularly with Lurie's framework. Ultimately, these advances have simplified the work here greatly; the main result of this paper is now an immediate consequence of the comparison between our theory of weak operads and Lurie's. Nevertheless, my efforts to bring this work in line with current technology have led to an embarrassing delay in the publication of this work. I apologize for this, especially to the students who have sought to employ aspects of the theory introduced here.

1 Operator categories

The objects of an operator category are finite sets equipped with some additional structure. Such an object will be regarded as an indexing set for some multiplication law. The structure of the operator category can thus be thought of determining the associativity and commutativity constraints on that law.

1.1 Notation For any ordinary category Φ with a terminal object 1 and for any object $K \in \Phi$, we write $|K| := \mathrm{Mor}_\Phi(1, K)$. For any $i \in |K|$, it will be convenient to denote the morphism $i \colon 1 \to K$ as $\{i\} \hookrightarrow K$. We call the elements of $|K|$ *points of K*, but a more familiar name (to some) might be *global element*.

1.2 Definition An *operator category* Φ is an essentially small category that satisfies the following three conditions:

(1.2.1) The category Φ has a terminal object.

(1.2.2) For any morphism $J \to I$ of Φ and for any point $i \in |I|$, there exists a fiber $J_i := \{i\} \times_I J$.

(1.2.3) For any pair of objects $I, J \in \Phi$, the set $\mathrm{Mor}_\Phi(J, I)$ is finite.

1.3 A note on terminology The notion we have defined here is distinct from the notion of a *category of operators* used in the brilliant work of May and Thomason [14], and it serves a distinct mathematical role. We hope that the obvious similarity in nomenclature will not lead to confusion.

There are very many interesting examples of operator categories, but for now, let us focus on a small number of these.

1.4 Example The following categories are operator categories:

(1.4.1) the trivial category $\{1\}$;

(1.4.2) the category **O** of ordered finite sets; and

(1.4.3) the category **F** of finite sets.

1.5 Example For any operator category Φ and for any integer $n \geq 1$, write $\Phi_{\leq n}$ for the full subcategory of Φ spanned by those objects $I \in \Phi$ such that $\#|I| \leq n$. Then the category $\Phi_{\leq n}$ is an operator category as well.

1.6 Example Suppose Ψ and Φ are two operator categories. Then we may define a category $\Psi \wr \Phi$ as follows. An object of $\Psi \wr \Phi$ will be a pair (I, M) consisting of

an object $I \in \Phi$ and a collection $M = \{M_i\}_{i \in |I|}$ of objects of Ψ, indexed by the points of I. A morphism $(\eta, \omega): (J, N) \to (I, M)$ of $\Psi \wr \Phi$ consists of a morphism $\eta: J \to I$ of Φ and a collection

$$\{\omega_j: N_j \to M_{\eta(j)}\}_{j \in |J|}$$

of morphisms of Ψ, indexed by the points of J. Then $\Psi \wr \Phi$ is an operator category. In [Section 3](#) we will give a more systematic discussion of this story, and we will show that this *wreath product* of operator categories in fact determines a monoidal structure on the collection of all operator categories.

1.7 Example Suppose Φ is any operator category. Then we may define a *semidirect product* category $\Phi \rtimes \mathbf{O}$ as follows. An object of $\Phi \rtimes \mathbf{O}$ will be a pair (I, M) consisting of a totally ordered finite set $I \in \mathbf{O}$ and a functor $M: I \to \Phi$. A morphism $(\eta, \omega): (J, N) \to (I, M)$ of $\Phi \rtimes \mathbf{O}$ consists of a morphism $\eta: J \to I$ of $\mathbf{O}$ and a natural transformation

$$\omega: N \to \eta^\star M$$

of functors $J \to \Phi$. Then $\Phi \rtimes \mathbf{O}$ is an operator category.

1.8 Example Denote by $\mathbf{C}$ the category of finite cyclically ordered sets and monotone maps, ie maps $\phi: J \to I$ such that for any $r, s, t \in J$, if $[\phi(r), \phi(s), \phi(t)]$ in I, then $[r, s, t]$ in J. Then $\mathbf{C}$ is an operator category.

1.9 Example Denote by $\mathbf{G}$ the category of finite simple graphs. Then $\mathbf{G}$ is an operator category.

1.10 Definition A functor $G: \Psi \to \Phi$ between operator categories will be said to be *admissible* if it preserves terminal objects and the formation of fibers. An admissible functor $G: \Psi \to \Phi$ will be said to be an *operator morphism* if in addition, for any object I of Ψ, the induced morphism $|I| \to |GI|$ is a surjection.

1.11 Example (1.11.1) Any equivalence $\Psi \xrightarrow{\sim} \Phi$ between operator categories is an operator morphism.

(1.11.2) For any operator category Φ, the assignment $I \mapsto |I|$ is an operator morphism $u: \Phi \to \mathbf{F}$.

(1.11.3) For any operator category Φ, the inclusion $\{1\} \hookrightarrow \Phi$ of the terminal object is an operator morphism.

(1.11.4) For any operator category Φ and for any positive integer n, the inclusion $\Phi_{\leq n} \hookrightarrow \Phi$ is an operator morphism.

(1.11.5) For any operator category Φ, the unique functor $\Phi \to \{1\}$ is an admissible functor, but it is generally not an operator morphism.

1.12 Proposition *Suppose $G\colon \Psi \to \Phi$ is an operator morphism. Then for any object I of Ψ, the induced map $|I| \to |GI|$ is a bijection.*

Proof Two points $i, j \in |I|$ are distinct if and only if the underlying set $|j^{-1}\{i\}| \cong |j|^{-1}\{i\}$ of the fiber of one point along the other is empty — or, equivalently, if and only if the intersection $i \cap j$ has no points, that is, $|i \cap j|$ is empty. Since G and $I \mapsto |I|$ are admissible, one also has a bijection $|G(j^{-1}\{i\})| \cong |G(j)|^{-1}\{G(i)\}$. Since G is an operator morphism, the map

$$|j^{-1}\{i\}| \to |G(j^{-1}\{i\})| \cong |G(j)|^{-1}\{G(i)\}$$

is surjective, so if the source is empty, then the target is as well. □

1.12.1 Corollary *Suppose*
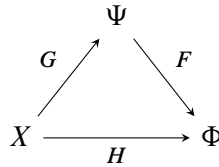
is a commutative triangle of admissible functors in which F is an operator morphism; then G is an operator morphism if and only if H is.

We organize the collection of operator categories into an ∞–category.

1.13 Notation Denote by Adm the (strict) 2–category in which the objects are small operator categories, the 1–morphisms are admissible functors, and the 2–morphisms are isomorphisms of functors. Denote by Op the sub-2–category of Adm in which the objects are small operator categories, the 1–morphisms are operator morphisms, and the 2–morphisms are isomorphisms of functors.

Applying the nerve to each Mor–groupoid in Adm and Op, we obtain categories enriched in fibrant simplicial sets, and we may apply the simplicial nerve to obtain ∞–categories that are 2–categories in the sense of [12, Section 2.3.4] (which could perhaps more precisely be called "$(2, 1)$–categories"). We will refer to these ∞–categories as **Adm** and **Op**.

As a result of Proposition 1.12, we have the following.

1.14 Proposition *The trivial operator category* $\{1\}$ *is initial in both* **Adm** *and* **Op**, *and it is terminal in* **Adm**. *The operator category* **F** *is terminal in* **Op**.

2 Complete Segal operads

Any operator category gives rise to a theory of operads (elsewhere called a colored operad or multicategory). Here we define a *weak* version of this theory, as well as its theory of algebras. To this end, we first single out an important class of morphisms of an operator category.

2.1 Definition A morphism $K \to J$ of an operator category Φ is a *fiber inclusion* if there exists a morphism $J \to I$ and a point $i \in |I|$ such that the square

$$
\begin{array}{ccc}
K & \longrightarrow & J \\
\downarrow & & \downarrow \\
\{i\} & \longrightarrow & I
\end{array}
$$

is a pullback square. A morphism $K \to J$ is an *interval inclusion* if it can be written as the composite of a finite sequence of fiber inclusions. An interval inclusion will be denoted by a hooked arrow $K \hookrightarrow J$.

2.2 Example (2.2.1) Any isomorphism of an operator category Φ is a fiber inclusion, and any point of any object of an operator category Φ is a fiber inclusion.

(2.2.2) Suppose that I is an object of **O**. Then for any elements $i_0, i_1 \in |I|$, write $[i_0, i_1]$ for the subset of elements $i \in |I|$ such that $i_0 \leq i \leq i_1$, with the induced ordering. A morphism $J \to I$ is a fiber inclusion if and only if it is a monomorphism whose image is precisely $[i_0, i_1]$ for some elements $i_0, i_1 \in |I|$. Interval inclusions and fibers inclusions coincide in **O**.

(2.2.3) In the category **F**, fiber inclusions are precisely monomorphisms. Again interval inclusions and fibers inclusions coincide.

2.3 We make the following observations:

(2.3.1) Interval inclusions in an operator category are monomorphisms.

(2.3.2) If Φ is an operator category, then the pullback of any morphism $L \to J$ along any interval inclusion $K \hookrightarrow J$ exists, and the canonical morphism $K \times_J L \to L$ is an interval inclusion.

(2.3.3) Suppose Φ is an operator category, $\phi\colon K \hookrightarrow J$ an interval inclusion in Φ, and $\psi\colon L \to K$ a morphism of Φ. Then ψ is an interval inclusion if and only if $\phi \circ \psi$ is.

(2.3.4) Suppose Φ is an operator category, $K \hookrightarrow J$ an interval inclusion, and $j \in |J|$ a point. Then either the fiber K_j is a terminal object or else $|K_j| = \varnothing$.

2.4 Definition Suppose Φ is an operator category. Then a Φ–*sequence* is a pair $(\mathbf{m}, I)$ consisting of an object $m \in \Delta$ and a functor $I\colon \mathbf{m} \to \Phi$. We will denote such an object by

$$[I_0 \to I_1 \to \cdots \to I_m].$$

A *morphism* $(\eta, \phi)\colon (\mathbf{n}, J) \to (\mathbf{m}, I)$ *of* Φ–*sequences* consists of a morphism of Δ $\eta\colon \mathbf{n} \to \mathbf{m}$ and a natural transformation $\phi\colon J \to I \circ \eta$ such that for any integer $0 \leq k \leq m$, the morphism $\phi_k\colon J_k \to I_{\eta(k)}$ is an interval inclusion, and for any pair of integers $0 \leq k \leq \ell \leq m$, the square

$$
\begin{array}{ccc}
J_k & \xrightarrow{\phi_k} & I_{\eta(k)} \\
\downarrow & & \downarrow \\
J_\ell & \xrightarrow[\phi_\ell]{} & I_{\eta(\ell)}
\end{array}
$$

is a pullback of Φ. Denote by Δ_Φ the category of Φ–sequences.

Any admissible functor $G\colon \Psi \to \Phi$ induces a functor $\Delta_\Psi \to \Delta_\Phi$ that is given by $(\mathbf{m}, I) \mapsto (\mathbf{m}, G \circ I)$.

2.5 Notation Suppose Φ is an operator category and $X\colon \Delta_\Phi^{\mathrm{op}} \to \mathbf{Kan}$ a functor. We study three classes of maps:

(2.5.1) For any Φ–sequence (m, I) and for any point $i \in |I_m|$, one obtains a map

$$X[I_0 \to I_1 \to \cdots \to I_m] \to X[I_{0,i} \to I_{1,i} \to \cdots \to \{i\}].$$

Consequently, one obtains a map

$$p_{(\mathbf{m}, I)}\colon X[I_0 \to I_1 \to \cdots \to I_m] \to \prod_{i \in |I_m|} X[I_{0,i} \to I_{1,i} \to \cdots \to \{i\}].$$

(2.5.2) For any Φ–sequence $(\mathbf{m}, I)$, one obtains a map

$$s_{\mathbf{m}, I}\colon X[I_0 \to \cdots \to I_m] \to X[I_0 \to I_1] \times^h_{X[I_1]} \cdots \times^h_{[I_{m-1}]} X[I_{m-1} \to I_m],$$

where the target is the homotopy fiber product.

(2.5.3) Lastly, the inclusion $\{1\} \hookrightarrow \Phi$ induces an inclusion $\Delta \cong \Delta_{\{1\}} \hookrightarrow \Delta_\Phi$; hence X restricts to a simplicial space $(X | \Delta^{\mathrm{op}})$, and one can define the map

$$r \colon (X | \Delta^{\mathrm{op}})_0 \to (X | \Delta^{\mathrm{op}})_K,$$

where $K = \Delta^3 / (\Delta^{\{0,2\}} \sqcup \Delta^{\{1,3\}})$, and $(X | \Delta^{\mathrm{op}})_K$ is the homotopy limit of the diagram

$$\Delta^{\mathrm{op}}_{/K} \to \Delta^{\mathrm{op}} \xrightarrow{(X | \Delta^{\mathrm{op}})} s\mathbf{Set}.$$

2.6 Definition Suppose Φ is an operator category. A *complete Segal Φ–operad* is a left fibration $q \colon X \to N\Delta^{\mathrm{op}}_\Phi$ such that any functor $\Delta^{\mathrm{op}}_\Phi \to \mathbf{Kan}$ that classifies q has the property that the maps $p_{(\mathbf{m},I)}$, $s_{(\mathbf{m},I)}$, and r (Notation 2.5) are all equivalences. A *morphism $g \colon X \to Y$ of complete Segal Φ–operads* is a commutative diagram:

A morphism $g \colon X \to Y$ will be said to be an *equivalence of complete Segal Φ–operads* if it is a covariant weak equivalence.

Denote by $\mathbf{Operad}^{\Phi,\Delta}_{\mathrm{CSS}}$ the full simplicial subcategory of $s\mathbf{Set}_{/N\Delta^{\mathrm{op}}_\Phi}$ spanned by the complete Segal Φ–operads. This is a fibrant simplicial category, so we can define an ∞–category $\mathbf{Operad}^\Phi_{\mathrm{CSS}}$ as the simplicial nerve of $\mathbf{Operad}^\Phi_{\mathrm{CSS}}$.

2.7 Suppose Φ is an operator category. We refer to the fiber of a complete Segal Φ–operad X over an object

$$[I_0 \to I_1 \to \cdots \to I_m] \in \Delta^{\mathrm{op}}_\Phi$$

as the *space of operations of type $[I_0 \to I_1 \to \cdots \to I_m]$*. The space of operations of type $\{1\}$ is the *space of objects* of X or *interior ∞–groupoid* of X, and the space of operations of type I is the *space of I–tuples of objects* of X. Let us denote by

$$[I_0/I_1/\cdots/I_m] \operatorname{Map}_X \big((x^0_{i_0})_{i_0 \in |I_0|}, \ldots, (x^m_{i_m})_{i_m \in |I_m|} \big)$$

the fiber of the map $X[I_0 \to I_1 \to \cdots \to I_m] \to X(I_0) \times \cdots \times X(I_m)$ over the vertex $((x^0_{i_0})_{i_0 \in |I_0|}, \ldots, (x^m_{i_m})_{i_m \in |I_m|})$. The condition that X is a complete Segal Φ–operad

then gives equivalences

$$[J/I/\{1\}] \operatorname{Map}_X \big((x_j)_{j \in |J|}, (y_i)_{i \in |I|}, z \big)$$

$$\simeq [I/\{1\}] \operatorname{Map}_X \big((y_i)_{i \in |I|}, z \big) \times [J/I] \operatorname{Map}_X \big((x_j)_{j \in |J|}, (y_i)_{i \in |I|} \big)$$

$$\simeq [I/\{1\}] \operatorname{Map}_X \big((y_i)_{i \in |I|}, z \big) \times \prod_{i \in |I|} [J_i/\{i\}] \operatorname{Map}_X \big((x_j)_{j \in |J_i|}, y_i \big),$$

and the map

$$[J/I/\{1\}] \operatorname{Map}_X \big((x_j)_{j \in |J|}, (y_i)_{i \in |I|}, z \big) \to [J/\{1\}] \operatorname{Map}_X \big((x_j)_{j \in |J|}, z \big)$$

induced by the map

$$
\begin{array}{ccc}
J & \longrightarrow & \{1\} \\
\| & & \| \\
J \longrightarrow I & \longrightarrow & \{1\}
\end{array}
$$

amounts to a *polycomposition map*, which is defined up to coherent homotopy. The functoriality in $N\Delta_\Phi^{\mathrm{op}}$ amounts to a coherent associativity condition.

We immediately obtain the following characterization of equivalences between complete Segal Φ–operads.

2.8 Proposition *Suppose Φ is an operator category. Then a morphism $g \colon X \to Y$ of complete Segal Φ–operads is an equivalence if and only if these conditions are satisfied:*

(2.8.1) **Essential surjectivity** *The map $\pi_0 X[\{1\}] \to \pi_0 Y[\{1\}]$ is surjective.*

(2.8.2) **Full faithfulness** *For any object $I \in \Phi$, any vertex $x \in X[I]_0$, and any vertex $y \in X[\{1\}]_0$, the induced map*

$$[I/\{1\}] \operatorname{Map}_X(x, y) \to [I/\{1\}] \operatorname{Map}_Y(g(x), g(y))$$

is an equivalence.

2.9 Example For any operator category Φ, the identity functor on the simplicial set $N\Delta_\Phi^{\mathrm{op}}$ is a complete Segal Φ–operad — the *terminal complete Segal Φ–operad*, which we denote by U_Φ. These complete Segal Φ–operads, for suitable choice of Φ, give rise to all the operads discussed in the introduction. In particular, when $\Phi = \mathbf{F}$, we show that the terminal complete Segal $\mathbf{F}$–operad $U_\mathbf{F}$ is equivalent to the operad E_∞ (Section 11).

2.10 Example When $\Phi = \{1\}$, we find that a complete Segal $\{1\}$–operad is a left fibration $X \to N\Delta^{\mathrm{op}}$ classified by a complete Segal *space* in the sense of Rezk [16].

(In particular, **Operad**$_{\text{CSS}}^{\{1\}}$ is a homotopy theory of $(\infty, 1)$–categories, in the sense of [4].) For simplicity, we will, by a small abuse, call such left fibrations *complete Segal spaces*.

2.11 Example   When $\Phi = \mathbf{F}$, we obtain a new homotopy theory of weak symmetric operads, which we will show is equivalent to Lurie's in Section 10.

Using the left Bousfield localization [9; 2] of the covariant model structure [12, Proposition 2.1.4.7] with respect to the set S_Φ of those morphisms that represent the morphisms in Notation 2.5, one obtains the following.

2.12 Proposition   *The category $s\mathbf{Set}_{/N\Delta_\Phi^{\mathrm{op}}}$ admits a left proper, tractable, simplicial model structure — called the **operadic model structure** — with these properties:*

(2.12.1)   *A map $X \to Y$ over $N\Delta_\Phi^{\mathrm{op}}$ is a cofibration if and only if it is a monomorphism.*

(2.12.2)   *An object $X \to N\Delta_\Phi^{\mathrm{op}}$ is fibrant if and only if it is a complete Segal Φ–operad.*

(2.12.3)   *A map $X \to Y$ of simplicial sets over $N\Delta_\Phi^{\mathrm{op}}$ is a weak equivalence if and only if for any complete Segal Φ–operad Z, the induced map*

$$\mathrm{Map}_{/N\Delta_\Phi^{\mathrm{op}}}(Y, Z) \to \mathrm{Map}_{/N\Delta_\Phi^{\mathrm{op}}}(X, Z)$$

is a weak equivalence.

(2.12.4)   *A map $X \to Y$ between complete Segal Φ–operads is a weak equivalence if and only if it is an equivalence of complete Segal Φ–operads.*

2.12.1 Corollary   *The ∞–category $\mathbf{Operad}_{\text{CSS}}^{\Phi}$ of complete Segal Φ–operads is an accessible localization of the functor ∞–category $\mathrm{Fun}(N\Delta_\Phi^{\mathrm{op}}, \mathbf{Kan})$; in particular, it is a presentable ∞–category.*

Critically, the homotopy theory of weak operads over operator categories is functorial with respect to operator morphisms.

2.13 Proposition   *For any operator morphism $G\colon \Psi \to \Phi$, the adjunction*

$$G_!\colon s\mathbf{Set}_{/N\Delta_\Psi^{\mathrm{op}}} \rightleftarrows s\mathbf{Set}_{/N\Delta_\Phi^{\mathrm{op}}} \colon G^\star$$

is a Quillen adjunction for the operadic model structure.

Proof   It is enough to note that the functor $G^\star$ is a right adjoint for the covariant model structure, and a left fibration $X \to N\Delta_\Phi^{\mathrm{op}}$ is a complete Segal Φ–operad only if $G^\star X \cong X \times_{N\Delta_\Phi^{\mathrm{op}}} N\Delta_\Psi^{\mathrm{op}} \to N\Delta_\Psi^{\mathrm{op}}$ is a complete Segal Ψ–operad. $\square$

2.13.1 Corollary *An operator morphism* $G\colon \Psi \to \Phi$ *induces an adjunction*

$$G_!\colon \mathbf{Operad}_{\mathrm{CSS}}^{\Psi} \rightleftarrows \mathbf{Operad}_{\mathrm{CSS}}^{\Phi} \colon G^{\star}$$

of ∞*–categories.*

2.14 Example For any operator category Ψ, the right adjoint

$$p^{\star}\colon \mathbf{Operad}^{\Psi} \to \mathbf{Operad}^{\{1\}}$$

induced by the inclusion $p\colon \{1\} \hookrightarrow \Psi$ carries any complete Segal Ψ–operad to its *underlying complete Segal space.*

2.15 Example When $\Phi = \mathbf{F}$, the notion of a complete Segal Φ–operad is closely related to the usual notion of a symmetric operad in simplicial sets. For any operator category Ψ, the right adjoint $u^{\star}\colon \mathbf{Operad}_{\mathrm{CSS}}^{\mathbf{F}} \to \mathbf{Operad}_{\mathrm{CSS}}^{\Psi}$ induced by the essentially unique operator morphism $u\colon \Psi \to \mathbf{F}$ carries any complete Segal $\mathbf{F}$–operad to its *underlying complete Segal Ψ–operad.* The left adjoint $\mathbf{Symm} := u_!$ carries any complete Segal Ψ–operad to its *symmetrization.*

2.16 Example When $\Psi = \mathbf{O}$, we show below that the symmetrization of the terminal complete Segal $\mathbf{O}$–operad $U_{\mathbf{O}}$ is equivalent to the operad E_1, or, equivalently, the operad A_{∞}. When $\Psi = \mathbf{O}_{\leq n}$ for some integer $n \geq 1$, we show below that the symmetrization of the terminal complete Segal $\mathbf{O}_{\leq n}$–operad $U_{\mathbf{O}_{\leq n}}$ is equivalent to the operad A_n (Section 11).

We note that it will sometimes be convenient to work with a straightened variant of the operadic model structure of Proposition 2.12, which is provided by the following left Bousfield localization of the injective model structure.

2.17 Proposition *The category* $\mathrm{Fun}(\Delta_{\Phi}^{\mathrm{op}}, s\mathbf{Set})$ *admits a left proper tractable model structure — called the **operadic model structure** — with the following properties:*

(2.17.1) *A natural transformation* $X \to Y$ *is a cofibration if and only if it is a monomorphism.*

(2.17.2) *An object X is fibrant if and only if it is valued in Kan complexes, and it classifies a complete Segal Φ–operad.*

(2.17.3) *A natural transformation* $X \to Y$ *is a weak equivalence if and only if for any fibrant object Z, the induced map*

$$\mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

is a weak equivalence.

2.18 For any operator category Φ, it is clear that the straightening/unstraightening Quillen equivalence of [12, Section 2.2.1] localizes to a Quillen equivalence

$$\text{St: } s\mathbf{Set}_{/N\Delta^{\mathrm{op}}} \rightleftarrows \text{Fun}(\Delta_{\Phi}^{\mathrm{op}}, s\mathbf{Set}) \text{ :Un}$$

between the operadic model structures on $\text{Fun}(\Delta_{\Phi}^{\mathrm{op}}, s\mathbf{Set})$ and $s\mathbf{Set}_{/N\Delta_{\Phi}^{\mathrm{op}}}$.

3 Wreath products

Roughly speaking, if the objects of an operator category index a certain sort of multiplications, then the objects of a wreath product of two operator categories Φ and Ψ index Φ–multiplications in objects that already possess Ψ–multiplications. This provides a wealth of new examples of operator categories, and with this insight, we can introduce the operator categories $\mathbf{O}^{(n)}$ ($1 \leq n < \infty$), which are key to the combinatorial gadgets that characterize the operads E_n.

3.1 Definition Suppose Φ is an operator category. Then a *coronal fibration*

$$p\colon X \to N\Phi$$

is a cartesian fibration such that, for any object $I \in \Phi$, the functors

$$\{X_I \to X_{\{i\}} \mid i \in |I|\}$$

together exhibit the fiber X_I as a product of the fibers $X_{\{i\}}$. In this situation, p will be said to *exhibit X as a wreath product of X_1 with Φ*.

3.2 Notation Denote by $\mathbb{O}^{\mathrm{cart}}(\mathbf{Cat}_\infty)$ the subcategory [12, Section 1.2.11] of the ∞–category

$$\mathbb{O}(\mathbf{Cat}_\infty) := \text{Fun}(\Delta^1, \mathbf{Cat}_\infty)$$

whose objects are cartesian fibrations and whose morphisms carry cartesian morphisms to cartesian morphisms. Let $\mathbf{Cat}_{\infty/S}^{\mathrm{cart}}$ denote the fiber of the target functor

$$t\colon \mathbb{O}^{\mathrm{cart}}(\mathbf{Cat}_\infty) \to \mathbf{Cat}_\infty$$

over an object $\{S\} \subset \mathbf{Cat}_\infty$. Note that $\mathbf{Cat}_{\infty/S}^{\mathrm{cart}}$ may be identified with the nerve of the cartesian simplicial model category of marked simplicial sets over S [12, Proposition 3.1.3.7], whence it is the relative nerve of the category of cartesian fibrations over S, equipped with the cartesian equivalences.

Now for any operator category Φ, denote by

$$\mathbf{Cor}_{/N\Phi} \subset \mathbf{Cat}_{\infty,/N\Phi}^{\mathrm{cart}}$$

the full subcategory spanned by the coronal fibrations.

3.3 Lemma *Suppose Φ is an operator category with a terminal object 1. Then the functor $\mathbf{Cor}_{/N\Phi} \to \mathbf{Cat}_\infty$ given by the extraction $X \mapsto X_1$ of the fiber over 1 is a trivial fibration.*

Proof It is a direct consequence of straightening [12, Section 3.2] that the ∞–category $\mathbf{Cat}_{\infty,/N\Phi}^{\mathrm{cart}}$ is equivalent to the functor ∞–category $\mathrm{Fun}(N\Phi^{\mathrm{op}}, \mathbf{Cat}_\infty)$, and the functor $\mathbf{Cat}_{\infty,/N\Phi}^{\mathrm{cart}} \to \mathbf{Cat}_\infty$ given by the assignment $X \mapsto X_1$ corresponds to evaluation at $1 \in N\Phi^{\mathrm{op}}$. Now the result follows from the observation that a cartesian fibration $p\colon X \to N\Phi$ is a coronal fibration if and only if the corresponding functor $F_p\colon N\Phi^{\mathrm{op}} \to \mathbf{Cat}_\infty$ is a right Kan extension of the restriction of F_p to $\{1\}$. $\qquad\square$

3.4 Notation For any ∞–category C and any operator category Φ, we write $C \wr \Phi$ for the total space X of a coronal fibration $p\colon X \to N\Phi$ such that C appears as the fiber over a terminal object. One can make the assignment $C \to C \wr \Phi$ into a functor by choosing a terminal object $1 \in \Phi$ and a section of the trivial fibration $\mathbf{Cor}_{/N\Phi} \to \mathbf{Cat}_\infty$ informally described as the assignment $X \mapsto X_1$. The space of these choices is contractible.

3.5 Lemma *If $F\colon \Psi \to \Phi$ is an operator morphism, then the pullback*

$$F^\star p\colon X \times_{N\Phi} N\Psi \to N\Psi$$

of a coronal fibration $p\colon X \to N\Phi$ is a coronal fibration.

Proof Suppose $I \in \Psi$. The maps $(X \times_{N\Phi} N\Psi)_I \to (X \times_{N\Phi} N\Psi)_{\{i\}}$ for $i \in |I|$ can be identified with the maps $X_{FI} \to X_{F\{i\}}$. Since the map $|I| \to |FI|$ is a bijection, these maps exhibit the fiber $(X \times_{N\Phi} N\Psi)_I$ as a product of the fibers $(X \times_{N\Phi} N\Psi)_{\{i\}}$. $\quad\square$

This lemma, combined with the previous one, implies that pullback along an operator morphism $F\colon \Psi \to \Phi$ induces an equivalence of ∞–categories

$$\mathbf{Cor}_{/N\Phi} \xrightarrow{\;\sim\;} \mathbf{Cor}_{/N\Psi};$$

they also imply that the whole ∞–category of coronal fibrations is equivalent to the ∞–category $\mathbf{Op} \times \mathbf{Cat}_\infty$. We will return momentarily to a special case of this observation.

3.6 Lemma *Suppose Φ is an operator category, and suppose $p\colon X \to N\Phi$ is a coronal fibration such that the fiber X_1 is the nerve of an operator category. Then X itself is also the nerve of an operator category, and p is the nerve of an admissible functor.*

Proof Since X_1 is a 1–category, every fiber $X_I \simeq X_1^{\times|I|}$ is a 1–category. It follows that X itself is a 1–category. The fact that X has a terminal object and all fibers follows directly from [12, Proposition 4.3.1.10]. $\square$

Informally, we conclude that the wreath product of two operator categories is again an operator category.

3.7 Lemma *Suppose Φ and Φ' are operator categories, and suppose $p\colon \Phi' \to \Phi$ is an admissible functor whose nerve is a coronal fibration. Then for any operator morphism $F\colon \Psi \to \Phi$ and any pullback diagram*

$$
\begin{array}{ccc}
\Psi' & \xrightarrow{\ F'\ } & \Phi' \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\
\Psi & \xrightarrow[\ F\ ]{} & \Phi
\end{array}
$$

the functor q is admissible, and the functor F' is an operator morphism.

Proof It is a simple matter to see that q and F' are admissible functors. To check that F' is an operator morphism, let us note first that F' induces an equivalence $\Psi'_* \xrightarrow{\sim} \Phi'_*$; hence for any object $K \in \Psi'_*$, the natural map $|K| \to |F'K|$ is a bijection. Now for any object $J \in \Psi$, the decomposition $\Psi'_J \simeq (\Psi'_\star)^{\times|J|}$ gives, for any object K of Ψ' lying over J, a decomposition of finite sets

$$
|K| \cong \coprod_{j \in |J|} |K_{\{j\}}|.
$$

After applying F', we similarly obtain a decomposition of finite sets

$$
|F'K| \cong \coprod_{i \in |FJ|} |(F'K)_{\{i\}}|.
$$

We thus conclude that the map $|K| \to |F'K|$ is a bijection since both the map $|J| \to |FJ|$ and all the maps $|K_{\{j\}}| \to |K_{F\{j\}}|$ are bijections. $\square$

We now set about showing that the wreath product defines a monoidal structure on the ∞–category **Op**.

3.8 Notation Denote by M the ordinary category whose objects are pairs (m, i) consisting of an integer $m \geq 0$ and an integer $0 \leq i \leq m$ and whose morphisms $(n, j) \to (m, i)$ are maps $\phi \colon \mathbf{m} \to \mathbf{n}$ of Δ such that $j \leq \phi(i)$. This category comes equipped with a natural projection $M \to \Delta^{\mathrm{op}}$.

Denote by $E(\mathbf{Adm})$ the simplicial set over $N\Delta$ specified by the following universal property. We require, for any simplicial set K and any map $\sigma \colon K \to N\Delta^{\mathrm{op}}$, a bijection

$$\mathrm{Mor}_{/N\Delta^{\mathrm{op}}}(K, E(\mathbf{Adm})^{\mathrm{op}}) \cong \mathrm{Mor}(K \times_{N\Delta^{\mathrm{op}}} N\mathrm{M}, \mathbf{Adm}^{\mathrm{op}}),$$

functorial in σ. Now by [12, Corollary 3.2.2.13], the map $E(\mathbf{Adm}) \to N\Delta$ is a cartesian fibration, and $E(\mathbf{Adm})$ is an ∞–category whose objects are pairs (m, X) consisting of an integer $m \geq 0$ and a functor $X \colon (\Delta^m)^{\mathrm{op}} \to \mathbf{Adm}$. A morphism $(\phi, g) \colon (n, Y) \to (m, X)$ can be regarded as a map $\phi \colon \mathbf{n} \to \mathbf{m}$ of Δ and an edge $g \colon Y \to \phi^{\star} X$ in $\mathrm{Fun}((\Delta^n)^{\mathrm{op}}, \mathbf{Adm})$.

Denote by $\mathbf{Op}^{\wr}$ the following subcategory of $E(\mathbf{Adm})$. The objects of $\mathbf{Op}^{\wr}$ are those pairs $(m, X) \in E(\mathbf{Adm})$ such that for any integer $1 \leq i \leq m$, the nerve of the admissible functor $X_i \to X_{i-1}$ is a coronal fibration, and the operator category X_0 is equivalent to $\{*\}$. The morphisms $(n, Y) \to (m, X)$ are those pairs (ϕ, g) such that for any integer $0 \leq i \leq n$, the admissible functor $Y_i \to X_{\phi(i)}$ is an operator morphism.

3.9 Proposition *The inner fibration $(\mathbf{Op}^{\wr})^{\mathrm{op}} \to N\Delta^{\mathrm{op}}$ is a monoidal ∞–category.*

Proof We first show that $\mathbf{Op}^{\wr} \to N\Delta$ is a cartesian fibration. Indeed, for any object (m, X) of $\mathbf{Op}^{\wr}$ and any edge $\phi \colon \mathbf{n} \to \mathbf{m}$ of $N\Delta$, there exists a morphism $Z \to \phi^{\star} X$ in which $Z \colon (\Delta^n)^{\mathrm{op}} \to \mathbf{Adm}$ is a diagram such that Z_0 is equivalent to $\{1\}$ and, for any integers $0 \leq i \leq j \leq n$, the following is a pullback square:

$$\begin{array}{ccc} Z_j & \longrightarrow & X_{\phi(j)} \\ \downarrow & & \downarrow \\ Z_i & \longrightarrow & X_{\phi(i)} \end{array}$$

It is straightforward now to check that the resulting edge $Z \to X$ is cartesian over ϕ. It now remains to show that for any integer $m \geq 0$, the natural map

$$\sigma_m \colon \mathbf{Op}^{\wr}_{\mathbf{m}} \to \prod_{i=1}^{m} \mathbf{Op}^{\wr}_{\{i-1, i\}}$$

is an equivalence of ∞–categories. This is the functor that assigns to any object (m, X) of $\mathbf{Op}^{\wr}_{\mathbf{m}}$ the tuple

$$(X(m)_1, X(m-1)_1, \ldots, X(2)_1, X(1)),$$

where $X(k)_1$ denotes the fiber of $X(k) \to X(k-1)$ over a terminal object of $X(k-1)$. It follows from Lemma 3.3 that this functor is indeed an equivalence. □

3.10 Example Of course we may form the wreath product of *any* two operator categories, but of particular import are the operator categories obtained by forming the iterated wreath product of **O** with itself:

$$\mathbf{O}^{(n)} := \mathbf{O} \wr \mathbf{O} \wr \cdots \wr \mathbf{O}.$$

As Clemens Berger has observed, the homotopy theory of complete Segal $\mathbf{O}^{(n)}$–operads is a homotopy-coherent variant of Batanin's notion of an $(n-1)$–*terminal* n–*operad* [5]. We prove below that the little n–cubes operad E_n is equivalent to the symmetrization **Symm**$(U_{\mathbf{O}^{(n)}})$ of the terminal complete Segal $\mathbf{O}^{(n)}$–operad (Section 11). This is a variant of Batanin's result in [5].

4 Perfect operator categories

The May–Thomason category of a symmetric operad [14] is a category that lives over Segal's category Γ^{op} of pointed finite sets. Lurie's theory of ∞–operads [13, Chapter 2] is built upon a generalization of this picture. How do we understand the relationship between the operator category **F** and the category Γ^{op}? For nonsymmetric operads, one has a similar May–Thomason construction over Δ^{op} [20]. How do we understand the relationship between the operator category **O** and the category Δ^{op}? Is it analogous to the relationship between the operator category **F** and the category Γ^{op}?

At first blush, the answer appears to be no: Γ^{op} is the category of pointed objects of **F**, and Δ^{op} is opposite of the category of nonempty objects of **O**. Expressed this way, Γ^{op} and Δ^{op} don't look the least bit similar. However, an insight that we inherited from Tom Leinster [11, pages 40–43] shows us how to think of them each as special cases of a general construction. This insight leads us to study a special class of operator categories, which we call *perfect*. We discuss the key properties of these operator categories here. In the next section we find that perfect operator categories admit a canonical monad, and in the section after that, we use that monad to define what we call the *Leinster category* $\Lambda(\Phi)$ of a perfect operator category, and we show that

$$\Lambda(\mathbf{F}) \simeq \Gamma^{\mathrm{op}} \quad \text{and} \quad \Lambda(\mathbf{O}) \simeq \Delta^{\mathrm{op}}.$$

These Leinster categories will be the foundation upon which our analogue of Lurie's theory of ∞–operads over operator categories is built.

The first property enjoyed by perfect operator categories is the existence of a *point classifier*. Point classifiers play a role in the theory of perfect operator categories that is in many respects analogous to the role played by the subobject classifier in topos theory.

4.1 Definition Suppose Φ is an operator category. Consider the category Φ^{cons} whose objects are *points* — that is, maps $\{i\} \to I$, which we shall denote by (I, i) — and whose morphisms $(I, i) \to (J, j)$ are pullback squares:

$$
\begin{array}{ccc}
\{i\} & \longrightarrow & I \\
\| & & \downarrow \\
\{j\} & \longrightarrow & J
\end{array}
$$

A *point classifier* for Φ is a terminal object T of Φ^{cons}.

If a point classifier in a pointed category exists, then it is essentially unique.

4.2 Proposition *Suppose Φ is an operator category, and suppose (T, t) is a point classifier for Φ. Then there is a functor $\chi\colon \Phi^{\mathrm{cons}} \to (\Phi/T)$ such that the square*

$$
\begin{array}{ccc}
\Phi^{\mathrm{cons}} & \xrightarrow{\ \chi\ } & (\Phi/T) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{fib}} \\
\{1\} & \longrightarrow & \Phi
\end{array}
$$

is a pullback square, where $\mathrm{fib}\colon (\Phi/T) \to \Phi$ *is the functor that assigns to any morphism* $I \to T$ *its fiber over* t.

Proof Compose the equivalence of categories $\Phi^{\mathrm{cons}} \xrightarrow{\sim} (\Phi^{\mathrm{cons}}/T)$ with the inclusion $(\Phi^{\mathrm{cons}}/T) \hookrightarrow (\Phi/T)$ to construct the functor χ. Now the result follows from the universal property of T. $\qquad\square$

4.3 Definition Suppose Φ is an operator category, and suppose (T, t) is a point classifier for Φ. Then for any object (I, i) of Φ_*, the unique conservative morphism $(I, i) \to (T, t)$ will be called the *classifying morphism* for i, and will be denoted by χ_i. We shall call the point $t \in |T|$ the *special point* of T, and for any morphism $I \to T$ of Φ, the fiber I_t will be called the *special fiber*. Write $\mathrm{fib}\colon (\Phi/T) \to \Phi$ for the *special fiber functor* $I \mapsto I_t$.

4.4 Example (4.4.1) The category $\{1\}$ trivially has a point classifier.

(4.4.2) The category $\mathbf{O}_*$ of pointed ordered finite sets has a point classifier, namely, $T_{\mathbf{O}} := \{0, 1, 2\}$, wherein 1 is the special point. Indeed, suppose (J, j) is a pointed ordered finite set. Then there is a morphism $\chi_j \colon J \to T_{\mathbf{O}}$ defined by the formula

$$\chi_j(k) := \begin{cases} 0 & \text{if } k < j, \\ 1 & \text{if } k = j, \\ 2 & \text{otherwise.} \end{cases}$$

The fiber of χ_j is of course the point j itself, and χ_j is moreover unique with this property.

(4.4.3) The category $\mathbf{F}_*$ of pointed finite sets has a point classifier, namely, the set $T_{\mathbf{F}} := \{0, 1\}$, wherein 1 is the special point. Indeed, this is a consequence of the observation that $T_{\mathbf{F}}$ is the subobject classifier for the topos of sets.

(4.4.4) The category $(\mathbf{O} \wr \mathbf{O})_*$ has a point classifier. This is the object

$$(T_{\mathbf{O}}, \{*, T_{\mathbf{O}}, *\}),$$

which may be pictured thus:

This trend continues: the category $(\mathbf{O}^{(n)})_*$ has a point classifier, which may be represented in $\mathbb{R}^n$ as the special point at the origin and $2n$ points at the intersection of the unit $(n-1)$–sphere and the coordinate axes.

4.5 Example Suppose $F \colon \Psi \to \Phi$ is a fully faithful operator morphism, and suppose (T, t) is an object of Ψ_* such that $F(T, t)$ is a point classifier for Φ_*. Then (T, t) is a point classifier for Ψ_* as well.

It follows that for any integer $m \geq 3$, the category $\mathbf{O}_{\leq m, *}$ has a point classifier, and for any integer $n \geq 2$, the category $\mathbf{F}_{\leq n, *}$ has a point classifier.

4.6 Definition An operator category Φ is *perfect* if these conditions are satisfied:

(4.6.1) The category Φ_* contains a point classifier (T_Φ, t_Φ).

(4.6.2) The special fiber functor fib: $(\Phi/T_\Phi) \to \Phi$ admits a right adjoint E_Φ.

One denotes the full subcategory of **Adm** (resp. of **Op**) spanned by the perfect operator categories by **Adm**$^{\text{perf}}$ (resp. **Op**$^{\text{perf}}$).

4.7 Notation As a rule, we drop the subscripts from the notation for the point classifier and the right adjoint of fib if it is clear from the context which operator category is under consideration.

If Φ is a perfect operator category, then let us abuse notation by writing T for the endofunctor of Φ obtained by composing $E\colon \Phi \to (\Phi/T_\Phi)$ with the forgetful functor $(\Phi/T_\Phi) \to \Phi$. Hence for some object I of Φ, the notation EI will denote TI along with the *structure morphism* $e_I\colon TI \to T$. This abuse is partly justified by the following observation.

4.8 Proposition *If Φ is a perfect operator category, then the structural morphism $e_1\colon T(1) \to T$ is an isomorphism.*

4.8.1 Corollary *If Φ is a perfect operator category, and I is an object of Φ, then $e_I = T(!)$, where $!\colon I \to 1$ is the canonical morphism.*

4.9 Example (4.9.1) Of course the operator category $\{1\}$ is perfect.

(4.9.2) The operator category **O** is perfect; the functor E assigns to any ordered finite set I the ordered finite set TI obtained by adding a single point at the beginning and a single point at the end, along with the unique map $e_I\colon TI \to T$ whose special fiber is precisely $I \subset TI$.

(4.9.3) The operator category **F** is perfect; the functor E assigns to any finite set I the finite set TI obtained by adding a disjoint basepoint to I, along with the unique map $e_I\colon TI \to T$ whose special fiber is precisely $I \subset TI$.

(4.9.4) For any integer $n \geq 1$, neither $\mathbf{O}_{\leq n}$ nor $\mathbf{F}_{\leq n}$ is perfect.

(4.9.5) The operator category $\mathbf{O} \wr \mathbf{O}$ is perfect. The functor $T_{\mathbf{O}\wr\mathbf{O}}$ carries an object (I, M) to an object

$$(T_{\mathbf{O}}I, \{S_\eta\}_{\eta \in |T_{\mathbf{O}}I|}),$$

where $S_\eta = 1$ if η is either of the endpoints in $T_{\mathbf{O}}I$ and $S_\eta = T_{\mathbf{O}}M_i$ if $\eta = i \in |I| \subset |T_{\mathbf{O}}I|$.

More generally, wreath products of perfect operator categories are perfect.

4.10 Proposition *Suppose Φ and Ψ are two perfect operator categories. Then the operator category $\Psi \wr \Phi$ is perfect as well.*

Proof Choose point classifiers (T_Φ, t_Φ) of Φ and (T_Ψ, t_Ψ) of Ψ and a terminal object $1 \in \Psi$. Consider an object $T_{\Psi \wr \Phi} = (T_\Phi, \{S_\eta\}_{\eta \in |T_\Phi|})$ of $\Psi \wr \Phi$ in which

$$S_\eta = \begin{cases} 1 & \text{if } \eta \neq t_\Phi, \\ T_\Psi & \text{if } \eta = t_\Phi. \end{cases}$$

Consider the point $t_{\Psi \wr \Phi} := (t_\Phi, t_\Psi) \in |T_{\Psi \wr \Phi}|$. Clearly the pair $(T_{\Psi \wr \Phi}, t_{\Psi \wr \Phi})$ is a point classifier of $\Psi \wr \Phi$.

The right adjoint $E_{\Psi \wr \Phi}$ of the special fiber functor fib is defined by carrying any object $(I, \{M_i\}_{i \in |I|})$ to the object $(T_\Phi I, \{N_j\}_{j \in |T_\Phi I|})$, where

$$N_j = \begin{cases} 1 & \text{if } j \notin |I| \subset |T_\Phi I|, \\ T_\Psi M_j & \text{if } j \in |I| \subset |T_\Phi I|, \end{cases}$$

along with the morphism

$$(T_\Phi I, \{N_j\}_{j \in |T_\Phi I|}) \to (T_\Phi, \{S_\eta\}_{\eta \in |T_\Phi|})$$

given by $e_I \colon T_\Phi I \to T_\Phi$ and, for any $j \in |T_\Phi I|$, the morphism $e_{M_j} \colon T_\Psi M_j \to T_\Psi$ when $j \in |I|$, and the identity map on 1 when $j \notin |I|$. $\qquad\square$

This allows us to restrict the monoidal structure of Proposition 3.9 to a monoidal structure on $\mathbf{Op}^{\text{perf}}$.

4.10.1 Corollary *Denote by $\mathbf{Op}^{\text{perf}, \wr}$ the full subcategory of $\mathbf{Op}^\wr$ spanned by those pairs (m, X) such that each X_i is perfect. Then the composite $(\mathbf{Op}^{\text{perf}, \wr})^{\text{op}} \to N\Delta^{\text{op}}$ is a monoidal ∞–category.*

Consequently, we obtain the following example.

4.11 Example The operator categories $\mathbf{O}^{(n)}$ are all perfect.

5 The canonical monad on a perfect operator category

A perfect operator category Φ always comes equipped with a monad. In effect, this monad adds points "in every direction" in Φ; these "directions" are indexed by the "nonspecial" points of the point classifier.

5.1 Observe that for any object I of an operator category Φ, and any point $i \in |I|$, the fiber functor $(-)_i$ is already right adjoint to the fully faithful functor $p_i \colon \Phi \to (\Phi/I)$ that sends any object J of Φ to the composite

$$J \to \{i\} \hookrightarrow I.$$

Observe that Φ thus has the structure of a localization of (Φ/I); that is, the unit $J \to p_i(J)_i$ is an isomorphism.

If Φ is perfect, there is a string of adjoints

$$\Phi \underset{E}{\overset{p_t}{\rightleftarrows}} (\Phi/T)$$

and as the following result shows, Φ is both a localization and a colocalization of (Φ/T).

5.2 Lemma *If Φ is a perfect operator $\mathbf{X}$–category, then the adjoint pair* (fib, E) *gives Φ the structure of a colocalization of (Φ/T).*

Proof The claim is simply that the counit $\kappa\colon \mathrm{fib} \circ E \to \mathrm{id}_\Phi$ is an isomorphism. The inverse to the unit $\mathrm{id}_\Phi \to \mathrm{fib} \circ p_t$ induces a morphism $\epsilon\colon p_t \to E$ of functors $\Phi \to (\Phi/T)$; this gives a morphism $\iota\colon \mathrm{id}_\Phi \to T$ of endofunctors of Φ.

Now for any object I of Φ, the resulting square

$$\begin{array}{ccc} I & \longrightarrow & TI \\ \downarrow & & \downarrow \\ \{t\} & \lhook\joinrel\longrightarrow & T \end{array}$$

is a pullback square; indeed, a commutative diagram

$$\begin{array}{ccc} I' & \longrightarrow & TI \\ \downarrow & & \downarrow \\ \{t\} & \lhook\joinrel\longrightarrow & T \end{array}$$

is the same data as a morphism $p_t I' \to EI$ of (Φ/T), which is in turn the same data as a morphism $I' \cong (p_t I')_t \to I$. $\qquad\square$

If Φ is a perfect operator category, then, in effect, the endofunctor $T\colon \Phi \to \Phi$, when applied to an object of Φ, adds as few points as possible in as many directions as possible. It turns out that this endofunctor is a monad; let us now construct a natural transformation $\mu\colon T^2 \to T$ that, together with the natural transformation $\iota\colon \mathrm{id}_\Phi \to T$ from the previous proof, will exhibit a monad structure on T.

5.3 Notation Suppose Φ is a perfect operator category. The embedding $\iota_T\colon T \hookrightarrow TT$ permits us to regard the special point $t \in |T|$ as a point of TT as well. Now consider

the classifying morphism $\chi_t \colon TT \to T$. Its special fiber is the point $\iota_T(t)$, so that the following diagram is a pullback:

$$
\begin{array}{ccc}
\{t\} \longrightarrow & T & \xrightarrow{\iota_T} TT \\
\| & & \downarrow{\scriptstyle \chi_t} \\
\{t\} \longrightarrow & & T
\end{array}
$$

5.4 Notation For any object I of a perfect operator category Φ, the functor E induces a functor

$$E_{/I} \colon (\Phi/I) \to (\Phi/TI),$$

which is right adjoint to the functor $\iota_I^{-1} := - \times_{TI} I \colon (\Phi/TI) \to (\Phi/I)$.

5.5 This feature of the endofunctor $T \colon \Phi \to \Phi$ is what makes it a *local* or *parametric right adjoint*; see [23, Definition 2.3].

5.6 Lemma *If Φ is a perfect operator category, then there is an isomorphism of functors*

$$\rho \colon \mathrm{fib} \xrightarrow{\ \sim\ } \mathrm{fib} \circ \chi_{t,!} \circ E_{/TT},$$

where $\chi_{t,!} \colon (\Phi/TT) \to (\Phi/T)$ is the functor given by composition with χ_t.

Proof For any morphism $\phi \colon J \to T$, every square of the diagram

$$
\begin{array}{ccccc}
J_t & \longrightarrow & J & \longrightarrow & TJ \\
\downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle T\phi} \\
\{t\} & \longrightarrow & T & \xrightarrow{\iota_T} & TT \\
\| & & & & \downarrow{\scriptstyle \chi_t} \\
\{t\} & \longrightarrow & & & T
\end{array}
$$

is a pullback square. $\qquad\square$

5.7 By adjunction, we obtain a natural transformation $\sigma \colon \chi_{t,!} \circ E_{/T} \to E \circ \mathrm{fib}$. If $\phi \colon I \to T$ is a morphism of Φ, then one can apply T to this morphism to obtain a morphism $T(\phi) \colon TI \to TT$. One can, alternatively, apply T to the special fiber I_t to

obtain a morphism $T(I_t) \to T$. The component σ_I then fits into a commutative square:

$$
\begin{array}{ccc}
TI & \xrightarrow{\ \sigma_J\ } & T(I_t) \\
{\scriptstyle T(\phi)}\downarrow & & \downarrow{\scriptstyle e_{I_t}} \\
TT & \xrightarrow[\ \chi_t\ ]{} & T
\end{array}
$$

Here, the special fiber of I inside TI is mapped isomorphically to the special fiber of $T(I_t)$ under σ_I.

5.8 Definition Suppose Φ is a perfect operator category. Define a morphism of endofunctors $\mu\colon T^2 \to T$ as the composition

$$T^2 = U \circ E \circ U \circ E = U \circ \chi_{t,!} \circ E_{/T} \circ E \xrightarrow{\ \mathrm{id}\circ\sigma\circ\mathrm{id}\ } U \circ E \circ \mathrm{fib} \circ E \xrightarrow{\ \mathrm{id}\circ\kappa\ } U \circ E = T,$$

where $\kappa\colon \mathrm{fib} \circ E \to \mathrm{id}_\Phi$ is the counit isomorphism (Lemma 5.2).

5.9 More explicitly, if I is an object of Φ, then one has, following 5.7, a commutative diagram:

(5.9.1)
$$
\begin{array}{ccccc}
T^2 I & \xrightarrow{\ \sigma_{TI}\ } & T((TI)_t) & \xrightarrow{\ \cong\ } & TI \\
{\scriptstyle T(e_I)}\downarrow & & & & \downarrow \\
TT & & \xrightarrow[\ \ \ \ \ \ \chi_t\ \ \ \ \ \ ]{} & & T
\end{array}
$$

The composite $T^2 I \to TI$ is the component μ_I.

5.10 Theorem *The endofunctor T on a perfect operator category Φ is a monad with unit $\iota\colon \mathrm{id}_\Phi \to T$ and multiplication $\mu\colon T^2 \to T$.*

The proof, though quite elementary, is a little involved, so we postpone it (Appendix B).

5.11 Example When $\Phi = \mathbf{F}$, the monad T is the *partial map monad*, so that the set of maps $J \to TI$ is precisely the set of partial maps $J \supseteq K \to I$. We emphasize, however, that this is not the case in general.

5.12 The *monad T* is a *local* or *parametric right adjoint* [23, Definition 2.3] as well. Work of Mark Weber [23, Proposition 2.6; 22, Proposition 5.9] shows that this implies instantly that the Kleisli category of this monad admits a factorization system, which is the inert/active factorization we describe in a more pedestrian way in Lemma 7.3 below. (See also [7, Section 2.6].)

Let us turn to the functoriality of this monad structure in admissible functors.

5.13 An admissible functor $F\colon \Psi \to \Phi$ between perfect operator categories induces a functor

$$F_{/T_\Psi}\colon (\Psi/T_\Psi) \to (\Phi/FT_\Psi),$$

and there is a unique conservative morphism $\chi_{F(t_\Psi)}\colon (FT_\Psi, F(t_\Psi)) \to (T_\Phi, t_\Phi)$ of Φ_*. So let $F_{/T}$ (with no subscript on T) denote the composite

$$(\Psi/T_\Psi) \xrightarrow{F_{/T_\Psi}} (\Phi/FT_\Psi) \xrightarrow{\chi_{F(t_\Psi),!}} (\Phi/T_\Phi).$$

We now have the following trivial observations.

5.14 Lemma *For any admissible functor $F\colon \Psi \to \Phi$ between perfect operator categories, there is a natural isomorphism $\beta_F\colon \mathrm{fib}_\Phi \circ F_{/T} \xrightarrow{\sim} F \circ \mathrm{fib}_\Psi$.*

5.15 Lemma *For any admissible functor $F\colon \Psi \to \Phi$ between perfect operator categories, the natural transformation $F \to \mathrm{fib}_\Phi \circ F_{/T} \circ E_\Psi$ corresponding to the isomorphism β_F by adjunction is itself an isomorphism.*

Adjoint to the inverse of this isomorphism is a natural transformation

$$\alpha_F\colon F_{/T} \circ E_\Psi \to E_\Phi \circ F.$$

It may be characterized as follows.

5.16 Proposition *For any object I of Ψ, there is a unique commutative square*

$$\begin{array}{ccc} FT_\Psi I & \xrightarrow{\alpha_{F,I}} & T_\Phi FI \\ \downarrow & & \downarrow \\ FT_\Psi & \xrightarrow[\chi_{F(t_\Psi)}]{} & T_\Phi \end{array}$$

of Φ whose special fiber is the square:

$$\begin{array}{ccc} FI & =\!=\!= & FI \\ \downarrow & & \downarrow \\ F\{t_\Psi\} & =\!=\!= & \{t_\Phi\} \end{array}$$

The natural transformation α_F is generally not an isomorphism, but it does behave well with respect to the monad structure.

5.17 More precisely, recall that if C and D are categories equipped with monads (T_C, ι_C, μ_C) and T_D, then a *colax morphism of monads* $(F, \eta): (C, T_C) \to (D, T_D)$ is a functor $F: C \to D$ equipped with a natural transformation $\eta: FT_C \to T_D F$ with the property that the following diagrams of $\mathrm{Fun}(C, D)$ commute:

(5.17.1)

Thus the interaction of the natural transformation α_F with the monad structures on Ψ and Φ is summarized by the following result.

5.18 Theorem *An admissible functor* $F: \Psi \to \Phi$ *of perfect operator categories induces a colax morphism of monads* $(F, \alpha_F): (\Psi, T_\Psi) \to (\Phi, T_\Phi)$.

Again we postpone the proof (Appendix C).

5.19 Observe that the uniqueness of conservative morphisms with target (T_Φ, t_Φ) implies that if $G: X \to \Psi$ is another admissible functor of perfect operator **X**–categories, then $(F \circ G)_{/T} = F_{/T} \circ G_{/T}$. Hence the assignment $\Phi \mapsto (\Phi/T)$ defines a functor **Adm**$^{\mathrm{perf}} \to$ **Cat**. On the other hand, there is a forgetful functor **Adm**$^{\mathrm{perf}} \to$ **Cat**.

Now the functor β of Lemma 5.14 can be regarded as a natural transformation from the functor $\Phi \mapsto (\Phi/T)$ to the forgetful functor **Adm**$^{\mathrm{perf}} \to$ **Cat** such that for any perfect operator category Φ, one has $\beta_\Phi = \mathrm{fib}_\Phi$. Similarly, α can be regarded as a lax natural transformation from the forgetful functor **Adm**$^{\mathrm{perf}} \to$ **Cat** to the functor $\Phi \mapsto (\Phi/T)$ such that for any perfect operator category Φ, one has $\alpha_\Phi = E_\Phi$.

5.20 A 2–morphism $\xi: (F, \eta) \to (F', \eta')$ of colax morphisms of monads is an isomorphism of functors $\xi: F \to F'$ such that the following square commutes:

$$
\begin{array}{ccc}
FT_C & \xrightarrow{\eta} & T_D F \\
\xi T_C \downarrow & & \downarrow T_D \xi \\
F'T_C & \xrightarrow{\eta} & T_D F'
\end{array}
$$

Composition and identities are defined in the obvious manner; hence this defines a 2–category $\mathrm{Mnd}_{\mathrm{colax}}$ of small categories with monads and colax functors. By applying the nerve of each Mor–groupoid and taking the simplicial nerve of the resulting simplicial category, we obtain a quasicategory $\mathbf{Mnd}_{\mathrm{colax}}$, which is in fact a 2–category in the sense of [12, Section 2.3.4].

We may summarize Theorems 5.10 and 5.18 together by stating that the assignment $\Phi \mapsto (\Phi, T)$ defines a functor $\mathbf{Adm}^{\mathrm{perf}} \to \mathbf{Mnd}_{\mathrm{colax}}$.

6 Leinster categories

The Kleisli category of the canonical monad on a perfect operator category Φ is the category of free algebras for this monad. It can be thought of as indexing both operations in Φ as well as projection maps in a coherent manner. In the examples of interest, this Kleisli category recovers a number of combinatorial categories familiar to homotopy theorists.

In particular, we'll find that the Kleisli category of the canonical monad on $\mathbf{F}$ is Segal's Γ^{op}, the Kleisli category of the canonical monad on $\mathbf{O}$ is Δ^{op}, and the Kleisli category of the canonical monad on $\mathbf{O}^{(n)}$ is Joyal's Θ_n^{op}

6.1 Definition   Suppose Φ is a perfect operator category. Then the *Leinster category* $\Lambda(\Phi)$ of Φ is the Kleisli category of the monad T_Φ.

6.2   That is, the objects of the Leinster category of a perfect operator category Φ are precisely those of Φ itself, and for any two objects I and J,

$$\mathrm{Mor}_{\Lambda(\Phi)}(J, I) := \mathrm{Mor}_\Phi(J, TI).$$

The identity at an object I is the unit $\iota_I \colon I \to TI$. The composition law is defined by the composite

$$\mathrm{Mor}_\Phi(K, TJ) \times \mathrm{Mor}_\Phi(J, TI) \xrightarrow{\mathrm{id} \times T_\Phi} \mathrm{Mor}_\Phi(K, TJ) \times \mathrm{Mor}_\Phi(TJ, T^2 I)$$
$$\xrightarrow{\circ} \mathrm{Mor}_\Phi(K, T^2 I) \xrightarrow{\mu_{I,*}} \mathrm{Mor}_\Phi(K, TI)$$

for objects I, J, and K.

6.3   Suppose now that $F \colon \Psi \to \Phi$ is an admissible functor of perfect operator categories. Since colax morphisms of monoids induce functors of their Kleisli categories, and in fact

this is a functor $\mathbf{Mnd}_{\mathrm{colax}} \to \mathbf{Cat}$, it follows that f induces a functor $\Lambda(F)\colon \Lambda(\Psi) \to \Lambda(\Phi)$ of the Leinster categories.

The admissible functor F induces a functor $\Lambda(F)\colon \Lambda(\Psi) \to \Lambda(\Phi)$ such that the map on objects is simply

$$\mathrm{Obj}\,\Lambda(F) = \mathrm{Obj}\,F\colon \mathrm{Obj}\,\Lambda(\Psi) = \mathrm{Obj}\,\Psi \to \mathrm{Obj}\,\Phi = \mathrm{Obj}\,\Lambda(\Phi),$$

and the map on morphisms is given by

$$\mathrm{Mor}_{\Psi}(J, T_{\Psi}I) \to \mathrm{Mor}_{\Phi}(FJ, FT_{\Psi}I) \xrightarrow{\alpha_{F,*}} \mathrm{Mor}_{\Phi}(FJ, T_{\Phi}FI)$$

(Proposition 5.16).

Consequently, the Leinster category construction defines a functor

$$\Lambda\colon \mathbf{Adm}^{\mathrm{perf}} \to \mathbf{Cat}.$$

6.4 Example For any object I of a perfect operator category Φ and for any point $i \in |I|$, the classifying morphism $\chi_i\colon I \to T$ of Φ is a morphism $I \to \{i\}$ of $\Lambda(\Phi)$. In particular, of course, $\{i\}$ is not terminal in $\Lambda(\Phi)$.

6.5 Example The Leinster category of $\mathbf{F}$ is Segal's category Γ^{op} of pointed finite sets [1; 18].

6.6 Example Let us study the Leinster category of $\mathbf{O}$. Denote by $\bot$ and $\top$ the two points of $T(\varnothing)$, so that

$$T(\varnothing) = \{\bot, \top\};$$

these correspond to two wide morphisms $T \to T(\varnothing)$, and for any object I of $\Lambda(\mathbf{O})$, they give rise to a map

$$c_I := (\bot_!, \top_!)\colon \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, *) \to \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \varnothing) \times \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \varnothing).$$

The map c_I is injective, so for any $\phi, \psi \in \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \varnothing)$, denote by $\phi \star \psi$ the unique element of $\mathrm{Mor}_{\Lambda(\mathbf{O})}(I, *)$ such that $c_I(\phi \star \psi) = (\phi, \psi)$, if it exists.

Then the subset $c_I(\mathrm{Mor}_{\Lambda(\mathbf{O})}(I, *)) \subset \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \varnothing) \times \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \varnothing)$ is a total ordering $\leq$ on $\mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \varnothing)$; in particular, for any $\phi, \psi \in \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \varnothing)$, the element ψ is the successor of ϕ if and only if $\phi \star \psi = \chi_i$ for some point $i \in |I|$.

The functor $\mathrm{Mor}_{\Lambda(\mathbf{O})}(-, \varnothing)$ thus defines a functor $\Lambda(\mathbf{O})^{\mathrm{op}} \to \Delta$, where Δ is defined as the full subcategory of $\mathbf{O}$ consisting of nonempty objects; a quasiinverse functor is given by

$$\mathbf{n} \mapsto \mathbf{n}^{\vee} := \mathrm{Mor}_{\Delta}(\mathbf{n}, \mathbf{1}).$$

We deduce that $\Lambda(\mathbf{O})$ is equivalent to Δ^{op}. (This observation goes back at least to Street [19].)

The following result is trivial to prove.

6.7 Proposition *Suppose* $p: \Phi' \to \Phi$ *is an admissible functor between perfect operator categories that is also a Grothendieck fibration classifying a functor* $\Phi^{\mathrm{op}} \to \mathbf{Adm}^{\mathrm{perf}}$. *Then the functor* $\Lambda(\Phi') \to \Lambda(\Phi)$ *is a Grothendieck fibration such that for any object* $I \in \Phi$, *one has* $(\Lambda(\Phi'))_I \simeq \Lambda(\Phi'_I)$.

6.8 Example The previous result now immediately implies that the Leinster category of the iterated wreath product $\mathbf{O}^{(n)}$ coincides with Berger's iterated wreath product $(\Delta \wr \Delta \wr \cdots \wr \Delta)^{\mathrm{op}}$ [6]. In particular, $\Lambda(\mathbf{O}^{(n)})$ is equivalent to Joyal's category Θ_n^{op}.

7 Quasioperads and their algebras

When Φ is a perfect operator category, the theory of complete Segal Φ–operads introduced above can also be codified in a manner similar to Lurie's theory of ∞–operads [13, Chapter 2]. Here, we explain how to generalize the basic elements of Lurie's theory to any perfect operator category; in most cases the proofs are trivial extensions of the proofs of loc. cit.

The following terminology was first introduced on Γ^{op} by Lurie.

7.1 Definition Suppose Φ is a perfect operator category. We call a morphism $J \to I$ of $\Lambda(\Phi)$ *inert* if the corresponding morphism $J \to TI$ in Φ has the property that the natural morphism

$$J \times_{TI} I \to I$$

is an isomorphism. Denote by $\Lambda^{\dagger}(\Phi)$ the collection of inert morphisms of $\Lambda(\Phi)$. We will simply write $N\Lambda(\Phi)$ for the marked simplicial set $(N\Lambda(\Phi), \Lambda^{\dagger}(\Phi))$.

Let us call a morphism $J \to I$ of $\Lambda(\Phi)$ *active* if the corresponding morphism $J \to TI$ of Φ factors as

$$J \hookrightarrow TJ \to TI,$$

where the morphism $TJ \to TI$ is of the form $T\phi$ for some morphism $\phi \colon J \to I$ of Φ. Denote by $\Lambda_{\dagger}(\Phi)$ the collection of active morphisms of $\Lambda(\Phi)$.

7.2 Suppose Φ is a perfect operator category. We observe the following:

(7.2.1) A morphism $J \to I$ of $\Lambda(\Phi)$ is active if and only if the corresponding morphism $J \to TI$ in Φ has the property that the natural morphism

$$J \times_{TI} I \to J$$

is an isomorphism.

(7.2.2) A morphism of $\Lambda(\Phi)$ is both inert and active if and only if it is an isomorphism.

Suppose $\psi \colon K \to J$ and $\phi \colon J \to I$ are morphisms of $\Lambda(\Phi)$. We further note:

(7.2.3) If ψ is inert, then $\phi \circ \psi$ is inert if and only if ϕ is.

(7.2.4) Dually, if ϕ is active, then $\phi \circ \psi$ is active if and only if ψ is.

It is an observation of Christopher Schommer-Pries that $\Lambda(\Phi)$ always admits an inert-active factorization system. In fact, Weber proves this as a natural consequence of the fact that T is a *local* or *parametric right adjoint*; see [23, Proposition 2.6] and [22, Proposition 5.9]. (See also [7, Section 2.6]. In their language, what we call "inert", Weber and others call "generic", and what we call "active", they call "free".) Berger has introduced the concept of a *moment category* that codifies the salient features of this factorization system, which we intend to explore elsewhere. For now, we simply record the observation of Schommer-Pries (with a perhaps suboptimal proof).

7.3 Lemma (Schommer-Pries) *Every morphism $J \to I$ of the Leinster category of a perfect operator category Φ admits a factorization $J \to K \to I$ into an inert morphism $J \to K$ followed by an active morphism $K \to I$. Moreover, this factorization is unique up to unique isomorphism.*

Proof Using the structural morphism $e_I \colon TI \to I$, regard the morphism $J \to TI$ as a morphism over T. Set $K := J \times_{TI} I$; the projection $K \to I$ in Φ induces an active morphism $K \to I$ in $\Lambda(\Phi)$. Now the universal property of T states that the set of maps $J \to TK$ over T is in bijection with the set of maps $J_t \to K$. Hence we

may choose the canonical isomorphism $J_t \cong K$, yielding a morphism $J \to TK$. It is obvious from the construction that this now gives the desired factorization $J \to K \to I$ of the original morphism, and the uniqueness follows from the observations above. □

7.4 Example When $\Phi = \mathbf{F}$, a pointed map $\phi: J_+ \to I_+$ of Γ^{op} corresponds to an inert morphism in our sense if and only if it is inert in the sense of Lurie, that is, if and only if the inverse image $\phi^{-1}(i)$ of any point $i \in I$ is a singleton. It corresponds to an active morphism in our sense if and only if it is active in the sense of Lurie, that is, if and only if the inverse image of the base point is a singleton.

7.5 Example When $\Phi = \mathbf{O}$, a morphism $\phi: \mathbf{m} \to \mathbf{n}$ of Δ^{op} corresponds to an inert morphism in our sense if and only if it corresponds to an injection $\mathbf{n} \to \mathbf{m}$ given by the formula $i \mapsto i + k$ for some fixed integer $k \geq 1$. It corresponds to an active morphism in our sense if and only if it corresponds to a map $\mathbf{n} \to \mathbf{m}$ that carries 0 to 0 and n to m.

7.6 Proposition *The functor* $\Lambda(\Psi) \to \Lambda(\Phi)$ *induced by an admissible functor* $F: \Psi \to \Phi$ *of perfect operator categories preserves inert morphisms.*

Proof Suppose $J \to I$ is an inert morphism of $\Lambda(\Phi)$. Consider the rectangle:

$$
\begin{array}{ccccc}
F(J \times_{TI} I) & \xrightarrow{\cong} & FI & = & FI \\
\downarrow & & \uparrow & & \downarrow \\
FJ & \longrightarrow & FTI & \xrightarrow{\alpha_{F,I}} & TFI
\end{array}
$$

The left-hand square is a pullback since F is admissible; the morphism

$$F(J \times_{TI} I) \to FI$$

is an isomorphism because $J \to I$ is inert; and the fact that right-hand square is a pullback follows from the characterization of $\alpha_{F,I}$ given in Proposition 5.16. □

7.7 Notation Suppose X is an ∞–category, S a 1–category, and $q: X \to S$ a functor. Suppose $x, y \in X$, and suppose $g: q(x) \to q(y)$ is a morphism of S. Denote by $\mathrm{Map}_X^g(x, y)$ the union of the connected components of $\mathrm{Map}_X(x, y)$ lying over the connected component of g in $\mathrm{Map}_S(q(x), q(y))$.

We can now define the notion of Φ–quasioperad in exact analogy with Lurie [13, Definition 2.1.1.10].

7.8 Definition Suppose Φ is an operator category. Then a Φ–*quasioperad* or ∞–*operad over* Φ is an inner fibration $p\colon X^{\otimes} \to N\Lambda(\Phi)$ satisfying these conditions:

(7.8.1) For every morphism $\phi\colon J \to I$ of $\Lambda^{\dagger}(\Phi)$ and every object $x \in X_J^{\otimes}$, there is a p–cocartesian edge $x \to y$ in $X^{\otimes}$ covering ϕ.

(7.8.2) For any objects $I, J \in \Phi$, any objects $x \in X_I^{\otimes}$ and $y \in X_J^{\otimes}$, any morphism $\phi\colon J \to I$ of $\Lambda(\Phi)$, and any p–cocartesian edges $\{y \to y_i \mid i \in |I|\}$ lying over the inert morphisms $\{\rho_i\colon I \to \{i\} \mid i \in |I|\}$, the induced map

$$\mathrm{Map}_{X^{\otimes}}^{\phi}(x, y) \to \prod_{i \in |I|} \mathrm{Map}_{X^{\otimes}}^{\rho_i \circ \phi}(x, y_i)$$

is an equivalence.

(7.8.3) For any object $I \in \Phi$, the p–cocartesian morphisms lying over the inert morphisms $\{I \to \{i\} \mid i \in |I|\}$ together induce an equivalence

$$X_I^{\otimes} \to \prod_{i \in |I|} X_{\{i\}}^{\otimes}.$$

7.9 Example When $\Phi = \mathbf{F}$, the above definition coincides with Lurie's definition of ∞–operad [13].

7.10 Example When $\Phi = \{1\}$, the conditions of the above definition are trivial, and we are simply left with the notion of a quasicategory.

In light of [13, Proposition 2.1.2.5], the inert-active factorization on the Leinster category of a perfect operator category lifts to any ∞–operad over it.

7.11 Definition Suppose Φ is a perfect operator category, and suppose

$$p\colon X^{\otimes} \to N\Lambda(\Phi)$$

is a Φ–quasioperad. Call a p–cocartesian edge of $X^{\otimes}$ that covers an inert morphism of $\Lambda(\Phi)$ *inert*. Dually, call any edge of $X^{\otimes}$ that covers an active morphism of $\Lambda(\Phi)$ *active*.

7.12 Proposition *For any perfect operator category and any Φ–quasioperad $X^{\otimes}$, the inert morphisms and the active morphisms determine a factorization system on $X^{\otimes}$.*

As in [13, Section 2.1.4], we can introduce a model category of ∞–preoperads over a perfect operator category Φ whose fibrant objects are ∞–operads over Φ.

7.13 Notation As in [13, Notation 2.1.4.5], for any operator category Φ and any ∞–operad $X^\otimes$ over Φ, denote by $X^{\otimes,\natural}$ the object $(X^\otimes, E)$ of $s\mathbf{Set}^+_{/N\Lambda(\Phi)}$, where E denotes the collection of inert morphisms of $X^\otimes$.

7.14 For any perfect operator category Φ, consider the *categorical pattern* (in the sense of Lurie; see [13, Appendix B])

$$\mathscr{P} = (M, T, \{p_\alpha \colon \Lambda_0^2 \to N\Lambda(\Phi)\}_{\alpha \in A})$$

on the simplicial set $N\Lambda(\Phi)$. The class M consists of all the inert morphisms of $\Lambda(\Phi)$; the class T is the class of all 2–simplices; and the set A is the set of diagrams $I \leftarrow J \to I'$ of $\Lambda(\Phi)$ in which both $J \to I$ and $J \to I'$ are inert, and

$$|J| = |J \times_{TI} I| \sqcup |J \times_{TI'} I'| \cong |I| \sqcup |I'|.$$

Now applying [13, Theorem B.0.19] to the categorical pattern $\mathscr{P}$, one deduces the following.

7.15 Theorem *Suppose Φ is a perfect operator category. There exists a left proper, tractable, simplicial model structure — called the **operadic model structure** — on $s\mathbf{Set}^+_{/N\Lambda(\Phi)}$ with the following properties:*

(7.15.1) *A marked map $X \to Y$ over $N\Lambda(\Phi)$ is a cofibration if and only if it is a monomorphism.*

(7.15.2) *A marked simplicial set over $N\Lambda(\Phi)$ is fibrant if and only if it is of the form $Z^{\otimes,\natural}$ for some Φ–quasioperad $Z^\otimes$.*

(7.15.3) *A marked map $X \to Y$ over $N\Lambda(\Phi)$ is a weak equivalence if and only if for any Φ–quasioperad $Z^\otimes$, the induced map*

$$\mathrm{Map}_{/N\Lambda(\Phi)}(Y, Z^{\otimes,\natural}) \to \mathrm{Map}_{/N\Lambda(\Phi)}(X, Z^{\otimes,\natural})$$

is a weak equivalence.

7.16 Notation For any perfect operator category Φ, denote by $\mathbf{Operad}^{\Phi,\Delta}_\infty$ the simplicial subcategory of $s\mathbf{Set}^+_{/N\Lambda(\Phi)}$ spanned by the fibrant objects for the operadic model structure. Since this is a fibrant simplicial category, we may apply the nerve to obtain an ∞–category $\mathbf{Operad}^\Phi_\infty$.

Using Lurie's characterization [13, Lemma B.2.4(3)] of $\mathscr{P}$–equivalences between fibrant objects, we obtain the following.

7.17 Proposition *Suppose Φ is a perfect operator category. Then a morphism of Φ–quasioperads $g\colon X \to Y$ is an equivalence if and only if the following conditions are satisfied:*

(7.17.1) **Essential surjectivity** *The functor $X_{\{1\}} \to Y_{\{1\}}$ is essentially surjective.*

(7.17.2) **Full faithfulness** *For any object $I \in \Phi$, any vertex $x \in X_I$, and any vertex $y \in X_{\{1\}}$, the induced map*

$$\mathrm{Map}_X^\alpha(x, y) \to \mathrm{Map}_Y^\alpha(g(x), g(y))$$

is an equivalence, where α is the unique active morphism $I \to \{1\}$ of $\Lambda(\Phi)$.

We may apply [13, Proposition B.2.9] to the functors $\Lambda(\Psi) \to \Lambda(\Phi)$ induced by operator morphisms $\Psi \to \Phi$ thanks to Proposition 7.6; it follows that the operadic model structure on $s\mathbf{Set}^+_{/N\Lambda(\Phi)}$ enjoys the same functoriality in Φ that is enjoyed by the operadic model structure on the category $s\mathbf{Set}_{/N\Delta_\Phi^{\mathrm{op}}}$ (Proposition 2.13):

7.18 Proposition *For any operator morphism $G\colon \Psi \to \Phi$, the adjunction*

$$G_!\colon s\mathbf{Set}^+_{/N\Lambda(\Psi)} \rightleftarrows s\mathbf{Set}^+_{/N\Lambda(\Phi)} \colon G^\star$$

is a Quillen adjunction for the operadic model structure.

8 Boardman–Vogt tensor products and weak algebras

It is well known that the classical Boardman–Vogt tensor product [8] exhibits better homotopical properties when it is extended to suitably weak operads, as in [15] and [13]. It can also be *externalized* over the wreath product construction given the previous section, as we now demonstrate.

8.1 Notation For any operator categories Φ and Ψ, consider the functor

$$\wr\colon \Psi \times \Phi \to \Psi \wr \Phi$$

that carries the pair (J, I) to the object $J \wr I := ((J_i)_{i \in |I|}; I)$, in which $J_i = J$ for each $i \in |I|$. This induces a functor

$$W\colon \Delta_\Psi \times_\Delta \Delta_\Phi \to \Delta_{\Psi \wr \Phi}$$

given by the assignment

$$([J_0 \to \cdots \to J_m], [I_0 \to \cdots \to I_m]) \mapsto [J_0 \wr I_0 \to \cdots \to J_m \wr I_m].$$

8.2 It is clear that for any operator morphisms $H\colon \Psi' \to \Psi$ and $G\colon \Phi' \to \Phi$, the following square commutes:

$$
\begin{array}{ccc}
\Delta_{\Psi'} \times_{\Delta} \Delta_{\Phi'} & \xrightarrow{\ W\ } & \Delta_{\Psi' \wr \Phi'} \\
\downarrow & & \downarrow \\
\Delta_{\Psi} \times_{\Delta} \Delta_{\Phi} & \xrightarrow[\ W\ ]{} & \Delta_{\Psi \wr \Phi}
\end{array}
$$

8.3 Definition Suppose Φ and Ψ are operator categories, and suppose $p\colon X \to N\Delta_{\Phi}^{\mathrm{op}}$ and $q\colon Y \to N\Delta_{\Psi}^{\mathrm{op}}$ are left fibrations. For any complete Segal $(\Psi \wr \Phi)$–operad Z, a *pairing* $(X, Y) \to Z$ is a commutative diagram:

$$
\begin{array}{ccc}
Y \times_{N\Delta^{\mathrm{op}}} X & \xrightarrow{\hspace{2cm}} & Z \\
{\scriptstyle (q,\,p)} \downarrow & & \downarrow \\
N\Delta_{\Psi}^{\mathrm{op}} \times_{N\Delta^{\mathrm{op}}} N\Delta_{\Phi}^{\mathrm{op}} & \xrightarrow[\ W\ ]{} & N\Delta_{\Psi \wr \Phi}^{\mathrm{op}}
\end{array}
$$

Write $\mathrm{Pair}^{\Psi,\Phi}(Y, X; Z)$ for the simplicial set $\mathrm{Map}_{/N\Delta_{\Psi \wr \Phi}^{\mathrm{op}}}(Y \times_{N\Delta^{\mathrm{op}}} X, Z)$ of pairings $(Y, X) \to Z$.

8.4 Proposition *Suppose Φ and Ψ are operator categories, and suppose $X \to N\Delta_{\Phi}^{\mathrm{op}}$ and $Y \to N\Delta_{\Psi}^{\mathrm{op}}$ are left fibrations. Then the functor*

$$
\mathrm{Pair}^{\Psi,\Phi}(Y, X; -)\colon \mathbf{Operad}^{\Psi \wr \Phi,\mathrm{op}} \to \mathbf{Kan}
$$

is corepresentable.

Proof Denote by $\mathbf{LFib}(S)$ the simplicial nerve of the simplicial category of left fibrations to a fixed simplicial set S. Denote by $L\colon \mathbf{LFib}(N\Delta_{\Psi \wr \Phi}^{\mathrm{op}}) \to \mathbf{Operad}^{\Psi \wr \Phi}$ the left adjoint to the inclusion. Then $\mathrm{Pair}^{\Psi,\Phi}(Y, X; -)$ is corepresented by the object $LW_!(Y \times_{N\Delta^{\mathrm{op}}} X)$. $\square$

8.5 Definition Suppose Φ and Ψ are operator categories, $X \to N\Delta_{\Phi}^{\mathrm{op}}$ and $Y \to N\Delta_{\Psi}^{\mathrm{op}}$ left fibrations, and Z a complete Segal $(\Psi \wr \Phi)$–operad. A pairing $(Y, X) \to Z$ will be said to *exhibit Z as the Boardman–Vogt tensor product of Y and X* if and only if for every complete Segal $(\Psi \wr \Phi)$–operad Z', it induces an equivalence

$$
\mathrm{Map}_{\mathbf{Operad}^{\Psi \wr \Phi}}(Z, Z') \xrightarrow{\sim} \mathrm{Pair}^{\Psi,\Phi}((Y, X), Z').
$$

In the presence of such a pairing, we write $Y^{\Psi} \otimes^{\Phi} X$ for Z. Using [12], one can organize this into a functor

$$
-^{\Psi} \otimes^{\Phi} -\colon \mathbf{LFib}(N\Delta_{\Phi}^{\mathrm{op}})^{\mathrm{op}} \times \mathbf{LFib}(N\Delta_{\Psi}^{\mathrm{op}})^{\mathrm{op}} \to \mathbf{Operad}^{\Psi \wr \Phi}.
$$

8.6 Warning In contrast with the Boardman–Vogt tensor product of ordinary symmetric operads, note that the order matters here: since the wreath product is noncommutative, the objects $Y^{\Psi}\otimes^{\Phi}X$ and $X^{\Phi}\otimes^{\Psi}Y$ are not even objects of the same ∞–category.

It follows immediately from 8.2 that the Boardman–Vogt tensor product is compatible with operator morphisms.

8.7 Proposition *For any operator morphisms $H\colon \Psi' \to \Psi$ and $G\colon \Phi' \to \Phi$ and for any left fibrations $X \to N\Delta^{\mathrm{op}}_{\Phi'}$ and $Y \to N\Delta^{\mathrm{op}}_{\Psi'}$, there is a canonical equivalence*

$$(H \wr G)_!(Y^{\Psi'}\otimes^{\Phi'}X) \simeq H_!Y^{\Psi}\otimes^{\Phi}G_!X$$

of complete Segal $(\Psi \wr \Phi)$–operads.

It is easy to see that the Boardman–Vogt tensor product preserves colimits separately in each variable. Consequently, we obtain the following.

8.8 Proposition *Suppose Φ and Ψ are operator categories, and suppose $X \to N\Delta^{\mathrm{op}}_{\Phi}$ and $Y \to N\Delta^{\mathrm{op}}_{\Psi}$ are left fibrations. Then the functors*

$$-^{\Psi}\otimes^{\Phi}X\colon \mathbf{LFib}(N\Delta^{\mathrm{op}}_{\Psi})^{\mathrm{op}} \to \mathbf{Operad}^{\Psi\wr\Phi}$$

and

$$Y^{\Psi}\otimes^{\Phi}-\colon \mathbf{LFib}(N\Delta^{\mathrm{op}}_{\Phi})^{\mathrm{op}} \to \mathbf{Operad}^{\Psi\wr\Phi}$$

each admit a right adjoint, denoted by $\mathbf{Alg}^{\Phi,\Psi\wr\Phi}(X,-)$ and $\mathbf{Alg}^{\Psi,\Psi\wr\Phi}(Y,-)$, respectively.

8.9 Hence for any two operator categories Φ and Ψ, we obtain a pair of functors

$$\mathbf{Alg}^{\Psi,\Psi\wr\Phi}\colon \mathbf{LFib}(N\Delta^{\mathrm{op}}_{\Psi})^{\mathrm{op}} \times \mathbf{Operad}^{\Psi\wr\Phi} \to \mathbf{LFib}(N\Delta^{\mathrm{op}}_{\Phi})$$

and

$$\mathbf{Alg}^{\Phi,\Psi\wr\Phi}\colon \mathbf{LFib}(N\Delta^{\mathrm{op}}_{\Phi})^{\mathrm{op}} \times \mathbf{Operad}^{\Psi\wr\Phi} \to \mathbf{LFib}(N\Delta^{\mathrm{op}}_{\Psi}).$$

In fact, these functors are valued in the ∞–category of weak operads. That is, suppose Φ and Ψ are operator categories, suppose

$$X \to N\Delta^{\mathrm{op}}_{\Phi} \quad \text{and} \quad Y \to N\Delta^{\mathrm{op}}_{\Psi}$$

are left fibrations, and suppose Z is a complete Segal $(\Psi \wr \Phi)$–operad; then the left fibration $\mathbf{Alg}^{\Psi,\Psi\wr\Phi}(Y,Z) \to N\Delta^{\mathrm{op}}_{\Phi}$ is a complete Segal Φ–operad, and the left fibration $\mathbf{Alg}^{\Phi,\Psi\wr\Phi}(X,Z) \to N\Delta^{\mathrm{op}}_{\Psi}$ is a complete Segal Ψ–operad.

This is not an entirely formal matter. It suffices to prove this for X and Y corepresentable. In this case, one shows that forming the Boardman–Vogt tensor product of any corepresentable left fibration with any element of the set S_Φ (or S_Ψ) of Proposition 2.12 itself lies in the strongly saturated class generated by $S_{\Psi \wr \Phi}$. We leave these details to the reader.

Now let's concentrate on the situation in which one of the two complete Segal operads is the terminal operad.

8.10 Notation Suppose Φ and Ψ are two operator categories. For any complete Segal $(\Psi \wr \Phi)$–operad Z, we will write $\mathbf{Mon}^{\Phi, \Psi \wr \Phi}(Z)$ for $\mathbf{Alg}^{\Phi, \Psi \wr \Phi}(U_\Phi, Z)$, and we will write $\mathbf{Mon}^{\Phi, \Psi \wr \Phi}(Z)$ for $\mathbf{Alg}^{\Psi, \Psi \wr \Phi}(U_\Psi, Z)$.

8.11 Suppose Φ is an operator category. Note that we have canonical equivalences $\Phi \wr \{1\} \simeq \Phi \simeq \{1\} \wr \Phi$, through which the functor $\mathbf{Alg}^{\Phi, \Phi \wr \{1\}}$ and $\mathbf{Alg}^{\Phi, \{1\} \wr \Phi}$ may be identified. We write

$$\mathbf{Alg}^\Phi \colon \mathbf{LFib}(N\Delta_\Phi^{\mathrm{op}})^{\mathrm{op}} \times \mathbf{Operad}^\Phi \to \mathbf{LFib}(N\Delta^{\mathrm{op}})$$

for the common functor. For any left fibration $Y \to N\Delta_\Phi^{\mathrm{op}}$ and any complete Segal Φ–operad Z, we will refer to objects of $\mathbf{Alg}^\Phi(Y, Z)$ (ie those 0–simplices that lie over $0 \in N\Delta^{\mathrm{op}}$) as Y–*algebras in* Z. If Y is the terminal Φ–operad U_Φ, then we will refer to Y–algebras in Z as *complete Segal Φ–monoids in* Z.

8.12 Theorem *For any two operator categories Φ and Ψ, the pairing*

$$(U_\Psi, U_\Phi) \to U_{\Psi \wr \Phi}$$

given by W exhibits $U_{\Psi \wr \Phi}$ as the Boardman–Vogt tensor product of U_Ψ and U_Φ.

The proof is a bit involved, so it appears in an appendix; see Appendix A.

8.12.1 Corollary *For any two operator categories Φ and Ψ, one has, for any $(\Psi \wr \Phi)$–operad Z, canonical equivalences*

$$\mathbf{Mon}^{\Psi \wr \Phi}(Z) \simeq \mathbf{Mon}^\Phi(\mathbf{Mon}^{\Psi, \Psi \wr \Phi}(Z)) \simeq \mathbf{Mon}^\Psi(\mathbf{Mon}^{\Phi, \Psi \wr \Phi}(Z)).$$

Roughly speaking, we have shown that complete Segal $(\Psi \wr \Phi)$–monoids are complete Segal Ψ–monoids in complete Segal Φ–monoids, which in turn are complete Segal Φ–monoids in complete Segal Ψ–monoids.

8.13 Example The $(\infty, 1)$–category of complete Segal $\mathbf{O}^{(n)}$–monoids (in some complete Segal $\mathbf{O}^{(n)}$–operad Z) is equivalent to the $(\infty, 1)$–category

$$\mathbf{Mon}^{\mathbf{O},\mathbf{O}}(\mathbf{Mon}^{\mathbf{O},\mathbf{O}\wr\mathbf{O}}(\cdots\mathbf{Mon}^{\mathbf{O},\mathbf{O}^{(n)}}(Z)\cdots))$$

of complete Segal $\mathbf{O}$–monoids in complete Segal $\mathbf{O}$–monoids in ... in complete Segal $\mathbf{O}$–monoids in Z. The assertion that the operad E_n is equivalent to the symmetrization of the terminal $\mathbf{O}^{(n)}$–operad (Proposition 11.4) thus states that if Z is a symmetric operad, then the homotopy theory of E_n–algebras in Z is equivalent to the homotopy theory of complete Segal $\mathbf{O}$–monoids in complete Segal $\mathbf{O}$–monoids in ... in complete Segal $\mathbf{O}$–monoids in the underlying complete Segal $\mathbf{O}^{(n)}$–operad of Z.

8.14 Example For any integers $n \geq m \geq 0$, we have an inclusion

$$s\colon \mathbf{O}^{(m)} \cong (\{1\}^{(n-m)} \wr \mathbf{O}^{(m)}) \hookrightarrow (\mathbf{O}^{(n-m)} \wr \mathbf{O}^{(m)}) \cong \mathbf{O}^{(n)},$$

which is a section of the coronal fibration $p\colon \mathbf{O}^{(n)} \to \mathbf{O}^{(m)}$. We can thus form the colimit $\mathbf{O}^{(\infty)} := \mathrm{colim}_{n\geq 0} \mathbf{O}^{(n)}$. This can be viewed as the category whose objects are sequences

$$(M_1, M_2, \dots)$$

with $M_n \in \mathbf{O}^{(n)}$ such that for every for each $n \geq 1$, one has $p(M_n) = M_{n-1}$ and for each $n \gg 1$, one has $s(M_n) = M_{n+1}$. Then $\mathbf{O}^{(\infty)}$ is an operator category.

9 Boardman–Vogt tensor products and ∞–algebras

We can define an analogue of the Boardman–Vogt tensor product introduced in Section 8 and study its interaction with the model structure introduced in Section 7. Once again, in most cases the proofs are trivial extensions of the proofs of [13, Chapter 2].

9.1 Notation For any perfect operator categories Φ and Ψ, define a natural transformation

$$\omega\colon \wr \circ (T_\Psi \times T_\Phi) \to T_{\Psi\wr\Phi} \circ \wr$$

as follows. For any pair $(K, I) \in \Psi \times \Phi$, let

$$\omega_{(K,I)} = ((\phi_j)_{j\in|T_\Phi I|}, \mathrm{id}_{T_\Phi I})\colon T_\Psi K \wr T_\Phi I \to T_{\Psi\wr\Phi}(K \wr I)$$

be the morphism in which

$$\phi_j = \begin{cases} \mathrm{id}_{T_\Psi K} & \text{if } j \in |I| \subset |T_\Phi I|, \\ ! & \text{if } j \notin |I|. \end{cases}$$

Using this, we obtain an induced functor $W \colon \Lambda(\Psi) \times \Lambda(\Phi) \to \Lambda(\Psi \wr \Phi)$ on the Leinster categories given by the assignment $(K, I) \mapsto K \wr I$.

9.2 Definition Suppose Φ and Ψ are two perfect operator categories, suppose $X \in s\mathbf{Set}^+_{/N\Lambda(\Phi)}$, and suppose $Y \in s\mathbf{Set}^+_{/N\Lambda(\Psi)}$. Write $Y^\Psi \otimes^\Phi X$ for the product $Y \times X$, regarded as a marked simplicial set over $N\Lambda(\Psi \wr \Phi)$ via

$$Y \times X \to N\Lambda(\Psi) \times N\Lambda(\Phi) \xrightarrow{W} N\Lambda(\Psi \wr \Phi).$$

We call this the *Boardman–Vogt tensor product* of Y and X.

9.3 Consider the operator category $\mathbf{F}$. To relate our Boardman–Vogt tensor product to the monoidal structure $\odot$ constructed in [13], we need only note that the functor $\wedge \colon \Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$ of [13] is isomorphic to the composition of the functor $W \colon \Lambda(\mathbf{F}) \times \Lambda(\mathbf{F}) \to \Lambda(\mathbf{F} \wr \mathbf{F})$ with the functor $\Lambda(\mathbf{F} \wr \mathbf{F}) \to \Lambda(\mathbf{F})$ induced by the unique operator morphism $U \colon \mathbf{F} \wr \mathbf{F} \to \mathbf{F}$. Consequently, we obtain an isomorphism $U_!(Y^\mathbf{F} \otimes^\mathbf{F} X) \cong Y \odot X$.

9.4 Proposition *For any operator morphisms $H \colon \Psi' \to \Psi$ and $G \colon \Phi' \to \Phi$ between perfect operator categories, and for any objects $Y \in s\mathbf{Set}_{/N\Lambda(\Psi')}$ and $X \in s\mathbf{Set}_{/N\Lambda(\Phi')}$, there is a canonical isomorphism*

$$(H \wr G)_!(Y^{\Psi'} \otimes^{\Phi'} X) \cong H_! Y^\Psi \otimes^\Phi G_! X$$

of simplicial sets over $N\Lambda(\Psi \wr \Phi)$.

The Boardman–Vogt tensor product preserves colimits separately in each variable. Consequently, we have the following.

9.5 Proposition *Suppose Φ and Ψ are two perfect operator categories, suppose $X \in s\mathbf{Set}_{/N\Lambda(\Phi)}$, and suppose $Y \in s\mathbf{Set}_{/N\Lambda(\Psi)}$. Then the functors*

$$Y^\Psi \otimes^\Phi - \colon s\mathbf{Set}^+_{/N\Lambda(\Phi)} \to s\mathbf{Set}^+_{N\Lambda(\Psi \wr \Phi)}$$

and

$$-^\Psi \otimes^\Phi X \colon s\mathbf{Set}^+_{/N\Lambda(\Psi)} \to s\mathbf{Set}^+_{N\Lambda(\Psi \wr \Phi)}$$

both admit right adjoints.

9.6 Notation For any two perfect operator categories Φ and Ψ, we obtain a pair of functors

$$\mathbf{Alg}_\infty^{\Psi,\Psi\wr\Phi}\colon (s\mathbf{Set}^+_{/N\Lambda(\Psi)})^{\mathrm{op}} \times s\mathbf{Set}^+_{/N\Lambda(\Psi\wr\Phi)} \to s\mathbf{Set}^+_{/N\Lambda(\Phi)}$$

and

$$\mathbf{Alg}_\infty^{\Phi,\Psi\wr\Phi}\colon (s\mathbf{Set}^+_{/N\Lambda(\Phi)})^{\mathrm{op}} \times s\mathbf{Set}^+_{/N\Lambda(\Psi\wr\Phi)} \to s\mathbf{Set}^+_{/N\Lambda(\Psi)}$$

such that $\mathbf{Alg}_\infty^{\Psi,\Psi\wr\Phi}(Y,-)$ is right adjoint to $Y^{\Psi}\!\otimes^{\Phi}-$ and $\mathbf{Alg}_\infty^{\Phi,\Psi\wr\Phi}(X,-)$ is right adjoint to $-^{\Psi}\!\otimes^{\Phi}X$.

It is clear that these functors, along with the Boardman–Vogt tensor product, comprise an adjunction of two variables $s\mathbf{Set}^+_{/N\Lambda(\Psi)} \times s\mathbf{Set}^+_{/N\Lambda(\Phi)} \to s\mathbf{Set}^+_{/N\Lambda(\Psi\wr\Phi)}$.

The interaction between this variant of the Boardman–Vogt tensor product and the operadic model structure is the most one could hope for. We apply [13, Remark B.2.5 and Proposition B.2.9] to the functors $W\colon \Lambda(\Psi)\times\Lambda(\Phi) \to \Lambda(\Psi\wr\Phi)$ to deduce the following.

9.7 Theorem *For any two perfect operator categories* Φ *and* Ψ, *the functors* $^{\Psi}\!\otimes^{\Phi}$, $\mathbf{Alg}_\infty^{\Psi,\Psi\wr\Phi}$, *and* $\mathbf{Alg}_\infty^{\Phi,\Psi\wr\Phi}$ *form a Quillen adjunction of two variables for the operadic model structures.*

9.8 Notation Suppose Φ and Ψ are two perfect operator categories. For any $(\Psi\wr\Phi)$–quasioperad Z, we will write $\mathbf{Mon}_\infty^{\Phi,\Psi\wr\Phi}(Z)$ for $\mathbf{Alg}_\infty^{\Phi,\Psi\wr\Phi}(U_\Phi, Z)$, and we will write $\mathbf{Mon}_\infty^{\Phi,\Psi\wr\Phi}(Z)$ for $\mathbf{Alg}_\infty^{\Psi,\Psi\wr\Phi}(U_\Psi, Z)$.

9.9 Suppose Φ is a perfect operator category. Note that we have canonical equivalences $\Phi\wr\{1\} \simeq \Phi \simeq \{1\}\wr\Phi$, through which the functors $\mathbf{Alg}_\infty^{\Phi,\Phi\wr\{1\}}$ and $\mathbf{Alg}_\infty^{\Phi,\{1\}\wr\Phi}$ may be identified. We write

$$\mathbf{Alg}_\infty^{\Phi}\colon (s\mathbf{Set}^+_{/N\Lambda(\Phi)})^{\mathrm{op}} \times s\mathbf{Set}^+_{/N\Lambda(\Phi)} \to s\mathbf{Set}^+$$

for the common functor. For any marked map $Y \to N\Lambda(\Phi)$ and any ∞–operad Z over Φ, we will refer to objects of the quasicategory $\mathbf{Alg}_\infty^{\Phi}(Y,Z)$ as ∞–*algebras over* Y *in* Z. If Y is the terminal Φ–operad U_Φ, then we will refer to Y–algebras in Z as ∞–*monoids over* Φ *in* Z.

The following theorem is proved exactly as in [13, Theorem 2.4.4.3].

9.10 Theorem *For any two operator categories* Φ *and* Ψ, *the functor* W *induces an equivalence* $U_\Psi{}^{\Psi}\!\otimes^{\Phi}U_\Phi \xrightarrow{\sim} U_{\Psi\wr\Phi}$.

9.10.1 Corollary *For any two perfect operator categories* Φ *and* Ψ, *one has, for any* $(\Psi \wr \Phi)$–*operad* Z, *canonical equivalences*

$$\mathbf{Mon}_\infty^{\Psi \wr \Phi}(Z) \simeq \mathbf{Mon}_\infty^\Phi(\mathbf{Mon}_\infty^{\Psi, \Psi \wr \Phi}(Z)) \simeq \mathbf{Mon}_\infty^\Psi(\mathbf{Mon}_\infty^{\Phi, \Psi \wr \Phi}(Z)).$$

10 Complete Segal operads versus quasioperads

We now compare our two approaches to the theory of complete Segal Φ–operads. Here is the main theorem.

Theorem *For any perfect operator category* Φ, *there exist inverse equivalences of* ∞–*categories*

$$P^\otimes \colon \mathbf{Operad}_{\mathrm{CSS}}^\Phi \simeq \mathbf{Operad}_\infty^\Phi \colon C.$$

We get these by obtaining homotopy equivalences of relative categories (Theorem 10.16).

We begin with the functor $P^\otimes$. To construct this functor, we must first relate the Leinster category $\Lambda(\Phi)$ to the category of Φ–sequences Δ_Φ.

10.1 Notation Suppose Φ is a perfect operator category. For any object $\mathbf{m} \in \Delta$, denote by $\widetilde{\mathbb{O}}(\mathbf{m})$ the twisted arrow category of $\mathbf{m}$; this is a poset whose objects are pairs of integers (i, j) such that $0 \le i \le j \le m$, and $(i', j') \le (i, j)$ if and only if $i \le i' \le j' \le j$.

Now for any m–simplex $\sigma \colon \mathbf{m} \to N\Lambda(\Phi)$ corresponding to a sequence of morphisms

$$I_0 \to I_1 \to \cdots \to I_m$$

in $\Lambda(\Phi)$, we obtain a functor $F_\sigma \colon \widetilde{\mathbb{O}}(\mathbf{m})^{\mathrm{op}} \to \Delta_\Phi$, given by the formula

$$F_\sigma(i, j) := [I_0 \times_{TI_j} I_j \to I_1 \times_{TI_j} I_j \to \cdots \to I_i \times_{TI_j} I_j].$$

We write Δ_Φ^σ for the colimit of the diagram $j \circ F_\sigma$, where j denotes the Yoneda embedding $\Delta_\Phi \hookrightarrow \mathrm{Fun}(\Delta_\Phi^{\mathrm{op}}, \mathbf{Set})$.

The assignment $\sigma \mapsto \Delta_\Phi^\sigma$ is functorial with respect to the category $\mathrm{Simp}(\Lambda(\Phi))$, whose objects are pairs $(\mathbf{m}, \sigma)$ consisting of an object $\mathbf{m} \in \Delta$ and an m–simplex $\sigma \colon \mathbf{m} \to N\Lambda(\Phi)$ of $\Lambda(\Phi)$ and whose morphisms $(\mathbf{m}, \sigma) \to (\mathbf{n}, \tau)$ are morphisms $\eta \colon \mathbf{m} \to \mathbf{n}$ of Δ and a natural isomorphism $\sigma \cong \tau \circ \eta$; in particular, we obtain a functor

$$\Delta_\Phi^* \colon \iota \, \mathrm{Fun}(\mathbf{m}, \Lambda(\Phi)) \to \iota \, \mathrm{Fun}(\Delta_\Phi^{\mathrm{op}}, \mathbf{Set}).$$

10.2 Construction Suppose that Φ is a perfect operator category, and suppose that $X: \Delta_\Phi^{\mathrm{op}} \to s\mathbf{Set}$ is a functor. For any integer $m \geq 0$ and any m–simplex

$$\sigma: \mathbf{m} \to N\Lambda(\Phi),$$

one may define a simplicial set $X(\Delta_\Phi^\sigma)$ via right Kan extension, so that

$$X(\Delta_\Phi^\sigma) \cong \lim_{(i,j) \in \widetilde{\mathbb{O}}(\mathbf{m})} X(F_\sigma(i,j)).$$

The assignment $\sigma \mapsto X(\Delta_\Phi^\sigma)$ defines a functor $\iota \operatorname{Fun}(\mathbf{m}, \Lambda(\Phi))^{\mathrm{op}} \to s\mathbf{Set}$. Apply the construction of [12, Definition 3.2.5.2] to this functor to obtain a map

$$\mathbf{P}^\otimes(X)_m \to N\iota \operatorname{Fun}(\mathbf{m}, \Lambda(\Phi))^{\mathrm{op}}.$$

This map is functorial in $\mathbf{m} \in \Delta^{\mathrm{op}}$, whence we obtain a morphism of simplicial spaces

$$\mathbf{P}^\otimes(X) \to \mathbf{N}\Lambda(\Phi),$$

where $\mathbf{N}\Lambda(\Phi)$ is the classifying diagram of the category $\Lambda(\Phi)$ in the sense of Rezk [16, 3.5] — except that in each degree we are taking the nerve of the opposite groupoid.

10.3 Proposition *Suppose Φ is a perfect operator category, and suppose that*

$$X: \Delta_\Phi^{\mathrm{op}} \to \mathbf{Kan}$$

is a functor. Assume that X that is fibrant for the injective model structure on the category $\operatorname{Fun}(\Delta_\Phi^{\mathrm{op}}, s\mathbf{Set})$, and that X classifies a complete Segal Φ–operad. Then $\mathbf{P}^\otimes(X)$ is a complete Segal space.

Proof To see that $\mathbf{P}^\otimes(X)$ is Reedy fibrant, consider any square:

$$
\begin{array}{ccc}
\Lambda_k^n & \longrightarrow & \mathbf{P}^\otimes(X)_m \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & \lim_{\mathbf{k} \subsetneq \mathbf{m}} \mathbf{P}^\otimes(X)_k
\end{array}
$$

Unwinding the definitions, we see that a lift $\Delta^n \to \mathbf{P}^\otimes(X)_m$ amounts to a lift σ of the diagram

$$
\begin{array}{ccc}
\Lambda_k^n & \longrightarrow & N\iota \operatorname{Fun}(\mathbf{m}, \Lambda(\Phi))^{\mathrm{op}} \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & \lim_{\mathbf{k} \subsetneq \mathbf{m}} N\iota \operatorname{Fun}(\mathbf{k}, \Lambda(\Phi))^{\mathrm{op}}
\end{array}
$$

along with a compatible collection of maps $\Delta^J \to X(\Delta_\Phi^{\sigma(\max J)})$, one for each nonempty subset $J \subset \mathbf{m}$. The existence of the lift σ follows from [16, Lemma 3.9], and the compatible collection of maps follows from the injective fibrancy.

Showing the Segal conditions on $\mathbf{P}^\otimes(X)$ reduces to showing that for any m–simplex $\sigma \colon \mathbf{m} \to \Lambda(\Phi)$, the induced map

$$X(\Delta_\Phi^\sigma) \to X(\Delta_\Phi^{\sigma|\{0,1\}}) \times_{X(\Delta_\Phi^{\sigma|\{1\}})} \cdots \times_{X(\Delta_\Phi^{\sigma|\{m-1\}})} X(\Delta_\Phi^{\sigma|\{m-1,m\}})$$

is a weak equivalence. But this follows from the fact that, for X itself, the maps $s_{\mathbf{n},J}$ are all equivalences.

The completeness conditions on $\mathbf{P}^\otimes(X)$ reduce to the assertion that for any object $I \in \Phi$, the natural map

$$X[I] \to X[I = I = I = I] \times_{X[I=I] \times X[I=I]} X[I]$$

is a weak equivalence, where the maps $X[I = I = I = I] \to X[I = I]$ are given by the inclusions $\{0,2\} \hookrightarrow \{0,1,2,3\}$ and $\{1,3\} \hookrightarrow \{0,1,2,3\}$. But this follows from the case when $I = \{1\}$ along with the decomposition

$$X[I] \simeq \prod_{i \in |I|} X[\{i\}]. \qquad\qquad \square$$

The results of [10] thus motivate the following.

10.4 Definition Suppose Φ is a perfect operator category, and suppose

$$X \colon \Delta_\Phi^{\mathrm{op}} \to s\mathbf{Set}$$

is a functor. Then we define a simplicial set $P^\otimes(X)$ in the following manner. For any $\mathbf{m} \in \Delta^{\mathrm{op}}$, an m–simplex (σ, x) of $P^\otimes(X)$ consists of a functor $\sigma \colon \mathbf{m} \to \Lambda(\Phi)$ and a vertex $x \in X(\Delta_\Phi^\sigma)_0$. This is obviously functorial in $\mathbf{m}$, and the assignment $(\sigma, x) \mapsto \sigma$ defines a projection $P^\otimes(X) \to N\Lambda(\Phi)$. Hence we obtain a functor $P^\otimes \colon \mathrm{Fun}(\Delta_\Phi^{\mathrm{op}}, s\mathbf{Set}) \to s\mathbf{Set}_{/N\Lambda(\Phi)}$.

10.5 Suppose Φ is a perfect operator category, and suppose

$$X \colon \Delta_\Phi^{\mathrm{op}} \to s\mathbf{Set}$$

is a functor. Note that the fiber of $P^\otimes(X)$ over an object $I \in \Lambda(\Phi)$ is the simplicial set whose m–simplices are the vertices of $X(m, I)$, where (m, I) denotes the constant

sequence $I = I = \cdots = I$ of length m. In particular, the fiber over $\{1\}$ is the simplicial set whose m–simplices are the vertices of the simplicial set $(X | \Delta^{\mathrm{op}})_m$.

10.6 Proposition *Suppose Φ is a perfect operator category, and suppose*

$$X \colon \Delta_\Phi^{\mathrm{op}} \to s\mathbf{Set}$$

is a functor. Assume that X is fibrant for the operadic model structure on the category $\mathrm{Fun}(\Delta_\Phi^{\mathrm{op}}, s\mathbf{Set})$. Then $P^\otimes(X)$ is a Φ–quasioperad.

Proof Since the m–simplices of $P^\otimes(X)$ can be identified with the 0–simplices of $\mathbf{P}^\otimes(X)_m$, Proposition 10.3 and [10, Theorem 4.11] together imply that $P^\otimes(X)$ is a quasicategory, whence by [12, Proposition 2.3.1.5], $P^\otimes(X) \to \Lambda(\Phi)$ is an inner fibration.

Suppose $J \to I$ is an inert morphism of $\Lambda(\Phi)$, and suppose that $x \in P^\otimes(X)_J$ is an object; hence $x \in X[J]_0$. Let $x' \in X[J \times_{TI} I]$ be its image under the map induced by the interval inclusion $J \times_{TI} I \hookrightarrow J$. Now we may use the inverse to the isomorphism $J \times_{TI} I \to I$ to define a morphism of Φ–sequences

$$[J \times_{TI} I \to I] \to [J \times_{TI} I],$$

and one may consider the image x'' of x' in $X[J \times_{TI} I \to I]$ under the induced map. The pair $(x, x'') \in X[J] \times_{X[J \times_{TI} I]} X[J \times_{TI} I \to I]$ is now a cocartesian edge covering ϕ.

Suppose that the following are given:

- two objects $I, J \in \Phi$,
- vertices $x \in P^\otimes(X)_I$ and $y \in P^\otimes(X)_J$,
- a morphism $\phi \colon J \to I$ of $\Lambda(\Phi)$, and
- p–cocartesian edges $\{y \to y_i \mid i \in |I|\}$ that lie over the inert morphisms $\{\rho_i \colon I \to \{i\} \mid i \in |I|\}$.

Now the simplicial set $\mathrm{Map}^\phi_{P^\otimes(X)}(x, y)$ can be identified with the fiber of the natural map

$$X[J] \times_{X[J \times_{TI} I]} X[J \times_{TI} I \to I] \to X[J] \times X[I]$$

over the point (x, y). The fact that the induced map

$$\mathrm{Map}^\phi_{X^\otimes}(x, y) \to \prod_{i \in |I|} \mathrm{Map}^{\rho_i \circ \phi}_{X^\otimes}(x, y_i)$$

is an equivalence thus follows from the decomposition

$$X[J] \times_{X[J \times_{TI} I]} X[J \times_{TI} I \to I] \simeq X[J] \times_{\prod_{i \in |I|} X[J_i]} \prod_{i \in |I|} X[J_i \to \{i\}]$$

$$\to X[J] \times \prod_{i \in |I|} X[\{i\}] \simeq X[J] \times X[I].$$

Finally, we must show that for any object $I \in \Phi$ and for any I–tuple

$$(y_i)_{i \in |I|} \in \prod_{i \in |I|} P^{\otimes}(X)_{\{i\}},$$

there exists an object $y \in P^{\otimes}(X)_I$ a collection of cocartesian edges $y \to y_i$ lying over the inert edges $I \to \{i\}$. For this, choose $y \in X[I]$ that maps to $(y_i)_{i \in |I|}$. (This is possible since the map $X[I] \to \prod_{i \in |I|} X[\{i\}]$ is a trivial fibration.) For any point $i \in |I|$, the image of y under the natural map

$$X[I] \cong X[I] \times_{X[\{i\}]} X[\{i\}] \to X[I] \times_{X[\{i\}]} X[\{i\} = \{i\}]$$

is the desired cocartesian edge $y \to y_i$. □

10.7 Example When $\Phi = \{1\}$, the functor $P^{\otimes}$ carries a complete Segal space X to the quasicategory whose m–simplices are the 0–simplices of X_m. By a theorem of Joyal and Tierney [10, Theorem 4.11], this is known to be an equivalence of homotopy theories.

It is obvious from the construction given here that $P^{\otimes}$ is compatible with changes of operator category, in the following sense.

10.8 Proposition *Suppose $G \colon \Psi \to \Phi$ is an operator morphism between perfect operator categories. Then we have a natural isomorphism of functors*

$$P^{\otimes} \circ G^{\star} \cong G^{\star} \circ P^{\otimes}.$$

10.9 Proposition *Suppose Φ is a perfect operator category. A morphism $X \to Y$ between injectively fibrant complete Segal Φ–operads is a weak equivalence if and only if the induced morphism $P^{\otimes}(X) \to P^{\otimes}(Y)$ is so.*

Proof Let us use the characterizations of Propositions 2.8 and 7.17. Combining the example and the proposition above, we deduce that a morphism $X \to Y$ of complete Segal Φ–operads induces an essentially surjective functor on underlying complete

Segal spaces if and only if the morphism $P^{\otimes}(X) \to P^{\otimes}(Y)$ induces an essentially surjective functor on underlying quasicategories.

On the other hand, if $I \in \Phi$, and if $\alpha \colon I \to \{1\}$ is the morphism of $\Lambda(\Phi)$ given by $I \to \{t\} \to T$, then for any complete Segal Φ–operad X, the mapping space $\mathrm{Map}_X^{\alpha}(x, y)$ is equivalent to the fiber of the map

$$X[I \to \{1\}] \to X[I] \times X[\{1\}].$$

In particular, a morphism $X \to Y$ of complete Segal Φ–operads is fully faithful if and only if the induced morphism $P^{\otimes}(X) \to P^{\otimes}(Y)$ of Φ–quasicategories is so. $\square$

In other words, $P^{\otimes}$ is a relative functor in the sense of [3], and it reflects weak equivalences.

To define a functor in the opposite direction, we introduce the following.

10.10 Construction Suppose Φ is a perfect operator category and $(\mathbf{m}, I)$ is a Φ–sequence. Define a poset $A(\mathbf{m}, I)$ as follows: the objects are triples (r, s, i), where r and s are integers such that $0 \le r \le s \le m$ and $i \in |I_s|$; we declare that $(r, s, i) \le (r', s', i')$ if and only if $r \le r' \le s' \le s$, and $i' \mapsto i$ under the map $|I_{s'}| \to |I_s|$.

Let $A(\mathbf{m}, I)^{\dagger}$ be the set of morphisms of $A(\mathbf{m}, I)$ of the form $(r, s, i) \to (r, s', i')$. Using the factorization system on $\Lambda(\Phi)$ of Lemma 7.3, we deduce that there is a functor $A(\mathbf{m}, I) \to \Lambda(\Phi)$ given by the assignment

$$(r, s, i) \mapsto I_{r,i} := I_r \times_{I_s} \{i\},$$

under which morphisms of $A(\mathbf{m}, I)^{\dagger}$ are carried to inert morphisms of $\Lambda(\Phi)$.

Now the assignment $(\mathbf{m}, I) \to (NA(\mathbf{m}, I), A(\mathbf{m}, I)^{\dagger})$ defines a functor

$$NA \colon \Delta_{\Phi} \to s\mathbf{Set}^{+}_{/N\Lambda(\Phi)}.$$

For any object $X^{\otimes} \in s\mathbf{Set}^{+}_{/N\Lambda(\Phi)}$, write $C(X)$ for the functor $\Delta_{\Phi}^{\mathrm{op}} \to s\mathbf{Set}$ given by the assignment

$$(\mathbf{m}, I) \mapsto \mathrm{Map}^{\sharp}_{N\Lambda(\Phi)}(NA(\mathbf{m}, I), X^{\otimes}).$$

10.11 Example When $\Phi = \{1\}$, the category $A(\mathbf{m}, \{1\})$ is the opposite of the twisted arrow category $\widetilde{\mathbb{O}}(\mathbf{m})$, and in $N\widetilde{\mathbb{O}}(\mathbf{m})^{\mathrm{op}}$, the marked edges are precisely the morphisms $(r, s) \to (r, s')$. Now the inclusion $\mathbf{m} \hookrightarrow N\widetilde{\mathbb{O}}(\mathbf{m})^{\mathrm{op}}$ given by $r \mapsto (r, m)$ induces a marked anodyne morphism on nerves. Consequently, for any quasicategory X, one

has $C(X)_m \simeq \iota \operatorname{Fun}(\Delta^m, X)$, which is an inverse homotopy equivalence of $P^\otimes$ by the theorem of Joyal and Tierney [10, Theorem 4.11].

The construction $(m, I) \mapsto NA(m, I)$ is compatible with changes of operator category. That is, for any operator morphism $G\colon \Psi \to \Phi$ between perfect operator categories, one has a canonical isomorphism $G_! NA(m, I) \cong NA(m, GI)$. Consequently, we have the following.

10.12 Lemma *For any operator morphism $G\colon \Psi \to \Phi$ between perfect operator categories, there is a natural isomorphism*

$$C \circ G^\star \cong G^\star \circ C.$$

10.13 Proposition *Suppose Φ is a perfect operator category, and suppose $X^\otimes$ is a Φ–quasioperad. Then $C(X^\otimes)$ is fibrant for the operadic model structure.*

Proof The fact that $C(X^\otimes)$ is fibrant for the injective model structure follows from [12, Remark A.2.9.27] and the observation that $(m, I) \mapsto (NA(m, I), A(m, I)^\dagger)$ is cofibrant for the projective model structure on $\operatorname{Fun}(\Delta_\Phi, s\mathbf{Set}^+_{/N\Lambda(\Phi)})$.

For any Φ–sequence $[I_0 \to \cdots \to I_m]$, the canonical map

$$\coprod_{i \in |I_m|} NA[I_{0,i} \to \cdots \to \{i\}] \to NA[I_0 \to \cdots \to I_m]$$

is an isomorphism by definition. Moreover, for any integer $0 \le k \le m$, an elementary computation shows that the inclusion

$$NA[I_0 \to \cdots \to I_k] \cup^{NA[I_k]} NA[I_k \to \cdots \to I_m] \hookrightarrow NA[I_0 \to \cdots \to I_m]$$

is $\mathcal{P}$–anodyne.

It thus remains only to observe that the underlying simplicial set $p^\star C(X^\otimes)$ is a complete Segal space, and this follows from Lemma 10.12 and Example 10.11. $\qquad\square$

10.14 Proposition *Suppose Φ is a perfect operator category. A morphism $X^\otimes \to Y^\otimes$ between Φ–quasioperads is a weak equivalence if and only if the induced morphism $C(X^\otimes) \to C(Y^\otimes)$ of complete Segal Φ–operads is so.*

Proof Let us use the characterizations of Propositions 2.8 and 7.17. Applying Lemma 10.12 and Example 10.11, we see that $X^\otimes \to Y^\otimes$ is essentially surjective if and only if the morphism $C(X^\otimes) \to C(Y^\otimes)$ is so.

To complete the proof, let us note that the inclusion $\Delta^1 \hookrightarrow NA[I \to \{1\}]$ given by $r \mapsto (r, m, 1)$ is $\mathscr{P}$–anodyne for any object $I \in \Phi$. Consequently, we obtain an equivalence

$$[I/\{1\}] \operatorname{Map}_{C(X^\otimes)}((x_i)_{i \in |I|}, y) \simeq \operatorname{Map}_{X^\otimes}^{\alpha}(x, y)$$

for any objects $x \in X_I^\otimes$, $y \in X_{\{1\}}^\otimes$, and any collection $\{x \to x_i \mid i \in |I|\}$ of inert morphisms. $\qquad \square$

In other words, C, when restricted to Φ–quasioperads is a relative functor in the sense of [3], and it reflects weak equivalences. In fact, it is part of a Quillen pair.

10.14.1 Corollary *Suppose Φ is a perfect operator category. Then the functors*

$$B \colon \operatorname{Fun}(\Delta_\Phi^{\mathrm{op}}, s\mathbf{Set}) \rightleftarrows s\mathbf{Set}_{/N\Lambda(\Phi)}^+ \colon C$$

given by the formulas

$$B(X) := \int^{(\mathbf{m}, I)} NA(\mathbf{m}, I) \times X(\mathbf{m}, I)^\sharp$$

and

$$C(Y^\otimes)(\mathbf{m}, I) := \operatorname{Map}_{N\Lambda(\Phi)}^\sharp (NA(\mathbf{m}, I), Y^\otimes)$$

form a Quillen pair for the operadic model structures.

10.15 Note that, using the functor C along with [13, Construction 2.1.1.7], we can now form the *nerve* of any symmetric operad enriched in Kan complexes as a complete Segal $\mathbf{F}$–operad.

We are now ready to state the main result of this section.

10.16 Theorem *For any perfect operator category Φ, the functors*

$$P^\otimes \colon \mathbf{Operad}_{\mathrm{CSS}}^\Phi \to \mathbf{Operad}_\infty^\Phi$$

and

$$C \colon \mathbf{Operad}_\infty^\Phi \to \mathbf{Operad}_{\mathrm{CSS}}^\Phi$$

are inverse equivalences of ∞–categories.

Proof Since we have shown that each functor is conservative, it is sufficient for us to furnish a natural transformation $\eta \colon \mathrm{id} \to P^\otimes \circ C$ that is objectwise essentially surjective and fully faithful.

For any object $\mathbf{m} \in \Delta$, any m–simplex $\sigma\colon \mathbf{m} \to \Lambda(\Phi)$, and any integers $0 \le i \le j \le m$, consider the functor $f_{\sigma,i,j}\colon AF_\sigma(i,j) \to \mathbf{m}$ over $\Lambda(\Phi)$ given by the assignment $(r,s,k) \mapsto r$; these are compatible with one another and hence define a map

$$f_\sigma\colon \operatorname{colim}_{(i,j)\in\widetilde{\mathbb{O}}(\mathbf{m})} NAF_\sigma(i,j) \to \Delta^m$$

in $s\mathbf{Set}^+_{/N\Lambda(\Phi)}$. This induces, for any Φ–quasioperad $X^\otimes$, a map

$$X^\otimes_m \to \operatorname{Mor}_{N\Lambda}\bigl(\operatorname{colim}_{(i,j)\in\widetilde{\mathbb{O}}(\mathbf{m})} NAF_\sigma(i,j), X^\otimes\bigr) \cong P^\otimes(C(X^\otimes))_m,$$

natural in $X^\otimes$, hence a natural transformation $\eta\colon \mathrm{id} \to P^\otimes \circ C$.

When $\Phi = \{1\}$, the functor $f_{\sigma,i,j}$ is the projection $\widetilde{\mathbb{O}}(\mathbf{m})^{\mathrm{op}} \to \mathbf{m}$ given by $(r,s) \to r$, which is a retract of the map considered in Example 10.11, which thus must induce an equivalence. Hence by functoriality in Φ, it follows that η is objectwise an equivalence on underlying quasicategories, and in particular is objectwise essentially surjective.

Suppose $I \in \Phi$, and consider the 1–simplex of $\Lambda(\Phi)$ given by the unique active morphism $\alpha\colon I \to \{1\}$. The map $f_\alpha\colon NA[I \to \{1\}] \to \Delta^1$ is a retract of the inclusion $\Delta^1 \hookrightarrow NA[I \to \{1\}]$ considered in the proof of Proposition 10.14. Consequently, it induces equivalences

$$\operatorname{Map}^\alpha_{X^\otimes}(x,y) \xrightarrow{\sim} \operatorname{Map}^\alpha_{P^\otimes(C(X^\otimes))}(x,y),$$

whence η is objectwise fully faithful. $\square$

10.17 Conjecture *We expect that the axiom system given in* [4] *should have an analogue for homotopy theories of weak operads over a fixed operator category Φ, and that a corresponding unicity theorem should hold.*

We conclude with a remark on the compatibility between the equivalences

$$\mathbf{Operad}^\Phi_{\mathrm{CSS}} \simeq \mathbf{Operad}^\Phi_\infty$$

and the Boardman–Vogt tensor product.

10.18 Proposition *Suppose Φ is a perfect operator category, and suppose X and Z are two Φ–quasioperads. Then one has a canonical equivalence*

$$C(\mathbf{Alg}^\Phi_\infty(X,Z)) \simeq \mathbf{Alg}^\Phi(CX,CZ).$$

Proof The claim is equivalent to the claim that for any simplicial space Y, one has a projection formula

$$B(Y)^{\{1\}} \otimes^{\Phi} X \simeq B(Y^{\{1\}} \otimes^{\Phi} CX).$$

Note that B preserves all weak equivalences (as it is left Quillen for the *injective* model structure on $\operatorname{Fun}(\Delta_{\Phi}^{\mathrm{op}}, s\mathbf{Set})$) and all colimits; hence it preserves all homotopy colimits. Consequently, we may assume that Y is either Δ^0 or Δ^1. In the former case, the result is obvious; in the latter, it follows from a computation. □

In general, we expect that the equivalence C is fully compatible with the Boardman–Vogt tensor products for any perfect operator categories, so that

$$C(Y^{\Psi} \otimes^{\Phi} X) \simeq C(Y)^{\Psi} \otimes^{\Phi} C(X).$$

This is true when both X and Y are terminal quasioperads by Theorems 8.12 and 9.10.

11 Some examples of complete Segal Φ–monoids

We may now prove the assertions stated in the introduction. First, we note that [13, Corollary 5.1.1.5] states the following.

11.1 Proposition *The ∞–operad E_{∞} over $\mathbf{F}$ is equivalent to the terminal ∞–operad over $\mathbf{F}$.*

Applying the functor C, we may also state this result in the context of complete Segal $\mathbf{F}$–operads.

11.1.1 Corollary *The complete Segal $\mathbf{F}$–operad $C(E_{\infty})$ is equivalent to the terminal complete Segal $\mathbf{F}$–operad $U_{\mathbf{F}}$.*

We may equally well state this result from the perspective of the ∞–categories of algebras.

11.1.2 Corollary *The ∞–category of E_{∞}–algebras in an ∞–operad Z over $\mathbf{F}$ is equivalent to the quasicategory associated with the complete Segal space $\mathbf{Mon}^{\mathbf{F}}(Z)$.*

Similarly, [13, Proposition 4.1.2.10 and Example 5.1.0.7], in light of the theory developed here, state the following.

11.2 Proposition *The associative ∞–operad over* **F** *and the ∞–operad E_1 over* **F** *are each equivalent to the symmetrization of the terminal ∞–operad over* **O**.

11.2.1 Corollary *The complete Segal* **F**–*operad $C(E_1)$ is equivalent to the symmetrization of the terminal complete Segal* **O**–*operad $U_{\mathbf{O}}$.*

11.2.2 Corollary *The ∞–category of E_1–algebras in an ∞–operad Z over* **F** *is equivalent to the quasicategory associated with the complete Segal space* $\mathbf{Mon}^{\mathbf{O}}(u^\star Z)$.

Since the operad A_n is the suboperad of E_1 generated by the operations of arity $\leq n$, we deduce the following.

11.3 Proposition *The complete Segal operad $C(A_n)$ is equivalent to the symmetrization of the terminal complete Segal* $\mathbf{O}_{\leq n}$–*operad.*

11.3.1 Corollary *The ∞–category of A_n–algebras in an ∞–operad Z over* **F** *is equivalent to the quasicategory associated with the complete Segal space* $\mathbf{Mon}^{\mathbf{O}_{\leq n}}(u^\star Z)$.

Finally, we have Proposition 11.4, which follows from [13, Theorem 5.1.2.2] and 9.3.

11.4 Proposition *For any integer $k \geq 0$, the ∞–operad E_k over* **F** *is equivalent to the symmetrization of the terminal ∞–operad over* $\mathbf{O}^{(k)}$.

11.4.1 Corollary *For any integer $k \geq 0$, the ∞–category of E_k–algebras in an ∞–operad Z over* **F** *is equivalent to the quasicategory associated with the complete Segal space* $\mathbf{Mon}^{\mathbf{O}^{(k)}}(U^\star Z)$.

11.4.2 Corollary *The ∞–category of E_k–algebras (in spaces) is equivalent to the ∞–category of left fibrations*

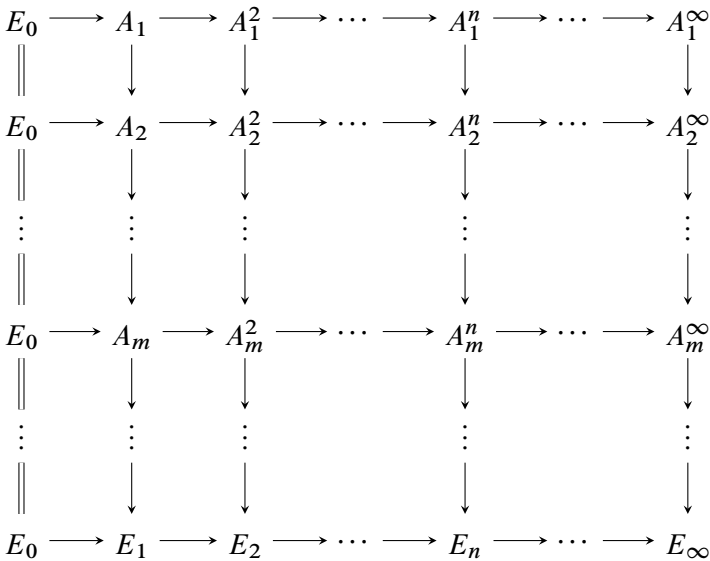$$X \to N\Lambda(\mathbf{O}^{(k)}) \simeq N\Theta_k^{\mathrm{op}}$$

satisfying the Segal condition, so that for any object $I \in \mathbf{O}^{(k)}$, the inert morphisms $\{\rho_i \colon I \to \{i\} \mid i \in |I|\}$ induce an equivalence

$$X_I \xrightarrow{\sim} \prod_{i \in |I|} X_{\{i\}}.$$

11.5 Example The formalism we have introduced invites the study of a wide range of new examples as well. In particular, let us contemplate the operator categories $\mathbf{O}^{(n)}_{\leq m} := (\mathbf{O}^{(n)})_{\leq m}$. They fit together in a diagram:

$$
\begin{array}{ccccccccc}
\{1\} & \longrightarrow & \mathbf{O}_{\leq 1} & \longrightarrow & (\mathbf{O} \wr \mathbf{O})_{\leq 1} & \longrightarrow \cdots \longrightarrow & \mathbf{O}^{(n)}_{\leq 1} & \longrightarrow \cdots \longrightarrow & \mathbf{F}_{\leq 1} \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\{1\} & \longrightarrow & \mathbf{O}_{\leq 2} & \longrightarrow & (\mathbf{O} \wr \mathbf{O})_{\leq 2} & \longrightarrow \cdots \longrightarrow & \mathbf{O}^{(n)}_{\leq 2} & \longrightarrow \cdots \longrightarrow & \mathbf{F}_{\leq 2} \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\{1\} & \longrightarrow & \mathbf{O}_{\leq m} & \longrightarrow & (\mathbf{O} \wr \mathbf{O})_{\leq m} & \longrightarrow \cdots \longrightarrow & \mathbf{O}^{(n)}_{\leq m} & \longrightarrow \cdots \longrightarrow & \mathbf{F}_{\leq m} \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\{1\} & \longrightarrow & \mathbf{O} & \longrightarrow & \mathbf{O} \wr \mathbf{O} & \longrightarrow \cdots \longrightarrow & \mathbf{O}^{(n)} & \longrightarrow \cdots \longrightarrow & \mathbf{F}
\end{array}
$$
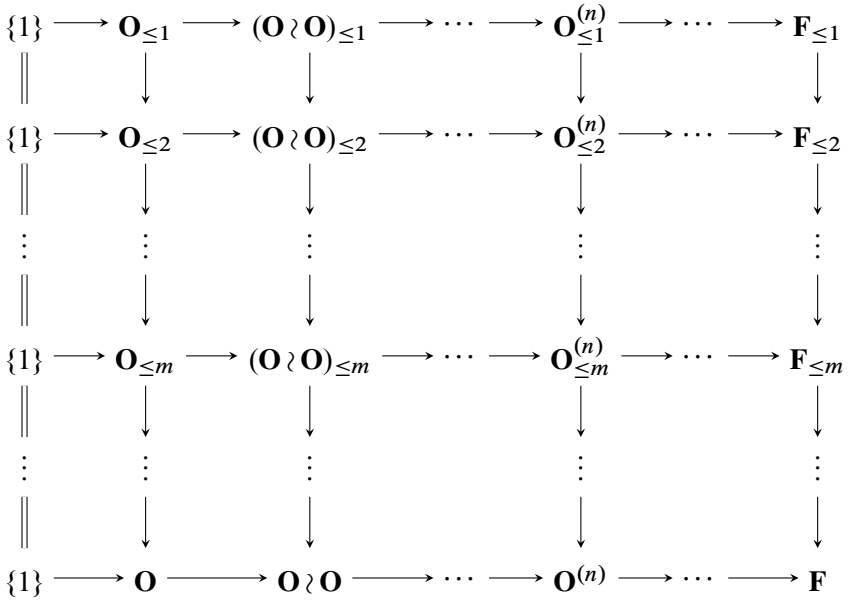
By forming the symmetrization A^n_m of the terminal complete Segal $\mathbf{O}^{(n)}_{\leq m}$–operad, we obtain a diagram of weak symmetric operads:

$$
\begin{array}{ccccccccc}
E_0 & \longrightarrow & A_1 & \longrightarrow & A_1^2 & \longrightarrow \cdots \longrightarrow & A_1^n & \longrightarrow \cdots \longrightarrow & A_1^\infty \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_0 & \longrightarrow & A_2 & \longrightarrow & A_2^2 & \longrightarrow \cdots \longrightarrow & A_2^n & \longrightarrow \cdots \longrightarrow & A_2^\infty \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_0 & \longrightarrow & A_m & \longrightarrow & A_m^2 & \longrightarrow \cdots \longrightarrow & A_m^n & \longrightarrow \cdots \longrightarrow & A_m^\infty \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow \cdots \longrightarrow & E_n & \longrightarrow \cdots \longrightarrow & E_\infty
\end{array}
$$

This is an interesting bifiltration $\{A_m^n\}_{m,n}$ of the E_∞–operad that incorporates both the E_n–operads as well as the A_n–operads. This filtration appears to include much that is already known about obstruction theories for finding E_∞ structures on spectra. In particular, when $n = 1$, we are simply looking at the filtration of $E_1 \simeq A_\infty$ by the operads A_n. When $n = \infty$, we are filtering E_∞ by the suboperads generated by the operations of arity $\leq m$, When $m = \infty$, we are looking at the filtration of E_∞ by the operads E_n. And finally, when $m = n + 1$, we expect that an algebra over A_{n+1}^n is the same thing as an n–stage E_∞ structure in the sense of Robinson [17, Section 5.2], thereby giving his "diagonal" filtration on E_∞, though we have not checked this.

Appendix A: A proof of Theorem 8.12

Suppose Φ and Ψ are two operator categories. The theorem claims that the functor

$$W \colon N\Delta_\Psi^{\mathrm{op}} \times_{N\Delta^{\mathrm{op}}} N\Delta_\Phi^{\mathrm{op}} \to N\Delta_{\Psi \wr \Phi}^{\mathrm{op}}$$

is a weak equivalence in the operadic model structure on $s\mathbf{Set}_{/N\Delta_{\Psi \wr \Phi}^{\mathrm{op}}}$. Note that this functor is faithful (in fact, pseudomonic), but not full. Let us call any object or morphism in its image *rectangular*.

The proof proceeds as follows: we find a full subcategory $\Delta_\Psi^{\mathrm{op}} \times_{\Delta^{\mathrm{op}}} \Delta_\Phi^{\mathrm{op}} \subset V^{\mathrm{op}} \subset \Delta_{\Psi \wr \Phi}^{\mathrm{op}}$ first, and we show that the inclusion $N\Delta_\Psi^{\mathrm{op}} \times_{N\Delta^{\mathrm{op}}} N\Delta_\Phi^{\mathrm{op}} \hookrightarrow NV^{\mathrm{op}}$ is an operadic weak equivalence. Then we replace the inclusion $NV^{\mathrm{op}} \hookrightarrow N\Delta_{\Psi \wr \Phi}^{\mathrm{op}}$ by a suitable left fibration $Y \to N\Delta_{\Psi \wr \Phi}^{\mathrm{op}}$, which we then show has contractible fibers by showing that every map in from a finite simplicial set can, up to homotopy, be "coned off".

A.1   Now let V denote the full subcategory of $\Delta_{\Psi \wr \Phi}^{\mathrm{op}}$ spanned by those objects $[K_0 \to \cdots \to K_m]$ such that for every integer $1 \leq i \leq m$, the fibers of the map $K_{i-1} \to K_i$ are all rectangular. Note that if Z is a $(\Psi \wr \Phi)$–operad, then for any object $[K_0 \to \cdots \to K_m] \in V$, we have a natural equivalence from $Z[K_0 \to \cdots \to K_m]$ to the (homotopy) limit of a diagram

$$\prod_{k_1 \in |K_1|} Z[K_{0,k_1} \to \{k_1\}] \quad \cdots \quad \prod_{k_m \in |K_m|} Z[K_{m-1,k_m} \to \{k_m\}]$$

$$\prod_{k_1 \in |K_i|} Z[\{k_1\}] \quad \prod_{k_{m-1} \in |K_{m-1}|} Z[\{k_{m-1}\}]$$

in which all the terms are the fiber of Z over rectangular objects, and all the maps that appear are induced by rectangular morphisms. Consequently, one may extend any

map of simplicial sets $N\Delta_\Psi^{\mathrm{op}} \times_{N\Delta^{\mathrm{op}}} N\Delta_\Phi^{\mathrm{op}} \to Z$ over $N\Delta_{\Psi\wr\Phi}^{\mathrm{op}}$, in a unique way up to homotopy, to a map $NV^{\mathrm{op}} \to Z$ over $N\Delta_{\Psi\wr\Phi}^{\mathrm{op}}$; in other words, the map

$$[NV^{\mathrm{op}}, Z]_{N\Delta_{\Psi\wr\Phi}^{\mathrm{op}}} \to [N\Delta_\Psi^{\mathrm{op}} \times_{N\Delta^{\mathrm{op}}} N\Delta_\Phi^{\mathrm{op}}, Z]_{N\Delta_{\Psi\wr\Phi}^{\mathrm{op}}}$$

is a bijection. Hence the functor W induces a weak equivalence

$$N\Delta_\Psi^{\mathrm{op}} \times_{N\Delta^{\mathrm{op}}} N\Delta_\Phi^{\mathrm{op}} \xrightarrow{\sim} NV^{\mathrm{op}}$$

in the operadic model structure on $s\mathbf{Set}_{/N\Delta_{\Psi\wr\Phi}^{\mathrm{op}}}$.

A.2 Now the inclusion $NV^{\mathrm{op}} \hookrightarrow (N\Delta_{\Psi\wr\Phi}^{\mathrm{op}})_{NV^{\mathrm{op}}/}$ is a trivial cofibration for the covariant model structure, and the target is fibrant [13, Corollary 2.1.2.2]. Hence it suffices now to prove that the natural map of left fibrations $(N\Delta_{\Psi\wr\Phi}^{\mathrm{op}})_{NV^{\mathrm{op}}/} \to N\Delta_{\Psi\wr\Phi}^{\mathrm{op}}$ is a pointwise weak equivalence. In other words, we aim to show that for any object $(\mathbf{m}, K) = [K_0 \to \cdots \to K_m]$, the nerve of the ordinary category $V_{(\mathbf{m},K)/}$ is contractible.

A.3 The objects of $V_{(\mathbf{m},K)/}$ can be described as triples

$$\big(\eta\colon \mathbf{m} \to \mathbf{n},\ L\colon \mathbf{n} \to \Psi\wr\Phi,\ \phi\colon K \to L\circ\eta\big)$$

consisting of a morphism η of Δ, a functor $L\colon \mathbf{n} \to \Psi\wr\Phi$ such that every fiber of each morphism $L_{j-1} \to L_j$ is rectangular, and a map $\phi\colon K \to L\circ\eta$ such that each morphism $\phi_i\colon K_i \hookrightarrow L_{\eta(i)}$ is an interval inclusion and the squares

$$\begin{array}{ccc}
K_{i-1} & \lhook\joinrel\longrightarrow & L_{\eta(i-1)} \\
\downarrow & & \downarrow \\
K_i & \lhook\joinrel\longrightarrow & L_{\eta(i)}
\end{array}$$

are all pullbacks. Now denote by $R_{(\mathbf{m},K)} \subset V_{(\mathbf{m},K)/}$ the full subcategory spanned by those objects $(\eta\colon \mathbf{m} \to \mathbf{n}, L, \phi)$ such that

- η has the property that $\eta(0) = 0$ and $\eta(m) = n$,
- $L_{\eta(i)} = K_i$ for any $0 \le i \le m$, and
- ϕ_i is the identity for any $0 \le i \le m$.

The inclusion i of this subcategory admits a retraction r given by the functor carrying any triple $(\theta\colon \mathbf{m} \to \mathbf{p}, M, \psi) \in V_{(\mathbf{m},K)/}$ to the triple $(\eta\colon \mathbf{m} \to \mathbf{n}, L, \phi) \in R_{(\mathbf{m},K)}$ in which

- $\eta\colon \mathbf{m} \to \mathbf{n} = \mathbf{p}_{\theta(0)/\ /\theta(m)}$ is the morphism induced by θ,
- $L_j = M_j \times_{M_{\theta(m)}} K_m$ for any $\theta(0) \le j \le \theta(m)$, and
- ϕ_i is the identity for any $0 \le i \le m$.

The factorization of ψ through ϕ provides a natural transformation $i \circ r \to \mathrm{id}$; hence we have a weak homotopy equivalence $NV_{(\mathbf{m},K)/} \simeq NR_{(\mathbf{m},K)}$. It thus remains to show that the nerve of $R_{(\mathbf{m},K)}$ is contractible.

A.4 If there exists an integer $1 \le i \le m$ such that the morphism $K_{i-1} \to K_i$ is an isomorphism, then it is easy to see that the natural functor

$$R_{[K_0 \to \cdots \to K_{i-1} \to K_{i+1} \to \cdots \to K_m]} \to R_{[K_0 \to \cdots \to K_m]}$$

induces an equivalence on nerves. Consequently, we may assume that none of the morphisms $K_{i-1} \to K_i$ is an isomorphism, and consequently, that for any object $(\eta \colon \mathbf{m} \to \mathbf{n}, L, \phi) \in R_{(\mathbf{m},K)}$, the map η is injective.

Now for any integer $m \ge 0$, we note that the natural functor

$$R_{[K_0 \to \cdots \to K_m]} \to R_{[K_0 \to K_1]} \times \cdots \times R_{[K_{m-1} \to K_m]}$$

is an equivalence of categories. Hence we reduce to the case in which $m = 1$.

A.5 So suppose that $m = 1$, and let us set about showing that the nerve of the category $R_{[K_0 \to K_1]}$ is contractible. By definition any morphism $g \colon L \to M$ of $\Psi \wr \Phi$ can be factored, in an essentially unique manner, as a map $L \to M(g)$ covering the identity morphism in Φ, followed by a morphism $M(g) \to M$ that is cocartesian for the coronal fibration $\Psi \wr \Phi \to \Phi$. Let us call this a *coronal factorization*. The resulting sequence $[L \to M(g) \to M]$ clearly lies in V.

Fix a coronal factorization $[K_0 \to K_{01} \to K_1] \in R_{[K_0 \to K_1]}$. Our aim is now to produce, for any finite simplicial set X and any map $g \colon X \to NR_{[K_0 \to K_1]}$, a weakly contractible simplicial set $P_g^* \supset X$ with a distinguished vertex v and an extension of g to a map $G \colon P_g^* \to NR_{[K_0 \to K_1]}$ such that $G|X = g$ and $G(v) = [K_0 \to K_{01} \to K_1] \in R_{[K_0 \to K_1]}$. This will complete the proof.

A.6 We now set about constructing the simplicial set P_g^*. Denote by $\mathbf{I}$ the category whose objects are objects $\mathbf{n} \in \Delta$ such that $n \ge 1$ and whose morphisms $\mathbf{n} \to \mathbf{n}'$ are morphisms $\mathbf{n} \to \mathbf{n}'$ of Δ that carry 0 to 0 and n to n'. We have an obvious projection $R_{[K_0 \to K_1]} \to \mathbf{I}$ given by $(\eta \colon \mathbf{1} \to \mathbf{n}, L, \phi) \mapsto \mathbf{n}$.

Let $\mathbf{P}$ be the following category. The objects will be triples $(\mathbf{n}, \mathbf{p}, h)$, where $\mathbf{m}, \mathbf{p} \in \mathbf{I}$, and h is a functor $\mathbf{p} \to \mathrm{Fun}(\mathbf{1}, \mathbf{n})$ such that $h(0) = 00$, $h(m) = nn$, and for any $1 \le i \le n$, the map $h(i-1) \to h(i)$ is of the form of one of the following:

$$ij \to ik, \quad ik \to jk, \quad \text{or} \quad ik \to (i+1)(k+1).$$

A morphism $(\mathbf{n}, \mathbf{p}, h) \to (\mathbf{n}', \mathbf{p}', h')$ of $\mathbf{P}$ is a morphism $\sigma \colon \mathbf{n} \to \mathbf{n}'$ of $\mathbf{I}$ such that one has an inclusion

$$\{h(i) \mid 0 \le i \le n\} \subset \{\sigma(h'(i')) \mid 0 \le i' \le n'\}.$$

We have the obvious projection $q \colon \mathbf{P} \to \mathbf{I}$ given by $(\mathbf{n}, \mathbf{p}, h) \mapsto \mathbf{n}$.

For any integer $n \ge 1$, one has a map $\mathbf{n} \to \mathrm{Fun}(\mathbf{1}, \mathbf{n})$ that carries each i to the identity map at i. This defines a section $i_0 \colon \mathbf{I} \to \mathbf{P}$ of q. There is another section $i_1 \colon \mathbf{I} \to \mathbf{P}$ of q, which carries an object $\mathbf{n}$ to the triple $(\mathbf{n}, \mathbf{2}, [00 \to 0n \to nn])$.

For any integer $N \ge 1$, we can consider the full subcategory $\mathbf{I}_{\le N} \subset \mathbf{I}$ spanned by those objects $\mathbf{n}$ such that $n \le N$, and we can consider the pullback $\mathbf{P}_{\le N} := \mathbf{I}_{\le N} \times_{\mathbf{I}} \mathbf{P}$. One shows easily that when restricted to $\mathbf{I}_N$, there exist a zigzag of natural transformations connecting i_0 and i_1 and a zigzag of natural transformations connecting $i_0 \circ q$ and the identity map on $\mathbf{P}_{\le N}$.

Now for any finite simplicial set X and any map $g \colon X \to NR_{[K_0 \to K_1]}$, let P_g denote the fiber product:

$$\begin{array}{ccc} P_g & \longrightarrow & N\mathbf{P} \\ \downarrow & & \downarrow \\ X \longrightarrow NR_{[K_0 \to K_1]} & \longrightarrow & N\mathbf{I} \end{array}$$

The sections i_0 and i_1 pull back to sections of $P_g \to X$. Since X is finite, there exists an integer $N \ge 1$ such that the composite $X \to NR_{[K_0 \to K_1]} \to N\mathbf{I}$ factors through $N\mathbf{I}_{\le N}$. Hence the zigzag of homotopies between i_0 and i_1 and between $i_0 \circ q$ and the identity map on $\mathbf{P}_{\le N}$ lift, and we deduce that $P_g \to X$ is an equivalence.

Now set $P_g^* = P_g / i_1(X)$; let v be the vertex corresponding to $i_1(X)$, and regard X as a simplicial subset of P_g^* via i_0. It is thus clear that P_g^* is weakly contractible.

A.7 Now we define the extension of g to a map $G \colon P_g^* \to NR_{[K_0 \to K_1]}$. Applying the coronal factorization repeatedly, one obtains, functorially, for any object

$$[K_0 = L_0 \to L_1 \to \cdots \to L_n = K_1] \in R_{[K_0 \to K_1]},$$

a functor $\bar{L} \colon \mathrm{Fun}(\mathbf{1}, \mathbf{n}) \to \Psi \wr \Phi$ such that

- for any integer $0 \le i \le n$, one has $\bar{L}_{ii} = L_i$,
- the factorization $\bar{L}_{ii} \to \bar{L}_{ij} \to \bar{L}_{jj}$ is a coronal factorization, and
- the factorization $K_0 = L_0 \to L_{0n} \to L_n = K_1$ is equal to our chosen coronal factorization $K_0 \to K_{01} \to K_1$.

It is easy to check that the morphisms

$$\bar{L}_{ij} \to \bar{L}_{ik}, \quad \bar{L}_{ik} \to \bar{L}_{jk}, \quad \text{and} \quad \bar{L}_{ik} \to \bar{L}_{(i+1)(k+1)}$$

all have rectangular fibers. This defines an extension of g to a map

$$P_g \to NR_{[K_0 \to K_1]},$$

which then factors through P_g^*, as desired.

Appendix B: A proof of Theorem 5.10

Suppose Φ is a perfect operator category. There are two unit axioms and an associativity axiom to be checked for the endofunctor T equipped with the natural transformations ι and μ.

B.1   The first unit axiom is the assertion that $(T\iota) \circ \mu = \mathrm{id}_T$. To verify this, we first claim that for any morphism $\phi\colon I \to T$ of Φ, one has $\sigma_I \circ \iota_I = U\lambda_I$, where $\lambda\colon \mathrm{id}_{(\Phi/T)} \to E \circ \mathrm{fib}$ is the unit natural transformation. Indeed, the diagram

commutes, and the special fiber of the composite is the identity on J_t. Hence one has

$$(T\iota) \circ \mu = (UE\kappa) \circ (U\lambda E).$$

The triangle identity for the adjunction (fib, E) now implies that

$$(T\iota) \circ \mu = U\,\mathrm{id}_E = \mathrm{id}_T,$$

as desired. The second unit axiom is analogous.

B.2   It remains to prove the associativity condition; that is, that for any object $I \in \Phi$, the following diagram commutes:

(B.2.1)
$$\begin{array}{ccc}
T^3 I & \xrightarrow{T\mu_I} & T^2 I \\
{\scriptstyle \mu_{TI}} \downarrow & & \downarrow {\scriptstyle \mu_I} \\
T^2 I & \xrightarrow[\mu_I]{} & TI
\end{array}$$

We begin with the following key technical lemma.

B.3 Lemma *For any object I of Φ, the following rectangle is a pullback:*

$$
\begin{array}{ccccc}
I & \xrightarrow{\iota_I} & TI & \xrightarrow{\iota_{TI}} & T^2I \\
\| & & & & \downarrow{\mu_I} \\
I & & \xrightarrow{\iota_I} & & TI
\end{array}
$$

Now we can prove the associativity.

Proof Form the larger diagram:

$$
\begin{array}{ccccc}
I & \xrightarrow{\iota_I} & TI & \xrightarrow{\iota_{TI}} & T^2I \\
\| & & & & \downarrow{\mu_I} \\
I & & \xrightarrow{\iota_I} & & TI \\
\downarrow & & & & \downarrow{e_I} \\
* & & \xrightarrow{t} & & T
\end{array}
$$

The lower diagram is a pullback diagram; hence it suffices to show that the exterior rectangle is a pullback. To verify this, observe that $\mu_* = \chi_t$; hence the naturality of μ implies that the square

$$
\begin{array}{ccc}
T^2I & \xrightarrow{Te_I} & TT \\
\mu_I \downarrow & & \downarrow{\chi_t} \\
TI & \xrightarrow{e_I} & T
\end{array}
$$

commutes, since $e_I = T(!)$. All the rectangles of the diagram

$$
\begin{array}{ccccc}
I & \xrightarrow{\iota_I} & TI & \xrightarrow{\iota_{TI}} & T^2I \\
\downarrow & & & & \downarrow{Te_I} \\
* & \xrightarrow{t} & T & \xrightarrow{\iota_T} & TT \\
\| & & & & \downarrow{\chi_t} \\
* & & \xrightarrow{t} & & T
\end{array}
$$

are pullback squares, whence the desired result. $\qquad\square$

B.4   The first claim is that, using the structure morphism $e_I \colon TI \to T$, one may view the square (B.2.1) as a square of (Φ/T); that is, we claim that the diagram

(B.4.1)

$$
\begin{array}{ccc}
 & T^2I \xrightarrow{\ \mu_I\ } TI & \\
T\mu_I \nearrow & & \searrow e_I \\
T^3I & & T \\
\mu_{TI} \searrow & & \nearrow e_I \\
 & T^2I \xrightarrow[\ \mu_I\ ]{} TI &
\end{array}
$$

commutes. To see this, consider the cube:

The outer square commutes since each composite $T^2T \to T$ is a conservative morphism $(T^2T, \iota_{TT}\iota_T(t)) \to (T, t)$. The top square commutes by naturality, and the commutativity of the two side faces and the bottom face follows from the commutativity of (5.9.1). Thus the outer rectangle

$$
\begin{array}{ccc}
T^3I & \xrightarrow{\ \mu_{TI}\ } & T^2I \\
\downarrow{\scriptstyle T^2e_I} & & \downarrow{\scriptstyle Te_I} \\
T^2T & \xrightarrow{\ \mu_T\ } & TT \\
\downarrow{\scriptstyle T\chi_t} & & \downarrow{\scriptstyle \chi_t} \\
TT & \xrightarrow[\ \chi_t\ ]{} & T
\end{array}
$$

commutes, and it follows that (B.4.1) does as well, since the bottom square of the

following diagram commutes:

$$
\begin{array}{ccc}
T^3I & \xrightarrow{\mu_{TI}} & T^2I \\
{\scriptstyle T\mu_I}\downarrow & & \downarrow{\scriptstyle \mu_I} \\
& T^2I \xrightarrow{\mu_I} TI & \\
{\scriptstyle Te_I}\downarrow & & \downarrow{\scriptstyle e_I} \\
TT & \xrightarrow{\chi_t} & T
\end{array}
$$

Now write K for the special fiber of the composite

$$
T^3I \xrightarrow{\mu_{TI}} T^2I \xrightarrow{\mu_I} TI \xrightarrow{e_I} T,
$$

and write L for the special fiber of the composite

$$
T^3I \xrightarrow{T\mu_I} T^2I \xrightarrow{\mu_I} TI \xrightarrow{e_I} T.
$$

By adjunction it suffices to show that the two morphisms of special fibers $K \to I$ and $L \to I$ are equal. To compute K, consider the following diagram:

$$
\begin{array}{ccccccc}
I & \xrightarrow{\iota_I} & TI & \xrightarrow{\iota_{TI}} & T^2I & \xrightarrow{\iota_{T^2I}} & T^3I \\
\| & & \| & & & & \downarrow{\scriptstyle \mu_{TI}} \\
I & \xrightarrow{\iota_I} & TI & \xrightarrow{\hspace{2em}\iota_{TI}\hspace{2em}} & & & T^2I \\
\| & & & & & & \downarrow{\scriptstyle \mu_I} \\
I & \xrightarrow{\hspace{4em}\iota_I\hspace{4em}} & & & & & TI \\
\downarrow & & & & & & \downarrow{\scriptstyle e_I} \\
* & \xrightarrow{\hspace{4em}t\hspace{4em}} & & & & & T
\end{array}
$$

By B.3, every rectangle of this diagram is a pullback. To compute L, consider the following diagram:

$$
\begin{array}{ccccccc}
I & \xrightarrow{\iota_I} & TI & \xrightarrow{\iota_{TI}} & T^2I & \xrightarrow{\iota_{T^2I}} & T^3I \\
\| & & \| & & & & \downarrow{\scriptstyle T\mu_I} \\
I & \xrightarrow{\iota_I} & TI & \xrightarrow{\hspace{2em}\iota_{TI}\hspace{2em}} & & & T^2I \\
\| & & & & & & \downarrow{\scriptstyle \mu_I} \\
I & \xrightarrow{\hspace{4em}\iota_I\hspace{4em}} & & & & & TI \\
\downarrow & & & & & & \downarrow{\scriptstyle e_I} \\
* & \xrightarrow{\hspace{4em}t\hspace{4em}} & & & & & T
\end{array}
$$

Again by B.3, and since T preserves all pullbacks, every rectangle of this diagram is a pullback. Hence $K = L = I$, and the morphisms $K \to I$ and $L \to I$ are each simply the identity.

This completes the proof of the theorem.

Appendix C: A proof of Theorem 5.18

Fix an admissible functor $F \colon \Psi \to \Phi$ between perfect operator categories. We have to show that the two diagrams (5.17.1) commute with $C = \Psi$, $D = \Phi$, and $\eta = \alpha_F$.

C.1 Note that F induces a functor

$$F_{/TT} \colon (\Psi/T_\Psi T_\Psi) \to (\Phi/T_\Phi T_\Phi)$$

that assigns to any object $I \to T_\Psi T_\Psi$ the composite

$$FI \to FT_\Psi T_\Psi \xrightarrow{\alpha_{F,T_\Psi}} T_\Phi FT_\Psi \xrightarrow{T_\Phi \chi_{F(t_\Psi)}} T_\Phi T_\Phi.$$

The commutativity of the first diagram of (5.17.1) follows directly from the following.

C.2 Lemma *The following diagram of natural transformations of functors*

$$(\Phi/T_\Phi) \to (\Psi/T_\Psi)$$

commutes:

$$
\begin{array}{ccccc}
F_{/T} \circ \chi_{t_\Psi,!} \circ E_{\Psi,/T_\Psi} & =\!=\!= & \chi_{t_\Phi,!} \circ F_{/TT} \circ E_{\Psi,/T_\Psi} & \xrightarrow{\chi_{t_\Phi,!} \circ \alpha_F} & \chi_{t_\Phi,!} \circ E_{\Phi,/T_\Phi} \circ F_{/T} \\
_{F_{/T} \circ \sigma_\Psi} \downarrow & & & & \downarrow _{\sigma_\Phi \circ F_{/T}} \\
F_{/T} \circ E_\Psi \circ \mathrm{fib}_\Psi & \xrightarrow{\alpha_F \circ \mathrm{fib}_\Psi} & E_\Phi \circ F \circ \mathrm{fib}_\Psi & =\!=\!= & E_\Phi \circ \mathrm{fib}_\Phi \circ F_{/T}
\end{array}
$$

Proof Suppose $\psi \colon J \to T_\Psi$ is a morphism. The claim is that the diagram

$$
\begin{array}{ccc}
FT_\Psi J & \longrightarrow & T_\Phi FJ \\
\downarrow & & \downarrow \\
FT_\Psi J_{t_\Psi} & \longrightarrow & T_\Phi (FJ)_{F(t_\Psi)}
\end{array}
$$

of (Φ/T_Φ) commutes, where the structure morphisms in question are

$$FT_\Psi J \xrightarrow{FT_\Psi\psi} FT_\Psi T_\Psi \xrightarrow{F(\chi_{t_\Psi})} FT_\Psi \xrightarrow{\chi_{F(t_\Psi)}} T_\Phi,$$
$$T_\Phi FJ \xrightarrow{T_\Phi F\psi} T_\Phi FT_\Psi \xrightarrow{T_\Phi\chi_{F(t_\Psi)}} T_\Phi T_\Phi \xrightarrow{\chi_{t_\Phi}} T_\Phi,$$
$$FT_\Psi J_{t_\Psi} \xrightarrow{Fe_{J_{t_\Psi}}} FT_\Psi \xrightarrow{\chi_{F(t_\Psi)}} T_\Phi,$$

and of course the usual structure morphism $e_{F(J_{t_\Psi})}\colon T_\Phi F(J_{t_\Psi}) \to T_\Phi$. By adjunction it suffices to show that the square of special fibers

$$
\begin{array}{ccc}
(FT_\Psi J)_{t_\Phi} & \longrightarrow & (T_\Phi FJ)_{t_\Phi} \\
\downarrow & & \downarrow \\
(FT_\Psi J_{t_\Psi})_{t_\Phi} & \longrightarrow & (T_\Phi(FJ)_{F(t_\Psi)})_{t_\Phi}
\end{array}
$$

commutes. But since the special fiber of the morphisms $\alpha_{F,J}$ and $\alpha_{F,J_{t_\Psi}}$ are each the identity, the special fiber square is in particular commutative. $\qquad\square$

C.3 We now prove the commutativity of the second diagram of (5.17.1). The claim is that for any object J of Ψ, the diagram

$$
\begin{array}{ccc}
 & & FT_\Psi J \\
 & \nearrow^{F\iota_J} & \\
FJ & & \downarrow{\alpha_{F,J}} \\
 & \searrow_{\iota_{FJ}F} & \\
 & & T_\Phi FJ
\end{array}
$$

commutes. By composing with the structure map $e_{FJ}\colon T_\Phi FJ \to T_\Phi$, this can be regarded as a diagram of (Φ/T_Φ). Hence it suffices to verify that the special fiber triangle

$$
\begin{array}{ccc}
 & & (FT_\Psi J)_{t_\Phi} \\
 & \nearrow & \\
(FJ)_{t_\Phi} & & \downarrow \\
 & \searrow & \\
 & & (T_\Phi FJ)_{t_\Phi}
\end{array}
$$

commutes. But since the special fiber of $\alpha_{F,J}\colon FT_\Psi J \to T_\Phi FJ$ is the identity on FJ, the special fiber triangle commutes.

This completes the proof of the theorem.

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