

# Unfolded Seiberg–Witten Floer spectra I: Definition and invariance

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Let  $Y$  be a closed and oriented 3–manifold. We define different versions of unfolded Seiberg–Witten Floer spectra for  $Y$ . These invariants generalize Manolescu’s Seiberg–Witten Floer spectrum for rational homology 3–spheres. We also compute some examples when  $Y$  is a Seifert space.

[57R57](#); [57R58](#)

## 1 Introduction

The Seiberg–Witten equations, introduced in Witten [35], and related theories have been playing a central role in the study of smooth 4–dimensional manifolds since the 1990s. Following the seminal work of Floer [9], Kronheimer and Mrowka [17] used Seiberg–Witten equations on 3–manifolds to construct monopole Floer homology. The monopole Floer homology and its counterparts are powerful invariants of 3–manifolds and became an important tool in the study of low-dimensional topology with many remarkable applications.

In the contexts of symplectic Floer theory and instanton Floer theory, Cohen, Jones and Segal [4] posed a question of constructing a “Floer spectrum”, an object whose homology recovers the Floer homology. In 2003, Manolescu [22] first constructed the Seiberg–Witten Floer spectrum for rational homology 3–spheres by incorporating Furuta’s technique [10] of finite-dimensional approximation in Seiberg–Witten theory and Conley index theory [5]. It has been just recently shown by Lidman and Manolescu [19] that the homology of this spectrum is isomorphic to the monopole Floer homology. In principle, the Seiberg–Witten Floer spectrum can be thought as a stable homotopy refinement of Floer homology. For example, one can apply the K-theory functor to this spectrum and define “Seiberg–Witten Floer K-theory” as well

as other generalized homology theories; see Manolescu [23], Furuta and Li [11] and Lin [21] for applications in this direction.

As monopole Floer homology is defined for general 3-manifolds, it is a natural question to extend Manolescu’s construction to any 3-manifold  $Y$  with  $b_1(Y) > 0$ . In the case where  $b_1(Y) = 1$ , Kronheimer and Manolescu [16] constructed a periodic pro-spectrum for such  $Y$  with nontorsion  $\text{spin}^c$  structure. The first author [13] gave an approach to construct Seiberg–Witten Floer spectrum for a general case.

The main goal of the current paper is to rigorously construct the “unfolded” version of Seiberg–Witten Floer spectrum for general 3-manifolds. Our invariants come with two variations: type-A invariant and type-R invariant. The letters “A” and “R” stand for attractor and repeller, which are notions in dynamical systems and play a role in our construction.

**Theorem 1.1** *Let  $Y$  be a closed, oriented 3-manifold and let  $\mathfrak{s}$  be a  $\text{spin}^c$  structure on  $Y$ . Given a Riemannian metric  $g$  on  $Y$  and a  $\text{spin}^c$  connection  $A_0$  which induces a connection on the determinant bundle of the spinor bundle with harmonic curvature, we can define*

$$\underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1) \quad \text{and} \quad \underline{\text{swf}}^R(Y, \mathfrak{s}, A_0, g; S^1)$$

as a direct system and an inverse system in the  $S^1$ -equivariant stable category. These objects are well defined up to canonical isomorphisms in the corresponding categories.

In the case where  $c_1(\mathfrak{s})$  is nontorsion and  $l = \text{gcd}\{(h \cup [c_1(\mathfrak{s})])[Y] \mid h \in H^1(Y; \mathbb{Z})\}$ , the objects  $\underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1)$  and  $\underline{\text{swf}}^R(Y, \mathfrak{s}, A_0, g; S^1)$  are  $l$ -periodic in the sense that

$$\begin{aligned} \Sigma^{(l/2)\mathbb{C}} \underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1) &\cong \underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1), \\ \Sigma^{(l/2)\mathbb{C}} \underline{\text{swf}}^R(Y, \mathfrak{s}, A_0, g; S^1) &\cong \underline{\text{swf}}^R(Y, \mathfrak{s}, A_0, g; S^1). \end{aligned}$$

When the metric  $g$  or the connection  $A_0$  changes, the objects  $\underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1)$  and  $\underline{\text{swf}}^R(Y, \mathfrak{s}, A_0, g; S^1)$  can change only by suspending or desuspending by copies of the complex representation  $\mathbb{C}$  of  $S^1$ .

In the case where  $c_1(\mathfrak{s})$  is torsion, we can normalize the above objects to obtain invariants

$$\underline{\text{SWF}}^A(Y, \mathfrak{s}; S^1) \quad \text{and} \quad \underline{\text{SWF}}^R(Y, \mathfrak{s}; S^1)$$

of the  $\text{spin}^c$  manifold  $(Y, \mathfrak{s})$ .

A portion of this paper is devoted to proving that our construction is well defined, ie it does not depend on choices involved in the construction up to canonical isomorphisms. Note that, for rational homology 3–spheres, the invariants  $\underline{\text{SWF}}^A$  and  $\underline{\text{SWF}}^R$  are the same and they agree with Manolescu’s spectrum. In the case where  $b_1(Y) = 1$  and  $\mathfrak{s}$  is nontorsion,  $\underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1)$  is equivalent to  $\text{SWF}_0(Y, \mathfrak{s}, g, A_0)$ , constructed by Kronheimer and Manolescu.

**Remark** According to Furuta (personal communication, 2015), one could set up a periodically graded category so that it is possible to define  $\underline{\text{SWF}}^A(Y, \mathfrak{s}; S^1)$  and  $\underline{\text{SWF}}^R(Y, \mathfrak{s}; S^1)$  as invariants of the manifold in the nontorsion case.

When  $\mathfrak{s}$  is a spin structure, there is an additional  $\text{Pin}(2)$ –symmetry on the Seiberg–Witten equations. The  $\text{Pin}(2)$ –equivariant Seiberg–Witten Floer spectrum for a rational homology sphere is instrumental in Manolescu’s solution [24] of the triangulation conjecture. For a general spin 3–manifold, we have the following generalization:

**Theorem 1.2** *Let  $Y$  be a closed, oriented 3–manifold and let  $\mathfrak{s}$  be a spin structure on  $Y$ . We can obtain*

$$\underline{\text{SWF}}^A(Y, \mathfrak{s}; \text{Pin}(2)) \quad \text{and} \quad \underline{\text{SWF}}^R(Y, \mathfrak{s}; \text{Pin}(2))$$

*as  $\text{Pin}(2)$ –equivariant analogs of  $\underline{\text{SWF}}^A(Y, \mathfrak{s}; S^1)$  and  $\underline{\text{SWF}}^R(Y, \mathfrak{s}; S^1)$ .*

Let us try to explain the motivation of our “unfolded” construction, which is inspired by [16]. Intuitively, the monopole Floer homology is a Morse–Floer homology of a quotient configuration space  $\text{Coul}(Y)/H^1(Y; \mathbb{Z})$ , where  $\text{Coul}(Y)$  is a Hilbert space of configurations with gauge fixing. We see that this is a Hilbert bundle when  $b_1(Y) > 0$ , and we cannot simply use vector spaces for finite-dimensional approximation. There is also a topological obstruction to finding a good sequence of subbundles for finite-dimensional approximation; see [16, Proposition 6]. Thus, we instead do finite-dimensional approximation on  $\text{Coul}(Y)$ . Since the Seiberg–Witten solutions and trajectories are no longer compact on  $\text{Coul}(Y)$ , we will consider spectra obtained from an increasing sequence of bounded sets with nice properties on  $\text{Coul}(Y)$ . Our unfolded spectrum is then obtained as a direct (or inverse) system from these spectra.

Recall that there is an equivalence between the homology (with twisted coefficients) of a topological space and the usual homology of its covering space; see, for example, Hatcher [12, Section 3H]. Since  $\text{Coul}(Y)$  is the universal covering space of

$\text{Coul}(Y)/H^1(Y; \mathbb{Z})$ , we expect the homology of our unfolded invariants to agree with monopole Floer homology with fully twisted coefficients, ie homology with a local system on the blown up configuration space whose fiber is the group ring  $\mathbb{Z}[H^1(Y; \mathbb{Z})]$ . By equivalence of monopole Floer homology and Heegaard Floer homology, the corresponding Heegaard Floer group with totally twisted coefficient  $\underline{\text{HF}}(Y, \mathfrak{s})$  is constructed by Ozsváth and Szabó [30, Section 8]. This inspires us to use underline notation  $\underline{\text{SWF}}$  for the unfolded spectrum. Moreover, it should be possible to give a rigorous proof of this speculation with techniques developed by Lidman and Manolescu [19]. However, this is not the aim of the present paper.

In another direction, the third author [34] defined a folded version of Seiberg–Witten Floer spectra in the case that the topological obstruction, as mentioned above, vanishes. The first author [13, Chapter 6] also gave an approach to define a folded invariant, called twisted Floer spectrum, for general 3–manifolds as a twisted parametrized spectrum. These theories will not be discussed here either.

One of the main complications to show well-definedness of our invariants is that we need to perturb the Chern–Simons–Dirac functional in the construction. First, we perturb the functional by a nonexact 2–form so that the functional is balanced; see Section 2. Second, we require that the set of critical points be discrete modulo gauge, otherwise we cannot construct a good sequence of bounded subsets to apply finite-dimensional approximation. As a result, the space of such perturbations may not be path connected and we cannot use a standard homotopy argument here. Note that this difficulty was avoided in Manolescu’s original construction because perturbations are not necessary in the case of homology spheres. Also note that our construction is flexible because it only requires the discreteness of critical points and avoids the need for the full transversality of Kronheimer–Mrowka’s monopole Floer homology.

In general, our invariants are quite difficult to compute. However, by using the explicit description of the Seiberg–Witten moduli space for Seifert manifolds by Mrowka, Ozsváth and Yu [26] and a refinement of the rescaling technique developed by the first author [13], we are able to give explicit computations of the invariants in torsion cases of the following manifolds:

- (1) the manifold  $S^2 \times S^1$ ;
- (2) large-degree circle bundles over surfaces;
- (3) all nil manifolds;
- (4) all flat manifolds except  $T^3$ .

At the end of this introductory section, we briefly mention further developments that we hope to cover in our subsequent papers:

- As an extension of Manolescu’s construction [22], we will define relative Bauer–Furuta invariants for a 4–manifold whose boundary can be an arbitrary 3–manifold.
- The relative invariants will give new inequalities regarding intersection forms of spin 4–manifolds with boundary as in [23].
- We will establish Spanier–Whitehead duality between  $\underline{\text{SWF}}^A$  and  $\underline{\text{SWF}}^R$ .
- We will prove a generalized gluing theorem for the relative Bauer–Furuta invariants.
- Various applications of the generalized gluing theorem include the following: behavior of the fiberwise Bauer–Furuta invariant under surgery along loops, generalization of Bauer’s connected sum theorem [3]; nonexistence of essential spheres with trivial normal bundle in a 4–manifold with nontrivial Bauer–Furuta invariant, and a Künneth formula for Manolescu’s spectrum.

The paper is organized as follows: Section 2 covers some of the basics of the Seiberg–Witten equations. Section 3 gives the analytical results which are needed in our constructions. Section 4 reviews some elementary facts about the Conley index theory. Section 5 constructs the spectrum invariants. Section 6 proves the invariance. Sections 7 and 8 are devoted to the calculation of the examples.

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## 2 The Chern–Simons–Dirac functional and Seiberg–Witten trajectories

Let  $Y$  be a closed, oriented (but not necessarily connected) 3–manifold endowed with a  $\text{spin}^c$  structure  $\mathfrak{s}$  and a Riemannian metric  $g$ . We denote its connected components by  $Y_1, \dots, Y_{b_0}$  and denote by  $b_1 = b_1(Y)$  its first Betti number. Let  $S_Y$  be the associated spinor bundle and  $\rho: TY \rightarrow \text{End}(S_Y)$  be the Clifford multiplication. After fixing a

base  $\text{spin}^c$  connection  $A_0$ , the space of  $\text{spin}^c$  connections on  $S_Y$  can be identified with  $i\Omega^1(Y)$  via the correspondence  $A \mapsto A - A_0$ .

Let  $A_0^t$  be the connection on  $\det(S_Y)$  induced by  $A_0$ . We choose  $A_0$  such that the curvature  $F_{A_0^t}$  equals  $2\pi i\nu_0$ , where  $\nu_0$  is the harmonic 2-form representing  $-c_1(\mathfrak{s})$ . For a 1-form  $a \in i\Omega^1(Y)$ , we let  $\not{D}_{A_0+a}$  be the Dirac operator associated with the connection  $A_0 + a$ . We also denote by  $\not{D} := \not{D}_{A_0}$  the Dirac operator corresponding to the base connection, so we have  $\not{D}_{A_0+a} = \not{D} + \rho(a)$ .

The gauge group  $\text{Map}(Y, S^1)$  acts on the space  $i\Omega^1(Y) \oplus \Gamma(S_Y)$  by

$$u \cdot (a, \phi) = (a - u^{-1}du, u\phi),$$

where  $u \in \text{Map}(Y, S^1)$  and  $(a, \phi) \in i\Omega^1(Y) \oplus \Gamma(S_Y)$ . In practice, we will work with the Sobolev completion of the spaces  $i\Omega^1(Y) \oplus \Gamma(S_Y)$  and  $\text{Map}(Y, S^1)$  by the  $L^2_k$  and  $L^2_{k+1}$  norms, respectively. We fix an integer  $k > 4$  throughout the paper and denote the completed spaces by  $\mathcal{C}_Y$  and  $\mathcal{G}_Y$ , respectively. We will also consider the following subgroups of  $\mathcal{G}_Y$ :

- $\mathcal{G}_Y^e := \{u \in \mathcal{G}_Y \mid u = e^\xi \text{ for some } \xi: Y \rightarrow i\mathbb{R}\}$ ;
- $\mathcal{G}_Y^{e,0} := \{u \in \mathcal{G}_Y^e \mid u = e^\xi \text{ with } \int_{Y_j} \xi \, d\text{vol} = 0 \text{ for } j = 1, \dots, b_0\}$ ;
- $\mathcal{G}_Y^h := \{u \in \mathcal{G}_Y \mid \Delta(\log u) = 0\}$  the harmonic gauge group, where  $\Delta = d^*d$ ;
- $\mathcal{G}_Y^{h,o} := \{u \in \mathcal{G}_Y^h \mid u(o_j) = 1 \text{ for } j = 1, \dots, b_0\}$  the based harmonic gauge group, where  $o_j$  is a chosen basepoint on  $Y_j$ .

Note that  $\mathcal{G}_Y^e \cong \mathcal{G}_Y^{e,0} \times (S^1)^{b_0}$  and  $\mathcal{G}_Y^h \cong \mathcal{G}_Y^{h,0} \times (S^1)^{b_0}$  and  $\mathcal{G}_Y^{h,0} \cong H^1(Y; \mathbb{Z})$ .

The balanced Chern–Simons–Dirac functional  $\text{CSD}_{\nu_0}: \mathcal{C}_Y \rightarrow \mathbb{R}$  is defined as

$$\text{CSD}_{\nu_0}(a, \phi) := -\frac{1}{2} \left( \int_Y a \wedge da - \int_Y \langle \phi, \not{D}_{A_0+a}(\phi) \rangle \, d\text{vol} \right).$$

Note that this is a perturbation of the standard Chern–Simons–Dirac functional by the closed but nonexact 2-form  $\nu_0$  so that  $\text{CSD}_{\nu_0}$  becomes invariant under the full gauge group; see [17, Definition 29.1.1]. The formal  $L^2$ -gradient is given by

$$(1) \quad \text{grad CSD}_{\nu_0}(a, \phi) = (*da + \rho^{-1}(\phi\phi^*)_0, \not{D}_{A_0+a}\phi),$$

where  $(\phi\phi^*)_0$  is the traceless part of the endomorphism  $\phi\phi^*$  on  $S_Y$ .

If we slightly perturb  $\text{CSD}_{\nu_0}$ , the critical points of  $\text{CSD}_{\nu_0}$  are discrete modulo gauge transformations. To ensure this property, we will need to pick a function  $f: \mathcal{C}_Y \rightarrow \mathbb{R}$

which is invariant under  $\mathcal{G}_Y$  and consider a twice-perturbed functional  $\text{CSD}_{v_0, f} := \text{CSD}_{v_0} + f$ . We will make use of a large Banach space of perturbations constructed by Kronheimer and Mrowka [17, Section 11].

**Definition 2.1** Let  $\{\widehat{f}_j\}_{j=1}^\infty$  be a countable collection of cylinder functions as in [17, page 193]. Given a sequence  $\{C_j\}_{j=1}^\infty$  of positive real numbers, we consider a separable Banach space

$$(2) \quad \mathcal{P} = \left\{ \sum_{j=1}^\infty \eta_j \widehat{f}_j \mid \eta_j \in \mathbb{R}, \sum_{j=1}^\infty C_j |\eta_j| < \infty \right\},$$

where the norm is defined by  $\|\sum_{j=1}^\infty \eta_j \widehat{f}_j\| = \sum_{j=1}^\infty |\eta_j| C_j$ . An element of  $\mathcal{P}$  will be called an *extended cylinder function*.

The Banach space  $\mathcal{P}$  will be fixed throughout the paper. In particular, we will choose a real sequence  $\{C_j\}_j$  satisfying our requirements as in the following result.

**Proposition 2.2** *The sequence  $\{C_j\}_j$  can be chosen so that any extended cylinder function  $\bar{f}$  in  $\mathcal{P}$  has the following properties:*

- (i) *The function  $\bar{f}$  is bounded.*
- (ii) *The formal  $L^2$ -gradient  $\text{grad } \bar{f}$  is a tame perturbation [17, Definition 10.5.1].*
- (iii) *For any positive integer  $m$ , the gradient  $\text{grad } \bar{f}$  defines a smooth vector field on the Hilbert space  $L_m^2(i\Omega^1(Y) \oplus \Gamma(S_Y))$ . Moreover, for each nonnegative integer  $n$ , we have*

$$\|\mathcal{D}_{(a,\phi)}^n \text{grad } \bar{f}\| \leq C p_{m,n}(\|(a, \phi)\|_{L_m^2}),$$

where  $p_{m,n}$  is a polynomial depending only on  $m$  and  $n$ , and  $C$  is a constant depending on  $m$ ,  $n$  and  $\bar{f}$ . The norm of  $\mathcal{D}_{(a,\phi)}^n \text{grad } \bar{f}$  is taken by considering  $\mathcal{D}_{(a,\phi)}^n \text{grad } \bar{f}$  as an element of

$$\text{Mult}^n(\times_n L_m^2(i\Omega^1(Y) \oplus \Gamma(S_Y)), L_m^2(i\Omega^1(Y) \oplus \Gamma(S_Y))).$$

- (iv)  $\{C_j\}_j$  is taken so that the statements of Lemmas 6.10 and 6.13 hold.

**Proof** By the definition of cylinder functions, each  $\widehat{f}_j$  is bounded. Therefore, property (i) can be ensured by taking  $\{C_j\}_j$  increasing fast enough. Property (ii) is a consequence of [17, Theorem 11.6.1]. For property (iii), let  $\widehat{f}_j$  be a cylinder function from the collection. By [17, Proposition 11.3.3], the gradient  $\text{grad } \widehat{f}_j$  defines a smooth

vector field over  $L_m^2(i\Omega^1(Y) \oplus \Gamma(S_Y))$  with the property that

$$\|\mathcal{D}_{(a,\phi)}^n \text{grad } \widehat{f}_j\| \leq C'_{j,m,n}(1 + \|\phi\|_{L^2})^n(1 + \|a\|_{L_{m-1}^2})^m(1 + \|\phi\|_{L_{m,A_0+a}^2}),$$

where  $C'_{j,m,n}$  is a constant and  $\|\cdot\|_{L_{m,A_0+a}^2}$  denotes the  $L_m^2$ -norm defined using the connection  $A_0 + a$ . Therefore, we only need to estimate  $\|\phi\|_{L_{m,A_0+a}^2}$  by a polynomial of  $\|(a, \phi)\|_{L_m^2}$ .

Notice that the expansion of  $\nabla_{A_0+a}^{(m)}\phi$  consists of terms of the form

$$\nabla^{(n_1)} a \cdot \nabla^{(n_2)} a \cdots \nabla^{(n_i)} a \cdot \nabla_{A_0}^{(n_{i+1})} \phi,$$

where  $\nabla$  denotes the Levi-Civita connection and  $i, n_1, \dots, n_{i+1}$  are nonnegative integers satisfying  $n_1 + n_2 + \cdots + n_{i+1} + i = m$ . As we want to control the  $L^2$ -norm of this term using  $\|(a, \phi)\|_{L_m^2}$ , there are three cases:

- **$i = 0$**  This is trivial since  $\|\phi\|_{L_m^2} \leq \|(a, \phi)\|_{L_m^2}$ .
- **$i = 1$  and  $n_1 = m - 1$**  We apply Sobolev multiplication  $L_1^2 \times L_m^2 \rightarrow L^2$  and obtain  $\|\nabla^{(m-1)} a \cdot \phi\|_{L^2} \leq C \|\nabla^{(m-1)} a\|_{L_1^2} \|\phi\|_{L_m^2} \leq C \|(a, \phi)\|_{L_m^2}^2$ . The case  $i = 1$  and  $n_2 = m - 1$  can be done in the same manner.
- **$i \geq 1$  and  $n_1, \dots, n_{i+1} < m - 1$**  We consider  $n_{\max} = \max\{n_1, \dots, n_{i+1}\}$  and apply Sobolev multiplication

$$L_{m-n_1}^2 \times \cdots \times L_{m-n_i}^2 \times L_{m-n_{i+1}}^2 \rightarrow L_{m-n_{\max}}^2 \hookrightarrow L^2.$$

Putting these together, we can find a polynomial  $p_{m,n}$  (independent of  $j$ ) such that

$$\|\mathcal{D}_{(a,\phi)}^n \text{grad } \widehat{f}_j\| \leq C'_{j,m,n} p_{m,n}(\|(a, \phi)\|_{L_m^2}).$$

For each  $j$ , take a constant  $C_j$  with

$$C_j \geq \max\{C'_{l_1, l_2, l_3} \mid 0 \leq l_1, l_2, l_3 \leq j\}.$$

We will prove that condition (iii) is satisfied. Take any element  $\bar{f} = \sum_j \eta_j \widehat{f}_j$  of  $\mathcal{P}$ . Then we have

$$\begin{aligned} \|\mathcal{D}_{(a,\phi)}^n \text{grad } \bar{f}\| &\leq \sum_j |\eta_j| \|\mathcal{D}_{(a,\phi)}^n \text{grad } \widehat{f}_j\| \\ &\leq \sum_j |\eta_j| C'_{j,m,n} p_{m,n}(\|(a, \phi)\|_{L_m^2}) \\ &\leq \left( \sum_{1 \leq j \leq N} |\eta_j| C'_{j,m,n} + \sum_{j \geq N} |\eta_j| C_j \right) p_{m,n}(\|(a, \phi)\|_{L_m^2}). \end{aligned}$$



Here  $N = \max\{m, n\}$ . Putting  $C := (\sum_{1 \leq j \leq N} |\eta_j| C'_{j,m,n} + \sum_{j \geq N} |\eta_j| C_j)$ , we obtain

$$\|D_{(a,\phi)}^n \text{grad } \bar{f}\| \leq Cp_{m,n}(\|(a, \phi)\|_{L_m^2}).$$

Thus  $\mathcal{P}$  satisfies (iii).

We may suppose that  $C_j$  satisfies Lemmas 6.10 and 6.13(2) by further shrinking  $C_j$ . That is, condition (iv) is satisfied. □

The perturbation we consider in the current paper will be of the form

$$f(a, \phi) = \bar{f}(a, \phi) + \frac{1}{2}\delta\|\phi\|_{L^2}^2,$$

where  $\bar{f}$  is an extended cylinder function and  $\delta$  is a real number. We sometimes write the above perturbation as a pair  $(\bar{f}, \delta)$ .

**Definition 2.3** A perturbation  $f = (\bar{f}, \delta)$  is called *good* if the critical points of  $\text{CSD}_{v_0, f}$  are discrete modulo gauge transformations.

When  $\delta = 0$ , we know that good perturbations are generic in  $\mathcal{P}$  by virtue of [17, Theorem 12.1.2]. It is immediate to extend the result to a general case and we only give a statement here.

**Lemma 2.4** For any real  $\delta$ , a subset of extended cylinder functions  $\bar{f}$  in  $\mathcal{P}$  such that  $(\delta, \bar{f})$  is a good perturbation is residual.

**Remark** To define our invariants, it is sufficient to take  $\delta = 0$ . We include the term  $\frac{1}{2}\delta\|\phi\|^2$  as it will facilitate computations of many examples in Section 8.

Our main object of interest is the negative gradient flow of the functional  $\text{CSD}_{v_0, f}$  on the space  $\mathcal{C}_Y$  modulo the gauge group. Let  $I \subset \mathbb{R}$  be an interval. A trajectory  $\gamma: I \rightarrow \mathcal{C}_Y$  of the negative gradient flow is described by the equation

$$-\frac{\partial}{\partial t}\gamma(t) = \text{grad } \text{CSD}_{v_0, f}(\gamma(t)).$$

As in [22; 13], it is more convenient to study the flow on the subspace called the Coulomb slice

$$\text{Coul}(Y) = \{(a, \phi) \mid d^*a = 0\} \subset \mathcal{C}_Y.$$

Since any configuration  $(a, \phi) \in \mathcal{C}_Y$  can be gauge transformed into  $\text{Coul}(Y)$  by a unique element of  $\mathcal{G}_Y^{e,0}$ , the Coulomb slice is isomorphic to the quotient  $\mathcal{C}_Y/\mathcal{G}_Y^{e,0}$  with residual action by the harmonic gauge group  $\mathcal{G}_Y^h$ .

Let  $\Pi: \mathcal{C}_Y \rightarrow \mathcal{C}_Y/\mathcal{G}_Y^{e,o} \cong \text{Coul}(Y)$  be the nonlinear Coulomb projection. The formula for  $\Pi$  is given by

$$(3) \quad \Pi(a, \phi) = (a - d\bar{\xi}(a), e^{\bar{\xi}(a)}\phi),$$

where  $\bar{\xi}(a): Y \rightarrow i\mathbb{R}$  is the unique function which solves

$$(4) \quad \Delta\bar{\xi}(a) = d^*a \quad \text{and} \quad \int_{Y_j} \bar{\xi}(a) = 0 \quad \text{for each } j = 1, \dots, b_0.$$

To describe the Seiberg–Witten vector field on  $\text{Coul}(Y)$ , we first consider a trivial bundle  $\mathcal{T}_{k-1}$  over  $\mathcal{C}_Y$  with fiber  $L^2_{k-1}(i\Omega^1(Y) \oplus \Gamma(S_Y))$ . Note that the vector field  $\text{grad CSD}_{v_0,f}$  is a section of  $\mathcal{T}_{k-1}$ . Similarly, we have a trivial bundle  $\text{Coul}_{k-1}$  over  $\text{Coul}(Y)$  whose fiber is the  $L^2_{k-1}$ -completion of  $\ker d^* \oplus \Gamma(S_Y)$ . At a point  $(a, \phi) \in \text{Coul}(Y)$ , the pushforward  $\Pi_*: \mathcal{T}_{k-1} \rightarrow \text{Coul}_{k-1}$  of the Coulomb projection  $\Pi$  is given by

$$(5) \quad \Pi_{*(a,\phi)}(b, \psi) = (b - d\bar{\xi}(b), \psi + \bar{\xi}(b)\phi).$$

We now project the negative gradient flow lines from  $\mathcal{C}_Y$  to  $\text{Coul}(Y)$  using  $\Pi$ . Such projected trajectories  $\gamma: I \rightarrow \text{Coul}(Y)$  are described by the equation

$$(6) \quad -\frac{\partial}{\partial t}\gamma(t) = \Pi_* \text{grad CSD}_{v_0,f}(\gamma(t)).$$

From (1) and (5), we can write down an explicit formula for the induced vector field on  $\text{Coul}(Y)$  as a section of  $\text{Coul}_{k-1}$ :

$$(7) \quad \Pi_* \text{grad CSD}_{v_0,f}(a, \phi) = l(a, \phi) + c(a, \phi),$$

where  $l = (*d, \mathcal{D})$  is a first-order elliptic operator and  $c = (c^1, c^2)$  is given by

$$(8) \quad c^1(a, \phi) = \rho^{-1}(\phi\phi^*)_0 + \text{grad}^1 f(a, \phi) - d\bar{\xi}(\rho^{-1}(\phi\phi^*)_0 + \text{grad}^1 f(a, \phi)),$$

$$(9) \quad c^2(a, \phi) = \rho(a)\phi + \text{grad}^2 f(a, \phi) + \bar{\xi}(\rho^{-1}(\phi\phi^*)_0 + \text{grad}^1 f(a, \phi))\phi.$$

Note that  $l$  is linear and the nonlinear term  $c$  has nice compactness properties which will be explored in Section 3. We will call those trajectories  $\gamma$  satisfying (6) the *Seiberg–Witten trajectories*. By the standard elliptic bootstrapping argument,  $\gamma$  is actually a smooth path in  $\text{Coul}(Y)$  when restricted to the interior of  $I$ .

We would also like to interpret the vector field  $\Pi_* \text{grad CSD}_{v_0,f}$  from (6) as a gradient vector field on  $\text{Coul}(Y)$ . However,  $\Pi_* \text{grad CSD}_{v_0,f}$  is not the gradient of the restriction  $\text{CSD}_{v_0,f}|_{\text{Coul}(Y)}$  with respect to the standard  $L^2$ -metric, and we need to introduce another metric on  $\text{Coul}(Y)$ . Roughly speaking, we have to measure only the

component of a vector on  $\text{Coul}(Y)$  which is orthogonal to the linearized gauge group action. More specifically, consider a bundle decomposition over  $\mathcal{C}_Y$

$$\mathcal{T}_{k-1} = \mathcal{J}_{k-1} \oplus \mathcal{K}_{k-1},$$

where the fiber of  $\mathcal{J}_{k-1}$  at  $(a, \phi)$  consists of a vector of the form  $(-d\xi, \xi\phi)$  where  $\xi \in L^2_k(Y; i\mathbb{R})$  with  $\int_{Y_j} \xi = 0$  and the fiber of  $\mathcal{K}_{k-1}$  is the  $L^2$ -orthogonal complement. Note that this decomposition is slightly different from the decomposition which appeared in [17, Section 9.3] since we use the derivative of the action of  $\mathcal{G}_Y^{e,0}$  rather than  $\mathcal{G}_Y^e$ . Let  $\tilde{\Pi}$  be the  $L^2$ -orthogonal projection onto  $\mathcal{K}_{k-1}$ . Explicitly, the projection  $\tilde{\Pi}$  at  $(a, \phi)$  is given by

$$\tilde{\Pi}_{(a,\phi)}(b, \psi) = (b - d\tilde{\xi}(b, \psi, \phi), \psi + \tilde{\xi}(b, \psi, \phi)\phi),$$

where  $\tilde{\xi}(b, \psi, \phi): Y \rightarrow i\mathbb{R}$  is a unique function such that  $-d^*(b - d\tilde{\xi}(b, \psi, \phi)) + i\text{Re}(i\phi, \psi + \tilde{\xi}(b, \psi, \phi)\phi)$  is a locally constant function and  $\int_{Y_j} \tilde{\xi}(b, \psi, \phi) = 0$ . It is not hard to see that we have a bundle isomorphism

$$\begin{array}{ccc} \text{Coul}_{k-1} & \begin{array}{c} \xleftarrow{\tilde{\Pi}} \\ \xrightarrow{\Pi_*} \end{array} & \mathcal{K}_{k-1} \\ & \searrow \quad \swarrow & \\ & \text{Coul}(Y) & \end{array}$$

since both are complementary to the derivative of the action of  $\mathcal{G}_Y^{e,0}$ .

We now define a metric  $\tilde{g}$  for the bundle  $\text{Coul}_{k-1}$  by setting

$$\langle (b_1, \psi_1), (b_2, \psi_2) \rangle_{\tilde{g}} := \langle \tilde{\Pi}(b_1, \psi_1), \tilde{\Pi}(b_2, \psi_2) \rangle_{L^2}.$$

Since  $\tilde{\Pi}$  and  $\Pi_*$  are inverses of each other and  $\tilde{\Pi}$  is an orthogonal projection, we have the identity

$$\langle \Pi_* v, w \rangle_{\tilde{g}} = \langle v, w \rangle_{L^2} \quad \text{whenever } v \in \mathcal{K}_{k-1}.$$

Since  $\text{CSD}_{v_0, f}$  is gauge-invariant,  $\text{grad CSD}_{v_0, f}$  lies in  $\mathcal{K}_{k-1}$ . From this point on, we will denote by  $\widetilde{\text{grad}}$  the gradient on  $\text{Coul}(Y)$  with respect to the metric  $\tilde{g}$  and put

$$\mathcal{L} := \text{CSD}_{v_0, f} |_{\text{Coul}(Y)}.$$

We then have

$$(10) \quad \widetilde{\text{grad}} \mathcal{L} = \Pi_* \text{grad CSD}_{v_0, f} = l + c \quad \text{and} \quad \|\widetilde{\text{grad}} \mathcal{L}\|_{\tilde{g}} = \|\text{grad CSD}_{v_0, f}\|_{L^2}.$$

Note that analogous results hold for any functional on  $\mathcal{C}_Y$  which is  $\mathcal{G}_Y^{e,0}$ -invariant.

### 3 Analysis of approximated Seiberg–Witten trajectories

In this section, we review some boundedness and convergence results relevant to finite-dimensional approximation which will be used in the main construction.

**Definition 3.1** A smooth path in  $\text{Coul}(Y)$  is said to be of *finite type* if it is contained in a fixed bounded set (in the  $L_k^2$ -norm).

It can be proved that a Seiberg–Witten trajectory  $\gamma(t) = (\alpha(t), \phi(t))$  is of finite type if and only if both  $\text{CSD}_{v_0, f}(\gamma(t))$  and  $\|\phi(t)\|_{C^0}$  are bounded; see [22, Definition 1].

Recall that the set of the Seiberg–Witten solutions is compact modulo the full gauge group. However, there is a residual action by the group  $\mathcal{G}_Y^{h, o} \cong H^1(Y; \mathbb{Z})$  on  $\text{Coul}(Y)$ . This motivates us to consider a strip of balls

$$\text{Str}(R) = \{x \in \text{Coul}(Y) \mid \text{there exists } h \in \mathcal{G}_Y^{h, o} \text{ such that } \|h \cdot x\|_{L_k^2} \leq R\},$$

where  $R$  is a positive real number.

Since  $\text{CSD}_{v_0, f}$  is invariant under the full gauge group  $\mathcal{G}_Y$ , we have a uniform bound for the topological energy of all finite-type trajectories; see [13, Proposition 10]. As a result, we have the following boundedness result.

**Theorem 3.2** [13] *There exists a constant  $R_0$  such that all finite-type Seiberg–Witten trajectories are contained in the interior of  $\text{Str}(R_0)$ . In particular, the set  $\text{Str}(R_0)$  contains all the critical points of  $\mathcal{L}$  and trajectories between them.*

We now discuss finite-dimensional approximation of Seiberg–Witten trajectories following [22; 13]. To describe various projections, we first specify the  $L_m^2$ -inner product ( $m \geq 1$ ) on  $i\Omega^1(Y) \oplus \Gamma(S_Y)$ . From the Hodge decomposition  $\Omega^1(Y) = \ker d^* \oplus \text{im } d$ , we will just define an inner product on each summand. On  $i\ker d^* \oplus \Gamma(S_Y)$ , we use the elliptic operator  $l = (*d, \mathcal{D})$ :

$$\langle (a_1, \phi_1), (a_2, \phi_2) \rangle_{L_m^2} := \langle (a_1, \phi_1), (a_2, \phi_2) \rangle_{L^2} + \langle l^m(a_1, \phi_1), l^m(a_2, \phi_2) \rangle_{L^2}.$$

For  $\beta_1, \beta_2 \in i\text{im } d$ , we define

$$\langle \beta_1, \beta_2 \rangle_{L_m^2} := \langle \beta_1, \beta_2 \rangle_{L^2} + \langle \Delta^m \beta_1, \beta_2 \rangle_{L^2}.$$

Consider the spectral decomposition of  $\text{Coul}(Y)$  with respect to the eigenspaces of  $l = (*d, \mathcal{D})$ . For any real numbers  $\lambda < 0 \leq \mu$ , let  $V_\lambda^\mu$  be the span of the eigenspaces

of  $l$  with eigenvalues in the interval  $(\lambda, \mu]$  and let  $p_\lambda^\mu$  be the  $L^2$ -orthogonal projection onto  $V_\lambda^\mu$ . We note that  $p_\lambda^\mu$  has the following properties:

- (i)  $p_\lambda^\mu$  is an  $L_m^2$ -orthogonal projection for any  $m \geq 0$ ;
- (ii)  $p_\lambda^\mu$  extends to a map on a cylinder  $I \times Y$  with  $\|p_\lambda^\mu\|_{L_m^2(I \times Y)} \leq 1$  for any  $m \geq 0$ .

Such properties allow one to deduce the boundedness property for approximated Seiberg–Witten trajectories from the corresponding property for actual Seiberg–Witten trajectories. As a result, they are sufficient for our proof of [Proposition 3.3](#).

Recall that a Seiberg–Witten trajectory is an integral curve of the vector field  $l + c$  on  $\text{Coul}(Y)$ . This leads us to consider a trajectory on a finite-dimensional subspace  $\gamma: I \rightarrow V_\lambda^\mu$  satisfying the equation

$$-\frac{d}{dt}\gamma(t) = (l + p_\lambda^\mu \circ c)(\gamma(t)).$$

Such a trajectory will be loosely called an approximated Seiberg–Witten trajectory. We will also call a sequence of approximated Seiberg–Witten trajectories  $\{\gamma_n: I \rightarrow V_{\lambda_n}^{\mu_n}\}_{n \in \mathbb{N}}$  an exhausting sequence when  $-\lambda_n, \mu_n \rightarrow \infty$ . The next proposition is the main convergence result of this section.

**Proposition 3.3** *Let  $\{\gamma_n: [a, b] \rightarrow V_{\lambda_n}^{\mu_n}\}$  be an exhausting sequence of approximated Seiberg–Witten trajectories whose  $L_k^2$ -norms are uniformly bounded. Then there exists a Seiberg–Witten trajectory  $\gamma_\infty: (a, b) \rightarrow \text{Coul}(Y)$  such that, after passing to a subsequence,  $\gamma_n(t) \rightarrow \gamma_\infty(t)$  uniformly in any Sobolev norm on any compact subset of  $(a, b)$ .*

The proof of this proposition will be at end of this section. We basically follow the same strategy as in the proofs of [22, Proposition 3; 13, Proposition 11]. Since our vector field  $l + c$  has an extra term coming from  $\text{grad } f$ , we need to assure that the nonlinear part  $c$  still has nice compactness properties similar to those of the quadratic term in the Seiberg–Witten equation. For this purpose, we recall the notion of “quadratic-like” map and related results in [13, Section 4.2]. Since our setting here is slightly different, we spell out some details for completeness.

**Definition 3.4** Let  $E$  be a vector bundle over  $Y$ . A smooth map  $Q: \text{Coul}(Y) \rightarrow L_k^2(\Gamma(E))$  is called *quadratic-like* if it has the following properties:

- (i) The map  $Q$  sends every bounded subset in  $L_k^2$  to a bounded subset in  $L_k^2$ .

- (ii) Let  $m$  be a nonnegative integer not greater than  $k - 1$ . If there is a convergence of paths  $(\frac{d}{dt})^s \gamma_n(t) \rightarrow (\frac{d}{dt})^s \gamma_\infty(t)$  over a compact interval that is uniform in  $L^2_{k-1-s}$  for each  $s = 0, 1, \dots, m$ , then  $(\frac{d}{dt})^m Q(\gamma_n(t)) \rightarrow (\frac{d}{dt})^m Q(\gamma_\infty(t))$  uniformly in  $L^2_{k-2-m}$ .
- (iii) The map  $Q$  extends to a continuous map from  $L^2_m(I \times Y)$  to  $L^2_m(I \times Y)$  (with suitable bundles understood) for each integer  $m \geq k - 1$ . Here  $I$  is a compact interval.

The sum of two quadratic-like maps is obviously quadratic-like. Furthermore, it can be shown that the pointwise tensor product of two quadratic-like maps is also quadratic-like; see [13, Lemma 10].

**Lemma 3.5** (compare [13, Lemma 9]) *Let  $f$  be a perturbation given by a pair  $(\delta, \bar{f})$  with  $\delta \in \mathbb{R}$  and  $\bar{f} \in \mathcal{P}$ . Then the map  $\text{grad } f: \text{Coul}(Y) \rightarrow L^2_k(i\Omega^1(Y) \oplus \Gamma(S_Y))$  is quadratic-like.*

**Proof** We see that  $\text{grad } f(a, \phi) = (0, \delta\phi) + \text{grad } \bar{f}(a, \phi)$  and the first term is obviously quadratic-like. We just need to show that  $\text{grad } \bar{f}$  is quadratic-like. First, we will check properties (i) and (ii) of Definition 3.4 when  $m = 0$ .

For two configurations  $(a_0, \phi_0)$  and  $(a_1, \phi_1)$ , we consider a straight segment  $(a_t, \phi_t) = (1-t)(a_0, \phi_0) + t(a_1, \phi_1)$  joining them and apply the fundamental theorem of calculus:

$$\begin{aligned} & \| \text{grad } \bar{f}(a_1, \phi_1) - \text{grad } \bar{f}(a_0, \phi_0) \|_{L^2_j} \\ &= \left\| \int_{[0,1]} \mathcal{D}_{(a_t, \phi_t)} \text{grad } \bar{f}(a_1 - a_0, \phi_1 - \phi_0) dt \right\|_{L^2_j} \\ &\leq C \int_{[0,1]} p_{j,1}(\|a_t, \phi_t\|_{L^2_j}) \| (a_1, \phi_1) - (a_0, \phi_0) \|_{L^2_j} dt, \end{aligned}$$

where the last inequality follows from Proposition 2.2(iii). When  $j = k$  and  $(a_0, \phi_0) = (0, 0)$ , this implies property (i) of Definition 3.4. Property (ii) when  $m = 0$  also follows from the above inequality when  $j = k - 1$ .

We now check property (ii) when  $1 \leq m \leq k - 2$ . Suppose that  $(\frac{d}{dt})^s \gamma_n(t) \rightarrow (\frac{d}{dt})^s \gamma_\infty(t)$  uniformly in  $L^2_{k-1-s}$  for each  $s = 0, 1, \dots, m$ . We observe that an expansion of  $(\frac{d}{dt})^m \text{grad } \bar{f}(\gamma(t))$  consists of terms of the form

$$\mathcal{D}^s_{\gamma(t)} \text{grad } \bar{f} \left( \left(\frac{d}{dt}\right)^{\alpha_1} \gamma(t), \dots, \left(\frac{d}{dt}\right)^{\alpha_s} \gamma(t) \right) \quad \text{with } \alpha_i \geq 1 \text{ and } \alpha_1 + \dots + \alpha_s = m.$$

From Proposition 2.2(iii),  $\|\mathcal{D}_{\gamma(t)}^s \text{grad } \bar{f}\| \leq Cp_{k-1-m,s}(\|\gamma(t)\|_{L^2_{k-1-m}})$  as an element of  $\text{Mult}^s(\times_s L^2_{k-1-m}, L^2_{k-1-m})$ . We see that  $\gamma_n$  is uniformly bounded in  $L^2_{k-1-m}$  and that the convergence  $(\frac{d}{dt})^{\alpha_i} \gamma_n(t) \rightarrow (\frac{d}{dt})^{\alpha_i} \gamma_\infty(t)$  is uniform in  $L^2_{k-1-m}$  as  $\alpha_i \leq m$ . These imply property (ii).

Property (iii) easily follows from the fact that  $\text{grad } \bar{f}$  is a tame perturbation. □

As a result, we can deduce compactness of the induced vector field on  $\text{Coul}(Y)$ .

**Corollary 3.6** *The nonlinear part  $c$  of the induced Seiberg–Witten vector field in (7) is quadratic-like.*

**Proof** It is clear that the composition of a quadratic-like map with a linear operator of nonpositive order is quadratic-like. Since the operator  $\xi$  in (3) is of order  $-1$ , Lemma 3.5 and closure under pointwise multiplication imply that the map  $c$  is quadratic-like. □

We are now ready to prove Proposition 3.3. However, we will only give an outline of the proof as the reader can find more details in [22; 13].

**Proof of Proposition 3.3** Let  $\{\gamma_n\}$  be an exhausting sequence of approximated trajectories which are all contained in a ball  $B(R)$  in  $L^2_k$ . The norm  $\|\frac{d}{dt}\gamma_n(t)\|_{L^2_{k-1}}$  is uniformly bounded by boundedness of the map  $l + c$ . By the Rellich lemma and the Arzela–Ascoli theorem, we can pass to a subsequence of  $\{\gamma_n\}$  which converges to a path  $\gamma_\infty$  uniformly in  $L^2_{k-1}$ . Moreover, it can be shown that  $\gamma_\infty$  is a Seiberg–Witten trajectory. By property (ii) of Definition 3.4 of  $c$ , we can inductively prove uniform convergence  $(\frac{d}{dt})^m(\gamma_n(t)) \rightarrow (\frac{d}{dt})^m(\gamma_\infty(t))$  in  $L^2_{k-1-m}$  for  $m = 1, \dots, k-1$ . This implies that  $\hat{\gamma}_n \rightarrow \hat{\gamma}_\infty$  in  $L^2_{k-1}([a, b] \times Y)$ . (Here we treat  $\gamma_n(t)$  and  $\gamma_\infty(t)$  as sections over  $I \times Y$  and denote them respectively by  $\hat{\gamma}_n$  and  $\hat{\gamma}_\infty$ .) Property (iii) of Definition 3.4 allows us to do the bootstrapping argument over any shorter cylinder  $I \times Y$ . This finishes the proof of the proposition. □

Proposition 3.3 has the following consequence.

**Corollary 3.7** *For a closed and bounded subset  $S$  of  $\text{Coul}(Y)$  in  $L^2_k$ , there exist large numbers  $-\bar{\lambda}, \bar{\mu}, \bar{T} \gg 0$  such that if  $\lambda < \bar{\lambda}$ ,  $\mu > \bar{\mu}$  and  $T > \bar{T}$ , then for any approximated Seiberg–Witten trajectory  $\gamma: [-T, T] \rightarrow V_\lambda^\mu$  contained in  $S$ , we have  $\gamma(0) \in \text{Str}(R_0)$ . (Here  $R_0$  is the universal constant from Theorem 3.2.)*

**Proof** Suppose the contrary; we can find an exhausting sequence of approximated trajectories  $\gamma_n: [-T_n, T_n] \rightarrow V_{\lambda_n}^{\mu_n} \cap S$  with  $T_n \rightarrow \infty$  and  $\gamma_n(0) \notin \text{Str}(R_0)$ . Since  $S$  is bounded, we can apply Proposition 3.3 and the diagonalization argument to find a Seiberg–Witten trajectory  $\gamma_\infty: \mathbb{R} \rightarrow S$  of finite type such that, after passing to a subsequence,  $\gamma_n(0) \rightarrow \gamma_\infty(0)$  in  $L_k^2$ . However,  $\gamma_\infty(0)$  is in the interior of  $\text{Str}(R_0)$  by Theorem 3.2. This is a contradiction.  $\square$

**Remark** In Corollary 3.7, we can also consider more generalized approximated trajectories. For example, we can use interpolation between two projections for approximation, ie a trajectory satisfying

$$-\frac{d}{dt}\gamma(t) = (l + ((1-s)p_\lambda^\mu + sp_{\lambda'}^{\mu'}) \circ c)(\gamma(t)) \quad \text{for } 0 \leq s \leq 1,$$

where  $\lambda' < \lambda < \bar{\lambda}$  and  $\mu' > \mu > \bar{\mu}$ .

## 4 Categorical and topological preliminaries

### 4.1 The stable categories

In this subsection, we will briefly review algebraic-topological constructions which will be needed later. In particular, we will define three stable categories  $\mathcal{C}$ ,  $\mathfrak{S}$  and  $\mathfrak{S}^*$  in which our invariants live as objects. The categories  $\mathfrak{S}$  and  $\mathfrak{S}^*$  are defined as direct systems and inverse systems of  $\mathcal{C}$ , respectively. Our treatment follows closely with [22; 23]. See [1; 25] for more systematic and detailed discussions regarding equivariant stable homotopy theory.

The category  $\mathcal{C}$ , which was defined in [22], is the  $S^1$ -equivariant analog of the classical Spanier–Whitehead category with  $\mathbb{R}^\infty \oplus \mathbb{C}^\infty$  as the universe. In other words, we will only consider suspensions involving the following two representations:

- (1)  $\mathbb{R}$ , the one-dimensional trivial representation;
- (2)  $\mathbb{C}$ , the two-dimensional representation where  $S^1 = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$  acts by complex multiplication.

For a representation  $V$ , we will denote by  $V^+$  its one-point compactification and by  $V^{S^1}$  its  $S^1$ -fixed point set. Note that the transposition  $(\mathbb{R}^{u_1})^+ \wedge (\mathbb{R}^{u_2})^+ \rightarrow (\mathbb{R}^{u_2})^+ \wedge (\mathbb{R}^{u_1})^+$  is homotopic to the identity only when  $u_1$  or  $u_2$  is even.

The objects of  $\mathcal{C}$  are triples  $(A, m, n)$  consisting of a pointed topological space  $A$  with an  $S^1$ -action, an even integer  $m$  and a rational number  $n$ . We require that  $A$



be  $S^1$ –homotopy equivalent to a finite  $S^1$ –CW complex. The set of morphisms between two objects is given by

$$\begin{aligned} \text{mor}_{\mathfrak{C}}((A, m, n), (A', m', n')) \\ := \text{colim}_{u, v \rightarrow \infty} [(\mathbb{R}^u \oplus \mathbb{C}^v)^+ \wedge A, (\mathbb{R}^{u+m-m'} \oplus \mathbb{C}^{v+n-n'})^+ \wedge A']_{S^1} \end{aligned}$$

if  $n - n' \in \mathbb{Z}$ , where  $[\cdot, \cdot]_{S^1}$  denotes the set of pointed  $S^1$ –equivariant homotopy classes. We define  $\text{mor}_{\mathfrak{C}}((A, m, n), (A', m', n'))$  to be the empty set if  $n - n' \notin \mathbb{Z}$ . As in [22], there is a full subcategory  $\mathfrak{C}_0$  inside of  $\mathfrak{C}$  consisting of objects of the form  $(A, 0, 0)$ , which we also denote by  $A$ . For an object  $Z = (A, m, n) \in \text{ob } \mathfrak{C}$ , an even integer  $m'$  and a rational number  $n'$ , we also write  $(Z, m', n')$  for  $(A, m + m', n + n')$ .

An  $S^1$ –representation  $E$  is called *admissible* if it is isomorphic to  $\mathbb{R}^a \oplus \mathbb{C}^b$  for some nonnegative integers  $a, b$ . For such a representation, we can define the suspension functor  $\Sigma^E: \mathfrak{C} \rightarrow \mathfrak{C}$  by setting  $\Sigma^E(A, m, n) := (\Sigma^E A, m, n)$ . For a morphism  $F$ , we define  $\Sigma^E F$  to be the  $U(1)$ –equivariant stable homotopy class represented by a composition

$$\begin{aligned} (11) \quad (\mathbb{R}^u \oplus \mathbb{C}^v)^+ \wedge E^+ \wedge A &\xrightarrow{\sigma_{1,2}} E^+ \wedge (\mathbb{R}^u \oplus \mathbb{C}^v)^+ \wedge A \\ &\xrightarrow{\text{id}_{E^+} \wedge f} E^+ \wedge (\mathbb{R}^{u+m-m'} \oplus \mathbb{C}^{v+n-n'})^+ \wedge A' \\ &\xrightarrow{\sigma_{1,2}} (\mathbb{R}^{u+m-m'} \oplus \mathbb{C}^{v+n-n'})^+ \wedge E^+ \wedge A', \end{aligned}$$

where  $f: (\mathbb{R}^u \oplus \mathbb{C}^v)^+ \wedge A \rightarrow (\mathbb{R}^{u+m-m'} \oplus \mathbb{C}^{v+n-n'})^+ \wedge A'$  is a  $U(1)$ –equivariant map representing  $F$  and  $\sigma_{1,2}$  is the interchanging map of the first and the second factor. We can also define the desuspension functor  $\Sigma^{-E}: \mathfrak{C} \rightarrow \mathfrak{C}$  by setting  $\Sigma^{-E}(A, m, n) := (\Sigma^{E^{S^1}} A, m + 2a, n + b)$  and define  $\Sigma^{-E} F$  as in (11) but replacing  $E^+$  with  $(E^{S^1})^+$ . The following lemma is straightforward and we omit the proof.

**Lemma 4.1** *As functors of  $\mathfrak{C}$ , we have*

$$\Sigma^{E_1} \circ \Sigma^{E_2} = \Sigma^{E_1 \oplus E_2}, \quad \Sigma^{-E_1} \circ \Sigma^{-E_2} = \Sigma^{-(E_1 \oplus E_2)}.$$

Furthermore, we show that suspension and desuspension are inverses of each other as functors in  $\mathfrak{C}$ .

**Proposition 4.2** *For an admissible representation  $E$ , there is a natural isomorphism  $\eta$  from the functor  $\Sigma^{-E} \circ \Sigma^E$  to  $\text{id}_{\mathfrak{C}}$ , where  $\text{id}_{\mathfrak{C}}$  is the identity functor on  $\mathfrak{C}$ .*

**Proof** For each object  $(A, m, n)$  of  $\mathfrak{C}$ , we want to construct an isomorphism  $\eta_{(A,m,n)}$  from  $\Sigma^{-E} \circ \Sigma^E(A, m, n) = ((E^{S^1} \oplus E)^+ \wedge A, m + 2a, n + b)$  to  $(A, m, n)$ . First, we choose an isomorphism  $\tau: E \rightarrow \mathbb{R}^a \oplus \mathbb{C}^b$ . Then  $\tau$  induces an isomorphism

$$\tilde{\tau}: (E \oplus E^{S^1})^+ \xrightarrow{\tau \oplus \tau^{S^1}} (\mathbb{R}^a \oplus \mathbb{C}^b \oplus \mathbb{R}^a)^+ \xrightarrow{\sigma^{2,3}} (\mathbb{R}^{2a} \oplus \mathbb{C}^b)^+.$$

Note that by the equivariant Hopf theorem (see [7, Section 2.4]), the homotopy class of  $\tilde{\tau}$  is independent of the choice of  $\tau$ . The isomorphism  $\eta_{(A,m,n)}: \Sigma^{-E} \circ \Sigma^E(A, m, n) \rightarrow (A, m, n)$  is the  $U(1)$ -equivariant stable homotopy class of

$$(\mathbb{R}^u \oplus \mathbb{C}^v)^+ \wedge (E^{S^1} \oplus E)^+ \wedge A \xrightarrow{\tilde{\tau}_{u,v} \wedge \text{id}_A} (\mathbb{R}^{u+2a} \oplus \mathbb{C}^{v+b})^+ \wedge A,$$

where  $\tilde{\tau}_{u,v}$  is the composition of  $\text{id}_{(\mathbb{R}^u \oplus \mathbb{C}^v)^+} \wedge \tilde{\tau}$  with the transposition

$$(\mathbb{R}^u \oplus \mathbb{C}^v)^+ \wedge (\mathbb{R}^{2a} \oplus \mathbb{C}^b)^+ \rightarrow (\mathbb{R}^{u+2a} \oplus \mathbb{C}^{v+b})^+.$$

For a map  $f: (\mathbb{R}^u \oplus \mathbb{C}^v)^+ \wedge A \rightarrow (\mathbb{R}^{u+m-m'} \oplus \mathbb{C}^{v+n-n'})^+ \wedge A'$ , we want to check that  $f \circ \eta_{(A,m,n)} = \eta_{(A',m',n')} \circ \Sigma^{-E} \circ \Sigma^E f$  up to stable homotopy. This follows from the following commutative diagram, a similar diagram for  $A'$ , and the fact that the transpositions in the diagrams are homotopic to the identity:

$$\begin{array}{ccc} (\mathbb{R}^u \oplus \mathbb{C}^v)^+ \wedge (E^{S^1} \oplus E)^+ \wedge A & \xrightarrow{\sigma^{1,2}} & (E^{S^1} \oplus E)^+ \wedge (\mathbb{R}^u \oplus \mathbb{C}^v)^+ \wedge A \\ \text{id}_{(\mathbb{R}^u \oplus \mathbb{C}^v)^+} \wedge \tilde{\tau} \wedge \text{id}_A \downarrow & & \downarrow \tilde{\tau} \wedge \text{id}_{(\mathbb{R}^u \oplus \mathbb{C}^v)^+} \wedge \text{id}_A \\ (\mathbb{R}^u \oplus \mathbb{C}^v)^+ \wedge (\mathbb{R}^{2a} \oplus \mathbb{C}^b)^+ \wedge A & \xrightarrow{\sigma^{1,2}} & (\mathbb{R}^{2a} \oplus \mathbb{C}^b)^+ \wedge (\mathbb{R}^u \oplus \mathbb{C}^v)^+ \wedge A \quad \square \end{array}$$

We now describe the category  $\mathfrak{S}$ . An object of  $\mathfrak{S}$  consists of a collection  $Z = (\{Z_p\}, \{i_p\})_{p \in \mathbb{N}}$  of objects  $\{Z_p\}_{p \in \mathbb{N}}$  of  $\mathfrak{C}$  and morphisms  $\{i_p \in \text{mor}_{\mathfrak{C}}(Z_p, Z_{p+1})\}_{p \in \mathbb{N}}$ . In other words, an object  $Z$  of  $\mathfrak{S}$  is a direct system

$$Z_1 \xrightarrow{i_1} Z_2 \xrightarrow{i_2} \dots$$

For two objects  $Z = (\{Z_p\}_p, \{i_p\}_p)$  and  $Z' = (\{Z'_p\}_p, \{i'_p\}_p)$  of  $\mathfrak{S}$ , we define the set of morphisms as

$$(12) \quad \text{mor}_{\mathfrak{S}}(Z, Z') := \lim_{\infty \leftarrow p} \lim_{q \rightarrow \infty} \text{mor}_{\mathfrak{C}}(Z_p, Z'_q).$$

The identity morphism and the composition law are defined in the obvious way. Notice that here we first take the direct limit and then the inverse limit. This order should not be changed.

As for the category  $\mathfrak{S}^*$ , its objects are the inverse systems

$$\bar{Z}_1 \xleftarrow{j_1} \bar{Z}_2 \xleftarrow{j_2} \dots,$$

where  $\bar{Z}_p \in \text{ob } \mathfrak{C}$  and  $j_p \in \text{mor}_{\mathfrak{C}}(\bar{Z}_{p+1}, \bar{Z}_p)$ . For two objects  $\bar{Z} = (\{\bar{Z}_p\}_p, \{j_p\}_p)$  and  $\bar{Z}' = (\{\bar{Z}'_p\}_p, \{j'_p\}_p)$  of  $\mathfrak{S}^*$ , we define the set of morphisms as

$$(13) \quad \text{mor}_{\mathfrak{S}^*}(\bar{Z}, \bar{Z}') := \lim_{\infty \leftarrow q} \lim_{p \rightarrow \infty} \text{mor}_{\mathfrak{C}}(\bar{Z}_p, \bar{Z}'_q).$$

Again, we first take the direct limit and then the inverse limit.

The suspension functor and the desuspension functor can be extended to functors on  $\mathfrak{S}$  and  $\mathfrak{S}^*$  in the obvious way. Lemma 4.1 and Proposition 4.2 continue to hold for these extended functors. For an object  $Z = (\{Z_p\}_p, \{i_p\}_p)$  of  $\mathfrak{S}$ , an even integer  $m$  and a rational number  $n$ , we write  $(Z, m, n)$  for  $(\{(Z_p, m, n)\}_p, \{i'_p\}_p)$ , where  $i'_p: (Z_p, m, n) \rightarrow (Z_{p+1}, m, n)$  is the morphism induced by  $i_p$ . For an object  $\bar{Z}$  of  $\mathfrak{S}^*$ , we define  $(\bar{Z}, m, n)$  similarly.

**Remark** The full subcategory of  $\mathfrak{C}$  consisting of objects  $\{(A, m, n) \mid m \in 2\mathbb{Z}, n \in \mathbb{Z}\}$  can be naturally embedded into the homotopy category of the  $S^1$ -equivariant spectra modeled on the standard universe  $\mathbb{R}^\infty \oplus \mathbb{C}^\infty$ . Thus an object  $(\{(A_p, m_p, n_p)\}_p, \{i_p\}_p)$  of  $\mathfrak{S}$  (resp.  $\mathfrak{S}^*$ ) with  $m_p \in 2\mathbb{Z}$  and  $n_p \in \mathbb{Z}$  corresponds to an inductive system (resp. projective system) of  $S^1$ -equivariant spectra. For this reason, we call an object of  $\mathfrak{S}$  an ind-spectrum and an object of  $\mathfrak{S}^*$  a pro-spectrum. However, this is not so accurate because, in the usual sense, an ind-spectrum (resp. pro-spectrum) refers to an inductive system (resp. projective system) in the category of spectra, not the homotopy category of spectra. Also, with a slight abuse of language, we call all our invariants spectrum invariants.

We end this subsection with the following useful lemma, which is directly implied by the definition of the direct limit and inverse limit.

**Lemma 4.3** *Let  $Z = (\{Z_p\}_{p \in \mathbb{N}}, \{i_p\}_{p \in \mathbb{N}})$  be an object of  $\mathfrak{S}$ . For any infinite sequence of positive integers  $0 < p_1 < p_2 < \dots$ , the subsystem*

$$Z_{p_1} \xrightarrow{i_{p_2-1} \circ \dots \circ i_{p_1}} Z_{p_2} \xrightarrow{i_{p_3-1} \circ \dots \circ i_{p_2}} Z_{p_3} \rightarrow \dots$$

*of  $Z$  is canonically isomorphic to  $Z$  as an object of  $\mathfrak{S}$ . Similarly, for an object  $\bar{Z} = (\{\bar{Z}_p\}_{p \in \mathbb{N}}, \{j_p\}_{p \in \mathbb{N}})$  of  $\mathfrak{S}^*$ , the subsystem*

$$\bar{Z}_{p_1} \xleftarrow{j_{p_1} \circ \dots \circ j_{p_2-1}} \bar{Z}_{p_2} \xleftarrow{j_{p_2} \circ \dots \circ j_{p_3-1}} \bar{Z}_{p_3} \leftarrow \dots$$

*of  $\bar{Z}$  is canonically isomorphic to  $\bar{Z}$  as an object of  $\mathfrak{S}^*$ .*

## 4.2 The Conley index

In this section, we recall basic facts regarding the Conley index theory. See [5; 22; 33] for more details.

Let  $V$  be a finite-dimensional manifold and  $\varphi$  be a smooth flow on  $V$ , ie a  $C^\infty$ -map  $\varphi: V \times \mathbb{R} \rightarrow V$  such that  $\varphi(x, 0) = x$  and  $\varphi(x, s + t) = \varphi(\varphi(x, s), t)$  for any  $x \in V$  and  $s, t \in \mathbb{R}$ . We denote by  $\text{inv}(\varphi, A) := \{x \in A \mid \varphi(x, \mathbb{R}) \subset A\}$  the maximal invariant set of  $A$ . We sometimes write  $\text{inv}(A)$  when the flow  $\varphi$  is obvious from the context.

A compact set  $A \subset V$  is called an *isolating neighborhood* if  $\text{inv}(A)$  lies in the interior of  $A$ . A compact set  $S \subset V$  is called an *isolated invariant set* if there exists an isolating neighborhood  $A$  such that  $\text{inv}(A) = S$ . In this situation, we also say that  $A$  is an isolating neighborhood of  $S$ . For an isolated invariant set  $S$ , a pair  $(N, L)$  of compact sets  $L \subset N$  is called an *index pair* of  $S$  if the following conditions hold:

- (i)  $\text{inv}(N \setminus L) = S \subset \text{int}(N \setminus L)$ , where  $\text{int}(N \setminus L)$  is the interior of  $N \setminus L$ ;
- (ii)  $L$  is an exit set for  $N$ , ie for any  $x \in N$  and  $t > 0$  such that  $\varphi(x, t) \notin N$ , there exists  $\tau \in [0, t)$  with  $\varphi(x, \tau) \in L$ ;
- (iii)  $L$  is positively invariant in  $N$ , ie for  $x \in L$  and  $t > 0$ , if we know  $\varphi(x, [0, t]) \subset N$ , then we have  $\varphi(x, [0, t]) \subset L$ .

We list two fundamental facts regarding index pairs:

- For an isolated invariant set  $S$  with an isolating neighborhood  $A$ , we can always find an index pair  $(N, L)$  of  $S$  such that  $L \subset N \subset A$ .
- The pointed homotopy type of  $N/L$  with  $[L]$  as a basepoint only depends on  $S$  and  $\varphi$ . More precisely, for any two index pairs  $(N, L)$  and  $(N', L')$  of  $S$ , there is a natural pointed homotopy equivalence  $N/L \rightarrow N'/L'$  induced by the flow.

These lead to us the definition of the Conley index.

**Definition 4.4** Given an isolated invariant set  $S$  of a flow  $\varphi$ , let  $I(\varphi, S, N, L)$  denote the pointed space of  $(N/L, [L])$ , where  $(N, L)$  is an index pair of  $S$ . This is called the *Conley index* of  $S$ . We will always suppress  $(N, L)$  from our notation and write  $I(\varphi, S)$  instead. We may also write  $I(S)$  when the flow is clear from the context.

**Remark** In [33], the Conley index was defined as a connected simple system of pointed spaces. That is, a collection of pointed spaces (given by different index pairs) together with natural homotopy equivalences between them (given by the flow map). In Definition 4.4, we actually pick a representative of this connected simple system

by making a choice of the index pair  $(N, L)$ . As we will see in the next section, we need to make choices of all kinds of index pairs in our construction of spectrum invariants. Just like the Riemannian metric  $g$  and the perturbation on  $f$ , these choices will be treated as auxiliary data involved in the construction and we will prove that our spectrum invariant is independent of this data up to canonical isomorphism.

We further provide relevant properties of the Conley index.

- (1) **Product flow** If  $\varphi_j$  is a flow on  $V_j$  for  $j = 1, 2$  and  $S_j$  is an isolated invariant set for  $\varphi_j$ , then we have a canonical homotopy equivalence  $I(\varphi_1 \times \varphi_2, S_1 \times S_2) \cong I(\varphi_1, S_1) \wedge I(\varphi_2, S_2)$ , where  $\wedge$  is the smash product.
- (2) **Continuation** Let  $\varphi_t$  be a continuous family of flows parametrized by  $t \in [0, 1]$ . Suppose that  $A$  is an isolating neighborhood of  $\varphi_t$  for any  $t \in [0, 1]$ , and let  $S_t$  be  $\text{inv}(\varphi_t, A)$ . Then we have a canonical homotopy equivalence  $I(\varphi_0, S_0) \cong I(\varphi_1, S_1)$ .

The following concept will be useful for explicitly computing the Conley index.

**Definition 4.5** [32] For a compact subset  $A$ , we consider the following subsets of its boundary:

$$n^+(A) := \{x \in \partial A \mid \text{there exists } \epsilon > 0 \text{ such that } \varphi(x, (-\epsilon, 0)) \cap A = \emptyset\},$$

$$n^-(A) := \{x \in \partial A \mid \text{there exists } \epsilon > 0 \text{ such that } \varphi(x, (0, \epsilon)) \cap A = \emptyset\}.$$

A compact subset  $N$  is called an *isolating block* if  $\partial N = n^+(N) \cup n^-(N)$ .

It is easy to verify that an isolating block is an isolating neighborhood. When  $N$  is an isolating block, its index pair can be given by  $(N, n^-(N))$ .

Next, we consider a situation when an isolated invariant set can be decomposed to smaller isolated invariant sets.

**Definition 4.6** (i) For a subset  $A \subset V$ , we define its  $\alpha$ -limit and  $\omega$ -limit sets as

$$\alpha(A) = \bigcap_{t < 0} \overline{\varphi(A, (-\infty, t])} \quad \text{and} \quad \omega(A) = \bigcap_{t > 0} \overline{\varphi(A, [t, +\infty))}.$$

- (ii) Let  $S$  be an isolated invariant set. A subset  $T \subset S$  is called an *attractor* (resp. *repeller*) if there exists a neighborhood  $U$  of  $T$  in  $S$  such that  $\omega(U) = T$  (resp.  $\alpha(U) = T$ ).
- (iii) When  $T$  is an attractor in  $S$ , we define the set  $T^* := \{x \in S \mid \omega(x) \cap T = \emptyset\}$ , which is a repeller in  $S$ . We call  $(T, T^*)$  an *attractor-repeller pair* in  $S$ .

Note that an attractor and a repeller are always isolated invariant sets. We give an important result relating Conley indices of an attractor-repeller pair.

**Proposition 4.7** (Salamon [33]) *Let  $S$  be an isolated invariant set with an isolating neighborhood  $A$ , and let  $(T, T^*)$  be an attractor-repeller pair in  $S$ . Then there exist compact sets  $\tilde{N}_3 \subset \tilde{N}_2 \subset \tilde{N}_1 \subset A$  such that the pairs  $(\tilde{N}_2, \tilde{N}_3)$ ,  $(\tilde{N}_1, \tilde{N}_3)$  and  $(\tilde{N}_1, \tilde{N}_2)$  are index pairs for  $T$ ,  $S$  and  $T^*$ , respectively. The maps induced by inclusions give a natural coexact sequence of Conley indices*

$$I(\varphi, T) \xrightarrow{i_1} I(\varphi, S) \xrightarrow{i_2} I(\varphi, T^*) \rightarrow \Sigma I(\varphi, T) \rightarrow \Sigma I(\varphi, S) \rightarrow \dots$$

We call the triple  $(\tilde{N}_3, \tilde{N}_2, \tilde{N}_1)$  an index triple for the pair  $(T, T^*)$ , and we call the maps  $i_1$  and  $i_2$  the attractor map and the repeller map, respectively.

By Corollary 4.4 of [33], the attractor maps are transitive in the following sense. Suppose that  $S_1$  is an attractor in  $S_2$  and  $S_2$  is an attractor in  $S_3$ . Then  $S_1$  is also an attractor in  $S_3$ . Moreover, the corresponding attractor maps

$$i_1: I(\varphi, S_1) \rightarrow I(\varphi, S_2), \quad i'_1: I(\varphi, S_2) \rightarrow I(\varphi, S_3) \quad \text{and} \quad i''_1: I(\varphi, S_1) \rightarrow I(\varphi, S_3)$$

satisfy the relation  $i''_1 = i'_1 \circ i_1$ . Similar statements hold for the repeller maps.

Lastly, we briefly discuss the equivariant Conley index theory, which has been developed in [8; 31]. Let  $G$  be a compact Lie group acting on  $V$  while preserving the flow  $\varphi$ . For a  $G$ -invariant isolated invariant set  $S$ , we can find a  $G$ -invariant isolating neighborhood as well as a  $G$ -invariant index pair  $(N, L)$ . As in the nonequivariant case, with the choice of  $(N, L)$ , we denote by  $I_G(\varphi, S)$  the pointed  $G$ -space  $(N/L, [L])$ , whose  $G$ -equivariant homotopy type only depends on  $S$  and  $\varphi$ . In particular,  $I_G(\varphi, S)$  is the  $G$ -equivariant Conley index of  $S$ . All the nonequivariant results stated above can be adapted to the  $G$ -equivariant setting. From now on, we will work on this equivariant setting with  $G = S^1$  or  $\text{Pin}(2)$ .

## 5 Construction of the spectrum invariants

In this section, we define different versions of unfolded Seiberg–Witten–Floer spectra for the  $\text{spin}^c$  manifold  $(Y, \mathfrak{s})$ . We first define the spectra  $\underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1)$  and  $\underline{\text{swf}}^R(Y, \mathfrak{s}, A_0, g; S^1)$  for a general  $\text{spin}^c$  structure  $\mathfrak{s}$ . In Section 5.2, we consider the situation when  $\mathfrak{s}$  is torsion and define normalized spectra  $\underline{\text{SWF}}^A(Y, \mathfrak{s}; S^1)$  and

$\underline{\text{SWF}}^R(Y, \mathfrak{s}; S^1)$  which are independent of the choices of base connection  $A_0$  and metric  $g$ . In Section 5.3, we deal with the  $\text{Pin}(2)$ –equivariant case for a spin structure  $\mathfrak{s}$  and define  $\underline{\text{SWF}}^A(Y, \mathfrak{s}; \text{Pin}(2))$  and  $\underline{\text{SWF}}^R(Y, \mathfrak{s}; \text{Pin}(2))$ .

### 5.1 The spectrum invariants for general $\text{spin}^c$ structures

The main idea of the construction follows [16; 13]. In summary, we want to apply finite-dimensional approximation of Conley indices to the set  $\text{Str}(R)$  which contains all critical points and flow lines between them. However, the set  $\text{Str}(R)$  is unbounded owing to the action of  $\mathcal{G}_Y^h$ . We then need to introduce transverse functions and use their level sets to obtain a collection of bounded subsets of  $\text{Str}(R)$ .

The space of imaginary-valued harmonic 1–forms, denoted by  $i\Omega_h^1(Y)$ , is a subspace of  $\text{Coul}(Y)$ . Let  $p_{\mathcal{H}}: \text{Coul}(Y) \rightarrow i\Omega_h^1(Y)$  be the  $L^2$ –orthogonal projection. Here, we identify  $i\Omega_h^1(Y)$  with  $\mathbb{R}^{b_1}$  by choosing harmonic forms  $\{h_1, h_2, \dots, h_{b_1}\} \subset i\Omega_h^1(Y)$  representing a set of free generators of the group

$$2\pi i \text{im}(H^1(Y; \mathbb{Z}) \rightarrow H^1(Y; \mathbb{R})) \cong \mathbb{Z}^{b_1}.$$

With this identification, we can write the projection as

$$p_{\mathcal{H}} = (p_{\mathcal{H},1}, \dots, p_{\mathcal{H},b_1}).$$

From now on, we assume that our perturbation  $f$  is good; see Definition 2.3. Together with the compactness result [17, Theorem 10.7.1], the set of critical points of  $\mathcal{L}$  in  $\text{Coul}(Y)$  is finite modulo the action of  $\mathcal{G}_Y^h$ . Consequently, we can find a small interval  $[r, s] \subset (0, 1)$  such that  $\bigcup_{j=1}^{b_1} p_{\mathcal{H},j}^{-1}([-s, -r] \cup [r, s])$  contains no critical point of  $\mathcal{L}$ . Pick a positive number  $\tilde{R}$  greater than the universal constant  $R_0$  from Theorem 3.2.

**Lemma 5.1** *There exists a positive number  $\tilde{\epsilon} > 0$  such that we have  $\|\widetilde{\text{grad}} \mathcal{L}(x)\|_{\tilde{g}} > \tilde{\epsilon}$  for any  $x \in (\bigcup_{j=1}^{b_1} p_{\mathcal{H},j}^{-1}([-s, -r] \cup [r, s])) \cap \text{Str}(\tilde{R})$ .*

**Proof** Suppose that the result is not true. We can then find a sequence  $\{x_n\}$  contained in  $(\bigcup_{j=1}^{b_1} p_{\mathcal{H},j}^{-1}([-s, -r] \cup [r, s])) \cap \text{Str}(\tilde{R})$  with  $\|\widetilde{\text{grad}} \mathcal{L}(x_n)\|_{\tilde{g}} \rightarrow 0$ . Notice that the sequence  $\{x_n\}$  is contained in  $p_{\mathcal{H}}^{-1}([-1, 1]^{b_1}) \cap \text{Str}(\tilde{R})$ , which is bounded in  $L_k^2$ . Hence, after passing to a subsequence,  $x_n$  converges to some point  $x_\infty$  of  $\text{Coul}(Y)$  weakly in  $L_k^2$  and strongly in  $L_{k-1}^2$  by the Rellich lemma. Consequently, we have  $p_{\mathcal{H}}(x_n) \rightarrow p_{\mathcal{H}}(x_\infty)$  and  $\widetilde{\text{grad}} \mathcal{L}(x_\infty) = 0$  by continuity. This is a contradiction since  $x_\infty$  is a critical point of  $\widetilde{\text{grad}} \mathcal{L}$  and lies in  $\bigcup_{j=1}^{b_1} p_{\mathcal{H},j}^{-1}([-s, -r] \cup [r, s])$ .  $\square$





and only if  $x$  is a critical point of  $\mathcal{L}$ . Otherwise,  $|p_{\mathcal{H},j}(x)| \in [n+r, n+s]$  for some integer  $n$ , and Lemma 5.1 implies

$$\begin{aligned} \langle \widetilde{\text{grad}} \mathcal{L}(x), \widetilde{\text{grad}} g_{j,+}(x) \rangle_{\widetilde{g}} &= \|\widetilde{\text{grad}} \mathcal{L}(x)\|_{\widetilde{g}}^2 + \langle \widetilde{\text{grad}} \mathcal{L}(x), \widetilde{\text{grad}} (\bar{g} \circ p_{\mathcal{H},j})(x) \rangle_{\widetilde{g}} \\ &\geq \|\widetilde{\text{grad}} \mathcal{L}(x)\|_{\widetilde{g}}^2 - \|\widetilde{\text{grad}} (\bar{g} \circ p_{\mathcal{H},j})(x)\|_{\widetilde{g}} \cdot \|\widetilde{\text{grad}} \mathcal{L}(x)\|_{\widetilde{g}} \\ &> \|\widetilde{\text{grad}} \mathcal{L}(x)\|_{\widetilde{g}} (\|\widetilde{\text{grad}} \mathcal{L}(x)\|_{\widetilde{g}} - \tilde{\epsilon}) > 0. \end{aligned}$$

The same argument applies to the inner product  $\langle \widetilde{\text{grad}} \mathcal{L}(x), \widetilde{\text{grad}} g_{j,-}(x) \rangle_{\widetilde{g}}$ . □

Since the number of critical points of  $\mathcal{L}$  is finite modulo gauge, we can find a real number  $\theta \in \mathbb{R}$  such that  $g_{j,\pm}(x) \neq \theta$  for any critical point  $x$  of  $\mathcal{L}$  and  $j \in \{1, 2, \dots, b_1\}$ . For convenience, we also choose a decreasing sequence of negative real numbers  $\{\lambda_n\}$  and an increasing sequence of positive real numbers  $\{\mu_n\}$  such that  $-\lambda_n, \mu_n \rightarrow \infty$ . We are now ready to define a collection of bounded sets in  $\text{Str}(\widetilde{R})$ .

**Definition 5.3** With the choice of  $\widetilde{R}, \bar{g}$  and  $\theta$  above, we define the sets

$$(15) \quad \begin{aligned} J_m^+ &:= \text{Str}(\widetilde{R}) \cap \bigcap_{1 \leq j \leq b_1} g_{j,+}^{-1}(-\infty, \theta + m], \\ J_m^- &:= \text{Str}(\widetilde{R}) \cap \bigcap_{1 \leq j \leq b_1} g_{j,-}^{-1}(-\infty, \theta + m] \end{aligned}$$

for each positive integer  $m$ . This collection of  $J_m^+$  (resp.  $J_m^-$ ) will be called a *positive* (resp. *negative*) *transverse system*. With the choice of  $\{\lambda_n\}$  and  $\{\mu_n\}$ , we also define

$$J_m^{n,\pm} := J_m^\pm \cap V_{\lambda_n}^{\mu_n}.$$

Notice that the functional  $\mathcal{L}$  is bounded on  $\text{Str}(\widetilde{R})$ , and the perturbed functional  $g_{j,\pm}$  is bounded below on  $\text{Str}(\widetilde{R})$ . Since a subset  $S \subset \text{Str}(\widetilde{R})$  is bounded if and only if  $p_{\mathcal{H}}(S)$  is bounded, we can see that the set  $J_m^\pm$  is bounded in the  $L_k^2$ -norm.

We will start to derive some properties of the finite-dimensional bounded sets  $J_m^{n,\pm}$ . Although some of the following results are slightly stronger than what we need to define the 3-dimensional invariants, they will be useful when we develop the 4-dimensional theory and prove the gluing theorem in [14; 15].

**Lemma 5.4** For any positive integer  $m$ , there exist positive real numbers  $\epsilon_m$  and  $\theta_m$  and an integer  $N_m \gg 0$  such that for any  $n > N_m$  and  $1 \leq j \leq b_1$ , we have

$$\begin{aligned} \langle (l + p_{\lambda_n}^{\mu_n} \circ c)(x), \widetilde{\text{grad}} g_{j,+}(x) \rangle_{\widetilde{g}} &> \epsilon_m \quad \text{for any } x \in J_m^{n,+} \cap g_{j,+}^{-1}[\theta + m - \theta_m, \theta + m], \\ \langle (l + p_{\lambda_n}^{\mu_n} \circ c)(x), \widetilde{\text{grad}} g_{j,-}(x) \rangle_{\widetilde{g}} &< -\epsilon_m \quad \text{for any } x \in J_m^{n,-} \cap g_{j,-}^{-1}[\theta + m - \theta_m, \theta + m]. \end{aligned}$$

**Proof** We only prove this lemma for  $g_{1,+}$ ; the other cases can be proved similarly. Suppose that the result is not true. Then we can find sequences  $n_i \rightarrow +\infty$ ,  $\epsilon_{m,i}, \theta_{m,i} \rightarrow 0$  and  $\{x_i\}$  with  $x_i \in J_m^{n_i,+} \cap g_{1,+}^{-1}[\theta + m - \theta_{m,i}, \theta + m]$  and

$$\langle (l + p_{\lambda_{n_i}}^{\mu_{n_i}} \circ c)(x_i), \widetilde{\text{grad}} g_{1,+}(x_i) \rangle_{\widetilde{g}} \leq \epsilon_{m,i}.$$

Since  $\{x_i\}$  is contained in the  $L_k^2$ -bounded set  $J_m^+$ , we can pass to a convergent subsequence  $x_i \rightarrow x_\infty$  in  $L_{k-1}^2$  by the Rellich lemma. By continuity, we have  $x_\infty \in g_{1,+}^{-1}(\theta + m)$  and  $\widetilde{\text{grad}} g_{1,+}(x_i) \rightarrow \widetilde{\text{grad}} g_{1,+}(x_\infty)$  in  $L_{k-2}^2$ . Since  $p_{\lambda_n}^{\mu_n}$  converges to the identity map pointwise, we also have  $(l + p_{\lambda_{n_i}}^{\mu_{n_i}} \circ c)(x_i) \rightarrow (l + c)(x_\infty) = \widetilde{\text{grad}} \mathcal{L}(x_\infty)$  in  $L_{k-2}^2$ . Therefore, we obtain

$$\langle (l + p_{\lambda_{n_i}}^{\mu_{n_i}} \circ c)(x_{n_i}), \widetilde{\text{grad}} g_{j,+}(x_{n_i}) \rangle_{\widetilde{g}} \rightarrow \langle \widetilde{\text{grad}} \mathcal{L}(x_\infty), \widetilde{\text{grad}} g_{j,+}(x_\infty) \rangle_{\widetilde{g}},$$

which implies that  $\langle \widetilde{\text{grad}} \mathcal{L}(x_\infty), \widetilde{\text{grad}} g_{j,+}(x_\infty) \rangle_{\widetilde{g}} \leq 0$  and  $x_\infty$  is a critical point by Lemma 5.2. This is a contradiction with the choice of  $\theta$ . □

Now we apply the Conley index theory to the flow on  $V_{\lambda_n}^{\mu_n}$  generated by the vector field  $-(l + p_{\lambda_n}^{\mu_n} \circ c)$ . There is a technical point here. Since  $V_{\lambda_n}^{\mu_n}$  is noncompact, this flow may go to infinity within a finite time. As in [22], we can fix this by choosing a bump function  $\iota_m: \text{Coul}(Y) \rightarrow [0, 1]$  for each  $m$  such that  $\iota_m$  is supported in a bounded subset of  $\text{Coul}(Y)$  and  $J_{m+1}^\pm$  is contained in the interior of  $\iota_m^{-1}(1)$ . We denote by  $\varphi_m^n$  the flow on  $V_{\lambda_n}^{\mu_n}$  generated by  $-\iota_m \cdot (l + p_{\lambda_n}^{\mu_n} \circ c)$ . The flow  $\varphi_{m'}^n$  on  $J_m^{n,\pm}$  does not depend on  $m'$  whenever  $m' \geq m - 1$ , so its invariant set and its Conley index remain unchanged.

**Lemma 5.5** *For a positive integer  $M$ , there exist large numbers  $N$  and  $T$  such that for any positive integers  $m \leq M$  and  $n \geq N$ , we have the following statements:*

- (a) *If  $\gamma: [-T, T] \rightarrow V_{\lambda_n}^{\mu_n}$  is an approximated Seiberg–Witten trajectory contained in  $J_m^{n,+}$ , then we have*

$$\gamma(0) \in \text{Str}(R_0) \cap \bigcap_{1 \leq j \leq b_1} g_{j,+}^{-1}(-\infty, \theta + m - \theta_m].$$

*In particular,  $J_m^{n,+}$  is an isolating neighborhood for the flow  $\varphi_m^n$ .*

- (b) *The set  $\text{inv}(\varphi_{m-1}^n, J_{m-1}^{n,+})$  is an attractor in  $\text{inv}(\varphi_m^n, J_m^{n,+})$  with respect to the flow  $\varphi_m^n$ .*

**Proof** Let  $\bar{T}$ ,  $\bar{\lambda}$  and  $\bar{\mu}$  be the large constants from Corollary 3.7 with  $S = J_M^+$ . Let  $\theta_m, \epsilon_m$  and  $N_m$  be the constants obtained from Lemma 5.4 for  $m = 1, \dots, M$ . Put  $T = \max\{\bar{T}, \theta_1/\epsilon_1, \dots, \theta_M/\epsilon_M\} + 1$ . We choose a positive integer  $N > \max\{N_1, \dots, N_M\}$  such that  $\lambda_N < \bar{\lambda}$  and  $\mu_N > \bar{\mu}$ . Let  $m \leq M$  and  $n > N$  be arbitrary positive integers.

(a) Let  $\gamma: [-T, T] \rightarrow V_{\lambda_n}^{\mu_n}$  be an approximated Seiberg–Witten trajectory contained in  $J_m^{n,+}$ . Corollary 3.7 and the choice of  $N$  and  $T$  ensure that  $\gamma(0) \in \text{Str}(R_0)$ . For the sake of contradiction, let us suppose that  $g_{j,+}(\gamma(0)) > \theta + m - \theta_m$  for some  $j \in \{1, \dots, b_1\}$ . By Lemma 5.4, the value of  $g_{j,+}(\gamma(t))$  decreases along the trajectory  $\gamma$  on  $[-T, 0]$  with

$$\frac{d}{dt} g_{j,+}(\gamma(t)) = \langle -(l + p_{\lambda_n}^{\mu_n} \circ c)(\gamma(t)), \widetilde{\text{grad}} g_{j,+}(\gamma(t)) \rangle_{\widetilde{g}} < -\epsilon_m.$$

Hence, we obtain  $g_{j,+}(\gamma(-T)) > g_{j,+}(\gamma(0)) + T\epsilon_m > \theta + m$  from the fundamental theorem of calculus. This is a contradiction with our assumption that  $\gamma(-T) \in J_m^+ \subset g_{j,+}^{-1}(-\infty, \theta + m]$ .

(b) From Lemma 5.4 and the choice of  $N$ , we have

$$\langle -(l + p_{\lambda_n}^{\mu_n} \circ c)(x), \widetilde{\text{grad}} g_{j,+}(x) \rangle_{\widetilde{g}} < 0$$

for any  $x \in J_{m-1}^{n,+} \cap g_{j,+}^{-1}(\theta + m - 1)$ . Consequently, the flow  $\varphi_m^n$  goes inside  $J_{m-1}^{n,+}$  along  $\partial J_{m-1}^{n,+} \setminus \partial \text{Str}(\widetilde{R})$ , and  $\text{inv}(\varphi_{m-1}^n, J_{m-1}^{n,+})$  is an attractor in  $\text{inv}(\varphi_m^n, J_m^{n,+})$  with respect to the flow  $\varphi_m^n$ .  $\square$

Consequently, we can acquire the  $S^1$ -equivariant Conley index  $I_{S^1}(\varphi_m^n, \text{inv}(J_m^{n,+}))$  from a compact finite-dimensional subset  $J_m^{n,+}$  when  $n$  is large enough relative to  $m$  as in Lemma 5.5. Using the orthogonal complement  $\overline{V}_\lambda^0$  of  $i\Omega_h^1(Y)$  in  $V_\lambda^0$ , we define

$$I_m^{n,+} := \Sigma^{-\overline{V}_{\lambda_n}^0} I_{S^1}(\varphi_m^n, \text{inv}(J_m^{n,+}))$$

as an object of  $\mathcal{C}$ . Note that here a choice of index pair for  $\text{inv}(J_m^{n,+})$  is made to get the Conley index; see the remark following Definition 4.4. Eventually, we will show that our invariants are independent of this choice up to canonical isomorphisms.

Let  $i_m^{n,+}: I_{S^1}(\varphi_m^n, \text{inv}(J_m^{n,+})) \rightarrow I_{S^1}(\varphi_m^n, \text{inv}(J_{m+1}^{n,+}))$  be the attractor map and denote by  $\widetilde{i}_m^{n,+}$  a morphism  $\Sigma^{-\overline{V}_{\lambda_n}^0} i_m^{n,+} \in \text{mor}_{\mathcal{C}}(I_m^{n,+}, I_{m+1}^{n,+})$ . We will show that the object  $I_m^{n,+}$  is stable in the following sense.

**Proposition 5.6** *For any positive integer  $M > 0$ , there exists a positive integer  $N$  such that for any positive integers  $m \leq M$  and  $n \geq N$ , there is a canonical isomorphism  $\widetilde{\rho}_m^{n,+} \in \text{mor}_{\mathcal{C}}(I_m^{n,+}, I_m^{n+1,+})$ . Moreover, we have the following commutative diagram:*

$$(16) \quad \begin{array}{ccc} I_{m-1}^{n,+} & \xrightarrow{\widetilde{i}_{m-1}^{n,+}} & I_m^{n,+} \\ \widetilde{\rho}_{m-1}^{n,+} \downarrow & & \downarrow \widetilde{\rho}_m^{n,+} \\ I_{m-1}^{n+1,+} & \xrightarrow{\widetilde{i}_{m-1}^{n+1,+}} & I_m^{n+1,+} \end{array}$$

**Proof** Following the remark after [Corollary 3.7](#), we can extend the result of [Lemma 5.5](#) to interpolated projections. With the integer  $N$  depending on  $M$  from [Lemma 5.5](#), we can deduce that  $J_m^{n+1,+}$  is an isolating neighborhood for the flow generated by  $-\iota_m \cdot (l + (s \cdot p_{\lambda_{n+1}}^{\mu_{n+1}} + (1-s)p_{\lambda_n}^{\mu_n}) \circ c)$  for any  $n > N$  and  $s \in [0, 1]$ .

The rest of proof follows from the arguments given in [[22](#), page 910; [13](#), Proposition 8]. By continuation property of the Conley index, we have a natural homotopy equivalence

$$\rho_m^{n,+} : \Sigma^{V_{\lambda_{n+1}}^{\lambda_n}} I_{S^1}(\varphi_m^n, \text{inv}(J_m^{n,+})) \rightarrow I_{S^1}(\varphi_m^{n+1}, \text{inv}(J_m^{n+1,+})).$$

The isomorphism  $\tilde{\rho}_m^{n,+}$  is then given by the composition

$$\begin{aligned} \Sigma^{-\bar{V}_{\lambda_n}^0} I_{S^1}(\varphi_m^n, \text{inv}(J_m^{n,+})) &\rightarrow \Sigma^{-\bar{V}_{\lambda_n}^0} \Sigma^{-V_{\lambda_{n+1}}^{\lambda_n}} \Sigma^{V_{\lambda_{n+1}}^{\lambda_n}} I_{S^1}(\varphi_m^n, \text{inv}(J_m^{n,+})) \\ &\rightarrow \Sigma^{-\bar{V}_{\lambda_n}^0} \Sigma^{-V_{\lambda_{n+1}}^{\lambda_n}} I_{S^1}(\varphi_m^{n+1}, \text{inv}(J_m^{n+1,+})) \\ &= \Sigma^{-\bar{V}_{\lambda_{n+1}}^0} I_{S^1}(\varphi_m^{n+1}, \text{inv}(J_m^{n+1,+})), \end{aligned}$$

where the first morphism is given by  $\Sigma^{-\bar{V}_{\lambda_n}^0} \eta^{-1}$  (see [Proposition 4.2](#)) and the second morphism equals  $\Sigma^{-\bar{V}_{\lambda_n}^0} \rho_m^{n,+}$ . The diagram ([16](#)) commutes because of the continuation property of attractor-repeller pairs [[33](#), Theorem 6.10]. □

For each positive integer  $M$ , we pick a positive integer  $n_M$  larger than the constant  $N$  from [Proposition 5.6](#), and we require that  $\{n_M\}$  be an increasing sequence. We are now ready to define the spectrum invariant.

**Definition 5.7** The  $S^1$ -equivariant ind-spectrum  $\underline{\text{swf}}^A(Y, \mathfrak{s}_Y, A_0, g; S^1)$  is defined to be an object of  $\mathfrak{S}$  given by

$$(17) \quad I_1^{n_1,+} \rightarrow I_2^{n_2,+} \rightarrow I_3^{n_3,+} \rightarrow \dots,$$

where the morphism from  $I_m^{n_m,+}$  to  $I_{m+1}^{n_{m+1},+}$  is a composition

$$\tilde{\iota}_m^{n_{m+1},+} \circ \tilde{\rho}_m^{n_{m+1}-1,+} \circ \dots \circ \tilde{\rho}_m^{n_m,+}$$

of the morphisms in [Proposition 5.6](#).

We will prove in the next section that this gives a well-defined object of the category  $\mathfrak{S}$  independent of the choices made in the construction up to canonical isomorphism.

To define another invariant  $\underline{\text{swf}}^R(Y, \mathfrak{s}_Y, A_0, g; S^1)$ , we follow almost the same steps for the construction of  $\underline{\text{swf}}^A$  except that there are two main differences. First, the

set  $\text{inv}(\varphi_m^n, J_m^{n,-})$  is a repeller in  $\text{inv}(\varphi_m^n, J_{m+1}^{n,-})$ , so the arrows in the system will be reversed. Second, we use  $V_{\lambda_n}^0$  for desuspension instead of  $\bar{V}_{\lambda_n}^0$ . We define

$$I_m^{n,-} := \Sigma^{-V_{\lambda_n}^0} I_{S^1}(\varphi_m^n, \text{inv}(J_m^{n,-})) \in \text{ob } \mathfrak{C},$$

where  $n$  is large enough relative to  $m$ , and we have a morphism

$$I_m^{n,-} \leftarrow I_{m+1}^{n,-}$$

induced by the repeller map. The following collection of results can be proved in the same way as the corresponding results for  $J_m^{n,+}$ .

**Proposition 5.8** *For a positive integer  $M$ , there exist large numbers  $N$  and  $T$  such that for any positive integers  $m \leq M$  and  $n \geq N$ , we have the following statements:*

- (a) *For any approximated Seiberg–Witten trajectory  $\gamma: [-T, T] \rightarrow V_{\lambda_n}^{\mu_n}$  which is contained in  $J_m^{n,-}$ , we have*

$$\gamma(0) \in \text{Str}(R_0) \cap \bigcap_{1 \leq j \leq b_1} g_{j,-}^{-1}(-\infty, \theta + m - \theta_m].$$

*In particular,  $J_m^{n,-}$  is an isolating neighborhood for the flow  $\varphi_n^m$ .*

- (b) *The set  $\text{inv}(\varphi_n^m, J_{m-1}^{n,-})$  is a repeller in  $\text{inv}(\varphi_n^m, J_m^{n,-})$  with respect to the flow  $\varphi_n^m$ . Consequently, we have the repeller map*

$$i_{m-1}^{n,-}: I_{S^1}(\varphi_n^m, \text{inv}(J_{m-1}^{n,-})) \rightarrow I_{S^1}(\varphi_n^m, \text{inv}(J_m^{n,-})).$$

- (c) *There is a canonical isomorphism  $\tilde{\rho}_m^{n,-} \in \text{mor}_{\mathfrak{C}}(I_m^{n,-}, I_m^{n+1,-})$  such that the diagram*

$$(18) \quad \begin{array}{ccc} I_{m-1}^{n,-} & \xleftarrow{\tilde{i}_{m-1}^{n,-}} & I_m^{n,-} \\ \tilde{\rho}_{m-1}^{n,-} \downarrow & & \downarrow \tilde{\rho}_m^{n,-} \\ I_{m-1}^{n+1,-} & \xleftarrow{\tilde{i}_{m-1}^{n+1,-}} & I_m^{n+1,-} \end{array}$$

*commutes, where  $\tilde{i}_{m-1}^{n,-}$  is given by  $\Sigma^{-V_{\lambda_n}^0} i_{m-1}^{n,-}$ .*

For each positive integer  $M$ , we also choose a positive integer  $n_M$ , larger than the constant  $N$  from Proposition 5.8, so that  $\{n_M\}$  is an increasing sequence.

**Definition 5.9** The  $S^1$ -equivariant pro-spectrum  $\underline{\text{swf}}^R(Y, \mathfrak{s}_Y, A_0, g; S^1)$  is defined to be an object of  $\mathfrak{S}^*$  given by

$$(19) \quad I_1^{n_1, -} \leftarrow I_2^{n_2, -} \leftarrow I_3^{n_3, -} \leftarrow \dots,$$

where the connecting morphisms are defined in the same manner as in Definition 5.7.

We will also prove well-definedness of  $\underline{\text{swf}}^R(Y, \mathfrak{s}_Y, A_0, g; S^1)$  in the next section.

### 5.2 The torsion case

When the  $\text{spin}^c$  structure  $\mathfrak{s}$  is torsion, we will be able to further normalize the spectrum invariants  $\underline{\text{swf}}^A$  and  $\underline{\text{swf}}^R$  following the idea of [22]. The resulting objects will not depend on  $A_0$  and  $g$ .

We will need to define a rational number  $n(Y, \mathfrak{s}_Y, A_0, g)$ . Choose a 4-manifold  $X$  with boundary  $Y$  with  $H^3(X, Y; \mathbb{Z}) \cong H_1(X; \mathbb{Z}) = 0$ . Such  $X$  always exists as we can construct  $X$  by attaching 2-handles on  $D^4$  according to the surgery diagram of  $Y$ . By the homology long exact sequence for the pair  $(X, Y)$ , we see that  $H^2(X, \mathbb{Z}) \rightarrow H^2(Y, \mathbb{Z})$  is surjective. Therefore, we can extend  $\mathfrak{s}$  to a  $\text{spin}^c$  structure  $\mathfrak{s}_X$  over  $X$  and extend  $A_0$  to a connection  $\hat{A}_0$  over  $X$ . Recall that we have a nondegenerate pairing

$$\cup: \text{im}(H^2(X, Y; \mathbb{Q}) \rightarrow H^2(X; \mathbb{Q})) \otimes \text{im}(H^2(X, Y; \mathbb{Q}) \rightarrow H^2(X; \mathbb{Q})) \longrightarrow \mathbb{Q}.$$

Denote by  $b^+(X)$  (resp.  $b^-(X)$ ) the dimension of a maximal positive (resp. negative) subspace with respect to this pairing and denote by  $\sigma(X)$  the signature of this pairing. Notice that we can define  $c_1(\mathfrak{s}_X)^2 = c_1(\mathfrak{s}_X) \cup c_1(\mathfrak{s}_X) \in \mathbb{Q}$  because  $c_1(\mathfrak{s}_X)|_Y = c_1(\mathfrak{s})$  is torsion. We define

$$(20) \quad n(Y, \mathfrak{s}, A_0, g) := \text{Ind}_{\mathbb{C}}(\hat{\mathcal{D}}_{\hat{A}_0}^+) - \frac{1}{8}(c_1(\mathfrak{s}_X)^2 - \sigma(X)),$$

where  $\hat{\mathcal{D}}_{\hat{A}_0}^+$  is the positive Dirac operator on  $X$  coupled with  $\hat{A}_0$ , and  $\text{Ind}_{\mathbb{C}}(\hat{\mathcal{D}}_{\hat{A}_0}^+)$  is its index defined by using spectral boundary condition as in [2]. It was proved in [22] that  $n(Y, \mathfrak{s}, A_0, g)$  does not depend on the choices of  $X, \mathfrak{s}_X$  and  $\hat{A}_0$  ([22] only considered a rational homology sphere  $Y$  but the proof works for a general 3-manifold  $Y$  without any changes). In fact, we have

$$(21) \quad n(Y, \mathfrak{s}, A_0, g) = \frac{1}{2}(\eta(\mathcal{D}) - \dim_{\mathbb{C}}(\ker \mathcal{D}) + \frac{1}{4}\eta_{\text{sign}}),$$

where  $\eta(\mathcal{D})$  and  $\eta_{\text{sign}}$  denote the eta-invariant of the Dirac operator and the odd signature operator, respectively; see [22; 2].

The normalized invariants  $\underline{\text{SWF}}^A$  and  $\underline{\text{SWF}}^R$  will be obtained by formally desuspending  $\underline{\text{swf}}^A$  and  $\underline{\text{swf}}^R$  with the rational number  $n(Y, \mathfrak{s}, A_0, g)$  as follows.

**Definition 5.10** We define the  $S^1$ –equivariant ind-spectrum and pro-spectrum by

$$\begin{aligned} \underline{\text{SWF}}^A(Y, \mathfrak{s}; S^1) &:= (\underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1), 0, n(Y, \mathfrak{s}, A_0, g)), \\ \underline{\text{SWF}}^R(Y, \mathfrak{s}; S^1) &:= (\underline{\text{swf}}^R(Y, \mathfrak{s}, A_0, g; S^1), 0, n(Y, \mathfrak{s}, A_0, g)) \end{aligned}$$

as objects of  $\mathfrak{S}$  and  $\mathfrak{S}^*$ , respectively.

The proof of invariance of  $\underline{\text{SWF}}^A$  and  $\underline{\text{SWF}}^R$  will also be in the next section.

### 5.3 The $\text{Pin}(2)$ –spectrum invariants for spin structures

In this subsection, we will define the  $\text{Pin}(2)$ –analog of the spectrum invariants for a 3–manifold  $Y$  equipped with a spin structure  $\mathfrak{s}$ . Since all the constructions are similar to the  $S^1$ –case, some of the discussions will be brief.

We define the group  $\text{Pin}(2)$  as the subgroup  $S^1 \cup jS^1 \subset \mathbb{H}$  of the algebra of quaternions containing  $S^1$  as the set of unit complex numbers. We are interested in the following real representations of  $\text{Pin}(2)$ :

- (1)  $\mathbb{R}$ , the trivial one-dimensional representation;
- (2)  $\tilde{\mathbb{R}}$ , the nontrivial one-dimensional representation where  $S^1$  acts trivially and  $j$  acts as multiplication by  $-1$ ;
- (3)  $\mathbb{H}$ , the four-dimensional representation where  $\text{Pin}(2)$  acts by left quaternionic multiplication.

We introduce the categories  $\mathfrak{C}_{\text{Pin}(2)}$ ,  $\mathfrak{S}_{\text{Pin}(2)}$  and  $\mathfrak{S}_{\text{Pin}(2)}^*$  which are the  $\text{Pin}(2)$ –versions of the categories  $\mathfrak{C}$ ,  $\mathfrak{S}$  and  $\mathfrak{S}^*$ . The objects of  $\mathfrak{C}_{\text{Pin}(2)}$  are triples  $(A, m, n)$  consisting of an even integer  $m$ , a rational number  $n$  and a pointed  $\text{Pin}(2)$ –space  $A$  which is  $\text{Pin}(2)$ –homotopy equivalent to a finite  $\text{Pin}(2)$ –CW complex. We define the set  $\text{mor}_{\mathfrak{C}_{\text{Pin}(2)}}((A, m, n), (A', m', n'))$  by

$$\text{colim}_{u,v,w \rightarrow \infty} [(\mathbb{R}^u \oplus \tilde{\mathbb{R}}^v \oplus \mathbb{H}^w)^+ \wedge A, (\mathbb{R}^u \oplus \tilde{\mathbb{R}}^{v+m-m'} \oplus \mathbb{H}^{w+n-n'})^+ \wedge A']_{\text{Pin}(2)}$$

when  $n - n' \in \mathbb{Z}$  and it is empty otherwise. The objects of  $\mathfrak{S}_{\text{Pin}(2)}$  (resp.  $\mathfrak{S}_{\text{Pin}(2)}^*$ ) are the sequential direct systems (resp. sequential inverse systems) in  $\mathfrak{C}_{\text{Pin}(2)}$ . We call an object

of  $\mathfrak{S}_{\text{Pin}(2)}$  a  $\text{Pin}(2)$ -equivariant ind-spectrum and call an object of  $\mathfrak{S}_{\text{Pin}(2)}^*$  a  $\text{Pin}(2)$ -equivariant pro-spectrum. The sets of morphisms are defined in the same way as (12) and (13). For an object  $W$  of  $\mathfrak{C}_{\text{Pin}(2)}$ ,  $\mathfrak{S}_{\text{Pin}(2)}$  and  $\mathfrak{S}_{\text{Pin}(2)}^*$ , the notation  $(W, m, n)$  will be used as in the  $S^1$ -case. A  $\text{Pin}(2)$ -representation  $E$  is called *admissible* if it is isomorphic to  $\tilde{\mathbb{R}}^a \oplus \mathbb{H}^b$ . For an admissible representation  $E$ , we can define the suspension functor  $\Sigma^E$  and the desuspension functor  $\Sigma^{-E}$  in the same manner, eg  $\Sigma^{-E}(A, m, n)$  is given by  $(\Sigma^{E^{S^1}}A, m + 2a, n + b)$ . All the results from Section 4 can be adapted to this setting.

We now turn to the Seiberg–Witten theory of a spin 3-manifold. Recall that the spin structure  $\mathfrak{s}$  induces a torsion  $\text{spin}^c$  structure on  $Y$ . With a slight abuse of notation, we also denote this  $\text{spin}^c$  structure by  $\mathfrak{s}$ . We will have the same setup from the  $\text{spin}^c$  structure  $\mathfrak{s}$  with the following new features coming from a spin structure:

- (1) The structure group of  $S_Y$  can be reduced to  $\text{SU}(2) \cong S(\mathbb{H})$ . Therefore,  $S_Y$  is a quaternionic bundle. Here we follow the convention of [24] and let the structure group act by the right multiplication.
- (2) The bundle  $\det(S_Y)$  has a canonical trivialization. The Levi-Civita connection on  $TY$  then induces a canonical spin connection  $A_0$  on  $S_Y$  with  $F_{A_0} = 0$ . We will always choose  $A_0$  for our base connection.
- (3) We have an additional action  $J: \mathcal{C}_Y \rightarrow \mathcal{C}_Y$  sending  $(a, \phi)$  to  $(-a, j\phi)$ . This action, together with the constant gauge group  $S^1$ , gives a  $\text{Pin}(2)$ -action on  $\mathcal{C}_Y$ . All the objects in the setup are  $\text{Pin}(2)$ -invariant, eg the functional  $\text{CSD}_{v_0}$ , the Coulomb slice  $\text{Coul}(Y)$  and the  $L_k^2$ -inner product, etc.

In order to respect the additional  $J$ -symmetry, we have two new requirements in our construction:

- (1) The perturbation  $f$  should be invariant under  $J$ . In other words, we should have  $f(a, \phi) = f(-a, j\phi)$ .
- (2) The sets  $J_m^{n, \pm}$  should be invariant under  $J$ .

A slight adaption of [20, Theorem 2.6] shows that for any real number  $\delta$ , we can find a  $J$ -invariant extended cylinder function  $\bar{f}$  such that  $(\delta, \bar{f})$  is a good perturbation. Since we required the staircase function  $\bar{g}$  from Section 5.1 to be even, it is not hard to see that  $J_m^{n, \pm}$  is  $J$ -invariant once the perturbation  $f$  is  $J$ -invariant.

We can now follow the construction from Section 5.1. In particular, the sets  $J_m^{n, \pm}$  are isolating neighborhoods for the  $\text{Pin}(2)$ -invariant flow  $\varphi_m^n$  when  $n$  is sufficiently large



relative to  $m$ , and we define

$$I_m^{n,+}(\text{Pin}(2)) := \Sigma^{-\bar{V}_{\lambda_n}^0} I_{\text{Pin}(2)}(\varphi_n^m, \text{inv}(J_m^{n,+})),$$

$$I_m^{n,-}(\text{Pin}(2)) := \Sigma^{-V_{\lambda_n}^0} I_{\text{Pin}(2)}(\varphi_n^m, \text{inv}(J_m^{n,-}))$$

as objects of  $\mathfrak{C}_{\text{Pin}(2)}$ . As before, we obtain an object  $\underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; \text{Pin}(2))$  of  $\mathfrak{S}_{\text{Pin}(2)}$  given by

$$I_1^{n_1,+}(\text{Pin}(2)) \rightarrow I_2^{n_2,+}(\text{Pin}(2)) \rightarrow \dots$$

and an object  $\underline{\text{swf}}^R(Y, \mathfrak{s}, A_0, g; \text{Pin}(2))$  of  $\mathfrak{S}_{\text{Pin}(2)}^*$  given by

$$I_1^{n_1,-}(\text{Pin}(2)) \leftarrow I_2^{n_2,-}(\text{Pin}(2)) \leftarrow \dots$$

for an increasing sequence of large positive integers  $\{n_i\}$ . We define spectrum invariants as in the torsion  $\text{spin}^c$  case.

**Definition 5.11** With the above setup, the  $\text{Pin}(2)$ –equivariant ind-spectrum and pro-spectrum are defined by

$$\underline{\text{SWF}}^A(Y, \mathfrak{s}; \text{Pin}(2)) := (\underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; \text{Pin}(2)), 0, \frac{1}{2}(n(Y, \mathfrak{s}, A_0, g))),$$

$$\underline{\text{SWF}}^R(Y, \mathfrak{s}; \text{Pin}(2)) := (\underline{\text{swf}}^R(Y, \mathfrak{s}, A_0, g; \text{Pin}(2)), 0, \frac{1}{2}(n(Y, \mathfrak{s}, A_0, g)))$$

as objects of  $\mathfrak{S}_{\text{Pin}(2)}$  and  $\mathfrak{S}_{\text{Pin}(2)}^*$ , respectively. Here  $n(Y, \mathfrak{s}, A_0, g)$  is the rational number defined in (20). As before, these objects are independent of the choices made in the construction up to canonical isomorphism.

## 6 The invariance for the spectrum

In this section, we will prove the invariance of our ind-spectrum (pro-spectrum). In other words, we will show that the spectra given by different choices of parameters are canonically isomorphic to each other (as objects of the category in which they are defined). We focus on the  $S^1$ –equivariant case, and the  $\text{Pin}(2)$ –case can be proved in the same way.

First, let us list the parameters in order of dependence (for example,  $\tilde{R}$  is any number greater than  $R_0$ , where  $R_0$  is the constant of [Theorem 3.2](#) depending on  $g$ ,  $A_0$  and  $f$ ):

- (I) the Riemannian metric  $g$  and the base connection  $A_0$ ;
- (II) the good perturbation  $f: \text{Coul}(Y) \rightarrow \mathbb{R}$ ;
- (III) the sequences of real numbers  $\{\lambda_n\}, \{\mu_n\}$ ;
- (IV) the number  $\tilde{R}$  (in the definition of  $\text{Str}(\tilde{R})$ );
- (V) the harmonic forms  $\{h_j\}$ , the cutting function  $\bar{g}$  and the cutting value  $\theta$ ;
- (VI) the positive integers  $n_m$  in (17) and (19);
- (VII) the index pairs for the isolated invariant sets.

The invariance for (VII) is a direct consequence of the invariance of the Conley index; see Section 4.2 and [33]. The commutative diagrams (16) and (18) imply the invariance for (VI).

In Section 6.1, we will make a digression into the discussion of the finite-dimensional approximation for a family of flows. In Section 6.2, we will prove the invariance for (III)–(V). The invariance for (II) (which is the most interesting one) and (I) will be proved in Sections 6.3 and 6.4, respectively.

### 6.1 The finite-dimensional approximation for a family of flows

In this subsection, we extend finite-dimensional approximation results in Section 3 for a continuous family of flows. This setup will be useful for proving the invariance and calculating examples.

Let  $S$  be a compact manifold (possibly with boundary) and consider a smooth family of Riemannian metrics  $\{g_s\}_{s \in S}$  and a smooth family of base connections  $\{A_{0,s}\}_{s \in S}$ . As before, we require that  $(i/2\pi)FA_{0,s}^t$  equals the harmonic form representing  $c_1(\mathfrak{s})$ . We denote by  $\text{Coul}(Y, s)$  the  $(L^2_k\text{-completed})$  Coulomb slice for  $(g_s, A_{0,s})$ . For each  $s$ , we have an elliptic operator  $l_s: \text{Coul}(Y, s) \rightarrow \text{Coul}(Y, s)$  given by  $(*_s d, \not{D}_{A_{0,s}})$ , where  $*_s$  is the Hodge operator of  $g_s$ . Although  $\{\text{Coul}(Y, s) \mid s \in S\}$  is a Hilbert bundle over  $S$ , by Kuiper’s theorem, this bundle is trivial, and we can identify it with  $S \times \text{Coul}(Y)$  by fixing a trivialization. We have the following generalization of Definition 3.4:

**Definition 6.1** Let  $E$  be a vector bundle over  $Y$ . A family of smooth and bounded maps  $\{Q_s: \text{Coul}(Y, s) \rightarrow L^2_k(\Gamma(E))\}_{s \in S}$  is called a *continuous family of quadratic-like maps* if  $Q_s$  is quadratic-like for each  $s \in S$  and, for each nonnegative integer  $m < k$ , we have a uniform convergence  $(\frac{d}{dt})^m Q_{s_n}(\gamma_n(t)) \rightarrow (\frac{d}{dt})^m Q_{s_\infty}(\gamma_\infty(t))$  in  $L^2_{k-2-m}$

whenever there is a uniformly convergent sequence of compact paths  $(\frac{d}{dt})^j \gamma_n(t) \rightarrow (\frac{d}{dt})^j \gamma_\infty(t)$  in  $L^2_{k-1-j}$  for each  $j = 0, 1, \dots, m$  with  $\gamma_n: I \rightarrow \text{Coul}(Y, s_n)$  and  $s_n \rightarrow s_\infty$ .

We now let  $\{Q_s: \text{Coul}(Y, s) \rightarrow L^2_k(\ker d^* \oplus \Gamma(S_Y))\}_{s \in S}$  be a continuous family of quadratic-like maps. As before, for real numbers  $\lambda < 0 \leq \mu$ , we define  $V^\mu_\lambda(s) \subset \text{Coul}(Y, s)$  to be the space spanned by the eigenvectors of  $l_s$  with eigenvalue in  $(\lambda, \mu]$ . We also consider  $\bar{V}^0_\lambda(s)$ , which is the orthogonal complement of  $i\Omega^1_h(Y)$  in  $V^0_\lambda(s)$ . Note that these spaces usually do not change continuously with  $s$  because the dimension can jump at eigenvalues of  $l_s$ .

Throughout this subsection, we say that for an interval  $I$ , a path  $\gamma: I \rightarrow \text{Coul}(Y, s)$  is an actual trajectory if it satisfies  $\frac{d}{dt}\gamma(t) = -(l + Q_s)\gamma(t)$ , and a path  $\gamma: I \rightarrow V^\mu_\lambda(s)$  is an approximated trajectory if it satisfies  $\frac{d}{dt}\gamma(t) = -(l + p^\mu_\lambda \circ Q_s)\gamma(t)$  for some  $\mu$  and  $\lambda$ . We denote by  $\varphi(\lambda, \mu, s)$  the flow generated by  $-\iota \cdot (l + p^\mu_\lambda \circ Q_s)$ , where  $\iota$  is a bump function which equals 1 on any bounded subset involved in our discussion.

**Theorem 6.2** *Let  $B$  be a closed and bounded subset of  $\text{Coul}(Y)$  and suppose that there exists a closed subset  $A \subset \text{int}(B)$  such that for any  $s \in S$  and any actual trajectory  $\gamma: \mathbb{R} \rightarrow \text{Coul}(Y, s)$  contained in  $B$ , we have  $\gamma$  contained in  $\text{int}(A)$ . Then there exist constants  $T, -\bar{\lambda}, \bar{\mu} \gg 0$  such that the following statements hold:*

- (i) *For any  $\lambda < \bar{\lambda}$ ,  $\mu > \bar{\mu}$  and  $s \in S$ , if an approximated trajectory  $\gamma: [-T, T] \rightarrow V^\mu_\lambda(s)$  is contained in  $B$ , then we have  $\gamma(0) \in A$ . In particular,  $B \cap V^\mu_\lambda$  is an isolating neighborhood for the flow  $\varphi(\lambda, \mu, s)$ .*
- (ii) *The spectra*

$$\begin{aligned} &\Sigma^{-V^0_\lambda(s)} I_{S^1}(\varphi(\lambda, \mu, s), \text{inv}(B \cap V^\mu_\lambda(s))), \\ &\Sigma^{-\bar{V}^0_\lambda(s)} I_{S^1}(\varphi(\lambda, \mu, s), \text{inv}(B \cap V^\mu_\lambda(s))) \end{aligned}$$

*do not depend on the choice of  $\lambda < \bar{\lambda}$  and  $\mu > \bar{\mu}$  up to canonical isomorphisms in  $\mathfrak{C}$ . We denote these objects by  $I(B, s)$  and  $\bar{I}(B, s)$ , respectively.*

- (iii) *For any path  $\alpha: [0, 1] \rightarrow S$ , we have well-defined isomorphisms*

$$\begin{aligned} \rho(B, \alpha): I(B, \alpha(0)) &\rightarrow \Sigma^{\text{sf}(-\not{D}, \alpha)^{\mathbb{C}}} I(B, \alpha(1)), \\ \bar{\rho}(B, \alpha): \bar{I}(B, \alpha(0)) &\rightarrow \Sigma^{\text{sf}(-\not{D}, \alpha)^{\mathbb{C}}} \bar{I}(B, \alpha(1)), \end{aligned}$$

*where  $\text{sf}(-\not{D}, \alpha)$  denotes the spectral flow of  $-\not{D}$  along the path  $\alpha$ . Moreover, the isomorphisms  $\rho$  and  $\bar{\rho}$  only depend on the homotopy class of  $\alpha$  relative to its endpoints.*

**Proof** For the first part, the proof is similar to that of [Corollary 3.7](#): we suppose there exists no such  $\bar{\lambda}$ ,  $\bar{\mu}$  and  $T$ . Then we can find a sequence of approximated trajectories  $\gamma_n: [-T_n, T_n] \rightarrow \text{Coul}(Y, s_n)$  with  $T_n, -\lambda_n, \mu_n \rightarrow +\infty$  such that  $\gamma_n$  is contained in  $B$  but  $\gamma_n(0) \notin A$ . Since  $S$  is compact, we can assume  $s_n \rightarrow s_\infty$  after passing to a subsequence. The properties in [Definition 6.1](#) allow us to repeat the argument in the proof of [Proposition 3.3](#) and find an actual trajectory  $\gamma_\infty: \mathbb{R} \rightarrow \text{Coul}(Y, s_\infty)$  as the limit of the  $\gamma_n$ . Consequently, we have  $\gamma$  contained in  $B$  and  $\gamma_\infty(0) \notin \text{int}(A)$ . This is a contradiction with our hypothesis. Thus, the proof of (i) is finished.

The proof of (ii) is a straight forward adaption of arguments from [Proposition 5.6](#), and we omit it. For (iii), we will focus on the case  $\rho(B, \alpha)$  as the other case can be proved similarly. For brevity, we will denote by  $E_\lambda^\mu(s)$  the Conley index  $I_{S^1}(\varphi(\lambda, \mu, s), \text{inv}(B \cap V_\lambda^\mu(s)))$ . The isomorphism  $\rho(B, \alpha)$  is constructed as follows: We consider the interval  $[0, 1]$  as the union of subintervals  $[t_j, t_{j+1}]$  with  $j = 1, \dots, m$  such that, for each  $j$ , we can find  $\mu_j > \bar{\mu}$  and  $\lambda_j < \bar{\lambda}$  which are not eigenvalues of  $l_{\alpha(t)}$  for any  $t \in [t_j, t_{j+1}]$ . Then  $V_{\lambda_j}^{\mu_j}(\alpha(t))$  from  $t = t_j$  to  $t = t_{j+1}$  is a continuous family of linear subspaces and  $\varphi(\lambda_j, \mu_j, \alpha(t))$  is a continuous family of flows on them. By the homotopy-invariance of the Conley index [[33](#), Section 6], we get an isomorphism

$$(22) \quad \rho_j: E_{\lambda_j}^{\mu_j}(\alpha(t_j)) \xrightarrow{\cong} E_{\lambda_j}^{\mu_j}(\alpha(t_{j+1})).$$

Notice that

$$[V_{\lambda_j}^0(\alpha(t_j))] + [\text{sf}(-\mathcal{D}, \alpha([t_j, t_{j+1}]))\mathbb{C}] = [V_{\lambda_j}^0(\alpha(t_{j+1}))]$$

as elements of the representation ring of  $S^1$ . We can desuspend both sides of (22) and get an isomorphism

$$I(B, \alpha(t_j)) \rightarrow \Sigma^{\text{sf}(-\mathcal{D}, \alpha([t_j, t_{j+1}]))\mathbb{C}} I(B, \alpha(t_{j+1})).$$

The isomorphism  $\rho(B, \alpha)$  is defined as the composition of the above isomorphisms for  $j = 1, \dots, m$ .

We will see that  $\rho(B, \alpha)$  is independent of the choices of  $t_j, \lambda_j$  and  $\mu_j$ . First, fix a choice of  $\{t_j\}$  and make different choices of  $\{\lambda'_j\}$  and  $\{\mu'_j\}$ . Without loss of generality, we may assume that  $\lambda_j < \lambda'_j$  and  $\mu_j > \mu'_j$ . As before, we have an isomorphism

$$\rho'_j: E_{\lambda'_j}^{\mu'_j}(\alpha(t_j)) \xrightarrow{\cong} E_{\lambda'_j}^{\mu'_j}(\alpha(t_{j+1})).$$

As in Proposition 5.6, we have isomorphisms for stability of Conley indices

$$\sigma_j: E_{\lambda_j}^{\mu_j}(\alpha(t_j)) \xrightarrow{\cong} \Sigma^{V_{\lambda_j}^{\lambda_j'}} E_{\lambda_j}^{\mu_j'}(\alpha(t_j)), \quad \sigma_{j+1}: E_{\lambda_{j+1}}^{\mu_{j+1}}(\alpha(t_{j+1})) \xrightarrow{\cong} \Sigma^{V_{\lambda_{j+1}}^{\lambda_{j+1}'}} E_{\lambda_{j+1}}^{\mu_{j+1}' }(\alpha(t_{j+1})).$$

Using the formula in [33, Theorem 6.7], we can easily see that  $\sigma_{j+1} \circ \rho_j$  is  $S^1$ -equivariantly homotopic to  $\rho_{j'} \circ \sigma_j$ . This implies that  $\Sigma^{-V_{\lambda_j}^0} \rho_j$  and  $\Sigma^{-V_{\lambda_{j'}}^0} \rho_{j'}$  are equal to each other as morphisms in  $\mathfrak{C}$ . Therefore  $\rho(B, \alpha)$  does not depend on the choices of  $\{\lambda_j\}$  and  $\{\mu_j\}$ . Next we prove the independence of the choice of  $\{t_j\}$ . Let us pick another sequence  $\{t'_j\}_{j=1}^{m'}$ . Without loss of generality, we will only work on the case where  $\{t'_j\} \subset \{t_j\}$ ; ie  $\{t_j\}$  is a finer subdivision. Let us suppose that

$$t_j = t'_{j'} < t_{j+1} < t_{j+2} = t'_{j'+1}$$

for some  $j' \in \{1, \dots, m'\}$ . An equivariant version of [33, Corollary 6.8] implies that  $\rho_{j+1} \circ \rho_j$  is  $S^1$ -equivariantly homotopic to  $\rho_{j'}$ . This discussion implies that  $\rho(B, \alpha)$  is independent of the choice of  $\{t_j\}$ .

Now suppose that we have two paths  $\alpha_0, \alpha_1$  which are homotopic to each other relative to their endpoints by a homotopy  $\alpha_u$  for  $u \in [0, 1]$ . For any  $(t_0, u_0) \in [0, 1]^2$ , one can also find  $\mu > \bar{\mu}$  and  $\lambda < \bar{\lambda}$  and a small neighborhood  $O$  of  $(t_0, u_0)$  such that  $\mu$  and  $\lambda$  are not eigenvalues of  $I_{\alpha_u}(t)$  for any  $(t, u)$  in  $O$ . By the definition of  $\rho$  and the homotopy-invariance of the Conley index, we see that  $\rho(B, \alpha_u)$  does not change as  $u$  varies inside  $O$ . By considering a finite cover of  $[0, 1]^2$  by such neighborhoods, we see that  $\rho(B, \alpha_0) = \rho(B, \alpha_1)$ . This finishes the proof of the theorem.  $\square$

The following corollary is directly implied by the homotopy-invariance of the attractor-repeller map.

**Corollary 6.3** *Let  $B_1 \subset B_2$  be two closed and bounded sets both satisfying the hypothesis of Theorem 6.2. Suppose that for any sufficiently large  $-\lambda, \mu$  and any  $s \in S$ , the set  $\text{inv}(\varphi(\lambda, \mu, s), B_1 \cap V_{\lambda}^{\mu}(s))$  is an attractor in  $\text{inv}(\varphi(\lambda, \mu, s), B_2 \cap V_{\lambda}^{\mu}(s))$ . Then the desuspensions of the corresponding attractor maps give well-defined morphisms  $i(s): I(B_1, s) \rightarrow I(B_2, s)$  and  $\bar{i}(s): \bar{I}(B_1, s) \rightarrow \bar{I}(B_2, s)$ . Moreover, for any path  $\alpha: [0, 1] \rightarrow S$ , we have*

$$\begin{aligned} \rho(B_2, \alpha) \circ i(\alpha(0)) &= (\Sigma^{\text{sf}(-\mathbb{D}, \alpha)^{\mathbb{C}}} i(\alpha(1))) \circ \rho(B_1, \alpha), \\ \bar{\rho}(B_2, \alpha) \circ \bar{i}(\alpha(0)) &= (\Sigma^{\text{sf}(-\mathbb{D}, \alpha)^{\mathbb{C}}} \bar{i}(\alpha(1))) \circ \bar{\rho}(B_1, \alpha). \end{aligned}$$

The repeller version of this result also holds given that  $\text{inv}(\varphi(\lambda, \mu, s), B_1 \cap V_{\lambda}^{\mu}(s))$  is a repeller in  $\text{inv}(\varphi(\lambda, \mu, s), B_2 \cap V_{\lambda}^{\mu}(s))$  for any  $s \in S$ .

### 6.2 The invariance for (III)–(V)

Notice that the three parameters in (V) only affect our results through the definition of the bounded set  $J_m^\pm$ . Suppose that we choose two different triples of parameters  $(\{h_j\}, \bar{g}, \theta)$  and  $(\{\tilde{h}_j\}, \tilde{g}, \tilde{\theta})$  and use them to define the sets  $J_m^+$  and  $\tilde{J}_m^+$ , respectively. From these subsets, we construct two direct systems, which we denote by (17) and (17)', respectively. Notice that  $J_m^+$  and  $\tilde{J}_m^+$  are bounded subsets of  $\text{Str}(\tilde{R})$ . We can find  $0 < m_1 < m_2 < \dots$  and  $0 < \tilde{m}_1 < \tilde{m}_2 < \dots$  such that

$$(23) \quad J_{m_1}^+ \subset \tilde{J}_{\tilde{m}_1}^+ \subset J_{m_2}^+ \subset \tilde{J}_{\tilde{m}_2}^+ \subset \dots,$$

which also implies the following inclusions for any positive integer  $n$ :

$$J_{m_1}^{n,+} \subset \tilde{J}_{\tilde{m}_1}^{n,+} \subset J_{m_2}^{n,+} \subset \tilde{J}_{\tilde{m}_2}^{n,+} \subset \dots$$

Notice that for any  $j > 0$  and any  $n$  and  $m$  large enough relative to  $m_j$  and  $\tilde{m}_j$ , the flow  $\varphi_m^n$  goes inside  $J_{m_j}^{n,+}$  and  $\tilde{J}_{\tilde{m}_j}^{n,+}$  along  $\partial J_{m_j}^{n,+} \setminus \partial \text{Str}(\tilde{R})$  and  $\partial \tilde{J}_{\tilde{m}_j}^{n,+} \setminus \partial \text{Str}(\tilde{R})$ , respectively. Therefore, the attractor maps, together with the isomorphisms  $\tilde{\rho}_*^{*,+}$  (as defined in Proposition 5.6) give a direct system

$$(24) \quad I_{m_1}^{n_1,+} \rightarrow \tilde{I}_{\tilde{m}_1}^{n_1,+} \rightarrow I_{m_2}^{n_2,+} \rightarrow \tilde{I}_{\tilde{m}_2}^{n_2,+} \rightarrow \dots$$

in the category  $\mathfrak{C}$  for suitable choices of  $n_1 < \tilde{n}_1 < n_2 < \tilde{n}_2 < \dots$ , where the connecting maps are defined in a similar way as (17). Since the attractor maps are transitive as mentioned after Proposition 4.7, the composition of the connecting morphisms  $I_{m_j}^{n_j,+} \rightarrow \tilde{I}_{\tilde{m}_j}^{n_j,+} \rightarrow I_{m_{j+1}}^{n_{j+1},+}$  is the same as the attractor map for  $\text{inv}(J_{m_j}^{n_j,+}) \subset \text{inv}(J_{m_{j+1}}^{n_{j+1},+})$ . Therefore, we see that (24) contains both a subsystem of (17) and a subsystem of (17)'. By Lemma 4.3, this implies that (17) and (17)' are canonically isomorphic as objects of  $\mathfrak{S}$ . In other words, up to canonical isomorphisms, the spectrum invariants  $\underline{\text{swf}}^A$  and  $\underline{\text{SWF}}^A$  do not depend on the choice of  $\{h_j\}, \bar{g}$  and  $\theta$ . The case of  $\underline{\text{swf}}^R$  and  $\underline{\text{SWF}}^R$  can be shown similarly. We have proved the invariance for (V).

The proof of the invariance for (IV) is easy: Let  $\tilde{R}_0 < \tilde{R}_1$  be two numbers which are both larger than the constant  $R_0$  from Theorem 3.2. Notice that when we choose a suitable choice of parameters  $(\{h_j\}, \bar{g}, \theta)$  for  $\tilde{R}_1$ , these parameters also work for  $\tilde{R}_0$  since  $\tilde{R}_0 < \tilde{R}_1$ . Denote by  $J_m^{n,\pm}(\tilde{R}_i)$  the corresponding bounded set corresponding for  $i = 0, 1$ . Then it is straightforward to see that for any positive integer  $m$  and any sufficiently large integer  $n$  (relative to  $m$ ), the sets  $J_m^{n,\pm}(\tilde{R}_0)$  and  $J_m^{n,\pm}(\tilde{R}_1)$  are both isolating neighborhoods of the same isolated invariant set. Therefore, their Conley

indices are related to each other by canonical isomorphisms which are compatible with attractor-repeller maps. This implies the invariance for (IV).

**Remark** Actually, from the above argument, we can replace  $\text{Str}(R)$  in our construction with any set  $C \subset \text{Coul}(Y)$  satisfying the following conditions:

- (1) for any bounded subset  $A \subset i\Omega_h^1(Y)$ , the set  $p_{\mathcal{H}}^{-1}(A) \cap C$  is also bounded;
- (2) any finite-type Seiberg–Witten trajectory is contained in the interior of  $C$ .

Also, we can define  $\{J_m^\pm\}$  to be any sequence of bounded, closed subsets of  $C$  such that  $J_m^\pm \subset J_{m+1}^\pm$ ,  $\bigcup_{m=1}^\infty J_m^\pm = C$ , and for any  $m > 0$  and  $n$  large enough relative to  $m$ , the flow  $\varphi_m^n$  goes inside (resp. outside)  $J_m^{n,+}$  (resp.  $J_m^{n,-}$ ) along  $\partial J_m^{n,+} \setminus \partial C$  (resp.  $\partial J_m^{n,-} \setminus \partial C$ ).

As for (III), we choose different sequences  $\{\lambda_n\}$ ,  $\{\mu_n\}$  and  $\{\tilde{\lambda}_n\}$ ,  $\{\tilde{\mu}_n\}$ . By Lemma 4.3, we can pass to their subsequences and assume that  $\lambda_{n+1} < \tilde{\lambda}_n < \lambda_n$  and  $\mu_n < \tilde{\mu}_n < \mu_{n+1}$  for any  $n$ . Let  $I_m^{n,+}$  and  $\tilde{I}_m^{n,+}$  be the objects of  $\mathfrak{C}$  obtained by desuspending the Conley indices corresponding to  $\{\lambda_n\}$ ,  $\{\mu_n\}$  and  $\{\tilde{\lambda}_n\}$ ,  $\{\tilde{\mu}_n\}$ , respectively. We can repeat the proof of Proposition 5.6 and establish canonical isomorphisms  $I_m^{n,+} \cong \tilde{I}_m^{n,+}$  and  $I_m^{n+1,+} \cong \tilde{I}_m^{n+1,+}$  for any positive integer  $m$  and any sufficiently large integer  $n$  (relative to  $m$ ). Moreover, they form commutative diagrams similar to (16). This implies that  $\underline{\text{swf}}^A$  and  $\underline{\text{SWF}}^A$  are independent of (III). The repeller case follows in the same manner.

### 6.3 The invariance for (II)

In this subsection, we will consider any choice of good perturbations  $f_j: \mathcal{C}_Y \rightarrow \mathbb{R}$  for  $j = 1, 2$ . Recall that  $f_j(a, \phi) = \frac{1}{2}\delta_j \|\phi\|_{L^2}^2 + \bar{f}_j(a, \phi)$ , where  $\delta_j$  is a real constant and  $\bar{f}_j$  is an extended cylinder function. We first assume that  $\delta_1 = \delta_2 = \delta$ . Since we do not know whether the space of good perturbations is path connected, the usual homotopy-invariance argument does not work. Therefore, we follow a different approach here. Because the whole argument is relatively long and technical, we first sketch the rough idea as follows.

Denote by  $\mathcal{L}_j$  the restriction of  $\text{CSD}_{v_0, f_j}$  to  $\text{Coul}(Y)$ . Recall that we identify  $i\Omega_h^1(Y)$  with  $\mathbb{R}^{b_1}$  by choosing independent harmonic forms  $\{h_j\}$ . For any real number  $e \geq 1$ , we will construct a family of “mixed” functionals  $\mathcal{E}_e^s$  for  $s \in [0, 1]$  such that  $\mathcal{E}_e^1 = \mathcal{L}_2$  and  $\mathcal{E}_e^0$  equals  $\mathcal{L}_1$  on  $p_{\mathcal{H}}^{-1}([-e + 1, e - 1]^{b_1})$  and equals  $\mathcal{L}_2$  on  $p_{\mathcal{H}}^{-1}(\mathbb{R}^{b_1} \setminus (-e, e)^{b_1})$ .

Suppose that all finite-type flow lines of  $\mathcal{L}_e^s$  are contained in  $\text{Str}(\tilde{R})$  and consider an increasing sequence of bounded subsets

$$J_{m_1}^+ \subset \tilde{J}_{\tilde{m}_1}^+ \subset J_{m_2}^+ \subset \tilde{J}_{\tilde{m}_2}^+ \subset \dots,$$

where  $J_{m_j}^+$  and  $\tilde{J}_{\tilde{m}_j}^+$  are the bounded subsets of  $\text{Str}(\tilde{R})$  corresponding to  $\mathcal{L}_1$  and  $\mathcal{L}_2$ , respectively. We will require that for each positive integer  $j$ , there exists a real number  $e_j \geq 1$  satisfying

$$J_{m_j}^+ \subset p_{\mathcal{H}}^{-1}([-e_j + 1, e_j - 1]^b) \cap \text{Str}(\tilde{R}) \subset p_{\mathcal{H}}^{-1}([-e_j, e_j]^b) \cap \text{Str}(\tilde{R}) \subset J_{\tilde{m}_j}^+.$$

Let  $\varphi^n(\mathcal{L})$  be the approximated gradient flow of  $\mathcal{L}$  on the compact set  $J_{m_j}^{n,+}$ . Since  $\mathcal{L}_1$  equals  $\mathcal{L}_{e_j}^0$  when restricted to  $J_{m_j}^+$  and the flow goes inside  $J_{m_j}^{n,+}$ , we have an attractor map

$$I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J_{m_j}^{n,+})) = I_{S^1}(\varphi^n(\mathcal{L}_{e_j}^0), \text{inv}(J_{m_j}^{n,+})) \rightarrow I_{S^1}(\varphi^n(\mathcal{L}_{e_j}^0), \text{inv}(\tilde{J}_{\tilde{m}_j}^{n,+})).$$

On the other hand, we have  $I_{S^1}(\varphi^n(\mathcal{L}_{e_j}^0), \text{inv}(\tilde{J}_{\tilde{m}_j}^{n,+})) \cong I_{S^1}(\varphi^n(\mathcal{L}_2), \text{inv}(\tilde{J}_{\tilde{m}_j}^{n,+}))$  by continuity of Conley indices. We combine these and obtain a map

$$I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J_{m_j}^{n,+})) \rightarrow I_{S^1}(\varphi^n(\mathcal{L}_2), \text{inv}(\tilde{J}_{\tilde{m}_j}^{n,+})).$$

We also construct another family of functionals  $\tilde{\mathcal{L}}_e^s$  to obtain a map

$$I_{S^1}(\varphi^n(\mathcal{L}_2), \text{inv}(\tilde{J}_{\tilde{m}_j}^{n,+})) \rightarrow I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J_{m_{j+1}}^{n,+})).$$

We will then prove that the composition

$$I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J_{m_j}^{n,+})) \rightarrow I_{S^1}(\varphi^n(\mathcal{L}_2), \text{inv}(\tilde{J}_{\tilde{m}_j}^{n,+})) \rightarrow I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J_{m_{j+1}}^{n,+}))$$

is just the attractor map corresponding to  $\mathcal{L}_1$ . A similar result holds for  $\mathcal{L}_2$ . Therefore, we have constructed a “mixed direct system” in the category  $\mathfrak{C}$ , and the spectra corresponding to  $f_1, f_2$  are both subsequential colimits of it. Therefore, the invariance of  $\text{swf}^A$  is implied by Lemma 4.3. The  $\text{swf}^R$  case can be proved similarly.

There is one technical difficulty here. We need to find a uniform constant  $R_2$  (independent of  $e, s$ ) such that  $\text{Str}(R_2)$  contains all the finite-type trajectories of  $\mathcal{L}_e^s$  and  $\tilde{\mathcal{L}}_e^s$ . This will be taken care of by Corollary 6.12 and Lemma 6.14, which generalize Theorem 3.2.

Let us prepare some general results regarding the perturbations. Recall that we have a canonical isomorphism

$$\pi_0(\mathcal{G}_Y) \cong \pi_0(\mathcal{G}_Y^h) \cong H^1(Y; \mathbb{Z}).$$



For any positive integer  $m$ , we denote by  $m\mathcal{G}_Y$  (resp.  $m\mathcal{G}_Y^h$ ) the subgroup of  $\mathcal{G}_Y$  (resp.  $\mathcal{G}_Y^h$ ) consisting of the connected components corresponding to  $m \cdot H^1(Y; \mathbb{Z})$ .

**Definition 6.4** For a positive integer  $m$ , a continuous function  $f: \text{Coul}(Y) \rightarrow \mathbb{R}$  is called  $m$ -periodic if  $f$  is invariant under the action of  $m\mathcal{G}_Y^h$ , which implies that  $f \circ \Pi$  is invariant under  $m\mathcal{G}_Y$ .

We will also need the following definition of tame functions.

**Definition 6.5** A smooth function  $f: \text{Coul}(Y) \rightarrow \mathbb{R}$  is called a tame function if the formal gradient  $\text{grad}(f \circ \Pi)$  satisfies all the conditions of the tame perturbations [17, Definition 10.5.1] except that it needs not be invariant under the full gauge group  $\mathcal{G}_Y$ , where  $\Pi: \mathcal{C}_Y \rightarrow \text{Coul}(Y)$  is the nonlinear Coulomb projection.

Furthermore, a continuous family of functions  $\{f_w\}$  parametrized by a compact manifold  $W$  (possibly with boundary) is called a continuous family of tame functions if each function is tame and  $\text{grad}(f_w \circ \Pi)$  extends to a continuous family of maps on the cylinder  $I \times Y$ . In addition, we require that the constant  $m_2$  and the function  $\mu_1$  from [17, Definition 10.5.1] be uniform with respect to  $w \in W$ .

Now we describe a way to construct a continuous family of tame functions from any pair of extended cylinder functions, given a family of smooth functions.

**Lemma 6.6** Let  $W$  be a compact manifold and  $\tilde{\tau}_w: i\Omega_h^1(Y) \cong \mathbb{R}^{b_1} \rightarrow \mathbb{R}$  a smooth family of smooth functions parametrized by  $w \in W$ . Then we can choose a sequence of constants  $\{C_j\}$  in the definition of the space of perturbations  $\mathcal{P}$  (see Definition 2.1) so that for any  $\delta \in \mathbb{R}$  and any  $\bar{f}_1, \bar{f}_2 \in \mathcal{P}$ , a family of functions  $\tilde{f}_w: \text{Coul}(Y) \rightarrow \mathbb{R}$  given by

$$(25) \quad \tilde{f}_w(a, \phi) := \frac{1}{2} \delta \|\phi\|_{L^2}^2 + (\tilde{\tau}_w \circ \pi_{\mathcal{H}}(a)) \cdot \bar{f}_1(a, \phi) + (1 - \tilde{\tau}_w \circ \pi_{\mathcal{H}}(a)) \cdot \bar{f}_2(a, \phi)$$

is a continuous family of tame functions. Moreover, if  $\tilde{\tau}_w$  is  $m\mathbb{Z}^{b_1}$ -periodic, then  $\tilde{f}_w$  is  $m$ -periodic.

**Proof** This is actually a parametrized version of [17, Theorem 11.6.1], and we will focus only on the term  $(\tilde{\tau}_w \circ \pi_{\mathcal{H}}(a)) \cdot \bar{f}_1(a, \phi)$ . To avoid repeating complicated analysis there, we introduce a trick turning a family of functions into a single function. Let  $Y'$  be another  $\text{spin}^c$  3-manifold with  $b_1(Y') > 2 \dim W$  so that we can embed  $W$  in the torus  $i\Omega_h^1(Y')/\mathcal{G}_{Y'}^{h,o}$ . We now consider the family  $\{\tilde{\tau}_w\}_{w \in W}$  as a single function

on  $i\Omega_h^1(Y) \times W$  and extend it to  $\bar{\tau}: i\Omega_h^1(Y) \times i\Omega_h^1(Y') \rightarrow \mathbb{R}$ . Recall that  $\bar{f}_1 = \sum_{j=1}^\infty \eta_j \hat{f}_j$ , where  $\hat{f}_j$  is a cylinder function of  $Y$  with  $\sum_{j=1}^\infty C_j |\eta_j| < \infty$ . We define a function

$$\hat{f}'_j: \mathcal{C}_Y \times \mathcal{C}_{Y'} \rightarrow \mathbb{R}$$

given by

$$(a, \phi) \times (a', \phi') \mapsto \bar{\tau}(\pi'_{\mathcal{H}}(a'), \pi_{\mathcal{H}}(a)) \cdot \hat{f}_j(a, \phi),$$

where  $\pi'_{\mathcal{H}}: i\Omega^1(Y') \rightarrow i\Omega_h^1(Y')$  denotes the projection onto harmonic forms on  $Y'$ . These functions  $\hat{f}'_j$  almost fit into the definition of cylinder functions (see Section 11 of [17]) on  $\mathcal{C}(Y) \times \mathcal{C}(Y')$  except that they are not invariant under the full gauge group. We can still repeat the argument the proof of [17, Theorem 11.6.1] and show that, by setting  $\{C_j\}$  to increase fast enough, the formal gradient  $\text{grad}(\sum_j \eta_j \hat{f}'_j)$  is a tame perturbation for the manifold  $Y \cup Y'$  except that it is not invariant under the full gauge group. As a result, it is not hard to see that this actually implies that  $(\bar{\tau}_w \circ \pi_{\mathcal{H}}(a)) \cdot \bar{f}_1(a, \phi)$  is a continuous family of tame functions.  $\square$

For a general functional  $\mathcal{L}: \text{Coul}(Y) \rightarrow \mathbb{R}$ , we can consider its negative gradient flow line  $\gamma: I \rightarrow \text{Coul}(Y)$ , described by the equation  $\frac{d}{dt}\gamma(t) = -\widetilde{\text{grad}} \mathcal{L}(\gamma(t))$ . Such a trajectory will be called an  $\mathcal{L}$ -trajectory. As before, we define the topological energy by

$$\mathcal{E}^{\text{top}}(\gamma, \mathcal{L}) := 2\left(\sup_{t \in I} \mathcal{L}(\gamma(t)) - \inf_{t \in I} \mathcal{L}(\gamma(t))\right).$$

Recall that a trajectory is said to be of finite type if it is contained in a bounded subset of  $\text{Coul}(Y)$ . We have the following uniform boundedness result for functionals perturbed by periodic functions.

**Proposition 6.7** *Let  $\{f_w\}$  be a continuous family of  $m$ -periodic tame functions parametrized by a compact manifold  $W$ , and consider a family of functionals  $\mathcal{L}_w = \text{CSD}_{v_0}|_{\text{Coul}(Y)} + f_w$ . Then for any  $C > 0$ , there exist constants  $R, C'$  such that for any  $w \in W$  and any  $\mathcal{L}_w$ -trajectory  $\gamma: [-1, 1] \rightarrow \text{Coul}(Y)$  with topological energy  $\mathcal{E}^{\text{top}}(\gamma, \mathcal{L}_w) \leq C$ , we have  $\gamma(0) \in \text{Str}(R)$  and  $|\mathcal{L}_w(\gamma(0))| \leq C'$ .*

**Proof** The proof is a slight adaption of [17, Theorem 10.7.1]. Suppose that the statement is not true. Then we can find a sequence  $\{\gamma_j\}$  of  $\mathcal{L}_{w_j}$ -trajectory  $\gamma_j: [-1, 1] \rightarrow \text{Coul}(Y)$  with  $w_j \in W$  such that at least one of the following two situations happens:

- $\limsup_{j \rightarrow \infty} \|u_j \cdot \gamma_j(0)\|_{L_k^2} = \infty$  for any sequence  $\{u_j\} \subset m\mathcal{G}_Y^h$ ;
- $\limsup_{j \rightarrow \infty} |\mathcal{L}_{w_j}(\gamma_j(0))| = \infty$ .

Since  $W$  is compact, after passing to a subsequence, we may assume that  $w_j \rightarrow w_\infty$ .

We lift  $\gamma_j$  to a path  $\tilde{\gamma}_j: [-\frac{1}{2}, \frac{1}{2}] \rightarrow \mathcal{C}_Y$ , which is the negative gradient flow line of  $\text{CSD}_{v_0} + f_{w_j} \circ \Pi$ . Note that we only consider an interior domain here to avoid a possible regularity issue. With  $X = [-\frac{1}{2}, \frac{1}{2}] \times Y$ , we treat  $\tilde{\gamma}_j$  as a section over  $X$  and denote it by  $(\hat{a}_j, \hat{\phi}_j)$ . We can find a gauge transformation  $\hat{u}_j$  over  $X$  whose restrictions to  $\{0\} \times Y$  belong to  $m\mathcal{G}_Y$  such that the following conditions hold:

- (1)  $\hat{d}^*(\hat{a}_j - \hat{u}_j^{-1}d\hat{u}_j) = 0$  on  $X$ ;
- (2)  $(\hat{a}_j - \hat{u}_j^{-1}d\hat{u}_j)(\mathbf{n}) = 0$  on  $\partial X$ , where  $\mathbf{n}$  is the outward normal vector;
- (3) for each for  $l = 1, \dots, b_1$ , we have  $\int_{\tilde{X}} (\hat{a}_j - \hat{u}_j^{-1}d\hat{u}_j) \wedge (*_4\hat{h}_l) \in [0, m)$ , where  $\hat{h}_l$  is the pullback of  $h_l$  on  $X$ .

The conditions in Definition 6.5 allow us to repeat the bootstrapping argument in the proof of [17, Theorem 10.7.1] and obtain the following statement. After passing to a further subsequence,  $(\hat{a}_j - \hat{u}_j^{-1}d\hat{u}_j, \hat{u}_j \cdot \hat{\phi}_j)$  is convergent in  $L^2_{k+1/2}$  when restricted to any interior cylinder. In particular, this implies that  $\Pi(\hat{u}_j|_{\{0\} \times Y} \cdot \tilde{\gamma}_j(0))$  is convergent in  $L^2_k$ . Notice that  $\Pi(\hat{u}_j|_{\{0\} \times Y} \cdot \tilde{\gamma}_j(0))$  equals  $u_j \cdot \gamma_j(0)$  for some  $u_j \in m\mathcal{G}_Y^h$ . Also  $\mathcal{L}_{w_j}(\gamma_j(0)) = \mathcal{L}_{w_j}(u_j \cdot \gamma_j(0))$  is a convergent sequence since  $\mathcal{L}_w(a, \phi)$  is continuous in  $w$  and  $(a, \phi)$ . Therefore, we arrive at a contradiction with the above two situations.  $\square$

We also have the following lemma, whose proof is essentially the same as Corollary 3.6, and we omit it.

**Lemma 6.8** *Let  $\{f_w\}$  be a continuous family of tame functions. For each  $w \in W$ , we define a nonlinear term  $c_w: \text{Coul}(Y) \rightarrow L^2_k(i \ker d^* \oplus \Gamma(S_Y))$  of the gradient of  $\text{CSD}_{v_0}|_{\text{Coul}(Y)} + f_w$  as in (8) and (9). Then  $\{c_w\}$  is a continuous family of quadratic-like maps.*

Now we explicitly construct “the mixed perturbation” as follows. Choose a smooth function  $\tau: \mathbb{R} \rightarrow [0, 1]$  satisfying  $\tau|_{(-\infty, 1/4]} \equiv 0$  and  $\tau|_{[1/2, \infty)} \equiv 1$ . For any real number  $e \geq 1$ , we define a bump function  $\tau_e: i\Omega^1_h(Y) \rightarrow [0, 1]$  from  $\tau$  by

$$\tau_e(x_1, x_2, \dots, x_{b_1}) = \prod_{1 \leq j \leq b_1} \tau(e + x_j)\tau(e - x_j).$$

Each  $\tau_e$  gives an induced tame function  $\tilde{f}_e^0: \text{Coul}(Y) \rightarrow \mathbb{R}$  as in (25), ie

$$\tilde{f}_e^0(a, \phi) := \frac{1}{2}\delta\|\phi\|_{L^2}^2 + (\tau_e \circ p_{\mathcal{H}}(a, \phi)) \cdot \bar{f}_1(a, \phi) + (1 - \tau_e \circ p_{\mathcal{H}}(a, \phi)) \cdot \bar{f}_2(a, \phi),$$

where  $\bar{f}_1, \bar{f}_2 \in \mathcal{P}$ . With  $f_j = \frac{1}{2}\delta\|\phi\|_{L^2}^2 + \bar{f}_j$ , we note that the function  $\tilde{f}_e^0$  equals  $f_1$  on  $p_{\mathcal{H}}^{-1}([-e + 1, e - 1]^b)$  and equals  $f_2$  on  $p_{\mathcal{H}}^{-1}(\mathbb{R}^b \setminus (-e, e)^b)$ . For  $s \in [0, 1]$ , we also consider an interpolation  $\tau_e^s = (1 - s)\tau_e$ , and define

$$(26) \quad \tilde{f}_e^s = (1 - s)\tilde{f}_e^0 + sf_2 \quad \text{and} \quad \mathcal{E}_e^s = \text{CSD}_{v_0}|_{\text{Coul}(Y)} + \tilde{f}_s^e.$$

Notice that  $\tilde{f}_e^s$  is essentially a tame function induced from  $\tau_e^s$  which is not  $m$ -periodic for any positive integer  $m$ . To utilize Proposition 6.7, we will introduce an explicit family of smooth periodic functions such that the induced periodic tame functions agree with  $\tilde{f}_e^s$  on desirable regions.

For any positive integer  $M$ , we consider a family of  $(6M+6)$ -periodic smooth functions parametrized by compact manifold  $W_M$  described as follows. The manifold  $W_M$  is of the form  $W_{M,1} \sqcup W_{M,2} \sqcup W_{M,3}$  where  $W_{M,1} := [1, M + 1] \times [0, 1]$ ,  $W_{M,2} := \{(B, \sigma) \mid \emptyset \neq B \subset \{1, 2, \dots, b_1\}, \sigma: B \rightarrow \{\pm 1\}\} \times (\mathbb{R}/(6M + 6)\mathbb{Z}) \times [0, 1]$  and  $W_{M,3} = \{1, 2\}$ . We construct a family of functions  $\{\tilde{\tau}_w\}$  as follows:

- For each positive integer  $M$  and  $(e, 0) \in W_{M,1}$ , assign the unique  $(6M+6)\mathbb{Z}^{b_1}$ -periodic function  $\tilde{\tau}_e: \mathbb{R}^{b_1} \rightarrow \mathbb{R}$  which extends  $\tau_e|_{[-3M-3, 3M+3]^{b_1}}$ .
- For each positive integer  $M$ , we pick a  $(6M+6)$ -periodic function  $\bar{\tau}_M: \mathbb{R} \rightarrow [0, 1]$  which extends  $\tau|_{[-2M-2, 2M+2]}$ . For each  $(B, \sigma, \theta, 0) \in W_{M,2}$ , we assign a function  $\tilde{\tau}_{B,\sigma,\theta}: \mathbb{R}^{b_1} \rightarrow [0, 1]$  given by

$$\tilde{\tau}_{(B,\sigma,\theta)}(x_1, \dots, x_{b_1}) := \prod_{j \in B} \bar{\tau}_M(\theta + \sigma(j)x_j).$$

- For general  $w = (w', s) \in W_{M,1} \sqcup W_{M,2}$ , we simply define  $\tilde{\tau}_{(w',s)} := (1 - s)\tilde{\tau}_{w'}$ .
- We set  $\tilde{\tau}_j \equiv 2 - j$  for  $j \in W_{M,3}$  so that  $\tilde{f}_j = f_j$ .

**Lemma 6.9** *For each positive integer  $M$ , any  $(e_1, e_2, \dots, e_{b_1}) \in \mathbb{R}^{b_1}$  and  $(s, e) \in [0, 1] \times [1, \infty)$ , there exists an element  $w \in W_M$  such that the induced function  $\tilde{f}_w$  equals  $\tilde{f}_e^s$  on  $p_{\mathcal{H}}^{-1}([e_1 - M, e_1 + M] \times \dots \times [e_{b_1} - M, e_{b_1} + M])$ .*

**Proof** For convenience, we let  $E = [e_1 - M, e_1 + M] \times \dots \times [e_{b_1} - M, e_{b_1} + M]$ . We will consider two main cases with several subcases:

**Case 1** ( $e \in [1, M+1]$ ) If  $E \cap [-M-1, M+1]^{b_1} \neq \emptyset$ , then  $E \subset [-3M-3, 3M+3]^{b_1}$ . This implies  $\tilde{\tau}_e|_A = \tau_e|_A$ . Therefore, we can just choose  $w = (e, s) \in W_{M,1}$ . If  $E \cap [-M-1, M+1]^{b_1} = \emptyset$ , then we have  $p_{\mathcal{H}}^{-1}(E) \subset p_{\mathcal{H}}^{-1}(\mathbb{R}^{b_1} \setminus (-e, e)^{b_1})$  and  $\tilde{f}_e^s = f_2$  on  $p_{\mathcal{H}}^{-1}(E)$ . We just take  $w = 2 \in W_{M,3}$  so that  $\tilde{f}_w = f_2$  in this case.

**Case 2** ( $e > M + 1$ ) We consider the following subsets of  $[1, 2, \dots, b]$ :

$$\begin{aligned} B_1 &= \{j \mid [e_j - M, e_j + M] \cap [e - 1, e] \neq \emptyset\}, \\ B_2 &= \{j \mid [e_j - M, e_j + M] \cap [-e, -e + 1] \neq \emptyset\}, \\ B_3 &= \{j \mid [e_j - M, e_j + M] \cap [-e, e] = \emptyset\}. \end{aligned}$$

If  $B_1 \cup B_2 = \emptyset$ , then  $E$  is either contained in  $[-e + 1, e - 1]^{b_1}$  or  $\mathbb{R}^{b_1} \setminus (-e, e)^{b_1}$ , and we can just take  $w \in W_{M,3}$ . If  $B_3 \neq \emptyset$ , then we have  $\tau_e|_E \equiv 0$  and  $\tilde{f}_e^s|_{p_{\mathcal{H}}^{-1}(E)} = f_2|_{p_{\mathcal{H}}^{-1}(E)}$ . We can take  $w = 2 \in W_{M,3}$  again in this subcase. We are now left with the case where  $B_1 \cup B_2 \neq \emptyset$  and  $B_3 = \emptyset$ . Notice that for any  $(x_1, \dots, x_{b_1}) \in E$ , we have

$$\begin{aligned} j \in B_1 &\implies e + x_j \geq 2e - 1 - 2M \geq 1 \quad \text{and} \quad e - x_j \in [-2M, 2M + 1], \\ j \in B_2 &\implies e - x_j \geq 2e - 1 - 2M \geq 1 \quad \text{and} \quad e + x_j \in [-2M, 2M + 1], \\ j \notin B_1 \cup B_2 &\implies e - |x_j| \geq 1. \end{aligned}$$

Therefore, for such  $(x_1, \dots, x_{b_1})$ , we have

$$\tau_e(x_1, \dots, x_b) = \prod_{j \in B_1} \tau(e - x_j) \cdot \prod_{j \in B_2} \tau(e + x_j) = \prod_{j \in B_1} \bar{\tau}_M(e - x_j) \cdot \prod_{j \in B_2} \bar{\tau}_M(e + x_j),$$

where we use the fact that  $\bar{\tau}_M|_{[-2M-2, 2M+2]} = \tau|_{[-2M-2, 2M+2]}$ . As a result, we see that  $\tilde{f}_e^s = \tilde{f}_w$  on  $p_{\mathcal{H}}^{-1}(E)$  when we set  $w = (B_1 \cup B_2, \sigma, e, s) \in W_{M,2}$  with  $\sigma: B_1 \cup B_2 \rightarrow \{\pm 1\}$  sending  $B_1$  to  $-1$  and  $B_2$  to  $1$ . Notice that  $B_1 \cap B_2 = \emptyset$  because  $e > M + 1$ . □

We also have the following extension of [Lemma 6.6](#) to a countable union of compact sets.

**Lemma 6.10** *We can choose a sequence of constants  $\{C_j\}$  in the definition of  $\mathcal{P}$  (see [Definition 2.1](#)) such that for any positive integer  $M$  and any  $\bar{f}_1, \bar{f}_2 \in \mathcal{P}$ , the induced family  $\{\tilde{f}_w\}_{w \in W_M}$  is a continuous family of  $(6M+6)$ -periodic tame functions.*

**Proof** For each  $W_M$ , there exists a sequence  $\{C_{M,j}\}_j$  such that, for any  $f_1, f_2 \in \mathcal{P}(\{C_{M,j}\}_j)$ , the family  $\{\tilde{f}_w\}_{w \in W_M}$  is a continuous family of  $(6M+6)$ -periodic tame functions. It is straightforward to see that a sequence of positive real numbers  $\{C_j\}$  such that

$$C_j \geq \max_{1 \leq M \leq j} C_{M,j}$$

satisfies our requirement. □

Next is the boundedness result for functionals with mixed perturbations.

**Lemma 6.11** For any  $C > 0$ , there exist constants  $R, C'$  such that for any  $(e, s) \in [1, \infty) \times [0, 1]$  and any  $\mathcal{L}_e^s$ -trajectory  $\gamma: [-2, 2] \rightarrow \text{Coul}(Y)$  with topological energy  $\mathcal{E}^{\text{top}}(\gamma; \mathcal{L}_e^s) \leq C$ , we have  $\gamma(0) \in \text{Str}(R)$  and  $|\mathcal{L}_e^s(\gamma(0))| < C'$ .

**Proof** We first write down  $\widetilde{\text{grad}} f_e^s$  as

$$\begin{aligned} \widetilde{\text{grad}} f_e^s(a, \phi) &= \delta\phi + (1-s)(\bar{f}_1(a, \phi) - \bar{f}_2(a, \phi)) \widetilde{\text{grad}}(\tau_e \circ p_{\mathcal{H}})(a, \phi) \\ &\quad + (1-s)(\tau_e \circ p_{\mathcal{H}}(a, \phi)) \widetilde{\text{grad}} \bar{f}_1(a, \phi) \\ &\quad + (1-(1-s)(\tau_e \circ p_{\mathcal{H}}(a, \phi))) \widetilde{\text{grad}} \bar{f}_2(a, \phi). \end{aligned}$$

By boundedness and tameness conditions of  $\bar{f}_j$ , we see that

$$\|\text{grad}(\xi_e^s \circ \Pi)(a, \phi)\|_{L^2} = \|\widetilde{\text{grad}} \xi_e^s(a, \phi)\|_{\tilde{g}} \leq m(1 + \|\phi\|_{L^2}),$$

where  $m$  is a constant independent of  $(e, s)$ . This implies

$$(27) \quad \|\text{grad}(\xi_e^s \circ \Pi)(a, \phi)\|_{L^2}^2 \leq 2m^2 + 2m^2 \|\phi\|_{L^2}^2.$$

We can lift  $\gamma|_{[-1,1]}$  to  $\tilde{\gamma}: [-1, 1] \rightarrow \mathcal{C}_Y$ , which is a negative gradient flow line for the functional  $\mathcal{L}_e^s \circ \Pi$ . Now we follow the argument on page 161 of [17]. Since  $\mathcal{L}_e^s \circ \Pi = \text{CSD}_{v_0} + \xi_e^s \circ \Pi$ , we have

$$\|\text{grad} \text{CSD}_{v_0}\|_{L^2}^2 - 2\|\text{grad}(\xi_e^s \circ \Pi)\|_{L^2}^2 \leq 2\|\text{grad}(\mathcal{L}_e^s \circ \Pi)\|_{L^2}^2.$$

By formula (27), this implies

$$\begin{aligned} (28) \quad &\int_{-1}^1 (\|\text{grad} \text{CSD}_{v_0}(\tilde{\gamma}(t))\|_{L^2}^2 + \|\tilde{\gamma}'(t)\|_{L^2}^2) dt - 2m^2 \int_{-1}^1 \|\phi(t)\|_{L^2}^2 dt - 4m^2 \\ &\leq 2 \int_{-1}^1 (\|\text{grad}(\mathcal{L}_e^s \circ \Pi)(\tilde{\gamma}(t))\|_{L^2}^2 + \|\tilde{\gamma}'(t)\|_{L^2}^2) dt < 2\mathcal{E}^{\text{top}}(\gamma, \mathcal{L}_e^s) \leq 2C. \end{aligned}$$

We can treat  $\tilde{\gamma}$  as a section over the 4-manifold  $[-1, 1] \times Y$  and denote it by  $(\hat{a}, \hat{\phi})$ . By Definition 4.5.4 and formula (4.19) of [17], the above estimate on the analytical energy actually implies

$$\frac{1}{4} \int_{[-1,1] \times Y} |d\hat{a}|^2 + \int_{[-1,1] \times Y} |\nabla_{\hat{A}} \hat{\phi}|^2 + \frac{1}{4} \int_{[-1,1] \times Y} ((\hat{\phi}|^2 - C_2)^2) \leq C_3,$$

where  $\hat{A}$  is the connection corresponding to  $\hat{a}$  and  $C_2$  is a constant independent of  $e$  and  $s$ . By Corollary 4.5.3 and Lemmas 5.1.2 and 5.1.3 of [17], we can find a gauge

transformation  $u: [-1, 1] \times Y \rightarrow S^1$  such that  $\|u \cdot \tilde{\gamma}\|_{L^2_1([-1,1] \times Y)}$  is bounded by a uniform constant  $C_4$ . Let  $u_t$  equals  $u|_{\{t\} \times Y}$ . Then there exists  $C_5$  such that for any  $t_1, t_2 \in [-1, 1]$ , we have

$$\|\Pi_{\mathcal{H}}(u_{t_1} \cdot \tilde{\gamma}(t_1)) - \Pi_{\mathcal{H}}(u_{t_2} \cdot \tilde{\gamma}(t_2))\|_{L^2} \leq \|u_{t_1} \cdot \tilde{\gamma}(t_1)\|_{L^2} + \|u_{t_2} \cdot \tilde{\gamma}(t_2)\|_{L^2} \leq C_5.$$

Recall that  $\Pi_{\mathcal{H}}: \mathcal{C}_Y \rightarrow i\Omega^1_h(Y)$  is just the orthogonal projection. Since  $u_{t_1}$  and  $u_{t_2}$  are in the same component of the gauge group  $\mathcal{G}_Y$ , we have

$$\|p_{\mathcal{H}}(\gamma(t_1)) - p_{\mathcal{H}}(\gamma(t_2))\|_{L^2} = \|\Pi_{\mathcal{H}}(u_{t_1} \cdot \tilde{\gamma}(t_1)) - \Pi_{\mathcal{H}}(u_{t_2} \cdot \tilde{\gamma}(t_2))\|_{L^2} \leq C_5.$$

This implies that  $\gamma([-1, 1])$  is contained in

$$p_{\mathcal{H}}^{-1}([e_1 - M_0, e_1 + M_0] \times \cdots \times [e_b - M_0, e_b + M_0])$$

for some  $(e_1, \dots, e_b) \in \mathbb{R}^b$  and some uniform constant  $M_0 \in \mathbb{Z}_{\geq 1}$ . By Lemma 6.9, we have

$$\xi_e^s|_{p_{\mathcal{H}}^{-1}([e_1 - M_0, e_1 + M_0] \times \cdots \times [e_b - M_0, e_b + M_0])} = f_w$$

for some  $w \in W_{M_0}$ . This implies that  $\gamma|_{[-1, 1]}$  is also a trajectory for  $\text{CSD}_{v_0}|_{\text{Coul}(Y)} + f_w$ . Notice that

$$\mathcal{E}^{\text{top}}(\gamma|_{[-1, 1]}, \text{CSD}_{v_0}|_{\text{Coul}(Y)} + f_w) < C.$$

Our result is directly implied by Proposition 6.7. □

The previous results imply uniform boundedness for finite-type trajectories for the family  $\{\mathcal{E}_e^s\}$ . For convenience, we will say that functional  $\mathcal{L}: \text{Coul}(Y) \rightarrow \mathbb{R}$  is called  $R$ -bounded if any finite-type  $\mathcal{L}$ -trajectory is contained in  $\text{Str}(R)$ .

**Corollary 6.12** *There exists a uniform constant  $R_1 > 0$  such that for any  $e \in \mathbb{R}_{\geq 1}$  and  $s \in [0, 1]$ , the functional  $\mathcal{L}_e^s$  is  $R_1$ -bounded.*

**Proof** Let  $\gamma: \mathbb{R} \rightarrow \text{Coul}(Y)$  be a finite-type  $\mathcal{L}_e^s$ -trajectory. Since  $\mathcal{E}^{\text{top}}(\gamma, \mathcal{L}_e^s) < \infty$ , we have  $\mathcal{E}^{\text{top}}(\gamma|_{[t-1, t+1]}, \mathcal{L}_e^s) < 1$  for any  $t$  with  $|t|$  sufficiently large. By Lemma 6.11 (with  $C = 1$ ), we have  $|\mathcal{L}_e^s(\gamma(t))| \leq C'$  for such  $t$ . Since  $\mathcal{L}_e^s$  is decreasing along  $\gamma$ , we see that  $\mathcal{L}_e^s(\gamma(t-1)) - \mathcal{L}_e^s(\gamma(t+1)) < 2C'$  for any  $t \in \mathbb{R}$ . We apply Lemma 6.11 again (now  $C = 2C'$ ), so there is a uniform constant  $R_1$  such that  $\gamma(t) \in \text{Str}(R_1)$  for any  $t \in \mathbb{R}$ . □

For the reader’s convenience, we summarize the functionals we will be dealing with. Two extended cylinder functions  $\bar{f}_1, \bar{f}_2$  are now fixed, along with their corresponding functionals  $\mathcal{L}_1, \mathcal{L}_2$ . We have the continuous family of functionals  $\{\mathcal{E}_e^s\}$  (see (26)) such that for each  $(e, s) \in [1, \infty) \times [0, 1]$ , they satisfy

$$\begin{aligned} \mathcal{E}_e^1 &= \mathcal{L}_2, \\ \mathcal{E}_e^0(x) &= \begin{cases} \mathcal{L}_1(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e + 1, e - 1]^{b_1}), \\ \mathcal{L}_2(x) & \text{if } x \in p_{\mathcal{H}}^{-1}(\mathbb{R}^{b_1} \setminus (-e, e)^{b_1}), \end{cases} \\ \mathcal{E}_e^s(x) &= \mathcal{L}_2(x) \quad \text{if } x \in p_{\mathcal{H}}^{-1}(\mathbb{R}^{b_1} \setminus (-e, e)^{b_1}). \end{aligned}$$

Since the above construction is asymmetrical in  $\bar{f}_1$  and  $\bar{f}_2$ , we also consider another family of functionals  $\{\tilde{\mathcal{L}}_e^s\}$  where the role of  $\bar{f}_1$  and  $\bar{f}_2$  are reversed. In other words, we have

$$\begin{aligned} \tilde{\mathcal{E}}_e^1 &= \mathcal{L}_1, \\ \tilde{\mathcal{E}}_e^0(x) &= \begin{cases} \mathcal{L}_2(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e + 1, e - 1]^b), \\ \mathcal{L}_1(x) & \text{if } x \in p_{\mathcal{H}}^{-1}(\mathbb{R}^b \setminus (-e, e)^b), \end{cases} \\ \tilde{\mathcal{E}}_e^s(x) &= \mathcal{L}_1(x) \quad \text{if } x \in p_{\mathcal{H}}^{-1}(\mathbb{R}^b \setminus (-e, e)^b). \end{aligned}$$

Roughly speaking, the family  $\{\mathcal{E}_e^s\}$  will give a morphism from Conley indices given by  $\mathcal{L}_1$  to Conley indices given by  $\mathcal{L}_2$  and vice versa.

To show equivalence, we need to introduce a (final) pair of families of functionals. For two real numbers  $e, e'$  with  $e - 1 \geq e' \geq 1$  and  $s \in [0, 1]$ , we define

$$\begin{aligned} \mathcal{E}_{e,e'}^s(x) &= \begin{cases} \tilde{\mathcal{E}}_{e'}^s(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e', e']^{b_1}), \\ \mathcal{E}_e^0(x) & \text{otherwise,} \end{cases} \\ \tilde{\mathcal{E}}_{e,e'}^s(x) &= \begin{cases} \mathcal{E}_{e'}^s(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e', e']^{b_1}), \\ \tilde{\mathcal{E}}_e^0(x) & \text{otherwise.} \end{cases} \end{aligned}$$

These functionals have the following properties:

- (1)  $\mathcal{E}_{e,e'}^1 = \mathcal{E}_e^0$  and  $\tilde{\mathcal{E}}_{e,e'}^1 = \tilde{\mathcal{E}}_e^0$ ,
- (2)  $\mathcal{E}_{e,e'}^0(x) = \begin{cases} \mathcal{L}_2(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e' + 1, e' - 1]^{b_1} \cup (\mathbb{R}^{b_1} \setminus (-e, e)^{b_1})), \\ \mathcal{L}_1(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e + 1, e - 1]^{b_1} \setminus (-e', e')^{b_1}), \end{cases}$
- (3)  $\tilde{\mathcal{E}}_{e,e'}^0(x) = \begin{cases} \mathcal{L}_1(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e' + 1, e' - 1]^{b_1} \cup (\mathbb{R}^{b_1} \setminus (-e, e)^{b_1})), \\ \mathcal{L}_2(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e + 1, e - 1]^{b_1} \setminus (-e', e')^{b_1}). \end{cases}$

We have the following extension of Lemmas 6.9 and 6.10. The proof is essentially the same, and we omit it.



**Lemma 6.13** (1) *Let  $M$  be any positive integer. Then we can find a smooth family of  $(6M+6)\mathbb{Z}^{b_1}$ -periodic functions  $\tilde{\tau}_w: \mathbb{R}^{b_1} \rightarrow [0, 1]$ , parametrized by a compact manifold  $W'_M$ , with the following property: for any functional in the family*

$$\{\mathcal{E}_{e,e'}^s \mid s \in [0, 1], e - 1 \geq e' \geq 1\}$$

and any  $(e_1, \dots, e_{b_1}) \in \mathbb{R}^{b_1}$ , we can find  $w \in W'_M$  such that

$$\mathcal{E}_{e,e'}^s = \text{CSD}_v|_{\text{Coul}(Y)} + f_w$$

when restricted to  $p_{\mathcal{H}}^{-1}([e_1 - M, e_1 + M] \times \dots \times [e_{b_1} - M, e_{b_1} + M])$ . Here  $f_w$  is the function on  $\text{Coul}(Y)$  induced by  $\tilde{\tau}_w$  (see (25)).

(2) *We can choose a sequence of constants  $\{C_j\}$  in the definition of  $\mathcal{P}$  (Definition 2.1) such that for any positive integer  $M$  and any  $\bar{f}_1, \bar{f}_2 \in \mathcal{P}$ , the induced family  $\{\tilde{f}_w\}_{w \in W'_M}$  is a continuous family of  $(6M+6)$ -periodic tame functions.*

(3) *Similar result holds if we consider any one of the following families instead:*

- $\{\tilde{\mathcal{E}}_{e,e'}^s \mid s \in [0, 1], e - 1 \geq e' \geq 1\}$ ;
- $\{(1-s)\mathcal{L}_2 + s\mathcal{E}_{e,e'}^{s'} \mid s, s' \in [0, 1], e - 1 \geq e' + 1\}$ ;
- $\{(1-s)\mathcal{L}_1 + s\tilde{\mathcal{E}}_{e,e'}^{s'} \mid s, s' \in [0, 1], e \geq e' + 1\}$ .

With Lemma 6.13 at hand, we can repeat the proof of Corollary 6.12 and get the following result.

**Lemma 6.14** *There exists a uniform constant  $R_2$  such that for  $e - 1 \geq e' \geq 1$  and  $s, s' \in [0, 1]$ , the functionals  $\mathcal{E}_{e,e'}^s, \tilde{\mathcal{E}}_{e,e'}^s, (1-s)\mathcal{L}_2 + s\mathcal{E}_{e,e'}^{s'}$  and  $(1-s)\mathcal{L}_1 + s\tilde{\mathcal{E}}_{e,e'}^{s'}$  are all  $R_2$ -bounded.*

Now we start constructing a mixed direct system relating the spectra given by  $\mathcal{L}_1$  and  $\mathcal{L}_2$ . As usual, we focus on the case of  $\text{swf}^A$ . We first choose a constant  $\tilde{R}$  greater than  $\max(R_1, R_2)$ , where  $R_1$  is the constant in Corollary 6.12 and  $R_2$  is a constant that we will specify later in Lemma 6.14. Let  $J_1^+ \subset J_2^+ \subset \dots$  and  $\tilde{J}_1^+ \subset \tilde{J}_2^+ \subset \dots$  be increasing sequences of bounded subsets corresponding to  $\mathcal{L}_1$  and  $\mathcal{L}_2$ , respectively; see (15). Although these bounded sets come from  $\text{Str}(\tilde{R})$ , they are different as we use different cutting functions and different cutting values. Since the sequences of subsets

are increasing, we can find increasing sequences of positive integers  $\{m_j\}$ ,  $\{\tilde{m}_j\}$ ,  $\{e_j\}$  and  $\{\tilde{e}_j\}$  such that

$$(29) \quad J_{m_j}^+ \subset p_{\mathcal{H}}^{-1}([-e_j + 1, e_j - 1]^b) \cap \text{Str}(\tilde{R}) \subset p_{\mathcal{H}}^{-1}([-e_j, e_j]^b) \cap \text{Str}(\tilde{R}) \subset \tilde{J}_{\tilde{m}_j}^+ \\ \subset p_{\mathcal{H}}^{-1}([-\tilde{e}_j + 1, \tilde{e}_j - 1]^b) \cap \text{Str}(\tilde{R}) \subset p_{\mathcal{H}}^{-1}([-\tilde{e}_j, \tilde{e}_j]^b) \cap \text{Str}(\tilde{R}) \subset J_{m_{j+1}}^+.$$

Let  $\{\mu_n\}$  and  $\{\lambda_n\}$  be an increasing sequence and a decreasing sequence of real numbers with  $-\lambda_n, \mu_n \rightarrow \infty$  and denote by  $V_{\lambda_n}^{\mu_n}$  the corresponding eigenspace. For a functional  $\mathcal{L}$  on  $\text{Coul}(Y)$ , we denote by  $\varphi^n(\mathcal{L})$  the flow generated by  $\iota \circ p_{\lambda_n}^{\mu_n} \text{grad } \mathcal{L}$  on  $V_{\lambda_n}^{\mu_n}$  where  $\iota$  is a bump function with value 1 on a specific bounded set. Since we are only interested in the Conley index which will be independent of  $\iota$ , we can drop  $\iota$  from our notation.

Consider  $J_{m_j}^{n,+} = J_{m_j}^+ \cap V_{\lambda_n}^{\mu_n}$  and  $\tilde{J}_{\tilde{m}_j}^{n,+} = \tilde{J}_{\tilde{m}_j}^+ \cap V_{\lambda_n}^{\mu_n}$ . By Theorem 6.2, we can fix a sufficiently large integer  $n$  so that  $J_{m_j}^{n,+}$ ,  $\tilde{J}_{\tilde{m}_j}^{n,+}$  are isolating neighborhoods for all of the above families of approximated flows. For the family  $\{\mathcal{E}_{e_j}^s\}$ , we get a homotopy equivalence from homotopy-invariance of Conley indices

$$\rho_1: I_{S^1}(\varphi^n(\mathcal{E}_{e_j}^0), \text{inv}(\tilde{J}_{\tilde{m}_j}^{n,+})) \xrightarrow{\cong} I_{S^1}(\varphi^n(\mathcal{L}_2), \text{inv}(\tilde{J}_{\tilde{m}_j}^{n,+})),$$

where we recall that  $\mathcal{E}_{e_j}^1 = \mathcal{L}_2$ . Since  $\mathcal{E}_{e_j}^0$  is equal to  $\mathcal{L}_1$  on  $p_{\mathcal{H}}^{-1}([-e_j + 1, e_j - 1]^{b_1})$ , which contains  $J_{m_j}^+$ , we see that the flow  $\varphi^n(\mathcal{E}_{e_j}^0)$  goes inside  $J_{m_j}^{n,+}$  along the boundary  $\partial J_{m_j}^{n,+} \setminus \partial \text{Str}_Y(\tilde{R})$ . Consequently, the subset  $J_{m_j}^{n,+} \subset \tilde{J}_{\tilde{m}_j}^{n,+}$  is an attractor with respect to  $\varphi^n(\mathcal{E}_{e_j}^0)$  and we obtain an attractor map

$$\rho_2: I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J_{m_j}^{n,+})) \rightarrow I_{S^1}(\varphi^n(\mathcal{E}_{e_j}^0), \text{inv}(\tilde{J}_{\tilde{m}_j}^{n,+})).$$

We combine the above two maps and obtain the map

$$(30) \quad \bar{i}_{\tilde{m}_j}^{n,+} := \rho_1 \circ \rho_2: I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J_{m_j}^{n,+})) \rightarrow I_{S^1}(\varphi^n(\mathcal{L}_2), \text{inv}(\tilde{J}_{\tilde{m}_j}^{n,+})).$$

Similarly, we use the family  $\{\tilde{\mathcal{E}}_{\tilde{e}_j}^s\}$  to get a homotopy equivalence

$$\tilde{\rho}_1: I_{S^1}(\varphi^n(\tilde{\mathcal{E}}_{\tilde{e}_j}^0), \text{inv}(J_{m_{j+1}}^{n,+})) \xrightarrow{\cong} I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J_{m_{j+1}}^{n,+})).$$

Since  $\tilde{J}_{\tilde{m}_j}^{n,+} \subset J_{m_{j+1}}^{n,+}$  is an attractor with respect to  $\tilde{\mathcal{E}}_{\tilde{e}_j}^0$ , we also get an attractor map

$$\tilde{\rho}_2: I_{S^1}(\varphi^n(\mathcal{L}_2), \text{inv}(\tilde{J}_{\tilde{m}_j}^{n,+})) \rightarrow I_{S^1}(\varphi^n(\tilde{\mathcal{E}}_{\tilde{e}_j}^0), \text{inv}(J_{m_{j+1}}^{n,+})).$$

We compose the above two maps and get the map

$$(31) \quad \hat{i}_{\tilde{m}_j}^{n,+} := \tilde{\rho}_1 \circ \tilde{\rho}_2: I_{S^1}(\varphi^n(\mathcal{L}_2), \text{inv}(\tilde{J}_{\tilde{m}_j}^{n,+})) \rightarrow I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J_{m_{j+1}}^{n,+})).$$

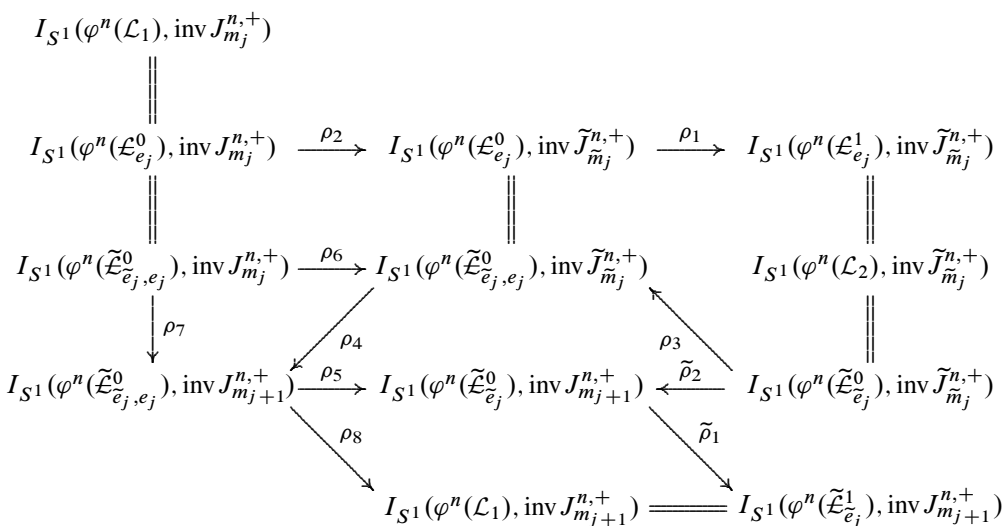
After appropriate desuspension, we obtain a direct system in the category  $\mathcal{C}$

$$(32) \quad I_{m_1}^{n_1,+} \rightarrow \tilde{I}_{\tilde{m}_1}^{\tilde{n}_1,+} \rightarrow I_{m_2}^{n_2,+} \rightarrow \tilde{I}_{\tilde{m}_2}^{\tilde{n}_2,+} \rightarrow \dots,$$

where  $I_m^{n,+}$  (resp.  $\tilde{I}_m^{n,+}$ ) be the object of  $\mathcal{C}$  obtained from desuspending the Conley indices of  $J_m^{n,+}$  (resp.  $\tilde{J}_m^{n,+}$ ) by  $\bar{V}_{-\lambda_n}^0$  and we can pick a suitable sequence of integers  $0 \ll n_1 < \tilde{n}_1 < n_2 < \tilde{n}_2 < \dots$ . The main result of this section follows from the following proposition.

**Proposition 6.15** *The map  $\hat{i}_{\tilde{m}_j}^{n,+} \circ \bar{i}_{m_j}^{n,+}$  is  $S^1$ -homotopic to attractor map for the attractor  $\text{inv}(\varphi^n(\mathcal{L}_1), J_{m_j}^{n,+}) \subset \text{inv}(\varphi^n(\mathcal{L}_1), J_{m_{j+1}}^{n,+})$ .*

**Proof** We consider the following commutative (up to  $S^1$ -homotopy) diagram:



The maps are defined as follows:

1. Different flows are generated by the same vector field when restricted to some isolating neighborhood. This defines all the identifications “=” in the diagram.
2. The maps  $\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2$  are defined as before.
3. The maps  $\rho_3, \rho_5$  are the homotopy equivalences given by the deformation  $\tilde{\mathcal{L}}_{e_j, e_j}^s$ ,  $s \in [0, 1]$ .
4. The maps  $\rho_4, \rho_6, \rho_7$  are the attractor maps for the flow  $\varphi^n(\tilde{\mathcal{L}}_{e_j, e_j}^0)$ .
5. The map  $\rho_8$  is homotopy equivalence given by the deformation

$$(33) \quad (1-s)\mathcal{L}_1 + s\tilde{\mathcal{L}}_{e_j, e_j}^0, \quad s \in [0, 1].$$

Now we check that the above diagram commutes:

1. The maps  $\rho_2$  and  $\rho_6$  are the defined as attractor maps for the flows  $\varphi^n(\mathcal{L}_{e_j}^0)$  and  $\varphi^n(\tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^0)$ , respectively. Since these two flows are generated by the same vector field when restricted to  $\tilde{J}_{\tilde{m}_j}^{n,+}$ , we see that  $\rho_2$  is  $S^1$ -homotopic to  $\rho_6$ , written as  $\rho_2 \cong \rho_6$ .
2. Because the attractor maps for the same flow are transitive, we have  $\rho_7 \cong \rho_4 \circ \rho_6$ .
3. We deform  $\tilde{\mathcal{L}}_{\tilde{e}_j}^0 = \tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^1$  to  $\tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^0$  through the family  $\tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^s$ . In the process of this deformation, nothing is changed on the set  $p_{\mathcal{H}}^{-1}(\mathbb{R}^b \setminus (-e_j, e_j)^b)$ , which contains both  $\partial J_{m_{j+1}}^{n,+} \setminus \partial \text{Str}(\tilde{R})$  and  $\partial \tilde{J}_{\tilde{m}_j}^{n,+} \setminus \partial \text{Str}(\tilde{R})$ . Therefore, we obtain a family of attractor maps: we get  $\rho_4$  when  $s = 0$  and get  $\tilde{\rho}_2$  when  $s = 1$ . Notice that  $\rho_3$  and  $\rho_5$  are the homotopy equivalences induced by this deformation. The identity  $\tilde{\rho}_2 \cong \rho_5 \circ \rho_4 \circ \rho_3$  can be proved using the homotopy-invariance of the attractor map.
4. The map  $\rho_3$  is induced by the deformation  $\tilde{\mathcal{L}}_{\tilde{e}_j, \mathcal{E}_j}^s$  with  $s$  going from 1 to 0. We just get  $\mathcal{L}_{e_j}^s$  if we restrict this deformation to the set  $\tilde{J}_{\tilde{m}_j}^{n,+}$ . Therefore, we have  $\rho_1 \cong \rho_3^{-1}$ .
5. Notice that  $\tilde{\rho}_1 \circ \rho_5$  is the homotopy equivalence induced by the deformation

$$(34) \quad \tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^0|_{J_{m_{j+1}}^+} \rightarrow \tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^1|_{J_{m_{j+1}}^+} = \tilde{\mathcal{L}}_{\tilde{e}_j}^0|_{J_{m_{j+1}}^+} \rightarrow \tilde{\mathcal{L}}_{\tilde{e}_j}^1|_{J_{m_{j+1}}^+} = \mathcal{L}_1|_{J_{m_{j+1}}^+}.$$

In order to prove the identity  $\rho_8 \cong \tilde{\rho}_1 \circ \rho_5$ , we just need to show that the homotopy equivalences  $I_{S^1}(\varphi^n(\tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^0), \text{inv}(J_{m_{j+1}}^{n,+})) \xrightarrow{\cong} I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J_{m_{j+1}}^{n,+}))$  which are induced by deformations (33) and (34) are  $S^1$ -homotopic to each other. To see this, for any  $r \in [0, 1]$ , we consider the following 2-step deformation.

- (a) Deform  $\tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^0$  to  $r\mathcal{L}_1 + (1-r)\tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^1 = \tilde{\mathcal{L}}_{\tilde{e}_j}^r$  through the family  $rs\mathcal{L}_1 + (1-rs)\tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^s$  with  $s$  going from 0 to 1.
- (b) Then deform  $\tilde{\mathcal{L}}_{\tilde{e}_j}^r$  to  $\tilde{\mathcal{L}}_{\tilde{e}_j}^1 = \mathcal{L}_1$  through the family  $\tilde{\mathcal{L}}_{\tilde{e}_j}^s$  with  $s$  going from  $r$  to 1.

Setting  $r$  to be 0 and 1 in the above deformation, we will get (34) and (33), respectively. As before, the flow near  $\partial J_{m_{j+1}}^+ \setminus \partial \text{Str}(\tilde{R})$  is not changed. By Lemma 6.14, all the functionals involved in the above deformation are  $R_2$ -bounded. Since  $\tilde{R} > R_2$ ,  $J_{m_{j+1}}^{n,+}$  is an isolating neighborhood for all these functionals when  $n$  is large enough. Therefore, as  $r$  goes from 0 to 1, we get a  $S^1$ -homotopy between the homotopy equivalences induced by (33) and (34).

We have proved that the diagram is commutative up to  $S^1$ -homotopy. As a corollary, the map  $\hat{i}_{\tilde{m}_j}^{n,+} \circ \hat{l}_{m_j}^{n,+} = \tilde{\rho}_1 \circ \tilde{\rho}_2 \circ \rho_1 \circ \rho_2$  is  $S^1$ -homotopic to  $\rho_8 \circ \rho_7$ . Now we consider the attractor map for the flow  $\mathcal{L}_1$ , which we denote by

$$i^+ : I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J_{m_j}^{n,+})) \rightarrow I_{S^1}(\varphi^n(\mathcal{L}_1), \text{inv}(J_{m_{j+1}}^{n,+})).$$

We deform  $\mathcal{L}_1$  to  $\tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^0$  through the family  $(1-s)\mathcal{L}_1 + s\tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^0$  ( $s \in [0, 1]$ ). For any  $s$ ,  $(1-s)\mathcal{L}_1 + s\tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^0$  equals  $\mathcal{L}_1$  on the set  $p_{\mathcal{H}}^{-1}([-e_j + 1, e_j - 1]^b \cup (\mathbb{R}^b \setminus (-\tilde{e}_j, \tilde{e}_j)^b))$ , which contains both  $\partial J_{m_j}^+ \setminus \partial \text{Str}(\tilde{R})$  and  $\partial J_{m_{j+1}}^+ \setminus \partial \text{Str}(\tilde{R})$ . Therefore, we get a family of attractors

$$\text{inv}(\varphi^n((1-s)\mathcal{L}_1 + s\tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^0), J_{m_j}^{n,+}) \subset \text{inv}(\varphi^n((1-s)\mathcal{L}_1 + s\tilde{\mathcal{L}}_{\tilde{e}_j, e_j}^0), J_{m_{j+1}}^{n,+}).$$

By the homotopy-invariance of the attractor maps, we see that  $i^+$  also is homotopic to  $\rho_8 \circ \rho_7$ . This finishes the proof of the proposition.  $\square$

**Proposition 6.15** actually implies that the direct system (32) contains a subsystem whose colimit gives the ind-spectrum  $\underline{\text{swf}}^A$  for the perturbation  $f_1$ . Similarly, we can prove that the ind-spectrum for the perturbation  $f_2$  is also a subsequential colimit of (32). Therefore, by Lemma 4.3, we see that  $f_1$  and  $f_2$  give the same ind-spectrum up to canonical isomorphism.

Finally, we address the situation when  $f_1(a, \phi) = \frac{1}{2}\delta_1\|\phi\|_{L^2}^2 + \bar{f}_1(a, \phi)$  and  $f_2(a, \phi) = \frac{1}{2}\delta_2\|\phi\|_{L^2}^2 + \bar{f}_2(a, \phi)$  with  $\delta_1 \neq \delta_2$ . This can now be proved by the standard homotopy-invariance argument as follows. We set  $\delta_t = (2-t)\delta_1 + (t-1)\delta_2$ . For each  $t_0 \in [1, 2]$ , we can find an extended cylinder function  $\bar{f}$  such that the pair  $(\delta_t, \bar{f})$  gives a perturbed Chern–Simons–Dirac functional whose critical points are all nondegenerate in the sense of [17, Definition 12.1.1] for any  $t$  near  $t_0$ . Here we essentially use the compactness result for critical points, which is a special case of [17, Proposition 11.6.4]. Hence, we can find a subdivision  $1 = t_1 < \dots < t_n = 2$  and  $\bar{f}'_1, \dots, \bar{f}'_{n-1} \in \mathcal{P}$  with  $\bar{f}'_1 = \bar{f}_1$  and  $\bar{f}'_{n-1} = \bar{f}_2$  such that the pair  $(\delta_t, \bar{f}'_j)$  gives a good perturbation for any  $t \in [t_j, t_{j+1}]$ . By homotopy-invariance of the Conley index, we see that  $(\delta_{t_j}, \bar{f}'_j)$  and  $(\delta_{t_{j+1}}, \bar{f}'_j)$  give the same ind-spectrum  $\underline{\text{swf}}^A$ . Since we already showed that the ind-spectrum does not depend on the choice of the extended cylinder function when  $\delta$  is fixed, we can conclude that  $f_1$  and  $f_2$  give the same  $\underline{\text{swf}}^A$  (up to canonical isomorphisms). This finishes the proof of the invariance for (II).

### 6.4 The invariance for (I)

Now we discuss what happens when we vary the metric  $g$  and the base connection  $A_0$ . Let  $(A_0, g_0)$  and  $(A_1, g_1)$  be two pairs of base connections and metrics. We can connect them by a smooth path  $\alpha(s) = (A_s, g_s)$  with  $s \in [0, 1]$ . As in the proof of the invariance for  $\delta$ , we can divide  $[0, 1]$  into small subintervals  $[s_j, s_{j+1}]$

such that, for each subinterval  $[s_j, s_{j+1}]$ , we can fix the choice of the auxiliary data  $(f, \bar{g}, \theta, \tilde{R}, \{\lambda_n\}, \{\mu_n\})$ .

As  $s$  varies between  $s_j$  and  $s_{j+1}$ , we get a continuous family of Coulomb slices  $\text{Coul}(Y, s)$  and a family of sequences of bounded sets

$$J_{1,s}^+ \subset J_{2,s}^+ \subset \dots$$

For any positive integer  $n$ , we have a (usually not continuous) family of finite-dimensional spaces  $V_{\lambda_n}^{\mu_n}(s)$ . As before, we denote by  $\bar{V}_{\mu_n}^0(s)$  the orthogonal complement of  $i\Omega_h^1(Y)$  in  $V_{\mu_n}^0(s)$ . Let  $J_{m,s}^{n,+} = J_{m,s}^+ \cap V_{\lambda_n}^{\mu_n}(s)$  and  $\varphi_{n,s}$  be the approximated Seiberg–Witten flow on  $V_{\lambda_n}^{\mu_n}(s)$ . The following lemma is a direct consequence of [Theorem 6.2](#).

**Lemma 6.16** *For any positive integer  $m$  and a sufficiently large integer  $n$  relative to  $m$ , we have*

$$\begin{aligned} \Sigma^{-\bar{V}_{\mu_n}^0(s_j)} I_{S^1}(\varphi_{n,s_j}, \text{inv}(J_{m,s_j}^{n,+})) \\ \cong \Sigma^{\text{sf}(-\not{D}, \alpha[s_j, s_{j+1}])\mathbb{C}} \Sigma^{-\bar{V}_{\mu_n}^0(s_{j+1})} I_{S^1}(\varphi_{n,s_{j+1}}, \text{inv}(J_{m,s_{j+1}}^{n,+})) \end{aligned}$$

as objects of  $\mathcal{C}$ .

The above isomorphisms induce an isomorphism

$$\underline{\text{swf}}^A(Y, \mathfrak{s}, A_{s_j}, g_{s_j}; S^1) \cong \Sigma^{\text{sf}(-\not{D}, \alpha[s_j, s_{j+1}])\mathbb{C}} \underline{\text{swf}}^A(Y, \mathfrak{s}, A_{s_{j+1}}, g_{s_{j+1}}; S^1).$$

By additivity of spectral flow, we can conclude that

$$(35) \quad \underline{\text{swf}}^A(Y, \mathfrak{s}_Y, A_0, g_0; S^1) \cong \Sigma^{\text{sf}(-\not{D}, \alpha)\mathbb{C}} \underline{\text{swf}}^A(Y, \mathfrak{s}_Y, A_1, g_1; S^1).$$

Therefore,  $\underline{\text{swf}}^A$  can only change by suspension or desuspension of copies of  $\mathbb{C}$  when we vary the pair  $(A_0, g_0)$ . Now we discuss the following two cases separately.

**Case 1** ( $\mathfrak{s}$  is torsion) In this case, we recall that there is a well-defined quantity  $n(Y, \mathfrak{s}, A_0, g)$ . By excision arguments as in [\[22\]](#), we have

$$n(Y, \mathfrak{s}, A_0, g_0) + \text{sf}(-\not{D}, \alpha) = n(Y, \mathfrak{s}, A_1, g_1).$$

This implies

$$\begin{aligned} (\underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g_0; S^1), 0, n(Y, \mathfrak{s}, A_0, g_0)) \\ \cong (\underline{\text{swf}}^A(Y, \mathfrak{s}, A_1, g_1; S^1), 0, n(Y, \mathfrak{s}, A_1, g_1)), \end{aligned}$$

and the same result holds for  $\underline{\text{swf}}^R$ . This proves invariance of  $\underline{\text{SWF}}^A(Y, \mathfrak{s}; S^1)$  and  $\underline{\text{SWF}}^R(Y, \mathfrak{s}; S^1)$  in the torsion case.

**Case 2** ( $\mathfrak{s}$  is nontorsion) Let  $l = \text{gcd}\{(c_1(\mathfrak{s}) \cup h)[Y] \mid h \in H^1(Y; \mathbb{Z})\}$ . We pick a harmonic gauge transformation  $u_0 \in \mathcal{G}_Y^{h,o} = H^1(Y; \mathbb{Z})$  such that  $(c_1(\mathfrak{s}) \cup [u_0])[Y] = l$  and denote by  $\text{Coul}(Y, A_0)$  and  $\text{Coul}(Y, u_0(A_0))$  the Coulomb slices with the base connections  $A_0$  and  $u_0(A_0) = A_0 - u_0^{-1}du_0$ , respectively. (Actually, these two slices correspond the same subspace of  $\mathcal{C}_Y$ . However, since the base connections are different, this subspace is identified with  $L_k^2(i \ker d^* \oplus \Gamma(S))$  in different ways. For this reason, we distinguish them for clarity.) The gauge transformation  $u_0: \text{Coul}(Y, A_0) \rightarrow \text{Coul}(Y, u_0(A_0))$  preserves the functional  $\text{CSD}_{v_0, f}$ , its formal gradient, the subspace  $i\Omega_h^1(Y)$ , the finite-dimensional subspaces  $V_{\lambda_n}^{\mu_n}$  and both the  $L^2$ -metric and the non-linear metric  $\|\cdot\|_{\tilde{g}}$ . From this fact, we get a natural isomorphism

$$(36) \quad \underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1) \cong \underline{\text{swf}}^A(Y, \mathfrak{s}, u_0(A_0), g; S^1).$$

Let  $\alpha$  be any path going from  $A_0$  to  $u_0(A_0)$ . As the spectral flow  $\text{sf}(-\not{D}_A, \alpha)$  can be calculated using excision and the Atiyah–Singer index theorem (see [17, Lemma 14.4.6]), it is not hard to check that  $\text{sf}(-\not{D}_A, \alpha) = \frac{1}{2}l$ . Combining the above two equivalences with (35) and (36), we get

$$\underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1) \cong \Sigma^{(l/2)\mathbb{C}} \underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1),$$

and similar results hold for  $\underline{\text{swf}}^R$ . This proves the periodicity result in the main theorem.

## 7 The linearized Seiberg–Witten flow and its associated spectra

Let  $(Y, \mathfrak{s}, A_0, g)$  be fixed as in the previous sections, except that  $\mathfrak{s}$  is assumed to be torsion. We first recall a decomposition on the Coulomb slice

$$(37) \quad \text{Coul}(Y) = L_k^2(i \text{im } d^* \oplus i\Omega_h^1(Y) \oplus \Gamma(S_Y)).$$

The linearized Seiberg–Witten flow we will consider is obtained by scaling the (perturbed) Seiberg–Witten flow in the direction of  $i \text{im } d^* \oplus \Gamma(S_Y)$ .

A general setup of the linearized Seiberg–Witten flow can be described as follows. Pick a real number  $\delta$  and consider  $\mathbf{D}(h) = \not{D}_{A_0+h} + \delta$  as a smooth family of self-adjoint elliptic operators on the spinor bundle parametrized by  $i\Omega_h^1(Y)$ . Choose an

even Morse function  $g_H: \mathbb{R} \rightarrow \mathbb{R}$  such that  $g_H(\theta + 1) = g_H(\theta)$  (usually we use  $g_H(\theta) = -\cos(2\pi\theta)$ ). After identifying  $i\Omega_h^1(Y)$  with  $\mathbb{R}^{b_1}$  and the action of  $\mathcal{G}_Y^{h,o}$  with addition by  $\mathbb{Z}^{b_1}$ , consider a function  $f_H: i\Omega_h^1(Y) \rightarrow \mathbb{R}$  given by  $f_H(\theta_1, \dots, \theta_{b_1}) = \sum_{i=1}^{b_1} g_H(\theta_i)$ . The function  $f_H$  can be viewed as a Morse function on the Picard torus of  $Y$ . The linearized Seiberg–Witten flow is given by a trajectory on  $\text{Coul}(Y)$  satisfying

$$(38) \quad -\frac{d}{dt}(\beta(t), h(t), \phi(t)) = (*d\beta(t), \text{grad } f_H(h(t)), \mathbf{D}(h(t))\phi(t)).$$

Recall that a trajectory  $\gamma$  is said to be of finite type if the image  $\gamma(\mathbb{R})$  is contained in a bounded subset of  $\text{Coul}(Y)$ . We will be interested in a particular situation where all the finite-type trajectories are reducible and lie in the space of harmonic 1-forms.

**Lemma 7.1** *Suppose that the family of self-adjoint operators  $2\mathbf{D}^2 + \rho(\text{grad } f_H)$  is positive definite for all  $h \in i\Omega_h^1(Y)$ . Then any finite-type trajectory of the linearized Seiberg–Witten flow is contained in  $i\Omega_h^1(Y)$ .*

**Proof** Let  $\gamma(t) = (\beta(t), h(t), \phi(t))$  be such a trajectory. We want to show that  $\beta(t) = 0$  and  $\phi(t) = 0$  for any  $t$ . Since  $\gamma$  is of finite type, we have  $\sup_{t \in \mathbb{R}} \|\beta(t)\|_{L^2}^2 < \infty$  and  $\sup_{t \in \mathbb{R}} \|\phi(t)\|_{L^2}^2 < \infty$ . Since  $*d$  has no kernel on  $i \text{im } d^*$ , it is easy to deduce that  $\beta \equiv 0$ .

For the spinor part, we compute the second derivative of its  $L^2$ -norm

$$\begin{aligned} & \frac{d^2}{dt^2} \|\phi(t)\|_{L^2}^2 \\ &= -2\frac{d}{dt} \langle \mathbf{D}(h(t))\phi(t), \phi(t) \rangle_{L^2} \\ &= -2\langle (\frac{d}{dt} \mathbf{D}(h(t)))\phi(t) + \mathbf{D}(h(t))(\frac{d}{dt} \phi(t)), \phi(t) \rangle - 2\langle \mathbf{D}(h(t))\phi(t), \frac{d}{dt} \phi(t) \rangle \\ &= 2\langle (\mathcal{D}_{\text{grad } f_H} \mathbf{D} + 2\mathbf{D}^2)(h(t))\phi(t), \phi(t) \rangle \\ &= 2\langle (\rho(\text{grad } f_H) + 2\mathbf{D}^2)(h(t))\phi(t), \phi(t) \rangle. \end{aligned}$$

By the assumption, we have  $\frac{d^2}{dt^2} \|\phi(t)\|_{L^2}^2 \geq 0$ , and the equality holds if and only if  $\phi(t) = 0$ . It is not hard to see that  $\phi \equiv 0$ . □

We can decompose the linearized Seiberg–Witten vector field (right-hand side of (38)) as  $l + c_0$ , where  $l(\beta, h, \phi) = (*d\beta, 0, \mathcal{D}_{A_0}\phi)$  is linear and  $c_0$  is a nonlinear part given by

$$(39) \quad c_0(\beta, h, \phi) := (0, \text{grad } f_H(h), (\rho(h) + \delta)\phi).$$

It is not hard to check that  $c_0$  is quadratic-like. As a result, we can obtain Conley indices from finite-dimensional approximation. We then construct associated spectra



as before, although some modification to the choices of a strip of balls and cut-off functions are made in order to simplify computation.

Recall  $V_\lambda^\mu$  is the span of the eigenspaces of  $l$  with eigenvalues in the interval  $(\lambda, \mu]$ . Furthermore, we will consider the space  $V_\lambda^{\mu, \nu}$  spanned by the eigenvectors of  $*d|_{\ker d^*}$  with eigenvalue in  $(\lambda, \mu]$  and the eigenvectors of  $\not{D}_{A_0}$  with eigenvalue in  $(\lambda, \nu]$ . The space  $\bar{V}_\lambda^\mu$  and  $\bar{V}_\lambda^{\mu, \nu}$  is the orthogonal complement of  $i\Omega_h^1(Y)$  in  $V_\lambda^\mu$  and  $V_\lambda^{\mu, \nu}$ , respectively. The construction proceeds as follows:

1. Choose real numbers  $\theta^\pm \in (0, 1)$  with  $\frac{d}{dt}g_H(\theta^+) > 0$  and  $\frac{d}{dt}g_H(\theta^-) < 0$ .
2. For a suitable real number  $\bar{\epsilon} > 0$ , we define a set

$$\bar{\text{Str}}(\bar{\epsilon}) := \{u \cdot (\beta, h, \phi) \mid u \in \mathcal{G}_Y^{h, o}, \|\beta\|_{L_k^2} \leq 1, h \in [0, 1]^{b_1}, \|\phi\|_{L_k^2} \leq \bar{\epsilon}\}$$

and let

$$\bar{J}_m^\pm = \bar{\text{Str}}(\bar{\epsilon}) \cap p_{\mathcal{H}}^{-1}([-\theta^\pm - m, \theta^\pm + m]^{b_1}).$$

3. For sequences  $\{\lambda_n\}, \{\mu_n\}$  with  $-\lambda_n, \mu_n \rightarrow \infty$ , consider  $\bar{J}_m^{n, \pm} = \bar{J}_m^\pm \cap V_{\lambda_n}^{\mu_n}$  as before. Let  $\bar{\varphi}_m^n$  be the flow on  $V_{\lambda_n}^{\mu_n}$  generated by the vector field  $\iota_m \cdot (l + p_{\lambda_n}^{\mu_n} \circ c_0)$ , where  $\iota_m$  is a bump function with value 1 on  $\bar{J}_{m+1}^\pm$ . When  $n$  is sufficiently large (depending on  $m$ ), the compact sets  $\bar{J}_m^{n, \pm}$  are isolating neighborhoods under the flow  $\bar{\varphi}_m^n$ . This allows us to define spectra

$$\bar{I}_m^{n, +} := \Sigma^{-\bar{V}_{\lambda_n}^{0, -\delta}} I_{S^1}(\bar{\varphi}_m^n, \text{inv}(\bar{J}_m^{n, +})), \quad \bar{I}_m^{n, -} := \Sigma^{-V_{\lambda_n}^{0, -\delta}} I_{S^1}(\bar{\varphi}_m^n, \text{inv}(\bar{J}_m^{n, -})).$$

As before, there are isomorphisms  $\bar{I}_m^{n, \pm} \cong \bar{I}_m^{n+1, \pm}$ .

4. It is straightforward to check the flow  $\bar{\varphi}_m^n$  goes inside  $\bar{J}_m^{n, +}$  along  $\partial\bar{J}_m^{n, +} \setminus \partial\bar{J}_{m+1}^{n, +}$  and goes outside  $\bar{J}_m^{n, -}$  along  $\partial\bar{J}_m^{n, -} \setminus \partial\bar{J}_{m+1}^{n, -}$ . Therefore, the attractor maps give a direct system

$$(40) \quad \bar{I}_1^{n_1, +} \rightarrow \bar{I}_2^{n_2, +} \rightarrow \dots$$

and the repeller maps give an inverse system

$$(41) \quad \bar{I}_1^{n_1, -} \leftarrow \bar{I}_2^{n_2, -} \leftarrow \dots$$

for a suitable increasing sequence of positive integers  $\{n_m\}$ .

5. When  $\mathfrak{s}$  is a spin structure, we define an direct system and inverse system of  $\text{Pin}(2)$ -equivariant spectra  $\bar{I}_m^{n, \pm}(\text{Pin}(2))$  in the same manner:

$$(42) \quad \begin{aligned} \bar{I}_1^{n_1, +}(\text{Pin}(2)) &\rightarrow \bar{I}_2^{n_2, +}(\text{Pin}(2)) \rightarrow \dots, \\ \bar{I}_1^{n_1, -}(\text{Pin}(2)) &\leftarrow \bar{I}_2^{n_2, -}(\text{Pin}(2)) \leftarrow \dots. \end{aligned}$$

To keep track of the number of eigenvalues near 0, we introduce a definition.

**Definition 7.2** Let  $L$  be a self-adjoint elliptic operator, we define a signed count with multiplicity of the eigenvalues of  $L$ , ie

$$m(L, \delta) := \begin{cases} \# \text{ of eigenvalues of } L \text{ in } (-\delta, 0] & \text{if } \delta \geq 0, \\ -(\# \text{ of eigenvalues of } L \text{ in } (0, -\delta)) & \text{if } \delta < 0. \end{cases}$$

The following theorem is the main tool for our calculations of the spectrum invariants when the Seiberg–Witten Floer spectra agree with the linearized ones.

**Theorem 7.3** Let  $Y$  be a 3–manifold equipped with a torsion  $\text{spin}^c$  structure  $\mathfrak{s}$ . Suppose that we can find an even Morse function  $g_H: \mathbb{R} \rightarrow \mathbb{R}$  with  $g_H(\theta + 1) = g_H(\theta)$  and an extended cylinder function  $\bar{f} \in \mathcal{P}$  satisfying the following conditions:

- (i) The function  $f_H: i\Omega_h^1(Y) \rightarrow \mathbb{R}$  given by  $f_H(\theta_1, \dots, \theta_{b_1}) = \sum_{i=1}^{b_1} g_H(\theta_i)$  satisfies the hypothesis of Lemma 7.1.
- (ii) There exists  $\epsilon' > 0$  such that  $\bar{f}(a, \phi) = 0$  for any  $(a, \phi)$  with  $\|\phi\|_{L^2} \leq \epsilon'$ .
- (iii) All critical points and finite-type gradient flow lines of the functional

$$(43) \quad \bar{\mathcal{L}}(a, \phi) := \text{CSD}|_{\text{Coul}(Y)}(a, \phi) + \frac{1}{2}\delta\|\phi\|_{L^2}^2 + f_H(p_{\mathcal{H}}(a, \phi)) + \bar{f}(a, \phi)$$

on the Coulomb slice  $\text{Coul}(Y)$  are contained in  $i\Omega_h^1(Y)$ .

Then we have

$$\begin{aligned} \underline{\text{SWF}}^A(Y, \mathfrak{s}; S^1) &\cong (\bar{I}^+, 0, n(\mathfrak{s}, A_0, g) + m(\mathcal{D}, \delta)), \\ \underline{\text{SWF}}^R(Y, \mathfrak{s}; S^1) &\cong (\bar{I}^-, 0, n(\mathfrak{s}, A_0, g) + m(\mathcal{D}, \delta)) \end{aligned}$$

as objects in  $\mathfrak{S}$  and  $\mathfrak{S}^*$ , respectively. Here  $\bar{I}^+$  and  $\bar{I}^-$  are the inductive system (40) and the inverse system (41) of the spectra associated to the linearized flow constructed above.

Furthermore, when  $\mathfrak{s}$  is a spin structure, we also have

$$\begin{aligned} \underline{\text{SWF}}^A(Y, \mathfrak{s}; \text{Pin}(2)) &\cong (\bar{I}^+(\text{Pin}(2)), 0, \frac{1}{2}(n(\mathfrak{s}, A_0, g) + m(\mathcal{D}, \delta))), \\ \underline{\text{SWF}}^R(Y, \mathfrak{s}; \text{Pin}(2)) &\cong (\bar{I}^-(\text{Pin}(2)), 0, \frac{1}{2}(n(\mathfrak{s}, A_0, g) + m(\mathcal{D}, \delta))) \end{aligned}$$

as objects in  $\mathfrak{S}_{\text{Pin}(2)}$  and  $\mathfrak{S}_{\text{Pin}(2)}^*$ , respectively. Here  $\bar{I}^+(\text{Pin}(2))$  and  $\bar{I}^-(\text{Pin}(2))$  are the direct system and inverse system of spectra in (42), respectively.

**Proof** We first consider the spectrum invariants  $\underline{\text{swf}}(Y, \mathfrak{s}, g, A_0; S^1)$  associated with the functional  $\bar{\mathcal{L}}$ . By the remark on page 2065 and the hypothesis, we can use the strip  $\overline{\text{Str}}(\bar{\epsilon})$  together with level sets  $p_{\mathcal{H}}^{-1}([-\theta^\pm - m, \theta^\pm + m]^{b_1})$  for constructing these spectrum invariants. Note that the set of critical points of this functional is discrete since they correspond to the critical points of the Morse function  $f_H$ .

Choose a basis  $\{h_j\}$  of  $i\Omega_h^1(Y)$ . Since the function  $\pi_{\mathcal{H}}(\rho^{-1}(\phi\phi^*)_0)$  is bounded on the strip and is homogeneous of degree 2, we can choose  $\bar{\epsilon}$  such that  $\|\pi_{\mathcal{H}}(\rho^{-1}(\phi\phi^*)_0)\| \leq \left| \frac{d}{dt} g_H(\theta^\pm) \right| / \sum \|h_j\|_{L^2}$  for any  $(a, \phi) \in \overline{\text{Str}}(\bar{\epsilon})$ . Furthermore, we require that  $\bar{\epsilon} < \epsilon'$ .

We now consider a continuous family of vector fields  $l + c_s$  parametrized by  $s \in [0, 1]$  on  $\overline{\text{Str}}(\bar{\epsilon})$  with  $l = (*d, 0, \not{D})$  and  $c_s$  is a family of quadratic-like maps given by

$$c_s(\beta, h, \phi) = (0, \text{grad } \tilde{f}(h), (\rho(h) + \delta)\phi) + s(\pi_{\text{im } d^*} \rho^{-1}(\phi\phi^*)_0, s\pi_{\mathcal{H}} \rho^{-1}(\phi\phi^*)_0, \rho(\beta)\phi + s\bar{\xi}(\rho^{-1}(\phi\phi^*)_0)\phi);$$

see (4) for the definition of  $\bar{\xi}$ . Observe that  $l + c_1$  is the gradient of  $\bar{\mathcal{L}}$  as  $\tilde{f}$  vanishes on  $\overline{\text{Str}}(\bar{\epsilon})$ , whereas  $l + c_0$  is exactly the linearized Seiberg–Witten vector field described earlier. For  $0 < s < 1$ , the scaling automorphism gives the commutative diagram

$$\begin{array}{ccc} \text{Coul}(Y) & \xrightarrow{\quad\quad\quad} & \text{Coul}(Y) \\ l+c \downarrow & & \downarrow l+c_s \\ L_{k-1}^2(i \ker(d^*) \oplus \Gamma(S_Y)) & \longrightarrow & L_{k-1}^2(i \ker(d^*) \oplus \Gamma(S_Y)) \end{array}$$

where both of the horizontal maps send  $(\beta, h, \phi)$  to  $(s \cdot \beta, h, s \cdot \phi)$ . In other words, the flow given by  $l + c_s$  on  $\overline{\text{Str}}(\bar{\epsilon})$  corresponds to the flow  $l + c_1$  on  $\overline{\text{Str}}(s^{-1}\bar{\epsilon})$ .

All that is left is to apply the homotopy-invariance of Conley index. We will focus on the case  $\underline{\text{swf}}^A$  as the repeller case can be done similarly. Since  $\left. \frac{d}{dt} \right|_{t=\theta^+} g_H(t) > 0$ , it is easy to find  $\epsilon_1 > 0$  such that any gradient trajectory of  $f_H$  in  $[-\theta^+ - m, \theta^+ + m]^{b_1}$  actually lies in  $[-\theta^+ - m + \epsilon_1, \theta^+ + m - \epsilon_1]^{b_1}$ . We have to check that the inner product  $\langle p_{\mathcal{H}} \circ (l + p_{\lambda_n}^{\mu_n} \circ c_s)(a, \phi), h_j \rangle$  is positive on  $p_{\mathcal{H}}^{-1}(\theta^+ h_j) \cap \overline{\text{Str}}(\bar{\epsilon})$ . We can see that

$$\begin{aligned} \langle p_{\mathcal{H}} \circ (l + p_{\lambda_n}^{\mu_n} \circ c_s)(a, \phi), h_j \rangle &= \langle \text{grad } f_H(\theta^+ h_j) + s\pi_{\mathcal{H}}(\rho^{-1}(\phi\phi^*)_0), h_j \rangle \\ &\geq \frac{d}{dt} g_H(\theta^+) - \|\pi_{\mathcal{H}}(\rho^{-1}(\phi\phi^*)_0)\| \|h_j\| \end{aligned}$$

is indeed positive by our choice of  $\bar{\epsilon}$ .

Therefore, we have an isomorphism  $\text{swf}^A(Y, \mathfrak{s}, g, A_0; S^1) \cong (\bar{I}^+, 0, m(\not{D}, \delta))$ . Note that  $m(\not{D}, \delta)$  appears because we desuspend the Conley index by  $-\bar{V}_{\lambda_n}^{0, -\delta}$  instead of  $\bar{V}_{\lambda_n}^0$  in the definition of  $\bar{I}_m^{n, +}$ .  $\square$

In practice, we will further deform the family of operators  $D$  to simplify the calculation of the Conley index of  $\text{inv}(\bar{J}_m^{n, \pm}, \bar{\varphi}_m^n)$  as follows: Let  $\{\mathbf{Q}_s(h)\}$  be a smooth family of  $0^{\text{th}}$ -order symmetric operators defined on  $\Gamma(S_Y)$  parametrized by  $(s, h) \in [0, 1] \times i\Omega_h^1(Y)$  with  $D = \not{D} + \mathbf{Q}_0$ . When  $2(\not{D} + \mathbf{Q}_s)^2 + \rho(\text{grad } f_H)$  is positive definite for any  $h \in [-\theta^\pm - m, \theta^\pm + m]^{b_1}$ , we can repeat our construction for the linearized Seiberg–Witten vector field to obtain spectra  $\bar{I}_{m'}^{n, \pm}(s)$  associated with  $\not{D} + \mathbf{Q}_s$  for any  $m' \leq m$  and  $n$  is sufficiently large. The following lemma is immediate from [Theorem 6.2](#).

**Lemma 7.4** *Suppose that  $2(\not{D} + \mathbf{Q}_s)^2 + \rho(\text{grad } f_H)$  is positive definite for any  $s \in [0, 1]$  and  $h \in [-\theta^\pm - m, \theta^\pm + m]^{b_1}$ . We also assume that  $\mathbf{Q}_s(0)$  does not depend on  $s$  so that the desuspension indices are constant. Then for any  $s, s' \in [0, 1]$ , we have isomorphisms  $\tau_{m'}^{n, \pm}(s, s') : \bar{I}_{m'}^{n, \pm}(s) \rightarrow \bar{I}_{m'}^{n, \pm}(s')$  when  $m' \leq m$  and  $n$  sufficiently large relative to  $m'$ . Furthermore, these isomorphisms satisfy the commutative diagram*

$$\begin{array}{ccc} \bar{I}_{m'-1}^{n, +}(s) & \longrightarrow & \bar{I}_{m'}^{n, +}(s) \\ \tau_{m'-1}^{n, +}(s, s') \downarrow & & \downarrow \tau_{m'}^{n, +}(s, s') \\ \bar{I}_{m'-1}^{n, +}(s') & \longrightarrow & \bar{I}_{m'}^{n, +}(s') \end{array}$$

where the horizontal maps are attractor maps. A similar diagram holds for the repeller maps. When  $\mathfrak{s}$  is spin, similar results hold for the  $\text{Pin}(2)$ -equivariant spectra.

Next, we describe a situation when [Theorem 7.3](#) implies that the spectrum invariants are just some suspensions of sphere spectra.

**Theorem 7.5** *Let  $Y$  be a 3-manifold with a torsion  $\text{spin}^c$  structure  $\mathfrak{s}$  and a Riemannian metric  $g$ . Suppose that there exists a real number  $\delta$  such that*

- (i) *the functional  $\text{CSD} + \frac{1}{2}\delta\|\phi\|_{L^2}^2$  has only reducible critical points,*
- (ii) *the operator  $\not{D}_A + \delta$  has no kernel for any  $A$  with  $F_{A^t} = 0$ .*

Then we have

$$\begin{aligned} \text{SWF}^A(Y, \mathfrak{s}; S^1) &\cong (S^0, 0, n(Y, \mathfrak{s}, A_0, g) + m(\not{D}_{A_0}, \delta)) \in \text{ob}(\mathfrak{S}), \\ \text{SWF}^R(Y, \mathfrak{s}; S^1) &\cong (S^0, 0, n(Y, \mathfrak{s}, A_0, g) + m(\not{D}_{A_0}, \delta)) \in \text{ob}(\mathfrak{S}^*), \end{aligned}$$

where  $A_0$  is any base connection with  $F_{A_0^t} = 0$ .

Moreover, if  $\mathfrak{s}$  is a spin structure, we also have

$$\begin{aligned} \underline{\text{SWF}}^A(Y, \mathfrak{s}; \text{Pin}(2)) &\cong (S^0, 0, \frac{1}{2}(n(Y, \mathfrak{s}, A_0, g) + m(\not{D}_{A_0}, \delta))) \in \text{ob}(\mathfrak{S}_{\text{Pin}(2)}), \\ \underline{\text{SWF}}^R(Y, \mathfrak{s}; \text{Pin}(2)) &\cong (S^0, 0, \frac{1}{2}(n(Y, \mathfrak{s}, A_0, g) + m(\not{D}_{A_0}, \delta))) \in \text{ob}(\mathfrak{S}_{\text{Pin}(2)}^*). \end{aligned}$$

**Proof** Since  $\not{D}_A + \delta$  has no kernel for any flat connection, we can find a positive number  $\sigma_0$  such that

$$(44) \quad \langle (\not{D}_{A_0+h} + \delta)^2 \phi, \phi \rangle_{L^2} \geq \sigma_0 \|\phi\|_{L^2}^2$$

for any  $h \in i\Omega_h^1(Y)$  and  $\phi \in L^2(\Gamma(S_Y))$ . We divide the proof into two steps.

**Step 1** We would like to show that there exists  $\epsilon_0 > 0$  such that, for any  $\epsilon \in [0, \epsilon_0]$ , all critical points and finite-type negative gradient flow lines of a functional

$$\mathcal{L}_{\delta, \epsilon} := \text{CSD} + \frac{1}{2} \delta \|\phi\|_{L^2}^2 + \epsilon f_H(p_{\mathcal{H}}(a, \phi))$$

on  $\text{Coul}(Y)$  are contained in  $i\Omega_h^1(Y)$  with  $f_H: i\Omega_h^1(Y) \cong \mathbb{R}^{b_1} \rightarrow \mathbb{R}$  given explicitly by  $f_H(\theta_1, \dots, \theta_{b_1}) = -\sum_{j=1}^{b_1} \cos(2\pi\theta_j)$ . This will hold once we can show that  $\frac{d^2}{dt^2} \|\phi(t)\|_{L^2}^2 > 0$  for any irreducible trajectory of finite type  $\gamma = (a, \phi): \mathbb{R} \rightarrow \text{Coul}(Y)$  of the gradient flow of the functional  $\mathcal{L}_{\delta, \epsilon}$ .

Suppose that there are a sequence of irreducible trajectories of finite type  $\gamma_n = (a_n, \phi_n)$  of the negative gradient flow of  $\mathcal{L}_{\delta, \epsilon_n}$  with positive  $\epsilon_n \rightarrow 0$  and a sequence of real numbers  $\{t_n\}$  such that

$$\frac{d^2}{dt^2} \Big|_{t=t_n} \|\phi_n(t)\|_{L^2}^2 \leq 0.$$

Since  $\gamma_n$  is of finite type, we have  $\lim_{t \rightarrow \pm\infty} \gamma_n(t) = \mathfrak{a}_n^\pm$  in  $C^\infty$ , where  $\mathfrak{a}_n^\pm$  are critical points of  $\mathcal{L}_{\delta, \epsilon_n}$ . After passing to a subsequence, we can assume that  $\mathfrak{a}_n^\pm \rightarrow \mathfrak{a}_\infty^\pm$ , where  $\mathfrak{a}_\infty^\pm$  are critical points of  $\mathcal{L}_{\delta, 0}$ . By hypothesis,  $\mathfrak{a}_\infty^\pm$  are reducible and  $\mathcal{L}_{\delta, 0}(\mathfrak{a}_\infty^+) = \mathcal{L}_{\delta, 0}(\mathfrak{a}_\infty^-)$ . This implies  $\mathcal{L}_{\delta, 0}(\mathfrak{a}_n^+) - \mathcal{L}_{\delta, 0}(\mathfrak{a}_n^-) \rightarrow 0$ . In other words, the topological energy of  $\gamma_n$  approaches zero.

Now we treat a finite trajectory  $\gamma_n|_{[t_n-1, t_n+1]}$ , denoted by  $\hat{\gamma}_n$ , as a configuration of a 4-manifold  $[-1, 1] \times Y$ . By the standard compactness result of the 4-dimensional Seiberg–Witten equations, after passing to a subsequence and applying suitable gauge transformations,  $\hat{\gamma}_n$  converges to a negative gradient flow line of  $\mathcal{L}_{\delta, 0}$  in  $C^\infty$  on any interior domain. It is not hard to check that the topological energy of  $\hat{\gamma}_n$  also approaches 0, so  $\hat{\gamma}_n$  in fact converges to a constant trajectory. In particular, there is  $u_n \in \mathcal{G}_Y^{h, o}$  such that  $u_n \cdot \gamma_n(t_n)$  to a critical point  $(h, 0)$  of  $\mathcal{L}_{\delta, 0}$  with  $h \in i\Omega_h^1(Y)$  by the hypothesis.

Since the quantity  $\|\phi_n(t_n)\|_{L^2}^2$  is gauge-invariant, we can instead consider a lift of  $\gamma_n$  to a path  $\tilde{\gamma}_n = (\tilde{a}_n, \tilde{\phi}_n): \mathbb{R} \rightarrow \mathcal{C}_Y$  which is a negative gradient flow line of the functional  $\mathcal{L}_{\delta, \epsilon_n}$  on  $\mathcal{C}_Y$  with  $\tilde{\gamma}_n(t_n) = u_n \cdot \gamma_n(t_n)$ . In particular, we have

$$\begin{aligned} -\frac{d}{dt} \tilde{\phi}_n(t) &= (\not{D}_{A_0 + \tilde{a}_n(t)} + \delta) \tilde{\phi}_n(t), \\ -\frac{d}{dt} \tilde{a}_n(t) &= *d(\tilde{a}_n(t)) + \rho^{-1}(\tilde{\phi}(t) \tilde{\phi}(t)^*)_0 + \epsilon_n \operatorname{grad} f_H(\tilde{h}_n(t)), \end{aligned}$$

where  $\tilde{h}_n(t)$  is the projection of  $\tilde{a}_n(t)$  onto  $i\Omega_h^1(Y)$ . By a calculation similar to one in the proof [Lemma 7.1](#), we obtain

$$(45) \quad \begin{aligned} \frac{d^2}{dt^2} \Big|_{t=t_n} \|\tilde{\phi}_n(t)\|_{L^2}^2 &= 4\langle (\not{D}_{A_0 + \tilde{a}_n(t_n)} + \delta)^2 \tilde{\phi}_n(t_n), \tilde{\phi}_n(t_n) \rangle_{L^2} + \|\tilde{\phi}_n(t_n)\|_{L^2}^4 \\ &\quad + 2\langle \rho(*d\tilde{a}_n(t_n) + \epsilon_n \operatorname{grad} f_H(\tilde{h}_n(t_n))) \tilde{\phi}_n(t_n), \tilde{\phi}_n(t_n) \rangle_{L^2}. \end{aligned}$$

Since  $\tilde{a}_n(t_n) \rightarrow h \in i\Omega_h^1(Y)$ , we have  $*d\tilde{a}_n(t_n) + \epsilon_n \operatorname{grad} f_H(\tilde{h}_n(t_n)) \rightarrow 0$ . Moreover, we can deduce from [\(44\)](#) that

$$\langle (\not{D}_{A_0 + \tilde{a}_n(t_n)} + \delta)^2 \tilde{\phi}_n(t_n), \tilde{\phi}_n(t_n) \rangle_{L^2} \geq \frac{1}{2} \sigma_0 \|\tilde{\phi}_n(t_n)\|_{L^2}^2$$

for  $n$  sufficiently large. Notice that  $\tilde{\phi}_n(t_n) \neq 0$  since  $\gamma_n$  is irreducible by the unique continuation property [\[17, Lemma 10.8.1\]](#). Therefore, for a sufficiently large  $n$ , we can conclude that  $\frac{d^2}{dt^2} \Big|_{t=t_n} \|\phi_n(t)\|_{L^2}^2 > 0$ , which is a contradiction.

**Step 2** Let  $\epsilon_0$  be a constant from step 1. We pick a positive real number  $\epsilon$  satisfying

$$\epsilon < \min\{\epsilon_0, \sigma_0 / (\sup_{h \in i\Omega_h^1(Y)} 2\|\operatorname{grad} f_H(h)\|_{C^0})\}$$

so that critical points and finite-type gradient trajectories of  $\mathcal{L}_{\delta, \epsilon}$  lie in  $i\Omega_h^1(Y)$  and the family of operators

$$2(\not{D}_{A_0 + h} + \delta)^2 + \epsilon \rho(\operatorname{grad} f_H(h))$$

is positive definite. Consequently, we can apply [Theorem 7.3](#) and compute Conley indices of the linearized flow.

Let us focus on the case of  $\underline{\operatorname{SWF}}^A(Y, \mathfrak{s}; S^1)$  as the other cases are almost identical. To simplify the calculation, we further deform the family of linear operators  $\not{D}_{A_0 + h} + \delta$  through a family  $\not{D}_{A_0 + (1-s)h} + \delta$  for  $0 \leq s \leq 1$ . We see that, in fact,

$$2(\not{D}_{A_0 + (1-s)h} + \delta)^2 + \epsilon \rho(\operatorname{grad} f_H(h))$$

is positive definite for all  $s \in [0, 1]$  and  $h \in i\Omega_h^1(Y)$ . Therefore, [Lemma 7.4](#) allows us to consider an approximated linearized flow  $\tilde{\varphi}_{n,1}^n$  associated with the constant family  $\not{D}_{A_0} + \delta$ . This flow actually splits into the product of the following three flows:

- (1) the negative gradient flow of  $\epsilon f_H$  on  $i\Omega_h^1(Y)$ ;
- (2) the linear flow on  $L_k^2(\text{im}(d^*)) \cap V_{\lambda_n}^{\mu_n}$  associated with the operator  $-(\ast d)$ ;
- (3) the linear flow on  $L_k^2(\Gamma(S_Y)) \cap V_{\lambda_n}^{\mu_n}$  associated with the operator  $-(\not{D} + \delta)$ .

From this, we see that

$$I_{S^1}(\bar{\varphi}_{m,1}^n, \text{inv}(\bar{J}_m^{n,+})) \cong (\bar{V}_{\lambda_n}^{0,-\delta})^+.$$

Moreover, the attractor map  $I_{S^1}(\bar{\varphi}_{m,0}^n, \text{inv}(\bar{J}_m^{n,+})) \rightarrow I_{S^1}(\bar{\varphi}_{m,1}^n, \text{inv}(\bar{J}_{m+1}^{n,+}))$  is just the identity map. Hence,  $\underline{\text{SWF}}^A(Y, \mathfrak{s}; S^1) \cong (S^0, 0, n(Y, \mathfrak{s}, A_0, g) + m(\delta))$ .  $\square$

**Example 7.6** ( $S^2 \times S^1$ ) Let  $\mathfrak{s}$  be the unique torsion  $\text{spin}^c$  structure on  $Y = S^2 \times S^1$  and  $g$  a Riemannian metric with constant positive scalar curvature. By the Weitzenböck formula, the triple  $(Y, \mathfrak{s}, g)$  satisfies the conditions in [Theorem 7.5](#) with  $\delta = 0$ . Therefore, we only need to compute the number  $n(Y, \mathfrak{s}, A_0, g)$  to completely describe the spectrum invariants. We choose the base connection  $A_0$  such that the induced connection  $A_0^t$  is flat with trivial holonomy (up to gauge transformations, there are two choices of such a connection and we pick any one of them). There exists an orientation reversing isometry on  $Y$  preserving  $(\mathfrak{s}, A_0)$ , so  $\mathfrak{s}$  and  $A_0$  correspond in a natural way to a  $\text{spin}^c$  structure and a  $\text{spin}^c$  connection on  $-Y$ . This implies that  $n(Y, \mathfrak{s}, A_0, g) = n(-Y, \mathfrak{s}, A_0, g)$ . On the other hand, by formula [\(21\)](#), we have  $n(Y, \mathfrak{s}, A_0, g) + n(-Y, \mathfrak{s}, A_0, g) = \dim_{\mathbb{C}}(\ker \not{D}) = 0$  as  $\not{D}$  has no kernel. Therefore, we get  $n(Y, \mathfrak{s}, A_0, g) = 0$  and conclude that

$$\underline{\text{SWF}}^A(S^2 \times S^1, \mathfrak{s}; S^1) \cong (S^0, 0, 0), \quad \underline{\text{SWF}}^R(S^2 \times S^1, \mathfrak{s}; S^1) \cong (S^0, 0, 0).$$

Notice that  $\mathfrak{s}$  can be lifted to a spin structure in two ways, denoted by  $\mathfrak{s}^j$  ( $j = 1, 2$ ). Then we have

$$\underline{\text{SWF}}^A(S^2 \times S^1, \mathfrak{s}^j; \text{Pin}(2)) \cong (S^0, 0, 0), \quad \underline{\text{SWF}}^R(S^2 \times S^1, \mathfrak{s}^j; \text{Pin}(2)) \cong (S^0, 0, 0).$$

## 8 More examples

### 8.1 The Seiberg–Witten monopoles on Seifert fibered spaces

When  $Y$  is a Seifert fibered space, the solutions of the Seiberg–Witten equations on  $Y$ , which are also called the Seiberg–Witten monopoles, are explicitly described by Mrowka, Ozsváth and Yu in [\[26\]](#). In this subsection, we will review their setup and some of their results that will be useful for us.

Let  $\Sigma$  be an oriented 2-dimensional orbifold of genus  $g$  with  $n$  marked points whose associated multiplicities are  $\alpha_1, \dots, \alpha_n$ . Its Euler characteristic is given by

$$\chi(\Sigma) = 2 - 2g - \sum_{j=1}^n \left( \frac{1}{\alpha_j} - 1 \right).$$

An orbifold Hermitian line bundle  $N$  over  $\Sigma$  can be specified by a set of integers  $(b, \beta_1, \dots, \beta_n)$  with  $0 \leq \beta_j < \alpha_j$ . The orbifold degree of this bundle is given by

$$\text{deg}(N) = b + \sum_{j=1}^n \frac{\beta_j}{\alpha_j}.$$

From now on, suppose that the integer  $\beta_j$  is coprime to  $\alpha_j$  for every  $j$ . Then the unit circle bundle  $S(N)$  is naturally a smooth 3-manifold, called a Seifert fibered space. We will also denote the manifold  $S(N)$  by  $Y$  and we have

$$\begin{aligned} H^1(Y; \mathbb{Z}) &= \begin{cases} H^1(\Sigma; \mathbb{Z}) & \text{if } \text{deg}(N) \neq 0, \\ H^1(\Sigma; \mathbb{Z}) \oplus \mathbb{Z} & \text{if } \text{deg}(N) = 0, \end{cases} \\ H^2(Y; \mathbb{Z}) &= (\text{Pic}^t(\Sigma) / \mathbb{Z}[N]) \oplus \mathbb{Z}^{2g}. \end{aligned}$$

Here  $\text{Pic}^t(\Sigma)$  denotes the topological Picard group of  $\Sigma$ , ie the group of topological isomorphism classes of orbifold line bundles over  $\Sigma$ , and the subgroup  $\text{Pic}^t(\Sigma) / \mathbb{Z}[N] \subset H^2(S(N); \mathbb{Z})$  is identified by the image of the pullback of the projection  $p: Y \rightarrow \Sigma$  via

$$\text{Pic}^t(\Sigma) \xrightarrow{p^*} [\text{line bundles over } Y] \xrightarrow{c_1} H^2(Y; \mathbb{Z}).$$

Let  $g_\Sigma$  be a constant-curvature metric on  $\Sigma$  with volume  $\pi$ . For a positive real  $r$ , we have a natural metric  $g_r$  on  $Y$  given by  $r^2\alpha^2 \oplus p^*(g_\Sigma)$ , where  $i\alpha \in i\Omega^1(Y)$  is a constant-curvature connection on  $S(N)$ . We will only have to pick  $r$  to be sufficiently small to use the computation in [27]. Instead of the Levi-Civita connection  $\nabla^{\text{LC}}$ , a connection  $\nabla^\circ$  on  $TY$  which is trivial in the fiber direction and equals the pullback of the Levi-Civita connection on  $\Sigma$  when restricted to  $\ker \alpha$  is used in [26]. For any  $\text{spin}^c$  structure  $\mathfrak{s}$  with spinor bundle  $S_Y$ , there is a natural one-to-one correspondence between connections on  $S_Y$  spinorial with respect to  $\nabla^{\text{LC}}$  and connections on  $S_Y$  spinorial with respect to  $\nabla^\circ$  by identifying those inducing the same connection on  $\det(S_Y)$ . From [26, Lemma 5.2.1], we have an identity

$$(46) \quad \tilde{D}_{\tilde{A}} = D_A + \delta_r,$$



where  $\tilde{D}$  is the Dirac operator induced by  $\nabla^\circ$  and  $\tilde{A}$  is the  $\nabla^\circ$ -spinorial connection corresponding to  $A$  and  $\delta_r = \frac{1}{2}r \deg(N)$  is a constant. Therefore, under this identification, solutions of the Seiberg–Witten equations described in [26] actually correspond to critical points of the functional  $\text{CSD} + \frac{1}{2}\delta_r \|\phi\|_{L^2}^2$ .

In addition, we have a canonical  $\text{spin}^c$  structure  $\mathfrak{s}_0$  on  $Y$  with spinor bundle  $S_Y^0 \cong p^*(K_\Sigma^{-1}) \oplus \mathbb{C}$ , where  $K_\Sigma$  is the canonical bundle of  $\Sigma$  (recall that  $\deg(K_\Sigma) = -\chi(\Sigma)$ ). When  $\deg(N)$  is nonzero, the  $\text{spin}^c$  structure  $\mathfrak{s}_0$  is torsion. Moreover,  $\text{Pic}^t(\Sigma)/\mathbb{Z}[N]$  is a finite abelian group and there is a one-to-one correspondence between  $\text{Pic}^t(\Sigma)/\mathbb{Z}[N]$  and the set of torsion  $\text{spin}^c$  structures by identifying  $[E_0]$  with  $p^*(E_0) \otimes S_Y^0$ . With this understood, the following proposition is a special case of Theorem 5.9.1 and Corollary 5.8.5 of [26].

**Proposition 8.1** *Let  $Y = S(N)$  be a Seifert fibered space corresponding to an orbifold line bundle  $N$  over  $\Sigma$  with nonzero degree. Let  $Y$  be equipped with a torsion  $\text{spin}^c$  structure induced by an orbifold line bundle  $E_0$  over  $\Sigma$ . Then we have the following:*

- (1) *The functional  $\text{CSD} + \frac{1}{2}\delta_r \|\phi\|_{L^2}^2$  has only reducible critical points if there is no orbifold line bundle  $E$  over  $\Sigma$  such that  $[E] \equiv [E_0] \pmod{\mathbb{Z}[N]}$  and  $0 \leq \deg(E) < -\frac{1}{2}\chi(\Sigma)$ .*
- (2) *The operator  $\tilde{D}_A + \delta_r$  has no kernel for any flat connection  $A$  if there is no orbifold line bundle  $E$  over  $\Sigma$  such that  $[E] \equiv [E_0] \pmod{\mathbb{Z}[N]}$  and  $\deg(E) = -\frac{1}{2}\chi(\Sigma)$ .*

Later, we will also consider some examples of  $S(N)$  with  $\deg(N) = 0$ , specifically, those manifolds with flat metric ( $\deg(N) = \chi(\Sigma) = 0$ ). By the Weitzenböck formula, the (unperturbed) Chern–Simons–Dirac functionals on these manifolds also have no irreducible critical points.

## 8.2 Large-degree circle bundles over a surface

Let  $\Sigma$  be a smooth surface of genus  $g > 0$  and  $N_d$  be the complex line bundle with degree  $d > 0$ . As we explained in the last subsection, the torsion  $\text{spin}^c$  structures on  $Y = S(N_d)$  can be parametrized by  $\mathbb{Z}/d\mathbb{Z}$  in a natural way. We denote them by  $\mathfrak{s}_0, \mathfrak{s}_1, \dots, \mathfrak{s}_{d-1}$  accordingly. In this subsection, we consider the torsion  $\text{spin}^c$  structures  $\mathfrak{s}_q$  such that  $q \geq g$ . By Proposition 8.1, we see that the triple  $(Y, \mathfrak{s}_q, g_r)$  satisfies the conditions of Theorem 7.5 with  $\delta = \delta_r = \frac{1}{2}rd$ .

To describe the spectrum invariants, we will calculate  $n(Y, \mathfrak{s}_q, A_0, g_r) + m(\not{D}, \delta_r)$  following the approach from [27; 28]. We introduce a family of connections on  $TY$  parametrized by  $s \in [0, 1]$  given by

$$\nabla^s = (1 - s)\nabla^\circ + s\nabla^{\text{LC}},$$

where  $\nabla^\circ$  and  $\nabla^{\text{LC}}$  are described earlier. Let  $A_{0,s}$  be the connection on  $S_Y$  which is spinorial with respect to  $\nabla^s$  and induces the same connection as  $A_0^t$  on  $\det(S_Y)$  and let  $\not{D}^s$  be the Dirac operator corresponding to  $A_{0,s}$ . From (46), we see that

$$(47) \quad \not{D}^s = (1 - s)\tilde{\not{D}} + s\not{D} = \not{D} + (1 - s)\delta_r.$$

Now consider the cylinder 4-manifold  $X = I \times Y$ . The family of connections  $\{\nabla^s\}$  induces a connection  $\hat{\nabla}$  on  $TX$  (with temporal gauge). Similarly, the family of connections  $\{A_{0,s}\}$  induces a connection  $\hat{A}$  on the spinor bundle  $S_X$ . Let  $\hat{\not{D}}^+$  be the positive Dirac operator coupled with  $\hat{A}$ . We have the following observations:

- (1) The operator  $\tilde{\not{D}}$  has no kernel (by Proposition 8.1).
- (2) Let  $\hat{A}^t$  be the induced connection on  $\det(S_X^+)$ . Then we see that  $F_{\hat{A}^t}|_{\{s\} \times Y} = 0$  for any  $s \in [0, 1]$ , which implies  $F_{\hat{A}^t} \wedge F_{\hat{A}^t} = 0$ .
- (3) As pointed out in [27, Remark 2.10], although  $\hat{\nabla}$  is not the Levi-Civita connection for the product metric on  $X$ , we can still use the Atiyah–Patodi–Singer index theorem [2] to calculate the index of  $\hat{\not{D}}^+$ .

As a result, the Atiyah–Patodi–Singer theorem<sup>1</sup> gives

$$(48) \quad \text{Ind}_{\mathbb{C}}(\hat{\not{D}}^+) = -\frac{1}{24} \int_X p_1(\hat{A}) + \frac{1}{2}(\eta(\not{D}) - \dim_{\mathbb{C}}(\ker \not{D})) - \frac{1}{2}\eta(\tilde{\not{D}}),$$

where  $p_1(\hat{A})$  is the first Pontryagin form for  $\hat{A}$ .

On the other hand, the operator  $\hat{\not{D}}^+$  can be written as  $\frac{d}{ds} + \not{D}^s$  under suitable bundle identification. Then the index equals the spectral flow

$$(49) \quad \text{Ind}_{\mathbb{C}}(\hat{\not{D}}^+) = -(\# \text{ of eigenvalues of } \not{D} \text{ in } [-\delta_r, 0]) = -m(\not{D}, \delta_r),$$

where we note that  $-\delta_r$  is not an eigenvalue of  $\not{D}$ . Recall that

$$n(Y, \mathfrak{s}_q, A_0, g_r) = \frac{1}{2}(\eta(\not{D}) - \dim_{\mathbb{C}}(\ker \not{D})) + \frac{1}{8}\eta_{\text{sign}}.$$

<sup>1</sup>The sign convention of this index formula in [22] is different from that in [2] and [27] because  $Y$  is oriented in different ways. For consistency, here we follow the convention in [22].

Combining this with (48) and (49), we get

$$m(\mathbb{D}, \delta_r) + n(Y, \mathfrak{s}_q, A_0, g_r) = \frac{1}{2}\eta(\tilde{\mathbb{D}}) + \frac{1}{24} \int_X p_1(\hat{A}) + \frac{1}{8} \eta_{\text{sign}}.$$

Here we assume that  $r$  is sufficiently small and apply Lemma 2.3, formula (2.22) and Appendix A of [27] to obtain

$$\eta(\tilde{\mathbb{D}}) = \frac{d}{6} + \frac{(g-1-q)(d+g-1-q)}{d},$$

$$\frac{1}{24} \int_X p_1(\hat{A}) = \frac{d}{12}(d^2r^4 - (2-2g)r^2).$$

From [27, page 108; 29], we also have

$$\frac{1}{8} \eta_{\text{sign}} = \frac{1}{24}(d-3) - \frac{1}{12}d(d^2r^4 - (2-2g)r^2).$$

Combining the above formulas, we can conclude

$$n(Y, \mathfrak{s}_q, A_0, g_r) + m(\delta_r) = \frac{d-1}{8} + \frac{(g-1-q)(d+g-1-q)}{2d}.$$

In summary, for  $0 < g \leq q < d$ , we have

$$\underline{\text{SWF}}^A(S(N_d), \mathfrak{s}_q; S^1) \cong (S^0, 0, c(g, d, q)),$$

$$\underline{\text{SWF}}^R(S(N_d), \mathfrak{s}_q; S^1) \cong (S^0, 0, c(g, d, q)),$$

where  $c(g, d, q)$  is the number  $\frac{1}{8}(d-1) + (g-1-q)(d+g-1-q)/(2d)$  above.

### 8.3 Circle bundles over torus and other nil manifolds

When  $Y = S(N)$  where  $p: N \rightarrow \Sigma$  is a complex line bundle over an orbifold  $\Sigma$  with  $\chi(\Sigma) = 0$  and  $\text{deg}(N) = d \neq 0$ , the manifold  $Y$  supports the so-called “nil-geometry”. For simplicity, will assume that  $d > 0$  and  $\Sigma$  is orientable. The case  $d < 0$  is similar (see the remark after formula (61)) and the nonorientable case can be done by passing to a suitable (orientable) double cover. Our main focus will be the case when  $\Sigma$  is smooth, ie  $Y$  is a circle bundle over a torus.

**8.3.1 Preparation** Let  $Y$  be a circle bundle over a torus. In this case, the torsion  $\text{spin}^c$  structures of  $Y$  can be parametrized by  $\mathbb{Z}/d\mathbb{Z}$  denoted by  $\mathfrak{s}_0, \dots, \mathfrak{s}_{d-1}$ . The spectrum invariants for  $\mathfrak{s}_q$  with  $1 \leq q \leq d-1$  are already calculated in Section 8.2 and we will only consider the  $\text{spin}^c$  structure  $\mathfrak{s}_0$  here. As earlier,  $Y$  is equipped with a canonical metric  $g_r$  and a canonical spinor bundle  $S_Y^0$ .

Let the torus  $\Sigma$  be given by  $(\mathbb{R}/\sqrt{\pi}\mathbb{Z}) \times (\mathbb{R}/\sqrt{\pi}\mathbb{Z})$  with coordinate  $(x_1, x_2)$ . To identify  $i\Omega_h^1(Y)$  with  $\mathbb{R}^2$ , we choose harmonic forms  $h_j$  to be  $2i\sqrt{\pi} p^*(dx_j)$ . We have a canonical trivialization of  $K_\Sigma$ , which also induces a trivialization of  $S_Y^0 \cong \underline{\mathbb{C}} \oplus p^*(K_\Sigma^{-1})$  and a trivialization of  $\det(S_Y^0) \cong \underline{\mathbb{C}}$ , where  $\underline{\mathbb{C}}$  is the trivial vector bundle on  $Y$  with fiber  $\mathbb{C}$ . Under this trivialization, the Clifford multiplication is given by

$$\rho(h_1) = \begin{pmatrix} 0 & -2i\sqrt{\pi} \\ 2i\sqrt{\pi} & 0 \end{pmatrix}, \quad \rho(h_2) = \begin{pmatrix} 0 & -2\sqrt{\pi} \\ -2\sqrt{\pi} & 0 \end{pmatrix}.$$

We set the base connection  $A_0$  to be the connection which is spinorial with respect to  $\nabla^{LC}$  and induces the trivial connection on  $\det(S_Y^0)$ . Then the corresponding connection  $\tilde{A}_0$  spinorial with respect to  $\nabla^\circ$  is just the trivial connection on  $S_Y^0$ .

Let  $\Gamma_c(S_Y^0)$  be the subspace of  $\Gamma(S_Y^0)$  consisting of sections which are constant along each fiber of  $Y = S(N)$ . We see that  $\Gamma_c(S_Y^0)$  is the same as a space of function from  $\mathbb{T}^2$  to  $\mathbb{C}^2$ , so Fourier series give an  $L^2$ -orthogonal decomposition

$$(50) \quad \Gamma_c(S_Y^0) = \bigoplus_{\vec{v} \in \mathbb{Z}^2} V_{\vec{v}},$$

where  $V_{\vec{v}}$  is the two-dimensional vector space spanned by  $\phi_{\vec{v},+}(x) = (e^{2\sqrt{\pi}i\langle x, \vec{v} \rangle}, 0)$  and  $\phi_{\vec{v},-}(x) = (0, e^{2\sqrt{\pi}i\langle x, \vec{v} \rangle})$ . We also have an orthogonal decomposition

$$(51) \quad \Gamma(S_Y^0) = \Gamma_c(S_Y^0) \oplus \Gamma_0(S_Y^0),$$

where  $\Gamma_0(S_Y^0)$  is the subspace of  $\Gamma(S_Y^0)$  consisting of sections which integrate to 0 along each fiber of  $S(N)$ . We have the following observations regarding the Dirac operators and these subspaces.

**Lemma 8.2** For any  $h \in i\Omega_h^1(Y)$ , we consider Dirac operators  $\tilde{D}_{\tilde{A}_0+h} = \tilde{D}_{A_0+h} + \delta_r$ .

- (1) The operator  $\tilde{D}_{\tilde{A}_0+h}$  preserves the subspaces  $\Gamma_0(S_Y^0)$  and  $V_{\vec{v}}$ .
- (2) The operator  $\tilde{D}_{\tilde{A}_0+h}$  has no kernel when restricted to  $L_k^2(\Gamma_0(S_Y^0))$ .
- (3) When  $h = \theta_1 h_1 + \theta_2 h_2$  and  $\vec{v} = (v_1, v_2)$ , the operator  $\tilde{D}_{\tilde{A}_0+h}$  restricted to  $V_{\vec{v}}$  is given by a matrix

$$\begin{pmatrix} 0 & -2\sqrt{\pi}(\theta_2 + v_2) - 2\sqrt{\pi}(\theta_1 + v_1)i \\ -2\sqrt{\pi}(\theta_2 + v_2) + 2\sqrt{\pi}(\theta_1 + v_1)i & 0 \end{pmatrix}.$$

**Proof** Statement (2) is implied by the proof of Proposition 5.8.4 of [26]. The other statements can be verified by simple calculation. □

As in Section 7, we will consider a functional of the form

$$\mathcal{L}_{\delta,\epsilon} := \text{CSD} + \frac{1}{2}\delta\|\phi\|_{L^2}^2 + \epsilon f_H(p_{\mathcal{H}}(a, \phi)),$$

where  $f_H(\theta_1, \theta_2) = -(1/(2\sqrt{2\pi}))(\cos 2\pi\theta_1 + \cos 2\pi\theta_2)$ . From Proposition 8.1, the functional  $\mathcal{L}_{\delta_r,0}$  only has reducible critical points (recall that  $\delta_r = \frac{1}{2}rd$ ). Although, the operator  $\tilde{\mathcal{D}}_{A_0+h} + \delta_r$  has nontrivial kernel for some  $h \in i\Omega_h^1(Y)$ , we can construct a suitable perturbation on  $\mathcal{L}_{\delta_r,0}$  which allows us to apply Theorem 7.3. The first step is the following lemma; the proof is almost identical to the proof of [17, Proposition 37.1.1], and we omit it.

**Lemma 8.3** *There exists  $\delta_1 > 0$  such that, for any  $\delta \in (0, \delta_1)$ , there exists  $\epsilon_1(\delta) > 0$  such that the functional*

$$\mathcal{L}_{\delta_r-\delta,\epsilon} = \mathcal{L}_{\delta_r,0} - \frac{1}{2}\delta\|\phi\|_{L^2}^2 + \epsilon f_H(p_{\mathcal{H}}(a, \phi))$$

*has only reducible critical points for all  $\epsilon \in (0, \epsilon_1(\delta))$ .*

We also have the following lemma:

**Lemma 8.4** *There exists  $\delta_2 > 0$  such that, for any  $\delta \in (0, \delta_2)$ , there exists  $\epsilon_2(\delta) > 0$  such that*

$$(52) \quad \langle (2(\tilde{\mathcal{D}}_{\tilde{A}_0+h} - \delta)^2 + \epsilon\rho(\text{grad } f_H(h)))\phi, \phi \rangle_{L^2} \geq C(\delta, \epsilon)\|\phi\|_{L^2}^2$$

*for all  $\epsilon \in (0, \epsilon_2(\delta))$ ,  $h \in i\Omega_h^1(Y)$  and  $\phi \in L_k^2(\Gamma(S_Y^0))$ , and  $C(\delta, \epsilon)$  is a positive constant depending only on  $\delta, \epsilon$ .*

**Proof** Since we have an  $L^2$ -orthogonal decomposition

$$\Gamma(S_Y^0) = \Gamma_0(S_Y^0) \oplus \bigoplus_{\vec{v} \in \mathbb{Z}^2} V_{\vec{v}},$$

which is preserved by both  $\tilde{\mathcal{D}}_{\tilde{A}_0+h} - \delta$  and  $\rho(\text{grad } f_H(h))$ , we just need to prove the statement on each of the summand.

The case  $L_k^2(\Gamma_0(S_Y^0))$  is easy. By Lemma 8.2 and the compactness of  $i\Omega_h^1(Y)/\mathcal{G}_Y^{h,o}$ , we can find a constant  $\sigma_1 > 0$  such that, for any  $h \in i\Omega_h^1(Y)$ , the restriction of  $\tilde{\mathcal{D}}_{\tilde{A}_0+h}$  to  $L_k^2(\Gamma_0(S_Y^0))$  has no eigenvalue lying in  $(-2\sigma_1, 2\sigma_1)$ . Therefore, for any  $0 < \delta < \sigma_1$  and  $0 < \epsilon < \sigma_1^2$ , we have

$$\langle (2(\tilde{\mathcal{D}}_{\tilde{A}_0+h} - \delta)^2 + \epsilon\rho(\text{grad } f_H(h)))\phi, \phi \rangle_{L^2} \geq \sigma_1^2\|\phi\|_{L^2}^2 \quad \text{for all } \phi \in L_k^2(\Gamma_0(S_Y^0)).$$

For the subspace  $V_{\vec{v}}$ , we first check the case  $\vec{v} = \vec{0}$ . For  $h = \theta_1 h_1 + \theta_2 h_2$ , by Lemma 8.2, the matrix of  $(\tilde{\mathcal{D}}_{\tilde{A}_0+h} - \delta)|_{V_{\vec{0}}}$  is given by

$$\begin{pmatrix} -\delta & -2\sqrt{\pi}(\theta_2 + \theta_1 i) \\ -2\sqrt{\pi}(\theta_2 - \theta_1 i) & -\delta \end{pmatrix}.$$

The eigenvalues of this matrix are  $-\delta \pm 2\sqrt{\pi(\theta_1^2 + \theta_2^2)}$ . The kernel of  $\tilde{\mathcal{D}}_{\tilde{A}_0+h} - \delta$  is nontrivial if and only if  $h$  is on the sphere  $S_{\vec{0}}$  of radius  $\delta/(2\sqrt{\pi})$  centered at 0. Let  $S(V_{\vec{0}})$  be the unit sphere in  $V_{\vec{0}}$ . We notice that

$$(53) \quad \langle 2(\tilde{\mathcal{D}}_{\tilde{A}_0+h} - \delta)^2 \phi, \phi \rangle_{L^2} \geq 0 \quad \text{for any } (h, \phi) \in i\Omega_h^1(Y) \times S(V_{\vec{0}}),$$

where equality holds if and only if  $(h, \phi)$  belongs to the compact set

$$K := \{(h, \phi) \in S_{\vec{0}} \times S(V_{\vec{0}}) \mid (\tilde{\mathcal{D}}_{\tilde{A}_0+h} - \delta)\phi = 0\}.$$

For any  $(h, \phi) \in K$ , we consider the negative gradient flow line  $\gamma$  of  $f_H$  with  $\gamma(0) = h$ . Here we choose  $\delta < \sqrt{\pi}$  so that  $\langle h, \gamma'(0) \rangle < 0$ , ie  $\gamma$  goes inside  $S_{\vec{0}}$  at  $h$ . For  $t > 0$ , we see that the operator  $(\tilde{\mathcal{D}}_{\tilde{A}_0+\gamma(t)} - \delta)|_{V_{\vec{0}}}$  has eigenvalues  $-\delta \pm 2\sqrt{\pi} \|\gamma(t)\|$ , which are both negative. Then the function  $-\delta + 2\sqrt{\pi} \|\gamma(t)\| - \langle (\tilde{\mathcal{D}}_{\tilde{A}_0+\gamma(t)} - \delta)\phi, \phi \rangle$  is nonnegative for  $t \geq 0$ . Since the value of this function is 0 at  $t = 0$ , its derivative at  $t = 0$  must be nonnegative as well. After computing the derivative, we have  $\langle \rho(\text{grad } f_H(h))\phi, \phi \rangle > -\langle h, \gamma'(0) \rangle$ . Therefore, we can conclude that

$$(54) \quad \langle \rho(\text{grad } f_H(h))\phi, \phi \rangle_{L^2} > 0 \quad \text{for any } (h, \phi) \in K.$$

Now we can find a small neighborhood  $U \subset i\Omega_h^1(Y) \times S(V_{\vec{0}})$  of  $K$  and a positive number  $\sigma_2$  such that

$$\langle \rho(\text{grad } f_H(h))\phi, \phi \rangle_{L^2} > \sigma_2 \quad \text{for any } (h, \phi) \in U.$$

Similar to the case of  $L_k^2(\Gamma_0(S_Y^0))$ , there exists  $\sigma_3 > 0$  such that

$$\langle 2(\tilde{\mathcal{D}}_{\tilde{A}_0+h} - \delta)^2 \phi, \phi \rangle_{L^2} \geq 2\sigma_3^2$$

for any  $\delta \in (0, \sigma_3)$  and  $(h, \phi) \in i\Omega_h^1(Y) \times S(V_{\vec{0}}) \setminus U$ . Let  $\epsilon > 0$  be a positive number such that

$$\epsilon \mid \langle \rho(\text{grad } f_H(h))\phi, \phi \rangle_{L^2} \mid < \sigma_3^2$$

for any  $(h, \phi) \in i\Omega_h^1(Y) \times S(V_{\vec{0}})$ . Then

$$\langle (2(\tilde{\mathcal{D}}_{\tilde{A}_0+h} - \delta)^2 + \epsilon \rho(\text{grad } f_H(h)))\phi, \phi \rangle_{L^2} \geq \epsilon \sigma_2$$

for any  $(h, \phi) \in i\Omega_h^1(Y) \times S(V_{\tilde{V}_0})$ . By applying elements in  $\mathcal{G}^{h,o}$ , we see that similar estimate (with the same constants) holds for general  $V_{\tilde{V}_0}$ . This proves the lemma.  $\square$

**Remark** For any function  $\xi: Y \rightarrow \mathbb{R}$ , by applying the gauge transformation  $u = e^{-i\xi}$  on (52), we also get  $\langle (2(\tilde{D}_{\tilde{A}_0+h+id\xi} - \delta)^2 + \epsilon\rho(\text{grad } f_H(h)))\phi, \phi \rangle_{L^2} > C(\delta, \epsilon)\|\phi\|_{L^2}^2$ . This observation will be useful soon.

Fix a choice of constants  $(\delta, \epsilon)$  with  $0 < \delta < \min(\delta_1, \delta_2)$  and  $0 < \epsilon < \min(\epsilon_1(\delta), \epsilon_2(\delta))$ , where  $\delta_j$  and  $\epsilon_j(\delta)$  are the constants from Lemma 8.3 and Lemma 8.4. All critical points  $\mathcal{L}_{\delta_r-\delta, \epsilon}$  are then reducible, so they are of the form  $(h_{p,q}, 0) \in \text{Coul}(Y)$  where  $h_{p,q} = \frac{1}{2}(ph_1 + qh_2)$  for each  $p, q \in \mathbb{Z}$ . Modulo the action of the whole gauge group, there are four equivalent classes  $[(h_{0,0}, 0)]$ ,  $[(h_{0,1}, 0)]$ ,  $[(h_{1,0}, 0)]$  and  $[(h_{1,1}, 0)]$ . The relative gradings between them are given by

$$(55) \quad \begin{aligned} \text{gr}([(h_{1,1}, 0)], [(h_{1,0}, 0)]) &= \text{gr}([(h_{1,1}, 0)], [(h_{0,1}, 0)]) = 1, \\ \text{gr}([(h_{1,0}, 0)], [(h_{0,0}, 0)]) &= \text{gr}([(h_{0,1}, 0)], [(h_{0,0}, 0)]) = -1. \end{aligned}$$

Notice that  $\text{gr}([(h_{1,0}, 0)], [(h_{0,0}, 0)])$  and  $\text{gr}([(h_{0,1}, 0)], [(h_{0,0}, 0)])$  do not coincide with the relative grading for the Morse function  $f_H$  (which is 1). This is because of the appearance of the spectral flow.

There may still be finite-type irreducible trajectories between these critical points. Our next aim is to find a suitable perturbation  $\bar{f}$  to eliminate these trajectories. To do this, we need to use the results regarding the “blown-up moduli space”. We quickly describe the situation in our case and refer to [17] for the Morse case and [20] for the Morse–Bott case.

The blown-up quotient configuration space  $\mathcal{B}^\sigma(Y)$  is obtained by blowing up the quotient space  $\mathcal{B}(Y) = \mathcal{C}(Y)/\mathcal{G}_Y$  along the reducible locus. The vector field  $-\text{grad } \mathcal{L}_{\delta_r-\delta, \epsilon}$  can be lifted to a vector field on  $\mathcal{B}^\sigma(Y)$ , which we denote by  $-\text{grad}^\sigma \mathcal{L}_{\delta_r-\delta, \epsilon}$ . The set of zero locus of this vector field is given by the union of the critical manifolds  $[C_{p,q,\mu}]$  where  $\mu$  is an eigenvalue of  $\tilde{D}_{\tilde{A}_0+h_{p,q}} - \delta$  and  $p, q \in \{0, 1\}$ . The critical manifold  $[C_{p,q,\mu}]$ , lies in the preimage of  $[(h_{p,q}, 0)]$  under the blow-down map, has (real) dimension  $2 \times (\text{multiplicity of } \mu) - 2$ . This manifold is called boundary stable (resp. boundary unstable) if  $\mu > 0$  (resp.  $\mu < 0$ ). Note that  $\tilde{D}_{\tilde{A}_0+h_{p,q}} - \delta$  has no kernel.

Denote by  $\check{\mathcal{M}}([C_{p,q,\mu}], [C_{p',q',\mu'}])$  the moduli space of unparametrized trajectories going from  $[C_{p,q,\mu}]$  to  $[C_{p',q',\mu'}]$ . We will be interested in the expected dimension of

the moduli space  $\check{\mathcal{M}}([C_{p,q,\mu}], [C_{p',q',\mu'}])$  in the case  $\mu < 0 < \mu'$ . From [20, Chapter 2, Proposition 3.12] (be careful with the notation there), we have

$$(56) \quad \dim(\check{\mathcal{M}}([C_{p,q,\mu}], [C_{p',q',\mu'}])) = \text{gr}([(h_{p,q}, 0)], [(h_{p',q'}, 0)]) - 2 \\ - 2 \#\{\text{eigenvalues of } \mathcal{D}_{A_0+h_{p,q}}+\delta_r-\delta \text{ in } (\mu, 0)\} \\ - 2 \#\{\text{eigenvalues of } \mathcal{D}_{A_0+h_{p,q}}+\delta_r-\delta \text{ in } (0, \mu')\}.$$

Observe that the dimension in this case is always negative by (55). We have the following result regarding these moduli spaces.

**Lemma 8.5** *There exists  $\epsilon' > 0$  such that, for any trajectory  $\gamma = (a, \phi): \mathbb{R} \rightarrow \text{Coul}(Y)$  representing a point of  $\check{\mathcal{M}}([C_{p,q,\mu}], [C_{p',q',\mu'}])$  with  $\mu < 0 < \mu'$ , we have*

$$\sup_{t \in \mathbb{R}} \|\phi(t)\|_{L^2} > 2\epsilon'.$$

**Proof** First, we show that a trajectory representing a point in  $\check{\mathcal{M}}([C_{p,q,\mu}], [C_{p',q',\mu'}])$  must be irreducible. A reducible point in  $\check{\mathcal{M}}([C_{p,q,\mu}], [C_{p',q',\mu'}])$  gives rise to a finite-type trajectory  $(\beta, h, \phi): \mathbb{R} \rightarrow i \text{im } d^* \oplus i \Omega_h^1(Y) \oplus \Gamma(S_Y^0)$  such that

$$(57) \quad \frac{d}{dt}(\beta(t), h(t), \phi(t)) = -( *d\beta(t), \epsilon \text{grad } f_H(h(t)), (\tilde{\mathcal{D}}_{\tilde{A}_0+h(t)} - \delta)\phi(t)),$$

where  $\|\phi(t)\|_{L^2} = 1$ . From Lemma 8.4, we can use Lemma 7.1 to conclude that there can be no such trajectory.

Now we can consider only irreducible trajectories. Suppose the contrary that there is a sequence of irreducible trajectories

$$\gamma_n(t) = (a_n(t), \phi_n(t)) \quad \text{with} \quad \lim_{n \rightarrow +\infty} (\sup_{t \in \mathbb{R}} \|\phi_n(t)\|_{L^2}) = 0.$$

As in the proof of Theorem 7.5, we can lift  $\gamma_n$  to a negative gradient flow line  $\tilde{\gamma}_n = (\tilde{a}_n, \tilde{\phi}_n): \mathbb{R} \rightarrow \mathcal{C}_Y$  of  $\mathcal{L}_{\delta_r - \delta_0, \epsilon}$ . Notice that the topological energy of  $\tilde{\gamma}_n$  is bounded above by  $\sqrt{2}\epsilon/\pi$ . By [17, Proposition 16.2.1], after passing to a subsequence and applying suitable gauge transformations, the sequence  $\tilde{\gamma}_n$  converges to a (possibly broken) flow line  $\tilde{\gamma}_\infty$ . From our assumption on the limit of  $\sup_{t \in \mathbb{R}} \|\phi_n(t)\|_{L^2}$ , the trajectory  $\tilde{\gamma}_\infty$  is reducible. Any such reducible flow line is contained in  $L_k^2(\ker d) \times \{0\}$ . In particular, if we decompose

$$\tilde{a}_n(t) = h_n(t) + \beta_n(t) + id\xi_n(t) \in i \Omega_h^1(Y) \oplus \text{im } d^* \oplus id\Omega^0(Y),$$

then  $\lim_{n \rightarrow +\infty} (\sup_{t \in \mathbb{R}} \|\beta_n(t)\|_{L_k^2}) = 0$ , and so  $\lim_{n \rightarrow +\infty} (\sup_{t \in \mathbb{R}} \|d\tilde{a}_n(t)\|_{C^0}) = 0$ .



From (45), we have

$$(58) \quad \frac{d^2}{dt^2} \|\phi_n(t)\|_{L^2}^2 = 2\langle (2(\tilde{\mathcal{D}}_{\tilde{A}_0 + \tilde{a}_n(t)} - \delta)^2 + \rho(\text{grad} \epsilon_{f_H}(h_n(t))))\tilde{\phi}_n(t), \tilde{\phi}_n(t) \rangle_{L^2} + \|\tilde{\phi}_n(t)\|_{L^2}^4 + 2\langle \rho(*d\tilde{a}_n(t))\tilde{\phi}_n(t), \tilde{\phi}_n(t) \rangle_{L^2}.$$

Notice that

$$\begin{aligned} & \langle (\tilde{\mathcal{D}}_{\tilde{A}_0 + \tilde{a}_n(t)} - \delta)^2 \tilde{\phi}_n(t), \tilde{\phi}_n(t) \rangle_{L^2} \\ &= \|(\tilde{\mathcal{D}}_{\tilde{A}_0 + \tilde{a}_n(t)} - \delta)\tilde{\phi}_n(t)\|_{L^2}^2 \\ &= \|(\tilde{\mathcal{D}}_{\tilde{A}_0 + h_n(t) + id\xi_n(t)} - \delta)\tilde{\phi}_n(t) + \rho(\beta_n(t))\tilde{\phi}_n(t)\|_{L^2}^2 \\ &\geq \langle (\tilde{\mathcal{D}}_{\tilde{A}_0 + h_n(t) + id\xi_n(t)} - \delta)^2 \tilde{\phi}_n(t), \tilde{\phi}_n(t) \rangle_{L^2} - \|\beta_n(t)\|_{C^0}^2 \|\tilde{\phi}_n(t)\|_{L^2}^2. \end{aligned}$$

This gives

$$(59) \quad \begin{aligned} & \frac{d^2}{dt^2} \|\phi_n(t)\|_{L^2}^2 \\ & \geq 2\langle (2(\tilde{\mathcal{D}}_{\tilde{A}_0 + h_n(t) + id\xi_n(t)} - \delta)^2 + \rho(\text{grad} \epsilon_{f_H}(h_n(t))))\tilde{\phi}_n(t), \tilde{\phi}_n(t) \rangle_{L^2} \\ & \quad + \|\tilde{\phi}_n(t)\|_{L^2}^4 + 2\langle \rho(*d\tilde{a}_n(t))\tilde{\phi}_n(t), \tilde{\phi}_n(t) \rangle_{L^2} - 4\|\beta_n(t)\|_{C^0}^2 \|\tilde{\phi}_n(t)\|_{L^2}^2 \\ & \geq (2C(\delta, \epsilon_3) - 2\|d\tilde{a}_n(t)\|_{C^0} - 4\|\beta_n(t)\|_{C^0}^2) \|\tilde{\phi}_n(t)\|_{L^2}^2, \end{aligned}$$

where we make use of remark after Lemma 8.4. Therefore, when  $n$  is sufficiently large, the above inequality implies that  $\frac{d^2}{dt^2} \|\phi_n(t)\|_{L^2}^2 > 0$  for any  $t$ . This is impossible because  $\sup_{t \in \mathbb{R}} \|\phi_n(t)\|_{L^2}^2 < \infty$ .  $\square$

To construct the final perturbation, we reintroduce the Banach space  $\mathcal{P}$  of extended cylinder functions (see Section 2). Define a subset  $O = \{(a, \phi) \in \mathcal{C}_Y \mid \|\phi\|_{L^2} < \epsilon'\}$ , where  $\epsilon'$  is the constant from Lemma 8.5, and  $\mathcal{P}_O = \{\bar{f} \in \mathcal{P} \mid \bar{f}|_O \equiv 0\}$  is a closed subspace of  $\mathcal{P}$ . By [17, Proposition 11.6.4], we can find an open neighborhood  $\mathcal{U}$  of 0 in  $\mathcal{P}_O$  such that for any  $\bar{f}$  in this neighborhood, the functional  $\mathcal{L}_{\delta_r - \delta, \epsilon} + \bar{f}$  has no critical points outside  $O$ . Therefore, the critical points are just  $(h_{p,q}, 0)$  for  $p, q \in \mathbb{Z}$  with the corresponding critical manifolds  $[C_{p,q,\mu}]$  as in the case of  $\mathcal{L}_{\delta_r - \delta, \epsilon}$ . Analogously, we denote by  $\check{\mathcal{M}}_{\bar{f}}([C_{p,q,\mu}], [C_{p',q',\mu'}])$  the moduli space of trajectories of  $\text{grad}^\sigma(\mathcal{L}_{\delta_r - \delta, \epsilon} + \bar{f})$ .

**Lemma 8.6** *For any critical manifolds  $[C_{p,q,\mu}]$  and  $[C_{p',q',\mu'}]$  with  $\mu < 0 < \mu'$ , there is a residue subset of  $\mathcal{P}_O$  such that the moduli space  $\check{\mathcal{M}}_{\bar{f}}([C_{p,q,\mu}], [C_{p',q',\mu'}])$  is empty.*

**Proof** Using the fact that the index of a Fredholm operator does not change under homotopy, it is easy to see that the expected dimension of  $\check{\mathcal{M}}_{\bar{f}}([C_{p,q,\mu}], [C_{p',q',\mu'}])$

does not depend on  $\bar{f}$ . In particular, it is always negative when  $\mu < 0 < \mu'$  by our discussion earlier. Therefore, we just need prove that  $\check{\mathcal{M}}_{\bar{f}}([C_{p,q,\mu}], [C_{p',q',\mu'}])$  is Smale regular; see [20, Definition 3.11]. The proof of this fact is very similar to the proofs of [17, Theorem 15.1.1; 20, Theorem 3.1.7]; ie one introduces a parametrized moduli space and then applies the Sard–Smale theorem.

The main difference is we require an extended cylinder function to satisfy  $\bar{f}(a, \phi) = 0$  as long as  $\|\phi\|_{L^2} < \epsilon'$  instead of asking  $\bar{f}$  to vanish in a small neighborhood of the critical manifolds. To see that this new requirement does not affect the result, we recall how the cylinder functions are constructed (see [17]): After choosing a set of sections  $\Phi_1, \dots, \Phi_m$  of the bundle  $\tilde{S}_Y := (i\Omega_h^1(Y) \times S_Y)/\mathcal{G}_Y^{h,o}$  over  $Y \times (H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z}))$  and a set of coexact forms  $a_1, \dots, a_n$ , we get a map

$$p_0: \mathcal{C}_Y \rightarrow \mathbb{C}^m \times (H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})) \times \mathbb{R}^n.$$

A cylinder function is obtained by composing  $p_0$  with a compactly supported function on  $\mathbb{C}^m \times (H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})) \times \mathbb{R}^n$ .

On the other hand, it is straightforward to deduce from Lemma 8.5 that any trajectory  $\gamma = (a, \phi): \mathbb{R} \rightarrow \mathcal{C}_Y$  representing a point in  $\check{\mathcal{M}}_{\bar{f}}([C_{p,q,\mu}], [C_{p',q',\mu'}])$  also satisfies  $\sup_{t \in \mathbb{R}} \|\phi(t)\|_{L^2} > \epsilon'$  for  $\bar{f} \in \mathcal{P}_O$ . We now pick a section of the bundle  $\tilde{S}_Y$  which equals  $\phi(t_0)/\|\phi(t_0)\|_{L^2}$  when restricted to  $Y \times \{\pi_{\mathcal{H}}a(t_0)\}$  and whose restriction to any  $Y \times \{*\}$  has unit  $L^2$ -norm. By specifying  $\Phi_1$  to be this section, we see that the image  $p_0(O)$  lies in the set

$$U := B(\epsilon') \times \mathbb{C}^{m-1} \times (H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})) \times \mathbb{R}^n,$$

where  $B(\epsilon')$  is the ball of radius  $\epsilon'$  in  $\mathbb{C}$ . Therefore, by composing  $p_0$  with a function which vanishes on  $U$ , we get a cylinder function which vanishes on  $O$ . Moreover, since  $p_0(\gamma) \not\subseteq U$ , this kind of cylinder functions are enough to repeat the argument on page 269 of [17] and prove the transversality result we need. □

The following result is now immediate from the previous lemma.

**Proposition 8.7** *There exists an extended cylinder function  $\bar{f} \in \mathcal{P}_O$  such that all finite-type gradient flow lines of the functional  $\mathcal{L}_{\delta_r, -\delta, \epsilon} + \bar{f}$  are contained in  $i\Omega_h^1(Y)$ .*

**Proof** Notice that any finite-type irreducible gradient flow line of  $\mathcal{L}_{\delta_0 - \delta, \epsilon_3} + \bar{f}$  gives a point in  $\check{\mathcal{M}}_{\bar{f}}([C_{p,q,\mu}], [C_{p',q',\mu'}])$  with  $\mu < 0 < \mu'$ . By Lemma 8.6, the moduli space is empty, so there are only reducible flow lines, which have to lie in  $i\Omega_h^1(Y)$ . □

**Remark** By reversing the orientation of  $Y$  and repeating the arguments above, we see that one can also choose  $\delta, \epsilon > 0$  and  $\bar{f}' \in \mathcal{P}_O$  such that gradient flow lines of  $\mathcal{L}_{\delta_r+\delta, -\epsilon} + \bar{f}'$  lie in  $i\Omega_h^1(Y)$ . This will be useful when we calculate  $\underline{\text{SWF}}^R$ .

**8.3.2 Computation of the invariants** We first consider the case  $\underline{\text{SWF}}^A(Y, \mathfrak{s}_0; S^1)$ .

**Proposition 8.7** and **Lemma 8.4** allow us to apply **Theorem 7.3** to work on the linearized flow given by

$$-\frac{d}{dt}(\beta(t), h(t), \phi(t)) = (*d\beta(t), \text{grad } \epsilon f_H(h(t)), \mathbf{D}(h(t))\phi(t)),$$

where  $\mathbf{D}(h) = \tilde{\mathcal{D}}_{\tilde{A}_0+h} - \delta$  with  $\delta, \epsilon$  and  $f_H$  previously chosen. Recall that we will have to compute Conley indices of finite-dimensional approximation of the flow on the bounded region  $\bar{J}_m^+ = \overline{\text{Str}}(\epsilon') \cap p_{\mathcal{H}}^{-1}([-\frac{1}{4}-m, \frac{1}{4}+m]^2)$ , where  $\epsilon'$  is from **Lemma 8.5** and we fix  $\theta^+ = \frac{1}{4}$ ; see **Section 7**.

To simplify the calculation of the corresponding Conley index, we consider subspaces

$$W = \bigoplus_{\tilde{v} \in \mathbb{Z}^2 \cap [-m, m]^2} V_{\tilde{v}}, \quad W' = \bigoplus_{\tilde{v} \in \mathbb{Z}^2 \setminus [-m, m]^2} V_{\tilde{v}},$$

From (50) and (51), we have an orthogonal decomposition  $\Gamma(S_Y^0) = W \oplus W' \oplus \Gamma_0(S_Y^0)$ . To deform  $\mathbf{D}(h)$ , we define a family of operators  $\mathbf{D}^s(h)$  parametrized by  $[0, 1] \times i\Omega_h^1(Y)$  as

$$\mathbf{D}^s(h)\phi = \begin{cases} (\tilde{\mathcal{D}}_{\tilde{A}_0+h} - \delta)\phi & \text{if } \phi \in W, \\ (\tilde{\mathcal{D}}_{\tilde{A}_0+sh} - \delta)\phi & \text{if } \phi \in W' \oplus \Gamma_0(S_Y^0). \end{cases}$$

Notice that for any  $h \in [-\frac{1}{4}-m, \frac{1}{4}+m]^2$ , the operator  $\mathbf{D}^s(h)$ , when restricted to  $W' \oplus \Gamma_0(S_Y^0)$ , has no eigenvalue in  $[-\epsilon_1, \epsilon_1]$ , where  $\epsilon_1$  is a constant independent of  $m$ . Therefore, by setting  $\epsilon$  small and applying **Lemma 7.4**, we can consider  $\mathbf{D}^0(h)$  instead of  $\mathbf{D}(h)$ .

Fix an integer  $n$  large enough so that  $W \subset V_{\lambda_n}^{\mu_n}$ . The approximated linearized flow on  $V_{\lambda_n}^{\mu_n}$  corresponding to  $\mathbf{D}^0(h)$  can be split into the product of the following flows:

- (1) the linear flow on  $(W' \oplus \Gamma_0(S_Y^0)) \cap V_{\lambda_n}^{\mu_n}$  given by the operator  $-(\tilde{\mathcal{D}}_{\tilde{A}_0} - \delta_3)$ ;
- (2) the linear flow on  $L_k^2(\text{im } d^*) \cap V_{\lambda_n}^{\mu_n}$  given by the operator  $-d^*$ ;
- (3) the flow  $\varphi$  on a finite-dimensional space  $i\Omega_h^1(Y) \oplus W$  generated by the vector field  $(-\epsilon \text{grad } f_H, -(\tilde{\mathcal{D}}_{\tilde{A}_0+h} - \delta))$ .

Since the first two flows are linear, the corresponding Conley indices are just  $S^0$  after suitable desuspension. Therefore, the stable Conley index  $\bar{I}_m^{n,+}$  is determined by the

third flow. More precisely, we have

$$\bar{I}_m^{n,+} = \Sigma^{-W^-} I_{S^1}(\varphi, \text{inv}(\bar{J}_m^+ \cap (i\Omega_h^1(Y) \oplus W))),$$

where  $W^- \subset W$  is spanned by the negative eigenvectors of  $(\tilde{D}_{\tilde{A}_0} - \delta)|_W$ .

By Lemma 8.4 and [13, Lemma 12], we can deduce that  $[-\frac{1}{4} - m, \frac{1}{4} + m]^2 \times B(W)$  is an isolating block for  $\text{inv}(\bar{J}_m^+ \cap (i\Omega_h^1(Y) \oplus W))$ , where  $B(W)$  is the unit ball inside  $W$ . Moreover, by [13, Lemma 4], we have  $I_{S^1}(\varphi, \text{inv}(\bar{J}_m^+ \cap (i\Omega_h^1(Y) \oplus W))) \cong \Sigma n^-$ , where  $n^-$  is the exit set of  $[-\frac{1}{4} - m, \frac{1}{4} + m]^2 \times B(W)$  with respect to the flow  $\varphi$ , and  $\Sigma$  denotes the unreduced suspension. By the definition of the flow  $\varphi$ , we see that

$$n^- = \{(h, \phi) \in [-\frac{1}{4} - m, \frac{1}{4} + m]^2 \times S(W) \mid \langle \phi, (\tilde{D}_{\tilde{A}_0+h} - \delta)\phi \rangle \leq 0\},$$

where  $S(W)$  is the unit sphere in  $W$ .

We now start to deform  $n^-$  to simpler spaces. Let  $W^-(h)$  be the space spanned by nonpositive eigenvectors of  $(\tilde{D}_{\tilde{A}_0+h} - \delta_3)|_W$ . We consider the subset

$$n_1^- := \{(h, \phi) \in n^- \mid \phi \in W^-(h)\}.$$

By Lemma 8.2, the operator  $(\tilde{D}_{\tilde{A}_0+h} - \delta)|_W$  can be represented by the matrix

$$\bigoplus_{-m \leq v_1, v_2 \leq m} \begin{pmatrix} lr - \delta & -2\sqrt{\pi}(\theta_2 + v_2) - 2\sqrt{\pi}(\theta_1 + v_1)i \\ -2\sqrt{\pi}(\theta_2 + v_2) + 2\sqrt{\pi}(\theta_1 + v_1)i & -\delta \end{pmatrix}.$$

Then we see that

$$S(W^-(h)) = \begin{cases} S(\mathbb{C}^{(2m+1)^2+1}) & \text{if } h \in B_{\bar{v}} \text{ for some } \bar{v} \in \mathbb{Z}^2 \cap [-m, m]^2, \\ S(\mathbb{C}^{(2m+1)^2}) & \text{otherwise,} \end{cases}$$

where  $B_{\bar{v}}$  denotes the ball in  $\mathbb{R}^2$  centered at  $\bar{v}$  with radius  $\delta/(2\sqrt{\pi})$ . A careful (but elementary) check shows that  $n_1^-$  is a deformation retract of  $n^-$ . Note that this deformation retraction does not preserve the projection  $p_{\mathcal{H}}$ .

To further deform  $n_1^-$ , we choose a point  $z_0 \in [-\frac{1}{4} - m, \frac{1}{4} + m]^2$  outside the union of balls  $\bigcup_{\bar{v} \in \mathbb{Z}^2 \cap [-m, m]^2} B_{\bar{v}}$  and connect  $z_0$  to each of  $B_{\bar{v}}$  by disjoint paths  $\gamma_{\bar{v}}$  containing in  $[-\frac{1}{4} - m, \frac{1}{4} + m]^2$ . It is not hard to see that the subset

$$(60) \quad \Lambda := \bigcup_{\bar{v} \in \mathbb{Z}^2 \cap [-m, m]^2} (\gamma_{\bar{v}} \cup B_{\bar{v}})$$

is a deformation retract of  $[-\frac{1}{4} - m, \frac{1}{4} + m]^2$ . This induces a deformation retract of  $n_1^-$  to  $n_2^- := n_1^- \cap p_1^{-1}(\Lambda)$ , where  $p_1: [-\frac{1}{4} - m, \frac{1}{4} + m]^2 \times B(W) \rightarrow [-\frac{1}{4} - m, \frac{1}{4} + m]^2$

is the projection. Since  $\Lambda$  is homotopy equivalent to the wedge sum of  $(2m + 1)^2$  balls, we see that  $n_2^-$  is homotopy equivalent to

$$n_3^- := \bigvee_{S(\{0\} \times \mathbb{C}^{(2m+1)^2})}^{(2m+1)^2} S(\mathbb{C}^{(2m+1)^2+1}).$$

The notation  $\bigvee_B^n X$  means the space obtained by gluing  $n$  copies of  $X$  along their subset  $B$ . Since  $W^- \cong \mathbb{C}^{(2m+1)^2+1}$ , we have

$$\bar{I}_m^{n,+} \cong \Sigma^{-\mathbb{C}^{(2m+1)^2+1}} \left( \bigvee_{(\{0\} \times \mathbb{C}^{(2m+1)^2})^+}^{(2m+1)^2} (\mathbb{C}^{(2m+1)^2+1})^+ \right) \cong \Sigma^{-\mathbb{C}} \left( \bigvee_{S^0}^{(2m+1)^2} \mathbb{C}^+ \right).$$

To determine the attractor maps, we notice that  $[-\frac{1}{4} - m + 1, \frac{1}{4} + m - 1]^2 \times B(W)$  is an isolating block for  $\text{inv}(J_{m-1}^+ \cap (i\Omega_h^1(Y) \oplus W))$  and the corresponding exiting set is just  $n^- \cap p_1^{-1}([-\frac{1}{4} - m + 1, \frac{1}{4} + m - 1]^2)$ . By this observation, we see that the morphism induced by the attractor map  $\bar{I}_{m-1}^{n,+} \rightarrow \bar{I}_m^{n,+}$  is just given by the desuspension of the inclusion  $\bigvee_{S^0}^{(2m-1)^2} \mathbb{C}^+ \rightarrow \bigvee_{S^0}^{(2m+1)^2} \mathbb{C}^+$ . Therefore, we have

$$\underline{\text{SWF}}^A(Y, \mathfrak{s}_0; S^1) \cong \left( \bigvee_{S^0}^{\infty} \mathbb{C}^+, 0, n(Y, \mathfrak{s}_0, A_0, g) + m(\not{D}_{A_0}, \delta_r - \delta) + 1 \right),$$

where  $\bigvee_{S^0}^{\infty} \mathbb{C}^+ \in \text{ob } \mathfrak{S}$  denotes the direct system

$$\mathbb{C}^+ \rightarrow \bigvee_{S^0}^2 \mathbb{C}^+ \rightarrow \bigvee_{S^0}^3 \mathbb{C}^+ \rightarrow \dots$$

whose connecting maps are given by the inclusions.

Since  $\delta_r, \delta > 0$  and  $\delta$  is small, we have  $m(\not{D}_{A_0}, \delta_r - \delta) = m(\not{D}_{A_0}, \delta_r)$ . We can repeat the calculation in Section 8.2 to find the value of  $n(Y, \mathfrak{s}_0, A_0, g) + m(\not{D}_{A_0}, \delta_r)$ . The only difference is that  $\not{D}^0 = \tilde{\not{D}}_{\tilde{A}_0}$  now has kernel of dimension 2. Therefore, (48) and (49) become

$$\text{Ind}_{\mathbb{C}}(\hat{\not{D}}^+) = -\frac{1}{24} \int_X p_1(\hat{A}) + \frac{1}{2}(\eta(\not{D}) - \dim_{\mathbb{C}}(\ker \not{D})) - \frac{1}{2}(\eta(\not{D}^0) - 2),$$

$$\text{Ind}_{\mathbb{C}}(\hat{\not{D}}^+) = -m(\delta_r) - 2.$$

Consequently, we obtain

$$n(\mathfrak{s}_0, A_0, g) + m(\delta_r) + 1 = c(1, d, 0) - 2 = \frac{1}{8}(d - 17).$$

Therefore, we can conclude that

$$\underline{\text{SWF}}^A(Y, \mathfrak{s}_0; S^1) \cong \left( \bigvee_{S^0}^{\infty} \mathbb{C}^+, 0, \frac{1}{8}(d - 17) \right) \quad \text{for } d > 0.$$

Next, we compute  $\underline{\text{SWF}}^R(Y, \mathfrak{s}_0; S^1)$ . To have a simpler description, we replace  $(\delta, \epsilon)$  in the  $\underline{\text{SWF}}^A$  case by  $(-\delta, -\epsilon)$  as discussed in the remark following Proposition 8.7). By setting  $\theta^- = \frac{1}{4}$ , we also consider  $\bar{J}_m^- = p_{\mathcal{H}}^{-1}([-\frac{1}{4} - m, \frac{1}{4} + m]^2) \cap \overline{\text{Str}}(\epsilon')$ . This linearized Seiberg–Witten flow goes outside  $\bar{J}_m^-$  along  $p_{\mathcal{H}}^{-1}(\partial[-\frac{1}{4} - m, \frac{1}{4} + m]^2)$ . Let  $W$  be as before and  $\tilde{W}^-$  be the subspace spanned by negative eigenvectors of  $(\tilde{\mathcal{D}}_{A_0} + \delta)|_W$ . By similar argument as in the previous case, we obtain

$$\bar{I}_m^{n,-} \cong \Sigma^{-(\tilde{W}^- \oplus i\Omega_h^1(Y))} I_{S^1}(\tilde{\varphi}, \text{inv}(\bar{J}_m^- \cap (i\Omega_h^1(Y) \oplus W))),$$

where  $\tilde{\varphi}$  is the flow on  $i\Omega_h^1(Y) \oplus W$  generated by  $(\epsilon \text{ grad } f_H(h), (\tilde{\mathcal{D}}_{\tilde{A}_0+h} + \delta)\phi)$ .

Instead of finding the Conley index  $I_{S^1}(\tilde{\varphi}, \text{inv}(\bar{J}_m^- \cap (i\Omega_h^1(Y) \oplus W)))$  directly, we consider the reverse flow  $-\tilde{\varphi}$  and use some duality results. Like  $\varphi$ , the flow  $-\tilde{\varphi}$  goes inside the isolating block  $[-\frac{1}{4} - m, \frac{1}{4} + m]^2 \times B(W)$  along  $(\partial[-\frac{1}{4} - m, \frac{1}{4} + m]^2) \times B(W)$ . With the same argument as in the previous case, we have that

$$I_{S^1}(-\tilde{\varphi}, \text{inv}(\bar{J}_m^- \cap (i\Omega_h^1(Y) \oplus W))) \cong \bigvee_{(\{0\} \times \mathbb{C}^{(2m+1)^2})^+}^{\mathbb{C}^{(2m+1)^2}} (\mathbb{C}^{(2m+1)^2+1})^+.$$

According to [6; 24] (see also [13, Proposition 3]), this space is the equivariant  $(i\Omega_h^1(Y) \oplus W)$ -dual of  $I_{S^1}(\tilde{\varphi}, \text{inv}(\bar{J}_m^- \cap (i\Omega_h^1(Y) \oplus W)))$ ; see [25, page 209] for the definition. Since we have a decomposition

$$i\Omega_h^1(Y) \oplus W \cong i\Omega_h^1(Y) \oplus \tilde{W}^- \oplus \mathbb{C}^{(2m+1)^2+1},$$

we can see that

$$\bar{I}_m^{n,-} \cong \Sigma^{\mathbb{C}} \left( \bigvee_{S^0}^{(2m+1)^2} \mathbb{C}^+ \right)^*,$$

where  $E^*$  denotes the equivariant Spanier–Whitehead dual spectrum of  $E$ . In order to calculate  $(\bigvee_{S^0}^{(2m+1)^2} \mathbb{C}^+)^*$ , we give the following lemma, which is a simple consequence of Theorem 4.1 and Lemma 4.9 of [18, Chapter III].

**Lemma 8.8** *Let  $E$  be a finite  $S^1$ -CW complex embedded into  $S(V)$ , the unit sphere of an  $S^1$ -representation space  $V$ . Then  $(V^+ \setminus E)$  and  $\Sigma E$  are equivariant  $V$ -dual to each other. Moreover, if  $E' \subset E$  is an inclusion of  $S^1$ -CW complex, then the inclusion  $\Sigma E' \rightarrow \Sigma E$  is dual to the inclusion  $(V^+ \setminus E) \rightarrow (V^+ \setminus E')$ . Similar results hold for the  $\text{Pin}(2)$ -equivariant case.*

Notice that

$$\bigvee_{S^0}^{(2m+1)^2} \mathbb{C}^+ \cong \Sigma \left( \bigsqcup^{(2m+1)^2} S^1 \right),$$

and we can embed the disjoint union of circles into  $S(\mathbb{C}^2)$ . By Lemma 8.8, we have

$$\Sigma \mathbb{C}^2 \left( \bigvee_{S^0}^{(2m+1)^2} \mathbb{C}^+ \right)^* = (\mathbb{C}^2)^+ \setminus \left( \bigsqcup^{(2m+1)^2} S^1 \right),$$

and we can obtain

$$\bar{I}_m^{n,-} \cong \Sigma^{-\mathbb{C}} \left( (\mathbb{C}^2)^+ \setminus \left( \bigsqcup^{(2m+1)^2} S^1 \right) \right).$$

Hence, we can conclude that

$$\underline{\text{SWF}}^R(Y, \mathfrak{s}_0; S^1) \cong \left( (\mathbb{C}^2)^+ \setminus \left( \bigsqcup^{\infty} S^1 \right), 0, n(Y, \mathfrak{s}_0, A_0, g) + m(\not{D}_{A_0}, \delta_r + \delta) + 1 \right),$$

where  $(\mathbb{C}^2)^+ \setminus \left( \bigsqcup^{\infty} S^1 \right)$  denotes the inverse system

$$(\mathbb{C}^2)^+ \setminus (S^1) \leftarrow (\mathbb{C}^2)^+ \setminus \left( \bigsqcup^2 S^1 \right) \leftarrow (\mathbb{C}^2)^+ \setminus \left( \bigsqcup^3 S^1 \right) \leftarrow \dots$$

whose connecting morphisms are given by the inclusions.

Since  $\delta_r, \delta > 0$  and  $-\delta_r$  is an eigenvalue of  $\not{D}$  with multiplicity 2, we get

$$m(\not{D}_{A_0}, \delta_r + \delta) = m(\not{D}_{A_0}, \delta) + 2,$$

which implies

$$(61) \quad \underline{\text{SWF}}^R(Y, \mathfrak{s}_0; S^1) \cong \left( (\mathbb{C}^2)^+ \setminus \left( \bigsqcup^{\infty} S^1 \right), 0, \frac{1}{8}(d-1) \right) \quad \text{for } d > 0.$$

**Remark** For the case  $d < 0$ , the results are

$$\underline{\text{SWF}}^A(Y, \mathfrak{s}_0; S^1) \cong \left( \bigvee_{S^0}^{\infty} \mathbb{C}^+, 0, \frac{1}{8}(d-15) \right),$$

$$\underline{\text{SWF}}^R(Y, \mathfrak{s}_0; S^1) \cong \left( (\mathbb{C}^2)^+ \setminus \left( \bigsqcup^{\infty} S^1 \right), 0, \frac{1}{8}(d+1) \right).$$

We have finished the calculation of the  $S^1$ -invariants. Since  $c_1(\mathfrak{s}_0) = 0$ , the  $\text{spin}^c$  structure  $\mathfrak{s}_0$  can be lifted to  $2^2 = 4$  different spin structures, whose spin connections are

given by  $A_0, A_0 + \frac{1}{2}h_1, A_0 + \frac{1}{2}h_2$  and  $A_0 + \frac{1}{2}(h_1 + h_2)$ . We denote the corresponding spin structures by  $\mathfrak{s}^0, \mathfrak{s}^1, \mathfrak{s}^2$  and  $\mathfrak{s}^3$ , respectively. Although we have to choose the base connection to be the corresponding spin connection, for consistency, we still identify  $\text{spin}^c$  connections with 1-forms by sending  $A$  to  $A - A_0$  instead.

Most of the argument in the  $S^1$ -equivariant case can be easily adapted to be  $\text{Pin}(2)$ -equivariant case. The only thing to be careful is that the set  $\Lambda$  (see (60)) should be invariant under the additional  $J$ -symmetry.

Now we start the calculation. It turns out that the invariants for  $\mathfrak{s}^1, \mathfrak{s}^2$  and  $\mathfrak{s}^3$  are isomorphic to each other, so we will just focus on  $\mathfrak{s}^1$ . With the same setup when computing  $\text{SWF}^A(Y, \mathfrak{s}_0; S^1)$ , the addition  $J$ -action is given by

$$J \cdot (h, \phi_{\vec{v},+}) = (h_1 - h, \phi_{(1,0)-\vec{v},-}) \quad \text{and} \quad J \cdot (h, \phi_{\vec{v},-}) = (h_1 - h, -\phi_{(1,0)-\vec{v},+}).$$

To preserve this symmetry, we consider

$$\begin{aligned} \widehat{J}_m^+ &= \overline{\text{Str}}(\epsilon') \cap p_{\mathcal{H}}^{-1} \left( \left[ -\frac{1}{4} - m + 1, \frac{1}{4} + m \right]^2 \right), \\ \widehat{W} &= \bigoplus_{\vec{v} \in \mathbb{Z}^2 \cap [-m+1, m]^2} V_{\vec{v}}, \end{aligned}$$

where we note that the basic interval is  $[-m + 1, m]$  instead of  $[-m, m]$ .

In a similar manner, we have that, for  $n$  sufficiently large,

$$\bar{I}_m^{n,-}(\text{Pin}(2)) \cong \Sigma^{-\widehat{W}^-} I_{\text{Pin}(2)}(\varphi, \text{inv}(\widehat{J}_m^+ \cap (i\Omega_h^1(Y) \oplus \widehat{W}))),$$

where  $\widehat{W}^-$  is spanned by the negative eigenvector of  $(\not{D}_{A_0+h_1/2} + \delta_r - \delta)|_W$ . The set  $\Lambda$  can be made  $J$ -invariant by choosing  $z_0 = \frac{1}{2}h_1$  and requiring that  $J \cdot \gamma_{\vec{v}} = \gamma_{(1,0)-\vec{v}}$ . Repeating the calculations, we can show that

$$I_{S^1}(\varphi, \text{inv}(\widehat{J}_m^+ \cap (i\Omega_h^1(Y) \oplus \widehat{W}))) \cong \bigvee_{(\{0\} \times \mathbb{C}^{4m^2})^+}^{4m^2} (\mathbb{C}^{4m^2+1})^+ \cong (\mathbb{C}^{4m^2})^+ \wedge \Sigma \left( \bigsqcup_{S^1}^{4m^2} S^1 \right).$$

Notice that these copies of  $S^1$  correspond to vertices  $\vec{v} \in \left[ -\frac{1}{4} - m + 1, \frac{1}{4} + m \right]^2$ , which are interchanged by  $J$ . Therefore, we see that

$$I_{\text{Pin}(2)}(\varphi, \text{inv}(\widehat{J}_m^+ \cap (i\Omega_h^1(Y) \oplus \widehat{W}))) \cong (\mathbb{H}^{2m^2})^+ \wedge \Sigma \left( \bigsqcup_{\text{Pin}(2)}^{2m^2} \text{Pin}(2) \right).$$



Since  $\widehat{W}^- \cong \mathbb{H}^{2m^2}$ , we can conclude

$$\underline{\text{SWF}}^A(Y, \mathfrak{s}^1; \text{Pin}(2)) \cong \left( \Sigma \left( \bigsqcup^{\infty} \text{Pin}(2) \right), 0, \frac{1}{16}(d - 17) \right),$$

where  $\Sigma(\bigsqcup^{\infty} \text{Pin}(2)) \in \text{ob}(\mathfrak{S}(\text{Pin}(2)))$  denotes the direct system

$$\Sigma \text{Pin}(2) \rightarrow \Sigma(\text{Pin}(2) \sqcup \text{Pin}(2)) \rightarrow \Sigma(\text{Pin}(2) \sqcup \text{Pin}(2) \sqcup \text{Pin}(2)) \rightarrow \dots$$

whose connecting morphisms are given by inclusions.

The calculation of  $\underline{\text{SWF}}^R(Y, \mathfrak{s}^1; \text{Pin}(2))$  is also very similar to the  $\underline{\text{SWF}}^R(Y, \mathfrak{s}_0; S^1)$  case. For example, we can show that  $\bar{I}_m^{n,-}(\text{Pin}(2))$  is the Spanier–Whitehead dual of  $\Sigma(\bigsqcup^{2m^2} \text{Pin}(2))$ . We can pick an embedding of  $\bigsqcup^{\infty} \text{Pin}(2)$  in  $S(\mathbb{H})$  and denote by  $\mathbb{H}^+ \setminus \bigsqcup^{\infty} \text{Pin}(2)$  the inverse system

$$\mathbb{H}^+ \setminus \text{Pin}(2) \leftarrow \mathbb{H}^+ \setminus \bigsqcup^2 \text{Pin}(2) \leftarrow \mathbb{H}^+ \setminus \bigsqcup^3 \text{Pin}(2) \leftarrow \dots,$$

whose connecting morphisms are given by inclusions. Then by [Lemma 8.8](#), the system  $\mathbb{H}^+ \setminus \bigsqcup^{\infty} \text{Pin}(2)$  is equivariant  $\mathbb{H}$ -dual to  $\Sigma(\bigsqcup^{\infty} \text{Pin}(2))$ . Hence we get

$$\underline{\text{SWF}}^R(Y, \mathfrak{s}^1; \text{Pin}(2)) \cong \left( \mathbb{H}^+ \setminus \bigsqcup^{\infty} \text{Pin}(2), 0, \frac{1}{16}(d - 1) \right).$$

Now we compute the invariants for  $\mathfrak{s}^0$ . The corresponding spin connection is  $A_0$ , and the additional  $J$ -symmetry is given by

$$J \cdot (h, \phi_{\vec{v},+}) = (-h, \phi_{-\vec{v},-}) \quad \text{and} \quad J \cdot (h, \phi_{\vec{v},-}) = (-h, -\phi_{-\vec{v},+}).$$

We want to compute the Conley index

$$\bar{I}_m^{n,+}(\text{Pin}(2)) \cong \Sigma^{-W^-} I_{\text{Pin}(2)}(\varphi, \text{inv}(\hat{J}_m^+ \cap (i\Omega_h^1(Y) \oplus W))).$$

Note that the set  $\Lambda$  defined in [\(60\)](#) can never be made  $J$ -invariant in this case. Instead, we can use the union of balls  $B_{\vec{v}}$  and paths  $\gamma_{\vec{v}}$  connecting  $B_{\vec{0}}$  to  $B_{\vec{v}}$  for each  $\vec{v} \neq 0$  with  $J \cdot \gamma_{\vec{v}} = \gamma_{-\vec{v}}$ . Using a deformation retract to this set, we can describe the Conley index  $I_{\text{Pin}(2)}(\varphi, \text{inv}(\hat{J}_m^+ \cap (i\Omega_h^1(Y) \oplus W)))$  as follows: the ball  $B_{\vec{v}}$  contributes to a copy of  $(\mathbb{H}^{2m^2+2m+1})^+ \cong \Sigma^{(2m^2+2m)\mathbb{H}} \Sigma S(\mathbb{H})$ . For  $\vec{v} \neq 0$ , each pair of  $B_{\pm\vec{v}}$  together contributes a copy of  $\Sigma^{(2m^2+2m)\mathbb{H}} \Sigma(\tilde{Z}_2 \times S(\mathbb{H}))$ , where  $\tilde{Z}_2 = \{\pm 1\}$  is the

two-point space with nontrivial  $\text{Pin}(2)$ -action. More precisely, we can write

$$I_{\text{Pin}(2)}(\varphi, \text{inv}(\bar{J}_m^+ \cap (i\Omega_h^1(Y) \oplus W))) \cong \Sigma^{(2m^2+2m)\mathbb{H}} \Sigma \left( S(\mathbb{H}) \vee_{\text{Pin}(2)} \bigvee_{\text{Pin}(2)}^{2m^2+2m} (\tilde{Z}_2 \times S(\mathbb{H})) \right).$$

Here we think of  $\text{Pin}(2)$  as the subset  $\{(1, e^{i\theta})\} \cup \{(-1, je^{i\theta})\}$  in  $\tilde{Z}_2 \times S(\mathbb{H})$ . Since  $W^- \cong \mathbb{H}^{2m^2+2m+1}$ , we obtain

$$\underline{\text{SWF}}^A(Y, \mathfrak{s}^0; \text{Pin}(2)) \cong \left( \Sigma \left( S(\mathbb{H}) \vee_{\text{Pin}(2)} \bigvee_{\text{Pin}(2)}^{\infty} (\tilde{Z}_2 \times S(\mathbb{H})) \right), 0, \frac{1}{16}(d-9) \right),$$

where  $S(\mathbb{H}) \vee_{\text{Pin}(2)} (\bigvee_{\text{Pin}(2)}^{\infty} (\tilde{Z}_2 \times S(\mathbb{H})))$  denotes the direct system

$$\begin{aligned} S(\mathbb{H}) \vee_{\text{Pin}(2)} (\tilde{Z}_2 \times S(\mathbb{H})) &\rightarrow S(\mathbb{H}) \vee_{\text{Pin}(2)} \left( \bigvee_{\text{Pin}(2)}^2 (\tilde{Z}_2 \times S(\mathbb{H})) \right) \\ &\rightarrow S(\mathbb{H}) \vee_{\text{Pin}(2)} \left( \bigvee_{\text{Pin}(2)}^3 (\tilde{Z}_2 \times S(\mathbb{H})) \right) \rightarrow \dots \end{aligned}$$

We are only left with the calculation of  $\underline{\text{SWF}}^R(Y, \mathfrak{s}^0; \text{Pin}(2))$ . To do this, we need to find the Spanier–Whitehead dual of  $S(\mathbb{H}) \vee_{\text{Pin}(2)} (\bigvee_{\text{Pin}(2)}^m (\tilde{Z}_2 \times S(\mathbb{H})))$ . It is not hard to check that this space can be embedded into  $S(\mathbb{H}^2)$  as

$$D_m := \bigcup_{n=0}^m \{(z_1 + jz_2, z_3 + jz_4) \in S(\mathbb{H}^2) \mid z_3 = -\bar{z}_4 = nz_1 \text{ or } z_3 = \bar{z}_4 = n\bar{z}_2\}.$$

Repeating the calculation we did for  $\underline{\text{SWF}}^R(Y, \mathfrak{s}^0; S^1)$ , we get

$$\underline{\text{SWF}}^R(Y, \mathfrak{s}^0; \text{Pin}(2)) \cong ((\mathbb{H}^2)^+ \setminus D_{\infty}, 0, \frac{1}{16}(d+7)),$$

where  $(\mathbb{H}^2)^+ \setminus D_{\infty}$  denotes the inverse system

$$(\mathbb{H}^2)^+ \setminus D_1 \leftarrow (\mathbb{H}^2)^+ \setminus D_2 \leftarrow (\mathbb{H}^2)^+ \setminus D_3 \leftarrow \dots$$

**8.3.3 Other nil manifolds** Suppose that  $\Sigma$  is not smooth. Then the genus of  $\Sigma$  is 0 and  $Y$  is a rational homology sphere. Without any further perturbation, the functional  $\mathcal{L}_{\delta_r, 0}$  has a unique critical point  $(A_0, 0)$ . This allows us to apply [Theorem 7.5](#) and obtain

$$\begin{aligned} \underline{\text{SWF}}^A(Y, \mathfrak{s}; S^1) &\cong (S^0, 0, n(Y, \mathfrak{s}, A_0, g) + m(\emptyset, \delta_r)), \\ \underline{\text{SWF}}^R(Y, \mathfrak{s}; S^1) &\cong (S^0, 0, n(Y, \mathfrak{s}, A_0, g) + m(\emptyset, \delta_r)). \end{aligned}$$

Moreover, when  $\mathfrak{s}$  is spin, we have

$$\begin{aligned} \underline{\text{SWF}}^A(Y, \mathfrak{s}; \text{Pin}(2)) &\cong (S^0, 0, \frac{1}{2}(n(Y, \mathfrak{s}, A_0, g) + m(\not{D}, \delta_r))), \\ \underline{\text{SWF}}^R(Y, \mathfrak{s}; \text{Pin}(2)) &\cong (S^0, 0, \frac{1}{2}(n(Y, \mathfrak{s}, A_0, g) + m(\not{D}, \delta_r))). \end{aligned}$$

The explicit formula for  $n(Y, \mathfrak{s}, A_0, g) + m(\not{D}, \delta_r)$  can be obtained in the same fashion as in Section 8.2 (see [27; 28]) and we omit it.

### 8.4 Flat manifolds except $T^3$

In this subsection, we calculate the spectrum invariants for manifolds  $Y$  supporting a flat metric  $g$  other than the 3–torus  $T^3$ . There are five manifolds belonging to this class: four of them are  $T^2$ –bundles over  $S^1$  with monodromy automorphism fixing a point and having orders 2, 3, 4 and 6, and the last of them is the Hantzsche–Wendt manifold. By the Weitzenböck formula, for any torsion  $\text{spin}^c$  structure  $\mathfrak{s}$  on  $Y$ , the functional CSD has only reducible critical points.

The Hantzsche–Wendt manifold is a rational homology sphere. Therefore, the functional  $\mathcal{L}_{0,0}$  has only one critical point  $(A_0, 0)$ , and we can just apply Theorem 7.5 to conclude that

$$\begin{aligned} \underline{\text{SWF}}^A(Y, \mathfrak{s}; S^1) &\cong (S^0, 0, n(Y, \mathfrak{s}, A_0)), \\ \underline{\text{SWF}}^R(Y, \mathfrak{s}; S^1) &\cong (S^0, 0, n(Y, \mathfrak{s}, A_0)) \end{aligned}$$

and analogous results for the spin case. The numbers  $n(Y, \mathfrak{s}, A_0, g)$  can be calculated using the method of [27; 28] and we omit the result.

Now we consider the  $T^2$ –bundles over  $S^1$  whose monodromies are automorphisms  $\tau: T^2 \rightarrow T^2$  of order 2 (ie the hyperelliptic involution on  $T^2$ ). The situations for the cases of order 3, 4 or 6 are very similar, so we will focus our attention on this case of order 2.

Let  $T^2$  be given by  $\mathbb{R}^2/\mathbb{Z}^2$  with  $\tau(\theta_1, \theta_2) = -(\theta_1, \theta_2)$ . Then the manifold  $Y$  can be obtained as the quotient of  $(\mathbb{R} \times T^2)/\mathbb{Z}$ , where the  $\mathbb{Z}$ –action is given by  $(\theta_0, \theta_1, \theta_2) \mapsto (\theta_0 + 1, -\theta_1, -\theta_2)$ . Since  $H_1(Y; \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$ , there are four torsion  $\text{spin}^c$  structures on  $Y$ . By [17, Lemma 37.4.1], only one of them admits a  $\text{spin}^c$  connection  $A$  with  $F_{A^i} = 0$  and  $\ker \not{D}_A \neq 0$ . We denote it by  $\mathfrak{s}_0$  and the other three by  $\mathfrak{s}_1, \mathfrak{s}_2$  and  $\mathfrak{s}_3$ .

Let us consider the  $\text{spin}^c$  structure  $\mathfrak{s}_0$  first. This  $\text{spin}^c$  structure can be identified with a quotient of the  $\text{spin}^c$  structure  $\tilde{\mathfrak{s}}_0$  on  $\mathbb{R} \times T^2$  whose spinor bundle is trivial and the

Clifford multiplication is given by

$$\rho(d\theta_0) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(d\theta_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho(d\theta_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Note that the generator of the  $\mathbb{Z}$ -action on the spinor bundle is given by the constant matrix  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

As in Section 8.3, we let  $\Gamma_c(S_Y)$  (resp.  $\Gamma_0(S_Y)$ ) be the space of sections of that is constant (resp. integrate to 0) along each fiber of the  $T^2$ -bundle. For each integer  $n$ , we define sections  $\phi_{n,\pm} \in \Gamma_c(S_Y)$  as

$$\phi_{n,+}(\theta_0, \theta_1, \theta_2) = (e^{\pi i \cdot 2n\theta_0}, 0), \quad \phi_{n,-}(\theta_0, \theta_1, \theta_2) = (0, e^{\pi i \cdot (2n+1)\theta_0}).$$

These give an  $L^2$ -orthonormal basis of  $\Gamma_c(S_Y)$ .

We choose as the base connection  $A_0$  induced from the trivial connection on  $\tilde{\mathfrak{s}}_0$  and we pick  $h_1 = 2\pi i \cdot d\theta_0$  as a basis for  $i\Omega^1_h(Y)$ . We have the following observation.

**Lemma 8.9** *For any  $\theta \in \mathbb{R}$ , the kernel of the operator  $\mathcal{D}_{A_0+\theta h_1}|_{L^2_k(\Gamma_0(S_Y))}$  is trivial. Moreover, we have*

$$\begin{aligned} \mathcal{D}_{A_0+\theta h_1}(\phi_{n,+}) &= -\pi(2n + 2\theta)\phi_{n,+}, \\ \mathcal{D}_{A_0+\theta h_1}(\phi_{n,-}) &= \pi(2n + 1 + 2\theta)\phi_{n,-}. \end{aligned}$$

**Proof** The first assertion can be proved by passing to the double cover of  $Y$ , which is  $T^3$ . The second assertion is easy to verify by direct calculation.  $\square$

Let  $f_H: i\Omega^1_h(Y) \cong \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(\theta) = -\cos(4\pi\theta)$ . By the same argument as in the proof of [17, Proposition 37.1.1] and Lemma 8.4, we have

**Lemma 8.10** *There exist a constant  $\delta_1 \in (0, \frac{\pi}{2})$  and a function  $\epsilon_1: (0, \delta_1) \rightarrow (0, \infty)$  such that for any  $\delta \in (0, \delta_1)$  and  $\epsilon \in (0, \epsilon_1(\delta_1))$ , we have the following:*

- (1) *The functional*

$$\mathcal{L}_{-\delta,\epsilon} = \text{CSD}|_{\text{Coul}(Y)} - \frac{1}{2}\delta\|\phi\|_{L^2}^2 + \epsilon f_H(p_{\mathcal{H}}(a, \phi))$$

*has only reducible critical points.*

- (2) *For any  $h \in i\Omega^1_h(Y)$  and  $\phi \in L^2_k(\Gamma(S_Y))$ , we have*

$$\langle (2(\mathcal{D}_{A_0+h} - \delta)^2 + \rho(\text{grad } \epsilon f_H(h)))\phi, \phi \rangle_{L^2} \geq C(\delta, \epsilon)\|\phi\|_{L^2}^2,$$

*where  $C(\delta, \epsilon)$  is a positive constant depending only on  $\delta, \epsilon$ .*

Let us fix a choice of  $\delta_2 \in (0, \delta_1)$  and  $\epsilon_2 \in (0, \epsilon_1(\delta_2))$ . The critical points of  $\mathcal{L}_{-\delta_2, \epsilon_2}$  are just  $(\theta h_1, 0)$  with  $4\theta \in \mathbb{Z}$ , and there are four gauge equivalent classes  $[(0, 0)]$ ,  $[(\frac{1}{4}h_1, 0)]$ ,  $[(\frac{1}{2}h_1, 0)]$  and  $[(\frac{3}{4}h_1, 0)]$ . Notice that the spectral flow of the operator  $\mathcal{D}_{A_0 + \theta h_1} - \delta_2$  is 0 when  $\theta$  goes from 0 to  $\frac{1}{4}$  or  $\theta$  goes from  $\frac{1}{4}$  to  $\frac{1}{2}$ , whereas the spectral flow is 1 when  $\theta$  goes from  $\frac{1}{2}$  to  $\frac{3}{4}$ . Consequently, we have

$$\begin{aligned} \text{gr}([(0, 0)], [(\frac{1}{4}h_1, 0)]) &= \text{gr}([\frac{1}{2}h_1, 0], [(\frac{1}{4}h_1, 0)]) = -1, \\ \text{gr}([(0, 0)], [(\frac{3}{4}h_1, 0)]) &= \text{gr}([\frac{1}{2}h_1, 0], [(\frac{3}{4}h_1, 0)]) = 1. \end{aligned}$$

As in the proof of Lemma 8.6, we can find an extended cylinder function  $\bar{f}$  satisfying the following requirements:

- (1) there exists  $\epsilon' > 0$  such that  $\bar{f}(a, \phi) = 0$  whenever  $\|\phi\|_{L^2} \geq \epsilon'$ ;
- (2) the functional  $\mathcal{L}_{-\delta_2, \epsilon_2} + \bar{f}$  has only reducible critical points;
- (3) for any boundary unstable reducible critical manifold  $[C]$  and any boundary stable reducible critical manifold  $[C']$ , the moduli space  $\check{\mathcal{M}}_{\bar{f}}([C], [C'])$  is Smale regular.

We now consider the perturbed functional  $\mathcal{L}_{-\delta_2, \epsilon_2} + \bar{f}$ . Let  $[C], [C']$  be critical manifolds whose corresponding critical points are  $[a], [b]$ , respectively. By a formula analogous to (56), we see that the expected dimension of  $\check{\mathcal{M}}_{\bar{f}}([C], [C'])$  is nonnegative only if  $\text{gr}([a], [b])$  is at least 2. This can happen only when  $[a] = [(\frac{1}{4}h_1, 0)]$  and  $[b] = [(\frac{3}{4}h_1, 0)]$ . However, since the functional  $\mathcal{L}_{-\delta_2, \epsilon_2} + \bar{f}$  takes the same value at these two points, there cannot be any trajectory connecting them. As a result, we have proved that the moduli space  $\check{\mathcal{M}}_{\bar{f}}([C], [C'])$  is actually empty. This implies that there are no irreducible trajectories for the functional  $\mathcal{L}_{-\delta_2, \epsilon_2} + \bar{f}$  and we can use Theorem 7.3 to calculate our invariants.

The computation of the Conley indices of the linearized Seiberg–Witten flow is exactly the same as in the case of nil manifolds and we will skip all the details. The number  $n(Y, \mathfrak{s}, A_0, g)$  can be calculated using the formula in [27]. Since  $Y$  admits an orientation reversing diffeomorphism preserving  $(\mathfrak{s}_0, A_0)$ , we see that  $\eta(\mathcal{D}) = \eta_{\text{sign}} = 0$ , which implies  $n(Y, g, \mathfrak{s}, A_0) = -\frac{1}{2} \dim_{\mathbb{C}}(\ker \mathcal{D}) = -\frac{1}{2}$ . Note that  $m(\mathcal{D}, -\delta_2) = 0$  as  $\delta_2$  is small. Therefore, we can conclude that

$$\begin{aligned} \underline{\text{SWF}}^A(Y, \mathfrak{s}_0; S^1) &\cong \left( \bigvee_{S^0}^{\infty} \mathbb{C}^+, 0, \frac{1}{2} \right), \\ \underline{\text{SWF}}^R(Y, \mathfrak{s}_0; S^1) &\cong \left( (\mathbb{C}^2)^+ \setminus \left( \bigsqcup^{\infty} S^1 \right), 0, \frac{3}{2} \right). \end{aligned}$$

As for the  $\text{Pin}(2)$ -invariants, notice that  $\mathfrak{s}_0$  can be lifted to two spin structures. Since the holonomy of  $A_0^t$  along the loop  $(0, 0, \theta)$  equals  $-1$ , we see that the spin connections are  $A_0 + \frac{1}{4}h_1$  and  $A_0 + \frac{1}{4}h_3$  and denote the corresponding spin structures by  $\mathfrak{s}_0^0$  and  $\mathfrak{s}_0^1$ , respectively. We have

$$\underline{\text{SWF}}^A(Y, \mathfrak{s}_0^0; \text{Pin}(2)) \cong \left( \Sigma \left( S(\mathbb{H}) \vee_{\text{Pin}(2)} \bigvee_{\text{Pin}(2)}^{\infty} (\tilde{Z}_2 \times S(\mathbb{H})) \right), 0, \frac{3}{4} \right),$$

$$\underline{\text{SWF}}^R(Y, \mathfrak{s}_0^0; \text{Pin}(2)) \cong ((\mathbb{H}^2)^+ \setminus D_{\infty}, 0, \frac{5}{4}),$$

$$\underline{\text{SWF}}^A(Y, \mathfrak{s}_0^1; \text{Pin}(2)) \cong \left( \Sigma \left( \bigsqcup^{\infty} \text{Pin}(2) \right), 0, \frac{1}{4} \right),$$

$$\underline{\text{SWF}}^R(Y, \mathfrak{s}_0^1; \text{Pin}(2)) \cong \left( \mathbb{H}^+ \setminus \bigsqcup^{\infty} \text{Pin}(2), 0, \frac{3}{4} \right).$$

As for the other  $\text{spin}^c$  structures  $\mathfrak{s}_1$ ,  $\mathfrak{s}_2$  and  $\mathfrak{s}_3$ , since the  $\ker(\mathcal{D}_{A_0+h}) = 0$  for any  $h \in i\Omega_h^1(Y)$ , we can just apply [Theorem 7.5](#) and get sphere spectra with suitable suspension. Notice, for  $j = 1, 2, 3$ , that we have  $n(Y, A_0, \mathfrak{s}_j, g) = 0$  because of existence of an orientation reversing diffeomorphism preserving  $(\mathfrak{s}_j, A_0)$ .

Finally, when  $Y$  is one of the other  $T^2$ -bundles, we can prove that the spectrum invariants of  $Y$  are just shifts in the suspension indices of the above results. The only difference comes from the change of the number  $n(Y, \mathfrak{s}, A_0, g)$ . Again, we refer to [\[27; 28\]](#) for the calculation of this quantity.

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