

A family of compact complex and symplectic Calabi–Yau manifolds that are non-Kähler

LIZHEN QIN
BOTONG WANG

We construct a family of 6–dimensional compact manifolds $M(A)$ which are simultaneously diffeomorphic to complex Calabi–Yau manifolds and symplectic Calabi–Yau manifolds. They have fundamental groups $\mathbb{Z} \oplus \mathbb{Z}$, their odd-degree Betti numbers are even, they satisfy the hard Lefschetz property, and their real homotopy types are formal. However, $M(A) \times Y$ is never homotopy equivalent to a compact Kähler manifold for any topological space Y . The main ingredient to show the non-Kählerness is a structure theorem of cohomology jump loci due to the second author.

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1 Introduction

A Kähler manifold is a symplectic manifold together with a compatible complex structure. Mathematicians have been interested in finding examples of compact complex or compact symplectic manifolds which are non-Kähler. In the 1940s, Hopf constructed complex manifolds $S^1 \times S^{2N-1}$ ($N > 1$) that are non-Kähler because their second Betti numbers vanish; see Morrow and Kodaira [31, page 14]. For similar reasons, it is easy to construct many compact complex non-Kähler manifolds. On the other hand, it is more difficult and hence more interesting to find the symplectic counterparts. The existence of compact symplectic non-Kähler manifolds had been an open question for many years until Thurston discovered the first example in [36]. Since then, there have been many works on compact symplectic non-Kähler manifolds; see eg Babenko and Taïmanov [4], Cordero, Fernández and Gray [10], Fernández, de León and Saralegui [13], Fernández and Muñoz [15], Gompf [20], McDuff [29], Park [34], and Tralle and Oprea [38]. It is particularly interesting, motivated by physics, to construct complex or symplectic non-Kähler manifolds satisfying the Calabi–Yau property; see eg Akhmedov [1], Baldrige and Kirk [5], Bogomolov [7], Fine and Panov [16; 17], Friedman [18], Goldstein and Prokushkin [19], Grantcharov [21],

Guan [23; 24], Gutowski, Ivanov and Papadopoulos [25], Lu and Tian [27], Smith, Thomas and Yau [35], Torres and Yazinski [37], and Tseng and Yau [39].

In this paper, we will construct a family of 6–dimensional compact smooth manifolds which are simultaneously diffeomorphic to complex and symplectic manifolds. They are Calabi–Yau with respect to both the complex structure and the symplectic structure. We call these manifolds complex and symplectic Calabi–Yau. Notice that our notion of “complex and symplectic” is different from the notion “holomorphic symplectic”; see Guan [23]. A holomorphic symplectic manifold is a complex manifold endowed with a holomorphic symplectic form. A holomorphic symplectic manifold certainly carries both complex and symplectic structures and is Calabi–Yau in either sense. However, such manifolds are always of real dimension $4n$, and hence never of dimension 6.

The examples in this paper are interesting to us because they share many properties with compact Kähler manifolds. However, they are non-Kähler in a strong sense. Before stating the main result, we want to make the notion “non-Kähler” precise. For a manifold M , we can interpret non-Kählerness in one of the following five ways:

The manifold M is (i) not homotopy equivalent, (ii) not homeomorphic, (iii) not diffeomorphic, (iv) not biholomorphic or (v) not symplectomorphic to any compact Kähler manifold.

Here, (iv) also means that M does not carry a Kähler structure compatible with its complex structure, and (v) also means that M does not carry a Kähler structure compatible with its symplectic structure. Clearly, (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) and (v). The non-Kählerness of our examples is in a sense even stronger than (i).

Let us focus on the non-Kählerness in the sense of (i) for now. Many authors prove a manifold is non-Kähler in the sense of (i) by showing the manifold does not satisfy one of the properties described here:

- (a) (fundamental groups) Only a small class of groups, named Kähler groups, can be realized as the fundamental groups of compact Kähler manifolds; see eg Amorós, Burger, Corlette, Kotschick and Toledo [2] and Burger [9] for a survey.
- (b) (Betti numbers) The odd-degree Betti numbers of compact Kähler manifolds are even; see Griffiths and Harris [22, page 117].
- (c) (hard Lefschetz property) For a compact Kähler manifold X of real dimension $2n$, there exists $\alpha \in H^2(X; \mathbb{R})$ such that for each j ,

$$L_{\alpha}^j: H^{n-j}(X; \mathbb{R}) \xrightarrow{\alpha^j \cup -} H^{n+j}(X; \mathbb{R})$$

is an isomorphism of cohomology groups. In fact, one can choose α to be the cohomology class represented by the Kähler form of X ; see Griffiths and Harris [22, page 122].

- (d) (formality) The real homotopy type of a compact Kähler manifold is formal; see Deligne, Griffiths, Morgan and Sullivan [11].

Now let us state our main theorem.

Theorem 1.1 *For each $A \in \text{SL}(2, \mathbb{Z}[\sqrt{-1}])$ such that the absolute value of its trace satisfies $|\text{tr}(A)| > 2$, there exists a 6-dimensional compact smooth manifold $M(A)$ (constructed in Proposition 3.1) which satisfies the following properties:*

- (1) *The manifold $M(A)$ is simultaneously C^∞ diffeomorphic to a complex manifold M_C and a smooth symplectic manifold M_S .*
- (2) *The complex manifold M_C is holomorphically Calabi–Yau, ie the canonical line bundle of M_C is trivial as a holomorphic line bundle.*
- (3) *The symplectic manifold M_S is integrally Calabi–Yau, ie $c_1(M_S) = 0$. Here $c_1(M_S) \in H^2(M_S; \mathbb{Z})$ is the first Chern class of M_S with respect to its symplectic structure.*
- (4) *The fundamental group satisfies $\pi_1(M(A)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Hence it is a Kähler group.*
- (5) *All odd-degree Betti numbers of $M(A)$ are even.*
- (6) *The map*

$$L_{[\omega]}^j: H^{3-j}(M_S; \mathbb{R}) \xrightarrow{[\omega]^j \cup -} H^{3+j}(M_S; \mathbb{R})$$

is an isomorphism for each j , where $[\omega] \in H^2(M_S; \mathbb{R})$ is the cohomology class represented by the symplectic form ω of M_S , and $L_{[\omega]}^j(\beta) = [\omega]^j \cup \beta$. Hence $M(A)$ satisfies the hard Lefschetz property.

- (7) *The real homotopy type of $M(A)$ is formal.*
- (8) *However, given any topological space Y , the product $M(A) \times Y$ is not homotopy equivalent to any compact Kähler manifold.*

The properties (4), (5), (6) and (7) in Theorem 1.1 show that $M(A)$ meets the above criteria (a), (b), (c) and (d). Furthermore, (1) shows $M(A)$ carries both the complex and symplectic structures. Nevertheless, property (8) shows that $M(A)$ is highly non-Kähler. Putting Y to be a point, (8) implies that $M(A)$ is not homotopy equivalent to any compact Kähler manifold.

Another advantage of (8) is that, by taking products of $M(A)$ with other manifolds, we get many non-Kähler examples in higher dimensions. For example, let K be a Kummer surface (see Definition 2.1). Then $M(A) \times \prod_{j=1}^n K$ are manifolds satisfying all the conclusions of Theorem 1.1 except that their dimensions are each $4n + 6$.

Notice that Calabi–Yau properties do depend on the choice of complex and symplectic structures. In general, the conclusions (2) and (3) in Theorem 1.1 are independent. So both of them are listed in the statement.

There are many matrices A satisfying the assumption of Theorem 1.1; see Example 3.13. Moreover, the manifolds $M(A)$ will indeed form a large family of homotopy types thanks to the following theorem.

Theorem 1.2 *Given two manifolds $M(A_1)$ and $M(A_2)$ as in Theorem 1.1, if A_1 and A_2 have different spectral radii, then $M(A_1)$ is not homotopy equivalent to $M(A_2)$.*

Our new ingredient to show non-Kählerness is from Hodge theory with twisted coefficients. More precisely, we prove the following theorem, which is essentially a consequence of the structure theorem of cohomology jump loci of compact Kähler manifolds by Wang [41].

Theorem 1.3 *Let $p: E \rightarrow S^1$ be a fiber bundle with path-connected fiber F . Suppose $H^j(F; \mathbb{C})$ is finite-dimensional for every j and suppose E is homotopy equivalent to a compact Kähler manifold. Then the eigenvalues of the monodromy action on $H^j(F; \mathbb{C})$ are roots of unity for every j .*

The construction of $M(A)$ is motivated by the example of Thurston [36] mentioned above. We sketch the construction here and defer more details to Section 3. Each $A \in \mathrm{SL}(2, \mathbb{Z}[\sqrt{-1}])$ yields a biholomorphic automorphism $A_K: K \rightarrow K$ of a Kummer surface (Proposition 2.11). Gluing the two ends of the cylinder $S^1 \times [0, 1] \times K$ by the relation $(a, 0, x) \sim (a, 1, A_K(x))$, we get $M(A) = (S^1 \times [0, 1] \times K)/\sim$. Note that K has a symplectic structure resulting from a holomorphic symplectic form on it. As A_K preserves the holomorphic symplectic form, $M(A)$ is both complex Calabi–Yau and symplectic Calabi–Yau. This manifold $M(A)$ is a fiber bundle over S^1 with fiber $S^1 \times K$. More generally, given any topological space Y , the product $M(A) \times Y$ is a fiber bundle over S^1 with fiber $S^1 \times K \times Y$. We will relate the monodromy actions on $H^j(S^1 \times K \times Y; \mathbb{C})$ to the matrix A . As far as A satisfies the assumption in Theorem 1.1, the monodromy actions will violate the conclusion of Theorem 1.3.

We would like to point out that our examples are similar to the ones by Magnússon in [28], where a weaker result was proved: those examples are not biholomorphic to any compact Kähler manifold.

The outline of this paper is as follows. In Section 2, we recall the definition and some properties of Kummer surfaces. In Sections 3 and 4, we construct examples $M(A)$ and prove the main theorem, Theorem 1.1. Finally, Theorems 1.2 and 1.3 are proved in Section 5.

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2 Kummer surfaces

In this section, we will study a Kummer surface which plays an important role in this paper. For simplicity, we will only use the Kummer surface defined by the standard lattice. Let us recall its definition first.

Let $\Lambda = \{(a_1 + a_2\sqrt{-1}, a_3 + a_4\sqrt{-1}) \mid a_i \in \mathbb{Z}\}$ be the standard lattice of \mathbb{C}^2 . Then \mathbb{C}^2/Λ is a torus of complex dimension 2, which we denote by T . The universal covering map $\pi: \mathbb{C}^2 \rightarrow T$ is also a homomorphism of complex Lie groups. Let

$$\hat{\Lambda} = \frac{1}{2}\Lambda = \left\{ \left(\frac{1}{2}a_1 + \frac{1}{2}a_2\sqrt{-1}, \frac{1}{2}a_3 + \frac{1}{2}a_4\sqrt{-1} \right) \mid a_i \in \mathbb{Z} \right\}.$$

Then $\pi(\hat{\Lambda})$ consists of 16 points w_j ($j = 1, \dots, 16$) in T . Let $w_1 = \pi(0)$.

Let $C_2 = \{\pm 1\}$ be the group with two elements. It acts on \mathbb{C}^2 by multiplication. This action descends to a holomorphic action on T with fixed points exactly the w_j ($j = 1, \dots, 16$). Denote the quotient T/C_2 by \bar{T} . Then \bar{T} is a complex orbifold with 16 singularities $[w_j]$, where $[w_j]$ is the image of w_j in the quotient space. We shall resolve these singularities and obtain a smooth complex surface.

Here we give a concrete description of the blowup map. Locally, each singularity in the orbifold is isomorphic to the singularity in the quotient orbifold \mathbb{C}^2/C_2 at the singular point $[0]$. Denote by \mathcal{H} the Hopf line bundle over $\mathbb{C}\mathbb{P}^1$, ie

$$\mathcal{H} = \{(v, l) \in \mathbb{C}^2 \times \mathbb{C}\mathbb{P}^1 \mid v \in l\}.$$

Define a holomorphic map $F_1: \mathcal{H} \rightarrow \mathbb{C}^2/C_2$ by $F_1(v, l) = [v]$, where $[v]$ is the image of v in \mathbb{C}^2/C_2 . By identifying the zero section of \mathcal{H} with \mathbb{CP}^1 , we have $F_1(\mathbb{CP}^1) = [0]$ and $F_1|_{\mathcal{H} \setminus \mathbb{CP}^1}$ is a holomorphic double covering of $(\mathbb{C}^2/C_2) \setminus \{[0]\}$. Let $\mathcal{H}^2 = \mathcal{H} \otimes \mathcal{H}$ be the tensor square of \mathcal{H} . We define another holomorphic map $F_2: \mathcal{H} \rightarrow \mathcal{H}^2$ by $F_2(v, l) = (v \otimes v, l)$. Also by identifying the zero section of \mathcal{H}^2 with \mathbb{CP}^1 , we have $F_2(\mathbb{CP}^1) = \mathbb{CP}^1$ and $F_2|_{\mathcal{H} \setminus \mathbb{CP}^1}: \mathcal{H} \setminus \mathbb{CP}^1 \rightarrow \mathcal{H}^2 \setminus \mathbb{CP}^1$ is a holomorphic double covering. It is straightforward to check that $F = F_1 \circ F_2^{-1}: \mathcal{H}^2 \rightarrow \mathbb{C}^2/C_2$ is a well-defined holomorphic map. Moreover, $F(\mathbb{CP}^1) = \{[0]\}$ and $F|_{\mathcal{H}^2 \setminus \mathbb{CP}^1}: \mathcal{H}^2 \setminus \mathbb{CP}^1 \rightarrow (\mathbb{C}^2/C_2) \setminus \{[0]\}$ is a biholomorphic map. Thus $F: \mathcal{H}^2 \rightarrow \mathbb{C}^2/C_2$ resolves the singularity $[0]$ of \mathbb{C}^2/C_2 . The map F is called the blowup of \mathbb{C}^2/C_2 at $[0]$.

Now we come back to \bar{T} . By blowing up all singular points $\{[w_j]\}$ in \bar{T} , we obtain a resolution of singularities $P: K \rightarrow \bar{T}$.

Definition 2.1 The complex surface K constructed above is called the Kummer surface defined by the standard lattice.

Immediately, we have the following lemma.

Lemma 2.2 *The map $P: K \rightarrow \bar{T}$ is continuous. Each $Y_j = P^{-1}([w_j])$ is a closed holomorphic submanifold of K . Furthermore,*

$$P|_{K \setminus \bigsqcup_{1 \leq j \leq 16} Y_j}: K \setminus \bigsqcup_{1 \leq j \leq 16} Y_j \rightarrow \bar{T} \setminus \bigsqcup_{1 \leq j \leq 16} \{[w_j]\}$$

is a biholomorphic map.

Each Y_j defined as above is called an exceptional divisor in K .

Remark 2.3 It is well known that every Kummer surface K is Calabi–Yau (see [6, page 241]), which means that its canonical line bundle $\mathcal{K}_K = \bigwedge^2(T^*K)$ is holomorphically trivial. Hence there exist holomorphic symplectic forms, ie nondegenerate holomorphic 2–forms, on K . In fact, we can give an explicit description of such a form. Let (z_1, z_2) be the standard coordinate of \mathbb{C}^2 . Then $dz_1 \wedge dz_2$ is a holomorphic symplectic form on T . Since $dz_1 \wedge dz_2$ is invariant under the C_2 action, it descends to a holomorphic form on the regular part of \bar{T} , which we also denote by $dz_1 \wedge dz_2$. One can easily check that $dz_1 \wedge dz_2$ extends to a nondegenerate holomorphic form on K . Denote this holomorphic symplectic form by ϖ , and denote its real part by $\text{Re } \varpi$. Then K is a real symplectic manifold with symplectic form $\text{Re } \varpi$.

Remark 2.4 A Kummer surface K carries many real symplectic forms compatible with its natural complex structure which define Kähler structures on K . However, $\text{Re } \varpi$ is not such a form. We shall exclusively consider the symplectic structure $(K, \text{Re } \varpi)$ of K throughout this paper.

Proposition 2.5 *The symplectic manifold $(K, \text{Re } \varpi)$ is integrally Calabi–Yau. In other words, $c_1(K, \text{Re } \varpi) = 0$, where $c_1(K, \text{Re } \varpi) \in H^2(K; \mathbb{Z})$ is the first Chern class of K with respect to the symplectic structure $\text{Re } \varpi$.*

Proof Since ϖ is holomorphic symplectic, the structural group $\text{Sp}(4, \mathbb{R})$ of the tangent bundle $(TK, \text{Re } \varpi)$ can be reduced to $\text{Sp}(2, \mathbb{C})$. Here $\text{Sp}(4, \mathbb{R})$ and $\text{Sp}(2, \mathbb{C})$ are the real and complex symplectic groups respectively. Therefore, the conclusion follows from the fact that $\text{Sp}(2, \mathbb{C})$ is simply connected. □

Next, we will prove some topological properties of Kummer surfaces.

Proposition 2.6 [6, page 257, (8.6)] *Every Kummer surface K is simply connected.*

The ring structure of $H^*(K; \mathbb{R})$ is well known (see eg [6, page 241]). However, most descriptions of $H^*(K; \mathbb{R})$ in the literature are rather abstract and algebraic, which is not enough for our purposes. We shall describe it more concretely and geometrically.

Lemma 2.7 $H^{2j+1}(K; \mathbb{R}) = 0$ for all j .

Proof This follows immediately from Proposition 2.6 and Poincaré duality. □

Lemma 2.8 *There exists a decomposition*

$$(2-1) \quad H^2(K; \mathbb{R}) = \bigoplus_{1 \leq j \leq 16} \mathbb{R} \langle \text{PD}(Y_j) \rangle \oplus H^2\left(K, \bigsqcup_{1 \leq j \leq 16} Y_j; \mathbb{R}\right)$$

of the cohomology group $H^2(K; \mathbb{R})$, where each Y_j is the exceptional divisor defined in Lemma 2.2, $\text{PD}(Y_j) \in H^2(K; \mathbb{Z})$ is the Poincaré dual of Y_j and $\mathbb{R} \langle \text{PD}(Y_j) \rangle$ is the real vector space generated by $\text{PD}(Y_j)$. All vector spaces on the right-hand side are naturally subspaces of $H^2(K; \mathbb{R})$.

Proof Given any Y_j , we can choose a small closed tubular neighborhood \mathcal{N}_j of Y_j . These \mathcal{N}_j are pairwise disjoint and they are homeomorphic to a tubular neighborhood (or a disk bundle) of $\mathbb{C}\mathbb{P}^1$ in \mathcal{H}^2 . Thus each $\partial\mathcal{N}_j$ is homeomorphic to $\mathbb{R}\mathbb{P}^3$.

Let $W = K \setminus \bigsqcup_{1 \leq j \leq 16} \text{Int } \mathcal{N}_j$, where $\text{Int } \mathcal{N}_j$ is the interior of \mathcal{N}_j . Using the Mayer–Vietoris sequence, we have

$$H^2(K; \mathbb{R}) = \left(\bigoplus_{1 \leq j \leq 16} H^2(\mathcal{N}_j, \partial \mathcal{N}_j; \mathbb{R}) \right) \oplus H^2(W, \partial W; \mathbb{R}),$$

where the vector spaces on the right side can be considered as subspaces of $H^2(K; \mathbb{R})$ by excision.

Since \mathcal{N}_j is a disk bundle associated with \mathcal{H}^2 over Y_j , we infer that the Thom class $\alpha_j \in H^2(\mathcal{N}_j, \partial \mathcal{N}_j; \mathbb{R})$ is a generator of $H^2(\mathcal{N}_j, \partial \mathcal{N}_j; \mathbb{R})$. Moreover, $\alpha_j = \text{PD}(Y_j)$ if we consider α_j as an element in $H^2(K; \mathbb{R})$; see [30, page 135, Problem 11-C] and [8, page 67]. Thus $H^2(\mathcal{N}_j, \partial \mathcal{N}_j; \mathbb{R}) = \mathbb{R} \langle \text{PD}(Y_j) \rangle$ as subspaces of $H^2(K; \mathbb{R})$.

By excision, $H^2(W, \partial W; \mathbb{R}) = H^2(K, \bigsqcup_{1 \leq j \leq 16} \mathcal{N}_j; \mathbb{R})$. Since Y_j is a deformation retract of \mathcal{N}_j , we infer that $H^2(K, \bigsqcup_{1 \leq j \leq 16} \mathcal{N}_j; \mathbb{R}) = H^2(K, \bigsqcup_{1 \leq j \leq 16} Y_j; \mathbb{R})$. Therefore (2-1) follows. □

Lemma 2.9 *The following induced maps on cohomology groups are all isomorphisms:*

$$H^2\left(K, \bigsqcup_{1 \leq j \leq 16} Y_j; \mathbb{R}\right) \xleftarrow{P^*} H^2\left(\bar{T}, \bigsqcup_{1 \leq j \leq 16} \{w_j\}; \mathbb{R}\right) \xrightarrow{Q^*} H^2(T; \mathbb{R}),$$

where $Q: T \rightarrow \bar{T}$ is the quotient map and P is the blowup map in Lemma 2.2.

Proof Clearly, $\bigsqcup_{1 \leq j \leq 16} Y_j$ is a neighborhood deformation retract of K , and also $\bigsqcup_{1 \leq j \leq 16} \{w_j\}$ is a neighborhood deformation retract of \bar{T} . According to Lemma 2.2, $P: K \setminus \bigsqcup_{1 \leq j \leq 16} Y_j \rightarrow \bar{T} \setminus \bigsqcup_{1 \leq j \leq 16} \{w_j\}$ is a homeomorphism. Therefore P^* is an isomorphism.

Consider the following commutative diagram:

$$\begin{array}{ccc} H^2(\bar{T}, \bigsqcup_{1 \leq j \leq 16} \{w_j\}; \mathbb{R}) & \xrightarrow{Q^*} & H^2(T, \bigsqcup_{1 \leq j \leq 16} \{w_j\}; \mathbb{R}) \\ \downarrow & & \downarrow \\ H^2(\bar{T}; \mathbb{R}) & \xrightarrow{Q^*} & H^2(T; \mathbb{R}) \\ \downarrow & & \downarrow \\ H^2(\bar{T} \setminus \bigsqcup_{1 \leq j \leq 16} \{w_j\}; \mathbb{R}) & \xrightarrow{\hat{Q}^*} & H^2(T \setminus \bigsqcup_{1 \leq j \leq 16} \{w_j\}; \mathbb{R}) \end{array}$$

where \hat{Q} is the restriction of Q to $T \setminus \bigcup_{j=1}^{16} \{w_j\}$, and all vertical maps are induced by the inclusions of spaces.

It is easy to check that all vertical maps are isomorphisms. Since

$$\hat{Q}: T \setminus \bigsqcup_{1 \leq j \leq 16} \{w_j\} \rightarrow \bar{T} \setminus \bigsqcup_{1 \leq j \leq 16} \{[w_j]\}$$

is a double covering, \hat{Q}^* is injective and its image contains exactly the elements fixed by the C_2 action of $H^2(T \setminus \bigsqcup_{1 \leq j \leq 16} \{w_j\}; \mathbb{R})$. Since $H^2(T \setminus \bigsqcup_{1 \leq j \leq 16} \{w_j\}; \mathbb{R}) = H^2(T; \mathbb{R})$ and the whole space $H^2(T; \mathbb{R})$ is fixed by C_2 , we can infer that \hat{Q}^* is surjective. Therefore,

$$Q^*: H^2\left(\bar{T}, \bigsqcup_{1 \leq j \leq 16} \{[w_j]\}; \mathbb{R}\right) \rightarrow H^2(T; \mathbb{R})$$

is an isomorphism. □

Suppose $A \in \text{SL}(2, \mathbb{Z}\sqrt{-1})$. Then $\det(A) = 1$, and A preserves the lattice Λ , ie $A(\Lambda) = \Lambda$. Hence A descends to a complex Lie group automorphism of T , which we denote by A_T .

The following lemma is obvious.

Lemma 2.10 *The following diagram commutes:*

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{A} & \mathbb{C}^2 \\ \pi \downarrow & & \pi \downarrow \\ T & \xrightarrow{A_T} & T \end{array}$$

Moreover, the automorphism $A_T: T \rightarrow T$ induces a permutation of the points w_j ($j = 1, \dots, 16$).

Since $A_T: T \rightarrow T$ commutes with the C_2 action, it descends further to an automorphism of complex orbifolds $A_{\bar{T}}: \bar{T} \rightarrow \bar{T}$. Thus we can blow up $A_{\bar{T}}: \bar{T} \rightarrow \bar{T}$ to get a holomorphic map $A_K: K \rightarrow K$.

Proposition 2.11 (1) *The following diagram commutes, where the map P is the blowup map defined in Lemma 2.2:*

$$\begin{array}{ccc} K & \xrightarrow{A_K} & K \\ P \downarrow & & \downarrow P \\ \bar{T} & \xrightarrow{A_{\bar{T}}} & \bar{T} \end{array}$$

Moreover, A_K is biholomorphic.

(2) *The map A_K permutes the exceptional divisors Y_j ($j = 1, \dots, 16$).*

Proof Part (1) follows immediately from the definition of the blowup map P , and part (2) follows from Lemmas 2.2 and 2.10. \square

Proposition 2.12 *Let ϖ be the holomorphic symplectic form on K defined in Remark 2.3. Then*

$$A_K^* \varpi = \varpi.$$

Proof Since ϖ is a holomorphic form, it suffices to show the equality holds on $K \setminus \bigsqcup_{1 \leq j \leq 16} Y_j$. By Lemma 2.2, Proposition 2.11(1) and the definition of ϖ , it further suffices to show the following equality on T :

$$A_T^*(dz_1 \wedge dz_2) = dz_1 \wedge dz_2.$$

Since $A_T^*(dz_1 \wedge dz_2) = \det(A) dz_1 \wedge dz_2$ and $\det(A) = 1$, the proposition follows. \square

Recall that $A \in \text{SL}(2, \mathbb{Z}\sqrt{-1})$. Considering A as an action on $\mathbb{C}^2 = \mathbb{R}^4$, we denote the corresponding element in $\text{GL}(4, \mathbb{R})$ by $A_{\mathbb{R}}$. Using (2-1), we can give an explicit description of the action of A_K^* on $H^2(K, \mathbb{R})$.

Proposition 2.13 *The subspaces $\bigoplus_{1 \leq j \leq 16} \mathbb{R}\langle \text{PD}(Y_j) \rangle$ and $H^2(K, \bigsqcup_{1 \leq j \leq 16} Y_j; \mathbb{R})$ in (2-1) are A_K^* -invariant subspaces of $H^2(K; \mathbb{R})$. Moreover, suppose $A_T(w_j) = w_k$ as in Lemma 2.10; then $A_K^*(\text{PD}(Y_k)) = \text{PD}(Y_j)$. The action of A_K^* on the space $H^2(K, \bigsqcup_{1 \leq j \leq 16} Y_j; \mathbb{R})$ is isomorphic to the induced action of $A_{\mathbb{R}}$ on $\wedge^2(4, \mathbb{R})$, where $\wedge^2(4, \mathbb{R})$ is the space of skew-symmetric bilinear forms on \mathbb{R}^4 .*

Proof By Proposition 2.11, we know that $A_K(Y_j) = Y_k$ if $A_T(w_j) = w_k$. Obviously, A_K preserves the orientations of all Y_j and K . Thus, $A_K^*(\text{PD}(Y_k)) = \text{PD}(Y_j)$, and hence $\bigoplus_{1 \leq j \leq 16} \mathbb{R}\langle \text{PD}(Y_j) \rangle$ is A_K^* -invariant.

Since A_K maps $(K, \bigsqcup_{1 \leq j \leq 16} Y_j)$ to itself, $H^2(K, \bigsqcup_{1 \leq j \leq 16} Y_j; \mathbb{R})$ is also A_K^* -invariant. We also have the following commutative diagram:

$$\begin{CD} H^2(K, \bigsqcup_{1 \leq j \leq 16} Y_j; \mathbb{R}) @<P^*<< H^2(\bar{T}, \bigcup_{1 \leq j \leq 16} \{w_j\}; \mathbb{R}) @>Q^*>> H^2(T; \mathbb{R}) \\ @V A_K^* VV @V A_T^* VV @V A_T^* VV \\ H^2(K, \bigsqcup_{1 \leq j \leq 16} Y_j; \mathbb{R}) @<P^*<< H^2(\bar{T}, \bigcup_{1 \leq j \leq 16} \{w_j\}; \mathbb{R}) @>Q^*>> H^2(T; \mathbb{R}) \end{CD}$$

By Lemma 2.9, all horizontal maps are isomorphisms. Therefore, the action of A_K^* on $H^2(K, \bigsqcup_{1 \leq j \leq 16} Y_j; \mathbb{R})$ is isomorphic to that of A_T^* on $H^2(T; \mathbb{R})$.

Obviously, the action of A_T^* on $H^2(T; \mathbb{R})$ is isomorphic to the action of $A_{\mathbb{R}}$ on $\wedge^2(4, \mathbb{R})$, which completes the proof. \square

Lemma 2.14 *If, as an element in $SL(2, \mathbb{Z}\sqrt{-1})$, the matrix A has eigenvalues λ and λ^{-1} (here we allow $\lambda = \lambda^{-1}$), then the set of eigenvalues of A_K^* on $H^2(K; \mathbb{C})$ consists of $|\lambda|^2$, $|\lambda|^{-2}$ and some complex numbers with absolute value 1.*

Proof By (2-1) and Proposition 2.13, we know that

$$H^2(K; \mathbb{C}) = \bigoplus_{1 \leq j \leq 16} \mathbb{C}\langle PD(Y_j) \rangle \oplus H^2\left(K, \bigsqcup_{1 \leq j \leq 16} Y_j; \mathbb{C}\right),$$

where $V_1 = \bigoplus_{1 \leq j \leq 16} \mathbb{C}\langle PD(Y_j) \rangle$ and $V_2 = H^2(K, \bigsqcup_{1 \leq j \leq 16} Y_j; \mathbb{C})$ are A_K^* -invariant subspaces. Moreover, A_K^* permutes $PD(Y_j)$. So $A_K^*|_{V_1}$ is unitary and hence its eigenvalues all have absolute value 1.

The action of A_K^* on V_2 is isomorphic to that of $A_{\mathbb{R}}$ on $\wedge^2(4, \mathbb{C}) \cong \wedge^2(4, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. Since $A_{\mathbb{R}} \in GL(4, \mathbb{R})$ is defined from $A \in GL(2, \mathbb{C})$, it has two invariant subspaces W and \overline{W} as an action on \mathbb{C}^4 such that $\mathbb{C}^4 = W \oplus \overline{W}$, $A_{\mathbb{R}}|_W \cong A$ and $A_{\mathbb{R}}|_{\overline{W}} \cong \overline{A}$. Here \overline{W} is the complex conjugate of W and $\overline{A} \in GL(2, \mathbb{C})$ is the complex conjugate of A . Thus the action of $A_{\mathbb{R}}$ on \mathbb{C}^4 exactly has eigenvalues λ , λ^{-1} , $\overline{\lambda}$ and $\overline{\lambda}^{-1}$. By Proposition 2.13, $A_K^*|_{V_2}$ exactly has eigenvalues $|\lambda|^2$, $|\lambda|^{-2}$, $\lambda\overline{\lambda}^{-1}$, $\overline{\lambda}\lambda^{-1}$ and 1. Thus, the lemma follows. \square

3 The main theorem

In this section, we will construct our examples of non-Kähler manifolds and prove the main theorem, Theorem 1.1. We will assume Theorem 1.3 and Proposition 4.1, and postpone their proofs to later sections.

Throughout this section A is a matrix in $SL(2, \mathbb{Z}[\sqrt{-1}])$. There is no additional restriction on A unless we state it explicitly.

Endow $S^1 \times \mathbb{R}^1$ with the standard complex structure and the standard symplectic structure so that $S^1 \times [0, 1]$ has symplectic area 1. By the complex structure and the symplectic structure (as in Proposition 2.5) of K , the manifold $S^1 \times \mathbb{R}^1 \times K$ has its product complex and symplectic structures.

For any A_K in Proposition 2.11, one can define a \mathbb{Z} action on $S^1 \times \mathbb{R}^1 \times K$ by

$$(3-1) \quad \begin{aligned} \mathbb{Z} \times S^1 \times \mathbb{R}^1 \times K &\rightarrow S^1 \times \mathbb{R}^1 \times K, \\ (n, a, b, c) &\mapsto (a, b + n, A_K^n(c)). \end{aligned}$$

Proposition 3.1 *The quotient space*

$$(3-2) \quad M(A) = (S^1 \times \mathbb{R}^1 \times K) / \mathbb{Z}$$

is a compact smooth manifold of dimension 6. It has a unique complex (resp. symplectic) structure such that the quotient map $S^1 \times \mathbb{R}^1 \times K \rightarrow M(A)$ is holomorphic (resp. symplectic). The above complex structure and symplectic structure of $M(A)$ are compatible with its smooth structure.

Proof Since the \mathbb{Z} action is smooth, proper and without fixed points, $M(A)$ is a smooth manifold. Notice that $M(A)$ is a fiber bundle over $S^1 \times S^1$ with compact fiber K . Hence it is compact and has dimension 6. Furthermore, the \mathbb{Z} action is both holomorphic (Proposition 2.11) and symplectic (Proposition 2.12). Thus $M(A)$ has the desired complex and symplectic structures. □

Remark 3.2 In fact, $M(A) = M(-A)$. This is because $A_{\bar{T}} = (-A)_{\bar{T}}$, and hence $A_K = (-A)_K$.

Proposition 3.1 defines the manifold $M(A)$ in Theorem 1.1. Now, we need to verify that $M(A)$ satisfies all the properties in Theorem 1.1 when $|\text{tr}(A)| > 2$.

In this section, we write $M_C(A)$ (or $M_S(A)$) when we want to emphasize the complex (or symplectic) structure on $M(A)$. Otherwise, we simply write $M(A)$.

Proposition 3.3 *The canonical line bundle $\mathcal{K}_{M_C(A)} = \wedge^3(T^*M_C(A))$ is holomorphically trivial.*

Proof It suffices to construct a holomorphic frame of $\mathcal{K}_{M_C(A)}$. Since \mathcal{K}_K is holomorphically trivial, there exist holomorphic frames of \mathcal{K}_K , for example, the holomorphic symplectic form ϖ in Proposition 2.12. Furthermore, since \mathbb{C}^1 is a holomorphic covering of $S^1 \times \mathbb{R}^1$, obviously, dz is a holomorphic frame of $\mathcal{K}_{S^1 \times \mathbb{R}^1}$, where z is the standard coordinate of \mathbb{C}^1 . Thus $dz \times \varpi$ is a holomorphic frame of $\mathcal{K}_{S^1 \times \mathbb{R}^1 \times K}$. By Proposition 2.12, $dz \times \varpi$ is invariant under the \mathbb{Z} action in (3-1). By (3-2), $dz \times \varpi$ descends to a desired holomorphic frame of $\mathcal{K}_{M_C(A)}$. □

Proposition 3.4 *The symplectic manifold $M_S(A)$ is integrally Calabi–Yau. In other words, $c_1 = 0$, where $c_1 \in H^2(M_S(A); \mathbb{Z})$ is the first Chern class of $M_S(A)$ with respect to its symplectic structure.*

Proof The tangent bundle of $S^1 \times \mathbb{R}^1 \times K$ splits symplectically into two subbundles:

$$T(S^1 \times \mathbb{R}^1 \times K) = \tilde{E}_H \oplus \tilde{E}_V,$$

where $\tilde{E}_H = T(S^1 \times \mathbb{R}^1) \times K$ and $\tilde{E}_V = S^1 \times \mathbb{R}^1 \times TK$. Since the \mathbb{Z} action in (3-1) preserves the above splitting, it descends to the splitting

$$TM_S(A) = E_H \oplus E_V,$$

where E_H comes from \tilde{E}_H , and E_V comes from \tilde{E}_V .

Let ϖ be the holomorphic symplectic form in Proposition 2.12. Clearly, the restriction of $\text{Re}(1 \times \varpi)$ on \tilde{E}_V yields the symplectic structure of \tilde{E}_V . Therefore, the structure group of \tilde{E}_V can be reduced from $\text{Sp}(4, \mathbb{R})$ to $\text{Sp}(2, \mathbb{C})$. Since the \mathbb{Z} action preserves $1 \times \varpi$, this form descends to a form on $M_S(A)$. Thus the structure group of E_V can be also reduced to $\text{Sp}(2, \mathbb{C})$, which implies the first Chern class of E_V vanishes.

On the other hand, $E_H = p^*T(S^1 \times S^1)$ as symplectic bundles, where $p: M_S(A) \rightarrow S^1 \times S^1$ is the bundle projection. Since $S^1 \times S^1$ is Calabi–Yau, the first Chern class of E_H also vanishes.

In summary, the first Chern class of $M_S(A)$ vanishes. □

Proposition 3.5 *The fundamental group satisfies $\pi_1(M(A)) = \mathbb{Z} \oplus \mathbb{Z}$, and the odd-degree Betti numbers of $M(A)$ are even.*

Proof Since $M(A)$ is a fiber bundle with base $S^1 \times S^1$ and fiber K , we have the homotopy long exact sequence of fibrations

$$\pi_1(K) \rightarrow \pi_1(M(A)) \rightarrow \pi_1(S^1 \times S^1) \rightarrow \pi_0(K).$$

According to Proposition 2.6, $\pi_1(K) = 0$. Obviously $\pi_0(K)$ is a one-point set. Therefore, $\pi_1(M(A)) \cong \pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$, and hence the first Betti number of $M(A)$ is 2. Since $\dim(M(A)) = 6$, by Poincaré duality, the fifth Betti number is also 2. It also follows from Poincaré duality that the cup product pairing

$$H^3(M(A); \mathbb{R}) \otimes H^3(M(A); \mathbb{R}) \xrightarrow{\cup} H^6(M(A); \mathbb{R}) \cong \mathbb{R}$$

is nondegenerate. This pairing is skew-symmetric. Therefore, the dimension of $H^3(M(A); \mathbb{R})$ is even. □

Suppose F is a topological space and $\eta: F \rightarrow F$ is a homeomorphism. Consider the \mathbb{Z} action on the topological space $\mathbb{R}^1 \times F$ defined by

$$(3-3) \quad \begin{aligned} \mathbb{Z} \times \mathbb{R}^1 \times F &\rightarrow \mathbb{R}^1 \times F, \\ (n, t, a) &\mapsto (n + t, \eta^n(a)). \end{aligned}$$

The quotient space $N = (\mathbb{R}^1 \times F)/\mathbb{Z}$ is a fiber bundle over S^1 with fiber F . Denote by $p_2: N \rightarrow S^1$ the projection. Let $i: F \rightarrow N$ be the natural inclusion map to the fiber over $0 \in \mathbb{R}^1/\mathbb{Z} = S^1$.

Lemma 3.6 *We have the long exact sequence*

$$\dots \rightarrow H^j(N; \mathbb{R}) \xrightarrow{i^*} H^j(F; \mathbb{R}) \xrightarrow{\eta^* - \text{Id}} H^j(F; \mathbb{R}) \xrightarrow{\delta} H^{j+1}(N; \mathbb{R}) \rightarrow \dots,$$

where Id is the identity map. If $\beta \in H^j(F; \mathbb{R})$ such that $i^*\alpha = \beta$ with $\alpha \in H^j(N; \mathbb{R})$, then $\delta(\beta) = p_2^*s \cup \alpha$. Here s is the generator in $H^1(S^1; \mathbb{Z})$ which represents the positive orientation of S^1 .

Lemma 3.6 is a special case of the Wang sequence which studies fibrations over spheres S^n ; see [42, page 319, (1.9)] and [12, page 212, Proposition 6.4.8]. In the literature, the Wang sequence is usually proved in the case of $n > 1$. Nevertheless, the proof of the case $n = 1$ using the Mayer–Vietoris sequence is similar. So we skip the proof here.

Taking $F = K$ and $\eta = A_K$ in (3-3), we get a fiber bundle $N = (\mathbb{R}^1 \times K)/\mathbb{Z}$ over S^1 with fiber K . By definition, $M(A) = S^1 \times N$.

Lemma 3.7 *The homomorphism $i^*: H^{2j}(N; \mathbb{R}) \rightarrow H^{2j}(K; \mathbb{R})$ is an isomorphism when $j = 0, 2$. When $j = 1$, the homomorphism i^* is injective, and its image satisfies $\text{Im } i^* = V$, where V is the set of fixed elements of A_K^* in $H^2(K; \mathbb{R})$.*

Proof By Lemma 3.6, we have the following Wang sequence:

$$H^{2j-1}(K; \mathbb{R}) \xrightarrow{\delta} H^{2j}(N; \mathbb{R}) \xrightarrow{i^*} H^{2j}(K; \mathbb{R}) \xrightarrow{A_K^* - \text{Id}} H^{2j}(K; \mathbb{R}).$$

By Lemma 2.7, $H^{2j-1}(K; \mathbb{R}) = 0$. Therefore, i^* is injective and its image is the subspace of $H^{2j}(K; \mathbb{R})$ fixed by A_K^* .

For $j = 0, 2$, the whole space $H^{2j}(K; \mathbb{R})$ is fixed by A_K^* . Thus i^* is an isomorphism for $j = 0, 2$. \square

Lemma 3.8 *If A is diagonalizable, then A_K^* is diagonalizable on $H^2(K; \mathbb{C})$.*

Proof By Proposition 2.13, it suffices to show that the restrictions of A_K^* to the direct summands $\bigoplus_{1 \leq j \leq 16} \mathbb{C} \langle \text{PD}(Y_j) \rangle$ and $H^2(K, \bigsqcup_{1 \leq j \leq 16} Y_j; \mathbb{C})$ are both diagonalizable. By Proposition 2.13, A_K^* permutes these $\text{PD}(Y_j)$. Thus the restriction of A_K^* on $\bigoplus_{1 \leq j \leq 16} \mathbb{R} \langle \text{PD}(Y_j) \rangle$ is unitary and hence diagonalizable. The restriction of A_K^* on $H^2(K, \bigsqcup_{1 \leq j \leq 16} Y_j; \mathbb{C})$ is isomorphic to the induced action of $A_{\mathbb{R}}$ on $\wedge^2(4, \mathbb{C})$, and hence diagonalizable too. \square

Lemma 3.9 *Let $s \in H^1(S^1; \mathbb{Z})$ be the generator corresponding to the positive orientation. Let $s_2 = p_2^*s \in H^1(N; \mathbb{Z})$.*

When $j = 0$ or 2 , the map

$$(3-4) \quad H^{2j}(N; \mathbb{R}) \xrightarrow{s_2 \cup -} H^{2j+1}(N; \mathbb{R})$$

is an isomorphism. Here $s_2 \cup -$ is the map defined by taking the cup product with s_2 . Assume additionally A is diagonalizable. Then (3-4) is also an isomorphism for $j = 1$.

Proof By Lemma 3.6, we have the following Wang sequence:

$$H^{2j}(K; \mathbb{R}) \xrightarrow{A_K^* - \text{Id}} H^{2j}(K; \mathbb{R}) \xrightarrow{\delta} H^{2j+1}(N; \mathbb{R}) \xrightarrow{i^*} H^{2j+1}(K; \mathbb{R}).$$

By Lemma 2.7, $H^{2j+1}(K; \mathbb{R}) = 0$, and hence δ is surjective.

Clearly, $A_K^* - \text{Id} = 0$ on $H^{2j}(K; \mathbb{R})$ when $j = 0$ or 2 . If additionally A is diagonalizable, then by Lemma 3.8, $A_K^* - \text{Id}$ is diagonalizable on $H^2(K; \mathbb{C})$. This assumption further implies that on $H^2(K; \mathbb{R})$, the kernel of $A_K^* - \text{Id}$ is a complement of the image of $A_K^* - \text{Id}$.

Thus, when $j = 0$ or 2 , the map

$$\delta: \text{Ker}[(A_K^* - \text{Id})|_{H^{2j}(K; \mathbb{R})}] \rightarrow H^{2j+1}(N; \mathbb{R})$$

is an isomorphism. When $j = 1$ and A is diagonalizable, δ is also an isomorphism.

On the other hand, by Lemma 3.7,

$$i^*: H^{2j}(N; \mathbb{R}) \rightarrow \text{Ker}[(A_K^* - \text{Id})|_{H^{2j}(K; \mathbb{R})}]$$

is an isomorphism. Combining the above two isomorphisms,

$$\delta \circ i^*: H^{2j}(N; \mathbb{R}) \rightarrow H^{2j+1}(N; \mathbb{R})$$

is also an isomorphism. By Lemma 3.6, $\delta \circ i^*(\alpha) = s_2 \cup \alpha$, which completes the proof. \square

By Lemmas 3.7 and 3.9 and the Künneth formula, we obtain the following description of $H^*(M(A); \mathbb{R})$.

Lemma 3.10 *Let $s_1 \in H^1(S^1; \mathbb{Z})$ and $[K]^* \in H^4(K; \mathbb{Z})$ be the generators corresponding to the positive orientations.*

For $M(A) = S^1 \times N$, we have

$$\begin{aligned} H^0(M(A); \mathbb{R}) &= \mathbb{R}, \\ H^1(M(A); \mathbb{R}) &= \mathbb{R}\langle s_1 \times 1 \rangle \oplus \mathbb{R}\langle 1 \times s_2 \rangle, \\ H^2(M(A); \mathbb{R}) &= \mathbb{R}\langle s_1 \times s_2 \rangle \oplus (1 \times V), \\ H^5(M(A); \mathbb{R}) &= \mathbb{R}\langle s_1 \times [K]^* \rangle \oplus \mathbb{R}\langle 1 \times (s_2 \cup [K]^*) \rangle, \\ H^6(M(A); \mathbb{R}) &= \mathbb{R}\langle s_1 \times (s_2 \cup [K]^*) \rangle. \end{aligned}$$

Assume additionally that A is diagonalizable. Then

$$\begin{aligned} H^3(M(A); \mathbb{R}) &= (s_1 \times V) \oplus [1 \times (s_2 \cup V)], \\ H^4(M(A); \mathbb{R}) &= [s_1 \times (s_2 \cup V)] \oplus \mathbb{R}\langle 1 \times [K]^* \rangle. \end{aligned}$$

Here $s_2 \in H^1(N; \mathbb{Z})$ is the element defined in Lemma 3.9, and V is the subspace of $H^2(K; \mathbb{R})$ defined in Lemma 3.7. We identify $H^2(N; \mathbb{R})$ with V via the isomorphism $i^*: H^2(N; \mathbb{R}) \rightarrow V$, and we also consider $[K]^*$ as an element in $H^4(N; \mathbb{R})$ via the isomorphism $i^*: H^4(N; \mathbb{R}) \rightarrow H^4(K; \mathbb{R})$.

Now we study the hard Lefschetz property of $M_S(A)$.

Lemma 3.11 *Let $[\omega] \in H^2(M_S(A); \mathbb{R})$ be the cohomology class represented by the symplectic form ω of $M_S(A)$. Then*

$$[\omega] = s_1 \times s_2 + 1 \times \theta,$$

where $\theta \in V$ and $\theta^2 = d[K]^*$ with $d > 0$. Here $s_1, s_2, [K]^*$ and V are as in Lemma 3.10.

Proof Putting together the standard symplectic form ω_1 on $S^1 \times \mathbb{R}^1$ and the form $\text{Re } \varpi$ on K , we obtain a product symplectic form on $S^1 \times \mathbb{R}^1 \times K$. This symplectic form descends to the symplectic form ω on $M_S(A)$. The form ω_1 also descends to a symplectic form on $S^1 \times S^1$, which we also denote by ω_1 . Then $\int_{S^1 \times S^1} \omega_1 = 1$.

Clearly, $1 \times \text{Re } \varpi$ is a closed form on $\mathbb{R}^1 \times K$. By Proposition 2.12, $1 \times \text{Re } \varpi$ descends to a closed form ω_2 on N . Then

$$\omega = p^* \omega_1 + 1 \times \omega_2,$$

where $p: M_S(A) \rightarrow S^1 \times S^1$ is the projection. This implies $[\omega] = s_1 \times s_2 + 1 \times \theta$, where $\theta = [\omega_2] \in H^2(N; \mathbb{R})$, and $H^2(N; \mathbb{R})$ is identified with the V in Lemma 3.10. Since $(\text{Re } \varpi)^2$ is a volume form on K , we have $[\text{Re } \varpi]^2 = d[K]^* \in H^4(K; \mathbb{R})$ with $d > 0$. Furthermore, the restriction of ω_2 on a fiber K is equal to $\text{Re } \varpi$. Therefore, $\theta^2 = d[K]^*$, where $H^4(N; \mathbb{R})$ is identified with $H^4(K; \mathbb{R})$ by Lemma 3.10. \square

Proposition 3.12 *Assume A is diagonalizable. Then, for any j ,*

$$L_{[\omega]}^j: H^{3-j}(M_S(A); \mathbb{R}) \xrightarrow{[\omega]^j \cup -} H^{3+j}(M_S(A); \mathbb{R})$$

is an isomorphism, where $[\omega] \in H^2(M_S(A); \mathbb{R})$ is the cohomology class represented by the symplectic form ω of $M_S(A)$.

Proof Since A is diagonalizable, we can use all the conclusions of Lemmas 3.9 and 3.10.

The statement is trivial if $j \neq 1, 2$ or 3 . Moreover, it suffices to show that $L_{[\omega]}^j$ is injective, because $\dim(H^{3-j}) = \dim(H^{3+j})$. By Lemma 3.11, we have $[\omega] = s_1 \times s_2 + 1 \times \theta$. Note that $s_1^2 = 0$, $s_2^2 = 0$, $\theta^3 = 0$ and $\theta^2 = d[K]^*$ with $d > 0$.

When $j = 3$, since ω^3 is a volume form of $M_S(A)$, the map $L_{[\omega]}^3$ is an isomorphism.

When $j = 2$,

$$[\omega]^2 = 2s_1 \times (s_2 \cup \theta) + 1 \times \theta^2 = 2s_1 \times (s_2 \cup \theta) + d \times [K]^*.$$

By Lemma 3.10, any $\beta \in H^1(M_S(A); \mathbb{R})$ is of the form $\beta = s_1 \times a + b \times s_2$, with $a, b \in \mathbb{R}$. Then

$$L_{[\omega]}^2(\beta) = ads_1 \times [K]^* + bd \times (s_2 \cup [K]^*).$$

By Lemma 3.9, $s_2 \cup [K]^* \neq 0$. By Lemma 3.10, $s_1 \times [K]^*$ and $1 \times (s_2 \cup [K]^*)$ are linearly independent in $H^5(K; \mathbb{R})$. Since $d \neq 0$, if $L_{[\omega]}^2(\beta) = 0$, then $a = b = 0$ and $\beta = 0$. Therefore, $L_{[\omega]}^2$ is injective.

When $j = 1$, note that (i) $\theta \in V$; (ii) $\theta \cup V \subseteq H^4(K; \mathbb{R})$; (iii) $\theta^2 \neq 0$; and (iv) $\dim(H^4(K; \mathbb{R})) = 1$. We infer that

$$V = \mathbb{R}(\theta) \oplus V_2, \quad \text{where } V_2 = \{\gamma \in V \mid \theta \cup \gamma = 0\}.$$

By Lemma 3.10, any $\beta \in H^2(M_S(A); \mathbb{R})$ is of the form $\beta = as_1 \times s_2 + b \times \theta + 1 \times \gamma$, where $a, b \in \mathbb{R}$ and $\gamma \in V_2$. Since $\theta^2 = d[K]^*$ and $\theta \cup \gamma = 0$, we have

$$\begin{aligned} L_{[\omega]}^1(\beta) &= (a + b)s_1 \times (s_2 \cup \theta) + s_1 \times (s_2 \cup \gamma) + b \times \theta^2 + 1 \times (\theta \cup \gamma) \\ &= (a + b)s_1 \times (s_2 \cup \theta) + s_1 \times (s_2 \cup \gamma) + bd \times [K]^*. \end{aligned}$$

Since $\theta \neq 0$ and by Lemma 3.9, $s_2 \cup \theta \neq 0$. By Lemma 3.10, $s_1 \times (s_2 \cup \theta)$ and $1 \times [K]^*$ are linearly independent in $H^4(K; \mathbb{R})$. If $\gamma \neq 0$, then $s_1 \times (s_2 \cup \theta)$, $s_1 \times (s_2 \cup \gamma)$ and $1 \times [K]^*$ are also linearly independent. Recall that $d \neq 0$. Suppose $L_{[\omega]}^1(\beta) = 0$. Then $a + b = 0$, $b = 0$ and $\gamma = 0$, and hence $\beta = 0$. Therefore, $L_{[\omega]}^1$ is injective. \square

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Since $|\text{tr}(A)| > 2$, we know that A has two distinct eigenvalues λ and λ^{-1} with $|\lambda| > 1$. Therefore, A is diagonalizable.

Take $M_C = M_C(A)$ and $M_S = M_S(A)$. Then the properties (1)–(7) follow from Propositions 3.1, 3.3, 3.4, 3.5, 3.12 and 4.1.

We prove (8) by contradiction. Suppose $M(A) \times Y$ is homotopy equivalent to a compact Kähler manifold X .

Since $H^*(M(A); \mathbb{C})$ is finite-dimensional, we have the Künneth formula

$$H^k(X; \mathbb{C}) \cong \bigoplus_{i+j=k} H^i(M(A); \mathbb{C}) \otimes H^j(Y; \mathbb{C}).$$

Since $H^*(X; \mathbb{C})$ is finite-dimensional, so is $H^*(Y; \mathbb{C})$. As the argument can be applied on each path component of Y , we may assume that Y is path-connected.

Recall that $M(A) = S^1 \times N$ and N is a fiber bundle over S^1 with fiber a Kummer surface K . Then $M(A) \times Y$ is a fiber bundle over S^1 with path-connected fiber $S^1 \times K \times Y$. By the Künneth formula, each $H^*(S^1 \times K \times Y; \mathbb{C})$ is finite-dimensional. Hence we can apply Theorem 1.3 to $\phi: M(A) \times Y \rightarrow S^1$, where ϕ is the composition of the projection $M(A) \times Y \rightarrow M(A)$ and the bundle map $M(A) \rightarrow S^1$. By (3-1), there is a monodromy map

$$\text{Id} \times A_K \times \text{Id}: S^1 \times K \times Y \rightarrow S^1 \times K \times Y$$

which yields a monodromy action on $H^*(S^1 \times K \times Y; \mathbb{C})$, where each Id is the identity map. By the Künneth formula again, $H^2(S^1 \times K \times Y; \mathbb{C})$ has an $(\text{Id} \times A_K \times \text{Id})^*$ -invariant subspace $H^0(S^1; \mathbb{C}) \otimes H^2(K; \mathbb{C}) \otimes H^0(Y; \mathbb{C})$, on which $(\text{Id} \times A_K \times \text{Id})^* = 1 \otimes A_K^* \otimes 1$. Since A has an eigenvalue λ , by Lemma 2.14, $(\text{Id} \times A_K \times \text{Id})^*$ has a real eigenvalue $|\lambda|^2$ on $H^2(S^1 \times K \times Y; \mathbb{C})$. Therefore, the monodromy action of the fiber bundle $\phi: M(A) \times Y \rightarrow S^1$ has an eigenvalue $|\lambda|^2$. Since $|\lambda| > 1$, we have $|\lambda|^2 > 1$.

However, we have assumed that $M(A) \times Y$ is homotopy equivalent to a compact Kähler manifold. By Theorem 1.3, the eigenvalues of this monodromy action have to be roots of unity. This is a contradiction to the fact that $|\lambda|^2 > 1$. \square

Example 3.13 There is a large family of matrices $A \in \text{SL}(2, \mathbb{Z}[\sqrt{-1}])$ satisfying the assumption of Theorem 1.1. For instance, we can choose A to be any of these matrices:

$$\begin{bmatrix} 1 + \sqrt{-1} & \sqrt{-1} \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 + 4\sqrt{-1} \\ 1 & 2\sqrt{-1} \end{bmatrix}, \quad \begin{bmatrix} 1 & n-2 \\ 1 & n-1 \end{bmatrix} \quad \text{for any integer } n > 2.$$

4 Formality

In this section, we prove the following proposition, which is Theorem 1.1(7).

Proposition 4.1 *The real homotopy type of $M(A)$ in Theorem 1.1 is formal.*

Let us first recall the definition of the real homotopy type of a manifold being formal. A commutative differential graded algebra (CDGA) over a field \mathbb{K} of characteristic 0, denoted by (\mathcal{A}, d) , is a graded algebra $\mathcal{A} = \bigoplus_{k \geq 0} \mathcal{A}^k$ with a differential d which satisfies the following conditions:

- (1) For all $x \in \mathcal{A}^k$ and $y \in \mathcal{A}^l$, we have $xy = (-1)^{kl}yx$.
- (2) The differential $d: \mathcal{A} \rightarrow \mathcal{A}$ has degree 1, ie for any k , we have $d(\mathcal{A}^k) \subseteq \mathcal{A}^{k+1}$, and $d^2 = 0$.
- (3) For all $x \in \mathcal{A}^k$ and $y \in \mathcal{A}$, we have $d(xy) = dx \cdot y + (-1)^k x \cdot dy$.

If $x \in \mathcal{A}^k$, we say x is a homogenous element with degree k , and write $\text{deg}(x) = k$.

With differential d , a CDGA (\mathcal{A}, d) is naturally a complex of \mathbb{K} -vector spaces. Hence we can define its cohomology groups $H^i(\mathcal{A}, d)$. Notice that with trivial differential, $(H^*(\mathcal{A}, d), 0)$ has a natural CDGA structure. A homomorphism of CDGAs is a homomorphism of graded algebras which is also a map of complexes.

Definition 4.2 A CDGA (\mathcal{M}, d) is called minimal if it satisfies these conditions:

- (1) (\mathcal{M}, d) is a free commutative graded algebra.
- (2) There exists a collection of homogeneous free generators $\{a_\tau \mid \tau \in \mathcal{I}\}$, for some well-ordered index set \mathcal{I} , such that $\text{deg}(a_\mu) \leq \text{deg}(a_\tau)$ if $\mu, \tau \in \mathcal{I}$ and $\mu < \tau$. Moreover, da_τ is expressed in terms of finitely many a_μ with $\mu < \tau$.

Here a free commutative graded algebra is the tensor product of the polynomial algebra generated by its free generators of even degrees and the exterior algebra generated by its free generators of odd degrees. For more discussions about minimal CDGAs, see [26, page 14] and [14, page 152].

Definition 4.3 Suppose \mathcal{A} is a CDGA and \mathcal{M} is a minimal CDGA. If there is a homomorphism of CDGAs $\varphi: \mathcal{M} \rightarrow \mathcal{A}$ which induces isomorphisms $\varphi^*: H^*(\mathcal{M}) \rightarrow H^*(\mathcal{A})$ of cohomology groups, then \mathcal{M} is a minimal model of \mathcal{A} .

A CDGA \mathcal{A} over \mathbb{K} is called connected if $H^0(\mathcal{A}) = \mathbb{K}$. (Here, $\mathbb{K} \subseteq \mathcal{A}^0$ as a subalgebra and the differential d vanishes on \mathbb{K} .) If \mathcal{A} is connected, then there exists a unique minimal model \mathcal{M} of \mathcal{A} up to isomorphisms; see [26, Chapter 6].

Definition 4.4 [11, page 260] Suppose \mathcal{M} is a minimal model of \mathcal{A} . If there exists a homomorphism of CDGAs $\psi: (\mathcal{M}, d) \rightarrow (H^*(\mathcal{A}), d=0)$ which induces the identity isomorphism of $H^*(\mathcal{A})$, then \mathcal{A} is called formal.

Suppose M is a smooth manifold. Denote by ΩM the de Rham complex of differential forms on M . Then ΩM is a CDGA over \mathbb{R} with cohomology $H^*(\Omega M) \cong H^*(M; \mathbb{R})$.

Definition 4.5 The real homotopy type of a smooth manifold M is formal, or briefly, M is formal if ΩM is formal.

We have explained the content of Proposition 4.1. To prove this proposition, we need the definition of s -formal, as in [14, Definition 2.2]. Suppose \mathcal{M} has homogeneous free generators a_τ ($\tau \in \mathcal{I}$). Denote by \mathcal{V}^i the linear space which is spanned by $\{a_\tau \mid \tau \in \mathcal{I}, \deg(a_\tau) = i\}$, ie \mathcal{V}^i is spanned by the homogeneous free generators of degree i . Clearly, $\mathcal{V}^i \subseteq \mathcal{M}^i$. It is necessary to point out that, for a fixed \mathcal{M} , there are different choices of homogeneous free generators. The space \mathcal{M}^i is independent of these choices. However, \mathcal{V}^i does depend on the choices. In fact, this observation has been employed in the proof of Theorem 4.9 below. Let X be a subset of \mathcal{M} . Denote by $\wedge(X)$ the subalgebra over \mathbb{K} generated by X in \mathcal{M} .

Definition 4.6 Suppose (\mathcal{A}, d) has a minimal model (\mathcal{M}, d) . We say \mathcal{A} is s -formal for some integer $s \geq 0$ or $s = +\infty$ if we can choose homogeneous free generators of \mathcal{M} such that $\mathcal{V}^i = \mathcal{C}^i \oplus \mathcal{N}^i$, where \mathcal{C}^i and \mathcal{N}^i satisfy the following conditions:

- (1) $d(\mathcal{C}^i) = 0$.
- (2) $d: \mathcal{N}^i \rightarrow \mathcal{M}$ is injective.
- (3) Any cocycle in the ideal $I(\bigoplus_{i \leq s} \mathcal{N}^i)$ generated by $\bigoplus_{i \leq s} \mathcal{N}^i$ in $\wedge(\bigoplus_{i \leq s} \mathcal{V}^i)$ is exact in \mathcal{M} .

We say a smooth manifold M is s -formal if its de Rham complex ΩM is s -formal.

Theorem 4.7 [11, (4.1)] *A CDGA \mathcal{A} is formal if and only if it is $+\infty$ -formal.*

For a manifold M of dimension n , we have $H^i(\Omega M) = 0$ for $i > n$. Therefore, when $i > n$, a cocycle of degree i in a minimal model of ΩM must be exact. Theorem 4.7 immediately implies the following result.

Theorem 4.8 [14, Lemma 2.10] *Suppose M is an n -dimensional smooth manifold. Then M is formal if and only if it is n -formal.*

Furthermore, for a connected and orientable compact manifold, Fernández and Muñoz proved in [14] the following powerful theorem which is important for our proof of Proposition 4.1. The key idea of their proof is that one can improve Theorem 4.8 by taking advantage of Poincaré duality.

Theorem 4.9 [14, Theorem 3.1] *Suppose M is a connected and orientable compact smooth manifold of dimension $2n$ or $2n - 1$. Then M is formal if and only if it is $(n-1)$ -formal.*

Recall that $M(A) = S^1 \times N$. We shall apply Theorem 4.9 to N by constructing a minimal model of ΩN . As mentioned above, every connected CDGA \mathcal{A} has a minimal model \mathcal{M} . Let us briefly describe the construction of such a minimal model \mathcal{M} . For more details, see [26, Chapter 6].

We shall inductively construct a minimal CDGA \mathcal{M}_n (where $n \in \mathbb{Z}$, $n \geq 0$) and a CDGA homomorphism $\varphi_n: \mathcal{M}_n \rightarrow \mathcal{A}$ such that φ_n is n -regular and $(\mathcal{M}_n, \varphi_n)$ is an extension of $(\mathcal{M}_{n-1}, \varphi_{n-1})$. Here n -regular means $\varphi_n^*: H^i(\mathcal{M}_n) \rightarrow H^i(\mathcal{A})$ is an isomorphism for $i \leq n$ and an injection for $i = n + 1$. Given such \mathcal{M}_n and φ_n , we define $\mathcal{M} = \bigcup_{n \geq 0} \mathcal{M}_n$, and $\varphi: \mathcal{M} \rightarrow \mathcal{A}$ by $\varphi|_{\mathcal{M}_n} = \varphi_n$. Certainly, (\mathcal{M}, φ) is a minimal model of \mathcal{A} .

To construct \mathcal{M}_n and φ_n , we start by putting $\mathcal{M}_0 = \mathcal{M}_0^0 = \mathbb{K}$ and by then setting $\varphi_0: \mathcal{M}_0^0 = \mathbb{K} \rightarrow \mathbb{K} \subseteq \mathcal{A}^0$ so that it is the identity isomorphism of \mathbb{K} . Then φ_0 is 0-regular.

Next, assuming the existence of $(\mathcal{M}_n, \varphi_n)$, we now construct $(\mathcal{M}_{n+1}, \varphi_{n+1})$. If $\varphi_n^*: H^{n+1}(\mathcal{M}_n) \rightarrow H^{n+1}(\mathcal{A})$ is surjective, then it is an isomorphism. Otherwise, choose a collection of cocycles $\{\alpha_{0,\mu} \mid \mu \in \mathcal{I}_0\} \subseteq \mathcal{A}^{n+1}$ which represent a basis of the cokernel of $\varphi_n^*: H^{n+1}(\mathcal{M}_n) \rightarrow H^{n+1}(\mathcal{A})$. Here \mathcal{I}_0 is an ordered index set. Introducing free homogeneous generators $a_{0,\mu}$ of degree $n + 1$ to \mathcal{M}_n , we get a CDGA \mathcal{X}_0 . More

precisely, $\mathcal{X}_0 = \mathcal{M}_n \otimes \mathcal{F}$, where \mathcal{F} is the free CDGA generated by these $a_{0,\mu}$ with zero differentials. We identify the new generators $1 \otimes a_{0,\mu} \in \mathcal{X}_0$ with $a_{0,\mu}$, and they succeed those old generators. Extend φ_n to $\psi_0: \mathcal{X}_0 \rightarrow \mathcal{A}$ so that $\psi_0(a_{0,\mu}) = \alpha_{0,\mu}$. Then $\psi_0^*: H^i(\mathcal{X}_0) \rightarrow H^i(\mathcal{A})$ is an isomorphism for $i \leq n + 1$. Denote by \mathcal{Y}_0 the kernel of $\psi_0^*: H^{n+2}(\mathcal{X}_0) \rightarrow H^{n+2}(\mathcal{A})$. If $\mathcal{Y}_0 \neq 0$, choose a collection of cocycles $\{b_{1,\mu} \mid \mu \in \mathcal{I}_1\} \subseteq \mathcal{X}_0^{n+2}$ which represent a basis of \mathcal{Y}_0 . Introducing free homogeneous generators $a_{1,\mu}$ of degree $n + 1$ to \mathcal{X}_0 , we get a CDGA extension \mathcal{X}_1 such that $da_{1,\mu} = b_{1,\mu}$. Here the elements in \mathcal{I}_1 succeed those in \mathcal{I}_0 . Since $\psi_0(b_{1,\mu})$ is exact in \mathcal{A} , we have $\psi_0(b_{1,\mu}) = d\alpha_{1,\mu}$ for some $\alpha_{1,\mu} \in \mathcal{A}^{n+1}$. Extend ψ_0 to $\psi_1: \mathcal{X}_1 \rightarrow \mathcal{A}$ so that $\psi_1(a_{1,\mu}) = \alpha_{1,\mu}$. Then \mathcal{Y}_0 is killed in $H^{n+2}(\mathcal{X}_1)$. Denote by \mathcal{Y}_1 the kernel of $\psi_1^*: H^{n+2}(\mathcal{X}_1) \rightarrow H^{n+2}(\mathcal{A})$. If $\mathcal{Y}_1 \neq 0$, then we extend (\mathcal{X}_1, ψ_1) to (\mathcal{X}_2, ψ_2) which kills \mathcal{Y}_1 . Repeating this procedure, we get $\mathcal{M}_{n+1} = \bigcup_{p \geq 0} \mathcal{X}_p$ and $\varphi_{n+1}: \mathcal{M}_{n+1} \rightarrow \mathcal{A}$ such that $\varphi_{n+1}|_{\mathcal{X}_p} = \psi_p$. Clearly, $\varphi_{n+1}^*: H^i(\mathcal{M}_{n+1}) \rightarrow H^i(\mathcal{A})$ is an isomorphism for $i \leq n + 1$. Suppose $b \in \mathcal{M}_{n+1}^{n+2}$ is a cocycle such that $\varphi_{n+1}(b)$ is exact in \mathcal{A} . This b must be in \mathcal{X}_p for some p . Then its cohomology class is killed in \mathcal{X}_{p+1} . Thus $\varphi_{n+1}^*: H^{n+2}(\mathcal{M}_{n+1}) \rightarrow H^{n+2}(\mathcal{A})$ is injective. We see that φ_{n+1} is $(n+1)$ -regular.

Proof of Proposition 4.1 Since $M(\mathcal{A}) = S^1 \times N$ and S^1 is formal, it suffices to show that N is formal. We shall first construct a minimal model of ΩN .

Since the matrix A is diagonalizable, by Lemma 3.9, $\dim H^1(\Omega N) = 1$ and for any i

$$s_2 \cup -: H^{2i}(\Omega N) \rightarrow H^{2i+1}(\Omega N)$$

is an isomorphism. Here s_2 is a basis of $H^1(\Omega N)$. Choose $\alpha \in (\Omega N)^1$ such that $[\alpha] = s_2$, and for $1 \leq j \leq k$, choose $\beta_j \in (\Omega N)^2$ such that β_1, \dots, β_k represent a basis of $H^2(\Omega N)$. Then $\alpha \wedge \beta_1, \dots, \alpha \wedge \beta_k$ represent a basis of $H^3(\Omega N)$.

Construct a minimal CDGA \mathcal{M}_2 which is generated by one homogeneous element a of degree 1 and homogeneous elements b_1, \dots, b_k of degree 2. Define the differential d of \mathcal{M}_2 to be zero. Then \mathcal{M}_2 is the tensor product of the exterior algebra $\wedge(a)$ and the polynomial algebra $\wedge(b_1, \dots, b_k)$. Since $(\wedge(a))^i = 0$ for $i > 1$, the linear space \mathcal{M}_2^1 has a basis a , the space \mathcal{M}_2^2 has a basis b_1, \dots, b_k , and \mathcal{M}_2^3 has a basis ab_1, \dots, ab_k . Define $\varphi_2: \mathcal{M}_2 \rightarrow \Omega N$ so that $\varphi_2(a) = \alpha$ and $\varphi_2(b_j) = \beta_j$. Then $\varphi_2(ab_j) = \alpha \wedge \beta_j$. We see that $\varphi_2^*: H^i(\mathcal{M}_2) \rightarrow H^i(\Omega N)$ is an isomorphism for $0 \leq i \leq 3$. In particular, φ_2 is 2-regular.

Applying the argument before this proof, we can extend \mathcal{M}_2 to be a minimal model \mathcal{M} for ΩN . Note that the extension only introduces new generators of degrees greater

than 2. Therefore, $\mathcal{M}^i = \mathcal{M}_2^i$ for $0 \leq i \leq 2$. In particular, the differential d vanishes on \mathcal{M}^i for $0 \leq i \leq 2$. We further infer that $\mathcal{N}^i = \{0\}$ for $0 \leq i \leq 2$, where \mathcal{N}^i is defined as in Definition 4.6. Thus N is 2-formal.

Since $\dim N = 5$, by Theorem 4.9, we conclude that N is formal. □

5 Cohomology jump loci

In this section, we will prove Theorem 1.3 and Theorem 1.2. The proofs will be based on the theory of cohomology jump loci. We shall first review some basic aspects of this theory and the main result of [41].

Let X be a connected topological space that is homotopy equivalent to a finite CW complex. Obviously, X is also path-connected. Define the character variety to be the set of rank-one characters of $\pi_1(X)$:

$$\text{Char}(X) = \text{Hom}(\pi_1(X), \mathbb{C}^*), \quad \text{where } \mathbb{C}^* = \mathbb{C} \setminus \{0\}.$$

Then $\text{Char}(X)$ is naturally isomorphic to the moduli space of rank-one local systems on X . (See for example [12] for the definition of local systems.) For each $\rho \in \text{Char}(X)$, there is a unique rank-one local system L_ρ whose monodromy action is isomorphic to ρ . The character variety $\text{Char}(X)$ is naturally an abelian complex linear algebraic group. In fact, since $\text{Hom}(\pi_1(X), \mathbb{C}^*) \cong \text{Hom}(H_1(X; \mathbb{Z}), \mathbb{C}^*)$, we infer $\text{Char}(X)$ is isomorphic to the product of $(\mathbb{C}^*)^{b_1(X)}$ and a finite abelian group, where $b_1(X)$ is the first Betti number of X .

The cohomology jump loci $\Sigma_k^j(X)$ of X are defined by

$$\Sigma_k^j(X) = \{\rho \in \text{Char}(X) \mid \dim_{\mathbb{C}} H^j(X; L_\rho) \geq k\}.$$

They are algebraic subsets of $\text{Char}(X)$. It is easy to see that both $\text{Char}(X)$ and $\Sigma_k^j(X)$ are homotopy invariants. More precisely, if $h: X \rightarrow Y$ is a homotopy equivalence, then h induces an isomorphism $h^*: \text{Char}(Y) \rightarrow \text{Char}(X)$ of algebraic groups which maps $\Sigma_k^j(Y)$ onto $\Sigma_k^j(X)$.

Theorem 5.1 [41, Theorem 1.3] *Suppose X is homotopy equivalent to a compact Kähler manifold. Then for any j and k , each irreducible component of $\Sigma_k^j(X)$ is of the form $\tau \cdot T$, where τ is a torsion element, ie an element of finite order, in $\text{Char}(X)$ and T is an irreducible linear subgroup of $\text{Char}(X)$.*

Remark 5.2 In [3], it was proved by Arapura that each irreducible component of $\Sigma_k^1(X)$ is of the form $\tau \cdot T$, where τ is a unitary character. Using this result, Papadima and Suciu [32] constructed a family of compact, orientable, formal 4-manifolds which have the real homotopy type of smooth projective surfaces, but they are not homotopy equivalent to any compact Kähler manifold.

Let $p: E \rightarrow S^1$ be a fiber bundle with fiber F . Suppose E is connected and suppose E is homotopy equivalent to a finite CW complex. Then p induces a homomorphism $p_*: \pi_1(E) \rightarrow \pi_1(S^1)$ which further induces a homomorphism $p^*: \text{Char}(S^1) \rightarrow \text{Char}(E)$.

Lemma 5.3 *Suppose F is path-connected. Then $p^*: \text{Char}(S^1) \rightarrow \text{Char}(E)$ is injective.*

Proof Since F is path-connected, $p_*: \pi_1(E) \rightarrow \pi_1(S^1)$ is surjective, and hence $p^*: \text{Char}(S^1) \rightarrow \text{Char}(E)$ is injective. □

Fix an orientation of S^1 . Then the monodromy action on F is well defined up to homotopy.

Proposition 5.4 *Suppose $H^j(F; \mathbb{C})$ is finite-dimensional for all j . Denote by Γ_j the set of eigenvalues of the monodromy action on $H^j(F; \mathbb{C})$. Define $\Gamma_j^{-1} = \{t^{-1} \mid t \in \Gamma_j\}$. Then*

$$\left(\bigcup_{j \geq 0} \Sigma_1^j(E) \right) \cap p^* \text{Char}(S^1) = p^* \left(\bigcup_{j \geq 0} \Gamma_j^{-1} \right).$$

Here $\text{Char}(S^1)$ is identified with \mathbb{C}^* via the fixed orientation of S^1 .

A more general form of Proposition 5.4 is proved in [33, Theorem 3.6]. For the reader's convenience, we give a proof here.

Proof of Proposition 5.4 Given any $\hat{\rho} \in \text{Char}(S^1)$, let $\rho = p^*(\hat{\rho})$. We have the following Wang sequence with the local system L_ρ , which is a generalized version of Lemma 3.6:

$$\dots \rightarrow H^{j-1}(F; L_\rho) \xrightarrow{\delta} H^j(E; L_\rho) \xrightarrow{i^*} H^j(F; L_\rho) \xrightarrow{\eta^{*,j} - \hat{\rho}^{-1} \text{Id}} H^j(F; L_\rho) \rightarrow \dots,$$

where the map $\eta^{*,j}: H^j(F; L_\rho) \rightarrow H^j(F; L_\rho)$ is induced by the monodromy action. This result can be easily checked using the Mayer-Vietoris sequence, and it is very similar to Proposition 6.4.8 in [12, page 212].

By the above Wang sequence, we immediately get

$$\dim_{\mathbb{C}} H^j(E; L_{\rho}) = \dim_{\mathbb{C}} \text{Ker}(\eta^{*,j} - \hat{\rho}^{-1} \text{Id}) + \dim_{\mathbb{C}} \text{Coker}(\eta^{*,j-1} - \hat{\rho}^{-1} \text{Id}).$$

By definition, $\rho \in \bigcup_{j \geq 0} \Sigma_1^j(E)$ if and only if $\dim_{\mathbb{C}} H^j(E; L_{\rho}) > 0$ for some j . By the above equation, $\dim_{\mathbb{C}} H^j(E; L_{\rho}) > 0$ if and only if $\hat{\rho}^{-1} \in \Gamma_j$ or $\hat{\rho}^{-1} \in \Gamma_{j-1}$. This completes the proof. \square

Proof of Theorem 1.3 If λ is an eigenvalue of this monodromy action, then by Proposition 5.4, $p^*(\lambda^{-1}) \in (\bigcup_{j \geq 0} \Sigma_1^j(E)) \cap p^* \text{Char}(S^1)$. By Lemma 5.3, p^* is injective. Thus it suffices to show that $(\bigcup_{j \geq 0} \Sigma_1^j(E)) \cap p^* \text{Char}(S^1)$ consists of torsion points, or equivalently $\Sigma_1^j(E) \cap p^* \text{Char}(S^1)$ consists of torsion points for any j .

By Theorem 5.1, an irreducible component of $\Sigma_1^j(E)$ is of the form $\tau \cdot T$, where τ is a torsion point. Suppose $V = (\tau \cdot T) \cap p^* \text{Char}(S^1)$ is nonempty. Then V is an algebraic subset. It suffices further to verify that V consists of torsion points.

By Proposition 5.4, V is a countable set. Thus V has dimension 0, and hence contains finitely many points. Suppose τ has order n . Then $G = \bigcup_{0 \leq r \leq n-1} \tau^r \cdot T$ is a subgroup, and hence so is $G \cap p^* \text{Char}(S^1)$. Let ξ be a point in V . It is easy to see that $G \cap p^* \text{Char}(S^1) = \bigcup_{0 \leq r \leq n-1} \xi^r \cdot V$. So $G \cap p^* \text{Char}(S^1)$ is a finite group. Certainly, $V \subseteq G \cap p^* \text{Char}(S^1)$. Therefore, every element of V is torsion. \square

It remains to prove Theorem 1.2.

Lemma 5.5 *Suppose G is an affine group and $g \in G$ is not a torsion element. Then g does not belong to two distinct irreducible affine subgroups with dimension 1.*

Proof Assume g belongs to two such subgroups G_1 and G_2 . Then $G_1 \cap G_2$ is an algebraic subset of G_1 . Since G_1 and G_2 are irreducible, $G_1 \neq G_2$ and $\dim(G_1) = \dim(G_2) = 1$, we infer $\dim(G_1 \cap G_2) = 0$. Therefore, $G_1 \cap G_2$ is a finite subgroup. Thus, $g \in G_1 \cap G_2$ is a torsion element, which yields a contraction. \square

Lemma 5.6 *The manifold $M(A)$ in Theorem 1.1 satisfies the following properties:*

- (1) $\text{Char}(M(A)) \cong \mathbb{C}^* \times \mathbb{C}^*$.
- (2) $\bigcup_{j \geq 0} \Sigma_1^j(M(A)) \subset \{1\} \times \mathbb{C}^*$ under the above isomorphism.
- (3) Suppose $|\rho| \neq 1$. Then $(1, \rho) \in \bigcup_{j \geq 0} \Sigma_1^j(M(A))$ if and only if $\rho = |\lambda|^2$ or $\rho = |\lambda|^{-2}$, where λ is an eigenvalue of A .

Proof By the assumption on A in Theorem 1.1, A has eigenvalues λ and λ^{-1} with $|\lambda| > 1$.

According to (3-2), $M(A) = S^1 \times N$, where N is a fiber bundle over S^1 with fiber K , the Kummer surface. Denote by $p: M(A) \rightarrow S^1 \times S^1$ the bundle projection. Since K is simply connected (Proposition 2.6), $p_*: \pi_1(M(A)) \rightarrow \pi_1(S^1 \times S^1)$ is an isomorphism (Proposition 3.5), and hence

$$\text{Char}(M(A)) = \text{Char}(S^1 \times N) = p^* \text{Char}(S^1 \times S^1) \cong \mathbb{C}^* \times \mathbb{C}^*.$$

By the fact that

$$H^0(S^1; L_\rho) \cong H^1(S^1; L_\rho) \cong \begin{cases} \mathbb{C} & \text{if } \rho = 1, \\ 0 & \text{if } \rho \neq 1 \end{cases}$$

and by the Künneth formula, we infer that

$$\bigcup_{j \geq 0} \Sigma_1^j(M(A)) = \left\{ (1, \rho) \mid \rho \in \bigcup_{j \geq 0} \Sigma_1^j(N) \right\}.$$

By (3-2), $A_K: K \rightarrow K$ in Proposition 2.11 is equal to the monodromy map of the bundle $N \rightarrow S^1$. By Proposition 5.4, $\rho \in \bigcup_{j \geq 0} \Sigma_1^j(N)$ if and only if ρ^{-1} is an eigenvalue of A_K^* on $H^*(K; \mathbb{C})$. The conclusion now follows immediately from Lemma 2.7 and Lemma 2.14. □

Proof of Theorem 1.2 By assumption, A_i has eigenvalues λ_i and λ_i^{-1} with $|\lambda_i| > 1$, where $i = 1, 2$. Since A_1 and A_2 have different spectral radii, $|\lambda_1| \neq |\lambda_2|$.

By Lemma 5.6, $\text{Char}(M(A_i)) \cong \mathbb{C}^* \times \mathbb{C}^*$. In addition, $\bigcup_{j \geq 0} \Sigma_1^j(M(A_i))$ is contained in $\{1\} \times \mathbb{C}^*$, and $(1, \rho) \in \bigcup_{j \geq 0} \Sigma_1^j(M(A_i))$ with $|\rho| \neq 1$ if and only if $\rho = |\lambda_i|^2$ or $\rho = |\lambda_i|^{-2}$.

We prove this theorem by contradiction. Assume $M(A_1)$ and $M(A_2)$ are of the same homotopy type. Since cohomology jump loci are homotopy invariants, we know that there is an isomorphism $\varphi: \mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathbb{C}^* \times \mathbb{C}^*$ of affine groups such that

$$\varphi \left(\bigcup_{j \geq 0} \Sigma_1^j(M(A_1)) \right) = \bigcup_{j \geq 0} \Sigma_1^j(M(A_2)).$$

Therefore, by Lemma 5.6(2),

$$\bigcup_{j \geq 0} \Sigma_1^j(M(A_2)) \subseteq (\{1\} \times \mathbb{C}^*) \cap \varphi(\{1\} \times \mathbb{C}^*).$$

As $\bigcup_{j \geq 0} \Sigma_1^j(M(A_2))$ contains a nontorsion element $(1, |\lambda_2|^2)$, by Lemma 5.5,

$$\varphi(\{1\} \times \mathbb{C}^*) = \{1\} \times \mathbb{C}^*.$$

Now, $\varphi|_{\{1\} \times \mathbb{C}^*}: \{1\} \times \mathbb{C}^* \rightarrow \{1\} \times \mathbb{C}^*$ is an isomorphism of linear groups. Thus, $\varphi(1, |\lambda_1|^2) = (1, |\lambda_1|^2)$ or $\varphi(1, |\lambda_1|^2) = (1, |\lambda_1|^{-2})$.

Since $\varphi(1, |\lambda_1|^2) \in \bigcup_{j \geq 0} \Sigma_1^j(M(A_2))$ and $|\lambda_1|^{\pm 2} \neq 1$, by the above arguments, we have $|\lambda_1|^2 = |\lambda_2|^2$ or $|\lambda_1|^{-2} = |\lambda_2|^2$. This contradicts the fact that $|\lambda_1| \neq |\lambda_2|$, $|\lambda_1| > 1$ and $|\lambda_2| > 1$. \square

Motivated by the work of Voisin [40] and Papadima and Suciu [32], we would like to end our paper with the following question.

Question Does there exist a compact 6-manifold which is of the same real homotopy type as a compact Kähler manifold and satisfies all the conclusions in Theorem 1.1?

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*Department of Mathematics, Nanjing University
Nanjing, Jiangsu, China*

*Department of Mathematics, University of Wisconsin
Madison, WI, United States*

qinlz@nju.edu.cn, bwang274@wisc.edu

Proposed: Yasha Eliashberg

Seconded: Richard Thomas, Dan Abramovich

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