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Eigenvalues of curvature, Lyapunov exponents and Harder–Narasimhan filtrations

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Inspired by the Katz–Mazur theorem on crystalline cohomology and by the numerical experiments of Eskin, Kontsevich and Zorich, we conjecture that the polygon of the Lyapunov spectrum lies above (or on) the Harder–Narasimhan polygon of the Hodge bundle over any Teichmüller curve. We also discuss the connections between the two polygons and the integral of eigenvalues of the curvature of the Hodge bundle by using the works of Atiyah and Bott, Forni, and Möller. We obtain several applications to Teichmüller dynamics conditional on the conjecture.

14H10, 30F60, 32G15; 37D25, 53C07

1 Introduction

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g, and let $\mathcal{H}_g \to \mathcal{M}_g$ be the bundle of pairs (X, ω), where $\omega \neq 0$ is a holomorphic 1–form on $X \in \mathcal{M}_g$. Denote by $\mathcal{H}_g(m_1, \dots, m_k) \hookrightarrow \mathcal{H}_g$ the stratum of pairs (X, ω) for which the nonzero holomorphic 1–form ω has k distinct zeros, of orders $m_1, \dots, m_k$, respectively; see Kontsevich and Zorich [30] for details.

There is a natural action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathcal{H}_g(m_1, \dots, m_k)$, whose orbits project to complex Teichmüller geodesics. The action of the one-parameter diagonal subgroup of $\mathrm{SL}_2(\mathbb{R})$ defines the *Teichmüller geodesic flow*; its orbits project to geodesics in the Teichmüller metric on $\mathcal{M}_g$. The projection of the orbit of almost every point is dense in the connected component of the ambient stratum. Teichmüller geodesic flow has strong connections with flat surfaces, billiards in polygons and interval exchange transformations; see Zorich [50] for a survey.

Fix an $\mathrm{SL}_2(\mathbb{R})$–invariant finite ergodic measure μ on $\mathcal{H}_g$. Zorich [48] initiated the study of the Lyapunov exponents for the Teichmüller geodesic flow on $\mathcal{H}_g$,

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq 0,$$

which measure the logarithm of the growth rate of the Hodge norm of cohomology classes under parallel transport along the geodesic flow.

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It is possible to evaluate Lyapunov exponents approximately through computer simulation of the corresponding dynamical system. Such experiments with Rauzy–Veech–Zorich induction (a discrete model of the Teichmüller geodesic flow), performed in Zorich [49], indicated a surprising rationality of the sums $\lambda_1 + \cdots + \lambda_g$ of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow on strata of abelian and quadratic differentials; see Kontsevich and Zorich [29]. An explanation of this phenomenon was given by Kontsevich in [27] and then developed by Forni [19]: this sum is, essentially, the characteristic number of the determinant of the Hodge bundle. Recently Eskin, Kontsevich and Zorich [14] have found the connection between the sum of Lyapunov exponents and Siegel–Veech constants by establishing an analytic Riemann–Roch formula.

Zorich conjectured the strict positivity of λ_g and simplicity of the spectrum of Lyapunov exponents for connected components of the strata of abelian differentials. Forni proved the first conjecture in [19]; Avila and Viana proved the second one in [3].

We reproduce in the tables in the appendix the approximate values of all individual Lyapunov exponents for connected components of the strata of small genera using [29] and [14] as a source. Though the *sum* of the top g Lyapunov exponents is always rational for the strata, for Teichmüller curves — and, conjecturally, for all $\mathrm{GL}(2,\mathbb{R})$–invariant orbifolds — the individual Lyapunov exponents seem to be completely transcendental and there are no tools which would allow us to evaluate them explicitly except in several very particular cases, which we describe below.

Exact values of individual Lyapunov exponents can be computed rigorously for certain invariant suborbifolds of the strata of abelian differentials. For example, Bainbridge [4] succeeded in performing such computations for suborbifolds in genus two. (Since λ_1 is identically equal to 1 for any $\mathrm{GL}(2,\mathbb{R})$–invariant orbifold, computation of λ_2 in genus 2 is equivalent to the computation of the sum $\lambda_1 + \lambda_2$.)

Such computation was also performed for certain special Teichmüller curves. For the Teichmüller curves related to triangle groups it was done by Bouw and Möller [8], and by Wright [44]; for square-tiled cyclic covers in Eskin, Kontsevich and Zorich [13] and in Forni, Matheus and Zorich [20]; for square-tiled abelian covers by Wright [43]; for some wind-tree models by Delecroix, Hubert and Lelièvre [11].

Recall the definition of a Teichmüller curve in $\mathcal{M}_g$. If the stabilizer $\mathrm{SL}(X,\omega) \subset \mathrm{SL}_2(\mathbb{R})$ of a given pair (X,ω) forms a lattice, then the projection of the orbit $\mathrm{SL}_2(\mathbb{R}) \cdot (X,\omega)$ to $\mathcal{M}_g$ gives a closed, algebraic curve called a *Teichmüller curve*. The relative canonical bundle over a Teichmüller curve has a particularly simple and elegant form as in Equation (2); see Chen and Möller [9] and Eskin, Kontsevich and Zorich [14]. For any Teichmüller curve, Kang Zuo and the author [47; 46] have introduced g numbers

$$1 = w_1 \geq w_2 \geq \cdots \geq w_g \geq 0\,,$$

where w_i is obtained by normalizing the slopes of the Harder–Narasimhan filtration of the Hodge bundle. We can get upper bounds of each w_i by using some filtrations of the Hodge bundle constructed using the special structure of the relative canonical bundle formula.

Now we have a collection of numbers λ_i, where $i = 1, \dots, g$, measuring the stability of the dynamical system, and a collection of numbers w_i, where $i = 1, \dots, g$, measuring the stability of algebraic geometry. Tables in the appendix provide the numerical data for the numbers λ_i corresponding to the low genera strata and for the numbers w_i corresponding to Teichmüller curves in the corresponding strata. It is natural to address a question: are there any relations between them?

Define the *Lyapunov polygon* of the Hodge bundle over a Teichmüller curve as the convex hull of the collection of points in $\mathbb{R}^2$ having coordinates

$$(0,0),\ (1,\lambda_1),\ (2,\lambda_1+\lambda_2),\ \dots,\ (g,\lambda_1+\cdots+\lambda_g).$$

Similarly, define the *Harder–Narasimhan polygon* of the Hodge bundle over a Teichmüller curve as the convex hull of the collection of points in $\mathbb{R}^2$ having coordinates

$$(0,0),\ (1,w_1),\ (2,w_1+w_2),\ \dots,\ (g,w_1+\cdots+w_g).$$

Inspired by the Katz–Mazur theorem (see Mazur [34; 35]), which tells us that the Hodge polygon lies above (or on) the Newton polygon of the crystalline cohomology, we make the following conjecture supported by all currently available numerical data.

Main Conjecture *For any Teichmüller curve, the Lyapunov polygon of the Hodge bundle lies above (or on) the Harder–Narasimhan polygon.*[1]

Warning Different articles have different definitions of "lie above" and "lie below" for convex polygons. In the context of the conjecture above, "lies above" is synonymous to "contains as a subset", since it is known that the two polygons share the rightmost and the leftmost vertices; see Figure 1. We discuss the notion of "lies above" in a more general context in Sections 3 and 4.

[1] This conjecture has been proved by Eskin, Kontsevich, Möller and Zorich in [12].

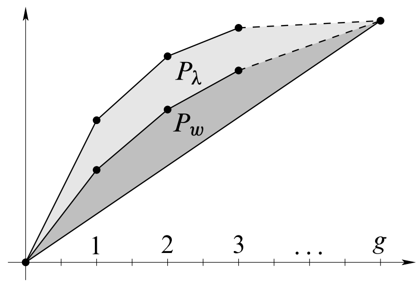

Figure 1: We conjecture that the Lyapunov polygon P_λ lies above (or on) the Harder–Narasimhan polygon P_w.

In analytic terms, our Main Conjecture claims that the following system of inequalities is valid for any Teichmüller curve:

$$\begin{cases} \sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} w_j & \text{for } i = 1, \dots, g-1, \\ \sum_{j=1}^{g} \lambda_j = \sum_{j=1}^{g} w_j, \end{cases}$$

where the equality for the last term $i = g$ is obtained by combining the Kontsevich formula for the sum of the Lyapunov exponents of the Hodge bundle over a Teichmüller curve (see Theorem 3.1 below) and the definition of the normalized Harder–Narasimhan slopes w_i; see Section 3.3 and also [47; 46]).

Equivalently, one can rewrite the latter system of inequalities as

$$\begin{cases} \sum_{j=i}^{g} \lambda_j \leq \sum_{j=i}^{g} w_j & \text{for } i = 2, \dots, g, \\ \sum_{j=1}^{g} \lambda_j = \sum_{j=1}^{g} w_j. \end{cases}$$

The Main Conjecture is stated for the Teichmüller curves. However, using the corollaries of recent rigidity theorems of Eskin, Mirzakhani and Mohammadi [16] the statement of the Main Conjecture implies analogous estimates for other $\mathrm{GL}(2,\mathbb{R})$–invariant suborbifolds, in particular for the connected components of the strata. To illustrate such applications we first recall the rigidity results.

Theorem [16, Theorem 2.3] *Let $\mathcal{N}_n$ be a sequence of affine $\mathrm{SL}(2,\mathbb{R})$–invariant manifolds, and suppose $\nu_{\mathcal{N}_n} \to \nu$. Then ν is a probability measure. Furthermore, ν is the affine $\mathrm{SL}(2,\mathbb{R})$–invariant measure $\nu_{\mathcal{N}}$, where $\mathcal{N}$ is the smallest submanifold with the property that there exists some $n_0 \in \mathbb{N}$ such that $\mathcal{N}_n \subset \mathcal{N}$ for all $n > n_0$.*

Bonatti, Eskin and Wilkinson [7] use this theorem and a theorem of Filip [17] to give the following affirmative answer to the question posed by Matheus, Möller and Yoccoz in [33]:

Theorem 1.1 [7, Theorem 2.8] *Let $\mathcal{N}_n$ be a sequence of affine $\mathrm{SL}(2,\mathbb{R})$–invariant manifolds, and suppose $\nu_{\mathcal{N}_n} \to \nu$. Then the Lyapunov exponents of $\nu_{\mathcal{N}_n}$ converge to the Lyapunov exponents of ν.*

As a corollary (conditional on the Main Conjecture) we prove the following conjecture of Kontsevich and Zorich [29].

Corollary 1.2 *The Main Conjecture implies,[2] in particular, that for any fixed positive integer k, the Lyapunov exponent λ_k of the Hodge bundle over the hyperelliptic connected components $\mathcal{H}_g^{\mathrm{hyp}}(2g-2)$ and $\mathcal{H}_g^{\mathrm{hyp}}(g-1,g-1)$ tends to 1 as the genus tends to infinity: $\lambda_k \to 1$ as $g \to \infty$ for any fixed $k \in \mathbb{N}$.*

Note that for all other components of all other strata of abelian differentials the Lyapunov exponent conjecturally tends to $\frac{1}{2}$ and not to 1; see [29]. It is a challenging problem to deduce these asymptotics from the Main Conjecture.

We prove Corollary 1.2, obtain further results conditional on the Main Conjecture, and state some further conjectures in Section 5.

The Main Conjecture was first announced by the author at the Oberwolfach conference [32]. After that we realized that this conjecture is analogous to the work of Atiyah and Bott [2] on hermitian Yang–Mills metrics. This analogy was independently noticed by Möller. Here we state the result of Atiyah and Bott in a form for which the analogy is more transparent. Let ε_j, where $1 \le j \le g$, be the normalized integral of the j^{th} eigenvalue of the curvature of the Hodge bundle over a Teichmüller curve; see Forni [19], who proves the bounds. It follows that

$$1 = \varepsilon_1 \ge \varepsilon_2 \ge \cdots \ge \varepsilon_g \ge 0.$$

It follows from [2] that

$$\begin{cases} \sum_{j=1}^{i} \varepsilon_j \ge \sum_{j=1}^{i} w_j & \text{for } i = 1, \dots, g-1, \\ \sum_{j=1}^{g} \varepsilon_j = \sum_{j=1}^{g} w_j. \end{cases}$$

[2] As we already mentioned, this conjecture has been proved in [12], so the corollary becomes unconditional.

Thus, upper bounds for w_i obtained in [46] provide some information about ε_i. Recall that partials sums of λ_i and of ε_i are also related; see [19] or Section 4.2 for an outline of these results.

In Section 2 we review the definition of Teichmüller curves, the formula for its relative canonical bundle and the structure of natural filtrations of the Hodge bundle over a Teichmüller curve. Section 3 summarizes facts about slope filtrations, especially Harder–Narasimhan filtrations. It also recalls necessary facts about the integrals of eigenvalues of the curvature. In Section 4 we discuss various manifestations of convexity in geometry and arithmetic. Section 4.1 compares polygons of eigenvalue spectrum and Harder–Narasimhan polygons. Section 4.2 studies the relation between polygons of eigenvalue spectrum and Lyapunov polygons. Finally, Section 4.3 compares Hodge and Newton polygons. We start Section 5 with more detailed discussion of the Main Conjecture. We proceed obtaining several applications (conditional on the Main Conjecture) to Teichmüller dynamics. In particular, we present the proof of the old conjecture of Kontsevich and Zorich stated in Corollary 1.2. We also prove a simple corollary, that $\lambda_i > 0$ implies $w_i > 0$, by using Higgs bundles and we reprove the Eskin–Kontsevich–Zorich formula for the difference between sums of λ^+ and λ^- Lyapunov exponents for Teichmüller curves and for connected components. We provide certain numerical evidence for the Main Conjecture in the appendix.

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2 Teichmüller curves

Teichmüller geodesic flow has close connections with flat surfaces, billiards in polygons and interval exchange transformations; see the survey [50] of Zorich covering many important ideas of this field, and also the survey [38] by Möller devoted to Teichmüller curves mainly from the view point of algebraic geometry.

Let $\mathcal{H}_g(m_1, \dots, m_k)$ denote the stratum parametrized by (X, ω), where X is a curve of genus g and ω is an abelian differential (ie a holomorphic 1–form) on X that has k distinct zeros of orders $m_1, \dots, m_k$. Let us denote by $\overline{\mathcal{H}}_g(m_1, \dots, m_k)$ the Deligne–Mumford compactification of $\mathcal{H}_g(m_1, \dots, m_k)$. Let $\mathcal{H}_g^{\mathrm{hyp}}(m_1, \dots, m_k)$, $\mathcal{H}_g^{\mathrm{odd}}(m_1, \dots, m_k)$ and $\mathcal{H}_g^{\mathrm{even}}(m_1, \dots, m_k))$ be, respectively, the hyperelliptic, the odd theta-characteristic and the even theta-characteristic connected components; see [30].

Let $\mathcal{Q}(d_1, \dots, d_n)$ be the stratum parametrizing (Y, q), where Y is a curve of genus h and q is a meromorphic quadratic differential with at most simple poles on Y that has k distinct zeros of orders $d_1, \dots, d_n$ respectively. If the quadratic differential is not a global square of a 1–form, there is a canonical double covering $\pi\colon X \to Y$ such that $\pi^* q = \omega^2$, where ω is already a holomorphic 1–form. This covering is ramified precisely at the zeros of odd order of q and at the poles. It induces a map

$$\phi\colon \mathcal{Q}(d_1, \dots, d_n) \to \mathcal{H}_g(m_1, \dots, m_k). \tag{1}$$

A singularity of order d_i of q gives rise to two zeros of degree $m = \frac{1}{2} d_i$ when d_i is even, and to a single zero of degree $m = d_i + 1$ when d_i is odd. In particular, any hyperelliptic locus in a stratum $\mathcal{H}_g(m_1, \dots, m_k)$ is induced from a stratum $\mathcal{Q}(d_1, \dots, d_n)$ satisfying $d_1 + \dots + d_n = -4$; see [14].

There is a natural action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathcal{H}_g(m_1, \dots, m_k)$, whose orbits project to complex geodesics[3] with respect to the Teichmüller metric on $\mathcal{M}_g$. The action of the one-parameter diagonal subgroup $\left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$, where $t \in \mathbb{R}$, defines the *Teichmüller geodesic flow*; its orbits project to real Teichmüller geodesics in $\mathcal{M}_g$.

It follows from the fundamental theorems of Masur [31] and Veech [40] that the $\mathrm{GL}_2^+(\mathbb{R})$–orbit of almost any point (X, ω) in any stratum $\mathcal{H}_g(m_1, \dots, m_k)$ of abelian differentials is dense in the ambient connected component of the stratum. The stabilizer $\mathrm{SL}(X, \omega) \subset \mathrm{SL}_2(\mathbb{R})$ of almost any point (X, ω) is trivial.

The situation with some exceptional points (X, ω) is the opposite: the stabilizer $\mathrm{SL}(X, \omega) \subset \mathrm{SL}_2(\mathbb{R})$ is as large as possible, namely it forms a lattice in $\mathrm{SL}_2(\mathbb{R})$. By the results of Smillie and Veech [41] this happens if and only if the $\mathrm{GL}_2^+(\mathbb{R})$–orbit of

[3] Developing the results of Eskin and Mirzakhani [15] and Eskin, Mirzakhani and Mohammadi [16], Filip proved in [17; 18] that the closure of any such complex geodesic is an algebraic variety.

(X,ω) is closed in the ambient stratum. The projection of such a closed orbit to the moduli space $\mathcal{M}_g$ gives a closed algebraic curve C called the *Teichmüller curve*

$$\rho\colon C = \mathbb{H}/\operatorname{SL}(X,\omega) \to \mathcal{M}_g.$$

In the expression of McMullen [36], Teichmüller curves represent *closed complex geodesics*, meaning that every Teichmüller curve is totally geodesic with respect to the Teichmüller metric on $\mathcal{M}_g$.

Following suitable base change and compactification, we can get a universal family $f\colon S \to C$, which is a relatively minimal semistable model with disjoint sections $D_1,\dots,D_k$, where the restriction $D_i|_X$ to each fiber X is a zero of order m_i of ω; see [9, page 11] and [37].

Let $\mathcal{L} \subset f_*\omega_{S/C}$ be the line bundle over the Teichmüller curve C whose fiber over the point corresponding to $(X,\omega)\in C$ is $\mathbb{C}\omega$, the generating differential of the Teichmüller curve C. This line bundle is known to be *maximal Higgs*; see [37]. Let $\Delta \subset C$ be the set of points with singular fibers. By definition of maximal Higgs bundle one has $\mathcal{L} \cong \mathcal{L}^{-1}\otimes\omega_C(\log\Delta)$; see [42]. The latter isomorphism implies the following equality for the degree of $\mathcal{L}$:

$$\chi := 2\deg\mathcal{L} = 2g(C) - 2 + |\Delta|.$$

The following particularly simple formula for the relative canonical bundle can be found in [9, pages 18–19], or in [14, page 33]:

$$\omega_{S/C} \simeq f^*\mathcal{L}\otimes\mathcal{O}_S\Big(\sum_{i=1}^{k} m_i D_i\Big). \tag{2}$$

By the adjunction formula we get

$$D_i^2 = -\omega_{S/C} D_i = -m_i D_i^2 - \deg\mathcal{L},$$

and the self-intersection number of D_i is

$$D_i^2 = -\frac{1}{m_i+1}\,\frac{\chi}{2}.$$

Let $h^0(\mathcal{V})$ be the dimension of $H^0(X,\mathcal{V})$. If $0\le d_i\le m_i$, then from the exact sequence

$$0 \to f_*\mathcal{O}(d_1D_1+\cdots+d_kD_k) \to f_*\mathcal{O}(m_1D_1+\cdots+m_kD_k) = f_*\omega_{S/C}\otimes\mathcal{L}^{-1}$$

and the fact that all subsheaves of a locally free sheaf on a curve are locally free, we deduce that

$$f_*\mathcal{O}(d_1D_1+\cdots+d_kD_k)$$

is a vector subbundle of rank

$$h^0(d_1 p_1 + \cdots + d_k p_k),$$

where p_i is the intersection point of the section D_i and a generic fiber F. Varying d_i in the vector subbundles as above, we have constructed in [47] numerous filtrations of the Hodge bundle.

Examining the fundamental exact sequence

$$0 \to f_*\mathcal{O}\big(\textstyle\sum(d_i - a_i)D_i\big) \to f_*\mathcal{O}\big(\sum d_i D_i\big) \to f_*\mathcal{O}_{\sum a_i D_i}\big(\sum d_i D_i\big)$$
$$\xrightarrow{\delta} R^1 f_*\mathcal{O}\big(\textstyle\sum(d_i - a_i)D_i\big) \to R^1 f_*\mathcal{O}\big(\sum d_i D_i\big) \to 0$$

one can deduce certain nice properties of these filtrations. In particular, we have:

Lemma 2.1 [47] *The Harder–Narasimhan filtration of $f_*\mathcal{O}_{aD}(dD)$ is*

$$0 \subset f_*\mathcal{O}_D((d-a+1)D) \subset \cdots \subset f_*\mathcal{O}_{(a-1)D}((d-1)D) \subset f_*\mathcal{O}_{aD}(dD),$$

and the direct sum of the graded quotient of this filtration is

$$\operatorname{grad}(\mathrm{HN}(f_*\mathcal{O}_{aD}(dD))) = \bigoplus_{i=0}^{a-1} \mathcal{O}_D((d-i)D).$$

By using those filtrations, we obtained Theorem 3.3 and Theorem 3.4 below, reproduced from [47] and [46]. They describe the Harder–Narasimhan polygon of the Hodge bundle over a Teichmüller curve.

3 Slope filtrations

Slope filtrations are present in algebraic and analytic geometry, in asymptotic analysis, in ramification theory, in p–adic theories, and in geometry of numbers; see the survey of André [1]. Five basic examples include the Harder–Narasimhan filtration of a holomorphic vector bundle over a smooth projective curve, the Dieudonné–Manin filtration of F–isocrystals over a p–adic point, the Turrittin–Levelt filtration of formal differential modules, the Hasse–Arf filtration of finite Galois representations of local fields, and the Grayson–Stuhler filtration of Euclidean lattices. Despite the variety of their origins, these filtrations share a lot of similar features.

Suppose that for some object N there is a unique descending *slope filtration*

$$0 \subset F^{\geq \lambda_1} N \subset \cdots \subset F^{\geq \lambda_r} N = N$$

for which $\lambda_1 > \cdots > \lambda_r$, such that there is some natural way to associate the *slope* λ_i to every graded piece $\mathrm{gr}^{\lambda_i} N = F^{\geq \lambda_i} N / F^{> \lambda_i} N$ (one says that the graded piece is *isoclinic of slope* λ_i). Let n_i denote $\mathrm{rk}(\mathrm{gr}^{\lambda_i} N)$, and let $n = \sum n_i$. We shall call the sequence of pairs $(n_i, n_i \lambda_i)$ for $i = 1, \ldots, r$ the *type* of N. It is sometimes convenient to describe the type equivalently by the single n–vector μ whose components are the slopes λ_i, each represented n_i times and arranged in decreasing order. Thus

$$\mu = (\mu_1, \ldots, \mu_n) = (\underbrace{\lambda_1, \ldots, \lambda_1}_{n_1}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n_2}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{n_r})$$

with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, where the first n_1 entries are equal to λ_1, the next n_2 entries are equal to λ_2 and so on.

We introduce a partial ordering on the vectors μ that parametrize our types. This partial ordering is defined for vectors μ', μ'' having the same number n of entries. We say that $\mu' \preceq \mu''$ when $\mu'_1 + \cdots + \mu'_i \leq \mu''_1 + \cdots + \mu''_i$ for all $i = 1, \ldots, n$. We also associate with every type μ a convex polygon P_μ in the coordinate plane $\mathbb{R}^2$, with vertices at the points having coordinates

$$(0,0),\ (1, \mu_1),\ (2, \mu_1 + \mu_2),\ \ldots,\ (n, \mu_1 + \cdots + \mu_n). \tag{3}$$

(See Figure 2.) It follows from our definition of the partial ordering that $\mu' \preceq \mu''$ if and only if for every pair of vertices sharing the same first coordinate, the vertex of $P_{\mu''}$ is located above the corresponding vertex of $P_{\mu'}$ or coincides with it.

Note that monotonicity of μ_i is equivalent to convexity of the polygon P_μ with vertices at the collection of points (3).

3.1 Eigenvalues of curvature: ε

Forni introduced in [19] the eigenvalues of curvature to study Lyapunov exponents of the Hodge bundle. Here we follow [21], whose setup is closer to the current paper.

Let

$$f\colon \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$$

be the natural forgetful map from the compactified moduli space $\mathcal{M}_{g,1}$ of pairs (X, p), where $p \in X$, to the compactified moduli space $\mathcal{M}_g$ of Riemann surfaces X of genus g.

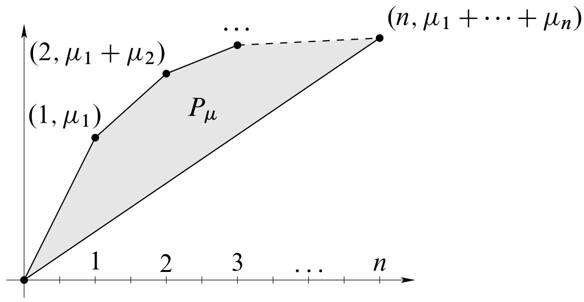

Figure 2: Convex polygon P_μ

For the weight-one $\mathbb{Q}$–VHS

$$(R^1 f_* \mathbb{Q}, H^{1,0} = f_* \omega_{\overline{\mathcal{M}}_{g,1}/\overline{\mathcal{M}}_g} \subset H = (R^1 f_* \mathbb{Q} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{M}_g})_{\mathrm{ext}}),$$

the flat Gauss–Manin connection ∇ composed with the inclusion and projection gives a map

$$A^{1,0}\colon H^{1,0} \to H \to H \otimes \Omega_{\overline{\mathcal{M}}_g}(\log(\overline{\mathcal{M}}_g \backslash \mathcal{M}_g)) \to (H/H^{1,0}) \otimes \Omega_{\overline{\mathcal{M}}_g}(\log(\overline{\mathcal{M}}_g \backslash \mathcal{M}_g)),$$

which is $\mathcal{O}_{\overline{\mathcal{M}}_g}$–linear.

The map $A^{1,0}$ is the second fundamental form of the Hodge bundle, which is also known as the Kodaira–Spencer map. Being restricted to a curve C in $\overline{\mathcal{M}}_g$, we have $A^{1,0} \wedge A^{1,0} = 0$. This condition (which is actually void for curves) defines a *Higgs field*, which is discussed in Section 5.2.

Denote by Θ_H, $\Theta_{H^{1,0}}$ and $\Theta_{H^{0,1}}$ the curvature tensors of the metric connections of the holomorphic hermitian bundles H, $H^{1,0}$ and $H^{0,1}$, respectively. By Cartan's structure equation,

$$\Theta_H = \begin{bmatrix} \Theta_{H^{1,0}} - \overline{A^{1,0}}^T \wedge A^{1,0} & * \\ * & \Theta_{H^{0,1}} - A^{1,0} \wedge \overline{A^{1,0}}^T \end{bmatrix}.$$

It follows that

$$\Theta_{H^{1,0}} = \Theta_H|_{H^{1,0}} + \overline{A^{1,0}}^T \wedge A^{1,0}.$$

Note that Θ_H is the curvature of the Gauss–Manin connection, which is flat. So Θ_H is null, and the curvature $\Theta_{H^{1,0}}$ of the Hodge bundle can be expressed as

$$\Theta_{H^{1,0}} = \overline{A^{1,0}}^T \wedge A^{1,0}.$$

We work with the pullbacks of the vector bundles H, $H^{1,0}$, $H^{0,1}$ to the moduli spaces $\mathcal{H}_g$ (resp. $\mathcal{Q}$) of abelian (resp. quadratic) differentials with respect to the natural projections $\rho\colon \mathcal{H}_g \to \mathcal{M}_g$ (resp. $\varrho\colon \mathcal{Q} \to \mathcal{M}_g$). For any pair (X,q) we can view the holomorphic quadratic differential q as the tangent vector $v = q$ to the moduli space $\mathcal{M}_g$ at the point X, under the identification between the bundle of holomorphic quadratic differentials and the tangent bundle of the moduli space of Riemann surfaces through Beltrami differentials. We can plug the vector v into the 1–form $A^{1,0}$ with values in linear maps to define a linear map

$$A_q\colon H^{1,0}(X) \to H^{0,1}(X)$$

for every point (X,q) of the moduli space $\mathcal{Q}$; see [21, page 8] for details. Analogously, for any abelian differential ω, let $A_\omega := A_q$ be the complex-linear map corresponding to the quadratic differential $q = \omega^2$.

Following Forni, for any $\alpha, \beta \in H^{1,0}(X)$, define

$$B_\omega(\alpha,\beta) := \frac{i}{2}\int_X \frac{\alpha\beta}{\omega}\,\overline{\omega}.$$

The complex-valued symmetric bilinear form B_ω depends continuously (actually, even real-analytically) on the abelian differential ω. The second fundamental form A_ω can be expressed in terms of the complex-valued symmetric bilinear form B_ω in the following way (see [19, page 27] and [21, Lemma 2.1]):

$$(A_\omega(\alpha), \overline{\beta}) = -B_\omega(\alpha,\beta).$$

It is related to the derivative of the period matrix along the Teichmüller geodesic flow.

For any abelian differential ω, let H_ω be the *negative* of the hermitian curvature form Θ_ω on $H^{1,0}(X)$. Let B be the matrix of the bilinear form B_ω on $H^{1,0}(X)$ with respect to some orthonormal basis $\Omega := \{\omega_1, \dots, \omega_g\}$ of holomorphic abelian differentials $\omega_1, \dots, \omega_g$ on X; that is,

$$B_{jk} := \frac{i}{2}\int_X \frac{\omega_j\omega_k}{\omega}\,\overline{\omega}.$$

The hermitian form H_ω is positive-semidefinite and its matrix H with respect to any Hodge-orthonormal basis Ω can be written (see [19, page 27; 21]) as

$$H = B \cdot \overline{B}^T.$$

Let $\mathrm{EV}(H_\omega)$ and $\mathrm{EV}(B_\omega)$ denote the set of eigenvalues of the forms H_ω and B_ω, respectively. We have the identity

$$\mathrm{EV}(H_\omega) = \{|\lambda|^2 : \lambda \in \mathrm{EV}(B_\omega)\}.$$

For every abelian differential ω, the eigenvalues of the positive semidefinite form H_ω on $H^{1,0}(X)$ will be denoted as

$$1 = \Lambda_1(\omega) > \Lambda_2(\omega) \geq \cdots \geq \Lambda_g(\omega) \geq 0,$$

where the identity $\Lambda_1(\omega) = 1$ is proved in [19, Theorem 0.2] and [21, page 16]. Every eigenvalue as above gives a well-defined continuous, nonnegative, bounded function on the moduli space of all (normalized) abelian differentials.

For a Teichmüller curve C, there is a natural volume form $\mathrm{d}\sigma$ which satisfies

$$\tfrac{i}{2\pi}\Theta_{H^{1,0}} = H_\omega \, \mathrm{d}\sigma.$$

This volume form coincides with the normalized hyperbolic area form

$$\mathrm{d}\sigma = \tfrac{1}{\pi}\, \mathrm{d}g_{\mathrm{hyp}}(\omega) \tag{4}$$

associated to the canonical hyperbolic metric of constant negative curvature -4 on the Teichmüller curve C used in [14, page 32]. Thus, we have

$$\int_C \Lambda_1(\omega)\, \mathrm{d}\sigma = \int_C \mathrm{d}\sigma = \tfrac{1}{2}\chi,$$

where $-\chi$ is the Euler characteristic of the Teichmüller curve C with punctures at the cusps, $\chi = 2g-2+|\Delta|$, and $|\Delta|$ is the number of cusps of C.

Following Forni [19, page 26], we define the integrals

$$\varepsilon_j = \frac{1}{\frac{1}{2}\chi} \int_C \Lambda_j(\omega)\, \mathrm{d}\sigma. \tag{5}$$

By definition, the numbers $\epsilon_1, \dots, \epsilon_g$ satisfy inequalities $1 = \varepsilon_1 \geq \cdots \geq \varepsilon_g \geq 0$. We define the *eigenvalue type* $\varepsilon(C)$ of a Teichmüller curve C as

$$\varepsilon(C) = (\varepsilon_1, \dots, \varepsilon_g).$$

3.2 Lyapunov exponents: λ

Zorich introduces the Lyapunov exponents of the Hodge bundle to study the Teichmüller geodesic flow [48]. The geometric meaning of these Lyapunov exponents is clearly explained in [50, Section 4].

A motivating example called the Ehrenfest wind-tree model for Lorenz gases appears in the work of Delecroix, Hubert and Lelièvre [11]. Consider a billiard on the plane with $\mathbb{Z}^2$–periodic rectangular obstacles as in Figure 3.

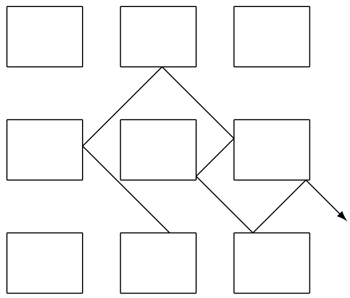

Figure 3: Ehrenfest wind-tree model for Lorenz gases: billiard in the plane with periodic rectangular obstacles.

It is shown in [11] that for all parameters (a, b) of the obstacle (ie for all pairs of lengths $a, b \in (0, 1)$ of the sides of the rectangular obstacles), for almost all initial directions θ, and for any starting point x, the diameter of the billiard trajectory grows with the rate $t^{2/3}$:

$$\lambda_2 = \limsup_{t \to \infty} \frac{\log\bigl(\text{distance between } x \text{ and } \phi_t^\theta(x)\bigr)}{\log t} = \frac{2}{3}.$$

The number $\frac{2}{3}$ here is the Lyapunov exponent of a certain renormalizing dynamical system associated to the initial one.

We recall now the definition of Lyapunov exponents of the Hodge bundle. Fix an $\mathrm{SL}_2(\mathbb{R})$–invariant, ergodic measure μ on $\mathcal{H}_g$. Let V be the restriction of the real Hodge bundle (ie the bundle with fibers $H^1(X, \mathbb{R})$) to the support $\mathcal{M} \subset \mathcal{H}_g$ of μ. Let S_t be the lift of the geodesic flow to V via the Gauss–Manin connection. Then *Oseledets' multiplicative ergodic theorem* guarantees the existence of a filtration

$$0 \subset V_{\lambda_g} \subset \cdots \subset V_{\lambda_1} = V$$

by measurable vector subbundles with the property that for almost all $m \in \mathcal{M}$ and all $v \in V_m \backslash \{0\}$, one has

$$\|S_t(v)\| = \exp(\lambda_i t + o(t)),$$

where i is the maximal index such that v is in the fiber of V_i over m (ie $v \in (V_i)_m$). The numbers λ_i for $i = 1, \ldots, k \le \mathrm{rk}(V)$ are called the *Lyapunov exponents* of the *Kontsevich–Zorich cocycle* S_t. Since V is symplectic, the spectrum of Lyapunov exponents is symmetric in the sense that $\lambda_{g+k} = -\lambda_{g-k+1}$. Moreover, from elementary geometric arguments it follows that one always has $\lambda_1 = 1$. Thus, the Lyapunov spectrum is completely determined by the nonnegative Lyapunov exponents

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq 0.$$

We will apply Oseledets' theorem in two instances. The first one corresponds to the Masur–Veech measures μ_{gen}. The support of such a measure coincides with the entire hypersurface of flat surfaces of area one in a connected component of a stratum of abelian or quadratic differentials. The second case corresponds to Teichmüller curves. When talking about Lyapunov exponents for Teichmüller curves C, we take μ to be the measure on the unit tangent bundle T^1C to a Teichmüller curve that stems from the Poincaré metric g_{hyp} on $\mathbb{H}$ with scalar curvature -4. In both cases, the integrability condition of Oseledets' theorem is known to be satisfied; see eg [38, page 38].

We define the *Lyapunov type* $\lambda(C)$ of a Teichmüller curve C as

$$\lambda(C) = (\lambda_1, \dots, \lambda_g).$$

A bridge between the "dynamical" definition of Lyapunov exponents and the "algebraic" method applied in the sequel originates from the following result. It was first formulated by Kontsevich [27] (in a slightly different form) and then extended by Forni [19, Corollary 5.3].

Theorem 3.1 [27; 19; 8] *If the VHS over the Teichmüller curve C contains a sub-VHS $\mathbb{W}$ of rank $2k$, then the sum of the k corresponding nonnegative Lyapunov exponents equals*

$$\sum_{i=1}^{k} \lambda_i^{\mathbb{W}} = \frac{2\deg \mathbb{W}^{(1,0)}}{2g(C) - 2 + |\Delta|},$$

where $\mathbb{W}^{(1,0)}$ is the $(1,0)$–part of the Hodge filtration of the vector bundle associated with $\mathbb{W}$ and $|\Delta|$ is the number of cusps of C. In particular, we have

$$\sum_{i=1}^{g} \lambda_i = \sum_{i=1}^{g} \varepsilon_i = \frac{2\deg f_*\omega_{S/C}}{2g(C) - 2 + |\Delta|}.$$

The formula immediately implies the Arakelov inequality for Teichmüller curves,

$$\deg f_*\omega_{S/C} = \left(\frac{1}{2}\sum_{i=1}^{g} \lambda_i\right)(2g(C) - 2 + |\Delta|) \leq \frac{g}{2}(2g(C) - 2 + |\Delta|).$$

Eskin, Kontsevich and Zorich have developed an appropriate analytic Riemann–Roch formula to compute the sum of Lyapunov exponents of the Hodge bundle along the Teichmüller geodesic flow on any $\mathrm{SL}(2,\mathbb{R})$–invariant suborbifold.

Theorem 3.2 [14, Theorem 1] *Let $\mathcal{M}_1$ be any closed connected $\mathrm{SL}(2,\mathbb{R})$–invariant suborbifold of some stratum $\mathcal{H}_g(m_1,\dots,m_n)$ of abelian differentials, where $m_1+\dots+m_n=2g-2$. The top g Lyapunov exponents of the Hodge bundle over $\mathcal{M}_1$ along the Teichmüller flow satisfy the relation*

$$\sum_{i=1}^{g}\lambda_i=\frac{1}{12}\sum_{i=1}^{k}\frac{m_i(m_i+2)}{m_i+1}+\frac{\pi^2}{3}\,c_{\mathrm{area}}(\mathcal{M}_1),$$

where $c_{\mathrm{area}}(\mathcal{M}_1)$ is the area Siegel–Veech constant corresponding to the suborbifold $\mathcal{M}_1$. The leading Lyapunov exponent λ_1 is equal to one.

3.3 Harder–Narasimhan filtrations: w

We refer the readers to [23], and to [25] for details about the Harder–Narasimhan filtration.

Consider a smooth curve C and a holomorphic vector bundle V over C. In order to recall the definition of stability one needs the normalized Chern class or *slope* $\mu(V)=\deg(V)/\operatorname{rk}(V)$ of the vector bundle V. A holomorphic bundle V is called *stable* if $\mu(W)<\mu(V)$ for every proper holomorphic subbundle W of V. A *semistable* bundle is defined similarly but we allow now the weak inequality $\mu(W)\le\mu(V)$.

Harder and Narasimhan show that every holomorphic bundle V has a canonical filtration

$$0=\mathrm{HN}_0(V)\subset\mathrm{HN}_1(V)\subset\dots\subset\mathrm{HN}_r(V)=V$$

satisfying the following two properties. Every graded quotient

$$\mathrm{gr}_i^{\mathrm{HN}}=\mathrm{HN}_i(V)/\,\mathrm{HN}_{i-1}(V)$$

for $i=1,\dots,r$ is semistable, and

$$\mu(\mathrm{gr}_1^{\mathrm{HN}})>\mu(\mathrm{gr}_2^{\mathrm{HN}})>\dots>\mu(\mathrm{gr}_r^{\mathrm{HN}}).$$

If $\mathrm{gr}_i^{\mathrm{HN}}$ has rank n_i and Chern number k_i, so that $n=\sum n_i$ and $k=\sum k_i$, we shall call the sequence of pairs (n_i,k_i) for $i=1,\dots,r$ the *slope type* of the holomorphic bundle V. As before, it is convenient to describe the type equivalently by a single

n–vector $\mu(V)$ whose components are the ratios k_i/n_i each represented n_i times and arranged in decreasing order:

$$\mu(V) = (\mu_1, \dots, \mu_n) = \Bigg(\underbrace{\frac{k_1}{n_1}, \dots, \frac{k_1}{n_1}}_{n_1}, \dots, \underbrace{\frac{k_r}{n_r}, \dots, \frac{k_r}{n_r}}_{n_r}\Bigg).$$

For a Teichmüller curve C, it is convenient to set $w_i = \mu_i(f_*\omega_{S/C})/(\chi/2)$ and to define the *Harder–Narasimhan type* of a Teichmüller curve C as the g–vector

$$w(C) = (w_1, \dots, w_g). \tag{6}$$

It follows from geometric arguments in [37] (see also [22] and [44]), that for any Teichmüller curve C in any stratum of abelian differentials the identity $w_1(C) = 1$ is valid.

The Harder–Narasimhan type of a Teichmüller curve is given by the next two theorems.

Theorem 3.3 [47] *Let C be a Teichmüller curve in the hyperelliptic locus of some stratum $\overline{\mathcal{H}}_g(m_1, \dots, m_k)$, and denote by $(d_1, \dots, d_n)$ the orders of singularities of underlying quadratic differentials. Then $w_i(C)$ is the i^{th} largest number in the set*

$$\{1\} \cup \left\{1 - \frac{2k}{d_j + 2} : 0 < 2k \le d_j + 1,\ j = 1, \dots, n\right\}.$$

In particular, the Harder–Narasimhan type $w(C)$ of a Teichmüller curve in any hyperelliptic locus of any stratum is constant and depends only on the locus.

The nonvarying property of the Harder–Narasimhan type of all Teichmüller curves is also valid for certain strata in low genera $g = 3, 4, 5$; see tables with explicit values of all $w_i(C)$ in the appendix. This observation provides an alternative proof [47] of the Kontsevich–Zorich conjecture on the nonvarying of the sums of the Lyapunov exponents of the Hodge bundle for the corresponding strata; see [9] for the original proof.

Since the action of $\mathrm{GL}^+(2, \mathbb{R})$ preserves the strata of meromorphic quadratic differentials with at most simple poles, it also preserves their images under the map (1). In particular, all hyperelliptic loci in the strata of abelian differentials are invariant under the action of $\mathrm{GL}^+(2, \mathbb{R})$. Thus, if some Teichmüller curve C intersects some hyperelliptic locus $\mathcal{M}^{\text{hyp}}$, it is entirely contained in it: $C \subset \mathcal{M}^{\text{hyp}}$.

Theorem 3.4 [46] *For any stratum $\mathcal{H}_g(m_1, \dots, m_k)$ of abelian differentials, order the entries of the set with multiplicities*

$$\left\{\frac{j}{m_l+1}\right\}_{1\le j\le m_l,\, 1\le l\le k}$$

to get an increasing sequence of $2g-2$ numbers $a_1 \le a_2 \le \dots \le a_{2g-2}$, where $2g-2 = m_1 + \dots + m_k$.

For any Teichmüller curve C in the stratum $\mathcal{H}_g(m_1, \dots, m_k)$, there exists a permutation P_C of the set $\{1, \dots, 2g-2\}$, satisfying the following properties. For $i = 2, \dots, g$, we have $P_C(i) \ge 2i-2$, where all inequalities for $i = 2, \dots, g-1$ are strict if C is not contained in some hyperelliptic locus. The normalized Harder–Narasimhan slopes $w_i(C)$ of the Hodge bundle over C satisfy, for $i = 2, \dots, g$, the system of inequalities

$$w_i \le 1 - a_{P_C(i)}. \tag{7}$$

Example 3.5 Consider the stratum $\mathcal{H}_5(6,1,1)$. Ordering the entries of the set with multiplicities

$$\left\{\tfrac{1}{7}, \tfrac{2}{7}, \tfrac{3}{7}, \tfrac{4}{7}, \tfrac{5}{7}, \tfrac{6}{7}, \tfrac{1}{2}, \tfrac{1}{2}\right\}$$

in increasing order we get an ordered set with multiplicities

$$\{a_1, a_2, \dots, a_8\} = \left\{\tfrac{1}{7}, \tfrac{2}{7}, \tfrac{3}{7}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{4}{7}, \tfrac{5}{7}, \tfrac{6}{7}\right\}.$$

For any Teichmüller curve in the stratum $\mathcal{H}_5(6,1,1)$, the permutation P_C of the set $\{1, \dots, 8\}$ satisfies $P_C(i) \ge 2i-2$ for $i = 1, \dots, 4$, so we have

$$P_C(2) \ge 2, \quad P_C(3) \ge 4, \quad P_C(4) \ge 6, \quad P_C(5) = 8.$$

Theorem 3.4 asserts that the normalized Harder–Narasimhan slopes $w_i(C)$ satisfy

$$\begin{aligned} w_2(C) &\le 1 - a_{P_C(2)} \le 1 - a_2 = \tfrac{5}{7}, \\ w_3(C) &\le 1 - a_{P_C(3)} \le 1 - a_4 = \tfrac{1}{2}, \\ w_4(C) &\le 1 - a_{P_C(4)} \le 1 - a_6 = \tfrac{3}{7}, \\ w_5(C) &\le 1 - a_{P_C(5)} \le 1 - a_8 = \tfrac{1}{7}. \end{aligned}$$

If C is not located in some hyperelliptic locus, then

$$P_C(2) \ge 3, \quad P_C(3) \ge 5, \quad P_C(4) \ge 7, \quad P_C(5) = 8.$$

Theorem 3.4 then asserts that the normalized Harder–Narasimhan slopes $w_i(C)$ satisfy

$$\begin{aligned} w_2(C) &\le 1 - a_{P_C(2)} \le 1 - a_3 = \tfrac{4}{7}, \\ w_3(C) &\le 1 - a_{P_C(3)} \le 1 - a_5 = \tfrac{1}{2}, \\ w_4(C) &\le 1 - a_{P_C(4)} \le 1 - a_7 = \tfrac{2}{7}, \\ w_5(C) &\le 1 - a_{P_C(5)} \le 1 - a_8 = \tfrac{1}{7}. \end{aligned}$$

A simple corollary of this theorem is:

Corollary 3.6 [46] *For a Teichmüller curve which lies in $\mathcal{H}_g(m_1, \dots, m_k)$, we have inequalities*

$$\sum_{i=1}^{g} \lambda_i = \sum_{i=1}^{g} \varepsilon_i = \sum_{i=1}^{g} w_i \le \frac{g+1}{2}.$$

The two equalities in the above formula are direct corollaries of Theorem 3.1.

4 Convexity

In [2, Section 12], Atiyah and Bott discuss the convexity of polygons and the relation with hermitian matrices. Shatz defines the partial ordering by

$$\lambda \succeq \mu \quad \text{if } P_\lambda \text{ is above } P_\mu.$$

If we consider P_μ as the graph of a concave function p_μ, then p_μ is defined on the integers by

$$p_\mu(i) = \sum_{j \le i} \mu_j,$$

and interpolates linearly between integers. Here the μ_j are the components of our n–vector μ. (See Figure 4.)

Hence, for our vector notation, it translates into the following partial ordering:

$$\lambda \succeq \mu \quad \Longleftrightarrow \quad \begin{cases} \displaystyle\sum_{j=1}^{i} \lambda_j \ge \sum_{j=1}^{i} \mu_j & \text{for } i = 1, \dots, n-1, \\ \displaystyle\sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{n} \mu_j. \end{cases}$$

This partial ordering on vectors in $\mathbb{R}^n$ is well known in various contexts.

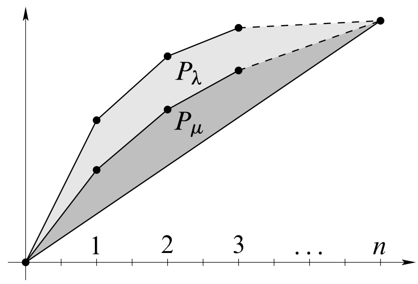

Figure 4: P_λ lies above (or on) P_μ.

This partial ordering occurs in Horn [24], where it is shown to be equivalent to each of the following properties:

(8)
- $\sum_j f(\mu_i) \leq \sum_j f(\lambda_j)$ for every convex function $f\colon \mathbb{R} \to \mathbb{R}$.
- $\mu = P\lambda$, where $\lambda, \mu \in \mathbb{R}^n$ and P is a doubly stochastic matrix.

We recall that a real square matrix is *stochastic* if $p_{ij} \geq 0$ and $\sum_j p_{ij} = 1$ for all i. If in addition the transposed matrix is also stochastic, then P is called *doubly stochastic*. A theorem of Birkhoff identifies doubly stochastic matrices in terms of permutation matrices, namely:

The doubly stochastic $n \times n$ matrices are the convex hull of the permutation matrices.

Now the equivalence relation can be replaced by

$$\widehat{\Sigma_n \mu} \subseteq \widehat{\Sigma_n \lambda},$$

where $\Sigma_n x$ denotes the orbit of any $x \in \mathbb{R}^n$ under the permutation group Σ_n, and $\widehat{C}$ denotes the convex hull of the set $C \subset \mathbb{R}^n$.

Schur showed that if μ_j (for $j = 1, \dots, n$) are the diagonal elements of a hermitian matrix whose eigenvalues are λ_j, then $\mu \preceq \lambda$. We give the proof for the largest eigenvalue; the proof is similar for the general case.

Lemma 4.1 (Schur) *For a hermitian matrix $H = [h_{ij}]$, let $\lambda_1 \geq \cdots \geq \lambda_n$ be its eigenvalues. Then*

$$\lambda_1 \geq h_{ii}.$$

Proof Let $U = [u_{ij}]$ be a unitary matrix such that

$$H = U \operatorname{diag}[\lambda_1, \dots, \lambda_n] \bar{U}^T.$$

Since $\sum_j u_{ij}\overline{u}_{ij} = 1$, we have

$$\lambda_1 = \lambda_1\Bigl(\sum_j u_{ij}\overline{u}_{ij}\Bigr) \ge \sum_j \lambda_j u_{ij}\overline{u}_{ij} = h_{ii}. \qquad \square$$

Horn proved the converse, giving another necessary and sufficient condition for the relation $\mu \preceq \lambda$.

Proposition *Vectors μ and λ satisfy the relation $\mu \preceq \lambda$ if and only if there exists a hermitian matrix with diagonal elements μ_j and eigenvalues λ_j.*

For a general compact Lie group G, the role of the hermitian (or rather skew-hermitian) matrices is played now by the Lie algebra $\mathfrak{g}$ of G. The diagonal matrices are replaced by the Lie algebra $\mathfrak{t}$ of a maximal torus T of G and Σ_n becomes the Weyl group W. Writing a set of λ_j in decreasing order corresponds to picking a (closed) positive Weyl chamber C in $\mathfrak{t}$: this is a fundamental domain for the action of W; see [2].

We also need the following linear algebra fact.

Lemma 4.2 *For $B = [b_{ij}]$ a complex symmetric matrix, $H = B\overline{B}^T = [h_{ij}]$, and $\alpha = [a_1, \dots, a_n]$ satisfying $\alpha\overline{\alpha}^T = 1$, we have*

$$\alpha H\overline{\alpha}^T \ge |\alpha B\alpha^T|^2.$$

Proof There is a decomposition for any complex symmetric matrix

$$B = U\,\mathrm{diag}[\lambda_1, \dots, \lambda_n]U^T,$$

where $U = [u_{ij}]$ is unitary, $\sum_i u_{ji}\overline{u}_{ji} = 1$, $\alpha = [a_1, \dots, a_n]$, $\sum_i a_i\overline{a}_i = 1$. Because

$$H = B\overline{B}^T = U\,\mathrm{diag}[|\lambda_1|^2, \dots, |\lambda_n|^2]\overline{U}^T,$$

we only need to show that

$$(\alpha U)\,\mathrm{diag}(|\lambda_1|^2, \dots, |\lambda_n|^2)\overline{(\alpha U)}^T \ge \bigl|(\alpha U)\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)(\alpha U)^T\bigr|^2.$$

Let $\beta = \alpha U = [b_1, \dots, b_n]$, then $\beta\overline{\beta}^T = 1$. We need to show that

$$\beta\,\mathrm{diag}(|\lambda_1|^2, \dots, |\lambda_n|^2)\overline{\beta}^T \ge \bigl|\beta\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)\beta^T\bigr|^2,$$

which is the same as

$$\sum |\lambda_i|^2\,|b_i|^2 \ge \Bigl|\sum \lambda_i b_i^2\Bigr|^2.$$

By the Cauchy inequality and $\sum |b_i|^2 = 1$, we have

$$\Big(\sum |\lambda_i|^2 |b_i|^2\Big)\Big(\sum |b_i|^2\Big) \geq \Big(\sum |\lambda_i||b_i|^2\Big)^2 \geq \Big|\sum \lambda_i b_i^2\Big|^2. \qquad \square$$

4.1 $\varepsilon \succeq w$

In this section we establish a relation between integrals of eigenvalue spectrum of the curvature of the Hodge bundle over a Teichmüller curve and its Harder–Narasimhan slopes.

Recall that by $\mu_i(E)$ we denote the Harder–Narasimhan slope of a holomorphic vector bundle over a curve C, and by $w_i(E)$ we denote the corresponding normalized Harder–Narasimhan slope, so that $\mu_i(E) = \big(\frac{1}{2}\chi\big) w_i(E)$, where $-\chi(C) = 2g - 2 + |\Delta|$ is the Euler characteristic of the underlying curve C. By $\mathrm{d}\sigma$ we denote the volume element (4) on C.

The following theorem is essentially contained in Atiyah and Bott [2, pages 573–575].

Theorem 4.3 *Let E be a hermitian vector bundle of rank n on a Riemann surface M with a volume form $\mathrm{d}\sigma$, and let*

$$\Lambda_1(E) \geq \cdots \geq \Lambda_n(E)$$

be the eigenvalues of $\frac{i}{2\pi}\Theta(E)$. For $1 \leq k \leq n$, we have

$$\sum_{j=1}^{k} \int_M \Lambda_j(E)\,\mathrm{d}\sigma \geq \sum_{j=1}^{k} \mu_j(E),$$

with equality when $k = n$.

Proof We shall begin by proving the simple case when

$$\mu_1 = \mu_2 = \cdots = \mu_r > \mu_{r+1} = \cdots = \mu_n,$$

so that the Harder–Narasimhan filtration of the bundle E has just two steps. We have an exact sequence of vector bundles

$$0 \to D_1 \to E \to D_2 \to 0,$$

where D_j has rank m_j Chern class k_j for $j = 1, 2$, so that $\mu_1 = k_1/m_1$ and $\mu_n = k_2/m_2$. For convenience we shall use the notation $\mu^j = k_j/m_j$ for $j = 1, 2$.

For the connection defined by the holomorphic structure and natural hermitian metric. The curvature $\Theta(E)$ can then be written as the form

$$\Theta(E) = \begin{bmatrix} F_1 - \eta \wedge \eta^* & d\eta \\ -d\eta^* & F_2 - \eta^* \wedge \eta \end{bmatrix},$$

where $\eta \in \Omega^{0,1}(M, \mathrm{Hom}(D_2, D_1))$, with η^* denoting its transposed conjugate and $d\eta$ the covariant differential, and F_j is the curvature of the metric connection of D_j. Now let f_j, α_j be scalar $m_j \times m_j$ matrices such that

$$\begin{aligned} &\mathrm{trace}\, f_j = \mathrm{trace}\, {*F_j}, \\ &\mathrm{trace}\, \alpha_1 = \mathrm{trace}\, {*(\eta \wedge \eta^*)} = -\,\mathrm{trace}\, {*(\eta^* \wedge \eta)} = -\,\mathrm{trace}\, \alpha_2. \end{aligned}$$

We know that $\frac{i}{2\pi} * \Theta(E)$ is a hermitian matrix. By the equivalent conditions (8) of convexity, some elementary inequalities concerning the convex invariant function ϕ show that

$$\phi(*\Theta(E)) \geq \phi \begin{bmatrix} f_1 - \alpha_1 & 0 \\ 0 & f_2 - \alpha_2 \end{bmatrix}.$$

In particular, by Lemma 4.1 it implies

$$\Lambda_1(E) \geq \frac{i}{2\pi} \frac{\mathrm{trace}(f_1 - \alpha_1)}{m_1}.$$

But the Chern class k_j of D_j is given by

$$k_j = \frac{i}{2\pi} \int_M \mathrm{trace}\, f_j \, \mathrm{d}\sigma.$$

Since f_j is a scalar matrix this means that $\int_M \mathrm{trace}\, f_j$ is scalar matrix whose diagonal entries are $-2\pi i k_j / m_j = -2\pi i \mu^j$. Also, since $\eta \in \Omega^{0,1}$, it follows that $-i\,\mathrm{trace}\, \alpha_1$ is nonnegative and so

$$\int_M \alpha_1 \, \mathrm{d}\sigma = 2\pi i a_1,$$

where a_1 is a nonnegative scalar $m_1 \times m_1$ matrix. Then

$$\int_M \alpha_2 \, \mathrm{d}\sigma = 2\pi i a_2,$$

where a_2 is nonnegative scalar $m_2 \times m_2$ matrix such that $\mathrm{trace}\; a_2 = \mathrm{trace}\; a_1$. Hence we have

$$\frac{i}{2\pi} \int_M \begin{bmatrix} f_1 - \alpha_1 & 0 \\ 0 & f_2 - \alpha_2 \end{bmatrix} \mathrm{d}\sigma = [\mu + a],$$

where [ ] denotes the diagonal matrix defined by a vector, so that $[a]$ denotes the matrix $\left[\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right]$.

But since $a_1 \geq 0$ and $a_2 \leq 0$, with $\operatorname{trace} a_1 = -\operatorname{trace} a_2$, it follows easily that $\mu + a \geq \mu$ with respect to the partial ordering. Hence we have

$$\int_M \Lambda_1(E)\,\mathrm{d}\sigma \geq \frac{i}{2\pi}\int_M \frac{\operatorname{trace}(f_1 - \alpha_1)}{m_1}\,\mathrm{d}\sigma \geq \frac{i}{2\pi}\int_M \frac{\operatorname{trace} f_1}{m_1}\,\mathrm{d}\sigma = \frac{k_1}{m_1}.$$

This completes the proof for the two-step case. The general case proceeds in the same manner and we simply have to keep track of the notation. The details are as follows.

We start with a holomorphic bundle E with its canonical filtration of type μ,

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = E,$$

where the quotients $D_j = E_j/E_{j-1}$ have normalized Chern classes μ^j with

$$\mu^1 > \mu^2 > \cdots > \mu^r.$$

The curvature $\Theta(E)$ can then be expressed in a generalized block form. For every $j < k$ we have an element

$$\eta_{jk} \in \Omega^{0,1}(M, \operatorname{Hom}(D_k, D_j)),$$

so that $d\eta_{jk}$ appears in the (j,k)–block. The η_{jk} are the components of the element

$$\eta_k \in \Omega^{0,1}(M, \operatorname{Hom}(D_k, E_{k-1}))$$

related to the exact sequence

$$0 \to E_{k-1} \to E_k \to D_k \to 0.$$

Now define scalar nonnegative $m_j \times m_j$ matrices a_{jk} for $j < k$ by

$$\operatorname{trace} a_{jk} = \frac{1}{2\pi i}\int_M \operatorname{trace}(\eta_{jk} \wedge \eta_{jk}^*)\,\mathrm{d}\sigma \geq 0,$$

and define a_{kk} by

$$\operatorname{trace}\ a_{kk} = \frac{1}{2\pi i}\int_M \operatorname{trace}(\eta_k^* \wedge \eta_k)\,\mathrm{d}\sigma \leq 0,$$

so that $\sum_{j \leq k} \operatorname{trace} a_{jk} = 0$. Then the convexity leads to the inequality

$$\int_M \Lambda_1(E)\,\mathrm{d}\sigma \geq \frac{i}{2\pi}\int_M \frac{\operatorname{trace}(f_1 - \alpha_1)}{m_1}\,\mathrm{d}\sigma,$$

where a stands for the vector (or diagonal matrix) whose j^{th} block is the scalar (matrix)

$$a^j = \sum_{k \geq j} a_{jk}.$$

Equivalently the vector a can be written as a sum

$$a = \sum b_k,$$

where b_k is the vector corresponding to the diagonal matrix whose j^{th} block is a_{jk} for $j \leq k$ (and zero for $j > k$). The fact that

$$\operatorname{trace} a_{jk} \geq 0 \quad \text{for } j < l \qquad \text{and} \qquad \sum_{j \leq k} \operatorname{trace} a_{jk} = 0$$

implies that $b_k \geq 0$, relative to the partial ordering. Hence $a = \sum b_k \geq 0$ and so $\mu + a \geq \mu$. As before this then implies that

$$\frac{i}{2\pi} \int_M \frac{\operatorname{trace}(f_1 - \alpha_1)}{m_1} \, \mathrm{d}\sigma \geq \frac{i}{2\pi} \int_M \frac{\operatorname{trace} f_1}{m_1} \, \mathrm{d}\sigma = \frac{k_1}{m_1},$$

and so completes the general proof. □

Of course, the theorem implies the following corollary, which was also noticed earlier by Möller.

Corollary 4.4 *For a Teichmüller curve C, we have*

$$\varepsilon(C) \succeq w(C).$$

Proof Because

$$\mu_1(f_* \omega_{S/C}) = \tfrac{1}{2}\chi,$$

by formulae (5), (6) and by Theorem 4.3 we have

$$\sum_{j=1}^{k} \varepsilon_i(C) = \frac{\int_C \Lambda_j(\omega) \, \mathrm{d}\sigma}{\frac{1}{2}\chi} \geq \sum_{j=1}^{k} \frac{\mu_i(f_* \omega_{S/C})}{\mu_1(f_* \omega_{S/C})} = \sum_{j=1}^{k} \mu_i(C). \qquad \square$$

4.2 $2\varepsilon \succeq \lambda$

The Lyapunov exponents of a vector bundle endowed with a connection also can be viewed as logarithms of mean eigenvalues of monodromy of the vector bundle along a flow on the base.

In the case of the Hodge bundle, we take a fiber of $H^1_{\mathbb{R}}$ and pull it along a Teichmüller geodesic flow on the moduli space. We wait till the geodesic (or Kähler random walk [28]) winds a lot and comes close to the initial point, and then compute the resulting monodromy matrix $A(t)$. Finally, letting $s_1(t) \geq \cdots \geq s_{2g}(t)$ be the eigenvalues of $A^T A$, we compute logarithms of $s_i(t)$ and normalize them by twice the length t of the geodesic,

$$\lambda_i = \lim_{t\to\infty} \frac{\log s_i(t)}{2t}.$$

By the Oseledets multiplicative ergodic theorem, for almost all choices of initial data (starting point, starting direction) the resulting $2g$ real numbers converge as $t \to \infty$, to limits which do not depend on the initial data within an ergodic component of the flow. These limits $\lambda_1 \geq \cdots \geq \lambda_{2g}$ are the Lyapunov exponents of the Hodge bundle along the Teichmüller geodesic flow [14].

A simple corollary of Lemma 4.1 is:

Corollary 4.5 *Let $a_1(t) \geq \cdots \geq a_{2g}(t)$ be the diagonal elements of $A^T A$. Then*

$$\lambda_1 \geq \lim_{t\to\infty} \sup \frac{\log a_1(t)}{2t}.$$

We confess that we do not know any geometric interpretation of the quantity on the right-hand side of the latter inequality.

Forni has shown that the eigenvalues of curvature are closely related to Lyapunov exponents. Let $h(c)$ be the unique holomorphic form such that c is the cohomology class of the closed 1–form $\operatorname{Re} h(c)$. By using the first variational formula [21, page 19]

$$\mathcal{L} \log \|c\|_\omega = -\frac{\operatorname{Re} B_\omega(h(c), h(c))}{\|c\|_\omega^2},$$

where $\mathcal{L}$ is the Lie derivative in direction $v = \omega^2$ and $\|\cdot\|_\omega$ is the Hodge norm on $H^{1,0}(X)$, he gets:

Corollary 4.6 [19, Corollary 2.2] *Let μ be any $\mathrm{SL}(2,\mathbb{R})$–invariant Borel probability ergodic measure on the moduli space $\mathcal{H}_g$ of normalized abelian differentials. The second Lyapunov exponent of the Kontsevich–Zorich cocycle with respect to the measure μ satisfies the inequality*

$$1 > \int_{\mathcal{H}_g} \sqrt{|\Lambda_2(\omega)|}\, d\mu(\omega) \geq \lambda_2^\mu.$$

Let $\{c_1, \dots, c_k\}$ be any Hodge-orthonormal basis of any isotropic subspace $I_k \subset H^1_{\mathbb{R}}$. His second variational formula [21, page 21] is

$$\Delta \log \|c_1 \wedge \cdots \wedge c_k\|_\omega = 2\Phi_k(\omega, I_k),$$

where Δ is the leafwise hyperbolic Laplacian for the metric of curvature -4 on the Teichmüller leaves, $\omega_i = h(c_i)$, and

$$\Phi_k(\omega, I_k) = 2\sum_{i=1}^{k} H_\omega(\omega_i, \omega_i) - \sum_{i,j=1}^{k} |B_\omega(\omega_i, \omega_j)|^2.$$

Then Theorem 3.1 can be deduced from the following.

Corollary 4.7 ([19, Corollary 5.5] and [21, Corollary 3.2]) *Let μ be any $\mathrm{SL}(2,\mathbb{R})$–invariant Borel probability ergodic measure on the moduli space $\mathcal{H}_g$ of normalized abelian differentials. Assume that there exists $k \in 1, \dots, g-1$ such that $\lambda_k^\mu > \lambda_{k+1}^\mu \geq 0$. Then the following formula holds:*

$$\lambda_1^\mu + \cdots + \lambda_k^\mu = \int_{\mathcal{H}_g} \Phi_k(\omega, E_k^+(\omega))\, \mathrm{d}\mu(\omega).$$

We also give an upper bound of λ_2:

Corollary 4.8 *Let μ be any $\mathrm{SL}(2,\mathbb{R})$–invariant Borel probability ergodic measure on the moduli space $\mathcal{H}_g$ of normalized abelian differentials. Then the following formula holds:*

$$2\int_{\mathcal{H}_g} \Lambda_2(\omega)\, \mathrm{d}\mu(\omega) \geq \lambda_2^\mu.$$

In particular, for a Teichmüller curve,

$$2\varepsilon_2 \geq \lambda_2.$$

Proof For any $c \in \langle[\mathrm{Re}(\omega)], [\mathrm{Im}(\omega)]\rangle^\perp$, Lemma 4.1 implies

$$2\Lambda_2(\omega) \geq \frac{2H_\omega(h(c), h(c))}{\|c\|_\omega^2} \geq \frac{2H_\omega(h(c), h(c))}{\|c\|_\omega^2} - \frac{|B_\omega(h(c), h(c))|^2}{\|c\|_\omega^4}. \tag{9}$$

Let c be a Kontsevich–Zorich cocycle with Lyapunov exponents λ_2^μ. If $\lambda_2^\mu > \lambda_3^\mu$, then by Corollary 4.7,

$$2\int_{\mathcal{H}_g} \Lambda_2(\omega)\, \mathrm{d}\mu(\omega) \geq \int_{\mathcal{H}_g} \left(\frac{2H_\omega(h(c), h(c))}{\|c\|_\omega^2} - \frac{|B_\omega(h(c), h(c))|^2}{\|c\|_\omega^4} \right) \mathrm{d}\mu(\omega) = \lambda_2^\mu.$$

If $\lambda_2^\mu = \cdots = \lambda_k^\mu > \lambda_{k+1}^\mu$, then by Corollary 4.7, the result also can be deduced from

$$\begin{aligned} 2(k-1)\int_{\mathcal{H}_g} \Lambda_2(\omega)\,\mathrm{d}\mu(\omega) &\geq 2\int_{\mathcal{H}_g} (\Lambda_2(\omega)+\cdots+\Lambda_k(\omega))\,\mathrm{d}\mu(\omega) \\ &\geq \lambda_2^\mu + \cdots + \lambda_k^\mu \\ &= (k-1)\lambda_2^\mu. \end{aligned} \qquad \square$$

The fiberwise inequality (9) does not imply $\varepsilon_2 \geq \lambda_2$. Since $\sqrt{|\Lambda_2(\omega)|} \geq 2\Lambda_2(\omega)$ if and only if $\frac{1}{4} \geq \Lambda_2(\omega)$, in contrast to Corollary 4.6, this corollary is useful when ε_2 is small. Similarly we can get

$$2\sum_{j=2}^{i} \varepsilon_j \geq \sum_{j=2}^{i} \lambda_j$$

for $i = 2, \ldots, n$. Or, in weak form,

$$2\varepsilon(C) \succeq \lambda(C).$$

(Warning: obviously $2\sum_{j=1}^{g} \varepsilon_j \neq 2\sum_{j=1}^{g} \lambda_j$. Here $\succeq$ only means "lies above".)

It would be interesting to get the best inequality between $\varepsilon(C)$ and $\lambda(C)$.

4.3 Hodge and Newton polygons

Since the Main Conjecture is originally inspired by the Katz–Mazur theorem, we briefly recall this theorem.

Let k be a finite field of $q = p^a$ elements. Let W denote its ring of Witt vectors, and K the field of fractions of W. Let X be projective and smooth over W, and such that the W–modules $H^r(X, \Omega^s_{X/W})$ are free (of rank $h^{s,r}$) for all s, r.

Form the polynomial

$$H_m(t) = \prod_{s=0}^{m} (1-q^s t)^{h^{s,m-s}} \in Z[t] \subset K[t],$$

which can be called the m–dimensional Hodge polynomial of X/W. Set

$$Z_m(t) = \det(1 - F^a|_{H^m_{\mathrm{DR}}(X/W)} t) \in K[t],$$

where F is the canonical lifting of Frobenius on de Rham cohomology.

Now, for any polynomial of the form $R(t) = 1 + R_1 t + R_2 t^2 + \cdots + R_\beta t^\beta \in K[t]$, Mazur defined the polygon of $R(t)$ to be the convex closure in the Euclidean plane of the finite set of points

$$\{(j, \operatorname{ord}_q(R_j)) \mid j = 0, 1, \ldots, \beta\},$$

where $\operatorname{ord}_q(q) = 1$. The left-most vertex of this polygon is the origin, while the right-most is $(\beta, \operatorname{ord}_q(R_\beta))$. The structure of this polygon is a measure of the p–adic valuations of the zeros of R.

According to our definition, the convex polygon has vertices

$$\{(j, \operatorname{ord}_q(R_{\beta-j+1})) \mid j = 0, 1, \ldots, \beta\}.$$

Now what Katz conjectured and Mazur proved is:

Theorem 4.9 [34; 35] *The convex polygon of $H_m(t)$ (ie Hodge polygon) lies above (or on) the convex polygon of $Z_m(t)$ (ie Newton polygon).*

A motivation for the main conjecture is to try to understand the sentence "Lyapunov exponents as dynamical Hodge decomposition"; see [29] and [50, page 37].

Corollary 4.10 [34; 35] *If the Hodge numbers $h^{s,m-s}$ vanish for $0 \le s < t$, then the eigenvalues of F^a acting on $H^m_{\mathrm{DR}}(X/W)$ are divisible by q^t.*

We think that Proposition 5.9 is an analogy of this corollary.

5 Conditional results and further conjectures

5.1 Is $\lambda \succeq w$?

Zorich asked the following question about the correspondence between Lyapunov exponents and characteristic numbers of some natural bundles. He set this as the last problem of his survey.

Problem 5.1 [50, page 135] To study individual Lyapunov exponents of the Teichmüller geodesic flow

- for all known $\mathrm{SL}(2; R)$–invariant subvarieties,
- for strata in large genera.

Are they related to characteristic numbers of some natural bundles over appropriate compactifications of the strata?

Originally inspired by the Katz–Mazur theorem (Theorem 4.9) in p–adic Hodge theory, we state our main conjecture after checking over all available numerical data:

Conjecture 5.2 (Main Conjecture) *For any Teichmüller curve, we have*

$$\lambda(C) \succeq w(C).$$

That is,

$$\begin{cases} \sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} w_j & \text{for } i = 1, \dots, g-1, \\ \sum_{j=1}^{g} \lambda_j = \sum_{j=1}^{g} w_j. \end{cases}$$

Or, equivalently,

$$\begin{cases} \sum_{j=i}^{g} \lambda_j \leq \sum_{j=i}^{g} w_j & \text{for } i = 2, \dots, g, \\ \sum_{j=1}^{g} \lambda_j = \sum_{j=1}^{g} w_j. \end{cases}$$

Remark 5.3 The result

$$\varepsilon(C) \succeq w(C)$$

in Corollary 4.4 has its roots in

$$\Lambda_1(\omega) \geq H_\omega(\omega_i, \omega_i).$$

The result

$$2\varepsilon(C) \succeq \lambda(C)$$

in Corollary 4.8 has its roots in

$$2\Lambda_1(\omega) \geq \frac{2H_\omega(h(c), h(c))}{\|c\|_\omega^2} - \frac{|B_\omega(h(c), h(c))|^2}{\|c\|_\omega^4}.$$

Lemma 4.2 or [21, page 22] gives us

$$\frac{2H_\omega(h(c), h(c))}{\|c\|_\omega^2} - \frac{|B_\omega(h(c), h(c))|^2}{\|c\|_\omega^4} \geq \frac{H_\omega(h(c), h(c))}{\|c\|_\omega^2}.$$

We suspect that the Main Conjecture might have its roots in the latter inequality.

When equality is attained, we also make the following rigidity conjecture.

Conjecture 5.4 *If for some Teichmüller curve in the moduli space of abelian differentials of genus g the equality*

$$\sum_{j=1}^{k} \lambda_j = \sum_{j=1}^{k} w_j$$

is achieved for some $k < g$ and

$$w_k \neq w_{k+1},$$

then the corresponding VHS contains a rank $2(g-k)$ local subsystem.

It can be considered as the inverse to the Kontsevich–Forni formula of Theorem 3.1. We will illustrate an example of a simple corollary of the two conjectures in Proposition 5.9.

Problem 5.5 (lower continuity conjecture) We also hope that one can define a g–vector $w(\mathcal{M})$ (of algebrogeometric origin) which would generalize the normalized Harder–Narasimhan slope type from Teichmüller curves to *all* $\mathrm{GL}(2,\mathbb{R})$–invariant submanifolds $\mathcal{M}$ in the moduli space of abelian differentials, and which would have the following natural properties:

(1) If for each $i = 1, \dots, g$ the normalized Harder–Narasimhan slope $w_i(C)$ is constant for all Teichmüller curves C in $\mathcal{M}$, then $w_i(\mathcal{M}) = w_i(C)$.

(2) If for some i, $1 \le i \le g$, one has $w_i(C) \le a$ for any Teichmüller curve C in $\mathcal{M}$, then $w_i(\mathcal{M}) \le a$.

We say that a family of Teichmüller curves is dense inside the ambient submanifold $\mathcal{N}$ of the moduli space of abelian differentials if the closure of the union of these curves coincides with $\mathcal{N}$. For example, the arithmetic Teichmüller curves form a dense family in any $\mathrm{GL}(2,\mathbb{R})$–invariant submanifold defined over $\mathbb{Q}$; in particular, in any connected component of any stratum, and in any hyperelliptic locus in any stratum.

Conditional Theorem 5.6 *Consider a $\mathrm{GL}(2,\mathbb{R})$–invariant submanifold $\mathcal{N}$ in the moduli space of abelian differentials. Suppose that $\mathcal{N}$ contains a dense family of Teichmüller curves and that the normalized Harder–Narasimhan slopes $w_i(C)$ of all but at most a finite number of Teichmüller curves $C \subset \mathcal{N}$ in this family satisfy*

$$\sum_{i=1}^{k} w_i(C) \ge a$$

for some $k \le g$. The Main Conjecture implies that the Lyapunov exponents $\lambda_i(\mathcal{N})$ of the Hodge bundle over the Teichmüller geodesic flow on $\mathcal{N}$ satisfy the inequality

$$\sum_{i=1}^{k} \lambda_i(\mathcal{N}) \ge a.$$

Proof The Main Conjecture implies that for any Teichmüller curve C satisfying

$$\sum_{i=1}^{k} w_i(C) \ge a,$$

we also have

$$\sum_{i=1}^{k} \lambda_i(C) \ge \sum_{i=1}^{k} w_i(C) \ge a.$$

The statement of the theorem now follows from Theorem 1.1. □

Proof of Corollary 1.2 It follows from Theorem 3.3 that all normalized Harder–Narasimhan slopes $w_i(C)$ are constant for all Teichmüller curves in the hyperelliptic connected components $\mathcal{H}_g^{\mathrm{hyp}}(2g-2)$ and $\mathcal{H}_g^{\mathrm{hyp}}(g-1,g-1)$, and the corresponding Harder–Narasimhan types $w(C)$ are given by the formula

$$w(C) = \begin{cases} \left(\frac{2g-1}{2g-1}, \frac{2g-3}{2g-1}, \dots, \frac{5}{2g-1}, \frac{3}{2g-1}, \frac{1}{2g-1}\right) & \text{for } C \subset \mathcal{H}_g^{\mathrm{hyp}}(2g-2), \\ \left(\frac{2g}{2g}, \frac{2g-2}{2g}, \dots, \frac{6}{2g}, \frac{4}{2g}, \frac{2}{2g}\right) & \text{for } C \subset \mathcal{H}_g^{\mathrm{hyp}}(g-1,g-1). \end{cases}$$

The above expressions imply that for any fixed $k \in \mathbb{N}$ one has $w_k(C) \to 1$ as $g \to \infty$, and that

$$\sum_{i=1}^{k} w_i(C) \ge a(g),$$

where $a(g) \to k$ as $g \to \infty$. By Conditional Theorem 5.6, the sum $\lambda_1 + \dots + \lambda_k$ of the top k Lyapunov exponents of the hyperelliptic components of the strata has the same asymptotic lower bound.

On the other hand, we have $1 = \lambda_1 \ge \lambda_i$ (actually, for $i > 1$ the inequality is strict; see [49] and [19, Theorem 0.2]) so for any fixed $k > 1$, as $g \to \infty$ we get

$$k > \sum_{i=1}^{k} \lambda_i \ge a(g) \to k.$$

Taking into consideration that $1 = \lambda_1 > \lambda_2 \ge \dots \ge \lambda_k$, this implies Corollary 1.2. □

We end this section with two more conditional statements in the spirit of Corollary 1.2. Though both results were already proved by alternative (and quite involved) methods, they serve as nice illustrations of potential further applications of the methods developed in this paper: the results immediately follow from combining the Main Conjecture and Theorem 1.1.

Conditional Theorem 5.7 *The assumption that the second Lyapunov exponent λ_2 of the principal stratum $\mathcal{H}_3(1,1,1,1)$ of abelian differentials in genus 3 is strictly greater than $\frac{1}{2}$ implies that algebraically primitive Teichmüller curves are not dense in this stratum.*

(Actually, the paper [5] contains an alternative unconditional proof that there is at most a finite number of algebraically primitive Teichmüller curves in this stratum.)

Proof It is proved in [46] that for any algebraically primitive Teichmüller curve C, the nonstrict inequalities in the Main Conjecture become equalities; in other words, the following system of equalities is valid:

$$w_i(C) = \lambda_i(C) \quad \text{for } i = 1, \dots, g.$$

An explicit calculation (see Table 1) shows that $w_2(C) \le \frac{1}{2}$ for any Teichmüller curve C in the stratum $\mathcal{H}_3(1,1,1,1)$. Thus for any algebraically primitive Teichmüller curve C in $\mathcal{H}_3(1,1,1,1)$, we have

$$\lambda_2(C) = w_2(C) \le \tfrac{1}{2}.$$

degrees of zeros	connected components	Lyapunov exponents λ_2	λ_3	normalized Harder–Narasimhan slopes w_2	w_3	sum $\sum_{j=1}^{g} \lambda_j = \sum_{j=1}^{g} w_j$
(4)	hyp	0.6156	0.1844	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{9}{5}$
(4)	odd	0.4179	0.1821	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{8}{5}$
(3,1)		0.5202	0.2298	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{7}{4}$
(2,2)	hyp	0.6883	0.3117	$\frac{2}{3}$	$\frac{1}{3}$	2
(2,2)	odd	0.4218	0.2449	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{3}$
(2,1,1)		0.5397	0.2936	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{11}{6}$
(1,1,1,1)		0.5517	0.3411	$\le\frac{1}{2}$	$\le\frac{1}{2}$	$\frac{53}{28}$

Table 1: All strata in genus 3

Thus, by Theorem 1.1, for any $\mathrm{GL}(2,\mathbb{R})$–submanifold $\mathcal{N} \subseteq \mathcal{H}_3(1,1,1,1)$ in which the algebraically primitive Teichmüller curves are dense, we also have $\lambda_2(\mathcal{N}) \leq \frac{1}{2}$, which implies that statement in the theorem. (Note that the experimental approximate value of the second Lyapunov exponent λ_2 of the principal stratum in genus 3 is $\lambda_2(\mathcal{H}_3(1,1,1,1)) \approx 0.5517$.) $\square$

Clearly, the same method applies to any stratum of abelian differentials and to any $\mathrm{GL}(2,\mathbb{R})$–submanifold $\mathcal{N}$ in it for which one can find lower bounds for some Lyapunov exponent $\lambda_i(\mathcal{N})$ exceeding the values of $w_i(C)$ for algebraically primitive Teichmüller curves in $\mathcal{N}$.

Conditional Theorem 5.8 *The Main Conjecture implies that the spectrum of Lyapunov exponents of the Hodge bundle under the Teichmüller geodesic flow on any Teichmüller curve in the hyperelliptic component $\mathcal{H}_3^{\mathrm{hyp}}(4)$, and on the component $\mathcal{H}_3^{\mathrm{hyp}}(4)$ itself, is simple, with $\lambda_1 > \lambda_2 > \lambda_3$.*

Proof The normalized Harder–Narasimhan slopes of all Teichmüller curves in the hyperelliptic connected component $\mathcal{H}_3^{\mathrm{hyp}}(4)$ are constant and have values

$$w_1 = 1, \quad w_2 = \tfrac{3}{5}, \quad w_3 = \tfrac{1}{5}.$$

Thus, the Main Conjecture implies that

$$\lambda_2 \geq \tfrac{3}{5} = w_2 > w_3 = \tfrac{1}{5} \geq \lambda_3,$$

and the Lyapunov spectrum of any Teichmüller curve in this component is simple. Applying Conditional Theorem 5.6 we obtain the same inequalities and, hence, the same conclusion for the entire component $\mathcal{H}_3^{\mathrm{hyp}}(4)$. $\square$

Note that simplicity of the spectrum of Lyapunov exponents for any connected component of any stratum of abelian differentials was proved in [3].

5.2 Higgs fields

For the weight-one $\mathbb{Q}$–VHS

$$\big(\mathbb{V}, H^{1,0} = f_*\omega_{S/C} \subset H = (\mathbb{V} \otimes_{\mathbb{Q}} \mathcal{O}_{(C\setminus\Delta)})_{\mathrm{ext}}\big),$$

which comes from the semistable family of curves $f\colon S \to C$, the connection ∇ composed with the inclusion and projection gives a $\mathcal{O}_C$–linear map

$$\theta^{1,0}\colon H^{1,0} \to H \to H \otimes \Omega_C(\log \Delta) \to (H/H^{1,0}) \otimes \Omega_C(\log \Delta).$$

If we extend $\theta^{1,0}$ by the zero mapping to the associated graded sheaf we get a Higgs bundle $(\mathrm{gr}(H), \theta) = (H^{1,0} \oplus H^{0,1}, \theta^{1,0} \oplus 0)$. By definition this is a vector bundle on C with a holomorphic map $\theta\colon F \to F \otimes \Omega_C(\log \Delta)$, the additional $\theta \wedge \theta$ being void if the base is a curve [42; 37].

Higgs subbundles of a Higgs bundle (F, θ) are subbundles $G \subset F$ such that $\theta(G) \subset G$. The Higgs bundle is stable if for any Higgs subbundle $(G, \theta|_G)$,

$$\frac{\deg(G)}{\mathrm{rk}(G)} < \frac{\deg(F)}{\mathrm{rk}(F)}.$$

A semistable Higgs bundle is defined similarly but we allow now the weak inequality

$$\frac{\deg(G)}{\mathrm{rk}(G)} \leq \frac{\deg(F)}{\mathrm{rk}(F)}.$$

A Higgs bundle (F, θ) is polystable if

$$(F, \theta) = \bigoplus_i (F_i, \theta_i),$$

where (F_i, θ_i) are stable Higgs bundles. Simpson shows that every stable Higgs bundle (F, θ) has a hermitian Yang–Mills metric. If $c_1(F) = 0$, $\theta \wedge \theta = 0$ and $c_2(F)[\omega]^{n-2} = 0$ then the connection is flat. The last two conditions are void if the base is a curve. Simpson also shows that for the complex variation of the Hodge structure H, the Higgs bundle $(\mathrm{gr}(H), \theta)$ is a polystable Higgs bundle such that each direct summand is of degree 0; see [39].

Simpson's correspondence allows us to switch back and forth between degree 0 stable Higgs subbundles of F and sublocal systems of $\mathbb{V}$.

The Higgs field $(\mathrm{gr}(H), \theta)$ is the edge morphism

$$H^{1,0} = f_*\omega_{S/C} \to R^1 f_*\mathcal{O}_S \otimes \Omega^1_C(\log \Delta) = H^{0,1} \otimes \Omega_C(\log \Delta)$$

of the tautological sequence

$$0 \to f^*\Omega^1_C(\log \Delta) \to \Omega^1_S(\log(f^{-1}\Delta)) \to \Omega^1_{S/C}(\log(f^{-1}\Delta)) \to 0.$$

By combining with some well-known results of Higgs bundles [26; 42], we have the following property, which says the nonuniform hyperbolicity in dynamical systems [19, Theorem 0.2] implies the positivity in algebraic geometry.

Proposition 5.9 *For any Teichmüller curve, we have*

$$\lambda_i > 0 \quad \Longrightarrow \quad w_i > 0.$$

If $w_k \neq 0$ and $w_{k+1} = \cdots = w_g = 0$, then the VHS contains a sublocal system of $\mathbb{V}$ of rank $2(n-k)$.

Proof For any stable quotient bundle

$$H^{1,0} = f_*\omega_{S/C} \to V \to 0,$$

by dualization we have

$$0 \to V^\vee \to f_*\omega_{S/C}^\vee = H^{0,1}.$$

Then we construct a stable Higgs subbundle

$$(0 \oplus V^\vee, 0) \subset (H^{1,0} \oplus H^{0,1}, \theta^{1,0} \oplus 0).$$

Since $(\mathrm{gr}(H), \theta)$ is a polystable Higgs bundle, this means that

$$-\deg(V) = \deg(0 \oplus V^\vee) \leq \deg(H^{1,0} \oplus H^{0,1}) = 0.$$

Denote by $\mathrm{HN}_{\min}(W)$ the last quotient in the Harder–Narasimhan filtration of a vector bundle W. Let $\mu_{\min}(W)$ be the slope $\mu(\mathrm{HN}_{\min}(W))$. Then

$$\mu_{\min}(f_*\omega_{S/C}) \geq 0.$$

Any quotient bundle $f_*\omega_{S/C} \xrightarrow{\varphi} Q \to 0$ induces a quotient bundle

$$f_*\omega_{S/C} \xrightarrow{\phi} \mathrm{HN}_{\min}(Q) \to 0,$$

we have $\mu_{\min}(Q) \geq \mu_{\min}(f_*\omega_{S/C})$ (otherwise the map ϕ is zero). We then obtain

$$\deg(Q) \geq \mu_{\min}(Q) \cdot \mathrm{rk}(Q) \geq \mu_{\min}(f_*\omega_{S/C}) \cdot \mathrm{rk}(Q) \geq 0.$$

If $w_k \neq 0$ and $w_{k+1} = \cdots = w_g = 0$, then $\mu_{\min}(f_*\omega_{S/C}) = 0$. Since $(\mathrm{gr}(H), \theta)$ is a polystable Higgs bundle, this means that

$$(0 \oplus \mathrm{HN}_{\min}^\vee(f_*\omega_{S/C}), 0)$$

is a direct summand of $(\mathrm{gr}(H), \theta)$. So $\mathrm{HN}_{\min}^\vee(f_*\omega_{S/C})$ is a direct summand of $H^{0,1}$ and $\mathrm{HN}_{\min}(f_*\omega_{S/C})$ is a direct summand of $f_*\omega_{S/C}$. Furthermore (see [42, page 1]),

$$(\mathrm{HN}_{\min}(f_*\omega_{S/C}) \oplus \mathrm{HN}_{\min}^\vee(f_*\omega_{S/C}), 0)$$

is a polystable Higgs bundle such that each direct summand has degree 0, so it comes from a sublocal system of $\mathbb{V}$ of rank $2(n-k)$.

By Theorem 3.1, we have

$$0 \le \lambda_{k+1} + \cdots + \lambda_g \le \frac{2\deg(\mathrm{HN}_{\min}(f_*\omega_{S/C}))}{2g(C)-2+|\Delta|} = 0,$$

and hence $\lambda_{k+1} = \cdots = \lambda_g = 0$. $\square$

The reader can compare this proposition with [21, Theorem 3]. It can be used to get some information on the zero eigenvalues of $\mathrm{EV}(H_\omega)$ and $\mathrm{EV}(B_\omega)$.

5.3 Quadratic differentials

For a Teichmüller curve C generated by (Y,q) in $\mathcal{Q}(d_1,\dots,d_s)$, let (X,ω) be the canonical double covering. The curve X comes with an involution τ. Its cohomology splits into the τ–invariant and τ–anti-invariant part. Adapting the notation of [14] we let $g = g(Y)$ and $g_{\mathrm{eff}} = g(X) - g$. Let λ_i^+ be the Lyapunov exponents of the τ–invariant part of $H^1(X,\mathbb{R})$ and λ_i^- be the Lyapunov exponents of the τ–anti-invariant part. The τ–invariant part descends to Y and hence the λ_i^+ are the Lyapunov exponents of (Y,q) that we are primarily interested in. Define

$$L^+ = \lambda_1^+ + \cdots + \lambda_g^+,$$
$$L^- = \lambda_1^- + \cdots + \lambda_{\mathrm{eff}}^-.$$

The role of λ_i^+ is analogous to the ordinary sum of Lyapunov exponents in the case of abelian differentials. We will reprove the following formula, found in [29, page 12] and [14, page 12], by using double-cover techniques [6, page 236]. It is largely based on the work of Chen and Möller [10].

Proposition 5.10 *For a Teichmüller curve C in $\mathcal{Q}(d_1,\dots,d_s)$, we have*

$$L^- - L^+ = \frac{1}{4} \sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} \frac{1}{d_j+2}.$$

Proof Note that

$$\phi\colon \mathcal{Q}(\dots,d_i,\dots,d_j,\dots) \to \mathcal{H}_g\big(\dots,\tfrac12 d_i,\tfrac12 d_i,\dots,d_j+1,\dots\big)$$

for d_i even and for d_j odd. Since the double cover is branched at the singularities of odd order. Restrict this to a Teichmüller curve C in $\mathcal{Q}(d_1,\dots,d_n)$. Then it gives rise to a Teichmüller curve isomorphic to C in the corresponding stratum of abelian differentials. After suitable base change and compactification, we can get two universal families f', f and we have the following commutative diagram:

$$\begin{tikzcd} S' \arrow[rr, "\sigma"] \arrow[dr, "f'"'] & & S \arrow[dl, "f"] \\ & C & \end{tikzcd}$$

Let D'_j be the section of $f'\colon S' \to C$ over D_j if d_j is odd, and $D_{j,1}, D_{j,2}$ be the sections over D_j if d_j is even.

In the case when d_j is odd, we have [10, page 14]

$$\sigma_* D'_j = D_j, \quad \sigma^* D_j = 2D'_j,$$

and the self-intersection number is

$$D_j^2 = (\sigma_* D'_j) D_j = 2(D'^2_j) = -\frac{\chi}{d_j + 2}.$$

In the case when d_j is even, we have

$$\sigma_*(D_{j,1} + D_{j,2}) = 2D_j, \quad \sigma^* D_j = D_{j,1} + D_{j,2},$$

and the self-intersection number is

$$D_j^2 = \tfrac{1}{2}(\sigma_*(D_{j,1} + D_{j,2})) D_j = \tfrac{1}{2}(D_{j,1}^2 + D_{j,2}^2) = -\frac{\chi}{d_j + 2}.$$

The relative canonical bundle formula for the fibration $f'\colon S' \to C$ is

$$\omega_{S'/C} = f'^* \mathcal{L} \otimes \mathcal{O}_{S'}\Bigg(\sum_{\substack{j \text{ such that} \\ d_j \text{ is even}}} \frac{d_j}{2}(D_{j,1} + D_{j,2}) \; + \sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} (d_j + 1) D'_j \Bigg),$$

where $\mathcal{L}$ is the line bundle on C corresponding to the generating abelian differential and $\deg \mathcal{L}$ equals $\frac{1}{2}\chi$.

The relative canonical bundle formula for the fibration $f\colon S \to C$ is

$$\omega^2_{S/C} = f^* \mathcal{F} \otimes \mathcal{O}_S\Big(\sum d_j D_j\Big),$$

where $\mathcal{F}$ is the line bundle on C corresponding to the generating quadratic differential and obviously $\mathcal{F}$ equals $\mathcal{L}^2$; see [10, page 42].

There is a smooth divisor

$$B = \sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} D_j$$

on S such that $B = 2D$ for some effective divisor D. The double covering $\sigma\colon S' \to S$ is ramified exactly over B. We have

$$\omega_{S'} = \sigma^*(\omega_S \otimes \mathcal{O}_S(D)),$$
$$\sigma_*(\mathcal{O}_{S'}) = \mathcal{O}_S \oplus \mathcal{O}_S(-D).$$

The direct image of the relative canonical bundle $f'_*\omega_{S'/C}$ decomposes into a direct sum

$$\begin{aligned} f'_*\omega_{S'/C} &= f'_*\omega_{S'} \otimes \omega_C^{-1} \\ &= f_*\sigma_*\omega_{S'} \otimes \omega_C^{-1} \\ &= f_*(\sigma_*(\mathcal{O}_{S'}) \otimes \omega_S \otimes \mathcal{O}_S(D)) \otimes \omega_C^{-1} \\ &= f_*(\omega_S \otimes \mathcal{O}_S(D) \oplus \omega_S) \otimes \omega_C^{-1} \\ &= f_*(\omega_{S/C}) \oplus f_*(\omega_{S/C} + D). \end{aligned}$$

We assume $D \neq 0$ (when $D = 0$, the result is trivial). Then all the higher direct images of $\omega_{S/C} + D$ are zero. By Grothendieck–Riemann–Roch we have

$$\begin{aligned} \mathrm{ch}(f_*(\omega_{S/C} + D)) &= f_*\left(\mathrm{ch}(\omega_{S/C}) \cdot \mathrm{ch}(D) \cdot \left(1 - \frac{\gamma}{2} + \frac{\gamma^2 + \eta}{12}\right)\right) \\ &= f_*\left(\left(1 + \gamma + \frac{\gamma^2}{2}\right) \cdot \left(1 + D + \frac{D^2}{2}\right) \cdot \left(1 - \frac{\gamma}{2} + \frac{\gamma^2 + \eta}{12}\right)\right) \\ &= f_*\left(1 + \left(\frac{\gamma}{2} + D\right) + \frac{D(D+\gamma)}{2} + \frac{\gamma^2 + \eta}{12}\right) \\ &= \mathrm{rk} + \left(f_*\frac{D(D+\gamma)}{2} + \lambda\right), \end{aligned}$$

where

- $\gamma = c_1(\omega_{S/C})$,
- $\lambda = c_1(f_*\omega_{S/C})$,
- η is the nodal locus in $f\colon S \to C$,

and they satisfy $\lambda = \frac{1}{12}(\gamma^2 + \eta)$ by Riemann–Roch. Now we have

$$c_1(f_*(\omega_{S/C} + D)) - c_1(f_*(\omega_{S/C})) = c_1(f_*(\omega_{S/C} + D)) - \lambda = f_*\tfrac{1}{2}D(D+\gamma),$$

because

$$\begin{aligned}\deg(f_*(\omega_{S/C}+D))-\deg(f_*(\omega_{S/C})) &= \frac{1}{2}D(\omega_{S/C}+D)\\ &= \frac{1}{8}\Bigg(\sum_{\substack{j \text{ such that}\\ d_j \text{ is odd}}} D_j\Bigg)\Bigg(\sum_{\substack{j \text{ such that}\\ d_j \text{ is odd}}} D_j+\sum d_j D_j+\mathcal{F}\Bigg)\\ &= \frac{1}{8}\Bigg(\sum_{\substack{j \text{ such that}\\ d_j \text{ is odd}}} (d_j+1)D_j^2+\chi\Bigg)\\ &= \frac{\chi}{2}\frac{1}{4}\sum_{\substack{j \text{ such that}\\ d_j \text{ is odd}}} \frac{1}{d_j+2},\end{aligned}$$

and $f_*(\omega_{S/C})$ is the σ–invariant part and $f_*(\omega_{S/C}+D)$ the σ–anti-invariant part. Hence we have

$$L^- = \frac{\deg(f_*(\omega_{S/C}+D))}{\frac{1}{2}\chi}, \quad L^+ = \frac{\deg(f_*(\omega_{S/C}))}{\frac{1}{2}\chi}.$$

We get the formula

$$L^- - L^+ = \frac{1}{4}\sum_{\substack{j \text{ such that}\\ d_j \text{ is odd}}} \frac{1}{d_j+2}. \qquad \square$$

Remark 5.11 A version of Theorem 1.1 allows us to generalize the proposition from Teichmüller curves to any $\mathrm{GL}^+(2,\mathbb{R})$–invariant submanifold in the moduli space of meromorphic quadratic differentials with at most simple poles defined over $\mathbb{Q}$; in particular, to every connected component of each stratum.

For the Teichmüller curve C in $\overline{\mathcal{H}_g}\big(\dots,\frac{1}{2}d_i,\frac{1}{2}d_i,\dots,d_j+1,\dots\big)$, we have defined a $(g+g_{\mathrm{eff}})$–vector

$$w(C) = (w_1,\dots,w_{g+g_{\mathrm{eff}}}),$$

and moreover we know the upper bound of each w_i. Because

$$\mathrm{grad}(\mathrm{HN}(f'_*\omega_{S'/C})) = \mathrm{grad}(\mathrm{HN}(f_*(\omega_{S/C}))) \oplus \mathrm{grad}(\mathrm{HN}(f_*(\omega_{S/C}+D))),$$

we can divide w_i into two parts,

$$w_1^+,\ \dots,\ w_g^+;\quad w_1^-,\ \dots,\ w_{\mathrm{eff}}^-.$$

Here $w_1^+,\dots,w_g^+$ and $w_1^-,\dots,w_{\mathrm{eff}}^-$ come, respectively, from the graded quotients $\mathrm{grad}(\mathrm{HN}(f_*(\omega_{S/C})))$ and $\mathrm{grad}(\mathrm{HN}(f_*(\omega_{S/C}+D)))$.

It is obvious, by definition, that

$$L^+ = w_1^+ + \cdots + w_g^+, \quad L^- = w_1^- + \cdots + w_{\mathrm{eff}}^-.$$

Example 5.12 Consider the map

$$\phi\colon \mathcal{Q}(1,2,-1,-1,-1) \to \mathcal{H}_3(2,1,1).$$

For a Teichmüller curve in $\mathcal{H}_3(2,1,1)$, we know that

$$w_1 = 1, \quad w_2 = \tfrac{1}{2}, \quad w_3 = \tfrac{1}{3}.$$

For a Teichmüller curve in $\mathcal{Q}(1,2,-1,-1,-1)$, we have

$$L^- - L^+ = \tfrac{5}{6} = \tfrac{1}{4}\left(\tfrac{1}{3}+1+1+1\right),$$
$$L^- + L^+ = w_1 + w_2 + w_3 = \tfrac{11}{6}.$$

We get $L^- = \frac{4}{3}$ and $L^+ = \frac{1}{2}$, which gives us

$$w_1^+ = \tfrac{1}{2}, \quad w_1^- = 1, \quad w_2^- = \tfrac{1}{3}.$$

Of course we can ask the same questions for $w_i^-(w_i^+)$ and $\lambda_i^-(\lambda_i^+)$.

Appendix

We present here approximate numerical values on the individual Lyapunov exponents λ_i for all strata in genera 3 and 4 and of some strata in genus 5. These values were computed experimentally in [29] and in [14]. The rational number representing the sum $\sum_{j=1}^{g} \lambda_j$ of the positive Lyapunov exponents in the right column of each table is exact; it is computed rigorously in [14]. (Note that the sum contains as a summand the top Lyapunov exponent $\lambda_1 = 1$, which is not present in the table.)

We present also numerical data for the normalized slopes w_i of the Harder–Narasimhan filtration of the Hodge bundle over Teichmüller curves in the corresponding strata, computed rigorously in [47] and in [46] and reproduced in Theorems 3.3 and 3.4 of the current paper.

When the numbers w_i might vary from one Teichmüller curve to another in the given stratum, we provide bounds (7) in the form of inequalities. When the inequality sign is missing, it means that the Harder–Narasimhan type $w(C)$ of any Teichmüller curve is constant for the stratum.

These data provide numerical evidence supporting the Main Conjecture. We hope that it will be also useful for applications (like evaluation of diffusion rates of some periodic polygonal billiards in the plane).

degrees of zeros	connected component	Lyapunov exponents λ_2	λ_3	λ_4	normalized Harder–Narasimhan slopes w_2	w_3	w_4	sum $\sum_{j=1}^{g}\lambda_j=\sum_{j=1}^{g}w_j$
(6)	hyp	0.7375	0.4284	0.1198	$\frac{5}{7}$	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{16}{7}$
(6)	even	0.5965	0.2924	0.1107	$\frac{4}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{14}{7}$
(6)	odd	0.4733	0.2755	0.1084	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{13}{7}$
(5, 1)		0.5459	0.3246	0.1297	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	2
(3, 3)	hyp	0.7726	0.5182	0.2097	$\frac{3}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{5}{2}$
(3, 3)	nonhyp	0.5380	0.3124	0.1500	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	2
(4, 2)	even	0.6310	0.3496	0.1527	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{32}{15}$
(4, 2)	odd	0.4789	0.3134	0.1412	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{29}{15}$
(2,2,2)	odd	0.4826	0.3423	0.1749	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	2
(3,2,1)		0.5558	0.3557	0.1718	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{25}{12}$
(2,2,2)	even	0.6420	0.3785	0.1928	$\leq\frac{2}{3}$	$\leq\frac{1}{3}$	$\leq\frac{1}{3}$	$\frac{737}{333}$
(1,1,1,3)		0.5600	0.3843	0.1849	$\leq\frac{1}{2}$	$\leq\frac{1}{2}$	$\leq\frac{1}{4}$	$\frac{66}{31}$
(1,1,2,2)		0.5604	0.3809	0.1982	$\leq\frac{2}{3}$	$\leq\frac{1}{2}$	$\leq\frac{1}{3}$	$\frac{5045}{2358}$
(1,1,1,1,2)		0.5632	0.4032	0.2168	$\leq\frac{1}{2}$	$\leq\frac{1}{2}$	$\leq\frac{1}{3}$	$\frac{131}{60}$
(1,1,1,1,1,1)		0.5652	0.4198	0.2403	$\leq\frac{1}{2}$	$\leq\frac{1}{2}$	$\leq\frac{1}{2}$	$\frac{839}{377}$

Table 2: All strata in genus 4

Example A.1 We illustrate the numerical evidence for the Main Conjecture taking the connected component with odd parity of the spin structure of the stratum $\mathcal{H}_5^{\mathrm{odd}}(6,2)$ as an example:

$$\begin{aligned}
\lambda_1 &= 1 = 1 = w_1,\\
\lambda_1+\lambda_2 &\approx 1.52 \geq \tfrac{10}{7} = w_1+w_2,\\
\lambda_1+\lambda_2+\lambda_3 &\approx 1.89 \geq \tfrac{37}{21} = w_1+w_2+w_3,\\
\lambda_1+\lambda_2+\lambda_3+\lambda_4 &\approx 2.1 \geq \tfrac{43}{21} = w_1+w_2+w_3+w_4,\\
\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5 &= \tfrac{46}{21} = \tfrac{46}{21} = w_1+w_2+w_3+w_4+w_5.
\end{aligned}$$

Or, equivalently,

$$\begin{aligned}
\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5 = \ & \tfrac{46}{21} \ = \tfrac{46}{21} = w_1+w_2+w_3+w_4+w_5,\\
\lambda_2+\lambda_3+\lambda_4+\lambda_5 \approx \ & \tfrac{25}{21} \ = \tfrac{25}{21} = w_2+w_3+w_4+w_5,\\
\lambda_3+\lambda_4+\lambda_5 \approx \ & 0.67 \le \tfrac{16}{21} = w_3+w_4+w_5,\\
\lambda_4+\lambda_5 \approx \ & 0.30 \le \ \tfrac{3}{7} \ = w_4+w_5,\\
\lambda_5 \approx \ & 0.09 \le \ \tfrac{1}{7} \ = w_5.
\end{aligned}$$

degrees of zeros	connected component	Lyapunov exponents λ_2	λ_3	λ_4	λ_5	normalized Harder–Narasimhan slopes w_2	w_3	w_4	w_5	sum $\sum_{j=1}^{g}\lambda_j=\sum_{j=1}^{g}w_j$
(8)	hyp	0.799	0.586	0.306	0.087	$\frac{7}{9}$	$\frac{5}{9}$	$\frac{3}{9}$	$\frac{1}{9}$	$\frac{25}{9}$
(8)	even	0.597	0.363	0.190	0.073	$\frac{5}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{20}{9}$
(8)	odd	0.515	0.343	0.181	0.071	$\frac{4}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{19}{9}$
(6, 2)	odd	0.521	0.369	0.212	0.089	$\frac{3}{7}$	$\frac{1}{3}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{46}{21}$
(5, 3)		0.562	0.376	0.216	0.096	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{9}{4}$
(4, 4)	hyp	0.819	0.639	0.390	0.152	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	3
(7, 1)		0.560	0.378	0.207	0.082	$\le\frac{3}{4}$	$\le\frac{1}{2}$	$\le\frac{3}{8}$	$\le\frac{1}{8}$	$\frac{2423}{1088}$
(6, 2)	even	0.604	0.386	0.221	0.092	$\le\frac{5}{7}$	$\le\frac{4}{7}$	$\le\frac{1}{7}$	$\le\frac{1}{7}$	$\frac{178429}{77511}$
(6, 1, 1)		0.563	0.397	0.230	0.094	$\le\frac{5}{7}$	$\le\frac{1}{2}$	$\le\frac{3}{7}$	$\le\frac{1}{7}$	$\frac{59332837}{25961866}$
(5, 2, 1)		0.564	0.396	0.237	0.103	$\le\frac{2}{3}$	$\le\frac{1}{2}$	$\le\frac{1}{3}$	$\le\frac{1}{6}$	$\frac{4493}{1953}$
(5, 1, 1, 1)		0.565	0.415	0.253	0.108	$\le\frac{2}{3}$	$\le\frac{1}{2}$	$\le\frac{1}{2}$	$\le\frac{1}{6}$	$\frac{103}{44}$

Table 3: Some strata in genus 5

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