

# Hyperbolic jigsaws and families of pseudomodular groups, I

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We show that there are infinitely many commensurability classes of pseudomodular groups, thus answering a question raised by Long and Reid. These are Fuchsian groups whose cusp set is all of the rationals but which are not commensurable to the modular group. We do this by introducing a general construction for the fundamental domains of Fuchsian groups obtained by gluing together marked ideal triangular tiles, which we call hyperbolic jigsaw groups.

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## 1 Introduction

We first recall the definition of a pseudomodular group, which is the main motivation for this paper. Recall that the cusp set of a Fuchsian group  $\Gamma \leq \mathrm{PSL}(2, \mathbb{R})$  is the set of all  $x \in \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}^2$  such that the stabilizer of  $x$  in  $\Gamma$  is generated by a parabolic element.

**Definition 1.1** Let  $\Gamma \leq \mathrm{PGL}(2, \mathbb{Q})$  be a discrete group such that  $\mathbb{H}^2/\Gamma$  is a complete hyperbolic surface of finite area. If  $\Gamma$  is not commensurable with  $\mathrm{PSL}(2, \mathbb{Z})$ , and  $\Gamma$  has cusp set precisely  $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ , then  $\mathbb{H}^2/\Gamma$  is called a pseudomodular surface and  $\Gamma$  a pseudomodular group.

These were introduced by Long and Reid in [4]; the main surprise and remarkable fact is that such groups exist. Clearly, the property of being pseudomodular carries over to the commensurability class, so we can consider commensurability classes of pseudomodular groups. Four different commensurability classes of pseudomodular groups were found in [4], although a fifth,  $\Delta(\frac{5}{13}, 4)$ , was also known to the authors (private communication), and found independently by Ayaka [1] and Junwei Tan [9], using ideas of [4]. Long and Reid asked if there were (in)infinitely many commensurability classes of pseudomodular groups (Question 2 of Section 6 in [4]).

So far, not much progress has been made towards this question. Some partial answers were provided by Proskin [7], and a couple of new pseudomodular groups have been found by Ayaka [1]. On the other hand, using methods from number theory, Fithian [2] was able to give conditions for a group to be nonpseudomodular. This however, does not answer the question if there are infinitely many commensurability classes of pseudomodular groups.

Our main result is the following answer to Long and Reid's question:

**Theorem 1.2** *There exist infinitely many commensurability classes of pseudomodular groups and surfaces.*

Theorem 1.2 follows from either Theorem 2.4 or Theorem 2.5. Our idea is a general way to construct subgroups of  $\mathrm{PGL}(2, \mathbb{Q})$  by constructing their fundamental domains by gluing together tiles from a finite collection  $\mathcal{S}$  of marked hyperbolic ideal triangle tiles which satisfy certain balancing and rationality conditions, and with matching conditions for the gluing. We call the polygons obtained from this construction  $\mathcal{S}$ -jigsaws, and the corresponding groups and surfaces  $\mathcal{S}$ -jigsaw groups and surfaces. In Theorem 2.4, we show that for the collection  $\mathcal{S} = \mathcal{S}(1, 2) = \{\Delta(1, 1, 1), \Delta(1, \frac{1}{2}, 2)\}$  — see Section 2 for the definition of  $\Delta(1, 1, 1)$  and  $\Delta(1, \frac{1}{2}, 2)$  — all hyperbolic  $\mathcal{S}$ -jigsaw groups and surfaces are pseudomodular. Furthermore, there are infinitely many commensurability classes of such groups. On the other hand, in Theorem 2.5, we show that for  $\mathcal{S} = \mathcal{S}(1, 3) = \{\Delta(1, 1, 1), \Delta(1, \frac{1}{3}, 3)\}$ , arithmetic, pseudomodular and nonmaximally cusped groups all can occur. There are also infinitely many (commensurability classes of)  $\mathcal{S}$ -jigsaw groups which are pseudomodular, and infinitely many (commensurability classes of)  $\mathcal{S}$ -jigsaw groups which are not pseudomodular. We give exact criteria for determining when such a group is arithmetic, pseudomodular or neither.

In some sense, the groups constructed in this way, in particular the groups arising from the jigsaws with two tiles,  $J = \Delta(1, 1, 1) \cup \Delta(1, \frac{1}{2}, 2)$  or  $J' = \Delta(1, 1, 1) \cup \Delta(1, \frac{1}{3}, 3)$ , are closest to being modular. We will explore properties of these groups, as well as general integral jigsaw groups in a future paper [5]. For example, we will describe an associated pseudoeuclidean algorithm and generalized continued fraction for some of these groups. We will also show that for any integral hyperbolic jigsaw set  $\mathcal{S}$ , there are infinitely many commensurability classes of pseudomodular  $\mathcal{S}$ -jigsaw groups and explore the question of arithmeticity for general integral jigsaw groups.

The rest of this paper is organized as follows. In Section 2, we give the definitions and some basic properties of  $\mathcal{S}$ -jigsaws, their associated surfaces and groups, and state the

main results. In Section 3, we describe Long and Reid's idea of covering a fundamental interval for the group by killer intervals to show that the cusp set is  $\mathbb{Q} \cup \{\infty\}$ . In Section 4, we describe integral hyperbolic jigsaw sets and groups and give some of their basic properties. In Sections 5 and 6, we specialize these results to  $\mathcal{S}(1, 2)$ - and  $\mathcal{S}(1, 3)$ -jigsaws and study the cusp sets of these jigsaw groups. In Section 7, we explore criteria for nonarithmeticity and show that all  $\mathcal{S}(1, 2)$ -jigsaw groups are nonarithmetic, hence pseudomodular. We also give exact criteria for an  $\mathcal{S}(1, 3)$ -jigsaw group to be arithmetic. In Section 8, we look at criteria for two nonarithmetic groups to be noncommensurable and use this to construct infinitely many commensurability classes of  $\mathcal{S}(1, 2)$  and  $\mathcal{S}(1, 3)$  pseudomodular groups, and infinitely many commensurability classes of nonpseudomodular  $\mathcal{S}(1, 3)$ -jigsaw groups. Finally, in Section 9 we put together the results of the previous sections to prove Theorems 2.4 and 2.5, and hence the main results of this paper, and conclude with some open questions.

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## 2 Hyperbolic jigsaws and main results

We will always use the upper half space model for the hyperbolic plane  $\mathbb{H}^2$ , with group of orientation-preserving isometries  $\mathrm{PSL}(2, \mathbb{R})$ . We also always denote translation to the right by 1 by

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let  $\Delta$  be a positively oriented ideal triangle with vertices  $v_1$ ,  $v_2$  and  $v_3$  and sides  $s_1 = [v_1, v_2]$ ,  $s_2 = [v_2, v_3]$  and  $s_3 = [v_3, v_1]$  and let  $x_i \in s_i$  for  $i = 1, 2, 3$  be marked points on the sides. Then  $\Delta$ , together with the points  $x_1$ ,  $x_2$  and  $x_3$ , is called a marked triangle; two such triangles are the same if they are isometric, as marked triangles. Let

$p_i$  be the point on  $s_i$  which is the projection of the opposite vertex of  $\Delta$  to  $s_i$ . Let  $d_i$  be the signed distance from  $p_i$  to  $x_i$  along  $s_i$ , and let  $k_i = e^{2d_i}$ . The marked triangle, up to isometry, is then determined by the triple  $(k_1, k_2, k_3) \in \mathbb{R}_+^3$ , and is denoted by  $\Delta(k_1, k_2, k_3)$ . Compare Figure 1. We use  $\Delta_{[v_1, v_2, v_3]}$  to denote the (unmarked) ideal triangle in  $\mathbb{H}^2$  with vertices  $v_1, v_2$  and  $v_3$ .

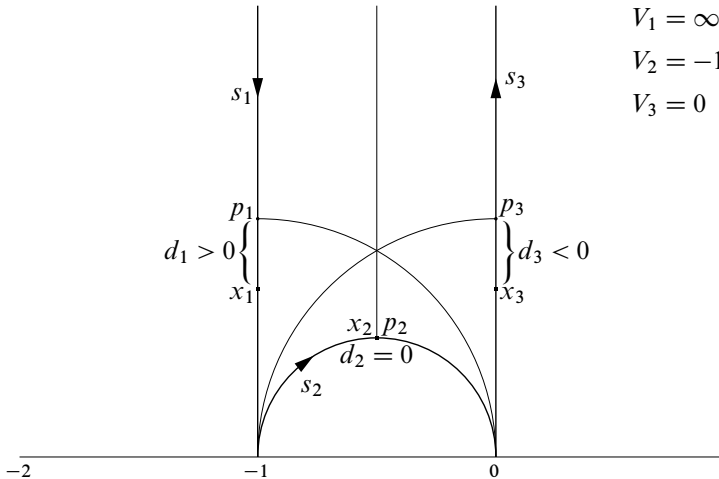


Figure 1: A marked triangle where  $v_1 = \infty, v_2 = -1, v_3 = 0, d_1 > 0, d_2 = 0$  and  $d_3 < 0$  ( $k_1 > 1, k_2 = 1$  and  $k_3 < 1$ )

Up to cyclic permutation,  $(k_1, k_2, k_3)$  determines the same marked triangle, that is,

$$\Delta(k_1, k_2, k_3) = \Delta(k_2, k_3, k_1) = \Delta(k_3, k_1, k_2).$$

Note however that, in general,  $\Delta(k_1, k_2, k_3)$  is not isometric to  $\Delta(k_2, k_1, k_3)$ .

We say that  $\Delta(k_1, k_2, k_3)$  is in standard position if  $v_1 = \infty, v_2 = -1$  and  $v_3 = 0$ , in which case we have

$$(1) \quad x_1 = -1 + \frac{i}{\sqrt{k_1}}, \quad x_2 = \frac{-1 + \sqrt{k_2}i}{1 + k_2}, \quad x_3 = \sqrt{k_3}i.$$

Fix an embedding of  $\Delta(k_1, k_2, k_3)$  into  $\mathbb{H}^2$  and let  $\iota_j$  for  $j = 1, 2, 3$  be the  $\pi$ -rotation about  $x_j$ . Then  $\iota_1, \iota_2$  and  $\iota_3$  generate a discrete subgroup, denoted by  $\Gamma(k_1, k_2, k_3) \leq \text{PSL}(2, \mathbb{R})$ . It is easy to see that  $\Gamma$  is a lattice; equivalently, the corresponding (orbifold) surface  $\mathbb{H}^2 / \Gamma(k_1, k_2, k_3)$  is finite-volume ( $\iota_1 \iota_2 \iota_3$  is parabolic) if and only if  $k_1 k_2 k_3 = 1$ . Furthermore, if  $k_j \in \mathbb{Q}$  for  $j = 1, 2, 3$ , and the triangle is embedded so the vertices  $v_j$  are in  $\mathbb{Q} \cup \{\infty\} \subset \partial \mathbb{H}^2$ , then  $\Gamma(k_1, k_2, k_3) < \text{PGL}(2, \mathbb{Q})$ . We have:

**Definition 2.1** (rational and balanced triangles and Weierstrass groups) The marked triangle  $\Delta(k_1, k_2, k_3)$  is balanced if  $k_1 k_2 k_3 = 1$  and rational if  $k_1, k_2, k_3 \in \mathbb{Q}$ . The group  $\Gamma(k_1, k_2, k_3)$ , defined up to conjugation by  $\text{PSL}(2, \mathbb{R})$ , is called the (Weierstrass) group generated by the involutions  $\iota_1, \iota_2$  and  $\iota_3$  about the marked points  $x_1, x_2$  and  $x_3$ , respectively. It is the index-two supergroup of the punctured torus group, where  $x_1, x_2$  and  $x_3$  correspond to the Weierstrass points on the torus.

For example, if  $\Delta(k_1, k_2, k_3)$  is in standard position, then  $\iota_1, \iota_2$  and  $\iota_3$  can be represented by the matrices (as elements of  $\text{PSL}(2, \mathbb{R})$ )

$$(2) \quad \frac{1}{\sqrt{k_1}} \begin{pmatrix} k_1 & 1+k_1 \\ -k_1 & -k_1 \end{pmatrix}, \quad \frac{1}{\sqrt{k_2}} \begin{pmatrix} 1 & 1 \\ -(k_2+1) & -1 \end{pmatrix}, \quad \frac{1}{\sqrt{k_3}} \begin{pmatrix} 0 & k_3 \\ -1 & 0 \end{pmatrix},$$

respectively. We remark that in the notation of Long and Reid in [4], the group  $\Gamma(k_1, k_2, k_3)$  corresponds to the index-two supergroup of the torus group  $\Delta(u^2, 2\tau)$ , where

$$(3) \quad u^2 = k_3, \quad \tau = \frac{1}{k_1} + 1 + k_3.$$

In particular, the group  $\Gamma(1, \frac{1}{n}, n)$  in standard position corresponds to the group  $\Delta(n, 2n+4)$  in their notation.

**Definition 2.2** (matching sides) Let  $\Delta = \Delta(k_1, k_2, k_3)$  and  $\Delta' = \Delta(k'_1, k'_2, k'_3)$  be two marked triangles, with sides  $s_1, s_2, s_3$  and  $s'_1, s'_2, s'_3$ , and marked points  $x_1, x_2, x_3$  and  $x'_1, x'_2, x'_3$ , respectively. We say that  $s_i$  and  $s'_j$  match if  $k_i = k'_j$ .

Geometrically, this means that if we glue  $\Delta$  to  $\Delta'$  by gluing  $s_i$  to  $s'_j$ , matching  $x_i$  to  $x'_j$ , then the  $\pi$ -rotation  $\iota_i$  about  $x_i = x'_j$  interchanges the unmarked triangles  $\Delta$  and  $\Delta'$ ; see Figure 2.

In terms of shear coordinates, this means that for the ideal quadrilateral formed by gluing  $\Delta$  to  $\Delta'$ , the shear coordinate along the diagonal formed from the identified side is  $\log k_i$ .

In this way, we may glue marked triangles along matching sides, obtaining a triangulated ideal polygon, unique up to isometry, with marked points on the interior and exterior sides.

We are now ready to define the objects needed to assemble a hyperbolic jigsaw and to state our main theorem.

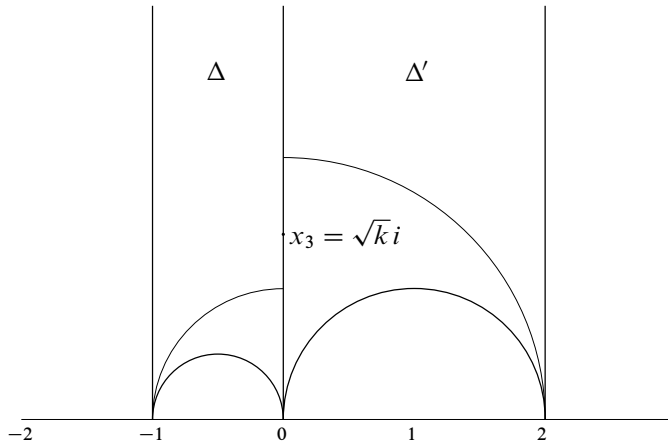


Figure 2: Gluing two marked triangles  $\Delta$  and  $\Delta'$  along the matching side with parameter  $k = 2$

**Definition 2.3** (1) **Jigsaw tiles** A jigsaw tile is a *rational, balanced, marked ideal triangle*  $\Delta(k_1, k_2, k_3)$ . That is,  $k_1, k_2, k_3 \in \mathbb{Q}$  and  $k_1 k_2 k_3 = 1$ .

(2) **Jigsaw sets** A jigsaw set is a finite (ordered) collection  $\mathcal{S} := \{\Delta_1, \dots, \Delta_s\}$  of distinct jigsaw tiles such that every tile  $\Delta_i \in \mathcal{S}$  has a side which matches with the side of some other tile  $\Delta_j \in \mathcal{S}$ .

(3) **Jigsaws** For a jigsaw set  $\mathcal{S}$ , an  $\mathcal{S}$ -jigsaw  $J$  is an ideal polygon, obtained by gluing a finite number of tiles from  $\mathcal{S}$  along matching sides, such that every  $\Delta_j \in \mathcal{S}$  occurs at least once in  $J$ . As a convention, we will usually normalize one of the  $\Delta_1$  tiles of  $J$  to be in standard position with vertices at  $\infty, -1$  and  $0$ , so that all other vertices of  $J$  are rational.

(4) **Size and signature of a jigsaw** The size of a jigsaw  $J$ , denoted by  $|J|$  is the number of tiles in it. The signature  $\text{sgn}(J)$  of  $J$  is  $(m_1, m_2, \dots, m_s)$ , where  $\mathcal{S} = \{\Delta_1, \dots, \Delta_s\}$  and  $m_j \in \mathbb{N}$  denotes the number of  $\Delta_j$  tiles in  $J$ . We have  $|J| = \sum_1^s m_j$ .

(5) **Jigsaw group and surface** The Fuchsian group associated to  $J$ , denoted by  $\Gamma_J$ , is the subgroup of  $\text{PGL}(2, \mathbb{Q})$  generated by the  $\pi$ -rotations about the marked points on the (exterior) sides of  $J$ . The hyperbolic (orbifold) surface associated to  $J$  is  $S_J := \mathbb{H}^2 / \Gamma_J$ . (By our convention of fixing some  $\Delta_1$  in standard position,  $\Gamma_J$  is a zonal Fuchsian group, that is, contains a parabolic which fixes  $\infty$ ).

(6) **Jigsaw tessellation** The tessellation  $\mathcal{Q}_J$  of  $\mathbb{H}^2$  is the tessellation of  $\mathbb{H}^2$  obtained by the translates by  $\Gamma_J$  of  $J$ .

- (7) **Jigsaw triangulation** The triangulation  $\mathcal{T}_J$  of  $\mathbb{H}^2$  is the triangulation of  $\mathbb{H}^2$  obtained from the translates by  $\Gamma_J$  of the triangulation of  $J$  by its tiles. There is a labeling of the triangles of  $\mathcal{T}_J$  by elements of the set  $\mathcal{S} := \{\Delta_1, \dots, \Delta_s\}$ .
- (8) **Exterior and interior sides** The set of sides of  $\mathcal{Q}_J$  is denoted by  $\mathcal{E}_J$  and the set of sides of  $\mathcal{T}_J$  is denoted by  $\mathcal{F}_J$ . Clearly,  $\mathcal{E}_J \subset \mathcal{F}_J$ . If  $e \in \mathcal{E}_J$ , it is called an *exterior side*; if  $e \in \mathcal{F}_J \setminus \mathcal{E}_J$ , it is called an *interior side*. There is a labeling of the sides of  $\mathcal{T}_J$  by the parameters  $k_i$ .
- (9) **Cutting sequences of closed geodesics** The  $V$ -cutting sequence of an oriented closed geodesic  $\gamma$  on the surface  $S_J$  is the (periodic) sequence  $\overline{\Delta_{i_1}, \Delta_{i_2}, \dots, \Delta_{i_t}}$ , or simply  $\overline{i_1, i_2, \dots, i_t}$ , induced from the labeling of the triangles. The  $W$ -cutting sequence of  $\gamma$  is  $\overline{k_1, k_2, \dots, k_t}$ , where  $k_i$  lies in the set of parameters of the sides of  $\Delta_i \in \mathcal{S}$ , induced from the labeling of the sides.

The following facts about hyperbolic jigsaws are clear:

- (1) An  $\mathcal{S}$ -jigsaw  $J$  is a fundamental polygon for  $\Gamma_J$ . The balancing condition ensures that  $S_J$  is a *complete* hyperbolic orbifold with one cusp and  $N + 2$  cone points of order 2, where  $N = |J|$ . Hence,  $\mathcal{Q}_J$  tessellates all of  $\mathbb{H}^2$ , and  $\mathcal{T}_J$  triangulates all of  $\mathbb{H}^2$ .
- (2) If  $\mathcal{S} = \{\Delta(k_1, k_2, k_3)\}$  and  $k_1, k_2$  and  $k_3$  are all distinct, then an  $\mathcal{S}$ -jigsaw group is a finite-index subgroup of the *Weierstrass group*  $\Gamma(k_1, k_2, k_3)$  and hence all such jigsaw groups belong to the commensurability class of  $\Gamma(k_1, k_2, k_3)$ . The question of which such groups are pseudomodular for the parameters  $k_1, k_2, k_3 \in \mathbb{Q}$  with  $k_1 k_2 k_3 = 1$  is essentially Question 1 of Section 6 of [4].
- (3) If  $\mathcal{S} = \{\Delta(k_1, k_2, k_3)\}$  and exactly two of  $k_1, k_2$  and  $k_3$  are the same, then the jigsaw group may not be a finite-index subgroup of  $\Gamma(k_1, k_2, k_3)$  as there is some choice involved in how the triangles are glued.
- (4) The  $\pi$ -rotation about the marked point of an exterior side  $e \in \mathcal{E}_J$  is an element of  $\Gamma_J$ . The  $\pi$ -rotation about the marked point of an interior side  $e \in \mathcal{F}_J \setminus \mathcal{E}_J$  is not in  $\Gamma_J$ .
- (5) The set of vertices of  $\mathcal{Q}_J$  coincide with the set of vertices of  $\mathcal{T}_J$ . This set is precisely the cusp set of  $\Gamma_J$  since  $S_J$  has one cusp.

We have the following results, either of which immediately implies Theorem 1.2:

**Theorem 2.4** *Let  $\mathcal{S} = \{\Delta(1, 1, 1), \Delta(1, \frac{1}{2}, 2)\}$ . Then  $\Gamma_J$  is a pseudomodular group for every  $\mathcal{S}$ -jigsaw  $J$ . Furthermore, these groups are in infinitely many commensurability classes.*

Recall from [4] that a *special* is a rational number which is fixed by a hyperbolic element of  $\Gamma$ . In particular, the existence of specials for  $\Gamma$  means that  $\Gamma$  cannot be pseudomodular since the fixed points of hyperbolic elements cannot be cusps. We have:

**Theorem 2.5** *Let  $\mathcal{S} = \{\Delta(1, 1, 1), \Delta(1, \frac{1}{3}, 3)\}$ . Let  $J_A$  be the jigsaw consisting of four tiles: one  $\Delta(1, 1, 1)$  tile in the middle glued to three  $\Delta(1, \frac{1}{3}, 3)$  tiles along its three sides. Let  $J$  be an  $\mathcal{S}$ -jigsaw with corresponding group  $\Gamma_J$  and surface  $S_J$ . Then:*

- $\Gamma_J$  is arithmetic if and only if  $J$  can be assembled from a finite number of copies of  $J_A$  glued along matching sides. In particular,  $\Gamma_{J_A}$  is arithmetic and all other arithmetic  $\mathcal{S}(1, 3)$ -jigsaw groups are subgroups of  $\Gamma_{J_A}$ .
- $\Gamma_J$  is pseudomodular if and only if the surface  $S_J$  does not contain a closed geodesic  $\gamma$  with  $W$ -cutting sequence  $3, 1, \frac{1}{3}, 1$  and it is not of the type above.
- $\Gamma_J$  is neither arithmetic nor pseudomodular if and only if the surface  $S_J$  contains a closed geodesic  $\gamma$  with  $W$ -cutting sequence  $3, 1, \frac{1}{3}, 1$ . In this case,  $\Gamma_J$  contains hyperbolic elements with integer fixed points (specials).
- There are infinitely many commensurability classes of pseudomodular  $\Gamma_J$ , and infinitely many commensurability classes of nonpseudomodular  $\Gamma_J$ .

Note that a geodesic  $\gamma$  in Theorem 2.5 with  $W$ -cutting sequence  $\overline{3, 1, \frac{1}{3}, 1}$  necessarily only intersects  $\Delta(1, \frac{1}{3}, 3)$  tiles, so the  $V$ -cutting sequence is  $\overline{\Delta(1, \frac{1}{3}, 3)}$ . In particular, any  $\mathcal{S}$ -jigsaw  $J$  such that every  $\Delta(1, \frac{1}{3}, 3)$  tile of  $J$  is matched to a  $\Delta(1, 1, 1)$  tile (along the side with parameter 1) is arithmetic or pseudomodular.

We will prove Theorems 2.4 and 2.5 by analyzing the relevant cusp sets, checking the arithmeticity criteria and using Margulis's criteria to show that there are infinitely many commensurability classes of pseudomodular groups coming from the constructions.

### 3 Killer intervals

We give a brief description of Long and Reid's method to show when a group has cusp set all of  $\mathbb{Q} \cup \{\infty\}$ .

Let  $\Gamma < \text{PGL}(2, \mathbb{Q})$  be a zonal Fuchsian group (that is,  $\infty$  is fixed by a parabolic subgroup of  $\Gamma$ ) such that  $\mathbb{H}^2/\Gamma$  is a complete, finite-area hyperbolic surface with exactly one cusp. We are interested in the cusp set of  $\Gamma$ , which in this case (since the surface has only one cusp) is

$$K_\Gamma := \{g(\infty) \mid g \in \Gamma\}.$$



Since  $\Gamma < \text{PGL}(2, \mathbb{Q})$ , we have  $K_\Gamma \subset \mathbb{Q} \cup \{\infty\}$ . The question is in what situation is  $K_\Gamma$  exactly equal to  $\mathbb{Q} \cup \{\infty\}$ . For this, Long and Reid introduced the concept of a killer interval, which generalizes the idea of “flipping” a rational number  $\frac{p}{q} \in (-1, 1)$  by  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})$ , to obtain a new rational number with strictly smaller (absolute) denominator. We have:

**Proposition 3.1** (see [4, Examples 1 and 2]) *Suppose*

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma < \text{PGL}(2, \mathbb{Q}), \quad \text{where } \alpha, \beta, \gamma, \delta \in \mathbb{Z} \text{ and } \gcd(\alpha, \beta, \gamma, \delta) = 1.$$

*Suppose further that  $\gcd(\alpha, \gamma) = k$ , so  $\alpha = k\alpha'$  and  $\gamma = k\gamma'$ , where  $\gcd(\alpha', \gamma') = 1$ . Then, for any  $\frac{p}{q} \in (\alpha'/\gamma' - 1/\gamma, \alpha'/\gamma' + 1/\gamma)$ , the flip  $g^{-1}(\frac{p}{q})$  has strictly smaller denominator than  $\frac{p}{q}$ .*

**Proof** We have

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{Z}, \quad \gcd(\alpha, \beta, \gamma, \delta) = 1.$$

Then  $\alpha/\gamma = \alpha'/\gamma' = g(\infty)$ , where  $\alpha = k\alpha'$ ,  $\gamma = k\gamma'$  and  $k = \gcd(\alpha, \gamma)$ . For any rational  $\frac{p}{q} \in \mathbb{Q}$ , where  $\gcd(p, q) = 1$ ,

$$g^{-1}\left(\frac{p}{q}\right) = \frac{\delta p - \beta q}{-\gamma p + \alpha q} = \frac{\delta p - \beta q}{k(-\gamma' p + \alpha' q)}.$$

Hence, the image has smaller denominator if

$$|-\gamma p + \alpha q| < |q|, \quad \text{that is, } \frac{p}{q} \in \left(\frac{\alpha'}{\gamma'} - \frac{1}{k\gamma'}, \frac{\alpha'}{\gamma'} + \frac{1}{k\gamma'}\right). \quad \square$$

**Remark** The assumption  $\gcd(\alpha, \beta, \gamma, \delta) = 1$  is used to obtain the largest possible interval about  $\alpha'/\gamma'$  such that the inequality is satisfied in the proof.

The interval obtained in Proposition 3.1 depends on the element  $g \in \Gamma$ . If we further assume that the stabilizer subgroup of the point infinity,  $\Gamma_\infty = \langle T^L \rangle$  for some  $L \in \mathbb{Z}$  (this will always be the case under consideration; see Proposition 4.5), then if  $h \in \Gamma$  is such that  $h(\infty) = g(\infty) = \alpha'/\gamma'$ , we have  $h^{-1}g = T^{RL}$  for some  $R \in \mathbb{Z}$  and it is easy to see that the interval obtained from  $g$  and  $h$  are the same. In this case, the interval depends only on the cusp  $\alpha'/\gamma'$  and not on  $g$  and we have the following definition:

**Definition 3.2** (killer intervals and contraction constant) When  $\Gamma_\infty = \langle T^L \rangle$  for some  $L \in \mathbb{Z}$ , the open interval  $I = (\alpha'/\gamma' - 1/k\gamma', \alpha'/\gamma' + 1/k\gamma')$  is called the *killer interval* about the cusp  $\alpha'/\gamma'$  and  $k \in \mathbb{N}$  the *contraction constant* of the cusp  $\alpha'/\gamma'$ .

**Remark** The definition can be easily amended in the case where  $\Gamma_\infty$  is generated by a fractional power of  $T$  (which is always the case since  $\Gamma < \text{PGL}(2, \mathbb{Q})$ ) as we only need to consider a finite number of possibilities for  $g$  and take the largest interval obtained.

We also have a useful variation of Proposition 3.1, where translation by an integer value does not affect the contraction constant:

**Proposition 3.3** *Suppose*

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{PGL}(2, \mathbb{Q}), \quad \text{where } \alpha, \beta, \gamma, \delta \in \mathbb{Z} \text{ and } \gcd(\alpha, \beta, \gamma, \delta) = 1.$$

*Suppose further that  $\gcd(\alpha, \gamma) = k$  and  $\alpha = k\alpha', \gamma = k\gamma'$ . Let  $h = T^n g T^{-n}$ , where  $n \in \mathbb{Z}$ .*

*Then, for any  $\frac{p}{q} \in (\alpha'/\gamma' + n - 1/k\gamma', \alpha'/\gamma' + n + 1/k\gamma')$ , the flip  $h^{-1}(\frac{p}{q})$  has strictly smaller denominator than  $\frac{p}{q}$ .*

**Proof** The proof is a simple computation and will be omitted. □

Now suppose that  $L$  is the smallest positive integer such that  $T^L \in \Gamma$ , for example, if  $\Gamma_\infty = \langle T^L \rangle$ . (More generally, since we assume that  $\Gamma < \text{PGL}(2, \mathbb{Q})$  and  $\Gamma$  is a zonal Fuchsian group, such an  $L$  always exists). Then every  $x \in \mathbb{Q}$  can be moved into a fundamental interval for the action of  $\langle T^L \rangle$  of length  $L$  (say  $[0, L]$ ), by a suitable power of  $T^L$ , without increasing its denominator. If we can cover the fundamental interval  $[0, L]$  for the action of  $\langle T^L \rangle$  by a finite set of killer intervals, then every rational number can be taken to  $\infty$  by a finite composition of elements of  $\Gamma$ , so that  $K_\Gamma = \mathbb{Q} \cup \{\infty\}$ . We can summarize the above discussion as:

**Proposition 3.4** (Long and Reid [4, Theorem 2.5]) *Let  $\Gamma < \text{PGL}(2, \mathbb{Q})$  be a zonal Fuchsian group such that  $\mathbb{H}^2/\Gamma$  is a complete, finite-volume (orbifold) surface with one cusp and suppose that  $L$  is the smallest positive integer such that  $\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \in \Gamma$ . If the fundamental interval  $[0, L]$  for the action of  $\langle T^L \rangle$  can be covered by a finite set of killer intervals, then the cusp set is  $K_\Gamma = \mathbb{Q} \cup \{\infty\}$ .*

Our basic strategy will be to construct jigsaw groups such that Proposition 3.4 applies.

## 4 Basic properties of hyperbolic jigsaw groups

In this section we explore some of the basic properties of hyperbolic jigsaws and their associated groups and surfaces. We will mostly be interested in jigsaws composed from

tiles which have some kind of integrality condition. We first establish some notation and convention.

### 4.1 Integral hyperbolic jigsaws

For  $n \in \mathbb{N}$ , let

$$\Delta^{(n)} := \Delta\left(1, \frac{1}{n}, n\right) = \Delta\left(\frac{1}{n}, n, 1\right) = \Delta\left(n, 1, \frac{1}{n}\right).$$

**Definition 4.1** An integral hyperbolic jigsaw set is a set of the form

$$\mathcal{S}(n_1, n_2, \dots, n_s) := \{\Delta^{(n_1)}, \Delta^{(n_2)}, \dots, \Delta^{(n_s)}\},$$

where  $1 = n_1 < n_2 < \dots < n_s$ . An integral jigsaw is an  $\mathcal{S}$ -jigsaw where  $\mathcal{S}$  is integral. It is in *standard position* if one of the  $\Delta^{(1)}$  tiles is in standard position, that is, has vertex set  $\{\infty, -1, 0\}$ .

Note that we have set  $\Delta^{(n_1)} = \Delta^{(1)}$ . This is convenient for our purposes although not completely necessary; we could have defined integral jigsaw sets which do not contain  $\Delta^{(1)}$ .

Henceforth,  $\mathcal{S}(n_1, n_2, \dots, n_s)$  will denote an integral jigsaw set with  $1 = n_1 < n_2 < \dots < n_s$ .

**Definition 4.2** For an integral jigsaw  $J$ , we will call sides of  $J$ ,  $\mathcal{Q}_J$  or  $\mathcal{T}_J$  with label/parameter in the set  $\{n_j, 1/n_j\}$  type  $n_j$  sides. So, for example,  $\mathcal{S}(1, 2)$ -jigsaws can have only type 1 or type 2 sides, and  $\mathcal{S}(1, 3)$ -jigsaws can have only type 1 or type 3 sides.

**Proposition 4.3** Let  $J$  be an integral  $\mathcal{S}(n_1, n_2, \dots, n_s)$ -jigsaw in standard position with  $\text{sgn}(J) = (m_1, \dots, m_s)$  and  $|J| = N$ . Let  $\Delta_{[v_1, v_2, v_3]} \subset \mathcal{T}_J$  with sides  $e_1 = [v_1, v_2]$ ,  $e_2 = [v_2, v_3]$  and  $e_3 = [v_3, v_1]$  labeled by  $k_1, k_2$  and  $k_3$ , respectively. Suppose that  $v_1 = \infty$ . Then:

- (1) If  $\Delta_{[\infty, v_2, v_3]}$  is a  $\Delta^{(1)}$  triangle, then  $v_2 = m$  and  $v_3 = m + 1$  for some integer  $m$ .
- (2) If  $\Delta_{[\infty, v_2, v_3]}$  is a  $\Delta^{(n)}$  triangle where  $n > 1$  and  $k_1 = 1$  — so  $k_2 = \frac{1}{n}$  and  $k_3 = n$  — then  $v_2 = m$  and  $v_3 = m + 1$  for some integer  $m$ . The triangle to the right of  $\Delta$  in  $\mathcal{T}_J$  is a  $\Delta^{(n)}$  triangle.
- (3) If  $\Delta_{[\infty, v_2, v_3]}$  is a  $\Delta^{(n)}$  triangle where  $n > 1$  and  $k_1 = n$  — so  $k_2 = 1$  and  $k_3 = \frac{1}{n}$  — then  $v_2 = m$  and  $v_3 = m + n$  for some integer  $m$ . The triangles to the left and right of  $\Delta$  in  $\mathcal{T}_J$  are also  $\Delta^{(n)}$  triangles.

- (4) If  $\Delta_{[\infty, v_2, v_3]}$  is a  $\Delta^{(n)}$  triangle where  $n > 1$  and  $k_1 = \frac{1}{n}$  — so  $k_2 = n$  and  $k_3 = 1$  — then  $v_2 = m$  and  $v_3 = m + 1$  for some integer  $m$ . The triangle to the left of  $\Delta$  in  $\mathcal{T}_J$  is a  $\Delta^{(n)}$  triangle.
- (5) If  $e = [\infty, m]$  is a side of  $\mathcal{T}_J$  of type  $n$  where  $n \in \{n_1, \dots, n_s\}$ , then the marked point on  $e$  is  $m + \sqrt{ni}$ , that is, it has height  $\sqrt{n}$  above the real line.

**Proof** By our normalization, the triangle  $\Delta_{[\infty, -1, 0]}$  is a  $\Delta^{(1)}$  triangle of  $\mathcal{T}_J$ , so satisfies condition (1). The proposition follows easily by induction moving to the right and left of this normalized triangle, using the matching conditions. □

It follows from the above that if  $\Delta_{[\infty, v_2, v_3]} \subset \mathcal{T}_J$ , then  $v_3 - v_2$  takes values in  $\{1, n_2, \dots, n_s\}$  and that the  $\Delta^{(n_i)}$  triangles where  $n_i > 1$  with  $\infty$  as a vertex occur in successive triples, where the middle triangle has width  $n_i$  and the other two width 1. We can use this to define a  $J$ -width at each vertex of a triangle, and hence a  $J$ -width for each vertex of the jigsaw  $J$ :

**Definition 4.4** (vertex  $J$ -widths) For any triangle  $\Delta^{(n)}$  with  $n \in \mathbb{N}$ , the  $J$ -width of the vertices between the type 1 side and the type  $n$  sides is 1. The  $J$ -width of the vertex between the two type  $n$  sides is  $n$ . For the jigsaw  $J$ , the  $J$ -width of a vertex  $v$  of  $J$ , denoted by  $JW(v)$ , is the sum of the  $J$ -widths of the vertices of the triangles at  $v$ .

**Remark** The  $J$ -width of a vertex is in general different from the weight of the vertex  $v$ , which is just the number of triangles at  $v$ . We denote the weight by  $wt(v)$ .

From this we easily get:

**Proposition 4.5** Let  $J$  be an  $\mathcal{S}(n_1, \dots, n_s)$ -jigsaw in standard position with size  $N$  and signature  $(m_1, m_2, \dots, m_s)$ , and let  $\Gamma_J = \langle \iota_0, \iota_1, \dots, \iota_{N+1} \rangle$ , where  $\iota_k$  for  $k = 0, \dots, N + 1$  is the  $\pi$ -rotation about the marked point  $x_k$  of the side  $s_k$  of  $J$  with  $s_k$  taken in anticlockwise cyclic order and  $s_0$  the leftmost vertical side of  $J$ . Then

$$\iota_{N+1}\iota_N \cdots \iota_0 = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix},$$

where

$$L = \sum_{i=1}^s m_i(2 + n_i) \in \mathbb{Z}.$$

**Proof** We only need to compute the total  $J$ -width about the cusp of  $\Gamma_J$ . Each  $\Delta^{(n)}$  triangle contributes a  $J$ -width of  $2 + n$ , so the result follows. □

To summarize, we have shown:

- All (normalized) integral hyperbolic jigsaw groups have a fundamental interval of integral length  $L = \sum_{i=1}^s m_i(2 + n_i)$ .
- If  $\Delta_{[\infty, v_2, v_3]} \subset \mathcal{T}_J$ , then  $v_2 \in \mathbb{Z}$  and  $v_3 - v_2$  takes values in  $\{1, n_2, \dots, n_s\}$ .
- If  $\Delta_{[\infty, v_2, v_3]} \subset \mathcal{T}_J$  has width  $n = v_3 - v_2 > 1$ , then  $\Delta_{[\infty, v_2, v_3]}$  is the lift of a  $\Delta^{(n)}$  tile, and the triangles to the left and right are also lifts of  $\Delta^{(n)}$  tiles with width one.

More importantly, the cusps at the end of vertical sides of  $\mathcal{T}_J$  have large killer intervals:

**Proposition 4.6** *Let  $J$  be an integral hyperbolic jigsaw in standard position. If  $\Delta_{[\infty, q, r]} \subset \mathcal{T}_J$ , then  $q \in \mathbb{Z}$  is a cusp of  $\Gamma_J$  with contraction constant one, so the killer interval about  $q$  is  $(q - 1, q + 1)$ .*

Note that it is possible that  $e = [\infty, q] \in \mathcal{F}_J \setminus \mathcal{E}_J$ , that is,  $e$  may be an interior side.

**Proof** This is basically a simple computation. We have  $\Delta_{[\infty, q, r]} \subset \mathcal{T}_J$  is the lift of some tile of  $J$ ; we consider another lift  $\Delta_{[w_1, w_2, w_3]}$  of the same tile of  $J$  such that  $w_2 = \infty$  (this is always possible since  $\Gamma_J$  has only one cusp). Then there is an element  $g \in \Gamma_J$  which takes the ordered triple  $(w_1, w_2, w_3) = (w_1, \infty, w_3)$  to  $(v_1, v_2, v_3) = (\infty, q, r)$ , and furthermore,  $g$  is completely determined by this pair of triples. We separate the analysis into the three cases, depending on the value of the parameter  $k$  at the side  $[\infty, q]$ .

**Case 1** ( $k = 1$ ) In this case,  $r = q + 1$ , and  $w_3 = m$  and  $w_1 = m + 1$  for some integer  $m$ . Then

$$g = \begin{pmatrix} q & -1 - q - mq \\ 1 & -1 - m \end{pmatrix}$$

and hence the killer interval about  $q$  is  $(q - 1, q + 1)$  by Proposition 3.1.

**Case 2** ( $k = n \in \{n_2, \dots, n_s\}$ ) In this case,  $r = q + n$ , and  $w_3 = m$  and  $w_1 = m + 1$  for some integer  $m$ . Then

$$g = \frac{1}{\sqrt{n}} \begin{pmatrix} q & -n - q - mq \\ 1 & -1 - m \end{pmatrix}$$

and hence the killer interval about  $q$  is  $(q - 1, q + 1)$  by Proposition 3.1.

**Case 3** ( $k = \frac{1}{n}$ , where  $n \in \{n_2, \dots, n_s\}$ ) In this case,  $r = q + 1$ , and  $w_3 = m$  and  $w_1 = m + n$  for some integer  $m$ . Then

$$g = \frac{1}{\sqrt{n}} \begin{pmatrix} q & -n - nq - mq \\ 1 & -n - m \end{pmatrix}$$

and hence the killer interval about  $q$  is  $(q - 1, q + 1)$  by Proposition 3.1. □

We note that any other lift  $\Delta_{[w_1, w_2, w_3]}$  of the same tile of  $J$  such that  $w_2 = \infty$  is a translation of the one we choose by some multiple of the fundamental interval  $L$ . Hence, if  $q$  is a cusp and  $g \in \Gamma_J$  maps  $\infty$  to  $q$ , then the set of  $A \in \Gamma_J$  mapping  $\infty$  to  $q$  is

$$\{A \in \text{PGL}(2, \mathbb{Q}) \mid A = gT^{LM}, M \in \mathbb{Z}\}.$$

We next analyze cusps at the next level away from  $\infty$ :

**Proposition 4.7** *Let  $J$  be a hyperbolic jigsaw in standard position. Suppose that  $\Delta_{[\infty, m, m+n]} \subset \mathcal{T}_J$ , where  $n > 1$  is the lift of a  $\Delta^{(n)}$  tile. Then  $\Delta_{[m+n, m, m+n/2]} \subset \mathcal{T}_J$ . The contraction constant of the cusp  $m + \frac{n}{2}$  is one if  $n$  is odd, and either one or two if  $n$  is even.*

**Proof** We first note that  $\Delta_{[v_1, v_2, v_3]} := \Delta_{[m+n, m, m+n/2]} \subset \mathcal{T}_J$  since the parameter at the side  $[m, m + n]$  is one. Then the parameters at the sides  $[m, m + \frac{n}{2}]$  and  $[m + \frac{n}{2}, m + n]$  are  $\frac{1}{k}$  and  $k$ , respectively, where  $k \in \{n_1, \dots, n_s\}$ . Let  $\Delta_{[w_1, w_2, w_3]}$  be another lift of the same tile as  $\Delta_{[v_1, v_2, v_3]}$ , where  $w_3 = \infty$  corresponds to  $v_3 = m + \frac{n}{2}$ . Then the parameter of the side  $[w_3, w_1]$  is  $k$ , so  $w_1 = r$  and  $w_2 = r + k$  for some  $r \in \mathbb{Z}$ . Hence, there is an element  $g \in \Gamma_J$  taking the ordered triple  $(\infty, r, r + k)$  to  $(m + \frac{n}{2}, m + n, m)$ . A direct computation gives

$$g = \frac{1}{\sqrt{kn}} \begin{pmatrix} 2m + n & -r(2m + n) - k(m + n) \\ 2 & -2r - k \end{pmatrix}$$

Hence, the contraction constant at  $m + \frac{n}{2}$  is one if  $n$  is odd, and either one or two if  $n$  is even, depending on whether  $k$  is odd or even, respectively. □

We next show that many of the sides of an integral  $\mathcal{S}$ -jigsaw cannot be type 1, which will be useful when we are determining if the group  $\Gamma_J$  is arithmetic or not.

**Proposition 4.8** *Let  $J$  be an  $\mathcal{S}(n_1, n_2, \dots, n_s)$ -jigsaw. For any  $n \in \{n_2, \dots, n_s\}$ , at least two of the sides of  $J$  are type  $n$ .*

**Proof** Let  $n \in \{n_2, \dots, n_s\}$ . By definition,  $J$  contains a  $\Delta^{(n)}$  tile. If its two type  $n$  sides are already sides of  $J$ , we are done. Otherwise, say one of the type  $n$  sides is not a side of  $J$ . It must be matched to another  $\Delta^{(n)}$  tile, which now has a new type  $n$  side. This is either a side of  $J$  or matched to another  $\Delta^{(n)}$  tile. Proceeding inductively, since  $|J|$  is finite, we eventually must have at least two sides of  $J$  which are type  $n$ .  $\square$

### 4.2 Nonintegral jigsaws

Much of the discussion above can be applied to nonintegral hyperbolic jigsaws, but without the integrality conditions, it is difficult to obtain useful conclusions. For example, the fundamental interval  $[0, L]$  may not have integral length, the vertical sides of  $\mathcal{T}_J$  may not have integer ends, and, typically, the contraction constant about a cusp at the end of a vertical side of  $\mathcal{T}_J$  may be greater than one. The analysis of such jigsaws is therefore more complicated.

## 5 Cusp set of $\mathcal{S}(1, 2)$ -jigsaw groups

We apply the results of Section 4.1 to  $\mathcal{S}(1, 2)$ -jigsaws which are particularly simple. What is interesting is that when we combine  $\Delta(1, 1, 1)$  tiles with  $\Delta(1, \frac{1}{2}, 2)$  tiles, the arithmetic property of the Weierstrass groups  $\Gamma(1, 1, 1)$  and  $\Gamma(1, \frac{1}{2}, 2)$  is destroyed while the property that the cusp set is  $\mathbb{Q} \cup \{\infty\}$  is preserved. In this section, we analyze the cusp set.

### 5.1 The Weierstrass groups $\Gamma(1, 1, 1)$ and $\Gamma(1, \frac{1}{2}, 2)$

We first note that the Weierstrass group  $\Gamma(1, 1, 1)$  associated to the tile  $\Delta(1, 1, 1)$  is just an index-three subgroup of the modular group  $\text{PSL}(2, \mathbb{Z})$ , hence arithmetic. In standard position, the three generators are

$$t_1 = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Weierstrass group  $\Gamma(1, \frac{1}{2}, 2)$  is also arithmetic. In standard position, the three generators are

$$t_1 = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \quad t_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 2 \\ -3 & -2 \end{pmatrix}, \quad t_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}.$$

It is commensurable to the group associated to the square torus and also the Hecke group  $G_4$ . In particular  $(\text{tr } t_1 t_2)^2$ ,  $(\text{tr } t_2 t_3)^2$  and  $(\text{tr } t_3 t_1)^2$  are all integers. Note that  $x_1$ ,

the fixed point of  $\iota_1$ , is at height one above the real axis and  $x_3$ , the fixed point of  $\iota_3$ , is at height  $\sqrt{2}$  above the real axis.

## 5.2 Killer intervals and cusp set of $\mathcal{S}(1, 2)$ -jigsaws

We have:

**Proposition 5.1** *If  $J$  is an  $\mathcal{S}(1, 2)$ -jigsaw, then  $K_J$ , the cusp set of  $\Gamma_J$ , is  $\mathbb{Q} \cup \{\infty\}$ .*

**Proof** This follows easily from Propositions 4.3, 4.6 and 4.7 as applied to  $\mathcal{S}(1, 2)$ -jigsaws. First, if  $\Delta := \Delta_{[\infty, v_2, v_3]} \subset \mathcal{T}_J$  and the parameter at  $s_1 = [\infty, v_2]$  is  $k_1 = 2$ , then, by Proposition 4.3,

- (i)  $v_2 = m$  and  $v_3 = m + 2$  for some integer  $m$ ,
- (ii)  $k_2 = 1$  and  $k_3 = \frac{1}{2}$ , and
- (iii) the triangles of  $\mathcal{T}_J$  to the left and right of  $\Delta$  have width one.

By Proposition 4.7,  $m + 1$  is a vertex of the triangle  $\Delta'$  of  $\mathcal{T}_J$  which shares the side  $[m, m + 2]$  with  $\Delta$  (since  $k_2 = 1$ ). All other types of triangles of  $\mathcal{T}_J$  with  $\infty$  as a vertex have width one. Hence, every  $m \in \mathbb{Z}$  is a cusp of  $\Gamma_J$ . Next, by Proposition 4.6, if  $[\infty, m]$  is a side of  $\mathcal{T}_J$ , then  $m \in \mathbb{Z}$  and the killer interval about  $m$  is  $(m - 1, m + 1)$ . It now follows that the set of killer intervals about the integers cover all of  $\mathbb{R}$  and Proposition 3.4 applies.  $\square$

## 6 Cusp set of $\mathcal{S}(1, 3)$ -jigsaws

In this section, we consider  $\mathcal{S}(1, 3)$ -jigsaws and show that either the cusp set is all of  $\mathbb{Q} \cup \{\infty\}$  or there are specials which are in the equivalence classes of certain integers. The latter case occurs if and only if there is a closed geodesic  $\gamma \subset S_J$  with  $W$ -cutting sequence  $\frac{1}{3}, 1, 3, 1$ .

### 6.1 The Weierstrass group $\Gamma(1, \frac{1}{3}, 3)$

In standard position, we get from (2) that  $\Gamma(1, \frac{1}{3}, 3)$  is generated by

$$(4) \quad \iota_1 = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \quad \iota_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 & 3 \\ -4 & -3 \end{pmatrix}, \quad \iota_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}.$$



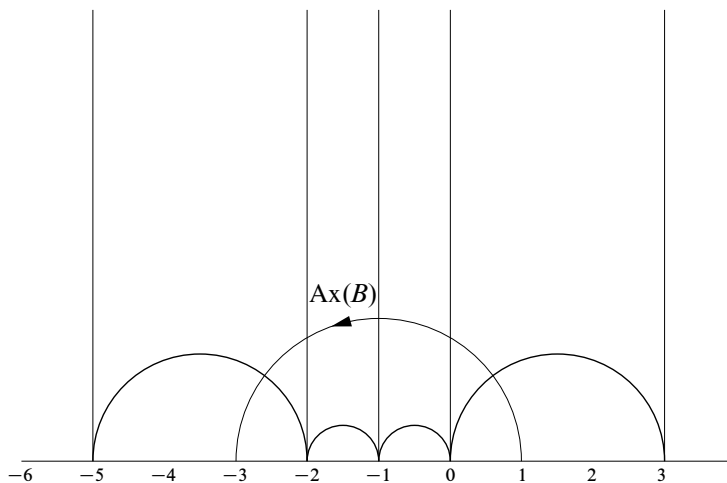


Figure 3: Tessellation associated to  $\Gamma(1, \frac{1}{3}, 3)$  and axis of the hyperbolic element  $B$  with endpoints at  $-3$  and  $1$

A fundamental interval for the stabilizer subgroup of infinity in  $\Gamma(1, \frac{1}{3}, 3)$  is of length 5; we may take this to be the closed interval  $[-2, 3]$ , with vertical sides  $[\infty, -2]$ ,  $[\infty, -1]$ ,  $[\infty, 0]$  and  $[\infty, 3]$  within this interval; see Figure 3. The killer intervals about  $-2, -1, 0$  and  $3$  all have width 2; they are  $(-3, -1)$ ,  $(-2, 0)$ ,  $(-1, 1)$  and  $(2, 4)$ , respectively. By Proposition 4.7, the triangle  $\Delta_{[0,3/2,3]}$  adjacent to  $\Delta_{[\infty,0,3]}$  is a triangle of  $\mathcal{T}$  and the contraction constant at  $\frac{3}{2}$  is 1, so the killer interval for the cusp  $\frac{3}{2}$  is  $(1, 2)$ . It follows that the killer intervals about  $-2, -1, 0, \frac{3}{2}$  and  $3$  cover all of  $[-2, 3]$  except possibly the points 1 and 2. In fact, these are specials and cannot be covered by any killer intervals. A direct computation shows that  $x_1 = -3$  and  $x_2 = 1$  are the attracting and repelling fixed points of the element

$$B = (t_1 t_2 t_1) t_3 = \frac{1}{3} \begin{pmatrix} 7 & -6 \\ -2 & 3 \end{pmatrix} \in \Gamma(1, \frac{1}{3}, 3),$$

and  $y_1 = 2$  and  $y_2 = 6$  are the attracting and repelling fixed points of

$$t_3 t_2 t_1 B (t_3 t_2 t_1)^{-1} = \frac{1}{3} \begin{pmatrix} -3 & 24 \\ -2 & 13 \end{pmatrix} \in \Gamma(1, \frac{1}{3}, 3).$$

Since  $B$  is the product of the two involutions  $t_1 t_2 t_1$  and  $t_3$ , its invariant axis  $Ax(B)$  (see Figure 3) passes through the fixed points  $-2 + \sqrt{3}i$  and  $\sqrt{3}i$  of these elements (in fact, it also passes through  $-1 + 2i$ ), and  $B$  translates by twice the distance between the points  $-2 + \sqrt{3}i$  and  $\sqrt{3}i$  along this axis. The  $W$ -cutting sequence of the axis is  $\frac{1}{3}, 1, 3, 1$ . Furthermore,  $t_3$  interchanges 1 and  $-3$ . It follows that every rational

number is either a cusp or is equivalent to the special 1. If we consider the punctured torus  $T$  which double covers the surface  $S = \mathbb{H}^2 / \Gamma(1, \frac{1}{3}, 3)$  (with the induced labeled triangulation), there is a (unique) simple closed geodesic  $\gamma \subset T$  which has  $W$ -cutting sequence  $\frac{1}{3}, 1, 3, 1$  whose fixed points are equivalent to 1 or  $-3$ .

### 6.2 Killer intervals and cusp set of $S(1, 3)$ -jigsaws

**Proposition 6.1** *Let  $J$  be an  $S(1, 3)$ -jigsaw. The cusp set  $K_\Gamma$  of  $\Gamma_J$  is a proper subset of  $\mathbb{Q} \cup \{\infty\}$  if and only if the surface  $S_J$  contains a closed geodesic  $\gamma$  with  $W$ -cutting sequence  $\frac{1}{3}, 1, 3, 1$ .*

**Proof** ( $\Leftarrow$ ) Suppose that  $S_J$  contains a geodesic  $\gamma$  with  $W$ -cutting sequence  $\frac{1}{3}, 1, 3, 1$ . Note that  $\gamma$  only intersects  $\Delta^{(3)}$  triangles, that is the  $V$ -cutting sequence of  $\gamma$  is  $\overline{\Delta^{(3)}}$ . We may lift one of these triangles to  $\Delta_{[\infty, m, m+3]} \subset \mathcal{T}_J$ , where  $m \in \mathbb{Z}$ . Without loss of generality, suppose that  $\tilde{\gamma}$  intersects the sides  $[\infty, m]$  and  $[m, m+3]$  of  $\Delta_{[\infty, m, m+3]}$  (which gives the 3, 1 part of the  $W$ -cutting sequence). From the analysis of the Weierstrass group  $\Gamma(1, \frac{1}{3}, 3)$ , since  $\tilde{\gamma}$  has  $W$ -cutting sequence  $\frac{1}{3}, 1, 3, 1$ , we see that the endpoints of  $\tilde{\gamma}$  are  $m-3$  and  $m+1$ , which are *specials* of  $\Gamma_J$ . Hence,  $K_\Gamma$  is a proper subset of  $\mathbb{Q} \cup \{\infty\}$ .

( $\Rightarrow$ ) Conversely, suppose that  $K_\Gamma$  is a proper subset of  $\mathbb{Q} \cup \{\infty\}$ . By Proposition 4.3, all triangles  $\Delta_{[\infty, q, r]} \subset \mathcal{T}_J$  have width either 1 or 3, and furthermore, the triangles with width 3 have triangles of width 1 on either side. The killer intervals about the cusps  $q$  are  $(q-1, q+1)$  by Proposition 4.6. Furthermore, if  $\Delta_{[\infty, q, q+3]} \subset \mathcal{T}_J$  has width 3, then by Proposition 4.7,  $q + \frac{3}{2}$  is a cusp with killer interval  $(q+1, q+2)$ . It follows that the fundamental interval  $[0, L]$  can be covered by the killer intervals except possibly for a finite number of points. These points are of the form  $m+1$  or  $m+2$ , where  $\Delta_{[\infty, m, m+3]} \subset \mathcal{T}_J$ . Since  $K_\Gamma$  is a proper subset of  $\mathbb{Q} \cup \{\infty\}$ , there is at least one such point which is not covered by any killer interval. Without loss of generality, by conjugating by a suitable translation, and to simplify the computations, we may assume that  $m = 0$ , and the point 1 is not covered by any killer interval where  $\Delta_{[\infty, 0, 3]} \subset \mathcal{T}_J$ . In particular, 1 is not a cusp of  $\Gamma_J$ . Let  $\alpha$  be the geodesic ray from  $y = -1 + 2i$  ending at 1. Note that  $\alpha$  is contained in the axis of

$$B = (t_1 t_2 t_1) t_3 = \frac{1}{3} \begin{pmatrix} 7 & -6 \\ -2 & 3 \end{pmatrix} \in \Gamma(1, \frac{1}{3}, 3).$$

The first two terms of the  $V$ -cutting sequence of  $\alpha$  are  $\Delta^{(3)}, \Delta^{(3)}, \dots$ , and the first three terms of the  $W$ -cutting sequence of  $\alpha$  is  $1, 3, 1, \dots$ , where both cutting sequences

are infinite as 1 is not a cusp. We claim the  $V$ -cutting sequence of  $\alpha$  cannot contain any  $\Delta^{(1)}$ ; if it does, it will be finite, which implies 1 is a cusp, giving a contradiction. First, suppose that  $\Delta_{[3,0,3/2]} \subset \mathcal{T}_J$  is a  $\Delta^{(1)}$  tile. Then a direct computation gives that the neighboring triangle sharing the side  $[0, \frac{3}{2}]$  with  $\Delta_{[3,0,3/2]}$  is  $\Delta_{[3/2,0,1]}$ , so that 1 is a cusp. So the  $V$ -cutting sequence of  $\alpha$  starts with at least four  $\Delta^{(3)}$  triangles: the fourth triangle is also a  $\Delta^{(3)}$  triangle since the side adjoining the third triangle has label in the set  $\{3, \frac{1}{3}\}$ . By a similar argument, if the  $V$ -cutting sequence of  $\alpha$  contains a  $\Delta^{(1)}$  triangle, the number of  $\Delta^{(3)}$  triangles before the first  $\Delta^{(1)}$  triangle in the cutting sequence is even, that is, either  $4k$  or  $4k + 2$  for some  $k \in \mathbb{N}$ . Consider the action of  $B^k$  on  $\alpha$  and the triangles of  $\mathcal{T}_J$  which it intersects. This maps the first  $\Delta^{(1)}$  triangle intersecting  $\alpha$  to either  $\Delta_{[\infty,-1,0]}$  or  $\Delta_{[3,0,3/2]}$ , while fixing 1. In either case, the next triangle intersected by  $\alpha$  has 1 as a vertex (either  $\Delta_{[\infty,0,1]}$  or  $\Delta_{[3/2,0,1]}$ ). Acting by  $B^{-k}$  to move back to the original configuration, we conclude that if  $\alpha$  intersects a  $\Delta^{(1)}$  triangle, then the next triangle it intersects must have 1 as a vertex. Hence, since 1 is not a cusp, we conclude that the  $V$ -cutting sequence of  $\alpha$  is  $\overline{\Delta(3)}$  and, from the analysis of  $\Gamma(1, \frac{1}{3}, 3)$  earlier, the  $W$ -cutting sequence of  $\alpha$  is  $1, 3, 1, \frac{1}{3}$ . Projecting back to the surface  $S_J$ , we get a geodesic ray  $\bar{\alpha} \subset S_J$  with cutting sequence  $1, 3, 1, \frac{1}{3}$ . Since there are only a finite number of sides in the triangulation of  $J$ , this ray  $\bar{\alpha}$  intersects some side  $e$  of the triangulation twice, say at  $y_1, y_2 \in \bar{\alpha}$ . Consider the closed curve on  $S_J$  consisting of the geodesic segment from  $y_1$  to  $y_2$  along  $\bar{\alpha}$  followed by the geodesic on  $e$  joining  $y_2$  back to  $y_1$ . Pulling this tight produces the closed geodesic  $\gamma$  on  $S_J$  with  $W$ -cutting sequence  $1, 3, 1, \frac{1}{3}$ , as required. This completes the proof.  $\square$

## 7 Criteria for (non)arithmeticity

We first establish some general results for integral jigsaw groups. By results of Takeuchi [8], to show that a noncompact Fuchsian group  $\Gamma \leq \mathrm{PSL}(2, \mathbb{R})$  of finite coarea with no elements of order 2 and with invariant trace field  $\mathbb{Q}$  is nonarithmetic, it suffices to find an element  $\gamma \in \Gamma$  such that  $\mathrm{tr} \gamma^2 \notin \mathbb{Z}$ . Conversely, to show that  $\Gamma$  is arithmetic, by [3], it suffices to show that  $\mathrm{tr} \gamma^2 \in \mathbb{Z}$  for all  $\gamma$  in a generating set of  $\Gamma$ . Since  $\mathrm{tr} \gamma^2 = (\mathrm{tr} \gamma)^2 - 2$ , we can check if  $(\mathrm{tr} \gamma)^2 \in \mathbb{Z}$  instead.

Recall that for any integral jigsaw  $J$  with  $|J| = N$ ,  $\Gamma_J$  is freely generated by  $\iota_0, \iota_1, \dots, \iota_{N+1}$ , where  $\iota_j$  is the  $\pi$ -rotation about the marked point  $x_j$  of the side  $s_j$  of  $J$ . So  $\Gamma_J \cong \mathbb{Z}_2 * \dots * \mathbb{Z}_2$ , the free product of  $N + 2$  copies of  $\mathbb{Z}_2$ . We will apply

results of [8; 3] to the index-two subgroup  $\Gamma_J^{(2)} < \Gamma_J$  consisting of elements of  $\Gamma_J$  with even word length. Then  $\Gamma_J^{(2)}$  is a free group of rank  $N + 1$  and

$$\Gamma_J^{(2)} = \langle \iota_0 \iota_1, \iota_0 \iota_2, \dots, \iota_0 \iota_{N+1} \rangle.$$

A fundamental domain for  $\Gamma_J^{(2)}$  is  $J \cup \iota_0(J)$  with opposite sides identified.

Let  $J$  be an  $\mathcal{S}(n_1, \dots, n_s)$ -jigsaw of size  $|J| = N$  in standard position. For  $i = 0, \dots, N + 1$ , let  $v_0 = \infty, v_1, \dots, v_{N+1}$  be the vertices of  $J$ , where  $v_1 < v_2 < \dots < v_{N+1}$ ,  $s_i = [v_i, v_{i+1}]$  be the sides of  $J$  and  $k_i \in \{1, n_2, 1/n_2, \dots, n_s, 1/n_s\}$  be the label (parameter) of the side  $s_i$ . For  $j \in \mathbb{Z}$ , let  $e_j := [\infty, m_j]$  be successive (unoriented) vertical sides of  $\mathcal{Q}_J$  indexed in the negative direction (so  $m_j > m_{j+1}$  for all  $j \in \mathbb{Z}$ ), normalized so  $e_0 = [\infty, m_0] = [\infty, v_1] = s_0$  (so  $e_{-1} = [\infty, v_{N+1}] = s_{N+1}$ ). By looking at the lifts of the fundamental domain  $J$  in the developing map, from the definition of the  $J$ -widths, and by Propositions 4.3 and 4.8, we easily get:

**Lemma 7.1** *Let  $J$  be an  $\mathcal{S}(n_1, \dots, n_s)$ -jigsaw with  $|J| = N$  and for  $i = 0, \dots, N + 1$  and  $j \in \mathbb{Z}$ , let  $v_i, s_i, k_i$  and  $e_j$  be defined as above. Then:*

- (1) *For each  $n \in \{n_2, \dots, n_s\}$ , at least two of the sides  $s_k$  and  $s_l$  of  $J$  are type  $n$ .*
- (2) *If  $j \equiv i \pmod{N + 2}$ ,  $e_j$  is a lift of the side  $s_i$  in  $\mathcal{Q}_J$  and so has label  $k_i$ .*
- (3) *If  $j \equiv i \pmod{N + 2}$ ,  $m_{j-1} - m_j$  is the  $J$ -width of  $v_i$ .*
- (4) *If  $j \equiv i \pmod{N + 2}$  and  $k_i \in \{n, \frac{1}{n}\}$ , where  $n \in \{n_1, \dots, n_2\}$ , the marked point  $y_j \in e_j$  is at height  $\sqrt{n}$  above the real line and the  $\pi$ -rotation about  $y_j$  is*

$$h_j := \frac{1}{\sqrt{n}} \begin{pmatrix} m_j & -(m_j^2 + n) \\ 1 & -m_j \end{pmatrix}.$$

*Note that  $h_j \in \Gamma_J$  has odd word length for all  $j \in \mathbb{Z}$  (it is the conjugate of a generator) and hence  $h_k h_l \in \Gamma_J^{(2)}$  for all  $k, l \in \mathbb{Z}$ .*

- (5)  *$L := [m_{N+2}, m_0]$  is a fundamental interval for  $\Gamma_J$ .*

### 7.1 Nonarithmeticity of $\mathcal{S}(1, 2)$ -jigsaws

We first recall that if  $\tau$  is a triangulation of a polygon  $P$  with  $n \geq 4$  sides, a triangle  $\Delta \subset \tau$  is an *ear* if two of its sides are sides of  $P$ . We call the vertex  $v$  between these two sides the *tip* of the ear, so  $\text{wt}(v) = 1$ . Every triangulation of  $P$  has at least two ears.

**Proposition 7.2** *For any  $\mathcal{S}(1, 2)$ -jigsaw  $J$ , the group  $\Gamma_J$  is nonarithmetic.*

**Proof** Let  $J$  be an  $S(1, 2)$ -jigsaw of size  $|J| = N$  and for  $i = 0, \dots, N + 1$  and  $j \in \mathbb{Z}$ , let  $v_i, s_i, k_i, e_j$  and  $h_j$  be defined as in Lemma 7.1. By Lemma 7.1(1)–(2), some side  $e_j$  is type 2. Hence, the rotation in the side  $e_j$  is

$$h_j = \frac{1}{\sqrt{2}} \begin{pmatrix} m_j & -m_j^2 - 2 \\ 1 & -m_j \end{pmatrix}.$$

Suppose that  $e_k$  is type 1 for  $k \neq j$ . A direct computation using Lemma 7.1(4) shows that the rotation in the side  $e_k$  is

$$h_k = \begin{pmatrix} m_k & -m_k^2 - 1 \\ 1 & -m_k \end{pmatrix}.$$

Then  $(\text{tr } h_j h_k)^2 = \frac{1}{2}(- (m_j - m_k)^2 - 3)^2 \in \mathbb{Z}$  if and only if  $(m_j - m_k)^2 \equiv 1 \pmod{2}$  if and only if  $m_j - m_k \equiv 1 \pmod{2}$ .

Similarly, if  $e_k$  is type 2 for  $k \neq j$ , then the rotation in the side  $e_k$  is

$$h_k = \frac{1}{\sqrt{2}} \begin{pmatrix} m_k & -m_k^2 - 2 \\ 1 & -m_k \end{pmatrix}$$

Then  $(\text{tr } h_j h_k)^2 = \frac{1}{4}(- (m_j - m_k)^2 - 4)^2 \in \mathbb{Z}$  if and only if  $(m_j - m_k)^2 \equiv 0 \pmod{4}$  if and only if  $m_j - m_k \equiv 0 \pmod{2}$ .

Hence, if  $\Gamma_J$  is arithmetic, the parity (mod 2) of the type 1 and type 2 vertical sides  $e_j$  must be different. This immediately shows that many jigsaws are not arithmetic. For example, if the jigsaw has an *ear* which is a  $\Delta^{(1)}$  tile, then it cannot be arithmetic since the  $J$ -width between the two unmatched sides is 1 and they are both type 1. By Lemma 7.1(3), this means there are two successive sides  $e_k$  and  $e_{k+1}$  of type 1 and  $m_k = m_{k+1} + 1$ . Then either  $m_k$  or  $m_{k+1}$  has the same parity as  $m_j$ , where  $e_j$  is type 2, so either  $(\text{tr } h_j h_k)^2$  or  $(\text{tr } h_j h_{k+1})^2$  is not integral.

So for a jigsaw  $J$  to be arithmetic, the  $J$ -width at each vertex  $v_i$  (which lies between  $s_{i-1}$  and  $s_i$ ) has to be

- even, if  $s_{i-1}$  and  $s_i$  are same types;
- odd, if  $s_{i-1}$  and  $s_i$  are different types.

We call an  $S(1, 2)$ -jigsaw  $J$  which satisfies the two conditions above admissible. Now, suppose for a contradiction that there exists an arithmetic  $\Gamma_J$ , so  $J$  is admissible. Let  $\Delta$  be a ear of  $J$ . By the earlier discussion,  $\Delta$  is a  $\Delta^{(2)}$  triangle. It is easy to check that  $J \setminus \Delta$  is again admissible. Proceeding inductively leads to a contradiction since  $J$  has a finite number of  $\Delta^{(2)}$  tiles and has at least one  $\Delta^{(1)}$  tile. □

### 7.2 (Non)arithmeticity of $\mathcal{S}(1, 3)$ -jigsaws

**Proposition 7.3** *For any  $\mathcal{S}(1, 3)$ -jigsaw  $J$ , if the group  $\Gamma_J$  is arithmetic, then all the sides  $s_i$  of  $J$  must be type 3, and the  $J$ -width at each vertex  $v_i$  of  $J$  is congruent to  $0 \pmod 3$ .*

**Proof** This follows easily from Lemma 7.1 applied to  $\mathcal{S}(1, 3)$ -jigsaws. Adopting the same notation, we first note that there exists a vertical side  $e_j = [\infty, m_j]$  of  $\mathcal{Q}_J$  of type 3. Then the  $\pi$ -rotation about the marked point on the side  $e_j$  is

$$h_j = \frac{1}{\sqrt{3}} \begin{pmatrix} m_j & -m_j^2 - 3 \\ 1 & -m_j \end{pmatrix}.$$

If some other vertical side  $e_k = [\infty, m_k]$  is type 1, then a direct calculation shows that the  $\pi$ -rotation about the marked point on the side  $e_k$  is

$$h_k = \begin{pmatrix} m_k & -m_k^2 - 1 \\ 1 & -m_k \end{pmatrix}.$$

Then  $(\text{tr } h_j h_k)^2 = \frac{1}{3}(-m_j - m_k)^2 - 4)^2 \notin \mathbb{Z}$  (this depends on the fact that  $3 \not\equiv 1 \pmod 4$ ).

Similarly, if some other  $e_k$  is type 3, then

$$h_k = \frac{1}{\sqrt{3}} \begin{pmatrix} m_k & -m_k^2 - 3 \\ 1 & -m_k \end{pmatrix}.$$

Hence,  $(\text{tr } h_j h_k)^2 = \frac{1}{9}(-m_j - m_k)^2 - 6)^2 \in \mathbb{Z}$  if and only if  $m_j - m_k \equiv 0 \pmod 3$ .

It follows that if  $\Gamma_J$  is arithmetic, all sides  $s_i$  of  $J$  must be of type 3, and  $\text{JW}(v_i) \equiv 0 \pmod 3$  for all vertices  $v_i$  of  $J$ . □

It turns out that arithmetic  $\mathcal{S}(1, 3)$ -jigsaws exist. We have:

**Proposition 7.4** *The  $\mathcal{S}(1, 3)$ -jigsaw  $J_A$  with signature  $(1, 3)$  consisting of one  $\Delta^{(1)}$  tile and three  $\Delta^{(3)}$  tiles, one attached to each of the three sides of the  $\Delta^{(1)}$  tile, is arithmetic.*

**Proof** We can show by direct calculation that  $(\text{tr } \iota_0 \iota_i)^2 \in \mathbb{Z}$  for  $i = 1, 2, \dots, 5$ , where  $\Gamma_J^{(2)} = \langle \iota_0 \iota_1, \dots, \iota_0 \iota_5 \rangle$ . Hence, by [3],  $\Gamma_J$  is arithmetic. We may also observe that  $S_J$  has a six-fold symmetry, under rotation by  $\frac{\pi}{3}$  about the centroid of the  $\Delta^{(1)}$  tile, and that the quotient surface corresponds to the Hecke group  $G_6$ , which is known to be arithmetic. □

It follows that jigsaws composed of copies of  $J_A$  glued along matching sides correspond to finite covers of  $S_{J_A}$ , or equivalently to finite-index subgroups of  $\Gamma_{J_A}$ , and are all arithmetic. In fact, the converse is also true. We will need the following:

**Lemma 7.5** *Let  $P$  be an ideal polygon with  $n \geq 4$  sides and  $T$  a triangulation of  $P$ . Then there exist at least two nonadjacent vertices  $u$  and  $v$  of  $P$  with  $\text{wt}(u), \text{wt}(v) \in \{2, 3\}$  with respect to the triangulation.*

**Proof** Without loss of generality, we may assume that  $n \geq 5$ ; the case  $n = 4$  is trivial. Let  $\Delta_1, \dots, \Delta_a$  be the ears of  $P$ , and let

$$P' = P \setminus \bigcup_{i=1}^a \Delta_i.$$

If  $P'$  consists of only one triangle, then  $a = 2$  or  $3$  and a direct check shows the result holds. If  $P'$  contains at least two triangles, then it has at least two ears (which are not ears of  $P$ ) with tips  $w_i$  and  $w_j$  which are not adjacent in  $P'$ . Then  $w_i$  and  $w_j$  are nonadjacent vertices of  $P = P' \cup \bigcup_{i=1}^a \Delta_i$  with  $\text{wt}(w_i), \text{wt}(w_j) \in \{2, 3\}$ , as required.  $\square$

**Proposition 7.6** *An  $\mathcal{S}(1, 3)$ -jigsaw  $J$  is arithmetic if and only if it consists of a finite number of  $J_A$  pieces glued along matching sides.*

**Proof** We will show that an arithmetic jigsaw  $J$  can be decomposed into polygons of certain types which we call blocks. Call a polygon  $B$  assembled from  $\Delta^{(1)}$  and  $\Delta^{(3)}$  tiles glued along matching sides a *block* if

- (1) all the sides of  $B$  are type 3; and
- (2) (i)  $B$  consists of only  $\Delta^{(3)}$  tiles, in which case  $B$  consists of quadrilaterals made up of pairs of  $\Delta^{(3)}$  triangles glued along the type 1 side, or
- (ii)  $B$  contains a core  $C$  consisting of  $n \geq 1$   $\Delta^{(1)}$  tiles, with one  $\Delta^{(3)}$  tile glued to each of the sides of  $C$ , in which case all the ears of  $B$  are  $\Delta^{(3)}$  tiles and there are  $n + 2$  of them, and the weights of the vertices alternate between 1 and  $m \geq 3$ .

We call blocks of the first type 0-blocks, and blocks of the second type  $n$ -blocks, where  $n$  is the number of tiles in the core  $C$ . In particular, a 1-block is just  $J_A$  and our aim will be to show that other types of blocks cannot occur. Note that by definition, all the sides of a block are of type 3.

We now show that an arithmetic  $J$  can be decomposed into blocks defined above. By Proposition 7.4, all the sides of  $J$  are of type 3 and by Proposition 7.3, the  $J$  widths of all vertices of  $J$  are congruent to  $0 \pmod 3$ . Also, by definition,  $J$  contains at least one  $\Delta^{(1)}$  tile. We thus have a unique  $n$ -block  $B_1$  (where  $n \geq 1$ ) which contains this tile and  $J \setminus B_1$  consists of a finite number of polygons whose sides are again all of type 3. Proceeding inductively, we see that  $J$  can be decomposed into a finite number of blocks  $B_1, B_2, \dots, B_n$ .

Now, for a block  $B$ , if  $v$  is a vertex of  $B$  with  $J$ -width  $JW(v)$ , we say that  $v$  is bad if  $JW(v) \not\equiv 0 \pmod 3$  and good if  $JW(v) \equiv 0 \pmod 3$ . Then a 1-block has only good vertices since all its six vertices have  $J$ -width 3.

We claim that an  $n$ -block with  $n \neq 1$  has at least two nonadjacent bad vertices.

This is certainly true for the 0-blocks since each quadrilateral consisting of two  $\Delta^{(3)}$  tiles glued along their one sides have two bad vertices which are not adjacent. Gluing a finite number of these quadrilaterals along matching sides will not reduce this number of bad vertices. Indeed, each gluing fixes at most one bad vertex but introduces at least one new bad vertex which will not be adjacent to the unfixed bad vertices.

For  $n$ -blocks  $B$  where  $n \geq 2$ , we note that by Lemma 7.5, the core  $C$  has at least two nonadjacent vertices  $u$  and  $v$  with  $J$ -width equal to 2 or 3. Then the  $J$ -width of the corresponding vertices on  $B$  will be 4 or 5 (we add 2 to the original widths coming from the  $J$ -widths of the ears). It follows that  $B$  has at least two bad vertices which are not adjacent.

Hence, if  $J$  has an  $n$ -block  $B$  with  $n \neq 1$ , then  $B$  has at least two nonadjacent bad vertices. If we glue a  $J_A$  block to  $B$ , then the two bad vertices persist and remain nonadjacent; if we glue an  $n$ -block with  $n \neq 1$  to  $B$ , then we fix at most one bad vertex of  $B$  but introduce at least one new bad vertex which will not be adjacent to the unfixed bad vertex of  $B$ . Thus, adjoining another block to  $B$  results in a jigsaw with at least two bad vertices which are not adjacent. The same argument holds when we glue another block to this new jigsaw. Hence, after a finite number of steps, we will always end up with a jigsaw  $J$  which has at least two nonadjacent bad vertices. By Proposition 7.3,  $J$  cannot be arithmetic. Hence, an arithmetic  $J$  consists only of  $J_A$  blocks and the result follows.  $\square$

## 8 Commensurability classes

By Margulis's result [6], if  $\Gamma < \mathrm{PSL}(2, \mathbb{R})$  is nonarithmetic, then  $\mathrm{Comm}(\Gamma)$  is the unique minimal element in the commensurability class  $[\Gamma]$ , where



$$\text{Comm}(\Gamma) = \{g \in \text{PSL}(2, \mathbb{R}) \mid \Gamma \text{ and } g\Gamma g^{-1} \text{ are commensurable}\},$$

and  $\Gamma < \text{Comm}(\Gamma)$  with finite index. We will show that for all  $\mathcal{S}(1, 2)$ - or  $\mathcal{S}(1, 3)$ -jigsaws  $J$  of signature  $(r, 1)$ ,

$$\text{Comm}(\Gamma_J) = \Gamma_J,$$

which will be enough to show that there are infinitely many commensurability classes of pseudomodular jigsaw groups.

If  $\Gamma_J$  is a pseudomodular hyperbolic jigsaw group, let  $\Gamma'_J := \text{Comm}(\Gamma_J)$  be the minimal element in the commensurability class of  $[\Gamma_J]$ ,  $S'_J$  the associated surface and  $C$  the maximal horocycle on  $S'_J$ . Let  $\tilde{C}$  be the lift of  $C$  to  $\mathbb{H}^2$ . By construction,  $\tilde{C}$  is invariant under the action of  $\Gamma'_J$ . By looking at the lift of  $C$  centered at  $\infty$  (that is, the horizontal lift), it is easy to see that the relative lengths of the intersection arcs of the horocycles with the triangles are equal to the relative  $J$ -widths (see Definition 4.4). Also, since  $S_J$  has only one cusp, the lift of the maximal horocycle of  $S_J$  to  $\mathbb{H}^2$  is also  $\tilde{C}$ . We have:

**Proposition 8.1** *There exist infinitely many commensurability classes of pseudomodular  $\mathcal{S}(1, 2)$ -jigsaw groups.*

**Proposition 8.2** *There exist infinitely many commensurability classes of pseudomodular  $\mathcal{S}(1, 3)$ -jigsaw groups and infinitely many commensurability classes of  $\mathcal{S}(1, 3)$ -jigsaw groups with specials (so they are neither arithmetic nor pseudomodular).*

**Proof of Propositions 8.1 and 8.2** First consider an  $\mathcal{S}(1, 2)$ -jigsaw  $J$  of signature  $(r, 1)$ . The fundamental interval has (euclidean) length  $3r + 4$  and  $T^{3r+4} \in \Gamma_J$ . The maximal horocycle  $C$  on  $S_J$  has (hyperbolic) length  $(3r + 4)/\sqrt{2}$  and intersects itself at the fixed points of the sides labeled by 2 and  $\frac{1}{2}$ . It decomposes into two pieces, a short piece of hyperbolic length  $\sqrt{2}$  (the length of the horocyclic segment joining  $a + i\sqrt{2}$  to  $a + 2 + i\sqrt{2}$ ), and a long piece of hyperbolic length  $(3r + 2)/\sqrt{2}$ , where  $(\infty, a)$  and  $(\infty, a + 2)$  are the lifts of two successive vertical sides of  $\mathcal{T}$  of type 2. Then  $\tilde{C}$  consists of the horizontal horocycle at height  $\sqrt{2}$ , “large horocycles” of diameter  $\frac{\sqrt{2}}{2}$  based at the points  $a + n(3r + 4)$ ,  $a + 2 + n(3r + 4) \in \mathbb{R}$ , where  $n \in \mathbb{Z}$ , which are tangent to the horizontal horocycle, and smaller horocycles based at the other cusps which are disjoint from the horizontal horocycle. In other words, the horizontal lift of  $C$  has exactly two tangency points in each fundamental interval dividing it into two pieces of length  $\sqrt{2}$  and  $(3r + 2)/\sqrt{2}$ , respectively. We claim

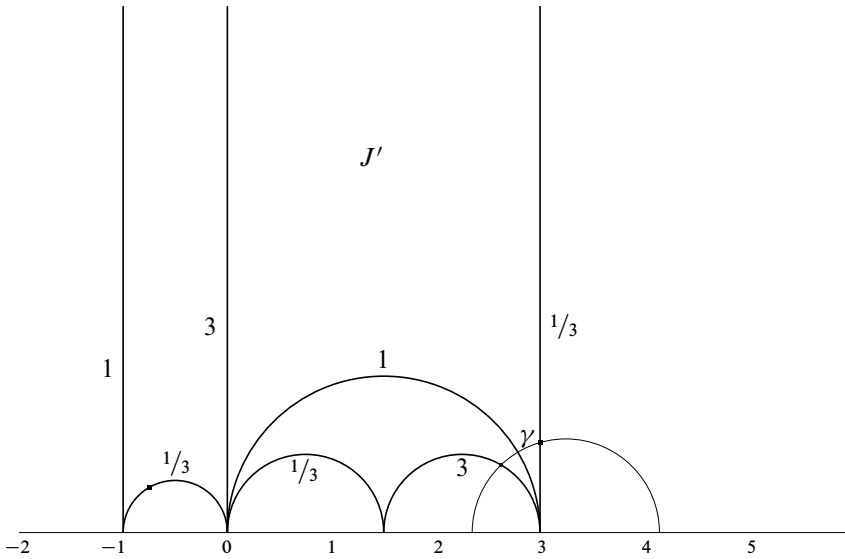


Figure 4: The hyperbolic jigsaw  $J'$  consisting of three  $\Delta^{(3)}$  tiles and the geodesic arc on  $\Gamma_{J'}$  which has cutting sequence  $3, 1, \frac{1}{3}, 1$

that  $\Gamma_J = \Gamma'_J$ . If not, since  $\Gamma_J$  has only one cusp,  $\tilde{C}$  is invariant under a horizontal translation by  $k$ , where  $0 < k < 3r + 4$ , but this is impossible as such a translation cannot preserve the pattern of tangency points at the horizontal lift of  $C$ . It follows that each  $\mathcal{S}(1, 2)$ -jigsaw group  $\Gamma_J$  of signature  $(r, 1)$  is the unique minimal element in its conjugacy class and that two nonisometric jigsaws with only one  $\Delta^{(2)}$  tile give rise to pseudomodular groups in different commensurability classes. This gives infinitely many commensurability classes of pseudomodular groups.

The same argument with minor modifications works for  $\mathcal{S}(1, 3)$ -jigsaws. We only need to note that if  $J$  is an  $\mathcal{S}(1, 3)$ -jigsaw of signature  $(r, 1)$ , then it must be pseudomodular since it does not contain a geodesic with cutting sequence  $1, 3, 1, \frac{1}{3}$ , and it has a side of type 1. To show the existence of infinitely many commensurability classes of nonpseudomodular  $\mathcal{S}(1, 3)$ -jigsaw groups, we only need to start with a jigsaw  $J'$  consisting of three  $\Delta^{(3)}$  triangles such that there exists a geodesic  $\gamma$  on  $S_{J'}$  with  $W$ -cutting sequence  $1, \frac{1}{3}, \frac{1}{3}$ , and  $J'$  has a side which does not intersect  $\gamma$  with label 1; see Figure 4. We then glue another jigsaw  $J_n$  to  $J'$  along this side, where  $J_n$  consists of  $n$   $\Delta^{(1)}$  tiles. The jigsaw groups constructed will all be nonarithmetic and nonpseudomodular. Again, the resulting jigsaw groups coming from  $J'_n = J' \cup J_n$  lie in infinitely many commensurability classes, by a similar argument.  $\square$

## 9 Conclusion

Propositions 5.1, 6.1, 7.2, 7.3, 8.1 and 8.2 now imply Theorems 2.4 and 2.5, which in turn both imply Theorem 1.2. One sees that integral hyperbolic jigsaws provide a rich source of pseudomodular groups. We will be exploring this in a subsequent paper [5], where we will show that for any integral hyperbolic jigsaw set  $\mathcal{S}$ , there are infinitely many commensurability classes of pseudomodular  $\mathcal{S}$ -jigsaw groups. We will also be exploring the related pseudoeuclidean algorithm as well as the generalized continued fraction expansion for these groups as well as for the integral Weierstrass groups. Many questions remain; we conclude with a partial list:

- (1) Can one give sharp invariants that will distinguish the commensurability classes of the pseudomodular groups, for example, by using the associated labeled graph  $G_J$  of  $J$ . In particular, we may define an  $\mathcal{S}$ -jigsaw to be prime if it cannot be decomposed into pieces all of which are isometric. Is it then true that  $\text{Comm}(\Gamma_J) = \Gamma_J$  if and only if  $J$  is prime?
- (2) Can one find integral jigsaw groups where the set of fixed points of the hyperbolics is exactly the set of quadratic irrationals? Or where some quadratic irrational does not lie in this set? More generally, what can be said about the fixed point set of the hyperbolics for a given jigsaw group?
- (3) Are there only finitely many pseudomodular Weierstrass groups  $\Gamma(1, \frac{1}{n}, n)$  where  $n \in \mathbb{N}$ ? Which Weierstrass groups are arithmetic?
- (4) What integral jigsaw groups can be arithmetic? It seems likely that the arithmetic integral jigsaw groups only come from a small finite collection of jigsaw sets. If so, can we determine this collection completely?
- (5) One can consider jigsaw tiles  $\Delta(k_1, k_2, k_3)$  where  $k_i \in \mathbb{Q}[i]$ , which results in groups  $\Gamma_J \in \text{PSL}(2, \mathbb{C})$ . What can we say about these groups? Is it possible to construct a jigsaw with cusp set  $\mathbb{Q}[i] \cup \{\infty\}$ ?

## References

- [1] **O Ayaka**, *A computer experiment on pseudomodular groups*, Master's thesis, Nara Women's University (2015)
- [2] **D Fithian**, *Congruence obstructions to pseudomodularity of Fricke groups*, C. R. Math. Acad. Sci. Paris 346 (2008) 603–606 MR

- [3] **H M Hilden, M T Lozano, J M Montesinos-Amilibia**, *A characterization of arithmetic subgroups of  $SL(2, \mathbb{R})$  and  $SL(2, \mathbb{C})$* , Math. Nachr. 159 (1992) 245–270 MR
- [4] **DD Long, A W Reid**, *Pseudomodular surfaces*, J. Reine Angew. Math. 552 (2002) 77–100 MR
- [5] **B Lou, S P Tan, A D Vo**, *Hyperbolic jigsaws and families of pseudomodular groups, II: Integral jigsaws, generalized continued fractions and arithmeticity*, in preparation
- [6] **G A Margulis**, *Discrete subgroups of semisimple Lie groups*, Ergeb. Math. Grenzgeb. 17, Springer (1991) MR
- [7] **H Proskin**, *Flat faces in punctured torus groups*, Rocky Mountain J. Math. 37 (2007) 2025–2051 MR
- [8] **K Takeuchi**, *A characterization of arithmetic Fuchsian groups*, J. Math. Soc. Japan 27 (1975) 600–612 MR
- [9] **J W Tan**, *The study of pseudomodular groups and the computations*, Undergraduate honours thesis, National University of Singapore (2016)

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