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Normalized entropy versus volume for pseudo-Anosovs

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Thanks to a recent result by Jean-Marc Schlenker, we establish an explicit linear inequality between the normalized entropies of pseudo-Anosov automorphisms and the hyperbolic volumes of their mapping tori. As corollaries, we give an improved lower bound for values of entropies of pseudo-Anosovs on a surface with fixed topology, and a proof of a slightly weaker version of the result by Farb, Leininger and Margalit first, and by Agol later, on finiteness of cusped manifolds generating surface automorphisms with small normalized entropies. Also, we present an analogous linear inequality between the Weil–Petersson translation distance of a pseudo-Anosov map (normalized by multiplying by the square root of the area of a surface) and the volume of its mapping torus, which leads to a better bound.

57M27; 37E30

1 Introduction

Let $\Sigma = \Sigma_{g,m}$ be an orientable surface of genus g with m punctures. We will suppose that $3g-3+m \geq 1$, so that Σ admits a Riemannian metric of constant curvature -1, a hyperbolic structure of finite area, which, by Gauss–Bonnet, satisfies $\operatorname{Area} \Sigma = 2\pi|\chi(\Sigma)| = 2\pi(2g-2+m)$ with respect to the hyperbolic metric.

The isotopy classes of orientation-preserving automorphisms of Σ, called mapping classes, were classified into three families by Nielsen and Thurston [24], namely periodic, reducible and pseudo-Anosov. Choose a representative h of a mapping class φ, and consider its mapping torus

$$\Sigma \times [0,1]/(x,1) \sim (h(x),0).$$

Since the topology of the mapping torus depends only on the mapping class φ, we denote its topological type by N_φ.

A celebrated theorem by Thurston [25] asserts that N_φ admits a hyperbolic structure if and only if φ is pseudo-Anosov. By Mostow–Prasad rigidity, a hyperbolic structure of

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finite volume in dimension 3 is unique, and geometric invariants are in fact topological invariants. Kin, Kojima and Takasawa [13] compared the hyperbolic volume of N_φ, denoted by $\operatorname{vol} N_\varphi$, with the entropy of φ, denoted by $\operatorname{ent}\varphi$. By *entropy*, we mean the infimum of the topological entropy of automorphisms isotopic to φ. In particular, they proved that there is a constant $C(g,m) > 0$ depending only on the topology of Σ such that

$$\operatorname{ent}\varphi \geq C(g,m) \operatorname{vol} N_\varphi.$$

This result only asserts the existence of a constant $C(g,m)$ since the proof is based on a result of Brock [5] involving several constants for which, a priori, it appears difficult to compute sharp values. On the other hand, it is well known that the infimum of $\operatorname{ent}\varphi / \operatorname{vol} N_\varphi$ is 0. In fact, Penner constructed examples [19] which demonstrate that, as the complexity of surface increases, the entropy of a pseudo-Anosov can be arbitrarily close to 0. By Jørgensen–Thurston theory [23], the infimum of volumes of hyperbolic 3–manifolds is strictly positive, so $C(g,m)$ necessarily tends to 0 when $g+m \to \infty$.

Our main theorem gives an explicit value for $C(g,m)$.

Theorem 1.1 *The inequality*

$$\operatorname{ent}\varphi \geq \frac{1}{3\pi|\chi(\Sigma)|} \operatorname{vol} N_\varphi, \tag{1-1}$$

or equivalently,

$$2\pi|\chi(\Sigma)| \operatorname{ent}\varphi \geq \tfrac{2}{3} \operatorname{vol} N_\varphi, \tag{1-2}$$

holds for any pseudo-Anosov φ.

The quantity appearing on the left-hand side of (1-2) is often referred to as the *normalized entropy*. The main theorem can thus be restated informally: the normalized entropy over the volume is bounded from below by a positive constant which does not depend on the topology of Σ.

The value of $C(g,m)$ above does not seem to be quite far from the sharp constant. For example, choose the case that the surface is the punctured torus, so that $g=1$, $m=1$ and $|\chi(\Sigma_{1,1})| = 1$. Then the inequality (1-1) becomes

$$\frac{\operatorname{ent}\varphi}{\operatorname{vol} N_\varphi} \geq \frac{1}{3\pi} = 0.10610\ldots.$$

In this particular case, it is conjectured (see Conjecture 6.10 in [13] with supporting evidence) that

$$\frac{\operatorname{ent}\varphi}{\operatorname{vol}N_\varphi} \geq \frac{\log((3+\sqrt{5})/2)}{2v_3} = 0.47412\ldots,$$

where $v_3 = 1.01494\ldots$ is the volume of the hyperbolic regular ideal simplex. The conjectured constant above is known to be attained by the figure-eight knot complement, which admits a unique $\Sigma_{1,1}$–fibration.

The first application of Theorem 1.1 is an amelioration of the lower bound

$$\operatorname{ent}\varphi \geq \frac{\log 2}{4(3g-3+m)}$$

for the entropy of pseudo-Anosovs on a surface due to Penner [19], provided that there is at least one puncture.

Corollary 1.2 *Let φ be a pseudo-Anosov on $\Sigma_{g,m}$ with $m \geq 1$. Then*

$$\operatorname{ent}\varphi \geq \frac{2v_3}{3\pi|\chi(\Sigma)|} = \frac{2v_3}{3\pi(2g-2+m)}.$$

Proof It is known by Cao and Meyerhoff [7] that the smallest volume of an orientable noncompact hyperbolic 3–manifold is attained by the figure-eight knot complement, and it is $2v_3$. Thus replacing $\operatorname{vol}N_\varphi$ in (1-1) by $2v_3$, we obtain the estimate. □

If a manifold admits a fibration over the circle, its first Betti number is necessarily positive. It is conjectured that the smallest volume of a hyperbolic 3–manifold with positive first Betti number is also $2v_3$. If it were true, then we could drop the assumption of $m \geq 1$ on the number of punctures in Corollary 1.2.

The second application of our main theorem is the proof of a slightly weaker form of Farb, Leininger and Margalit's finiteness theorem for small dilatation pseudo-Anosovs.

Corollary 1.3 (Farb, Leininger and Margalit [10]; Agol [1]) *For any $C > 0$, there are finitely many cusped hyperbolic 3–manifolds M_k such that any pseudo-Anosov φ on Σ with $|\chi(\Sigma)|\operatorname{ent}\varphi < C$ can be realized as the monodromy of a fibration on a manifold obtained from one of the M_k by an appropriate Dehn filling.*

We note that Farb, Leininger and Margalit are also able to obtain that the M_k are in fact fibered, and that the surgeries (fillings) are along suspensions of punctures of the fiber.

Proof If $|\chi(\Sigma)|\operatorname{ent}\varphi$ is bounded from above by a constant C, then it certainly bounds the volume of N_φ by Theorem 1.1. Recall that the thin part of N_φ consists of neighborhoods of (rank-2) cusps and Margulis tubes around short geodesics, all pairwise disjoint. Thus the boundary of the *thick part*, that is, the complement of the thin part, consists of finitely many tori, and N_φ is obtained from the thick part by Dehn filling these. The volume of N_φ bounds the volume of the thick part and, using a covering by (finitely many) metric balls, Jørgensen and Thurston [23] have shown that there are only finitely many possibilities for its topological type. □

Finally, replacing the entropy by Weil–Petersson translation distance in the proof of Theorem 1.1 yields an explicit value for the constant appearing in the upper bound for volume of the mapping torus in Brock's [5, Theorem 1.1].

Theorem 1.4 *If Σ is compact, then the inequality*

$$\|\varphi\|_{\mathrm{WP}} \geq \frac{2}{3\sqrt{2\pi|\chi(\Sigma)|}} \operatorname{vol} N_\varphi,$$

or equivalently,

$$\sqrt{2\pi|\chi(\Sigma)|}\,\|\varphi\|_{\mathrm{WP}} \geq \tfrac{2}{3} \operatorname{vol} N_\varphi,$$

holds for any pseudo-Anosov φ, where $\|\cdot\|_{\mathrm{WP}}$ is the Weil–Petersson translation distance of φ.

It was pointed out to the authors by McMullen that there is a family of pseudo-Anosov automorphisms φ_k such that $\|\varphi_k\|_{\mathrm{WP}}$ are bounded whilst the entropy of φ_k, which is just the translation distance for the Teichmüller metric, diverges. In fact, one can construct such examples so that the mapping tori N_{φ_k} converge to a cusped hyperbolic 3–manifold (though this limit may not be a surface bundle). This can be interpreted as showing that the relationship between volume and Weil–Petersson distance is stronger than that with the Teichmüller distance. Indeed, Brock has shown that there is a lower bound for volume in terms of $\|\varphi\|_{\mathrm{WP}}$, but the method that we present does not as yet extend to prove this.

The proof of Theorem 1.1 has two main ingredients: The first is the recent results by Krasnov and Schlenker [15] and Schlenker [21] on the renormalized volumes of quasi-Fuchsian manifolds. The second is the work of McMullen [16] and Brock and Bromberg [6] on geometric inflexibility to obtain convergence in Thurston's double limit theorem. In the next section, we review briefly the requisite results of Krasnov and

Schlenker and obtain an intermediate inequality (Corollary 2.8) between the volume of the convex core of a quasi-Fuchsian manifold and Teichmüller distance. In Section 3, we prove Theorem 1.1 and Theorem 1.4. In the appendix, we present a simplified exposition of the ideas behind geometric inflexibility and a proof of the convergence result stated in Brock [5] which leads to a proof of the main theorem.

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2 Preliminaries

2.1 Differentials

Let R be a Riemann surface, and let $T^{1,0}R$ and $T^{0,1}R$ respectively denote the holomorphic and the antiholomorphic parts of the complex cotangent bundle, a canonical bundle over R.

A quadratic differential q on R locally expressed by $q(z)\,dz^2$ is a section of the line bundle $(T^{1,0}R)^{\otimes 2}$. A Beltrami differential μ on R locally expressed by $\mu(z)\,d\bar{z}/dz$ is a section of the line bundle $T^{0,1}R \otimes (T^{1,0})^* R$. They are main players in Teichmüller theory. A Beltrami differential μ can be interpreted as a representative of an infinitesimal deformation of the complex structure on R. Hence to each μ, there is a corresponding tangent vector to Teichmüller space $\mathcal{T} = \mathcal{T}_{g,m}$ at R; however, there is an infinite-dimensional subspace that represents the trivial deformation, so the correspondence is not injective. We now describe a construction which allows one to eliminate this ambiguity.

Let $Q(R)$ be the space of *holomorphic* quadratic differentials. By Riemann–Roch, the dimension of $Q(R)$ is $3g-3+m$, that is, equal to that of $\mathcal{T}$. We define the L^1–norm on $Q(R)$ by

$$\|q\|_1 = \int_R |q|.$$

We note that, since it is finite-dimensional, all norms on $Q(R)$ are equivalent, and we shall compare the L^1–norm to another norm, the L^∞–norm, in Section 2.6.

Given a Beltrami differential μ and a quadratic differential q, the product of μ and q is a section of the line bundle $T^{0,1}R \otimes T^{1,0}R$, and there is a natural pairing defined by

$$(q,\mu) = \int_R \mu q.$$

Let $L_\infty(R)$ be the space of uniformly bounded Beltrami differentials with respect to the norm

$$\|\mu\|_\infty = \sup_{\|q\|_1 = 1} |(q,\mu)|;$$

namely, $L_\infty(R) = \{\mu : \|\mu\|_\infty < \infty\}$. Define K to be the subspace of $L_\infty(R)$

$$K = \{\mu : (q,\mu) = 0 \text{ for all } q \in Q(R)\}.$$

Then $L_\infty(R)/K$ can be identified with the tangent space $T_R\mathcal{T}$ of the Teichmüller space $\mathcal{T}$ at R, and moreover, the pairing induces an isomorphism of $Q(R)^*$ with $L_\infty(R)/K \cong T_R\mathcal{T}$. Thus $Q(R)$ can be regarded as a cotangent space $T_R^*\mathcal{T}$ of $\mathcal{T}$ at R, and $(\cdot,\cdot)$ induces the duality pairing

$$(\cdot,\cdot)\colon Q(R) \times L_\infty(R)/K \to \mathbb{C}. \tag{2-1}$$

2.2 Projective structures

A projective structure on a surface Σ is a special type of complex structure locally modeled on the geometry of the complex projective line $(\widehat{\mathbb{C}}, \mathrm{PSL}(2,\mathbb{C}))$, where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. The projective structure on Σ is given by an atlas such that each transition map is the restriction of some element in $\mathrm{PSL}(2,\mathbb{C})$. Clearly these transition maps are holomorphic, and so there is a unique complex structure naturally associated to a given projective structure. Let X be a surface homeomorphic to Σ together with a projective structure, and let R be its underlying Riemann surface. When $\chi(\Sigma) < 0$, there is a bijection between projective structures and holomorphic quadratic differentials. To see this, recall that by the uniformization theorem, the universal cover of X can be identified with the Poincaré disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The developing map $f\colon \widetilde{X} \to \widehat{\mathbb{C}}$ can be regarded as a meromorphic function on $\mathbb{D}$. Now the Schwarzian derivative $S(f)$ of f defines a holomorphic quadratic differential q on R. Conversely, given a holomorphic quadratic differential q on R, the Schwarzian differential equation

$$S(f) = q$$

has a solution which gives rise to a developing map of some complex projective structure on R. Thus, there is a one-to-one correspondence between the set of all complex projective structures on R and $Q(R)$.

2.3 Quasi-Fuchsian manifolds

A quasi-Fuchsian group is defined to be a discrete subgroup Γ of $\mathrm{PSL}(2,\mathbb{C})$ such that its limit set L_Γ of Γ on the boundary $\partial\mathbb{H}^3 = S^2_\infty$ is either a circle or a quasicircle (an embedded copy of the circle with Hausdorff dimension strictly greater than 1). This definition implies many consequences. For example, the domain of discontinuity $\Omega_\Gamma = S^2_\infty - L_\Gamma$ consists of exactly two simply connected domains, denoted by Ω^+_Γ and Ω^-_Γ, the quotients of $\Omega^\pm_\Gamma$ by Γ are the same 2–orbifold O but with opposite orientations, and $\mathbb{H}^3/\Gamma$ is a geometrically finite hyperbolic 3–orbifold homeomorphic to $O\times\mathbb{R}$. If Γ is torsion free, then O becomes a surface Σ, and in particular, Γ is isomorphic to the fundamental group of Σ. In this case, we say that the quotient of $\mathbb{H}^3$ by the quasi-Fuchsian group Γ is a *quasi-Fuchsian manifold.*

The action of Γ on Ω_Γ is holomorphic, so $X = \Omega^+_\Gamma/\Gamma$ and $Y = \Omega^-_\Gamma/\Gamma$ are marked Riemann surfaces. Thus, there is a well-defined map taking a quasi-Fuchsian manifold $\mathbb{H}^3/\Gamma$ to a pair of marked Riemann surfaces X, Y in the Teichmüller space $\mathcal{T}$ of Σ. In [2], Bers showed that this map has an inverse. He obtains as a corollary a parametrization of the set of quasi-Fuchsian manifolds by $\mathcal{T}\times\overline{\mathcal{T}}$. In other words, for any pair $(X,Y)\in\mathcal{T}\times\overline{\mathcal{T}}$, there is a unique quasi-Fuchsian manifold $\mathrm{QF}(X,Y)$. As noted above, the limit set of a quasi-Fuchsian group is either a circle or a quasicircle, and the quotient of its convex hull in $\mathbb{H}^3$ by Γ, denoted by $C(X,Y)$, is homeomorphic to Σ or $\Sigma\times[0,1]$ accordingly, called the *convex core*. It is known to be the smallest convex subset homotopy equivalent to $\mathrm{QF}(X,Y)$.

Since the action of a quasi-Fuchsian group Γ on Ω_Γ is linear fractional, the Riemann surfaces X and Y are equipped not only with complex structures but also with complex projective structures. Thus we have associated holomorphic quadratic differentials q_X and q_Y. Let q denote the unique holomorphic quadratic differential on $X\sqcup Y$ such that its restriction to X is q_X and to Y is q_Y.

The notation may be a bit misleading since q_X and q_Y could both vary even if one of the complex structures of X or Y stays constant. However, as long as we are discussing quasi-Fuchsian deformations, we regard q_X as a complex projective structure of the Riemann surface on the left and q_Y on the right. This convention resolves any possible confusion of notation.

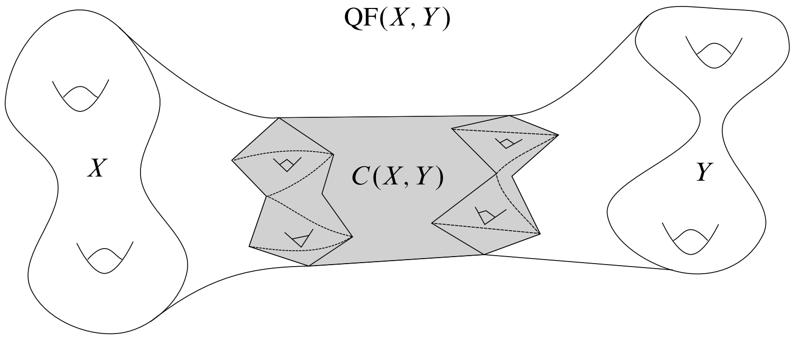

Figure 1: The quasi-Fuchsian manifold, the surfaces at infinity and the convex core

2.4 Renormalization of volume

Renormalization of the volume of convex cocompact hyperbolic 3–manifolds were studied extensively by Krasnov and Schlenker in [15]. In the following sections, we recall Krasnov and Schlenker's results focusing on the quasi-Fuchsian case. Note that the surface at infinity Ω_Γ/Γ has two connected components.

Throughout the rest of this section, we assume that Σ is compact. Let M be a quasi-Fuchsian manifold $\mathbb{H}^3/\Gamma$ homeomorphic to $\Sigma \times \mathbb{R}$. Following [15], we say that a codimension-zero smooth compact convex submanifold $N \subset M$ is *strongly convex* if the normal hyperbolic Gauss map from $\partial N = \partial_+ N \sqcup \partial_- N$ to the boundary at infinity Ω_Γ/Γ is a homeomorphism. For example, a closed ε–neighborhood of the convex core of a quasi-Fuchsian manifold is strongly convex. Let $S_0 = \partial N$ then there is a family of surfaces $\{S_r\}_{r \geq 0}$ equidistant to S_0 foliating the ends of M. If g_r denotes the induced metric on S_r, then define a metric at infinity associated to the family $\{S_r\}_{r \geq 0}$ by

$$g = \lim_{r \to \infty} 2e^{-2r} g_r.$$

The resulting metric g in fact belongs to the *conformal class at infinity* that is the conformal structure determined by the complex structure on Ω_Γ/Γ. It is easy to see that if we start with a strongly convex submanifold bounded by S_{r_0} for some $r_0 > 0$, then the limiting metric is $e^{2r_0} g$. Namely, if we shift the parametrization of an equidistant foliation by r_0, then the limiting metric changes only by scaling e^{2r_0}.

Conversely, if g is a Riemannian metric in the conformal class at infinity, then [15, Theorem 5.8] shows that there is a unique foliation of the ends of M by equidistant surfaces with compatible parametrization of leaves starting $r_0 \geq 0$ such that the associated metric at infinity is equal to g. Notice that the parametrization may have to start with a positive r_0. The construction of a foliation is due to Epstein [8]. Then, a natural quantity to study in the context of strongly convex submanifolds $N \subset M$ is the *W–volume* defined by

$$W(M, g) := \operatorname{vol} N_r - \frac{1}{4}\int_{S_r} H_r \, da_r + \pi r \chi(\partial M), \tag{2-2}$$

where the parametrization r is induced by g, N_r is a strongly convex submanifold bounded by the associated leaf S_r, H_r is the mean curvature of S_r and da_r is the induced area form of S_r. A simple computation which can be found in [21] shows that the W–volume depends only on the metric at infinity g, justifying the notation.

The renormalized volume of M is now defined by

$$\operatorname{Rvol}(M) := \sup_g W(M, g),$$

where the supremum is taken over all metrics g in the conformal class at infinity such that the area of each surface at infinity Ω_Γ/Γ with respect to g is $2\pi|\chi(\Sigma)|$. Section 7 in [15] presents an argument, based on the variational formula stated in Corollary 6.2 in [15], that the supremum is in fact uniquely attained by the metric of constant curvature -1.

We can now state, in a slightly modified form, that one half of Theorem 1.1 in [21] is as follows.

Theorem 2.1 ([21, Theorem 1.1] and its revised version in [22]) *Assume that Σ is compact. Then there exists a constant $D = D(\Sigma) > 0$ depending only on the topology of Σ such that the inequality*

$$\operatorname{vol} C(X, Y) \le \operatorname{Rvol} \mathrm{QF}(X, Y) + D \tag{2-3}$$

holds for any $X, Y \in \mathcal{T}$.

2.5 Variational formula

In [21], the metric at infinity of a quasi-Fuchsian manifold M which attains the renormalized volume is denoted by I^*. In particular, the curvature of I^* is constant -1. The notation is consistent with the standard one for the first fundamental form. There is also an analogous notion II^* of the second fundamental form on the surface at infinity with the same parametrization with appropriate scaling factor. More precisely, there is a unique bundle morphism B^* of the tangent space of the boundary, corresponding to the shape operator, which is self-adjoint for I^* and such that

$$\mathrm{II}^* = \mathrm{I}^*(B^* \cdot, \cdot).$$

The variational formula of the renormalized volume involves II^*, $\dot{\mathrm{I}}^*$ and a Riemannian metric of constant curvature -1 in the conformal class of the boundary. They all are symmetric 2–tensors. In general, if we choose a local complex coordinate $z = x + iy$, then we can express a symmetric 2–tensor using the associated real coordinate (x, y) by $a(dx \otimes dx) + b(dx \otimes dy + dy \otimes dx) + c(dy \otimes dy)$ and hence by a symmetric matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Corollary 6.2 in [15] states the variational formula of the W–volume as follows.

Lemma 2.2 ([21, Proposition 3.10] and its revised version in [22]) *Under a first-order deformation of the hyperbolic structure on N,*

$$d\mathrm{Rvol} = -\tfrac{1}{4}\langle \mathrm{II}_0^*, \dot{\mathrm{I}}^* \rangle$$

holds, where $\dot{\mathrm{I}}^*$ is a variation of the metric. Here $\langle \cdot, \cdot \rangle$ is the extension to symmetric 2–tensors of the Riemannian metric $\rho^2 |dz|^2$ of constant curvature -1 defined by

$$\langle A, B \rangle = \int_R \mathrm{tr}(G^{-1} A G^{-1} B) \rho^2 \, dx \, dy,$$

where G is the metric tensor, and II_0^* is a trace-free part of II^*.

Krasnov and Schlenker found a remarkable relation between II_0^*, the trace-free part of II^*, and the holomorphic quadratic differential q corresponding to the projective structure of the boundary. To see this more precisely, recall that q has a local expression $q = q(z)\, dz^2$ and $q(z) = f(z) + i h(z)$. Then

$$\mathrm{Re}\, q = f\, dx \otimes dx - f\, dy \otimes dy - h(dx \otimes dy + dy \otimes dx)$$

can be expressed as a symmetric 2–tensor by a trace-free symmetric matrix

$$\begin{pmatrix} f & -h \\ -h & -f \end{pmatrix}.$$

The identity below follows directly from explicit formulae for the holomorphic quadratic differential q in question. Another more geometric proof can also be found in the appendix of [15].

Lemma 2.3 [15, Lemma 8.3] $\mathrm{II}_0^* = -\operatorname{Re} q$.

Notice that $q = (f + ih)\,dz^2$ can be recovered from $\operatorname{Re} q$.

Passing through the identification "between 2–tensors and holomorphic quadratic differentials" and using the identification of $Q(R)$ with $T_R\mathcal{T}$ with respect to the metric $\rho^2|dz|^2$ (see for instance Lemma 7.7.5 in [12]), the variation of the metric

$$\dot{\mathrm{I}}^* = \begin{pmatrix} 2\varphi & -2\psi \\ -2\psi & -2\varphi \end{pmatrix}$$

can be transformed to the Beltrami differential

$$\mu = \frac{\varphi - i\psi}{\rho^2}\frac{d\bar{z}}{dz}. \tag{2-4}$$

This leads us to the reinterpretation of the variational formula in Lemma 2.2 in terms of the duality pairing $(\cdot,\cdot)$ in (2-1).

Lemma 2.4 *Under a first-order deformation of the hyperbolic structure on N, we have*

$$d\mathrm{Rvol} = -\operatorname{Re}(q,\mu) = -\int_R \operatorname{Re}\mu q.$$

Proof Fix a local coordinate $z = x + iy$ and let $\rho^2|dz|^2$ denote the hyperbolic metric. Then using (2-4), one obtains

$$\begin{aligned}\langle \mathrm{II}_0^*, \dot{\mathrm{I}}^*\rangle &= \int_R \operatorname{tr}(G^{-1}\mathrm{II}_0^* G^{-1}\dot{\mathrm{I}}^*)\rho^2\,dx\,dy \\ &= 4\int_R (f\varphi + h\psi)\rho^{-2}\,dx\,dy \\ &= 4\int_R \operatorname{Re}\Big(\frac{\varphi - i\psi}{\rho^2}(f+ih)\Big)\,dx\,dy \\ &= 4\int_R \operatorname{Re}\mu q.\end{aligned}$$

□

2.6 Rvol versus Teichmüller distance

We start with a quasi-Fuchsian manifold $\mathbb{H}^3/\Gamma = \mathrm{QF}(X,Y)$. Fix a conformal structure X on the left boundary component, and regard a conformal structure Y on the right as a variable. To each Y, we assign an associated complex projective structure on X and therefore a holomorphic quadratic differential q_X. This defines a map called a Bers embedding:

$$B_X\colon \mathcal{T} \to Q(X).$$

Using the hyperbolic metric in the conformal class of X, we can measure at each point of Σ the norm of q_X. Let $Q^\infty(X)$ be $Q(X)$ endowed with the L^∞–norm, namely

$$\|q\|_\infty = \sup_{x \in X} \frac{|q(x)|}{\rho^2(x)},$$

where $\rho|dz|$ defines the hyperbolic metric of constant curvature -1.

The following theorem with respect to the hyperbolic metric of constant curvature -1, due to Nehari, can be found in a standard textbook of the Teichmüller theory such as [11, Theorem 1, page 134].

Theorem 2.5 [18] *The image of B_X in $Q^\infty(X)$ is contained in the ball of radius $\frac{3}{2}$.*

Now consider the L^1–norm on $Q(X)$, and denote by $Q^1(X)$ the vector space $Q(X)$ endowed with the L^1–norm.

Corollary 2.6 *The image of B_X in $Q^1(X)$ is contained in the ball of radius $3\pi|\chi(\Sigma)|$.*

Proof The inequality

$$\|q\|_1 = \int_X |q| = \int_X \frac{|q|}{\rho^2}\rho^2 \le \|q\|_\infty \int_X \rho^2 = 2\pi|\chi(\Sigma)|\|q\|_\infty$$

immediately implies the conclusion. □

The proof of the following comparison result is the same as that of Theorem 1.2 in [21] by Schlenker with a different norm.

Proposition 2.7 *Suppose Σ is compact. The inequality*

$$\operatorname{Rvol} \mathrm{QF}(X,Y) \le 3\pi|\chi(\Sigma)|\, d_{\mathrm{T}}(X,Y) \tag{2-5}$$

holds for any quasi-Fuchsian manifold $\mathrm{QF}(X,Y)$, where d_{T} is the Teichmüller distance on $\mathcal{T}$.

Proof Let $Y\colon [0,d] \to \mathcal{T}$ be the unit speed Teichmüller geodesic joining X and Y, so that, in particular, $Y(0) = X$, $Y(d) = Y$ and $d = d_{\mathrm{T}}(X,Y)$. Then consider a one-parameter family of quasi-Fuchsian manifolds $\{\mathrm{QF}(X, Y(t))\}_{0 \le t \le d}$. By Lemma 2.4, the variation under the first-order deformation at time t is given by

$$d\operatorname{Rvol} = \operatorname{Re}\bigl((q_X(t), \dot{X}) + (q_{Y(t)}(t), \dot{Y}(t))\bigr),$$

where $\dot{X}$, $\dot{Y}(t)$ are tangent vectors of the deformation of complex structures on the two ideal boundary components. Since $\dot{X}=0$, integrating the variation of Rvol along the path $Y(t)$ ($t\in[0,d]$), yields an expression for the renormalized volume

$$\mathrm{Rvol}\,\mathrm{QF}(X,Y)=\mathrm{Re}\int_{t=0}^{d}(q_{Y(t)}(t),\dot{Y}(t))\,dt.$$

On the other hand,

$$\begin{aligned}|(q_{Y(t)}(t),\dot{Y}(t))| &= \left|\int_R q_{Y(t)}(t)\,\dot{Y}(t)\right| \\ &= \int_R |q_{Y(t)}(t)\,\dot{Y}(t)| \\ &\le \|q_{Y(t)}(t)\|_1\,\|\dot{Y}(t)\|_\infty,\end{aligned}$$

where $\|\cdot\|_\infty$ is the supremum norm on $L_\infty(Y(t))$ which is the dual to the L^1–norm on $Q(Y(t))$ and hence an infinitesimal form of the Teichmüller metric. By Corollary 2.6, for all $t\in[0,d]$ one has

$$\|q_{Y(t)}(t)\|_1\le 3\pi|\chi(\Sigma)|,$$

and by definition, $\|\dot{Y}(t)\|_\infty=1$, so the inequality now follows. □

Replacing (2-5) in the inequality in Theorem 2.1, one has:

Corollary 2.8 *With the notation above,*

$$\mathrm{vol}\,C(X,Y)\le 3\pi|\chi(\Sigma)|\,d_{\mathrm{T}}(X,Y)+D. \tag{2-6}$$

3 Proofs

We first deal with the case that Σ is compact. Let φ be a pseudo-Anosov automorphism on Σ and choose a marked Riemann surface $X\in\mathcal{T}$ on the Teichmüller geodesic invariant by φ. Remember that φ acts naturally on $\mathcal{T}$ by precomposing φ^{-1} to the marking of $X\in\mathcal{T}$, and consider a family of quasi-Fuchsian manifolds $\{\mathrm{QF}(\varphi^{-n}X,\varphi^{n}X):n\in\mathbb{Z}\}$. These manifolds are quite close to the infinite cyclic covering space of N_φ if n is sufficiently large. Applying Corollary 2.8 and dividing by $2n$, we obtain the estimate

$$\frac{1}{2n}\,\mathrm{vol}\,C(\varphi^{-n}X,\varphi^{n}X)\le\frac{1}{2n}\big(3\pi|\chi(\Sigma)|\,d_{\mathrm{T}}(\varphi^{-n}X,\varphi^{n}X)+D\big). \tag{3-1}$$

We consider the limit as $n\to\infty$, beginning with the right-hand side.

By a result of Bers in [3] (compare [14]), we know that

$$\lim_{n\to\infty} \frac{1}{2n}\, d_{\mathrm{T}}(\varphi^{-n}X, \varphi^{n}X) = \|\varphi\|_{\mathrm{T}} = \operatorname{ent}\varphi, \tag{3-2}$$

where $\|\varphi\|_{\mathrm{T}}$ is the Teichmüller translation distance of φ defined by

$$\|\varphi\|_{\mathrm{T}} = \inf_{R\in\mathcal{T}} d_{\mathrm{T}}(R, \varphi R).$$

Since the constant D in (3-1) does not depend on n, the limit of the right-hand side is just a multiple of the entropy.

We now consider the left-hand side of (3-1). Brock [5] states that geometric inflexibility should yield a proof that the limit as $n \to \infty$ exists and is equal to $\operatorname{vol} N_\varphi$. However, at the time of writing, it appears that there is no written proof of this fact. So for completeness, we give its proof in the appendix, and we proceed with the argument assuming this fact.

Proof of Theorem 1.1 for a compact surface Σ The inequality in Theorem 1.1 follows immediately from (3-1), (3-2) and the asymptotic behavior of the left-hand side of (3-1), for which we give a proof in the appendix; see Theorem A.2. □

We now deal with the case that Σ has m punctures, where $m \geq 1$. Ian Agol suggested the argument below reducing the proof to the compact case.

Lemma 3.1 *Suppose $m \geq 2$; then there is a finite cover of Σ of sufficiently high degree such that the number of punctures is exactly m.*

Proof We construct the required family of coverings as follows. Choose an increasing sequence of distinct primes $\{p_i\}$ such that each p_i is coprime to $m-1$. Then, there is a homomorphism $\theta_i\colon \pi_1(\Sigma) \to \mathbb{Z}/p_i\mathbb{Z}$ such that every element represented by a simple loop around a puncture is mapped to the same nontrivial element in $\mathbb{Z}/p_i\mathbb{Z}$. The family of coverings associated to $\{\operatorname{Ker}\theta_i\}$ has the property in the statement. □

Proof of Theorem 1.1 for noncompact Σ Suppose $m \geq 2$. By the preceding lemma, there is a family of coverings $\widetilde{\Sigma}_i \to \Sigma$ of increasing degree d_i such that the number of punctures of each cover is the constant m. For each i, there exists k_i such that the action of φ^{k_i} leaves $\pi_1(\widetilde{\Sigma}_i)$ invariant in $\pi_1(\Sigma)$, and so φ^{k_i} lifts to an automorphism $\widetilde{\varphi}_i$ of $\widetilde{\Sigma}_i$. The mapping torus $N_{\widetilde{\varphi}_i}$ of $\widetilde{\varphi}_i$ is a $k_i d_i$–sheeted cover of N_φ.

Then by Thurston's orbifold Dehn filling theorem [23], N_φ can be Dehn filled on its cusps along punctures of Σ as a hyperbolic orbifold N_i singular along core curves with cone angle $2\pi/d_i$ if d_i is large enough. It induces the filling $N_{\overline{\varphi}_i}$ on the cusps of N_{φ_i} upstairs, where $\overline{\varphi}_i$ is a pseudo-Anosov, extending φ_i on the compact surface $\overline{\Sigma}_i$ obtained by filling punctures of Σ_i. The construction guarantees that we have the following relation between the entropies of the automorphisms:

$$\operatorname{ent} \overline{\varphi}_i = \operatorname{ent} \varphi^{k_i} = k_i \operatorname{ent} \varphi.$$

Applying the estimate for compact surfaces obtained above, we have

$$2\pi(d_i|\chi(\Sigma)| + m) \operatorname{ent} \overline{\varphi}_i \geq \tfrac{2}{3} \operatorname{vol} N_{\overline{\varphi}_i}.$$

Now, dividing both sides by $k_i d_i$ and letting $i \to \infty$, the limit of the left-hand side is $2\pi|\chi(\varphi)| \operatorname{ent} \varphi$, while since $(\operatorname{vol} N_{\overline{\varphi}_i})/(k_i d_i) = \operatorname{vol} N_i$, the limit of the right-hand side is $\frac{2}{3} \operatorname{vol} N_\varphi$, and N_i geometrically converges to N_φ again by Thurston's hyperbolic Dehn filling theorem. □

It remains to prove Theorem 1.4, so consider the L^2 inner product on $Q(R) \cong T^*_R\mathcal{T}$ defined by

$$\langle q, q' \rangle = \int_R \frac{\overline{q} q'}{\rho^2},$$

and recall that the Weil–Petersson metric d_{WP} on the Teichmüller space $\mathcal{T}$ is a Riemannian part of the dual Hermitian metric to the above cometric on the cotangent space. Then, Weil–Peterson translation distance is defined by

$$\|\varphi\|_{\mathrm{WP}} = \inf_{R \in \mathcal{T}} d_{\mathrm{WP}}(R, \varphi R).$$

Proof of Theorem 1.4 If we start the proof of Theorem 1.1 for compact surfaces with Theorem 1.2 in [21], which asserts

$$\operatorname{Rvol} \mathrm{QF}(X, Y) \leq \frac{3\sqrt{2\pi|\chi(\Sigma)|}}{2} d_{\mathrm{WP}}(X, Y),$$

instead of Proposition 2.7, and applies

$$\|q\|_2^2 = \int_R \frac{|q|^2}{\rho^2} \leq \|q\|_1 \|q\|_\infty \leq 2\pi|\chi(\Sigma)| \|q\|_\infty^2,$$

where $\|\cdot\|_2$ is the L^2–norm induced by $\langle \cdot, \cdot \rangle$, we obtain the desired estimate. □

Remark 3.2 If we prove Theorem 1.4 first, then Theorem 1.1 for the compact case follows immediately by the Cauchy–Schwarz inequality:

$$\|q\|_1^2 = \left(\int_R |q|\right)^2 = \left(\int_R \rho \cdot \frac{|q|}{\rho}\right)^2 \leq \int_R \rho^2 \cdot \int_R \frac{\overline{q}q}{\rho^2} = 2\pi|\chi(\Sigma)|\|q\|_2^2.$$

Appendix

We now consider the left-hand side of (3-1). Brock [5] states that geometric inflexibility should yield a proof that the limit as $n \to \infty$ is

$$\operatorname{vol} C(\varphi^{-n}X, \varphi^n X) = 2n \operatorname{vol} N_\varphi + O(1).$$

We prove this using Minsky's work on bilipschitz models to simplify certain points.

It is useful to think of the convex hull C_n of $Q_n = \mathrm{QF}(\varphi^{-n}X, \varphi^n X)$ as being "modeled" on another 3–manifold M_n as follows. The boundary ∂C_n consists of a pair of surfaces $\partial^\pm C_n$, each homeomorphic to Σ. By the work of Epstein, Marden and Markovic [9] and Bridgeman [4], these surfaces (equipped with their path metrics) are 5–bilipschitz to $\varphi^{\pm n}X$, respectively (equipped with their Poincaré metrics and the markings $\varphi^{\mp n}$). Thus the boundary components of the convex core are modeled on the surfaces at infinity in that they have roughly the same geometry. Ideally one would like to extend this equivalence allowing us to think of the convex hull C_n as being modeled on a 3–manifold, which we denote by M_n, that is none other than the portion of the universal curve above the axis of φ between the points $\varphi^{-n}X$ and $\varphi^n X$. In fact, by a theorem of Minsky [17] and additional work of Rafi [20], the convex core C_n is uniformly bilipschitz to M_n equipped with a metric which we now describe. We parametrize the Teichmüller geodesic g between $\varphi^{-n}X$ and $\varphi^n X$ by arc length and identify its source with $[-n \operatorname{ent}\varphi, n \operatorname{ent}\varphi]$. For $t \in [-n \operatorname{ent}\varphi, n \operatorname{ent}\varphi]$ the metric on $\Sigma \times \{t\}$ is the metric assigned by $g(t)$. The distance between $\Sigma \times \{t\}$ and $\Sigma \times \{s\}$ is $|s-t|$. There are three important consequences of the existence of Minsky's model M_n:

(1) **Bounded geometry** Recall a hyperbolic 3–manifold has *bounded geometry* if the injectivity radius is bounded below by a positive constant. All the manifolds that we consider have *uniformly bounded geometry*. That is, the injectivity radius of the sequence Q_n is bounded from below by a constant $\epsilon_0 > 0$. Under this hypothesis, a version of the Morse lemma says that a closed curve γ of length $l(\gamma)$ is contained in an R–regular neighborhood of the closed geodesic in its homotopy class, where $R = l(\gamma)/2 + \cosh^{-1}(l(\gamma)/(2\epsilon_0))$.

(2) **Bounded lengths** Let γ_* be a (short, simple) closed curve on X, and we denote the geodesic representative of $\varphi^k(\gamma_*)$ in C_n by $\varphi^k(\gamma)$. Then the lengths of the geodesics $\varphi^k(\gamma)$ for $-n \le k \le n$ are uniformly bounded.

(3) **Roughly linear spacing** If $-n \le i, j \le n$, then the distance between geodesics $\varphi^i(\gamma)$ and $\varphi^j(\gamma)$ in C_n is *roughly* $|i-j|$. More precisely, there are constants $E > 1$ and $F > 0$ such that

$$\frac{1}{E}|i-j| \operatorname{ent}\varphi - F \le d(\varphi^i(\gamma), \varphi^j(\gamma)) \le E|i-j| \operatorname{ent}\varphi + F. \tag{A-1}$$

To see this, we identify the curve $\varphi^k(\gamma_*)$ with the obvious curve in the fiber $\Sigma \times \{k \operatorname{ent}\varphi\}$ to obtain a family of curves in the model manifold M_n, all of which have the same length, say L. The distance between $\varphi^i(\gamma_*)$ and $\varphi^j(\gamma_*)$ in the metric on the model is *exactly* $|i-j| \operatorname{ent}\varphi$. Push forward using the E–bilipschitz homeomorphism that Minsky constructs (hence the factors $E^{\pm 1}$) to obtain a pair of curves homotopic to the geodesics $\varphi^i(\gamma)$ and $\varphi^j(\gamma)$, respectively. Observe that the lengths of these curves are bounded by EL, so they are at bounded distance from the closed geodesics by the Morse lemma (hence the terms $\pm F$).

An important notion, due to McMullen, is that of *depth* in the convex core which, for a set of points, is defined to be the minimum distance to the boundary of the convex core. The inequality (A-1) can be used to prove an estimate for the depth of the geodesic $\varphi^k(\gamma)$:

Lemma A.1 *With the notation above, there exists $F_1 > 0$, which does not depend on n, such that for $-n \le k \le n$,*

$$\frac{1}{E}(n-|k|) \operatorname{ent}\varphi - F_1 \le d(\partial C_n, \varphi^k(\gamma)) \le E(n-|k|) \operatorname{ent}\varphi + F_1. \tag{A-2}$$

Brock and Bromberg [6] give a proof of an analogous inequality without the hypothesis of bounded geometry. As we will use (A-2) in an essential way in the proof of Theorem A.2, we give a short proof.

Proof The inequality follows from (A-1) provided the distances $d(\partial^+ C_n, \varphi^{-n}(\gamma))$ and $d(\partial^- C_n, \varphi^n(\gamma))$ are uniformly bounded.

The geodesic $\varphi^{-n}(\gamma)$ in Q_n represents the curve $\varphi^{-n}(\gamma_*)$ on the surface at infinity $\varphi^n(X)$. The length of $\varphi^{-n}(\gamma_*)$ on $\varphi^n(X)$ is equal to the length of γ_* on X and so is less than L. By Epstein–Marden–Markovic and Bridgeman, the nearest point retraction to $\partial^+ C_n$ is 5–bilipschitz, and applying this to γ_*, we obtain $\gamma_{**} \subset \partial^+ C_n$ of length at most $5L$. Now, since the injectivity radius is bounded below by ϵ_0, the Morse lemma tells us that there is a some constant $R > 0$ independent of n such that γ_{**} stays within an R–regular neighborhood of the closed geodesic in its homotopy class, namely $\varphi^{-n}(\gamma)$, so

$$d(\partial^+ C_n, \varphi^{-n}(\gamma)) \leq d(\gamma_{**}, \varphi^{-n}(\gamma)) \leq R.$$

To prove the lower bound, one chooses a piecewise geodesic path joining $\varphi^{-n}(\gamma)$ to $\varphi^{-k}(\gamma)$ and passing via $\partial^+ C_n$. The length of this path gives the upper bound

$$d(\varphi^{-n}(\gamma), \varphi^{-k}(\gamma)) \leq d(\partial^+ C_n, \varphi^{-n}(\gamma)) + \operatorname{diam} \partial^+ C_n + d(\partial^+ C_n, \varphi^{-k}(\gamma)) + L,$$

so we have

$$\begin{aligned} d(\partial^+ C_n, \varphi^{-k}(\gamma)) &\geq d(\varphi^{-n}(\gamma), \varphi^{-k}(\gamma)) - R - \operatorname{diam} \partial^+ C_n - L \\ &\geq \left(\frac{1}{E}|n-k| \operatorname{ent} \varphi - F\right) - R - 5 \operatorname{diam} X - L, \end{aligned}$$

using (A-1) and the fact that X and $\partial^+ C_n$ are 5–bilipschitz. Thus the distance from $\varphi^{-k}(\gamma)$ to $\partial^+ C_n$ is roughly $|n-k|$. Replacing $\varphi^{-n}(\gamma)$ by $\varphi^n(\gamma)$ and applying the same reasoning, one obtains an analogous lower bound for the distance from $\varphi^{-k}(\gamma)$ to $\partial^- C_n$ in terms of $|n+k|$. Combining the two bounds yields the required lower bound for the depth of $\varphi^k(\gamma)$ in terms of $\min(|n-k|, |n+k|) = n - |k|$.

The upper bound is proved in the same way. □

We will now apply this to prove:

Theorem A.2 (see Brock [5]) *If Σ is compact, then*

(A-3) $$|\operatorname{vol} C(\varphi^{-n} X, \varphi^n X) - 2n \operatorname{vol} N_\varphi|$$

is uniformly bounded.

Our strategy is, following McMullen and Brock–Bromberg, to decompose the convex core into a *deep part* and a *shallow part*. We first show that the shallow part is "negligible". Then, by geometric inflexibility, we see that the deep part is "almost isometric" to a large chunk of the infinite cyclic cover $\widetilde{N}_\varphi$, which we can explicitly describe. Consequently, the volume of the deep part grows like $2n \operatorname{vol} N_\varphi$.

Proof The work of Minsky and Rafi on bounded geometry (see above) for the convex core guarantees a universal lower bound for injectivity radii of $Q_n = \mathrm{QF}(\varphi^{-n} X, \varphi^n X)$ (ie one which does not depend on n). This will simplify the argument below.

Let $d > 0$, and define the *d–deep part of Q_n* to be

$$D_n(d) := \{x \in C_n : d(x, \partial C_n) \geq d\}.$$

Note that this is a proper subset of the convex core $C_n \subset Q_n$. Moreover, it follows from (A-2) that, for fixed d and n sufficiently large, $D_n(d)$ is nonempty and that its width grows linearly in n (by *width* we mean the minimum distance between connected components of $Q_n \setminus D_n(d)$). Finally, we define the *shallow part of C_n* to be the complement of the deep part, that is, $C_n - D_n(d)$.

Let $X \in \mathcal{T}$ be a point on the axis of φ. A slight modification of the proof of Theorem 8.3 of Brock and Bromberg [6] yields: given d sufficiently large, there are constants $N, K_1, K_2 > 0$ such that for all $n > N(d)$, there is a diffeomorphism $g_n \colon D_n(d) \to \widetilde{N}_\varphi$ with bilipschitz distortion at a point $x \in D_n(d)$ less than

$$1 + \exp(-K_1 d(\partial C_n, x) + K_2), \tag{A-4}$$

and where the constants K_1, K_2 depend on ϵ_0, that is, the lower bound on the injectivity radii of the Q_n, and $\chi(\Sigma)$. We have given a simplified statement of a more general result they obtain because we are working in a geometrically bounded context. In order to show that (A-3) is uniformly bounded, we must obtain a description of $E_n := g_n(D_n(d))$ and, in particular, estimate the number of translates of a well-chosen fundamental domain that are contained in E_n. We begin by estimating how many copies of a given (short, simple) closed curve are contained in E_n. To facilitate the exposition, we will define

$$\epsilon(H) := \exp(-K_1 d(\partial C_n, H) + K_2),$$

where H is a subset of C_n.

Choose a homotopy class γ_* of a simple closed curve on X, and denote the geodesic representative of $\varphi^k(\gamma_*)$ in C_n by $\varphi^k(\gamma)$. In particular, there is a collection of $2n+1$ closed geodesics in C_n:

$$\Gamma_n := \{\varphi^k(\gamma) : -n \leq k \leq n\}.$$

By Minsky [17], the lengths of the geodesics in Γ_n are uniformly bounded (ie not depending on n) from above by some $L > 0$.

Consider the subset of geodesics belonging to Γ_n that are not contained in the deep part $D_n(d)$, or equivalently, the values of k such that $d(\partial C_n, \varphi^k(\gamma)) < d$. By (A-2),

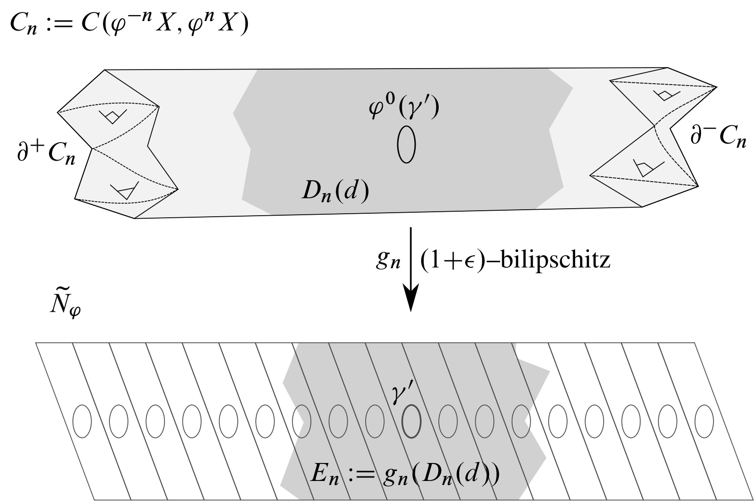

Figure 2: The map from the deep part to the infinite cyclic cover, which is shown tessellated by φ–translates of a fundamental domain (depicted as parallelograms), each containing a closed geodesic $\varphi^k(\gamma')$.

one has the inequality

$$\frac{1}{E}(n-|k|)\,\mathrm{ent}\,\varphi - F_1 \le d(\partial C_n, \varphi^k(\gamma)) < d \tag{A-5}$$

for $F_1 > 0$ and $E > 1$ which do not depend on n. One can explicitly compute $A_1 > 0$ depending on the depth d, φ and X (but not on n) such that the number of values of k which satisfy this inequality, hence the number of curves in Γ_n not contained in $D_n(d)$, is less than A_1.

Thus the deep part $D_n(d)$ contains at least $2n+1-A_1$ members of Γ_n, and the image E_n contains the same number of $g_n(\varphi^k(\gamma))$. Our objective is to "promote" each of these latter curves to a fundamental domain for the action of φ by translation on $\widetilde{N}_\varphi$ contained in E_n. The curve $g_n(\gamma)$ is homotopic to a closed geodesic $\gamma' \subset \widetilde{N}_\varphi$. Since the length of γ' is bounded below (N_φ is compact) and the length of $g_n(\gamma)$ uniformly bounded above, by the Morse lemma stated in point (1) above, the curve $g_n(\gamma)$ is contained in an R–neighborhood of γ'. It follows that there exists $A_2 \ge A_1$ such that if $n-|k| > A_2$, then $\varphi^k(\gamma)$ and $g_n^{-1}(\varphi^k(\gamma'))$ are contained in $D_n(d)$.

Now define

$$\Delta := \{x : d(x,\gamma') \le d(x, \varphi^{\pm 1}(\gamma'))\} \subset \widetilde{N}_\varphi. \tag{A-6}$$

Clearly, the interior of Δ is a fundamental domain for the action of φ on $\widetilde{N}_\varphi$, and the diameter of Δ is bounded since N_φ is compact. Moreover,

$$d(\partial C_n, \varphi^k(\gamma)) \le d(\partial C_n, g_n^{-1}(\varphi^k(\Delta)\cap E_n)) \\ + \operatorname{diam} g_n^{-1}(\varphi^k(\Delta)\cap E_n) + d(g_n^{-1}(\varphi^k(\gamma')), \varphi^k(\gamma)).$$

So if $g_n^{-1}(\varphi^k(\gamma)) \subset D_n(d)$, then

$$\begin{aligned} d(\partial C_n, g_n^{-1}(\varphi^k(\Delta)\cap E_n)) & \\ &\ge d(\partial C_n, \varphi^k(\gamma)) - (1+\epsilon)\operatorname{diam}\Delta - d(g_n^{-1}(\varphi^k(\gamma')), \varphi^k(\gamma)) \\ &\ge d(\partial C_n, \varphi^k(\gamma)) - (1+\epsilon)(\operatorname{diam}\Delta + R), \end{aligned}$$

where $\epsilon = \epsilon(D_n(d))$. Consequently, there exists $A_3 \ge A_2$ such that if $n-|k| \ge A_3$, then $g_n^{-1}(\varphi^k(\Delta)) \subset D_n(d)$, that is,

$$\bigcup_{|k|\le n-A_3} g_n^{-1}(\varphi^k(\Delta)) \subset D_n(d). \tag{A-7}$$

From the above estimate, one also obtains an explicit "linear" lower bound for the depth by using the information contained in Minsky's model, $g_n^{-1}(\varphi^k(\Delta)\cap E_n)$:

$$d(\partial C_n, g_n^{-1}(\varphi^k(\Delta)\cap E_n)) \ge \frac{1}{E}(n-|k|)\operatorname{ent}\varphi - F_1. \tag{A-8}$$

The inclusion (A-7) yields a lower bound for the volume as follows:

$$\begin{aligned} \operatorname{vol} D_n(d) &\ge \sum_{|k|\le n-A_3} \operatorname{vol} g_n^{-1}(\varphi^k(\Delta)) \\ &\ge \sum_{|k|\le n-A_3} (1+\epsilon(g_n^{-1}(\varphi^k(\Delta))))^{-3} \operatorname{vol}\varphi^k(\Delta) \\ &\ge \sum_{|k|\le n-A_3} (1-4\epsilon(g_n^{-1}(\varphi^k(\Delta))))\operatorname{vol}\varphi^k(\Delta) \\ &\ge 2(n-A_3)\operatorname{vol}\Delta - 4(\operatorname{vol}\Delta)\sum_{|k|\le n-A_3}\epsilon(g_n^{-1}(\varphi^k(\Delta))). \end{aligned}$$

The first term is the sum of $2n\operatorname{vol}\Delta$ plus a quantity which does not depend on n, and by (A-8), the second is bounded above by the sum of a geometric series.

In order to bound the volume from above, we cover the convex core by two sets. Define

$$V_n := \bigcup_{|k|\le n-A_3} g_n^{-1}(\varphi^k(\overline{\Delta})) \quad\text{and}\quad S_n := g_n^{-1}(\varphi^{-n+A_3}(\Delta)) \cup g_n^{-1}(\varphi^{n-A_3}(\Delta)),$$

so that one obtains the convex core as the union $V_n \cup (S_n \cup (C_n \setminus V_n))$. Observe that Δ separates N_φ, so S_n separates C_n, and the complement of consists of three components: one containing $\partial^+ C_n$, another containing $\partial^- C_n$ and the remaining component containing $g^{-1}(\varphi^k)$ for $k < |n| - A_3$. Clearly, the depth of a point contained in the components that contain the boundary ∂C_n is bounded from above by the maximum depth of a point in S_n. Let us give an explicit upper bound using (A-2). For $x \in g^{-1}(\varphi^k(\Delta))$, one has

$$\begin{aligned} d(\partial^- C_n, x) &\leq d(\partial^- C_n, \varphi^k(\gamma)) + d(\varphi^k(\gamma), x) \\ &\leq E(n - |k|)\,\mathrm{ent}\,\varphi + F_1 + R + (1+\epsilon)\,\mathrm{diam}\,\Delta. \end{aligned}$$

Setting $k = A_3$ in the expression on the right, one obtains the required uniform bound (it is easy to see that $g^{-1}(\varphi^k(\Delta))$ satisfies the same upper bound). It follows that the complement in the convex core of V_n is contained in a T–regular neighborhood of the boundary ∂C_n; that is,

$$C_n = N_T(\partial C_n) \cup V_n.$$

From this, we obtain the upper bound for volume

$$\mathrm{vol}\, C_n \leq \mathrm{vol}\, N_T(\partial C_n) + \mathrm{vol}\, V_n.$$

The diameter of each component of $N_T(\partial C_n)$ is uniformly bounded (by $T + 5\,\mathrm{diam}\, X$), so the first term is uniformly bounded in n, and by a similar calculation to the above for the lower bound, the second term is seen to be bounded from above by $2n\,\mathrm{vol}\,\Delta$ plus a constant. □

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