Geometry & Topology 22 (2018) 2465–2510

Classification and arithmeticity of toroidal compactifications with $3\overline{c}_2 = \overline{c}_1^2 = 3$

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We classify the minimum-volume smooth complex hyperbolic surfaces that admit smooth toroidal compactifications, and we explicitly construct their compactifications. There are five such surfaces, and they are all arithmetic; ie they are associated with quotients of the ball by an arithmetic lattice. Moreover, the associated lattices are all commensurable. The first compactification, originally discovered by Hirzebruch, is the blowup of an abelian surface at one point. The others are bielliptic surfaces blown up at one point. The bielliptic examples are new and are the first known examples of smooth toroidal compactifications birational to bielliptic surfaces.

22E40, 20H10, 57M50; 14M27, 32Q45, 11F60

1 Introduction

A classical and important problem in algebraic geometry is to classify surfaces of general type with given numerical invariants. For some aspects of this fascinating and long-standing problem, we refer to the recent survey of Bauer–Catanese–Pignatelli [4]. A situation that has attracted much recent interest is the case $c_1^2 = 3c_2 = 9$, where c_1^2 is the self-intersection of the canonical bundle and c_2 the Euler number of the underlying surface. This problem is particularly interesting because of its connection with low-dimensional geometry. In fact, Hirzebruch proportionality, Yau's solution to the Calabi conjecture [40], and the work of Miyaoka [29] imply that this is equivalent to classifying the minimum-volume quotients of the unit ball in $\mathbb{C}^2$ by a torsion-free cocompact lattice in $\mathrm{PU}(2,1)$. Most famously, this includes the classification of fake projective planes by Prasad–Yeung [34] and Cartwright–Steger [9]. Recall that Mumford [33] constructed the first example of a fake projective plane by p–adic uniformization, and explicitly raised the problem of classifying them.

This paper considers the corresponding problem in the noncompact or logarithmic setting. For example, given r and s, one can ask for a classification of the smooth

Published: 11 April 2018 DOI: 10.2140/gt.2018.22.2465

projective surfaces X containing a normal-crossing divisor D for which the logarithmic Chern numbers satisfy

$$\bar{c}_1^2(X, D) = r, \quad \bar{c}_2(X, D) = s,$$

where $\bar{c}_1^2(X, D)$ is the self-intersection of the log-canonical divisor $K_X + D$, and $\bar{c}_2(X, D)$ is the Euler number of $X \smallsetminus D$. We refer to Sakai [35] for the foundation of the theory of logarithmic surfaces. For some more recent results, see Urzúa [39]. In this setting, $\bar{c}_1^2 = 3\bar{c}_2$ now implies that $X \smallsetminus D$ is a noncompact finite-volume quotient of the ball by a torsion-free lattice by Tian–Yau [38], and X is then a smooth toroidal compactification of a ball quotient. In fact, Tian–Yau derive a logarithmic Bogomolov–Miyaoka–Yau inequality for surfaces of log-general type and then characterize the pairs that attain equality as smooth toroidal compactifications of ball quotients.

In this paper, we classify all smooth toroidal compactifications with $\bar{c}_1^2 = 3\bar{c}_2 = 3$. Equivalently, we classify the minimum-volume complex hyperbolic surfaces that admit smooth toroidal compactifications. Our main result is the following.

Theorem 1.1 *There are exactly five toroidal compactifications with* $3\bar{c}_2 = \bar{c}_1^2 = 3$. *One has underlying space an abelian variety blown up at one point. The other four have underlying space a bielliptic surface blown up once. The associated lattices in* $\mathrm{PU}(2,1)$ *are arithmetic and commensurable.*

The blown-up abelian surface in Theorem 1.1 was first studied by Hirzebruch [22] and later shown to be the compactification of a Picard modular surface by Holzapfel [23]. The other four bielliptic examples are new and are the first examples of smooth toroidal compactifications of ball quotients birational to bielliptic surfaces. All previously known examples not of general type (Hirzebruch's [22] and an additional example found by Holzapfel [25]) are birational to an abelian surface. They are also the first completely explicit examples, in the sense that they are complements of a specific divisor on a given surface, not of abelian type.

In contrast, we note that it is one of the central results in the study of toroidal compactifications that any lattice Γ in $\mathrm{PU}(2,1)$ contains a subgroup Γ' of finite index such that the associated ball quotient admits a smooth toroidal compactification of general type; see Ash–Mumford–Rapoport–Tai [1, Theorem IV.1.4]. This result has been refined by Di Cerbo in [13], where it is shown that, up to finite covers, all such compactifications have ample canonical divisor. In fact, it was previously believed that any smooth toroidal compactification not of general type must be birational to an abelian surface (see Momot [31], which unfortunately contains a critical error), and our results explicitly refute this. For applications of the existence of bielliptic smooth toroidal compactifications to problems on volumes of complex hyperbolic manifolds and group-theoretic properties of lattices in $\mathrm{PU}(2,1)$, see Di Cerbo–Stover [16].

In light of the analytical results contained in Tian–Yau [38], Theorem 1.1 implies the following result of interest for the geography of surfaces of log-general type.

Corollary 1.2 *There are exactly five surfaces* (X, D) *of log-general type satisfying* $3\bar{c}_2 = \bar{c}_1^2 = 3$.

Theorem 1.1 has also the following corollary regarding the arithmeticity of the fundamental groups of minimum-volume ball quotients. Recall that a noncompact ball quotient with finite volume admits a smooth toroidal compactification when the parabolic elements of its fundamental group have no rotational part; see Section 2.

Corollary 1.3 *Let* Γ *be a torsion-free lattice in* $\mathrm{PU}(2,1)$ *with minimum covolume. If the parabolic elements in* Γ *have no rotational part, then* Γ *is arithmetic.*

This gives further evidence toward the folklore conjecture that minimum-volume locally symmetric manifolds and orbifolds are arithmetic; see eg Belolipetsky [6]. More precisely, Corollary 1.3 proves this conjecture for torsion-free nonuniform lattices in $\mathrm{PU}(2,1)$ with rotation-free parabolic elements. In this language, Theorem 1.1 contributes to the wide literature on classification of minimal covolume lattices in semisimple Lie groups. For example, see the important very recent work of Gabai–Meyerhoff–Milley [19, Corollary 1.2] for the solution to the analogous problem for cusped hyperbolic 3–manifolds.

The proofs of the above results follow an algebro-geometric approach. More precisely, we fully exploit the implications of the Kodaira–Enriques classification for smooth toroidal compactifications of ball quotients with Euler number 1, following a program outlined by the first author [14]. All five examples are associated with arithmetic lattices and they appear in the appendix to [37], where the second author gave several examples of noncompact arithmetic ball quotients of Euler number 1. We note that in [37], there are three other ball quotients of Euler number 1 that do not admit smooth toroidal compactifications. See Section 7 for more on this.

We briefly touch upon an analogy between our work and the classification of fake projective planes by Klingler [28], Prasad–Yeung [34], and Cartwright–Steger [9].

On the one hand, fake projective planes are the minimum-volume closed complex hyperbolic 2–manifolds with first betti number zero. On the other hand, they are precisely the minimal surfaces of general type with irregularity 0 and $c_1^2 = 3c_2 = 9$. The techniques used in this paper are different from those used in the classification of fake projective planes. Crucial to the classification of fake projective planes is the proof that their fundamental groups are arithmetic by Klingler [28] and, independently, Yeung [41]. Recall that nonarithmetic lattices are known to exist by the work of Mostow and Deligne–Mostow [11]. From that point, fake projective planes are then classified by enumerating the torsion-free arithmetic lattices in $\mathrm{PU}(2,1)$ of the appropriate covolume. Our proof is of a completely different nature. We first classify the possible smooth toroidal compactifications using algebro-geometric techniques, then deduce arithmeticity from commensurability with Hirzebruch's ball quotient.

We now outline the organization of the paper. Section 2 starts with a short review of the geometry of smooth toroidal compactifications. The problem of finding the smallest toroidal compactifications is formulated in terms of logarithmic Chern numbers. In Section 3, we show that a toroidal compactification with $\bar{c}_2 = 1$ cannot have Kodaira dimension two, one, or $-\infty$. In light of the Kodaira–Enriques classification, this reduces the problem to the Kodaira dimension-zero case.

In Section 4, we study the Kodaira dimension-zero case in detail. It is shown that the minimal model of a toroidal compactification with $\bar{c}_2 = 1$ that is not a bielliptic surface must be the product of two elliptic curves with large automorphism group. Finally, an explicit example is constructed and its uniqueness is proved.

In Section 5, we then classify all toroidal compactifications with $\bar{c}_2 = 1$ that are birational to a bielliptic surface. There are exactly four of them. We briefly recap the constructions of the five examples in Section 6, and in Section 7, we show that they are commensurable and arithmetic.

Acknowledgements Di Cerbo would like to thank the Mathematics Department at Notre Dame University for the nice research environment he enjoyed while carrying out most of this project. Di Cerbo was supported by the S-S Chern grant at ICTP. Stover was supported by the National Science Foundation under Grant Number NSF 1361000. Stover also acknowledges support from US National Science Foundation grants DMS 1107452, 1107263, 1107367 "RNMS: GEometric structures And Representation varieties" (the GEAR Network). The authors also thank the referee for a very careful reading of the paper.

2 Preliminaries

The theory of compactifications of locally symmetric varieties has been extensively studied. For example, see Borel–Ji [7]. Let $\mathcal{H}^n$ be n–dimensional complex hyperbolic space. Noncompact finite-volume complex hyperbolic manifolds then correspond with conjugacy classes of torsion-free nonuniform lattices in $\mathrm{PU}(n,1)$. Let Γ be any such lattice. It is well known that when the parabolic elements in Γ have no rotational part, the manifold $\mathcal{H}^n/\Gamma$ has a particularly nice compactification (X, D) consisting of a smooth projective variety X and boundary divisor D. In other words, we require the subgroup of $\mathbb{C}$ generated by the eigenvalues of parabolic elements of Γ to be torsion-free. Under these assumptions, the divisor D is the union of smooth disjoint abelian varieties, each having normal bundle of negative degree. The pair (X, D) is called a *toroidal* compactification of $\mathcal{H}^n/\Gamma$. For more details about this construction, see Hummel [26], Ash et al [1], and Mok [30]. Note that in [30], this construction is carried out without any arithmeticity assumption on Γ.

We now describe the 2–dimensional case in more detail. Let $\mathcal{H}^2/\Gamma$ be a complex hyperbolic surface with cusps that admits a smooth toroidal compactification (X, D). It is known (see Sakai [35] and Di Cerbo [13, Section 4]) that (X, D) is D–minimal of log-general type, where (X, D) is D–minimal if X does not contain any exceptional curve E of the first kind such that $D \cdot E \leq 1$. Moreover, by the Hirzebruch–Mumford proportionality principle [32], we have

$$3\bar{c}_2 = \bar{c}_1^2,$$

where $\bar{c}_1$ and $\bar{c}_2$ are the logarithmic Chern numbers of the pair (X, D).

For the standard properties of logarithmic Chern classes, we refer to Kawamata [27]. Recall that $\bar{c}_1^2$ is equal to the self-intersection of the log-canonical divisor $K_X + D$, while $\bar{c}_2$ is simply the topological Euler characteristic of $X \smallsetminus D$. Since D consists of smooth disjoint elliptic curves, we have

$$\bar{c}_2(X) = \chi(X) - \chi(D) = \chi(X) = c_2(X).$$

By construction, $X \smallsetminus D$ is equipped with a complete metric with pinched negative sectional curvature. For this class of metrics, it is well known that the Pfaffian of the curvature matrix is pointwise strictly positive; see Di Cerbo [13, Section 2]. Thus, Gromov–Harder's generalization of Gauss–Bonnet [20] implies that $\bar{c}_2$ must be a strictly positive integer. In particular, we see that $\bar{c}_2 = 1$ is the minimum possible value, and hence the Chern–Gauss–Bonnet theorem implies that the smooth toroidal compactifications with $3\bar{c}_2 = \bar{c}_1^2 = 3$ are minimum-volume complex hyperbolic 2–manifolds.

We close this section with the following fact, which will be useful throughout the paper.

Theorem 2.1 [12, Theorem 3.18] *Let (X, D) be a toroidal compactification of dimension $n \geq 2$. Let q be the number of components of D. Then $q < \rho(X)$, where $\rho(X)$ is the Picard number of X.*

In particular, the Picard number of X strictly bounds the number of cusps from above. We briefly give the argument. The divisor D gives q disjoint elliptic curves of negative self-intersection, which, along with the class of an ample divisor, generate a subspace of dimension $q+1$ in the Néron–Severi group of X, and the result follows.

3 The case $\kappa \neq 0$

In this section, we show that a toroidal compactification (X, D) with $\bar{c}_2 = 1$ must have Kodaira dimension $\kappa(X)$ equal to zero. Let us start by showing that X cannot be of general type.

Lemma 3.1 *Let (X, D) be a toroidal compactification with $\bar{c}_2 = 1$. Then X cannot have $\kappa(X) = 2$.*

Proof Assume for a contradiction that such a pair (X, D) exists. Recall that, given a surface Y and $\mathrm{Bl}_k(Y)$ the blowup of Y at k points, the second Chern number of $\mathrm{Bl}_k(Y)$ is given by

$$c_2(\mathrm{Bl}_k(Y)) = c_2(Y) + k.$$

It is well known that the Euler number of a minimal surface of general type is strictly positive; see [3, Proposition VII.2.4]. Since $c_2(X) = 1$, we conclude that X must be minimal.

Next, observe that the adjunction formula implies that $K_X \cdot D = -D^2$, and so

$$\bar{c}_1^2 = c_1^2 - D^2 = 3\bar{c}_2 = 3c_2,$$

which implies that

$$0 < c_1^2 < 3c_2$$

since $D^2 < 0$ and $c_1^2 > 0$ for any minimal surface of general type. We then have $c_1^2 \in \{1, 2\}$. However, for any complex surface we must have

$$c_1^2 + c_2 \equiv 0 \mod 12$$

by Noether's formula; see Friedman [18, page 9]. We therefore conclude that (X, D) cannot have X be of general type. □

Now we rule out the case of Kodaira dimension one.

Lemma 3.2 *Let* (X, D) *be a toroidal compactification with* $\overline{c}_2 = 1$*. Then* X *cannot have* $\kappa(X) = 1$*.*

Proof Given X, there exists a unique minimal model Y such that $c_1^2(Y) = 0$ and $c_2(Y) \geq 0$; see Barth et al [3, Theorem VI.1.1]. Noether's formula then implies that

$$c_2(Y) = 12d$$

for some $d \in \mathbb{Z}_{\geq 0}$. It follows that a surface with $\kappa(X) = 1$ must satisfy

$$c_2(X) = 12d + k$$

with $d, k \in \mathbb{Z}_{\geq 0}$. Therefore, if we want $c_2(X) = 1$, we must have $d = 0$ and $k = 1$. In other words, X is the blowup at just one point of a minimal elliptic surface Y with Euler number zero.

For a minimal elliptic fibration

$$\pi \colon Y \to B$$

with multiple fibers $F_1, \dots, F_k$ of multiplicities $m_1, \dots, m_k$, we have

$$K_Y = \pi^*(K_B \otimes L) \otimes \mathcal{O}_Y\Bigg(\sum_{i=1}^{k}(m_1 - 1)F_i\Bigg), \tag{1}$$

where $L = (R^1\pi_*\mathcal{O}_Y)^{-1}$. Note that $d = \deg(L)$; see Friedman [18, Corollary 16, page 177]. In the case under consideration, $c_2(Y) = 0$, so all the singular fibers of the elliptic fibration are multiple fibers with smooth reduction [18, Corollary 17, page 177].

Now, consider

$$f \colon X \to B,$$

where $f = \pi \circ \mathrm{Bl}$ and $\mathrm{Bl} \colon X \to Y$ is the blowup map. We claim that some irreducible component D_i of D must map onto B under f. If every D_i were contained in a fiber, then there would be a general fiber of f that does not meet the divisor D, which means that there would exist an irreducible smooth elliptic curve E in $X \smallsetminus D$. The existence of such a holomorphic curve E is impossible because $X \smallsetminus D$ is by construction hyperbolic. This proves the claim.

Since $f(D_i) = B$ for some i and D_i is an elliptic curve, the Hurwitz formula then implies that the genus of B must be 0 or 1. Indeed, D_i cannot be contained in a fiber of the fibration, since fibers are rational. Therefore the elliptic curve D_i maps onto B, and so B must have genus at most one. Equation (1) implies that if $\kappa(Y) = 1$, we must assume the existence of multiple fibers. Indeed, otherwise $\kappa(Y) = -\infty$ when $g(B) = 0$ and $\kappa(Y) = 0$ when $g(B) = 1$; see Barth et al [3, Section V.12].

Let (Y, C) be the blowdown configuration of (X, D). We first study the case $g(B) = 1$. We then have that some irreducible component C_i of C is a holomorphic n–section of the elliptic fibration; ie the map $C_i \to B$ is generically n-to-one. Moreover, C_i is normalized by an irreducible component D_i of D, which is a smooth elliptic curve. Let Y' be the fiber product $Y \times_B D_i$, so $Y' \to Y$ is an étale covering of degree n. However, Y' then has a holomorphic 1–section, which implies that $\kappa(Y') = 0$, since there cannot be multiple fibers and all fibers are nonsingular. Then $\kappa(Y) = 0$ by invariance of κ under unramified coverings, which is a contradiction.

Now assume that $g(B) = 0$. In this case, $\deg(L) = 0$, so L is trivial. Again, there is a holomorphic n–section C_i that is normalized by a smooth elliptic curve D_i. Following a base change argument as in the setup for the last exercise on Friedman [18, page 193], there is a finite unramified cover $\pi'\colon Y' \to D_i$ of $\pi\colon Y \to B$ with a holomorphic section and such that $L' = \mathcal{O}_{D_i}$. We then have that $Y' = D_i \times E$ for some elliptic curve E. By Barth et al [3, Section VI.1], the Kodaira dimension of Y cannot be one. □

We now show that X cannot be birational to a rational or ruled surface.

Lemma 3.3 *Let (X, D) be a toroidal compactification with $\bar{c}_2 = 1$. Then X cannot have $\kappa(X) = -\infty$.*

Proof Recall that the minimal models of surfaces with negative Kodaira dimension are $\mathbb{P}^2$, the Hirzebruch surfaces X_e, and ruled surfaces over Riemann surfaces of genus $g \geq 1$; see Barth et al [3, Chapter VI]. Since

$$c_2(\mathbb{P}^2) = 3, \quad c_2(X_e) = 4,$$

and c_2 only increases under blowup, a toroidal compactification with $\bar{c}_2 = 1$ must have minimal model a ruled surface with base of genus $g \geq 1$.

Then $c_2(X) = 4(1-g) + k$, where k is the number of blowup points, so X must be the blowup at exactly one point of a surface Y ruled over an elliptic curve. Indeed, similar to the argument in Lemma 3.2, any elliptic curve D_i in the compactification must map onto the base, which has genus $g \geq 1$, and hence $g = 1$. This fact combined with the formula for $c_2(X)$ given above implies that $k = 1$. Therefore, $c_2(Y) = 0$. Since the rank of the Picard group of X is three, by Theorem 2.1, we have that X can have at most two cusps.

Let (Y, C) denote the blowdown configuration of (X, D), and assume that C consists of exactly one irreducible component. We then have by an argument similar to the $\kappa = 1$ case that C must be an n–section of the ruling of Y. It is easily seen that C cannot be a smooth n–section of the ruling for any $n \geq 1$, since this implies that $X \smallsetminus D$ contains a $\mathbb{P}^1$ with just one puncture, namely the exceptional fiber of the blowup, which contradicts hyperbolicity of the metric on $X \smallsetminus D$. This argument implies that C must be singular at some point p. In fact, p is the unique singular point since $\bar{c}_2(X) = 1$, and hence X is the blowup of Y at p.

Consider the composition of the blowdown $X \to Y$ with the map of Y to the base B of the ruling. Then D is a smooth elliptic curve on X not contained in a fiber of the map to B, hence $D \to B$ is a surjective map between elliptic curves. Such a map must be étale, but it factors through the map from D onto the singular curve C. This is a contradiction.

Let us conclude by studying the case when (X, D) has two cusps. In this situation, (Y, C) is such that C consists of two irreducible components, say C_1 and C_2, intersecting in a point p, where p is the point of Y blown up to obtain X. Considering the exceptional divisor of the blowup, if both C_1 and C_2 were smooth multisections then we would obtain a twice-punctured $\mathbb{P}^1$ in $X \smallsetminus D$, which is impossible. They are also clearly not sections of the fibration, since their proper transform to X would not be an elliptic curve. It follows that at least one of C_1 and C_2 must be singular at the blowup point p. Moreover, the tangents lines of C_1 and C_2 at p must be all distinct for the proper transforms to be disjoint. We can then proceed as in the one cusp case to obtain a contradiction. □

We summarize the results of this section as a proposition.

Proposition 3.4 *Let (X, D) be a toroidal compactification with $\bar{c}_2 = 1$. Then the Kodaira dimension of X is zero.*

Of course, it remains to be seen if any such example actually exists. The next two sections completely solve this problem.

4 The case $\kappa = 0$

In light of Proposition 3.4, a toroidal compactification with $\bar{c}_2 = 1$ must be birational to a minimal surface of Kodaira dimension zero. Recall that minimal surfaces with Kodaira dimension zero are given as follows:

- K3 surfaces, $c_2 = 24$;
- Enriques surfaces, $c_2 = 12$;
- abelian surfaces, $c_2 = 0$;
- bielliptic surfaces, $c_2 = 0$;

for details, see Barth et al [3, Chapter VI]. Thus, let (X, D) be as in Proposition 3.4. Since $\bar{c}_2 = c_2 = 1$, we have that X is the blowup an abelian or bielliptic surface at exactly one point.

Now, let $D_1, \dots, D_k$ be the irreducible components of the compactifying divisor D. Since each D_i is a smooth elliptic curve with negative self-intersection, adjunction implies that $K_X \cdot D = -D_i^2$, so we have

$$\bar{c}_1^2(X) = \Bigl(K_X + \sum_i D_i\Bigr)^2 = K_X^2 - \sum_i D_i^2 = -1 - \sum_i D_i^2.$$

Then $3\bar{c}_2(X) = \bar{c}_1^2(X)$, which implies

$$-D_1^2 - \cdots - D_k^2 = 4.$$

Therefore, we have the following finite list of configurations:

- 1 cusp, $D_1^2 = -4$;
- 2 cusps, $D_1^2 = -1$, $D_2^2 = -3$ or $D_1^2 = -2$, $D_2^2 = -2$;
- 3 cusps, $D_1^2 = -1$, $D_2^2 = -1$, $D_3^2 = -2$;
- 4 cusps, $D_1^2 = D_2^2 = D_3^2 = D_4^2 = -1$.

Let (Y, C) denote the blowdown configuration of (X, D). Since Y is an abelian or bielliptic surface, $K_Y = 0$. Thus, if C_i is an irreducible component of C in Y, adjunction implies that the arithmetic genus p_a of C_i satisfies

$$p_a(C_i) = 1 + \tfrac{1}{2}C_i^2;$$

see for example, Friedman [18, page 13]. Note that $C_i^2 \geq -2$ and it is even. If $C_i^2 = -2$, then C_i is a smooth rational curve, which is impossible since Y has universal cover $\mathbb{C}^2$. If $C_i^2 = 0$, with C_i nonsmooth, then the genus-degree formula implies that C_i is a rational curve with a single node or a cusp. This is again impossible, since in both of these cases C_i is normalized by $\mathbb{P}^1$. In conclusion, either C_i is a smooth elliptic curve with trivial self-intersection, or C_i has a singular point p and $C_i^2 = 2n$ for some $n \geq 1$.

We study the singular case first. Let

$$\pi\colon X \to Y$$

be the blowup map at p. We then have

$$\pi^* C_i = D_i + rE,$$

where D_i is the proper transform of C_i in X, E is the exceptional divisor, and r is the multiplicity of the singular point p. Moreover, we have $D_i \cdot E = r$, $D_i^2 = C_i^2 - r^2$ and

$$2p_a(D_i) - 2 = 2p_a(C_i) - 2 - r(r-1). \tag{2}$$

If we want $D_i^2 \leq -1$ with C_i not smooth, we must have

$$D_i^2 = 2n - r^2 \leq -1.$$

Since D_i is a smooth elliptic curve, (2) simplifies to the quadratic equation

$$r^2 - r - 2n = 0,$$

whose solutions are given by

$$r_{1,2} = \tfrac{1}{2}(1 \pm \sqrt{1+8n}).$$

Since r is a positive integer, we only need to consider the positive square root case in the above formula. Therefore, the self-intersection of D_i is given by

$$2n - \left(\tfrac{1}{2}(1 + \sqrt{1+8n})\right)^2$$

for $n \geq 1$. This self-intersection is easily seen to be decreasing in n and less than -4 for $n \geq 7$. All the possibilities for $1 \leq n \leq 6$ are then given by the following list:

$$\begin{aligned} n &= 1, \quad C_i^2 = 2, \quad r = 2; \\ n &= 3, \quad C_i^2 = 6, \quad r = 3; \\ n &= 6, \quad C_i^2 = 12, \quad r = 4. \end{aligned} \tag{3}$$

In conclusion, we must understand whether or not an abelian or bielliptic surface can support a curve with only one singular point of order r and self-intersection as in (3).

4.1 The abelian case and the first example

We now study the case when Y is an abelian surface in detail. First, observe that the line bundle associated with a curve as in (3) must be ample.

Lemma 4.1 *Let C be an irreducible divisor on an abelian surface Y such that $C^2 > 0$. Then $L = \mathcal{O}_Y(C)$ is ample.*

Proof For any curve E on Y, we would like to show that $C \cdot E > 0$. Since $C^2 > 0$, we need to study curves $E \neq C$. For these curves, we clearly have $C \cdot E \geq 0$. Assume then that $C \cdot E = 0$. Let $t_y(E)$ denote the translate of E by an element $y \in Y$. Choosing $y \in Y$ appropriately, we can assume that $t_y(E) \cap C \neq \{0\}$. Since the curve $t_y(E)$ is numerically equivalent to E, we obtain a contradiction. We therefore conclude that L is a strictly nef line bundle with positive self-intersection. The lemma is now a consequence of Nakai's criterion for ampleness of divisors on surfaces [3, Corollary 6.4, page 161]. □

Next, we show that curves as in (3) cannot exist on an abelian surface. The proof of this fact uses standard properties of theta functions. Recall that any effective divisor on a complex torus is the divisor of a theta function by Debarre [10, Theorem 3.1]. Let C be a reduced divisor as in Lemma 4.1. Then, if we let $V = \mathbb{C}^2$ and $\pi \colon V \to V/\Gamma$ be the universal covering map, we have that

$$\pi^* C = (\theta) \tag{4}$$

for some theta function θ on V.

More precisely, we can find a hermitian form H, a character $\alpha \colon \Gamma \to U(1)$, and a theta function satisfying (4) and the "normalized" functional equation

$$\theta(z+\gamma) = \alpha(\gamma) e^{\pi H(\gamma, z) + \frac{\pi}{2} H(\gamma,\gamma)} \theta(z) = e_\gamma(z)\theta(z) \tag{5}$$

for any $z \in V$ and $\gamma \in \Gamma$. Note that e_γ is the factor of holomorphy for the line bundle $L = \mathcal{O}_Y(C)$, and we use the convention that the first variable of H is the antiholomorphic variable. Then there is an identification between the space of sections of L and the vector space of theta functions of type (H, α) on V.

Considering the list obtained in (3), we are interested in the case when C has exactly one singular point. Thus, let $C \in |L|$ be a reduced divisor and denote by $C^* = C \backslash \{p\}$ the smooth part of C. For every $q \in C^*$, the tangent space $T_q C$ is a well-defined 1–dimensional subspace of $T_q Y$. Therefore, if we let z_1, z_2 be coordinate functions for V, the equation for $T_q C$ is given by

$$\sum_{i=1}^{2} \partial_{z_i} \theta(q)(z_i - q_i) = 0.$$

We can then consider a Gauss type map

$$G\colon C^* \to \mathbb{P}^1,$$

where

$$G(q) = (\partial_{z_1} \theta(q) : \partial_{z_2} \theta(q)).$$

We claim that since C is reduced and L is ample, the Gauss map cannot be constant. We proceed by contradiction. Suppose that the image of the Gauss map is the point $(x_1 : x_2) \in \mathbb{P}^1$. If $x_2 \neq 0$, define the derivation

$$\partial_w := \partial_{z_1} - k \partial_{z_2},$$

where $k = x_1/x_2$. If $x_2 = 0$, simply consider the derivative along the second coordinate function, in other words, $\partial_w = \partial_{z_2}$. By construction, we have $\partial_w \theta = 0$ for all $q \in C^*$. Since C is reduced, the function

$$f = \partial_w \theta / \theta$$

is holomorphic on V except at the singular points of $\pi^* C$. By the Hartogs extension theorem, we know that f can be extended to a holomorphic function on V. First, notice that

$$\begin{aligned} \partial_w \theta(z + \gamma) &= \partial_w e_\gamma(z) \theta(z) + e_\gamma(z) \partial_w \theta(z) \\ &= \pi H(\gamma, v) e_\gamma(z) \theta(z) + e_\gamma(z) \partial_w \theta(z), \end{aligned}$$

where

$$v = \partial_w \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Therefore, the functional equation (5) implies that

$$f(z+\gamma)-f(z)=\pi H(\gamma,v)$$

for any $\gamma\in\Gamma$, which further implies that

$$f(z)=\pi H(z,v)+K$$

for some constant K. Since f is holomorphic and H is antiholomorphic in z, we have therefore reached a contradiction. To summarize, we have shown that for any derivation ∂_w, the function $\partial_w\theta$ cannot be identically zero on C^*.

Now, by the functional equation given in (4), the restriction of $\partial_w\theta$ to π^*C can be considered as a section of the line bundle L restricted to C. Thus, the intersection number $(\partial_w\theta)\cdot C$ coincides with the self-intersection C^2. Now consider a derivation ∂_w with parameter w determined by a generic point in the image of the Gauss map, and suppose that the multiplicity of the singular point p is r_p. The intersection number of $(\partial_w\theta)$ and C at the singular point p is then $r_p(r_p-1)$. Moreover, by construction, $\partial_w\theta$ vanishes somewhere on C^*. We conclude that

$$C^2\geq r_p(r_p-1)+1. \tag{6}$$

Remark 4.2 The same argument shows that, if C is an irreducible curve on an abelian surface with $C^2>0$ and singular points p_j of multiplicities r_j, then $C^2>\sum_j r_j(r_j-1)$.

Next, we observe that in all of the cases given in (3), we have

$$C_i^2=r(r-1),$$

so we can rule out the cases of one, two, and three cusps using (6). Indeed, these are precisely the cases for which C would contain an irreducible component that is forbidden by the above discussion. We summarize this as a lemma.

Lemma 4.3 *Let (X,D) be a toroidal compactification with $\bar{c}_2=1$ and $\kappa(X)=0$. If X is not birational to a bielliptic surface, then X is the blowup of an abelian surface. Moreover, D consists of four disjoint smooth elliptic curves with $D_1^2=D_2^2=D_3^2=D_4^2=-1$.*

Thus, by Lemma 4.3, we must classify the pairs (Y,C) where Y is an abelian surface and C consists of four smooth elliptic curves intersecting in just one point. We will show that, up to isomorphism, there is only one such pair. This result follows from a few geometric facts.

Fact 4.4 *Let* $Y = \mathbb{C}^2/\Gamma$ *be an abelian surface containing two smooth elliptic curves* C_1*,* C_2 *such that* $C_1 \cdot C_2 = 1$*. Then* Y *is isomorphic to the product* $C_1 \times C_2$*.*

Proof By translating the curves C_1 and C_2, we can assume that $C_1 \cap C_2 = \{(0,0)\}$. The curves C_i for $i = 1, 2$ are then subgroups of Y. Thus, we can define the map

$$\varphi\colon C_1 \times C_2 \to Y$$

that sends the point $(p,q) \in C_1 \times C_2$ to $p - q \in Y$. The map φ is clearly one-to-one. □

Fact 4.5 *Let* $Y = \mathbb{C}^2/\Gamma$ *be an abelian surface containing three smooth elliptic curves* C_i *for* $i = 1, 2, 3$ *such that* $C_1 \cap C_2 \cap C_3 = \{(0,0)\}$ *and such that* $C_i \cdot C_j = 1$ *for any* $i \neq j$*. Then* Y *is isomorphic to the product* $C \times C$*, where* $C_i = C$ *for any* i*.*

Proof By Fact 4.4, we have that $Y = C_1 \times C_2$. Since $C_3 \cdot C_1 = 1$, we conclude that $C_3 = C_i$ for $i = 1, 2$. □

Fact 4.6 *Let* Y *be an abelian surface that is the product of two identical elliptic curves, say* $C = \mathbb{C}/\Lambda$*. Let* (w, z) *be the natural product coordinates on* Y*. Then any smooth elliptic curve in* Y*, passing through the point* $(0,0)$*, is given by an equation of the form* $w = \alpha z$*, with* α *such that* $\alpha\Lambda \subseteq \Lambda$*.*

Proof A subgroup in $\mathbb{C}^2$ is given by an equation of the form $w = \alpha z$. This equation descends to Y precisely when $\alpha\Lambda \subseteq \Lambda$. □

Fact 4.7 *Let* C_α *denote the curve in* $Y = C \times C$ *given by the equation* $w = \alpha z$ *with* $\alpha\Lambda \subseteq \Lambda$ *and* $\alpha \neq 0$*. Then* $C_0 \cdot C_\alpha = 1$ *if and only if* $\alpha\Lambda = \Lambda$*.*

Proof The intersection $C_0 \cap C_\alpha$ consists of $[\Lambda : \alpha\Lambda]$ distinct points, where $[\Lambda : \alpha\Lambda]$ denotes the index of the subgroup $\alpha\Lambda$ in Λ. □

Let us now return to our original problem of classifying all configurations of four elliptic curves C_i for $i = 1, 2, 3, 4$ on an abelian surface Y such that

$$C_1 \cap C_2 \cap C_3 \cap C_4 = \{p\}$$

for a point $p \in Y$ and

$$C_i \cdot C_j = 1$$

for any $i \neq j \in \{1, 2, 3, 4\}$. Any such configuration will be called a *good configuration*. By translating the C_i, we can assume the point p coincides with the origin in Y. By

Facts 4.4 and 4.5, we can assume $Y = C \times C$ with the curves C_1 and C_2 being the factors in this splitting of Y. Then Facts 4.6 and 4.7 imply that we must look for values of α, say α_1 and α_2, such that

$$C_3 = C_{\alpha_1}, \quad C_4 = C_{\alpha_2}.$$

For a generic elliptic curve $C = \mathbb{C}/\Lambda$, the only values of α such that $\alpha\Lambda = \Lambda$ are given by $\alpha = \pm 1$. If this is the case, note that $C_1 \cap C_{-1}$ consists of four distinct points. These points are exactly the 2–torsion points of the lattice Λ. In conclusion, for a generic elliptic curve C, the abelian surface $Y = C \times C$ cannot support a good configuration.

It remains to treat the case of a nongeneric elliptic curve C. Recall that there are only two elliptic curves with nongeneric automorphism group; see Hartshorne [21, Section IV.4]. These elliptic curves correspond to the lattices $\mathbb{Z}[1,i] = \mathbb{Z} + \mathbb{Z}i$ and $\mathbb{Z}[1,\zeta] = \mathbb{Z} + \mathbb{Z}\zeta$, where $\zeta = e^{\frac{\pi i}{3}}$.

For the lattice $\mathbb{Z}[1,i]$, we have four choices of the value of α so that $\alpha\mathbb{Z}[1,i] = \mathbb{Z}[1,i]$:

$$\alpha = 1, i, i^2, i^3.$$

It turns out that none of the possible choices involving these parameters gives a good configuration. To see this, it suffices to observe that the configuration

$$w = 0, \quad z = 0, \quad w = z, \quad w = iz$$

is such that

$$C_1 \cap C_i = \left\{(0,0),\ \left(\tfrac{1}{2} + \tfrac{1}{2}i, \tfrac{1}{2} + \tfrac{1}{2}i\right)\right\}.$$

Notice that $C_1 \cap C_i$ is two points precisely because $1 - i \in \mathbb{Z}[i]$ has norm 2. Any other configuration is either equivalent to the one above by a self-isomorphism of Y, or fails to be a good configuration by completely analogous reasons in the sense that some pair of curves will intersect in at least two points because the difference between their slopes will not be a unit of $\mathbb{Z}[i]$.

For the lattice $\mathbb{Z}[1,\zeta]$, we have six choices of the value of α so that $\alpha\mathbb{Z}[1,\zeta] = \mathbb{Z}[1,\zeta]$:

$$\alpha = 1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5.$$

Observe that

$$w = 0, \quad z = 0, \quad w = z, \quad w = \zeta z$$

is a good configuration. In fact, the curves C_1 and C_ζ intersect at the points whose z–values satisfy

$$(\zeta - 1)z = 0 \mod \mathbb{Z}[1,\zeta]. \tag{7}$$

Since $(\zeta - 1) = \zeta^2$, we conclude that

$$C_1 \cap C_\zeta = \{(0,0)\}.$$

We claim that this is the only good configuration (note this is implicit in Hirzebruch's work [22]). First, consider the configuration given by

$$w = 0, \quad z = 0, \quad w = z, \quad w = \zeta^2 z.$$

Observe that the curves C_1 and C_{ζ^2}, not only meet at the origin, but also in other two distinct points. These points are the two distinct zeros of the Weierstrass $\wp$–function associated with the lattice $\mathbb{Z}[1,\zeta]$. More precisely, we have

$$C_1 \cap C_{\zeta^2} = \left\{(0,0),\ \left(\tfrac{1}{3}(1-\zeta^2), \tfrac{1}{3}(1-\zeta^2)\right),\ \left(\tfrac{1}{3}(\zeta^2-1), \tfrac{1}{3}(\zeta^2-1)\right)\right\}.$$

As the reader can easily verify by finding an explicit self-isomorphism of the abelian surface, any other configuration can be reduced to the above two or to the configuration

$$w = 0, \quad z = 0, \quad w = z, \quad w = -z,$$

which we already know not to be good.

In conclusion, we have the following result, which proves the uniqueness of Hirzebruch's example among abelian surfaces that are a smooth toroidal compactification of a ball quotient of Euler number one.

Theorem 4.8 *Let (X, D) be a toroidal compactification with $\bar{c}_2 = 1$ and $\kappa(X) = 0$ for which X is not birational to a bielliptic surface. Then X is the blowup of an abelian surface $Y = \mathbb{C}^2/\Gamma$ with $\Gamma = \mathbb{Z}[1,\zeta] \times \mathbb{Z}[1,\zeta]$ and $\zeta = e^{\frac{\pi i}{3}}$. Moreover, up to a self-isomorphism of Y, the blowdown divisor C of D is given by*

$$w = 0, \quad z = 0, \quad w = z, \quad w = \zeta z,$$

where (w, z) are the natural product coordinates on Y. In other words, (X, D) is the toroidal compactification with $\bar{c}_2 = 1$ described by Hirzebruch.

5 The bielliptic case and four more examples

Recall that a *bielliptic* surface is a minimal surface of Kodaira dimension zero and irregularity one. As shown at the beginning of the last century by Bagnera and de Franchis [2], all such surfaces are finite quotients of products of elliptic curves. More precisely, we have the following classification theorem. For a modern treatment, we refer to Beauville [5, Chapter VI].

Theorem 5.1 (Bagnera–de Franchis, 1907) *Let X be a bielliptic surface. Then there are elliptic curves E_λ and E_τ associated with the lattices $\mathbb{Z}[1,\lambda]$ and $\mathbb{Z}[1,\tau]$ such that X is biholomorphic to $(E_\lambda \times E_\tau)/G$, where G is a group of translations of E_τ that acts on E_λ with $E_\lambda/G = \mathbb{P}^1$. Moreover, G has one of the following types:*

(1) $G = \mathbb{Z}/2\mathbb{Z}$ *acting on* E_λ *by* $x \to -x$;

(2) $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *acting on* E_λ *by*

$$x \to -x \quad \text{and} \quad x \to x + \alpha_2,$$

where α_2 *is a* 2*–torsion point;*

(3) $G = \mathbb{Z}/4\mathbb{Z}$ *acting on* E_λ *by* $x \to \lambda x$, *where* $\lambda = i$;

(4) $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *acting on* E_λ *by*

$$x \to \lambda x \quad \text{and} \quad x \to x + \tfrac{1}{2}(1+\lambda),$$

where $\lambda = i$;

(5) $G = \mathbb{Z}/3\mathbb{Z}$ *acting on* E_λ *by* $x \to \lambda x$, *where* $\lambda = e^{\frac{2\pi i}{3}}$;

(6) $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ *acting on* E_λ *by*

$$x \to \lambda x \quad \text{and} \quad x \to x + \tfrac{1}{3}(1-\lambda),$$

where $\lambda = e^{\frac{2\pi i}{3}}$;

(7) $G = \mathbb{Z}/6\mathbb{Z}$ *acting on* E_λ *by* $x \to \zeta x$, *where* $\lambda = e^{\frac{2\pi i}{3}}$ *and* $\zeta = e^{\frac{\pi i}{3}}$.

Note that the action of G on $Y = E_\lambda \times E_\tau$ is clearly free, since G acts on E_τ by translations, so the map $Y \to X$ is an étale cover. We now address the existence and uniqueness of toroidal compactifications of Euler number one that are birational to a bielliptic surface. First, observe that if such examples exist then they must be the blowup at just one point of a bielliptic surface by arguments in Section 4. Second, we have the following.

Lemma 5.2 *Suppose that X is the blowup of a bielliptic surface at exactly one point, and that (X, D) is a smooth toroidal compactification of a complex hyperbolic manifold $M = X \smallsetminus D$. Then M has either one or two cusps.*

Proof From Section 4, M has between one and four cusps. Since the Picard number of a bielliptic surface is 2, the Picard number of the blowup is then 3. Since the rank of the Picard group of X is 3, by Theorem 2.1 we have that X can at most have two cusps. □

The problem is then reduced to the study of the existence of certain singular elliptic curves C_i as in (3) on a bielliptic surface. Regarding these possibilities, let us observe the following, which will allow us to make further reductions.

Lemma 5.3 *Let Y be bielliptic and $\pi\colon E_\lambda \times E_\tau \to Y$ the associated étale cover as in Theorem 5.1. Suppose that C_i is a curve from* (3) *with a unique regular singular point of order $r \geq 2$. Then $\pi^{-1}(C_i)$ is the union of distinct smooth elliptic curves, and exactly r of them pass through each lift of the singular point of C_i. In fact, $\pi^{-1}(C_i)$ contains exactly r distinct irreducible components, and the stabilizer $G_{C_i'}$ of any irreducible component of $\pi^{-1}(C_i)$ has order d/r, where d is the degree of π. In particular, r divides d.*

Proof Let $d = |G|$ be the degree of π. Then there are exactly d points on $E_\lambda \times E_\tau$ above the singular point p of C_i. Consider an irreducible component C_i' of $\pi^{-1}(C_i)$. Then Remark 4.2 implies that C_i' cannot be singular. Since C_i is normalized by an elliptic curve, we see that C_i' must be a smooth elliptic curve, hence C_i' is an étale cover of the normalization of C_i.

Smoothness of each C_i' implies there are exactly r irreducible components of $\pi^{-1}(C_i)$ passing through any given point in $\pi^{-1}(p)$. To obtain a singularity of order exactly r in the quotient, we see that the G–action on $E_\lambda \times E_\tau$ must identify exactly r distinct smooth curves in each connected component of $\pi^{-1}(C_i)$. Thus, to prove the lemma, it suffices to show that $\pi^{-1}(C_i)$ is connected.

Since C_i is an irreducible curve on Y of positive self-intersection, it follows that $\pi^{-1}(C_i)$ has positive self-intersection. In fact, our assumption that C_i has positive self-intersection, along with the fact that π is étale, implies that $\pi^{-1}(C_i)$ is reduced and intersects each irreducible component C_i' positively. Then one sees exactly as in the proof of Lemma 4.1 that Nakai's criterion implies that $\pi^{-1}(C_i)$ is a reduced ample divisor. It follows that the support of $\pi^{-1}(C_i)$ must be topologically connected by Zariski's main theorem (eg applying Hartshorne [21, Corollary III.11.3] to the linear system determined by the ample divisor), which proves the lemma. □

Now, we use the structure of the group $\mathrm{Num}(Y)$ of divisors modulo numerical equivalence to restrict the possible number of cusps even further. Let Y be a bielliptic surface with associated group

$$G = \mathbb{Z}/s\mathbb{Z} \times \mathbb{Z}/t\mathbb{Z}$$

with $s \geq 2$ and $t \geq 1$. Set $d = |G| = st$. Recall from Serrano [36] that $\mathrm{Num}(Y)$ has a $\mathbb{Z}$–basis consisting of $(1/s)A$ and $(1/t)B$, where A and B are the general fibers of the two fibrations of Y associated with the coordinate projections of the abelian variety from Theorem 5.1. Note that one projection is onto $E_\lambda/G = \mathbb{P}^1$. Projection onto the elliptic curve E_τ/G is the Albanese fibration of Y; see Beauville [5, Chapters V, VIII].

Lemma 5.4 *The case $C_1^2 = C_2^2 = 2$ associated with $D_1^2 = D_2^2 = -2$ on a two-cusped manifold cannot occur.*

Proof From the list given in (3), we must have two curves C_1 and C_2 with self-intersection two, each with a singular point of order two. Since C_1 and C_2 only intersect at the singular point, which has order two on each, $C_1 \cdot C_2 = 4$. Up to numerical equivalence, any curve C_i in Y can be written as $C_i = (k_1/s)A + (k_2/t)B$ with $k_1, k_2 \in \mathbb{Z}$, and then

$$C_i^2 = 2\frac{k_1 k_2}{st} A \cdot B = 2k_1 k_2. \tag{8}$$

For $C_i^2 = 2$, the only possibility is then $k_1 = k_2 = \pm 1$, the negative case being excluded by $C_i \cdot A \geq 0$. Therefore, we must have

$$C_1 = C_2 = \frac{1}{s}A + \frac{1}{t}B$$

in $\mathrm{Num}(Y)$. Then $C_1 \cdot C_2 = 2 \neq 4$, which is a contradiction. □

Lemma 5.5 *The case $C_1^2 = 0$, $C_2^2 = 6$ associated with $D_1^2 = -1$ and $D_2^2 = -3$ on a two-cusped manifold cannot occur unless 3 divides $|G|$.*

Proof In this case, the curve C_2 has a regular singular point of order three. By Lemma 5.3, $\pi^{-1}(C_2)$ is the union of exactly $3k$ smooth irreducible elliptic curves for some $k \geq 1$. Since G must act transitively on these curves, the lemma follows. □

Lemma 5.6 *The case $C_1^2 = 12$ associated with $D_1^2 = -4$ on a one-cusped manifold cannot occur unless $|G|$ is divisible by 4.*

Proof In this case, the curve C_1 has a regular singular point of order four. The lemma follows exactly as Lemma 5.5. □

This covers all three possible cases with one or two cusps. We now have two more lemmas that will prove useful.

Lemma 5.7 *Let Y be a bielliptic surface with associated group G, and let C_1 be a curve with a singular point p of order r. Suppose that (X, D_1) is a smooth toroidal compactification, where X is the blowup of Y at p, and D_1 is the proper transform of C_1. Choose generators $(1/s)A$ and $(1/t)B$ for $\mathrm{Num}(Y)$ as above. Given $(k_1/s)A + (k_2/t)B \in \mathrm{Num}(Y)$ representing the numerical class of a curve C_1 with a singular point p of order r, we must have $k_1 > rs/|G|$ and $k_2 > rt/|G|$.*

Proof We have $A \cdot B = |G|$. Recall that B is numerically equivalent to a general fiber of the Albanese map. Then, considering a fiber of the Albanese map through p,

$$C_1 \cdot B = \frac{k_1|G|}{s} \geq r,$$

where r is the order of the singular point.

Now consider the case of equality. There must be a smooth fiber B_0 (ie not a multiple fiber) of the Albanese map passing through p, and this fiber will intersect C_1 transversally with intersection number r, and B_0 is disjoint from C_1 away from p. Now, suppose that (X, D_1) is a smooth toroidal compactification, where X is the blowup of Y at p and D_1 is the proper transform of C_1. Then the proper transform E_0 of B_0 to X is an elliptic curve disjoint from C_1 in X. In particular, it defines an elliptic curve on the complex hyperbolic manifold $X \smallsetminus D$, which is a contradiction. The analogous argument for the fibration of the bielliptic surface over $\mathbb{P}^1$ gives the bound on k_2. □

Now we proceed to analyze the remaining possibilities.

5.1 Initial reductions

In this section, we show that the only bielliptic surfaces whose blowup can produce a smooth toroidal compactification of Euler number one are associated with the groups $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. The first case is immediate from the above.

Corollary 5.8 *No $\mathbb{Z}/2\mathbb{Z}$ bielliptic surface can produce a smooth toroidal compactification of Euler number one.*

Proof Lemmas 5.4–5.6 imply that 3 or 4 divides $|G|$, a contradiction. □

We now proceed to rule out the remaining cases.

Proposition 5.9 *No blowup of a $\mathbb{Z}/4\mathbb{Z}$ bielliptic surface leads to a smooth toroidal compactification of Euler number one.*

Proof By Lemmas 5.4–5.6, we must consider the one-cusped case. Here, the curve C_1 on Y satisfies $C_1^2 = 12$, and has a regular singular point of order four. Write

$$C_1 = \tfrac{1}{4} k_1 A + k_2 B$$

as above. Then $C_1^2 = 2k_1k_2$, so we have four possibilities:

$$\begin{aligned} k_1 &= 1, \quad k_2 = 6, \quad C_1 = \tfrac{1}{4}A + 6B; \\ k_1 &= 2, \quad k_2 = 3, \quad C_1 = \tfrac{1}{2}A + 3B; \\ k_1 &= 3, \quad k_2 = 2, \quad C_1 = \tfrac{3}{4}A + 2B; \\ k_1 &= 6, \quad k_2 = 1, \quad C_1 = \tfrac{3}{2}A + B. \end{aligned}$$

All four cases are impossible by applying Lemma 5.7. More precisely, we apply the bound on k_1 to the first three cases and the bound on k_2 to the last case. This proves the proposition. □

Proposition 5.10 *No blowup of a $\mathbb{Z}/6\mathbb{Z}$ bielliptic surface defines a smooth toroidal compactification of Euler number one.*

Proof In this case, we must consider the case where there are two curves C_1, C_2 on Y such that C_1 is a smooth elliptic curve with $C_1^2 = 0$, $C_2^2 = 6$, and C_2 has a regular singular point of order three. Write

$$C_2 = \tfrac{1}{6} k_1 A + k_2 B$$

with the above notation. Then, $C_2^2 = 2k_1k_2$, so $C_2^2 = 6$ leaves us with the following possibilities:

$$\begin{aligned} k_1 &= 1, \quad k_2 = 3, \quad C_2 = \tfrac{1}{6}A + 3B; \\ k_1 &= 3, \quad k_2 = 1, \quad C_2 = \tfrac{1}{2}A + B. \end{aligned}$$

The first case is impossible by Lemma 5.7.

In the second case, $C_2 \cdot B = 3$. Thus, assume we can find a curve C_2 with a singular point of order three in the numerical class of $\frac{1}{2}A + B$. Let $\pi\colon E_\rho \times E_\tau \to Y$ be the covering of degree 6. Moreover, we can assume that the $\mathbb{Z}/6\mathbb{Z}$ action is given by the group generated by

$$\varphi(w, z) = \left(\zeta w, z + \tfrac{1}{6}\gamma\right)$$

for some $\gamma \in \mathbb{Z}[1, \tau]$ and $\zeta = e^{\frac{\pi i}{3}}$.

Using Lemma 5.3, $\pi^{-1}(C_2)$ must consist of a union of smooth elliptic curves. Moreover, $\pi^{-1}(C_2)$ consists of three smooth elliptic curves E_1, E_2, E_3 intersecting in the six lifts of the points on Y where C_1 and C_2 meet. Thus, the automorphism group $\mathbb{Z}/6\mathbb{Z}$ must act transitively on these elliptic curves with isotropy group $\{1, \varphi^3\}$. Next, observe that, since $C_2 \cdot B = 3$, we have that $E_i \cdot F = 1$ for any $i = 1, 2, 3$ and any fiber F of the map $E_\rho \times E_\tau \to E_\tau$.

Therefore, for any $i = 1, 2, 3$ we can write $E_i = \{w = \alpha_i z + a_i\}$ for some appropriate complex numbers α_i and a_i. We then must have $\varphi^3 E_1 = E_1$, which then implies that

$$\left(-\alpha_1 z - a_1,\, z + \tfrac{1}{2}\gamma\right) = \left(\alpha_1\left(z + \tfrac{1}{2}\gamma\right) + a_1,\, z + \tfrac{1}{2}\gamma\right).$$

We then have $2\alpha_1 z = -\alpha_1\left(\frac{1}{2}\gamma\right) - 2a_1$ modulo $\mathbb{Z}[1, \rho]$ for any $z \in E_\tau$, which is impossible unless $\alpha_1 = a_1 = 0$. Reiterating this argument for the three elliptic curves we get that $\alpha_i = a_i = 0$ for any $i = 1, 2, 3$. This is impossible. □

Proposition 5.11 *No* $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ *bielliptic surface can determine a smooth toroidal compactification of Euler number one.*

Proof By Lemmas 5.4–5.6, it suffices to consider the one-cusp case, where

$$C_1 = \tfrac{1}{2}k_1 A + \tfrac{1}{2}k_2 B,$$

$C_1^2 = 12$, and C_1 has a regular singular point of order four. We have four possibilities:

$$\begin{aligned} k_1 &= 1, \quad k_2 = 6, \quad C_1 = \tfrac{1}{2}A + 3B;\\ k_1 &= 2, \quad k_2 = 3, \quad C_1 = A + \tfrac{3}{2}B;\\ k_1 &= 3, \quad k_2 = 2, \quad C_1 = \tfrac{3}{2}A + B;\\ k_1 &= 6, \quad k_2 = 1, \quad C_1 = 3A + \tfrac{1}{2}B. \end{aligned}$$

The first three cases are eliminated by Lemma 5.7. For the fourth case, note that $C_1 \cdot A = 2$, which is a contradiction because we assumed that C_1 has a singular point of order four. □

Proposition 5.12 *No* $\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ *bielliptic surface can determine a smooth toroidal compactification of Euler number one.*

Proof By Lemmas 5.4–5.6, it suffices to consider the one cusp case, where $C_1^2 = 12$ and it has a regular singular point of order four. If

$$C_1 = \tfrac{1}{4}k_1 A + \tfrac{1}{2}k_2 B$$

with $C_1^2 = 12$, we have four possibilities:

$$\begin{aligned} k_1 &= 1, \quad k_2 = 6, \quad C_1 = \tfrac{1}{4}A + 3B; \\ k_1 &= 2, \quad k_2 = 3, \quad C_1 = \tfrac{1}{2}A + \tfrac{3}{2}B; \\ k_1 &= 3, \quad k_2 = 2, \quad C_1 = \tfrac{3}{4}A + B; \\ k_1 &= 6, \quad k_2 = 1, \quad C_1 = \tfrac{3}{2}A + \tfrac{1}{2}B. \end{aligned}$$

The first two cases are impossible by Lemma 5.7, as is the fourth.

For the third case, let us observe that $C_1 \cdot B = 6$, so the curve C_1 is a 6–section of the Albanese map $\pi_2\colon Y \to E_\tau/(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$. Let $\pi\colon E_i \times E_\tau \to Y$ be the degree-8 étale cover and observe that $\pi^{-1}(B) = 8E_i$ (here i is a square root of -1). Next, let H_1, H_2, H_3, H_4 denote the irreducible components of $\pi^{-1}(C_1)$. Since the automorphism group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts transitively on the H_j and trivially on the numerical class of E_i, for any $j = 1, 2, 3, 4$ we obtain that H_j is an s–section for a fixed integer s. This implies the contradiction $4s = 6$. □

This leaves us with only $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. In fact, both will produce examples of smooth toroidal compactifications. We now proceed to analyze these cases and completely classify the examples they determine.

5.2 The classification in the case of $\mathbb{Z}/3\mathbb{Z}$ quotients

Let Y be a $\mathbb{Z}/3\mathbb{Z}$ bielliptic quotient. By Theorem 5.1, we can find two elliptic curves E_ρ and E_τ associated with the lattices $\mathbb{Z}[1, \rho]$ and $\mathbb{Z}[1, \tau]$, respectively, such that

$$Y = (E_\rho \times E_\tau)/(\mathbb{Z}/3\mathbb{Z}).$$

More precisely, $\rho = e^{\frac{2\pi i}{3}}$ while τ is arbitrary, and the $\mathbb{Z}/3\mathbb{Z}$ group of automorphisms of $E_\rho \times E_\tau$ is generated by the automorphism $\varphi(w, z) = \left(\rho w, z + \tfrac{1}{3}\gamma\right)$ for some $\gamma \in \mathbb{Z}[1, \tau]$.

Consider the group $\mathrm{Num}(Y)$ of divisors on Y up to numerical equivalence. Given the bielliptic quotient $\pi\colon E_\rho \times E_\tau \to Y$, we have two elliptic fibrations

$$\begin{aligned} \pi_1&\colon Y \to E_\rho/(\mathbb{Z}/3\mathbb{Z}) = \mathbb{P}^1, \\ \pi_2&\colon Y \to E_\tau/(\mathbb{Z}/3\mathbb{Z}) \end{aligned}$$

with generic fibers A and B. Recall from above that, up to numerical equivalence, we can write any curve $C \in \mathrm{Num}(Y)$ as

$$C = \tfrac{1}{3}k_1 A + k_2 B$$

for $k_1, k_2 \in \mathbb{Z}$. Notice that all of the fibers of π_2 are smooth and reduced and that this map is none other than the Albanese map. The class $\frac{1}{3}A$ represents a multiple fiber of the map π_1 counted with multiplicity one. Moreover, we have $A \cdot B = 3$.

By Lemmas 5.4–5.6, the only possibility is a two-cusped manifold with $D_1^2 = -1$ and $D_2^2 = -3$, the latter of which determines a curve C_2 on Y with $C_2^2 = 6$. Up to numerical equivalence, we have two possibilities:

$$k_1 = 1, \quad k_2 = 3, \quad C_2 = \tfrac{1}{3}A + 3B;$$
$$k_1 = 3, \quad k_2 = 1, \quad C_2 = A + B.$$

The first case is ruled out by Lemma 5.7.

It remains to discuss the case when the curve C_2 in Y is numerically equivalent to $A + B$. Notice that in this case $C_2 \cdot A = C_2 \cdot B = 3$. Let E_1, E_2, and E_3 denote the three smooth elliptic curves on $E_\rho \times E_\tau$ that are the irreducible components of $\pi^{-1}(C_2)$. Since $C_2 \cdot A = 3$, we have a one-to-one map from E_i to E_ρ for each $i = 1, 2, 3$. Next, since $C_2 \cdot B = 3$ we also have a one-to-one map from each E_i to E_τ. Indeed, it follows that each E_i has intersection number one with the general fiber of each factor projection. We therefore conclude that $E_\rho \cong E_\tau$ and Y is a quotient of $E_\rho \times E_\rho$ by the group of automorphisms generated by the order-three automorphism

$$\varphi(w, z) = \big(\rho w,\ z + \tfrac{1}{3}\gamma\big)$$

for some $\gamma \in \mathbb{Z}[1, \rho]$.

The next step is to determine the admissible values for γ. First, observe that, up to a translation in the w–direction, we can assume that

$$E_1 = (\alpha_1 z, z), \quad E_2 = (\alpha_2 z + a_2, z), \quad E_3 = (\alpha_3 z + a_3, z),$$

where the α_i and a_i are complex numbers to be determined. In particular, the E_i cannot be three φ–translates of a curve with second coordinate zero, since they would project to a smooth curve on the quotient. Then, up to renumbering we have

$$\varphi(E_1) = E_2, \quad \varphi(E_2) = E_3, \quad \varphi(E_3) = E_1.$$

Analytically, this means that

$$\begin{aligned} \rho\alpha_1 z &= \alpha_2\big(z + \tfrac{1}{3}\gamma\big) + a_2, \\ \rho\alpha_2 z + \rho a_2 &= \alpha_3\big(z + \tfrac{1}{3}\gamma\big) + a_3, \\ \rho\alpha_3 z + \rho a_3 &= \alpha_1\big(z + \tfrac{1}{3}\gamma\big) \end{aligned}$$

(recall that our coordinates are on the abelian surface, so in $\mathbb{C}^2$ our equations must be taken modulo $\mathbb{Z}[1,\rho]$), and we then obtain

$$\begin{aligned} \rho\alpha_1 &= \alpha_2, & \rho\alpha_2 &= \alpha_3, & \rho\alpha_3 &= \alpha_1, \\ \tfrac{1}{3}\alpha_2\gamma + a_2 &= 0, & \rho a_2 &= \tfrac{1}{3}\alpha_3\gamma + a_3, & \rho a_3 &= \tfrac{1}{3}\alpha_1\gamma. \end{aligned}$$

It follows immediately that $\rho^3 = 1$, ie that ρ is a cube root of unity.

Since each E_i is a 1–section of the map $E_\rho \times E_\rho \to E_\rho$ that intersects $\{w = 0\}$ in exactly one point, we have that $\{\alpha_1, \alpha_2, \alpha_3\} \in \{1, \zeta, \dots, \zeta^5\}$, where $\zeta = e^{\frac{\pi i}{3}}$; see Fact 4.7. We also want these three sections to intersect in three distinct points, so $\{\alpha_1, \alpha_2, \alpha_3\}$ is of the form $\{u, u\rho, u\rho^2\}$ for some 6^{th} root of unity u (not necessarily primitive).

Since these choices of slopes all differ by an automorphism of the abelian surface, namely complex multiplication by u on the first factor, it suffices to consider the first choice, $\alpha_1 = 1$, $\alpha_2 = \rho$, $\alpha_3 = \rho^2$. Note that complex multiplication also changes the a_i, but that is also forced by the above relationship between the α_i and a_i. Next, we want the automorphism group generated by φ not only to act transitively on the curves E_i, but also on their intersection points

$$E_1 \cap E_2 \cap E_3 = \{(z_1, z_1),\ (z_2, z_2),\ (z_3, z_3)\}.$$

This is necessary because we want the curve C_2 in Y to have a unique singular point. Therefore, we have

$$z_2 - z_1 = \tfrac{1}{3}\gamma, \quad z_3 - z_2 = \tfrac{1}{3}\gamma, \quad z_3 - z_1 = \tfrac{2}{3}\gamma.$$

On the other hand, we have the identities

$$z_1 = \rho z_1 + a_2, \quad z_2 = \rho z_2 + a_2, \quad z_3 = \rho z_3 + a_2,$$

which imply that

$$\rho(z_2 - z_1) = z_2 - z_1, \quad \rho(z_3 - z_1) = z_3 - z_1, \quad \rho(z_3 - z_2) = z_3 - z_2.$$

It follows that $\frac{1}{3}\gamma = \frac{1}{3}(1-\rho)$ or $\frac{1}{3}\gamma = \frac{2}{3}(1-\rho)$, since $z_2 - z_1$ is a nonzero point on the elliptic curve E_ρ stable under complex multiplication by ρ. We will later see that each choice gives an isomorphic quotient, so we assume for now that $\frac{1}{3}\gamma = \frac{1}{3}(1-\rho)$.

Using this information together with the previously derived formulas, we obtain

$$a_2 = -\tfrac{1}{3}(1-\rho), \quad a_3 = -\tfrac{2}{3}(1-\rho),$$

so

$$E_1 = (z, z), \quad E_2 = \bigl(\rho z - \tfrac{1}{3}(1-\rho), z\bigr), \quad E_3 = \bigl(\rho^2 z - \tfrac{2}{3}(1-\rho), z\bigr).$$

Next, we compute that

$$E_1 \cap E_2 \cap E_3 = \bigl\{\bigl(\tfrac{2}{3}, \tfrac{2}{3}\bigr), \bigl(\tfrac{2}{3}\rho, \tfrac{2}{3}\rho\bigr), \bigl(\tfrac{2}{3}\rho^2, \tfrac{2}{3}\rho^2\bigr) = \bigl(\tfrac{1}{3}(1+\rho), \tfrac{1}{3}(1+\rho)\bigr)\bigr\}.$$

In conclusion, the curves E_1, E_2, E_3 are uniquely determined. Thus, the bielliptic surface Y and the curve C_2 are also uniquely determined. We now show that Y is independent of our choice of the 3–torsion point.

Lemma 5.13 *The bielliptic surface Y and the curve C_2 described above are independent of the choice of $\frac{1}{3}\gamma$.*

Proof The two possibilities for the α_i and $\frac{1}{3}\gamma$ are

- (z, z), $\bigl(\rho z - \tfrac{1}{3}(1-\rho), z\bigr)$, $\bigl(\rho^2 z - \tfrac{2}{3}(1-\rho), z\bigr)$;
- (z, z), $\bigl(\rho z - \tfrac{2}{3}(1-\rho), z\bigr)$, $\bigl(\rho^2 z - \tfrac{1}{3}(1-\rho), z\bigr)$.

Let X and X' be the bielliptic surfaces obtained by taking the quotient of $E_\rho \times E_\rho$ by the automorphism groups generated by

$$\varphi(w, z) = \bigl(\rho w, z + \tfrac{1}{3}(1-\rho)\bigr) \quad \text{and} \quad \varphi'(w, z) = \bigl(\rho w, z + \tfrac{2}{3}(1-\rho)\bigr),$$

respectively. The first configuration must be considered in X, while the latter must be considered in X'. The self-isomorphism of $E_\rho \times E_\rho$ given by $\phi\colon (w, z) \to (-w, -z)$ takes the first configuration to the second and descends to an isomorphism between X and X'. This proves that the two choices determine the same isomorphism class of bielliptic quotient X with the same singular curve C_2. □

Next, we must find the possible elliptic curves C_1 in Y intersecting C_2 in its unique singular point such that $C_1 \cdot C_2 = 3$; see the list given in (3). The first and most obvious choice is to let C_1 be the unique fiber B of the Albanese map passing through the singular point of C_2. Notice that since $B \cdot C_2 = 3$, the intersection is transverse. Thus, let $C_1 = B$ and consider the pair (Y, C), where $C = B + C_2$, and let X denote the blowup of Y at the singular point p of C_2. Let D_i be the proper transform of C_i and $D = D_1 + D_2$. We then claim that the pair (X, D) is a smooth toroidal compactification. Indeed, it saturates the logarithmic Bogomolov–Miyaoka–Yau inequality:

$$\bar{c}_1^2 = (K_X + D)^2 = K_X^2 - D_1^2 - D_2^2 = -1 + 3 + 1 = 3\bar{c}_2.$$

Next, we can take C_1' as the unique fiber of π_2 passing through the singular point of C_2. Notice that this is not a multiple fiber and, since $C_2 \cdot A = 3$, the intersection is transverse. Consider the pair (Y, C'), where $C' = C_1' + C_2$, and blow up the singular point p of C_2. Let X denote the blowup of Y, let D_1' and D_2 denote the proper transforms of C_1' and C_2, respectively, and set $D' = D_1' + D_2$. Again, it is easy to check that it saturates the logarithmic Bogomolov–Miyaoka–Yau inequality, so the pair (X, D') is a smooth toroidal compactification.

Finally, we argue that these are the unique smooth toroidal compactifications coming from a $\mathbb{Z}/3\mathbb{Z}$ bielliptic surface. If $\frac{1}{3}k_1 A + k_2 B$ is another possible choice, it is a smooth elliptic curve of self-intersection zero, so $2k_1k_2 = 0$. In other words, it must be a multiple of $\frac{1}{3}A$ or B. This curve also must have intersection number 3 with C_2, which leaves us only with the above two choices, C_1 and C_1'.

5.3 Discussion of the second and third examples

Let (X, D) and (X, D') be the toroidal compactification found in Section 5.2. We want to show that those compactifications are associated with two distinct complex hyperbolic surfaces. Assume this is not the case. There exists an automorphism $\Psi\colon X \to X$ sending $D_1' + D_2$ to $D_1 + D_2$, since the map of complex hyperbolic surfaces takes cusps to cusps. The first claim is that we necessarily must have $\Psi(D_2) = D_2$ and $\Psi(D_1') = D_1$. Observe that X is the blowup at a single point of a bielliptic surface. Thus, inside X there is a unique rational curve that is the exceptional divisor, say E. This implies that $\Psi(E) = E$. Now $E \cdot D_2 = 3$ while $E \cdot D_1 = E \cdot D_1' = 1$. The claim then follows.

Let Y denote the bielliptic surface obtained by contracting the exceptional divisor E. Also, let C_1, C_1', and C_2 denote the blowdown transform of the curves D_1, D_1', and C_2. Next, let $\psi\colon Y \to Y$ denote the automorphism on Y induced by Ψ on X, which exists since Ψ must preserve the exceptional curve of the blowup. Observe that $\psi(C_2) = C_2$ and $\psi(C_1') = C_1$.

Fact 5.14 *There exists no such automorphism of X.*

Proof Recall that C_1 is numerically equivalent to B and C_1' is numerically equivalent to A. Let $\psi_*\colon \mathrm{Num}(Y) \to \mathrm{Num}(Y)$ be the induced automorphism. Since $\psi(A) = B$, we have

$$\psi_*\big(\tfrac{1}{3}A\big) = \tfrac{1}{3}B \notin \mathrm{Num}(Y),$$

which is a contradiction. □

5.4 The classification in the case of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ quotients

Let Y be a $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ bielliptic quotient. By Theorem 5.1, we can find two elliptic curves E_ρ and E_τ, respectively associated with the lattices $\mathbb{Z}[1,\rho]$ and $\mathbb{Z}[1,\tau]$, such that

$$Y = (E_\rho \times E_\tau)/(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}).$$

More precisely, $\rho = e^{\frac{2\pi i}{3}}$ while τ is arbitrary, and the $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ group of automorphisms of $E_\rho \times E_\tau$ is generated by the commuting order-three automorphisms

$$\varphi_1(w,z) = \bigl(\rho w, z + \tfrac{1}{3}\gamma\bigr), \quad \varphi_2(w,z) = \bigl(w + k \cdot \tfrac{1}{3}(1-\rho), z + \tfrac{1}{3}\gamma'\bigr)$$

for some $\gamma, \gamma' \in \mathbb{Z}[1,\tau]$ and $k = 1$ or 2. Consider the group $\mathrm{Num}(Y)$ of divisors on Y up to numerical equivalence. If A and B are the generic fibers of the two elliptic fibrations $\pi_1\colon Y \to \mathbb{P}^1 = E_\rho/(\mathbb{Z}/3\mathbb{Z})$ and $\pi_2\colon Y \to E_\tau/(\mathbb{Z}/3\mathbb{Z})$, then $\frac{1}{3}A$ and $\frac{1}{3}B$ form a basis of $\mathrm{Num}(Y)$. Therefore, up to numerical equivalence we can write any curve C as $\frac{1}{3}k_1 A + \frac{1}{3}k_2 B$ for $k_1, k_2 \in \mathbb{Z}$. Notice that π_2 is the Albanese map, and all its fibers are generic. The class $\frac{1}{3}A$ represents a multiple fiber of the map π_1 with multiplicity one. Finally, we have $A \cdot B = 9$.

As in the $\mathbb{Z}/3\mathbb{Z}$ case, it suffices to consider the case where C_1 is an elliptic curve of self-intersection zero and $C_2^2 = 6$. Up to numerical equivalence, we can write $C_2 = \frac{1}{3}k_1 A + \frac{1}{3}k_2 B$ where $\frac{1}{3}A$ and $\frac{1}{3}B$ are the above basis of $\mathrm{Num}(Y)$. There are two possibilities:

$$\begin{aligned} k_1 &= 1, \quad k_2 = 3, \quad C_2 = \tfrac{1}{3}A + B; \\ k_1 &= 3, \quad k_2 = 1, \quad C_2 = A + \tfrac{1}{3}B. \end{aligned}$$

Proposition 5.15 *The case* $C_2 = A + \frac{1}{3}B$ *cannot occur.*

Proof In this case, $C_2 \cdot A = 3$ and $C_2 \cdot B = 9$. Then $\pi^*(C_2)$ is numerically equivalent to $3E_\rho + 9E_\tau$, and Lemma 5.3 implies that it has three irreducible components, each of which is a smooth elliptic curve. Let E_1, E_2 and E_3 denote the three elliptic curves. We have a one-to-one map from E_i to E_ρ and a three-to-one map from E_i to E_τ for each $i = 1, 2, 3$.

There are two cases to consider, associated with the two isomorphism classes of degree-3 quotients of E_ρ:

$$E_\rho \times E_{\rho/(1-\rho)}, \quad E_\rho \times E_{\rho/3},$$

where $E_{\rho/(1-\rho)}$ is the quotient of $\mathbb{C}$ by $(1/(1-\rho))\mathbb{Z}[1,\rho]$, and $E_{\rho/3}$ denotes the quotient of $\mathbb{C}$ by $\mathbb{Z}\bigl[1, \frac{1}{3}\rho\bigr]$. In the first case, notice that $\mathbb{Z}[3, 1-\rho] = (1-\rho)\mathbb{Z}[1,\rho]$ is an index-3 subring of $\mathbb{Z}[1,\rho]$, and $E_\rho \simeq E_{1-\rho} = \mathbb{C}/\mathbb{Z}[3, 1-\rho]$ under its self-isogeny of degree 3. In order to simplify the computation, we can replace E_ρ with $E_{1-\rho}$, and the first case becomes $E_{1-\rho} \times E_\rho$. We first rule out that situation.

Claim *The case $E_{1-\rho} \times E_\rho$ cannot occur.*

Proof We work with coordinates (w, z). Complex multiplication by ρ on the curve $E_{1-\rho}$ has fixed points $0, 1, 2$. Therefore, the automorphisms of $E_{1-\rho} \times E_\rho$ of interest for bielliptic quotients are

$$\varphi_1(w,z) = \big(\rho w, z + \tfrac{1}{3}\gamma\big), \quad \varphi_2(w,z) = \big(w + k, z + \tfrac{1}{3}\gamma'\big)$$

for some $\gamma, \gamma' \in \mathbb{Z}[1,\rho]$ and $k = 1$ or 2.

Notice that no E_i can be numerically equivalent to the second factor of the product, since the E_i must all intersect in a point and the $(\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z})$–orbit the second factor is a collection of disjoint curves. Since each E_i is isomorphic to the first factor, up to translation in the z–direction we can assume

$$E_1 = (w, \alpha_1 w), \quad E_2 = (w, \alpha_2 w + a_2), \quad E_3 = (w, \alpha_3 w + a_3),$$

where the α_i and a_i are, as of yet, unknown. To be well defined, we only need $\alpha_i \in (1/(1-\rho))\mathbb{Z}[1,\rho]$, but since each E_i has intersection number 3 with the curve $(w, 0)$, we see that α_i is a unit of $\mathbb{Z}[1,\rho]$; ie each α_i is a power of $\zeta = e^{\frac{\pi i}{3}}$.

We now claim that the $\mathbb{Z}/3\mathbb{Z}$ isotropy group of each E_i is generated by φ_2. To prove this, by Lemma 5.3, the isotropy group of each E_i in $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Since $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ acts transitively on the E_i, the stabilizers of each E_i are all the same subgroup. We must rule out φ_1, $\varphi_1\varphi_2$, and $\varphi_1\varphi_2^2$ from being in the isotropy group of E_i, and it suffices to focus on E_1.

If $\varphi_1(E_1) = E_1$, for every $w \in E_1$ we would have some $w' \in E_1$ such that

$$\varphi_1(w, \alpha_1 w) = \big(\rho w, \alpha_1 w + \tfrac{1}{3}\gamma\big) = (w', \alpha_1 w'),$$

which then implies that

$$\begin{aligned} \rho w - w' &\equiv 0 \quad \mod \mathbb{Z}[3, 1-\rho], \\ \alpha_1(w - w') &\equiv \tfrac{1}{3}\gamma \quad \mod \mathbb{Z}[1,\rho] \end{aligned}$$

for all $w \in E_{1-\rho}$, and hence

$$\alpha_1(\rho - 1)w \equiv \tfrac{1}{3}\gamma \quad \mod \mathbb{Z}[1,\rho]$$

for all w. This is clearly impossible for transcendental w.

Similarly, one can show that the isotropy group cannot be generated by $\varphi_1\varphi_2$. In fact, the identity $\varphi_1(\varphi_2(E_1)) = E_1$ gives

$$\alpha_1(\rho-1)w = -\alpha_1 k + \tfrac{1}{3}\gamma + \tfrac{1}{3}\gamma' \mod \mathbb{Z}[1,\rho]$$

for all $w \in E_{1-\rho}$. This again cannot hold for all w. The same argument rules out $\varphi_1\varphi_2^2$ after replacing k with $2k$ and $\frac{1}{3}\gamma'$ with $\frac{2}{3}\gamma'$. Thus the isotropy group of each E_i must be generated by φ_2.

Consequently, φ_1 acts transitively on the E_i, so up to renumbering we can assume

$$\varphi_1(E_1) = E_2, \quad \varphi_1(E_2) = E_3, \quad \varphi_1(E_3) = E_1.$$

From $\varphi_1(E_1) = E_2$, we see that for all $w \in \mathbb{C}$, there is a $w' \in \mathbb{C}$ such that

$$\big(\rho w, \alpha_1 w + \tfrac{1}{3}\gamma\big) = (w', \alpha_2 w' + a_2).$$

In other words, $\rho w - w' \in \mathbb{Z}[3, 1-\rho]$ and

$$\alpha_1 w + \tfrac{1}{3}\gamma - \alpha_2 w' - a_2 \in \mathbb{Z}[1,\rho].$$

Combining these two congruences gives

$$(\alpha_1 - \alpha_2\rho)w + \tfrac{1}{3}\gamma - a_2 \in \mathbb{Z}[1,\rho].$$

Since this holds for all w, we must have $(\alpha_2\rho - \alpha_1) \equiv 0 \mod \mathbb{Z}[1,\rho]$. It then follows that $\frac{1}{3}\gamma - a_2 \in \mathbb{Z}[1,\rho]$. Analogous arguments show that

$$\alpha_3\rho - \alpha_2,\ \tfrac{2}{3}\gamma - a_3,\ \alpha_1\rho - \alpha_3 \in \mathbb{Z}[1,\rho].$$

Enumerating all the possibilities for the α_i, we see that there is always some α_i that is either ± 1. First, assume $\alpha_i = 1$. From $\varphi_2(E_i) = E_i$, we then see that

$$\varphi_2(w, w + a_i) = \big(w + k, w + a_i + \tfrac{1}{3}\gamma'\big)$$

(with $a_1 = 0$) for all $w \in \mathbb{C}$, where the first coordinate is taken modulo $\mathbb{Z}[3, 1-\rho]$ and the second is modulo $\mathbb{Z}[1,\rho]$. If this equals $(w', w' + a_i)$, then $w \equiv w' \mod \mathbb{Z}[1,\rho]$ follows from $w + k \equiv w' \mod \mathbb{Z}[3, 1-\rho]$ and $k = 1$ or 2. Combining this with

$$w + a_i + \tfrac{1}{3}\gamma' - w' - a_i \in \mathbb{Z}[1,\rho]$$

allows one to conclude that $\frac{1}{3}\gamma' \in \mathbb{Z}[1,\rho]$, which is a contradiction. Taking $\alpha_i = -1$ leads to the exact same contradiction. This rules out the case $E_{1-\rho} \times E_\rho$. □

Claim *The case $E_\rho \times E_{\rho/3}$ also cannot occur.*

Proof Consider the automorphisms

$$\varphi_1(w,z) = \big(\rho w, z + \tfrac{1}{3}\gamma\big), \quad \varphi_2(w,z) = \big(w + k\cdot\tfrac{1}{3}(1-\rho), z + \tfrac{1}{3}\gamma'\big)$$

that define our bielliptic quotient, where $\gamma, \gamma' \in \mathbb{Z}\big[1, \tfrac{1}{3}\rho\big]$ and $k = 1$ or 2. Since E_i is isomorphic to the first factor of the product, up to translation in the z–direction we can write

$$E_1 = (w, \alpha_1 w), \quad E_2 = (w, \alpha_2 w + a_2), \quad E_3 = (w, \alpha_3 w + a_3),$$

where the α_i and a_i are complex numbers to be determined. First, to give a well-defined elliptic curve, α_i must have the property that $\alpha_i\lambda \in \mathbb{Z}\big[1, \tfrac{1}{3}\rho\big]$ for all $\lambda \in \mathbb{Z}[1,\rho]$, which actually implies that $\alpha_i \in \mathbb{Z}[1,\rho]$. Next, since for each $i = 1,2,3$ we want $E_i \cdot (w,0) = 3$ to obtain the desired configuration on the bielliptic surface, α_i must be a root of unity.

We now claim that the group generated by $\varphi_1\varphi_2^r$ cannot be the isotropy group of the $(\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z})$–action for $r = 0,1,2$. If $\varphi_1\varphi_2^r(E_1) = E_1$, then for all $w \in \mathbb{C}$, there is a $w' \in \mathbb{C}$ such that

$$\big(\rho w + rk\cdot\tfrac{1}{3}(1-\rho),\ \alpha_1 w + \tfrac{1}{3}\gamma + r\cdot\tfrac{1}{3}\gamma'\big) = (w', \alpha_1 w')$$

on $E_\rho \times E_{\rho/3}$. This implies that $\rho w - w'$ is congruent to $rk\cdot\frac{1}{3}(1-\rho)$ modulo $\mathbb{Z}[1,\rho]$, and we can conclude that

$$\alpha_1(1-\rho)w + \tfrac{1}{3}\gamma + r\cdot\tfrac{1}{3}\gamma' \in \mathbb{Z}\big[1, \tfrac{1}{3}\rho\big].$$

This cannot hold for all $w \in \mathbb{C}$, so $\varphi_1\varphi_2^r$ cannot stabilize E_1.

Then, up to renumbering, we can assume

$$\varphi_1(E_1) = E_2, \quad \varphi_1(E_2) = E_3, \quad \varphi_1(E_3) = E_1.$$

This means that, as points on $E_\rho \times E_{\rho/3}$, we must have

$$(\alpha_2\rho - \alpha_1) = 0, \quad a_2 = \tfrac{1}{3}\gamma, \quad (\alpha_3\rho - \alpha_2) = 0, \quad a_3 = \tfrac{2}{3}\gamma, \quad (\alpha_1\rho - \alpha_3) = 0.$$

This implies that $\{\alpha_1, \alpha_2, \alpha_3\}$ is of the form $\{u, u\rho^2, u\rho\}$ for some (possibly not primitive) 6^{th} root of unity u. Therefore, up to the automorphism of $E_\rho \times E_{\rho/3}$ generated by complex multiplication by u on E_ρ, we can assume that $u = 1$.

Since $\varphi_2(E_1) = E_1$ and E_1 is the curve (w,w), we see that

$$k\cdot\tfrac{1}{3}(1-\rho) = \tfrac{1}{3}\gamma' \mod \mathbb{Z}\big[1, \tfrac{1}{3}\rho\big].$$

Since $\frac{1}{3}(1-\rho) = \frac{1}{3} \bmod \mathbb{Z}\big[1, \frac{1}{3}\rho\big]$, we obtain $\gamma' = k$, which is consistent with $\varphi_2(E_i) = E_i$ for all $i = 1, 2, 3$.

In conclusion, we have the configuration

$$E_1 = (w, w), \quad E_2 = \big(w, \rho^2 w + \tfrac{1}{3}\gamma\big), \quad E_3 = \big(w, \rho w + \tfrac{2}{3}\gamma\big)$$

for some $\gamma \in \mathbb{Z}\big[1, \frac{1}{3}\rho\big]$. Recall that we must have

$$E_1 \cap E_2 = E_1 \cap E_3 = E_2 \cap E_3.$$

For a point (w, w) on $E_1 \cap E_2$, we have

$$(1-\rho^2)w - \tfrac{1}{3}\gamma \in \mathbb{Z}\big[1, \tfrac{1}{3}\rho\big],$$

and for a point (w, w) on $E_1 \cap E_3$ we similarly have

$$(1-\rho)w - \tfrac{2}{3}\gamma \in \mathbb{Z}\big[1, \tfrac{1}{3}\rho\big].$$

Consider the point $w = \gamma/(3(1-\rho^2))$ on $E_1 \cap E_2$. For this point to lie on $E_1 \cap E_3$, we have

$$(1-\rho)\left(\frac{\gamma}{3(1-\rho^2)}\right) - \tfrac{2}{3}\gamma = -(\rho+2)\cdot\tfrac{1}{3}\gamma \in \mathbb{Z}\big[1, \tfrac{1}{3}\rho\big].$$

Write $\gamma = a + b\cdot\frac{1}{3}\rho$ for $a, b \in \{0, 1, 2\}$, which we can do because we only care about $\frac{1}{3}\gamma$ as a 3–torsion point on $E_{\rho/3}$. Then

$$(-\rho-2)\cdot\tfrac{1}{3}\gamma = -\tfrac{1}{9}(6a-b) - \tfrac{1}{9}(3a+b)\rho \in \mathbb{Z}\big[1, \tfrac{1}{3}\rho\big].$$

However, then $6a-b$ is divisible by 9 for $a, b \in \{0, 1, 2\}$. This is only possible if $a = b = 0$, which implies that we can take $\gamma = 0$ in the definition of φ_1. However, this is impossible, since φ_1 then cannot act freely on $E_\rho \times E_{\rho/3}$. This contradiction proves the claim. □

This rules out all possibilities where C_2 is numerically equivalent to $A + \frac{1}{3}B$, which proves the proposition. □

We are then left with the case $C_2 = \frac{1}{3}A + B$. Let E_1, E_2, and E_3 denote the three smooth elliptic curves in $\pi^{-1}(C_2)$. Since $C_2\cdot A = 9$, we have that for each $i = 1, 2, 3$ there is a three-to-one map from E_i to E_ρ. Next, since $C_2\cdot B = 3$ we have a one-to-one map from E_i to E_τ for each $i = 1, 2, 3$. We therefore conclude that Y is a quotient of $E_\rho \times E_\tau$, where E_τ is a degree-3 cover of E_ρ.

As in previous cases, up to translation in the w–direction we can write

$$E_1 = (\alpha_1 z, z), \quad E_2 = (\alpha_2 z + a_2, z), \quad E_3 = (\alpha_3 z + a_3, z).$$

Here, each E_i is a 1–section of the map $E_\rho \times E_\tau \to E_\tau$ and $E_i \cdot E_\tau = 3$, and it follows that each α_i is a power of $\zeta = e^{\frac{\pi i}{3}}$.

Up to isomorphism, the degree-3 covers of E_ρ are associated with the lattices $\mathbb{Z}[3, 1-\rho]$ and $\mathbb{Z}[3, \rho]$, as one can easily see by enumerating the index-3 subgroups of $\mathbb{Z}[1, \rho]$. For simplicity, we let E_τ denote one of these degree-3 covers of E_ρ. The $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ automorphism group of $E_\rho \times E_\tau$ is then generated by the following commuting automorphisms of order three:

$$\varphi_1(w, z) = \left(\rho w, z + \tfrac{1}{3}\gamma\right), \quad \varphi_2(w, z) = \left(w + k \cdot \tfrac{1}{3}(1-\rho), z + \tfrac{1}{3}\gamma'\right),$$

where $\gamma, \gamma' \in \mathbb{Z}[3, \rho]$ or $\gamma, \gamma' \in \mathbb{Z}[3, 1-\rho]$ and k is an integer.

Next, the $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ group of automorphism must act transitively on the three elliptic curves E_1, E_2, E_3 with isotropy group $\mathbb{Z}/3\mathbb{Z}$. We claim that the isotropy group must be generated by φ_2. If not, assume that $\varphi_2(E_1) = E_i$ for some $i \neq 1$. This implies that for all $w \in \mathbb{C}$, there is a $w' \in \mathbb{C}$ such that

$$\begin{aligned} w' &\equiv w + \tfrac{1}{3}\gamma' && \mod \mathbb{Z}[1, \rho], \\ \alpha_i w' + a_i &\equiv \alpha_1 w + k \cdot \tfrac{1}{3}(1-\rho) && \mod \mathbb{Z}[1, \rho]. \end{aligned}$$

Combining these gives

$$(\alpha_1 - \alpha_i)w - \alpha_i \cdot \tfrac{1}{3}\gamma' + k \cdot \tfrac{1}{3}(1-\rho) - a_i \in \mathbb{Z}[1, \rho]$$

for all $w \in \mathbb{C}$. Taking a transcendental w, this is impossible unless $\alpha_i = \alpha_1$, which is a contradiction. Therefore φ_2 is contained in the isotropy group of E_1, and hence generates the isotropy group of every E_i.

Now, up to renumbering we can assume $\varphi_1(E_i) = E_{i+1}$, where the index i is considered modulo 3. We then have that

$$\rho\alpha_1 - \alpha_2, \ \rho\alpha_2 - \alpha_3, \ \rho\alpha_3 - \alpha_1 \in \mathbb{Z}[1, \rho].$$

As in previous cases, this implies that $\{\alpha_1, \alpha_2, \alpha_3\}$ is of the form $\{u, u\rho, u\rho^2\}$ for some 6^{th} root of unity u, and up to an automorphism of the abelian surface, we can assume $u = 1$.

Then, $\varphi_2(E_1) = E_1$ implies that $\frac{1}{3}\gamma' \equiv k \cdot \frac{1}{3}(1-\rho)$ modulo $\mathbb{Z}[1, \rho]$. Assume that $\gamma' \in \mathbb{Z}[3, \rho]$. We then have $\gamma' = 3a + b\rho$ with $a, b \in \mathbb{Z}$, which implies that k is 3.

However, k must be congruent to 1 or 2 modulo 3, since the $(\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z})$–action by definition has nontrivial translation part on the first factor, so this is a contradiction.

We therefore conclude that $E_\tau = E_{1-\rho}$ is associated with the lattice $\mathbb{Z}[3, 1-\rho]$ and that $\gamma' = k(1-\rho)$ for k an integer, since

$$k(1-\rho) - \gamma' \in 3\mathbb{Z}[1, \rho] \subset \mathbb{Z}[3, 1-\rho].$$

From the fact that $\varphi_1(E_1) = E_2$ we obtain $a_2 = -\rho\cdot\frac{1}{3}\gamma$. Similarly, since $\varphi_1(E_2) = E_3$, we have $a_3 = -\rho\cdot\frac{2}{3}\gamma$. In conclusion, we have that the curves E_i are uniquely determined by the equations

$$E_1 = (z, z), \quad E_2 = \bigl(\rho z - \rho\cdot\tfrac{1}{3}\gamma, z\bigr), \quad E_3 = \bigl(\rho^2 z - 2\rho\cdot\tfrac{1}{3}\gamma, z\bigr).$$

Suppose first that $\gamma = 3 \in \mathbb{Z}[3, 1-\rho]$. We then have

$$E_1 = (z, z), \quad E_2 = (\rho z, z), \quad E_3 = (\rho^2 z, z),$$

and see that $E_1 \cap E_2 = E_1 \cap E_3 = E_2 \cap E_3$ and $E_1 \cap E_2$ equals

$$\bigl\{(0,0),\ (0,1),\ (0,2),\ \bigl(\tfrac{1}{3}1-\rho, \tfrac{1}{3}1-\rho\bigr),\ \bigl(\tfrac{1}{3}1-\rho, 1+\tfrac{1}{3}1-\rho\bigr),\ \bigl(\tfrac{1}{3}1-\rho, 2+\tfrac{1}{3}1-\rho\bigr),$$
$$\bigl(\tfrac{2}{3}(1-\rho), \tfrac{2}{3}(1-\rho)\bigr),\ \bigl(\tfrac{2}{3}(1-\rho), 1+\tfrac{2}{3}(1-\rho)\bigr),\ \bigl(\tfrac{2}{3}(1-\rho), 2+\tfrac{2}{3}(1-\rho)\bigr)\bigr\}.$$

The $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ action acts transitively on these nine points, so we obtain a curve with a unique singular point of order 3 in the bielliptic quotient. In other words, the choice $\gamma = 3$ determines a bielliptic surface Y with a curve C_2 numerically equivalent to $\frac{1}{3}A + B$ with a unique singular point of order 3.

Proposition 5.16 *Any other choice of* $\gamma \in \mathbb{Z}[3, 1-\rho]$ *either gives an isomorphic configuration as* $\gamma = 3$ *or gives a configuration that descends to a curve in the quotient with three singular points* (*and hence cannot determine a smooth toroidal compactification of Euler number one*).

Proof Define automorphisms

$$\psi_1(w, z) = (-w, -z), \quad \psi_2(w, z) = \bigl(w + \tfrac{2}{3}, z + \tfrac{2}{3}\bigr)$$

of $E_\rho \times E_{1-\rho}$. Suppose that $\frac{1}{3}\gamma$ a 3–torsion point on $E_{1-\rho}$, and consider the curves

$$E_1 = (z, z), \quad E_2(\gamma) = \bigl(\rho z + \rho\cdot\tfrac{1}{3}\gamma\bigr), \quad E_3(\gamma) = \bigl(\rho^2 + \rho\cdot\tfrac{2}{3}\gamma\bigr).$$

A direct calculation shows that

$$\begin{array}{c} \psi_i(E_1) = E_1 \\ \text{for } i = 1,2, \end{array} \quad \begin{array}{ll} \psi_1(E_2(\gamma)) = E_2(-\gamma), & \psi_1(E_3(\gamma)) = E_3(-\gamma), \\ \psi_2(E_2(\gamma)) = E_2(\gamma+2(1-\rho)), & \psi_2(E_3(\gamma)) = E_3(\gamma+2(1-\rho)), \\ \psi_2^2(E_2(\gamma)) = E_2(\gamma+(1-\rho)), & \psi_2^2(E_3(\gamma)) = E_3(\gamma+(1-\rho)). \end{array}$$

These maps descend to isomorphisms of the associated bielliptic quotients, and the orbit of $\gamma = 3$ under this action has cardinality six. In particular, six of the eight nontrivial 3–torsion points on $E_{1-\rho}$ all determine the same pair (Y, C_2).

The remaining points not in the orbit of $\gamma = 3$ under the group generated by ψ_1 and ψ_2 are $\gamma = (1-\rho)$ and $2(1-\rho)$. We now show that these cases cannot give rise to a smooth toroidal compactification of Euler number one.

For $\gamma = (1-\rho)$, we have

$$E_1 = (z, z), \quad E_2 = \left(\rho z - \tfrac{1}{3}(1-\rho), z\right), \quad E_3 = \left(\rho^2 z - \tfrac{2}{3}(1-\rho), z\right).$$

We see that $E_1 \cap E_2 = E_1 \cap E_3 = E_2 \cap E_3$, and $E_1 \cap E_2$ equals

$$\begin{aligned} &\big\{\left(\tfrac{2}{3}, \tfrac{2}{3}\right), \left(\tfrac{2}{3}, \tfrac{2}{3}+1\right), \left(\tfrac{2}{3}, \tfrac{2}{3}+2\right), \\ &\left(\tfrac{2}{3}+\tfrac{1}{3}(1-\rho), \tfrac{2}{3}+\tfrac{1}{3}(1-\rho)\right), \left(\tfrac{2}{3}+\tfrac{1}{3}(1-\rho), \tfrac{2}{3}+1+\tfrac{1}{3}(1-\rho)\right), \\ &\left(\tfrac{2}{3}+\tfrac{1}{3}(1-\rho), \tfrac{2}{3}+2+\tfrac{1}{3}(1-\rho)\right), \left(\tfrac{2}{3}+\tfrac{2}{3}(1-\rho), \tfrac{2}{3}+\tfrac{2}{3}(1-\rho)\right), \\ &\left(\tfrac{2}{3}+\tfrac{2}{3}(1-\rho), \tfrac{2}{3}+1+\tfrac{2}{3}(1-\rho)\right), \left(\tfrac{2}{3}+\tfrac{2}{3}(1-\rho), \tfrac{2}{3}+2+\tfrac{2}{3}(1-\rho)\right)\big\}. \end{aligned}$$

Next, observe that the orbit of the point $\left(\tfrac{2}{3}, \tfrac{2}{3}\right)$ under the action of the order-three isomorphism φ_1 is

$$\big\{\left(\tfrac{2}{3}, \tfrac{2}{3}\right), \left(\tfrac{2}{3}+\tfrac{1}{3}(1-\rho), \tfrac{2}{3}+\tfrac{1}{3}(1-\rho)\right), \left(\tfrac{2}{3}+\tfrac{2}{3}(1-\rho), \tfrac{2}{3}+\tfrac{2}{3}(1-\rho)\right)\big\}.$$

Now,

$$\varphi_2(w, z) = \left(w + k \cdot \tfrac{1}{3}(1-\rho),\ z + k \cdot \tfrac{1}{3}(1-\rho)\right),$$

so the orbit of the point $\left(\tfrac{2}{3}, \tfrac{2}{3}\right)$ under this automorphism is the same as above. Therefore, the curve C determined by the images of the E_i in the bielliptic quotient determined by φ_1 and φ_2 has three order-3 singular points. In particular, it does not satisfy the criteria necessary to determine a smooth toroidal compactification of Euler number one. Indeed, the proper transform of this curve under the blowup at one point will remain singular, and hence cannot be one of the smooth elliptic curves in a compactification divisor.

The case $\gamma = 2(1-\rho)$ is isomorphic to the first under ψ_1, and hence also cannot occur. This rules out $\gamma = k(1-\rho)$ for $k = 1, 2$ from consideration, and completes the proof of the proposition. □

Thus the bielliptic surface Y and the curve C_2 are therefore uniquely determined. Next, we have to find smooth elliptic curves E in Y intersecting C_2 in its unique singular point such that $C_2 \cdot E = 3$.

First, consider the unique fiber C_1 of the Albanese map, which is numerically equivalent to B, passing through the singular point of C_2. Since $C_2 \cdot B = 3$, the intersection is transverse. Thus, consider the pair (Y, C), where $C = C_1 + C_2$, and let X be the blowup of Y at the singular point p of C_2. Let D_1 and D_2 be the proper transforms of C_1 and C_2, respectively, and set $D = D_1 + D_2$. Then (X, D) saturates the logarithmic Bogomolov–Miyaoka–Yau inequality, and hence defines a smooth toroidal compactification.

Next, notice that the singular point $p \in C_2$ corresponds to the point $(0,0) \in E_\rho \times E_\tau$. Therefore, the fiber C_1' of π_2 passing through the point p must be a multiple fiber, and C_1' is numerically equivalent to $\frac{1}{3}A$. Since $C_2 \cdot \frac{1}{3}A = 3$, the intersection of C_1' with C_2 is transverse. Consequently, consider the pair (Y, C) with $C = C_1' + C_2$, let X be the blowup of Y at the singular point p of C_2, and let D_1' and D_2 be the proper transforms of C_1' and C_2, respectively. Again, (X, D'), where $D' = C_1' + C_2$, saturates the logarithmic Bogomolov–Miyaoka–Yau inequality.

The exact same argument as in Section 5.2 shows that (X, D) and (X, D') are the unique smooth toroidal compactifications arising from a $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ bielliptic surface. Indeed, we again see that such a curve C_1 numerically equivalent to $\frac{1}{3}k_1 A + \frac{1}{3}k_2 B$ has self-intersection $2k_1k_2 = 0$, so C_1 is a multiple of $\frac{1}{3}A$ or $\frac{1}{3}B$. Then $C_1 \cdot C_2 = 3$ leaves us with only the two possibilities considered above.

5.5 Discussion of the fourth and fifth examples

Let (X, D) and (X, D') be the toroidal compactifications found in Section 5.4. We want to show that these compactifications are associated with two distinct complex hyperbolic surfaces. Assume this is not the case. There exists an automorphism $\Psi\colon X \to X$ sending $D_1' + D_2$ to $D_1 + D_2$. The first claim is that we necessarily must have $\Psi(D_2) = D_2$ and $\Psi(D_1') = D_1$. Observe that X is the blowup at a single point of a bielliptic surface. Thus, inside X there is a unique rational curve that is the exceptional divisor say E. This implies that $\Psi(E) = E$. Now $E \cdot D_2 = 3$ while $E \cdot D_1 = E \cdot D'_1 = 1$. The claim then follows.

Let Y denote the bielliptic surface obtained by contracting the exceptional divisor E. Also, let C_1, C'_1, and C_2 denote the blowdown transform of the curves D_1, D'_1, and D_2. Next, let $\psi\colon Y \to Y$ denote the automorphism on Y induced by Ψ on X. Observe that $\psi(C_2) = C_2$ and $\psi(C'_1) = C_1$.

Fact 5.17 *There exists no such automorphism of X.*

Proof Recall that C'_1 is numerically equivalent to $\frac{1}{3}A$ and C_1 is numerically equivalent to B. Let $\psi_*\colon \mathrm{Num}(Y) \to \mathrm{Num}(Y)$ be the induced automorphism. Since $\psi(C_2) = C_2$ and $\psi(C'_1) = C_1$, we have $\psi_*(B) = \frac{1}{3}A$ and then

$$\psi_*\bigl(\tfrac{1}{3}B\bigr) = \tfrac{1}{9}A \notin \mathrm{Num}(Y),$$

which is a contradiction. □

6 Recap of the five examples

We now give a concise recap of the five examples constructed above. Recall that $\rho = e^{\frac{2\pi i}{3}}$, $\zeta = e^{\frac{\pi i}{3}}$, and E_ρ is the elliptic curve $\mathbb{C}/\mathbb{Z}[1,\rho]$.

Example 1 Consider $E_\rho \times E_\rho$ with coordinates (w, z), and consider the curves $C_1^{(1)}, \dots, C_4^{(1)}$ on Y_1 defined by

$$w = 0, \quad z = 0, \quad w = z, \quad w = \zeta z.$$

Then $C_1^{(1)} \cap \dots \cap C_4^{(1)} = \{(0,0)\}$. Let X_1 be the blowup of $E_\rho \times E_\rho$ at $(0,0)$, $D_i^{(1)}$ the proper transform of $C_i^{(1)}$ to X_1, and $D_1 = \sum D_i^{(1)}$. Our first example (originally due to Hirzebruch [22]), is the pair (X_1, D_1).

Example 2 Let Y_2 be bielliptic quotient of $E_\rho \times E_\rho$ defined by the automorphism

$$\varphi(w, z) = \bigl(\rho w, z + \tfrac{1}{3}(1-\rho)\bigr)$$

of order 3. Let $C_1^{(2)}$ be the image on Y_2 of the curve $w = 0$ on $E_\rho \times E_\rho$, ie a fiber of the Albanese fibration of Y_2, and $C_2^{(2)}$ the curve on Y_2 defined by the images of the curves

$$E_1 = (z, z), \quad E_2 = \bigl(\rho z - \tfrac{1}{3}(1-\rho), z\bigr), \quad E_3 = \bigl(\rho^2 z - \tfrac{2}{3}(1-\rho), z\bigr)$$

on $E_\rho \times E_\rho$. Then $C_1^{(2)} \cap C_2^{(2)}$ is the image on Y_2 of the origin in $E_\rho \times E_\rho$. This point is a singular point on $C_2^{(2)}$ of order 3 and is the unique singular point on that curve. Let X_2 be the blowup of Y_2 at this point. For $i = 1, 2$, let $D_i^{(2)}$ be the proper transform of $C_i^{(2)}$ in X_2, and define $D_2 = D_1^{(2)} + D_2^{(2)}$. The pair (X_2, D_2) is our second example.

Example 3 Let $Y_3 = Y_2$, $X_3 = X_2$, and $C_2^{(3)} = C_2^{(2)}$ be as in the second example. Consider the fibration $X_3 \to \mathbb{P}^1$ associated with the first coordinate projection of $E_\rho \times E_\rho$ and let $C_1^{(3)}$ be the fiber passing through the unique singular point of $C_2^{(3)}$. Note $C_1^{(3)}$ is not a multiple fiber of the fibration $X_3 \to \mathbb{P}^1$. For $i = 1, 2$, let $D_i^{(3)}$ be the proper transform of $C_i^{(3)}$ in X_3, and $D_3 = D_1^{(3)} + D_2^{(3)}$. Then (X_3, D_3) is our third example.

Example 4 Let $E_{1-\rho}$ be the quotient of $\mathbb{C}$ by $\mathbb{Z}[3, 1-\rho] = (1-\rho)\mathbb{Z}[1, \rho]$, consider $E_\rho \times E_{1-\rho}$, and let Y_4 be the bielliptic quotient defined by the automorphisms

$$\varphi_1(w, z) = (\rho w, z + 1), \quad \varphi_2(w, z) = \big(w + \tfrac{1}{3}(1-\rho), z + \tfrac{1}{3}(1-\rho)\big),$$

which have order 3 and generate an abelian group of order 9. Suppose that $C_2^{(4)}$ is the curve on Y_4 defined by the images of the curves

$$E_1 = (z, z), \quad E_2 = (\rho z, z), \quad E_3 = (\rho^2 z, z)$$

on $E_\rho \times E_{1-\rho}$, and $C_1^{(4)}$ is the fiber of the Albanese map of Y_4 passing through the unique singular point p of $C_2^{(4)}$. Then, let X_4 be the blowup of Y_4 at the point p. For $i = 1, 2$, let $D_i^{(4)}$ be the proper transform of $C_i^{(4)}$ in X_4. Finally, let $D_4 = D_1^{(4)} + D_2^{(4)}$. The fourth example is the pair (X_4, D_4).

Example 5 We take $Y_5 = Y_4$, $C_2^{(5)} = C_2^{(4)}$, and $X_5 = X_4$. Let $Y_5 \to \mathbb{P}^1$ be the fibration associated with the first coordinate projection of $E_\rho \times E_{1-\rho}$. Let $C_1^{(5)}$ be the fiber of such a fibration passing through the unique singular point of $C_2^{(5)}$. More precisely, $C_1^{(5)}$ is the support of a multiple fiber of the fibration $X_5 \to \mathbb{P}^1$. For $i = 1, 2$, let $D_i^{(5)}$ be the proper transform of $C_i^{(5)}$ in X_5. Finally, define $D_5 = D_1^{(5)} + D_2^{(5)}$. Our fifth and final example is the pair (X_5, D_5).

Lemma 6.1 *The above examples are mutually distinct.*

Proof We already know from Section 5.3 that the second and third are distinct, and from Section 5.5 that the fourth and fifth are distinct. For the remaining distinctions, it suffices to compute the first homology groups of the compactifications. Recall that the blowup operation leaves the fundamental group, and hence the first homology group, unchanged. The first example is the blowup at one point of an abelian surface $Y = E_\rho \times E_\rho$, so $H_1(Y, \mathbb{Z}) = \mathbb{Z}^4$. The second and third examples are the blowup of a $\mathbb{Z}/3\mathbb{Z}$ bielliptic surface Y_2, where $H_1(Y_2, \mathbb{Z}) = \mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$; see Serrano [36, page 531]. On the other hand, the fourth and fifth examples are the blowup of a $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ bielliptic surface Y_4. In this case, one can compute that $H_1(Y_4, \mathbb{Z}) = \mathbb{Z}^2$; see [36, page 531]. We then have that all five examples are distinct. □

Remark 6.2 Although the complex hyperbolic surfaces with cusps identified in Examples 2 and 3 (or Examples 4 and 5) are not isomorphic, they nevertheless have biholomorphic smooth toroidal compactifications. We have recently constructed arbitrarily large families of distinct ball quotients with biholomorphic smooth toroidal compactifications. For more details, we refer to Di Cerbo and Stover [15].

7 Proof of Theorem 1.1

We showed above that there are exactly five complex hyperbolic 2–manifolds of Euler number one that admit a smooth toroidal compactification. It remains to show that these five manifolds are commensurable, ie that they all share a common finite-sheeted covering. Since Hirzebruch's ball quotient is arithmetic by Holzapfel [23], arithmeticity of the other four examples follows immediately.

Let $\rho = e^{\frac{2\pi i}{3}}$, $k = \mathbb{Q}(\rho)$, and $\mathcal{O}_k = \mathbb{Z}[1, \rho]$ be its ring of integers. Holzapfel showed that Hirzebruch's example (X_0, D_0) has fundamental group

$$\Gamma_0 = \pi_1(X_0 \smallsetminus D_0),$$

a subgroup of index 72 in the *Picard modular group* $\Gamma = \mathrm{PU}(2, 1; \mathcal{O}_k)$ associated with a hermitian form on k^3 of signature $(2, 1)$. Considering the volume of the Picard modular orbifold $\mathcal{H}^2/\Gamma$, it follows that any subgroup $\Gamma' \subset \Gamma$ of index 72 determines a quotient of $\mathcal{H}^2$ with Euler–Poincaré characteristic one. Consequently, if such a Γ' is torsion-free and every parabolic element of Γ' is rotation-free, then it defines a complex hyperbolic manifold $\mathcal{H}^2/\Gamma'$ that admits a smooth toroidal compactification. Since we classified all such complex hyperbolic manifolds above, any Γ' with these properties defines one of our five smooth toroidal compactifications.

In Stover [37], the appendix contains eight nonisomorphic torsion-free subgroups of the Picard modular group Γ of index 72. In particular, the associated quotients of $\mathcal{H}^2$ are distinct, smooth, and have Euler number one. If we show that exactly five of these subgroups have rotation-free parabolic elements, then these lattices must determine the five smooth toroidal compactifications described in this paper. In particular, the complex hyperbolic manifolds associated with these five surfaces must be commensurable and arithmetic, which proves Theorem 1.1.

The strategy of proof is computational, using the presentation for Γ given by Falbel and Parker [17]. They showed that Γ has a presentation on generators R, P, and Q. Representatives in $\mathrm{GL}_3(\mathbb{Z}[1,\rho])$ for $R, P, Q \in \mathrm{PU}(2,1)$ are given by

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & \rho \\ 0 & \rho & -\rho \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 & \rho \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, $\mathcal{H}^2/\Gamma$ has one cusp, and the unique conjugacy class of parabolic subgroups associated with the cusp is represented by $\Delta = \langle P, Q \rangle$, and Δ fits into an exact sequence

$$1 \to \mathbb{Z} \to \Delta \to \Delta(2,3,6) \to 1,$$

where $\Delta(2,3,6)$ is the $(2,3,6)$ triangle group, and

$$\Delta = \langle P, Q \mid (PQ^{-1})^6, P^3Q^{-2} \rangle.$$

Given a finite-index subgroup $\Gamma' \subset \Gamma$, the conjugacy classes of parabolic subgroups of Γ' are then represented (perhaps with repetition) by the groups

$$\Delta_\sigma = \sigma\Delta\sigma^{-1} \cap \Gamma',$$

where σ runs over all coset representatives of Γ' in Γ. To check that a given Δ_σ contains only rotation-free elements, it suffices to check generators for Δ_σ. Indeed, to check that a parabolic group is rotation-free, it suffices to check that its generators in an appropriate basis are strictly upper-triangular (ie have all 1 on the diagonal).

Using Magma [8], we enumerated the eight torsion-free lattices in Stover [37], and calculated generators for a representative of each conjugacy class of parabolic subgroups (a Magma routine that describes these lattices and finds the conjugacy classes of parabolic subgroups was written for [37], and is available on the second author's website). See Section 7.1 for more details. Using the explicit matrices given by Falbel and Parker's generators, we see that exactly five of these lattices determine smooth toroidal compactifications, namely the third, fourth, fifth, seventh, and eighth examples from the appendix to [37]. These five manifolds must be the five examples described in this paper, and this completes the proof of Theorem 1.1. □

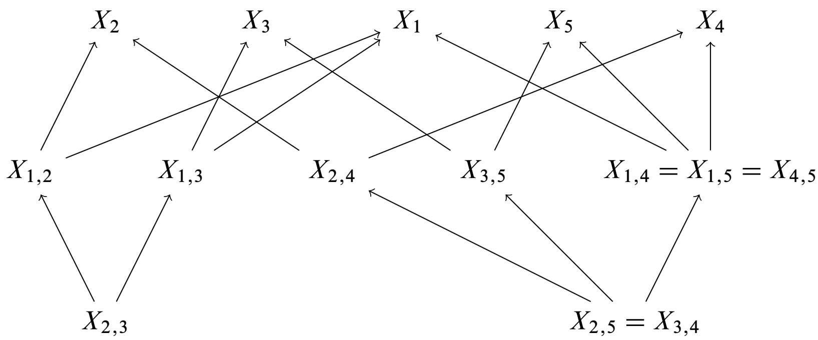

Figure 1: Commensurability relations between the five examples. Each arrow represents a 3–fold covering.

Remark 7.1 Considering first homology groups, it is clear that the third example in [37] is Hirzebruch's example, the fourth and seventh arise from the $\mathbb{Z}/3\mathbb{Z}$ bielliptic surface, and the fifth and eighth arise from the $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ bielliptic surface.

In Figure 1, we give the commensurability relations between the five manifolds, where $X_1, \dots, X_5$ are the manifolds described in Section 6. These were computed with the Magma [8] code from [37].

7.1 More on parabolic subgroups

In this section, we give a few remarks that explicitly connect the cusps of our examples to the group structure of the associated Picard modular group and its unique cusp subgroup. Retaining the notation from the proof of Theorem 1.1, we consider the following subgroups of Δ:

$$\begin{aligned}
\Delta_6 &= \langle [Q, P], [P^{-1}, Q] \rangle, \\
\Delta_{18,a} &= \langle P^3, [QPQ, P], [QP^{-1}Q, P] \rangle, \\
\Delta_{18,b} &= \langle P^3, PQPQ^{-1}P, [QPQ, P^{-1}QP] \rangle, \\
\Delta_{54} &= \langle [QPQ, P], P^{-1}(QP)^2 QP^{-1}Q^{-1} \rangle.
\end{aligned}$$

Here, $[x, y] = xyx^{-1}y^{-1}$, which we note is the opposite of Magma's notation. The integer part of the subscript denotes the index in Δ, and $\Delta_{18,a}$ is not conjugate to $\Delta_{18,b}$. It is easy to check that each generator is a parabolic with trivial rotational part, which implies that any conjugate in $\mathrm{PU}(2,1)$ of a Δ_i is rotation-free.

compactification	parabolic subgroups
Hirzebruch's example	Δ_6, Δ_{54}
$\mathbb{Z}/3\mathbb{Z}$ bielliptic #1 [37, #4]	$\Delta_{18,a}$, Δ_{54}
$\mathbb{Z}/3\mathbb{Z}$ bielliptic #2 [37, #7]	$\Delta_{18,a}$, Δ_{54}
$(\mathbb{Z}/3\mathbb{Z})^2$ bielliptic #1 [37, #5]	$\Delta_{18,b}$, Δ_{54}
$(\mathbb{Z}/3\mathbb{Z})^2$ bielliptic #2 [37, #8]	$\Delta_{18,b}$, Δ_{54}

Table 1: There are three distinct conjugacy classes for Δ_6, and one conjugacy class for every other parabolic subgroup in the right column.

Also, for each integer $k \geq 1$, define abstract nil 3–manifold groups

$$N_k = \langle a, b, c \mid [a,c], [b,c], [a,b]c^{-k} \rangle.$$

One can check with Magma that

$$\Delta_6 \cong \Delta_{54} \cong N_1,$$
$$\Delta_{18,a} \cong \Delta_{18,b} \cong N_3.$$

Suppose that N_k is the maximal parabolic subgroup of a lattice Γ in $\mathrm{PU}(2,1)$, and it is rotation-free. Then $\mathcal{H}^2/\Gamma$ has a cusp associated with N_k, and this cusp can be smoothly compactified by an elliptic curve of self-intersection $-k$; see [24, Section 4.2]. In particular, Δ_6, Δ_{54} will be associated with cusps of self-intersection -1 and $\Delta_{18,a}$, $\Delta_{18,b}$ with cusps of self-intersection -3.

In Table 1, we give the conjugacy classes of parabolic subgroups for each of the five examples in this paper. We identify the Δ_i for which some conjugate of Δ_i in the Picard modular group appears as a maximal parabolic subgroup of the lattice in $\mathrm{PU}(2,1)$, but leave it to the reader to calculate the exact conjugates.

Notice that the sum of the indices is always 72, since the cusp associated with some Δ_i is i-to-one over the unique cusp of the Picard modular surface and the total covering degree is 72.

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Seconded: Dan Abramovich, Ian Agol — Revised: 20 June 2017