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On the unstable intersection conjecture

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Compacta X and Y are said to admit a stable intersection in $\mathbb{R}^n$ if there are maps $f\colon X \to \mathbb{R}^n$ and $g\colon Y \to \mathbb{R}^n$ such that for every sufficiently close continuous approximations $f'\colon X \to \mathbb{R}^n$ and $g'\colon Y \to \mathbb{R}^n$ of f and g, we have $f'(X) \cap g'(Y) \neq \varnothing$. The unstable intersection conjecture asserts that X and Y do not admit a stable intersection in $\mathbb{R}^n$ if and only if $\dim X \times Y \leq n-1$. This conjecture was intensively studied and confirmed in many cases. we prove the unstable intersection conjecture in all the remaining cases except the case $\dim X = \dim Y = 3$, $\dim X \times Y = 4$ and $n = 5$, which still remains open.

55M10; 54F45, 55N45

1 Introduction

All the spaces are assumed to be separable metrizable. A map means a continuous map and a compactum means a compact metric space. Compacta X and Y are said to admit a stable intersection in $\mathbb{R}^n$ if there are maps $f\colon X \to \mathbb{R}^n$ and $g\colon Y \to \mathbb{R}^n$ such that for all sufficiently close continuous approximations $f'\colon X \to \mathbb{R}^n$ and $g'\colon Y \to \mathbb{R}^n$ of f and g, we have $f'(X) \cap g'(Y) \neq \varnothing$. The modern research on this subject was initiated by work of D McCullough and L Rubin [17], who refined the classical Nöbeling–Pontrjagin theorem by constructing for every $n \geq 2$ an n–dimensional Boltyanskii compactum X such that every map from X to $\mathbb{R}^{2n}$ can be arbitrarily closely approximated by an embedding. Recall that a compactum X is called a Boltyanskii compactum if $\dim X^2 < 2\dim X$ (Boltyanskii's compacta emphasize the phenomenon that, in general, Lebesgue's covering dimension does not obey the logarithmic law $\dim X \times Y = \dim X + \dim Y$ even for compacta; this was first shown by Pontrjagin in 1930). One can easily observe that any map from a compactum X to $\mathbb{R}^{2n}$ can be approximated by embeddings if and only if X does not admit a stable intersection with itself in $\mathbb{R}^{2n}$. This motivated the following well-known conjecture:

Conjecture 1.1 (unstable intersection conjecture) *Compacta X and Y do not admit a stable intersection in $\mathbb{R}^n$ if and only if $\dim X \times Y \leq n-1$.*

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Extensive work on this conjecture culminated in the following results:

Theorem 1.2 (Dranishnikov and West [8]; see also Sternfeld [21]) *Let X and Y be compacta such that $\dim X \times Y \geq n$. Then X and Y admit a stable intersection in $\mathbb{R}^n$.*

Theorem 1.3 (Dranishnikov, Repovs and Schepin [9]; Torunczyk and Spiez [20]) *Let X and Y be compacta such that $2\dim X + \dim Y \leq 2n-2$ and $\dim X \times Y \leq n-1$. Then X and Y do not admit a stable intersection in $\mathbb{R}^n$.*

Theorem 1.4 (Dranishnikov [4]) *Let X and Y be compacta such that $\dim X \leq n-3$, $\dim Y \leq n-3$ and $\dim X \times Y \leq n-1$. Then X and Y do not admit a stable intersection in $\mathbb{R}^n$.*

The goal of this paper is to settle all the remaining open cases of Conjecture 1.1, except only one, which still remains open.

Theorem 1.5 *Conjecture 1.1 holds in all the cases except the following one, which still remains open: $\dim X = \dim Y = 3$, $\dim X \times Y = 4$ and $n = 5$.*

It is difficult to overestimate the impact of the unstable intersection conjecture on the development of dimension theory. It gave rise to extension theory (in particular to Dranishnikov's extension criterion (Theorem 3.2) and a generalization of the Menger–Urysohn formula for cohomological dimension (Theorems 2.7 and 2.8)), which, by now, is considered as one of the major tools in dimension theory. This led to A Dranishnikov's breakthrough result (Theorem 1.4), which took care of the unstable intersection conjecture for compacta of codimension larger than 2. In [4], Dranishnikov expressed his opinion regarding the codimension-2 case, which is not covered by Theorem 1.4: *"The difficulties there look enormous, and they are basically due to the presence of the fundamental groups. The problem with the fundamental group is that basically the extension theory for nonsimple spaces is not constructed."* Indeed, as Dranishnikov predicted, a development of extension theory was needed to advance the codimension-2 case. It came in two papers by Dydak and the author [12; 13], where it was first shown in [12] that Dranishnikov's extension criterion holds for the projective plane $\mathbb{R}P^2$ and then this result was generalized in [13] to all Moore spaces $M(\mathbb{Z}_m, 1)$. Nevertheless, this partial development of extension theory covered only specific CW–complexes that are not simply connected and did not seem to be much use for the unstable intersection conjecture. The crucial step in proving Theorem 1.5 is establishing a link between the results of [13] and general CW–complexes that are not simply connected. It is done through a version of the plus construction (Proposition 4.4) and an appropriate factorization theorem (Theorem 3.1).

The paper is organized as follows: basics of cohomological dimension are presented in Section 2, some results of extension theory are discussed in Section 3 and applied to obtain a factorization theorem, Theorem 1.5 is proved in Section 4 and a few remarks related to the paper's results are given in the last section.

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2 Cohomological dimension

Let us review basic facts of cohomological dimension. By cohomology we always mean the Cech cohomology. Let G be an abelian group. The cohomological dimension $\dim_G X$ of a space X with respect to the coefficient group G does not exceed n, that is, $\dim_G X \le n$, if $H^{n+1}(X,A;G)=0$ for every closed $A\subset X$. We note that this condition implies that $H^{n+k}(X,A;G)=0$ for all $k\ge 1$ [14; 5]. Thus, $\dim_G X$ is the smallest integer $n\ge 0$ satisfying $\dim_G X\le n$ and $\dim_G X=\infty$ if such an integer does not exist. Clearly, $\dim_G X\le \dim_{\mathbb{Z}} X\le \dim X$.

Theorem 2.1 (Alexandroff) $\dim X=\dim_{\mathbb{Z}} X$ *if* X *is a finite-dimensional space.*

Let $\mathcal{P}$ denote the set of all primes. The *Bockstein basis* is the collection of groups $\sigma=\{\mathbb{Q},\mathbb{Z}_p,\mathbb{Z}_{p^\infty},\mathbb{Z}_{(p)}\mid p\in\mathcal{P}\}$, where $\mathbb{Z}_p=\mathbb{Z}/p\mathbb{Z}$ is the p–cyclic group, $\mathbb{Z}_{p^\infty}=\operatorname{dirlim}\mathbb{Z}_{p^k}$ is the p–adic circle and $\mathbb{Z}_{(p)}=\{m/n\mid n \text{ is not divisible by } p\}\subset\mathbb{Q}$ is the p–localization of integers.

The Bockstein basis of an abelian group G is the collection $\sigma(G)\subset\sigma$ determined by the rule

- $\mathbb{Z}_{(p)}\in\sigma(G)$ if $G/\operatorname{Tor}G$ is not divisible by p;
- $\mathbb{Z}_p\in\sigma(G)$ if p–$\operatorname{Tor}G$ is not divisible by p;
- $\mathbb{Z}_{p^\infty}\in\sigma(G)$ if p–$\operatorname{Tor}G$ is nontrivial and divisible by p;
- $\mathbb{Q}\in\sigma(G)$ if $G/\operatorname{Tor}G$ is nontrivial and divisible by all $p\in\mathcal{P}$.

Thus, $\sigma(\mathbb{Z})=\{\mathbb{Z}_{(p)}\mid p\in\mathcal{P}\}$.

Theorem 2.2 (Bockstein theorem) *For a compactum X,*

$$\dim_G X = \sup\{\dim_H X : H \in \sigma(G)\}.$$

Suggested by the Bockstein inequalities, we say that a function $D\colon \sigma \to \mathbb{N} \cup \{0, \infty\}$ is *p–regular* if $D(\mathbb{Z}_{(p)}) = D(\mathbb{Z}_p) = D(\mathbb{Z}_{p^\infty}) = D(\mathbb{Q})$ and it is *p–singular* if $D(\mathbb{Z}_{(p)}) = \max\{D(\mathbb{Q}), D(\mathbb{Z}_{p^\infty})+1\}$. A p–singular function D is called p^+–singular if $D(\mathbb{Z}_{p^\infty}) = D(\mathbb{Z}_p)$ and it is called *p^-–singular* if $D(\mathbb{Z}_{p^\infty}) = D(\mathbb{Z}_p)-1$. A function $D\colon \sigma \to \mathbb{N} \cup \{0, \infty\}$ is called a *dimension type* if for every prime p it is either p–regular or $p^\pm$–singular. Thus, the values of $D(F)$ for the Bockstein fields $F \in \{\mathbb{Z}_p, \mathbb{Q}\}$ together with p–singularity types of D determine the value $D(G)$ for all groups in σ. For a dimension type D, denote $\dim D = \sup\{D(G) : G \in \sigma\}$. We impose on every dimension type D the following restriction: $D(G) \geq 1$ for every $G \in \sigma$ if $\dim D > 0$.

Theorem 2.3 (Bockstein inequalities [14; 5]) *For every space X the function $d_X\colon \sigma \to \mathbb{N} \cup \{0, \infty\}$ defined by $d_X(G) = \dim_G X$ is a dimension type.*

The function d_X is called the *dimension type of X*.

Theorem 2.4 (Dranishnikov realization theorem [1; 3]) *For every dimension type D there is a compactum X with $d_X = D$ and $\dim X = \dim D$.*

Theorem 2.5 (Olszewski completion theorem [18]) *For every space X there is a complete space X' such that $X \subset X'$ and $d_X = d_{X'}$.*

Let D be a dimension type. We will use the abbreviations $D(0) = D(\mathbb{Q})$ and $D(p) = D(\mathbb{Z}_p)$. Additionally, if $D(p) = n \in \mathbb{N}$, we will write $D(p) = n^+$ if D is p^+–singular and $D(p) = n^-$ if it is p^-–singular. For a p–regular D we leave it without decoration: $D(p) = n$. Thus, any sequence of decorated numbers $D(p) \in \mathbb{N}$, where $p \in \mathcal{P} \cup \{0\}$, defines a unique dimension type. There is a natural order on decorated numbers

$$\cdots < n^- < n < n^+ < (n+1)^- < \cdots.$$

Note that the inequality of dimension types $D \leq D'$ as functions on σ is equivalent to the family of inequalities $D(p) \leq D'(p)$ for the above order for all $p \in \mathcal{P} \cup \{0\}$. Also note that 0 has no decoration, 1 does not have the "$-$" decoration and $D(0) = D(\mathbb{Q})$ has no decoration.

Let ϵ be a decoration. We define the reversed decoration $-\epsilon$ and the commutative product of decorations $\otimes$ as follows:

$$-(-) = +, \quad -(+) = -, \quad -(\text{no decoration}) = \text{no decoration},$$
$$\epsilon \otimes (\text{no decoration}) = \epsilon, \quad \epsilon \otimes \epsilon = \epsilon \quad \text{and} \quad + \otimes - = -.$$

For dimension types D_1 and D_2 we define the dimension types $D_1 \boxplus D_2$ and $D_1 \oplus D_2$ as follows: if $D_1(p) = n^{\epsilon_1}$ and $D_2(p) = m^{\epsilon_2}$, where ϵ_i is a decoration, then

$$(D_1 \boxplus D_2)(p) = (n+m)^{\epsilon_1 \otimes \epsilon_2}, \quad (D_1 \oplus D_2)(p) = (n+m)^{-((-\epsilon_1)\otimes(-\epsilon_2))}.$$

For an integer $n \geq 0$ we denote by n the dimension type which sends every $G \in \sigma$ to n and for a dimension type D we denote by $D+n$ the dimension type which is the ordinary sum of D and n as functions. Note that $D+n$ preserves the decorations of D. Also note that $d_{\mathbb{R}^n} = n$.

The operations $D_1 \boxplus D_2$ and $D_1 \oplus D_2$ are motivated by the following properties:

Theorem 2.6 (Bockstein product theorem [5; 19; 11]) *For any two compacta X and Y,*

$$d_{X\times Y} = d_X \boxplus d_Y.$$

Theorem 2.7 (Dydak union theorem [10; 7]) *Let X be a compactum and D_1 and D_2 dimension types and let $X = A \cup B$ be a decomposition with $d_A \leq D_1$ and $d_B \leq D_2$. Then $d_X \leq D_1 \oplus D_2 + 1$.*

Theorem 2.8 (Dranishnikov decomposition theorem [3; 7]) *Let X be a finite-dimensional compactum and D_1 and D_2 dimension types such that $d_X \leq D_1 \oplus D_2 + 1$. Then there is a decomposition $X = A \cup B$ such that $d_A \leq D_1$ and $d_B \leq D_2$.*

For a dimension type D and $n \geq \dim D$, we define the dimension type $n+1 \ominus D$ by

$$(n+1 \ominus D)(p) = (n+1-m)^{-\epsilon} \quad \text{if } D(p) = m^{\epsilon}.$$

Note that $n+1 \ominus D$ is indeed a dimension type and if $\dim D > 0$ then $\dim(n+1 \ominus D) \leq n$ and $n+1 \ominus (n+1 \ominus D) = D$. One can also easily verify the following properties:

Proposition 2.9 *Let D be a dimension type and $n = \dim D$. Then:*

(i) *$D \boxplus (n+1 \ominus D) \leq n+1$. Moreover, for a dimension type D' the condition $\dim(D \boxplus D') \leq n+1$ implies that $D' \leq n+1 \ominus D$;*

(ii) $n+1 \le D \oplus (n+1 \ominus D)$. *Moreover, for dimension types* D' *and* D'' *the conditions* $D' \le D$, $D'' \le n+1 \ominus D$ *and* $n+1 \le D' \oplus D''$ *imply that* $D' = D$ *and* $D'' = n+1 \ominus D$.

In particular, Proposition 2.9(ii) implies Theorem 2.4. Indeed, let D be a dimension type and $n = \dim D$. By Theorem 2.8, consider a decomposition $A \cup B = \mathbb{R}^{n+2}$ with $d_A \le D$ derived from the inequality $n+2 \le D \oplus (n+1 \ominus D) + 1$. Then, by Proposition 2.9(ii) and Theorems 2.5 and 2.7, one can easily conclude that $\mathbb{R}^{n+2}$ contains a compact subset of dimension type D.

3 Maps to CW–complexes

The goal of this section is to prove:

Theorem 3.1 *Any map from a finite-dimensional compactum* X *to a finite CW–complex* L *with* $\dim L \le 3$ *can be arbitrarily closely approximated by a map which factors through a compactum* Z *with* $d_Z \le d_X$ *and* $\dim Z \le 3$.

For proving this theorem we need a few facts from extension theory.

Cohomological dimension is characterized by the following basic property: $\dim_G X \le n$ if and only for every closed $A \subset X$ and map $f\colon A \to K(G,n)$, f continuously extends over X, where $K(G,n)$ is the Eilenberg–Mac Lane complex of type (G,n) (we assume that $K(G,0) = G$ with the discrete topology and $K(G,\infty)$ is a singleton). This extension characterization of cohomological dimension gives a rise to extension theory (more general than cohomological dimension theory) and the notion of extension dimension. The *extension dimension* of a space X is said to be dominated by a CW–complex K, written $\text{e-dim}\, X \le K$, if every map $f\colon A \to K$ from a closed subset A of X continuously extends over X. Thus, $\dim_G X \le n$ is equivalent to $\text{e-dim}\, X \le K(G,n)$ and $\dim X \le n$ is equivalent to $\text{e-dim}\, X \le S^n$.

The following theorem shows a close connection between extension and cohomological dimensions.

Theorem 3.2 (Dranishnikov extension theorem [2; 10]) *Let* K *be a CW–complex and* X *a metric space. Denote by* $H_*(K)$ *the reduced integral homology of* K. *Then:*

(i) $\dim_{H_n(K)} X \le n$ *for every* $n \ge 0$ *if* $\text{e-dim}\, X \le K$;

(ii) e-dim $X \le K$ *if* K *is simply connected,* X *is finite-dimensional and*

$$\dim_{H_n(K)} X \le n$$

for every $n \ge 0$.

Let G be an abelian group. We always assume that a Moore space $M(G,n)$ of type (G,n) is an $(n-1)$–connected CW–complex. Theorem 3.2 implies that for a finite-dimensional compactum X and $n > 1$, $\dim_G X \le n$ if and only if e-dim $X \le M(G,n)$. We will refer to this property of $M(G,n)$ as being a classifying space for finite-dimensional compacta X for which $\dim_G X \le n$. This property can be extended to some Moore spaces $M(G,1)$ for the groups G in the Bockstein basis σ.

We will consider the following standard models of $M(G,1)$ for $G \in \sigma$:

- $M(\mathbb{Q},1)$ is the infinite telescope of a sequence of maps from $S^1 \to S^1$ of all possible nonzero degrees.
- $M(\mathbb{Z}_{(p)},1)$ is the infinite telescope of a sequence of maps $S^1 \to S^1$ of all possible nonzero degrees not divisible by p.
- $M(\mathbb{Z}_{p^\infty},1)$ is the infinite telescope of a p–fold covering map $S^1 \to S^1$ with a disk attached to the first circle of the telescope by the identity map of the disk boundary.
- $M(\mathbb{Z}_p,1)$ is a disk attached to S^1 by a p–fold covering map from the disk boundary to S^1.

Note that $M(\mathbb{Q},1) = K(\mathbb{Q},1)$ and $M(\mathbb{Z}_{(p)},1) = K(\mathbb{Z}_{(p)},1)$ and hence $M(\mathbb{Q},1)$ and $M(\mathbb{Z}_{(p)},1)$ are classifying spaces for compacta X for which $\dim_{\mathbb{Q}} X \le 1$ and $\dim_{\mathbb{Z}_{(p)}} X \le 1$, respectively. The case of $M(\mathbb{Z}_p,1)$ was settled in [13].

Theorem 3.3 (Dydak and Levin [13]) *A Moore space* $M(\mathbb{Z}_p,1)$ *is a classifying space for finite-dimensional compacta* X *for which* $\dim_{\mathbb{Z}_p} X \le 1$.

The case of $M(\mathbb{Z}_{p^\infty},1)$ still remains open.

Problem 3.4 *Is a Moore space* $M(\mathbb{Z}_{p^\infty},1)$ *a classifying space for finite-dimensional compacta* X *for which* $\dim_{\mathbb{Z}_{p^\infty}} X \le 1$?

Problem 3.4 leads to:

Definition 3.5 A finite-dimensional compactum X is said to be extensionally regular if for every prime p we have either $\dim_{\mathbb{Z}_{p^\infty}} X \ne 1$, or $\dim_{\mathbb{Z}_{p^\infty}} X = 1$ and e-dim $X \le M(\mathbb{Z}_{p^\infty},1)$.

Note that every $M(G,1)$ for $G \in \sigma$ is a 2–dimensional CW–complex and $M(G,n) = \Sigma^{n-1} M(G,1)$. Thus, we obtain:

Corollary 3.6 *For every n and $G \in \sigma$ there is an $(n-1)$–connected, $(n+1)$–dimensional countable CW–complex which is a classifying space for finite-dimensional extensionally regular compacta X for which $\dim_G X \le n$.*

By a partial map of a space L to a CW–complex M we mean a map from a closed subset of L to M. A collection $\mathcal{F}$ of partial maps from L to M is said to be representative if for every closed subset F' of L and every map $f'\colon F' \to M$ with $M \in \mathcal{M}$, there is $f\colon F \to M$ in $\mathcal{F}$ such that $F' \subset F$ and $f|_{F'}$ is homotopic to f'. If $\mathcal{M}$ is a collection of CW–complexes then a collection of partial maps of L to the CW–complexes of $\mathcal{M}$ is said to be representative for $\mathcal{M}$ if it contains a representative collection of partial maps to each $M \in \mathcal{M}$. Note that if L is a compactum and M is a countable CW–complex, then there is a countable representative collection $\mathcal{F}$ of partial maps from L to M and a closed subset A of L is of extension dimension $\le M$ if and only if every map $f\colon F \to M$ in $\mathcal{F}$ extends over $A \cup F$ (see Proposition 2.2 of [3]).

In the proof of Theorem 3.1 we will use the following construction from [15; 16] for resolving partial maps. A map between CW–complexes is said to be combinatorial if the preimage of every subcomplex of the range is a subcomplex of the domain. Let L be a simplicial complex and let $L^{[m]}$ be the m–skeleton of L (that is, the union of all simplexes of L of dimension $\le m$). By a resolution $\mathrm{EW}(L,m)$ of L we mean a CW–complex $\mathrm{EW}(L,m)$ and a combinatorial map $\omega\colon \mathrm{EW}(L,m) \to L$ such that ω is one-to-one over $L^{[m]}$. Let $f\colon N \to M$ be a map of a subcomplex N of L into a CW–complex M. A resolution $\omega\colon \mathrm{EW}(L,m) \to L$ is said to resolve the map f if the map $f \circ \omega|_{\omega^{-1}(N)}$ extends to a map $f'\colon \mathrm{EW}(L,m) \to M$. We will call f' a resolving map for f. The resolution is said to be suitable for a compactum X if for every simplex Δ of L, $\text{e-dim}\, X \le \omega^{-1}(\Delta)$. Note that if $\omega\colon \mathrm{EW}(L,m) \to L$ is a resolution suitable for X then for every map $\phi\colon X \to L$ there is a map $\psi\colon X \to \mathrm{EW}(L,m)$ such that $(\omega \circ \psi)(\phi^{-1}(\Delta)) \subset \Delta$ for every simplex Δ of L. We will call ψ a combinatorial lifting of ϕ.

Let L be a finite simplicial complex. Let $f\colon N \to M$ be a cellular map from a subcomplex N of L to a CW–complex M such that $L^{[m]} \subset N$. Now we will construct a resolution $\omega\colon \mathrm{EW}(L,m) \to L$ of L resolving f and we will refer to this resolution as the standard resolution for f. We will associate to the standard resolution a cellular resolving map $f'\colon \mathrm{EW}(L,m) \to M$, which will be called the standard resolving map. The standard resolution is constructed by induction on $l = \dim(L \setminus N)$.

For $L = N$ set $\mathrm{EW}(L,m) = L$ and let $\omega\colon \mathrm{EW}(L,m) \to L$ be the identity map with the standard resolving map $f' = f$. Let $l > m$. Denote $L' = N \cup L^{[l-1]}$ and assume that $\omega'\colon \mathrm{EW}(L',m) \to L'$ is the standard resolution of L' for f with the standard resolving map $f'\colon \mathrm{EW}(L',m) \to M$. The standard resolution $\omega\colon \mathrm{EW}(L,m) \to L$ is constructed as follows.

The CW–complex $\mathrm{EW}(L,m)$ is obtained from $\mathrm{EW}(L',m)$ by attaching the mapping cylinder of $f'|_{\omega'^{-1}(\partial\Delta)}$ to $\omega'^{-1}(\partial\Delta)$ for every l–simplex Δ of L which is not contained in L'. Let $\omega\colon \mathrm{EW}(L,m) \to L$ be the projection which extends ω' by sending each mapping cylinder to the corresponding l–simplex Δ such that the M–part of the cylinder is sent to the barycenter of Δ and each interval connecting a point of $\omega'^{-1}(\partial\Delta)$ with the corresponding point of the M–part of the cylinder is sent linearly to the interval connecting the corresponding point of $\partial\Delta$ with the barycenter of Δ. We can naturally define the extension of $f'|_{\omega'^{-1}(\partial\Delta)}$ over its mapping cylinder by sending each interval of the cylinder to the corresponding point of M. Thus, we define the standard resolving map which extends f' over $\mathrm{EW}(L,m)$. The CW–structure of $\mathrm{EW}(L,m)$ is induced by the CW–structure of $\mathrm{EW}(L',m)$ and the natural CW–structures of the mapping cylinders in $\mathrm{EW}(L,m)$. Then, with respect to this CW–structure, the standard resolving map is cellular and ω is combinatorial.

It is easy to see from the construction of the standard resolution that $\omega^{-1}(\Delta)$, for each simplex Δ of L, is either contractible or homotopy equivalent to M and $\dim \mathrm{EW}(L,m) \le \dim L$ if $\dim M \le m+1$.

Theorem 3.7 *Any map from a finite-dimensional, extensionally regular compactum X to a finite CW–complex L can be arbitrarily closely approximated by a map which factors through a compactum Z with $d_Z \le d_X$ and $\dim Z \le \dim L$.*

Proof Recall that a finite CW–complex is a compact ANR and hence the identity map of a finite CW–complex can be arbitrarily closely approximated by a map which factors through a finite simplicial complex of the same dimension. Thus we may assume that L is a finite simplicial complex.

Let $g\colon X \to L$ be a map. Set $L_0 = L$ and $g_0 = g\colon X \to L_0$. Fix $\epsilon > 0$ and denote $l = \dim L$. We will construct by induction a sequence of finite simplicial complexes L_i with $\dim L_i \le l$, bonding maps $\omega_i^{i+1}\colon L_{i+1} \to L_i$ and maps $g_i\colon X \to L_i$ such that g_i and $\omega_i^j \circ g_j$ are $\epsilon/2^j$–close for every $j > i$, where $\omega_i^j\colon L_j \to L_i$ is the composition of the bonding maps between $L_j, \dots, L_i$ and ω_i^i is the identity map of L_i. Denote $Z = \operatorname{invlim}(L_i, \omega_i^{i+1})$ and note that $\dim Z \le l$. Also denote $g_i' = \lim_{j\to\infty} \omega_i^j \circ g_j\colon X \to L_i$ and note that g_i' is a well-defined map and $\omega_i^{i+1} \circ g_{i+1}' = g_i'$. Hence, the maps g_i' determine the corresponding map $g'\colon X \to Z$ and for the projection $\omega_0\colon Z \to L_0 = L$ we have that g_0 and $\omega_0 \circ g'$ are ϵ–close. The construction will be carried out in such a way that $d_Z \le d_X$. Thus, Z and $\omega_0 \circ g'$ will provide the compactum and the approximation required in the theorem.

Assume that the construction is completed for i and proceed to $i+1$ as follows. Let $m = \dim_G X$ for a group $G \in \sigma$. Since the theorem is obvious for $\dim X = 0$ we may assume that $m \ge 1$. By Corollary 3.6 there is an $(m-1)$–connected, $(m+1)$–dimensional countable CW–complex M classifying for the finite-dimensional, extensionally regular compacta X for which $\dim_G X \le m$. Take a map $\alpha\colon F \to M$ from a closed subset F of L_i. Replace the triangulation of L_i by a sufficiently fine barycentric subdivision such that α extends over a subcomplex N of L_i to a map $f\colon N \to M$ and, for every simplex Δ of L_i and every $j \le i$, we have $\operatorname{diam} \omega_j^i(\Delta) \le \epsilon/2^{i+1}$. Since M is $(m-1)$–connected we may assume that N contains the m–skeleton of L_i and, replacing f by a cellular approximation, we also assume that $f\colon N \to M$ is a cellular map. Let $\omega\colon \mathrm{EW}(L_i, m) \to L_i$ be the standard resolution resolving the map f. Since $\text{e-dim}\, X \le M$ there is a combinatorial lifting $g_{i+1}\colon X \to \mathrm{EW}(L_i, m)$ of g_i. Set L_{i+1} to be a finite subcomplex of $\mathrm{EW}(L_i, m)$ containing $g_{i+1}(X)$ and ω_i^{i+1} to be ω restricted to L_{i+1}. Since the identity map of L_{i+1} can be arbitrarily closely approximated by a map which factors through a finite simplicial complex of dimension $\le \dim L_{i+1}$, we may assume that L_{i+1} is a simplicial complex and the construction is completed. Recall that ω resolves the map f and hence $\alpha \circ \omega_i^{i+1}|_{(\dots)}\colon (\alpha \circ \omega_i^{i+1})^{-1}(F) \to M$ extends over L_{i+1} as well.

Now we will show that the map α on the inductive step of the construction from i to $i+1$ can be chosen in a way that will lead to $d_Z \le d_X$. Denote by $\mathcal{M}$ all the classifying spaces mentioned in Corollary 3.6 such that $\text{e-dim}\, X \le M$. Note that $\mathcal{M}$ is a countable collection of countable CW–complexes. Once L_i is constructed, take a countable representative collection $\mathcal{B}_i$ of partial maps $\beta\colon B \to M$ from closed subsets B of L_i to the CW–complexes M of $\mathcal{M}$ and fix a surjection $\tau_i\colon \mathbb{N} \to \mathcal{B}_i$. Take any bijection $\tau\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that for every i and $\tau(i) = (j,k)$ we have $i \ge j$. Let $\tau(i) = (j,k)$ and $\beta = \tau_j(k) \in \mathcal{B}_j$. Recall that $i \ge j$ and hence $\mathcal{B}_j$ is already constructed. Thus $\beta\colon B \to M$, where B is a closed subset of L_j and $M \in \mathcal{M}$. Denote

$F=(\omega^i_j)^{-1}(B)$ and set $\alpha=\beta\circ\omega^j_i\colon F\to M$. One can easily verify that choosing in this way the map α for constructing L_{i+1} leads to $\text{e-dim}\,Z\le M$ for every $M\in\mathcal{M}$ and hence $d_Z\le d_X$. The theorem is proved. □

In order to derive Theorem 3.1 from Theorem 3.7 we need to bypass the difficulties imposed by Problem 3.4. We will need the extension versions of Theorems 2.5 and 2.8.

Theorem 3.8 [18] *Let K be a countable CW–complex and X a space such that $\text{e-dim}\,X\le K$. Then there is a complete space X' such that $X\subset X'$ and $\text{e-dim}\,X'\le K$.*

Theorem 3.9 [3] *Let K_1 and K_2 be countable CW–complexes, $K=K_1*K_2$ the join of K_1 and K_2, and X a compactum such that $\text{e-dim}\,X\le K$. Then X decomposes into $X=A\cup B$ such that $\text{e-dim}\,A\le K_1$ and $\text{e-dim}\,B\le K_2$.*

In order to avoid confusion with our previous use of the letter L we will denote the infinite-dimensional lens space model for $K(\mathbb{Z}_m,1)$ by $\mathcal{L}_m$ and, as usual, $\mathcal{L}_m^{[n]}$ stands for the n–skeleton of $\mathcal{L}_m$ and p for a prime number. By a projection from $\mathcal{L}_{p^i}$ to $\mathcal{L}_{p^j}$ with $j\ge i$, we mean a cellular map realizing the standard monomorphism of $\mathbb{Z}_{p^i}$ into $\mathbb{Z}_{p^j}$. The restrictions of a projection to the skeletons of $\mathcal{L}_{p^i}$ and $\mathcal{L}_{p^j}$ will also be called projections. Note that $\mathcal{L}_m^{[2]}=M(\mathbb{Z}_m,1)$ and $\mathcal{L}_m^{[3]}$ is a 3–dimensional lens space.

We assume that $K(\mathbb{Z}_{p^\infty},1)$ is represented as the infinite telescope of a sequence of projections from $\mathcal{L}_{p^i}$ to $\mathcal{L}_{p^{i+1}}$ and we consider the lens spaces $\mathcal{L}_{p^i}$ as subcomplexes of $K(\mathbb{Z}_{p^\infty},1)$.

Theorem 3.10 [13, Theorem 7.1] *Let X be a finite-dimensional metric space with $\dim_{\mathbb{Z}_m}X\le 2$ and $f\colon X\to\mathcal{L}_m^{[n]}$ a map. Then there is a map $f'\colon X\to\mathcal{L}_m^{[3]}$ such that f and f' coincide on $f^{-1}(\mathcal{L}_m^{[2]})$.*

Proposition 3.11 *Let X be a finite-dimensional compactum with $\dim_{\mathbb{Z}_{p^\infty}}X\le 1$. Then for every i and every map $f\colon F\to\mathcal{L}_{p^i}^{[2]}$ from a closed subset F of X there is $j\ge i$ such that f followed by a projection of $\mathcal{L}_{p^i}^{[2]}$ to $\mathcal{L}_{p^j}^{[2]}$ extends over X as a map to $\mathcal{L}_{p^j}^{[3]}$.*

Proof Take a map $f\colon F\to\mathcal{L}_{p^i}^{[2]}$ from a closed subset F of X and extend f to a map $h\colon X\to K(\mathbb{Z}_{p^\infty},1)$. Since X is compact, $h(X)$ is contained in a finite subtelescope of $K(\mathbb{Z}_{p^\infty}, 1)$ and hence there is $j \geq i$ for which h can be homotoped to a map $g\colon X \to \mathcal{L}_{p^j}$ such that g on F coincides with f followed by a projection to $\mathcal{L}_{p^j}^{[2]}$. Again by the compactness of X there is n such that $g(X) \subset \mathcal{L}_{p^j}^{[n]}$. By the Bockstein theorem and inequalities, $\dim_{\mathbb{Z}_{p^j}} X \leq 2$. Then, by Theorem 3.10, g can be replaced by a map to $\mathcal{L}_{p^j}^{[3]}$ which coincides with g on F and the proposition follows. □

Let L be a simplicial complex. By the star of a subset A of L, written $\operatorname{st} A$, we mean the union of the simplexes of L which intersect A. Let us say that for maps $\phi\colon X \to L$ and $\omega\colon Y \to L$, a map $\psi\colon X \to Y$ is an almost combinatorial lifting of ϕ to Y if $(\omega \circ \psi)(\phi^{-1}(\Delta)) \subset \operatorname{st} \Delta$ for every simplex Δ in L.

Proposition 3.12 *Let X be a finite-dimensional compactum with $\dim_{\mathbb{Z}_{p^\infty}} X \leq 1$, L a finite simplicial complex with $\dim L \leq 3$, N a subcomplex of L, and $\phi\colon X \to L$ and $f\colon N \to K(\mathbb{Z}_{p^\infty}, 1)$ maps. Then there is a resolution $\omega\colon \mathrm{EW}(L,1) \to L$ such that $\dim \mathrm{EW}(L,1) \leq 3$, ω resolves the map f and the map ϕ admits an almost combinatorial lifting to $\mathrm{EW}(L,1)$.*

Proof Extending f over the 1–skeleton of L, we assume that $L^{[1]} \subset N$. Since f can be homotoped into $\mathcal{L}_{p^i} \subset K(\mathbb{Z}_{p^\infty}, 1)$, we may assume that $f(N) \subset \mathcal{L}_{p^i}$. By the Bockstein theorem and inequalities, $\dim_{\mathbb{Z}_{p^i}} X \leq 2$. Then, by Theorem 3.2, $\text{e-dim}\, X \leq \Sigma M(\mathbb{Z}_{p^i}, 1) = S^0 * M(\mathbb{Z}_{p^i}, 1)$ and hence, by Theorem 3.9, X decomposes into $X = A \cup B$ such that $\dim A \leq 0$ and $\text{e-dim}\, B \leq M(\mathbb{Z}_{p^i}, 1)$, and by Theorem 3.8 (or Corollary 2 of [3]) we may assume that B is G_δ and A is σ–compact.

Now replace the triangulation of L by its sufficiently fine subdivision. Let $\mathcal{R}$ be the collection of the stars of the vertices of L with respect to the barycentric subdivision L_β of L and let v_R be the vertex of L contained in $R \in \mathcal{R}$. Note that $\mathcal{R}$ partitions L_β into contractible subcomplexes with nonintersecting interiors.

By ∂R for $R \in \mathcal{R}$, we denote the topological boundary of R in L. Clearly we may assume that the triangulation of L is so fine that f can be extended to a map $f'\colon N' \to \mathcal{L}_{p^i}$ over a subcomplex N' of L_β such that $N \subset N'$ and for every $R \in \mathcal{R}$ we have either $R \subset N'$ or $(R \setminus \partial R) \cap N' = \varnothing$. Furthermore, we may assume that N' contains every 1–simplex of L_β contained in ∂R for every $R \in \mathcal{R}$. Denote by L' the subcomplex of L_β which is the union of N' with ∂R for all $R \in \mathcal{R}$. Let $\mathcal{R}'$ be the collection of $R \in \mathcal{R}$ such that R is contained in N', $\mathcal{R}'' = \mathcal{R} \setminus \mathcal{R}'$, and let L'' be the subcomplex of L' which is the union of ∂R for all $R \in \mathcal{R}''$, and $N'' = N' \cap L''$.

Replace f' by its cellular approximation to $\mathcal{L}_{p^i}$. Then

$$f'(N') \subset \mathcal{L}_{p^i}^{[3]} \quad \text{and} \quad f'(N'') \subset \mathcal{L}_{p^i}^{[2]}.$$

Denote $f'' = f'|_{N''}\colon N'' \to \mathcal{L}_{p^i}^{[2]}$. Let $\omega''\colon \mathrm{EW}(L'', 1) \to L''$ be the standard resolution resolving the map f'' and let $\rho''\colon \mathrm{EW}(L'', 1) \to \mathcal{L}_{p^i}^{[2]}$ be the standard resolving map for f''. Extend ω'' to the resolution $\omega'\colon \mathrm{EW}(L', 1) \to L'$ such that ω' is one-to-one over N' and extend ρ'' to the map $\rho'\colon \mathrm{EW}(L', 1) \to \mathcal{L}_{p^i}^{[3]}$ defined by $f' \circ \omega'$ on $\omega'^{-1}(N')$. Note that $\dim \mathrm{EW}(L', 1) \le 3$ and ρ' resolves the map f' over L'.

Recall that the set A is σ–compact and 0–dimensional. Then one can replace ϕ by an arbitrarily close approximation by changing ϕ only on the preimages of the interiors of the 3–simplexes of L and assume that for every 3–simplex Δ of L and every 2–simplex Δ' of the barycentric subdivision of Δ we have that $\phi(A) \cap (\Delta' \setminus \partial\Delta) = \varnothing$ and hence $\phi^{-1}(\Delta' \setminus \partial\Delta) \subset B$. Thus, for every 2–simplex Δ' of L' not contained in N' we have that $\phi^{-1}(\Delta' \setminus N') \subset B$ and $\omega'^{-1}(\Delta')$ is homotopy equivalent to $\mathcal{L}_{p^i}^{[2]}$. Then, since $\text{e-dim}\, B \le M(\mathbb{Z}_{p^i}, 1) = \mathcal{L}_{p^i}^{[2]}$, we conclude that ϕ restricted to $X' = \phi^{-1}(L')$ admits a combinatorial lifting $\psi'\colon X' \to \mathrm{EW}(L', 1)$.

Note that $(\rho' \circ \psi')(\phi^{-1}(\partial R)) \subset \mathcal{L}_{p^i}^{[2]}$ for $R \in \mathcal{R}''$. Then, by Proposition 3.11, for every $R \in \mathcal{R}''$ there is $j \ge i$ such that $\rho' \circ \psi'$ restricted to $\phi^{-1}(\partial R)$ and followed by a projection $\tau\colon \mathcal{L}_{p^i} \to \mathcal{L}_{p^j}$ extends over $\phi^{-1}(R)$ as a map to $\mathcal{L}_{p^j}^{[3]}$. Clearly j can be replaced by any larger integer and hence we can find an integer j that fits every $R \in \mathcal{R}''$. Since f' followed by τ is homotopic to f' as a map to $K(\mathbb{Z}_{p^\infty}, 1)$, we can replace f' and ρ' by their compositions with τ and assume that f' and ρ' are maps to $\mathcal{L}_{p^j}^{[3]}$.

Now define $\mathrm{EW}(L, 1)$ as the CW–complex obtained from $\mathrm{EW}(L', 1)$ by attaching to $\omega'^{-1}(\partial R)$ the mapping cylinder of ρ' restricted to $\omega'^{-1}(\partial R)$ for every $R \in \mathcal{R}''$. Note that each such mapping cylinder is of dimension ≤ 3 since $\dim \omega'^{-1}(\partial R) \le 2$ for every $R \in \mathcal{R}''$, and hence $\dim \mathrm{EW}(L, 1) \le 3$. Define $\omega\colon \mathrm{EW}(L, 1) \to L$ as the map that extends ω' by sending the $\mathcal{L}_{p^j}^{[3]}$–part of every attached mapping cylinder to the vertex v_R of L contained in the corresponding set $R \in \mathcal{R}''$ and the intervals of the mapping cylinder to the corresponding intervals connecting the points of ∂R with v_R. Clearly ω is combinatorial. Let $\rho\colon \mathrm{EW}(L, 1) \to \mathcal{L}_{p^j}^{[3]}$ be the map naturally extending ρ' over each mapping cylinder. Then ρ resolves f' and hence ρ resolves f as well.

Finally, note that for every $R \in \mathcal{R}''$ the map ψ' restricted to $\phi^{-1}(\partial R)$ and considered as a map to the mapping cylinder attached to $\omega'^{-1}(\partial R)$ can be homotoped to the $\mathcal{L}_{p^j}^{[3]}$–part of the mapping cylinder and then extended over $\phi^{-1}(R)$ as a map to $\mathcal{L}_{p^j}^{[3]}$. This way, we can extend ψ' to a map $\psi\colon X \to \mathrm{EW}(L,1)$ such that $(\omega\circ\psi)(\phi^{-1}(R)) \subset R$ for every $R \in \mathcal{R}$.

Thus, with respect to the original triangulation of L, the map $\omega\colon \mathrm{EW}(L,1) \to L$ is a resolution resolving the map f and admitting an almost combinatorial lifting $\psi\colon X \to \mathrm{EW}(L,1)$ of ϕ, and the proposition follows. □

Proof of Theorem 3.1 The proof of Theorem 3.7 applies to prove Theorem 3.1 with the following minor adjustment. Recall that the compactum Z in Theorem 3.7 is constructed as the inverse limit of maps $\omega_i^{i+1}\colon L_{i+1} \to L_i$ of finite simplicial complexes, where ω_i^{i+1} comes from the standard resolution $\omega\colon \mathrm{EW}(L_i,m) \to L_i$ resolving a partial map $f\colon N \to M(G,m)$ from a subcomplex N of L_i to a Moore space $M(G,m)$ with $G \in \sigma$ and $m = \dim_G X$. Since in Theorem 3.1 the compactum X is not assumed to be extensionally regular, we cannot use the Moore space $M(\mathbb{Z}_{p^\infty},1)$ as the classifying space any more, and therefore we replace $M(\mathbb{Z}_{p^\infty},1)$ by the Eilenberg–Mac Lane complex $K(\mathbb{Z}_{p^\infty},1)$. Then, for $K(\mathbb{Z}_{p^\infty},1)$, instead of the standard resolution we use the resolution from Proposition 3.12. We should mention here that Proposition 3.12 provides a resolution that admits an almost combinatorial lifting to $\mathrm{EW}(L_i,1)$ in contrast to the standard resolution which provides a resolution that admits a combinatorial lifting to $\mathrm{EW}(L_i,1)$. However, we can assume that the triangulation of L_i is as fine as we wish, and therefore the difference between a combinatorial lifting and an almost combinatorial lifting does not affect at all the construction and the proof of the theorem. □

4 Main result

The goal of this section is to prove Theorem 1.5. The following results will be used in the proof.

Using Alexander duality and the Künneth formula in the Leray form it was shown in [2] that:

Theorem 4.1 [2] *Let X and Y be compacta such that $Y \subset \mathbb{R}^n$ and $\dim X\times Y \le n-1$. Then, for every open ball U in $\mathbb{R}^n$ and $i \ge 0$, we have $\dim_{H_i(U\setminus Y)} X \le i$, where $H_*(U\setminus Y)$ is the reduced integral homology.*

Let us recall that a tame compactum $X \subset \mathbb{R}^n$ with $\dim X \le n-3$ is characterized by the following property: for every open ball U in $\mathbb{R}^n$, the complement $U \setminus X$ is simply connected. It was also shown in [2] that Theorems 4.1 and 3.2 lead to:

Theorem 4.2 [2] *Let X be a finite-dimensional compactum and $Y \subset \mathbb{R}^n$ a tame compactum of $\dim Y \le n-3$ such that $\dim X \times Y \le n-1$. Then every map $f\colon X \to \mathbb{R}^n$ can be arbitrarily closely approximated by a map $f'\colon X \to \mathbb{R}^n$ such that $f'(X) \cap Y = \varnothing$.*

One of the problems this section is concerned with (and whose importance for the unstable intersection conjecture was realized by Dranishnikov) is under what conditions any map $g\colon X \to \mathbb{R}^n$ from a finite-dimensional compactum X admits an arbitrarily close approximation by a map $g'\colon X \to \mathbb{R}^n$ with $d_{g'(X)} \le d_X$. Let us observe that such an approximation exists if, for every partial map $\alpha\colon F \to K(G,m)$ of $\mathbb{R}^n$ from a compact subset F of $\mathbb{R}^n$ to an Eilenberg–Mac Lane complex $K(G,m)$ with $\dim_G X \le m$ and G in the Bockstein basis σ, every map $g\colon X \to \mathbb{R}^n$ can be arbitrarily closely approximated by a map $g'\colon X \to \mathbb{R}^n$ such that α extends over $F \cup g'(X)$.

Indeed, since every partial map to a CW–complex extends over a neighborhood of its domain, the collection $\mathcal{G}_\alpha$ of the maps $g' \in C(X,\mathbb{R}^n)$ such that α extends over $F \cup g'(X)$ is open in $C(X,\mathbb{R}^n)$. Consider a countable representative collection $\mathcal{A}$ of partial maps of $\mathbb{R}^n$ from compact subsets of $\mathbb{R}^n$ to all of $K(G,m)$ with $\dim_G X \le m$ and $G \in \sigma$. Then, if each $\mathcal{G}_\alpha$ is dense in $C(X,\mathbb{R}^n)$, the intersection of all $\mathcal{G}_\alpha$ for $\alpha \in \mathcal{A}$ is also dense in $C(X,\mathbb{R}^n)$ and any map g' from this intersection has the property that $d_{g'(X)} \le d_X$.

Proposition 4.3 *Let X be a compactum with $\dim X \le 3$ and $n \ge 6$. Then any map from $g\colon X \to \mathbb{R}^n$ can be arbitrarily closely approximated by a map $g'\colon X \to \mathbb{R}^n$ such that $d_{g'(X)} \le d_X$ and $g'(X)$ is tame in $\mathbb{R}^n$.*

Proof By Stanko's reembedding theorem [23] it suffices to construct g' such that $d_{g'(X)} \le d_X$. The cases $n > 6$ or $\dim X \le 2$ are trivial since any map from X to $\mathbb{R}^n$ can be approximated by an embedding. So the only case we need to consider is $\dim X = 3$ and $n = 6$.

Let $\alpha\colon F \to K(G,m)$ with $m \ge 1$ be a map from a compact subset F of $\mathbb{R}^n$ to $K(G,m)$ such that $\dim_G X \le m$ and G is a group in the Bockstein basis σ. Extend α over a closed neighborhood F^+ of F to a map $\alpha_F^+\colon F^+ \to K(G,m)$ and extend g restricted to $g^{-1}(F^+)$ and followed by α_F^+ to a map $\beta\colon X \to K(G,m)$. Note that, since X is compact, the map β can be considered as a map to a finite subcomplex K of $K(G,m)$. Let $\epsilon > 0$ be such that any two 2ϵ–close maps to K are homotopic. Approximate g through a 3–dimensional finite simplicial complex L and maps $\gamma\colon X \to L$ and $g_L\colon L \to \mathbb{R}^n$ such that γ is surjective and:

(i) g and $g_L \circ \gamma$ are so close that $g(\gamma^{-1}(F_L)) \subset F^+$ for $F_L = g_L^{-1}(F)$ and the maps $\alpha_F^+ \circ g$ and $\alpha_F^+ \circ g_L \circ \gamma$ restricted to $\gamma^{-1}(F_L)$ are ϵ–close.

(ii) The fibers of γ are so small that there is a map $\beta_L\colon L \to K$ such that β and $\beta_L \circ \gamma$ are ϵ–close.

Take $y \in F_L$ and let $x \in X$ be such that $\gamma(x) = y$. By (ii), $\beta(x)$ and $(\beta_L \circ \gamma)(x) = \beta_L(y)$ are ϵ–close. By (i), $\beta(x) = (\alpha_F^+ \circ g)(x)$ and $(\alpha_F^+ \circ g_L \circ \gamma)(x) = (\alpha_F^+ \circ g_L)(y) = (\alpha \circ g_L)(y)$ are ϵ–close. Thus, we get that β_L and $\alpha \circ g_L$ are 2ϵ–close on F_L and hence, replacing β_L by a homotopic map, we may assume that β_L coincides on F_L with g_L followed by α.

Since $\dim L \leq 3$ and $n \geq 6$, we can in addition assume that g_L is finite-to-one and g_L is not one-to-one over only finitely many points of $g_L(L)$. Then, since $K(G,m)$ is a connected CW–complex, we can change β_L (up to homotopy) outside the set F_L so that β_L will be constant on each fiber of g_L which is not over F, and hence we can assume that β_L factors through a map $\alpha'\colon g_L(L) \to K(G,m)$ such that α' coincides with α on $g_L(L) \cap F$. Thus the map g can be arbitrarily closely approximated by a map $g' = g_L \circ \gamma\colon X \to \mathbb{R}^n$ such that α extends over $F \cup g'(X) = F \cup g_L(L)$. As we observed before, this property implies the proposition. □

Note that the use of Stanko's reembedding theorem in Proposition 4.3 can be easily avoided by constructing $f(X)$ to be the intersection of a decreasing sequence of sufficiently close PL–regular neighborhoods of 3–dimensional finite simplicial complexes in $\mathbb{R}^n$.

Proposition 4.4 (also see [6; 22]) *Let K be a CW–complex. Then one can attach to K cells of dimensions ≤ 3 to obtain a simply connected CW–complex K^+ such that the inclusion of K into K^+ induces an isomorphism of the integral homology in dimensions > 1.*

Proof Clearly by attaching intervals to K we can turn K into a connected CW–complex preserving the integral homology of K in dimensions > 0. Thus, we may assume that K is connected.

Let a simply connected CW–complex K' be obtained from K by attaching 2–cells to kill the fundamental group of K. Consider the inclusion $i\colon K \to K'$ and the quotient map $p\colon K' \to K'/K$. Note that $i_*(H_2(K)) = \ker p_*$ and $H_2(K'/K)$ is a free group since K'/K is a bouquet of 2–spheres. Then $p_*(H_2(K'))$ is a free group as well. Take a collection ω_j for $j \in J$ of 2–cycles of K' such that $p_*[\omega_j]$ for $j \in J$ are free generators of $p_*(H_2(K'))$. Since K' is simply connected, one can enlarge K' to a CW–complex K'' by attaching for every j a 3–cell Ω_j such that $\partial\Omega_j$ is homologous in K' to ω_j (we consider the cellular homology). Replacing ω_j by $\partial\Omega_j$, assume that $\omega_j = \partial\Omega_j$. Let us show that the inclusion of K into K'' induces an isomorphism of the homology groups in dimensions > 1.

Take a 2–cycle α of K''. Then α lies in K' and $p_*[\alpha] = \sum n_j p_*[\omega_j]$, with $n_j \in \mathbb{Z}$. Thus, $\alpha - \sum n_j\omega_j$ is homologous to a cycle in K and since $\omega_j = \partial\Omega_j$ we get that α is homologous in K'' to a cycle in K. Hence, the inclusion of K into K'' induces an epimorphism of 2–homology.

Take a 2–cycle α in K homologous to 0 in K''. Then $\alpha = \partial(\beta + \sum n_j\Omega_j)$ with β being a 3–chain in K. Thus, $\alpha = \partial\beta + \sum n_j\omega_j$ and $p_*[\alpha] = \sum n_j p_*[\omega_j] = 0$, and hence $n_j = 0$ for every j. Thus, $\alpha = \partial\beta$ is homologous to 0 in K and hence the inclusion of K into K'' induces a monomorphism of 2–homology.

Since we attached to K only cells of dimension ≤ 3 the inclusion of K into K'' induces a monomorphism of 3–homology. Take a 3–cycle α in K''. Then $\alpha = \beta + \sum n_j\Omega_j$, where β is a 3–chain in K. Hence, $p_*[\partial\alpha] = p_*[\partial\beta + \sum n_j\omega_j] = \sum n_j p_*[\omega_j] = 0$ and we get that $n_j = 0$ for every j. Thus, $\alpha = \beta$ and hence the inclusion of K into K'' induces an epimorphism of 3–homology.

Clearly the homology groups of K and K'' coincide in $\dim > 3$. Thus, the proposition holds with $K^+ = K''$. □

Definition 4.5 Let us say that a map $f\colon X \to \mathbb{R}^n$ with $n \geq 6$ from a finite-dimensional compactum X is almost supported by an open subset $V \subset \mathbb{R}^n$ if there is an open subset $V^- \subset V$ such that the closure of V^- is contained in V and $f(X) \setminus V^-$ is contained in a tame compactum in $\mathbb{R}^n$ with dimension ≤ 3 and dimension type $\leq d_X$.

Proposition 4.6 *Let $Y \subset \mathbb{R}^n$ with $n \geq 6$ be a compactum, $U \subset \mathbb{R}^n$ an open ball in $\mathbb{R}^n$ and let X be a compactum such that $\dim X \times Y \leq n-1$. Then any map from a closed subset of X to $V = U \setminus Y$ extends over X to a map to U almost supported by V.*

Proof Let $f\colon F \to V$ be a map from a closed subset F of X. By Theorem 4.1 and Proposition 2.9 we have $\dim_{H_i(V)} X \leq i$ for every i. Take a triangulation of V and, by Proposition 4.4, attach to V cells of dimensions ≤ 3 to obtain a simply connected CW–complex V^+ preserving the homology of V in dimensions > 1. Then, by Theorem 3.2, e-dim $X \leq V^+$ and hence here is a map $\phi\colon X \to V^+$ extending f.

Let L be a finite subcomplex of V^+ containing $\phi(X)$. Denote $L^- = V \cap L$, let L^+ be the 3–skeleton of K, $X^- = (\phi)^{-1}(L^-)$ and $X^+ = (\phi)^{-1}(L^+)$. Clearly L^- and L^+ are subcomplexes of L, $\dim L^+ \le 3$ and $L = L^- \cup L^+$. Take any map $\psi\colon L \to U$ such that ψ does not move the points of $L^-$and take an open neighborhood V^- of L^- in V such that the closure of V^- is contained in V. By Theorem 3.1, approximate ϕ restricted to X^+ by the composition of maps $\phi_Z^+\colon X^+ \to Z$ and $\phi_L^+\colon Z \to L^+$ such that $d_Z \le d_X$ and $\dim Z \le 3$. By Proposition 4.3, the map ϕ_L^+ followed by ψ can be approximated by a map $g\colon Z \to U$ with a tame image of dimension ≤ 3 and dimension type $\le d_Z \le d_X$.

Consider the maps $f^+ = g \circ \phi_Z^+\colon X^+ \to U$ and $f^- = \phi|_{X^-}\colon X^- \to L^- \subset V^-$. Note that f^- coincides with f on F. Also note that if $\phi_L^+ \circ \phi_Z^+$ is close enough to ϕ restricted to X^+ and g is close enough to ϕ_L^+ followed by ψ, we may assume that f^- and f^+ restricted to $X^- \cap X^+$ are as close as we wish. Then one can find a map $f'\colon X \to U$ such that f' coincides with f^- on X^-, $f'^{-1}(U \setminus V^-) = (f^+)^{-1}(U \setminus V^-)$ and f' coincides with f^+ on $(f^+)^{-1}(U \setminus V^-) \subset X^+$. Thus, f' extends f and $f'(X) \setminus V^- \subset g(Z)$ and, hence, f' is almost supported by V. □

Theorem 4.7 *Let X be a compactum with $\dim X \le n-2$ and $n \ge 6$. Then any map $f\colon X \to \mathbb{R}^n$ can be arbitrarily closely approximated by a map $f'\colon X \to \mathbb{R}^n$ such that $d_{f'(X)} \le d_X$.*

Proof We prove the proposition by induction on $\dim X$. Clearly the theorem holds if $\dim X = 0$. Assume that the theorem is proved for compacta of dimension $\le \dim X - 1$. Fix $\epsilon > 0$ and partition X into finitely many closed subsets $X = \bigcup X_j$ with disjoint interiors such that $\dim \partial X_j \le \dim X - 1$ and $f(X_j)$ is contained in an open ϵ–ball U_j in $\mathbb{R}^n$. By the induction hypothesis we can assume that $d_{f(\partial X_j)} \le d_{X_j} \le d_X$.

By Proposition 2.9 and Theorem 2.8, decompose $\mathbb{R}^n$ into $\mathbb{R}^n = A \cup B$ with $d_A \le d_X$ and $d_B \le n-1 \ominus d_X$. Clearly we may assume that $f(\partial X_j) \subset A$. Take any map $\alpha\colon F \to K(G,m)$ from a compact subset F of $\mathbb{R}^n$ to an Eilenberg–Mac Lane complex with $\dim_G X \le m$ for some $G \in \sigma$ and extend α to a map $\alpha_W\colon W \to K(G,m)$ over an open subset W of $\mathbb{R}^n$ such that $F \cup A \subset W$. Let Y_j be the closure of $U_j \setminus W$ in $\mathbb{R}^n$ and $V_j = U_j \setminus Y_j = U_j \cap W$. Then Y_j is a compact subset of B and, by Proposition 2.9, we have that $\dim X_j \times Y_j \le n-1$. Note that $f(\partial X_j) \subset V_j$, apply Proposition 4.6 to extend $f_j = f|_{\partial X_j}\colon \partial X_j \to V_j$ to a map $f'_j\colon X_j \to U_j$ almost supported by V_j, and take an open set $V_j^- \subset V_j$ witnessing that f'_j is almost supported by V_j.

Set $f'\colon X \to \mathbb{R}^n$ to be the map defined by the maps f'_j and V^- to be the union of all V_j^-. Note that f' is ϵ–close to f, the closure $\operatorname{cl} V^-$ of V^- is contained in W, $f'(X) \setminus V^-$ is contained in the union of $f'_j(X_j) \setminus V_j^-$, and, hence, $f'(X) \setminus V^-$ is of dimension type $\le d_X$. Then α_W restricted to $F \cup \operatorname{cl} V^-$ extends over $F \cup f'(X)$ and, as we observed before Proposition 4.3, this implies the result required in the theorem. □

Theorem 4.8 *Let X and Y be compacta such that $\dim X \le n-2$, $\dim Y \le n-2$, $\dim X \times Y \le n-1$ and $n \ge 6$. Then X and Y do not admit a stable intersection in $\mathbb{R}^n$.*

Proof We prove the theorem by induction on $\dim X$. The case $\dim X = 0$ is trivial. Assume that $\dim X > 0$. Take maps $f\colon X \to \mathbb{R}^n$ and $g\colon Y \to \mathbb{R}^n$. Fix $\epsilon > 0$ and partition X into finitely many closed subsets $X = \bigcup X_j$ with disjoint interiors such that $\dim \partial X_j \le \dim X - 1$ and $f(X_j)$ is contained in an open ϵ–ball U_j in $\mathbb{R}^n$. By the induction hypothesis and Theorem 4.7, we can replace f and g by arbitrarily close approximations and assume that $d_{g(Y)} \le d_Y$ and $f(\partial X_j) \cap g(Y) = \varnothing$ for every j.

Set $V_j = U_j \setminus g(Y)$. Since $d_{g(Y)} \le d_Y$, we have $\dim X \times g(Y) \le n-1$. Note that $f(\partial X_j) \subset V_j$, apply Proposition 4.6 to extend $f_j = f|_{\partial X_j}\colon \partial X_j \to V_j$ to a map $f'_j\colon X_j \to U_j$ almost supported by V_j, and take an open set $V_j^- \subset V_j$ witnessing that f'_j is almost supported by V_j.

Set $f'\colon X \to \mathbb{R}^n$ to be the map defined by the maps f'_j and V^- to be the union of all V_j^-. Note that f' is ϵ–close to f, the closure $\operatorname{cl} V^-$ of V^- does not meet $g(Y)$, $f'(X) \setminus V^-$ is contained in the union of $f'_j(X_j) \setminus V_j^-$, and, hence, $f'(X) \setminus V^-$ is contained in a finite union of tame compacta of dimension $\le 3 \le n-3$ and of dimension type $\le d_X$. Then, by Theorem 4.2, the map g can be arbitrarily closely approximated by a map $g'\colon Y \to \mathbb{R}^n$ such that $f'(X) \cap g'(Y) = \varnothing$ and the theorem follows. □

Proof of Theorem 1.5 Theorem 1.5 follows from Theorems 4.8, 1.2 and 1.3. □

5 Remarks

Problem 3.4 can be considered in a more general context.

Problem 5.1 *Let K be a connected CW–complex whose fundamental group is abelian and X a finite-dimensional compactum such that $\dim_{H_n(K)} X \le n$ for every $n > 0$. Does this imply that $\operatorname{e-dim} X \le K$?*

It turns out that this problem reduces to 3–dimensional compacta.

Proposition 5.2 *Problem 5.1 is equivalent to the same problem with the additional assumption that* $\dim X \le 3$.

Proof Let $f\colon F \to K$ be a map from a closed subset F of X. By Proposition 4.4, K can be enlarged to a simply connected CW–complex K^+ preserving the integral homology of K in dimensions > 1 with $\dim K^+ \setminus K \le 3$. Then, by Theorem 3.2, f extends to a map $g\colon X \to K^+$. Let L be the 3–skeleton of K^+, $X_L = g^{-1}(L)$ and $X_K = g^{-1}(K)$. By, Theorem 3.1, g restricted to X_L can be arbitrarily closely approximated by a map that factors through maps $\phi\colon X_L \to Z$ and $\psi\colon Z \to L$ with a compactum Z such that $\dim Z \le 3$ and $d_Z \le d_{X_L} \le d_X$. Thus, we can assume that g and $\psi \circ \phi$ restricted to $X_L \cap X_K$ are as close as we wish. Then one can replace the map ψ by a homotopic map and assume that $\psi(\phi(X_L \cap X_K)) \subset K$, and g and $\psi \circ \phi$ restricted to $X_L \cap X_K$ are homotopic in K. Now, assuming that Problem 5.1 has the affirmative answer for compacta of dimension ≤ 3, one can extend ψ restricted to $\phi(X_L \cap X_K)$ over Z as a map to K and, hence, g restricted to X_K extends over X as a map to K. Recall that g coincides with f on F, and the proposition follows. □

Let us state without proof two more results related to the techniques presented in this paper.

- Theorem 3.1 can be extended to the following result: Let X be a finite-dimensional compactum, L a finite CW–complex and $f\colon X \to L$ a map. Then f can be arbitrarily closely approximated by a map that factors through a compactum Z with $\dim Z \le \max\{\dim L, 3\}$ and $d_Z \le d_X$.
- It turns out that Theorem 3.7 and Problem 3.4 are closely related. Namely, a Moore space $M(\mathbb{Z}_{p^\infty}, 1)$ is a classifying space for finite-dimensional compacta with $\dim_{\mathbb{Z}_{p^\infty}} = 1$ if and only if, for every 3–dimensional compactum X with $\dim_{\mathbb{Z}_{p^\infty}} X = 1$ and every map $f\colon X \to \mathbb{R}^2$, we have that f can be arbitrarily closely approximated by a map that factors through a compactum Z with $\dim Z \le 2$ and $\dim_{\mathbb{Z}_{p^\infty}} Z \le 1$.

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