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Exotic open 4–manifolds which are nonleaves

CARLOS MENIÑO COTÓN
PAUL A SCHWEITZER

We study the possibility of realizing exotic smooth structures on finitely punctured simply connected closed 4–manifolds as leaves of a codimension-one foliation on a compact manifold. In particular, we show the existence of uncountably many smooth open 4–manifolds which are not diffeomorphic to any leaf of a codimension-one C^2 foliation on a compact manifold. These examples include some exotic $\mathbb{R}^4$'s and exotic cylinders $S^3 \times \mathbb{R}$.

37C85, 53C12, 57R30; 57R55

Introduction

The stunning results of Donaldson [9] and Freedman [10] provided the existence of exotic smooth structures on $\mathbb{R}^4$, which is known to be the unique euclidean space with this property. This is in fact also true for an open 4–manifold with a collarable end; see Bižaca and Etnyre [3]. The fact that these structures can arise in 4–dimensional manifolds has implications for physics (see eg Asselmeyer-Maluga and Brans [1] and Król [25]): what if our space-time carries an exotic structure? Since the discovery of the exotic family in the 1980s, nobody has been able to find an explicit and useful exotic atlas. It is worthy of interest to obtain alternative explicit descriptions of these exotica.

An open manifold which is realizable as a leaf of a foliation in a compact manifold must satisfy some restrictions. Since the ambient manifold is compact, an open manifold has to accumulate somewhere, and this induces recurrence and "some periodicity" on its ends.

Before reviewing the history of realizability of open manifolds as leaves, we now state our main results. In Section 2 we shall define a class $\mathcal{Y}$ of smooth open manifolds (up to diffeomorphism) whose underlying topological manifolds are obtained by removing a finite nonzero number of points from a closed, connected, simply connected topological 4–manifold. In fact (see Example 2.2 and Remark 2.3), every such topological manifold is homeomorphic to uncountably many elements of $\mathcal{Y}$.

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Theorem 1 *No manifold $Y \in \mathcal{Y}$ is diffeomorphic to any leaf of a C^2 codimension-one foliation of a compact manifold.*

Theorem 1 and all our results and proofs hold for the slightly weaker assumption of $C^{1+\mathrm{Lip}}$ regularity. For the sake of readability and coherence with the references we have decided to state this theorem for C^2 foliations. As a consequence of Theorem 1, for every $Y \in \mathcal{Y}$ there are uncountably many diffeomorphically distinct smooth manifolds homeomorphic to Y that cannot be leaves in any C^2 foliation of a compact 5–manifold. Also note that if Z is obtained by puncturing a *smooth* closed simply connected manifold M, then the induced smooth manifold can easily be realized as a leaf of a C^∞ codimension-one foliation; just insert Reeb components along transverse closed curves in the product foliation of $M \times S^1$.

It is not known whether any element of $\mathcal{Y}$ is diffeomorphic to a leaf of a $C^{1,0}$ codimension-one foliation on a compact manifold. If that can happen, we show that some restrictions must appear as to how it is realized; this is summarized in the next proposition, which is the main step in proving Theorem 1.

Proposition 2 *If there exists a leaf diffeomorphic to $Y \in \mathcal{Y}$ in a $C^{1,0}$ codimension-one foliation of a closed 5–manifold, then it is a proper leaf and each connected component of the union of the leaves diffeomorphic to Y fibers over the circle with the leaves as fibers.*

Next we review some of the history of leaves and nonleaves. It was shown by J Cantwell and L Conlon [6] that every orientable open surface is homeomorphic (in fact, diffeomorphic) to a leaf of a foliation on each closed 3–manifold, and nonorientable open surfaces are homeomorphic to leaves in nonorientable 3–manifolds. The first examples of topological nonleaves were due to E Ghys [16] and T Inaba, T Nishimori, M Takamura and N Tsuchiya [23]; these are highly topologically nonperiodic open 3–manifolds which cannot be homeomorphic to leaves in a codimension-one foliation in a compact manifold. Years later, O Attie and S Hurder [2], in a deep analysis of the question, found simply connected 6–dimensional examples of nonleaves, nonleaves which are homotopy equivalent to leaves and even a Riemannian manifold which is not quasi-isometric to a leaf in arbitrary codimension. These examples follow the line of the work of A Phillips and D Sullivan [28] and T Januszkiewicz [24] and led to other examples of Zeghib [36] and Schweitzer [30].

C L Taubes [32] showed that the smooth structure of some of the exotic $\mathbb{R}^4$'s is, in some sense, nonperiodic at infinity, and this leads to the existence of uncountably many nondiffeomorphic smooth structures on $\mathbb{R}^4$. It is an open problem whether any exotic $\mathbb{R}^4$ — and, by extension, any given open manifold with a similar exotic smooth end structure — can be diffeomorphic to a leaf of a foliation on a compact manifold. By a simple cardinality argument, most exotic $\mathbb{R}^4$'s cannot be covering spaces of closed smooth 4–manifolds by smooth covering maps since the diffeomorphism classes of smooth closed manifolds are countable. All these results motivated a folklore conjecture in foliation theory suggesting that these exotic structures cannot occur on leaves of a smooth foliation in a compact manifold.

The main difference between some exotic $\mathbb{R}^4$'s (called *large*) and the standard $\mathbb{R}^4$ is the fact that they cannot embed smoothly in a standard $\mathbb{R}^4$. An important question for a large exotic $\mathbb{R}^4$ is to describe what the simplest spin manifolds (in the sense of the second Betti number) in which it can be embedded are; this is measured by the invariant defined by L Taylor [33], which provided the first direct tool to show that some exotic $\mathbb{R}^4$'s cannot be nontrivial covering spaces. We shall show that these exotica are also nonleaves.

For all the other finitely punctured simply connected closed 4–manifolds we shall see that a *Taubes-like* end (see Definition 1.11) suffices to show that they are nonleaves. In Proposition 2 we adapt Ghys' procedure in [16] to show some necessary conditions for such structures to be leaves of a codimension-one foliation on a compact manifold. In Theorem 1, which is an easy corollary of Proposition 2, we complete this analysis in the case of C^2 foliations (those where the transverse coordinate changes are C^2 maps).

The paper is organized as follows:

- The first section is devoted to exotic structures on open 4–manifolds, particularly on $\mathbb{R}^4$. This is in fact a brief exposition of results in Taubes [32] and Taylor [33]. Here we define the particular exotic structures that we consider on $\mathbb{R}^4$ and show some of their properties.
- In the second section we prove Proposition 2, which gives necessary conditions for certain exotic punctured simply connected closed 4–manifolds to be diffeomorphic to leaves, following Ghys' method of proof [16], and we derive its corollary, Theorem 1.
- The last section includes some last remarks and open questions.

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1 Exotic structures on $\mathbb{R}^4$

In this section we construct uncountably many exotic structures in $\mathbb{R}^4$ which are nonperiodic by Taubes' work. Later we shall need a better control of this structure, which is provided by the invariant defined by Taylor [33]. This introduction begins with a brief reminder of some known facts in 4–dimensional topology.

1a Background

Theorem 1.1 (Freedman [10]) *Two simply connected closed 4–manifolds are homeomorphic if and only if their intersection forms are isomorphic and they have the same modulo 2 Kirby–Siebenmann invariant. In particular, simply connected smooth closed 4–manifolds are homeomorphic if and only if their intersection forms are isomorphic.*

Theorem 1.2 (Donaldson [9]) *If a smooth closed simply connected 4–manifold has a definite intersection form then it is isomorphic to a diagonal form.*

Definite symmetric bilinear unimodular forms are not classified and it is known that the number of isomorphism classes grows at least exponentially with the range. Indefinite unimodular forms are classified [31]: two indefinite forms are isomorphic if they have the same range, signature, and parity. There are canonical representatives for the indefinite forms; in the odd case the form is diagonal and in the even case it splits into invariant subspaces where the intersection form is either $\pm E_8$ or H; see Figure 1. These canonical representatives are denoted as usual with the notation $m[+1]\oplus n[-1]$ for the odd case and $\pm mE_8 \oplus nH$ with $n>0$ for the even one.

For each symmetric bilinear unimodular form there exists at least one topological simply connected closed 4–manifold with an isomorphic intersection form. But this is no longer true for the smooth case, as Donaldson's theorem asserts. It is an open problem which unimodular forms can be realized in smooth simply connected closed 4–manifolds. Recall that a (not necessarily closed) simply connected smooth 4–manifold is spin if

$$E_8 = \begin{pmatrix} 2&1&0&0&0&0&0&0\\ 1&2&1&0&0&0&0&0\\ 0&1&2&1&0&0&0&0\\ 0&0&1&2&1&0&0&0\\ 0&0&0&1&2&1&0&1\\ 0&0&0&0&1&2&1&0\\ 0&0&0&0&0&1&2&0\\ 0&0&0&0&1&0&0&2 \end{pmatrix} \qquad H = \begin{pmatrix} 0&1\\ 1&0 \end{pmatrix}$$

Figure 1: The intersection forms E_8 (left) and H (right)

and only if its intersection form is even. It is known that for a smooth simply connected spin 4–manifold with indefinite intersection form the number of "E_8 blocks" must be even (Rokhlin's theorem). It is possible to say more, as in Furuta's theorem [13], which will be useful in this section.

Theorem 1.3 (Furuta [13]) *If M is a smooth closed spin* (*not necessarily simply connected*) *4–manifold with an intersection form equivalent to $\pm 2mE_8 \oplus nH$ and $m > 0$, then $n \geq 2m+1$.*

Let us recall an important theorem of M H Freedman, which is the main tool to determine when a manifold is homeomorphic to $\mathbb{R}^4$.

Theorem 1.4 (Freedman [10]) *An open 4–manifold is homeomorphic to $\mathbb{R}^4$ if and only if it is contractible and simply connected at infinity.*

Definition 1.5 Two ends $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ of smooth (resp. topological) manifolds are *diffeomorphic* (resp. *homeomorphic*) if they have diffeomorphic (resp. homeomorphic) neighborhoods $\boldsymbol{X}_{\boldsymbol{e}_1}$ and $\boldsymbol{X}_{\boldsymbol{e}_2}$. It will always be assumed that orientation is preserved by that diffeomorphism (resp. homeomorphism). Two manifolds with one end are *end-diffeomorphic* (resp. *end-homeomorphic*) if their ends are diffeomorphic (resp. homeomorphic).

The main tool for measuring the wildness of some exotica will be the *Taylor index*, introduced by Taylor in [33].

Definition 1.6 (Taylor [33]) Let E be a smoothing of $\mathbb{R}^4$. Let $\mathrm{Sp}(E)$ be the set of closed smooth spin 4–manifolds N with trivial or hyperbolic intersection form (a sum of copies of H) in which E embeds smoothly. Define $b_E = \infty$ if $\mathrm{Sp}(E) = \varnothing$; otherwise

$$2b_E = \min_{N \in \mathrm{Sp}(E)} \{b_2(N)\},$$

where $b_2(N)$ is the second Betti number of N.

Let $\mathcal{E}(E)$ be the set of topological embeddings $e\colon D^4 \to E$ such that e is smooth in the neighborhood of some point of the boundary and $e(\partial D^4)$ is (topologically) bicollared. Set $b_e = b_{e(\mathring{D}^4)}$ where $e(\mathring{D}^4)$ has the smooth structure induced by E. The *Taylor index* of E is defined to be

$$\gamma(E) = \max_{e\in\mathcal{E}(E)} \{b_e\}.$$

For a spin manifold M, the Taylor index of M is the supremum of the Taylor indices of all the exotic $\mathbb{R}^4$'s embedded in M.

Remark 1.7 For the definition of b_E the regularity of the embedding is not important since every C^1 manifold admits a C^∞ structure by [35, Theorem 1], and every C^1 map (resp. diffeomorphism) of C^∞ manifolds can be approximated arbitrarily closely by a C^∞ map (resp. diffeomorphism).

Another important tool for this section is the "end-sum" construction. For open manifolds this is analogous to the connected sum of closed manifolds. Given two open smooth oriented manifolds M and N with the same dimension we choose two smooth properly embedded paths $c_1\colon [0,\infty) \to M$ and $c_2\colon [0,\infty) \to N$, each of them defining one end in M and N respectively. Let V_1 and V_2 be smooth tubular neighborhoods of $c_1([0,\infty))$ and $c_2([0,\infty))$. The boundaries of these neighborhoods are clearly diffeomorphic to $\mathbb{R}^3$ and we can obtain a smooth sum by removing the interiors of these neighborhoods and identifying their boundaries so as to produce a manifold with an orientation respecting the orientations of M and N. This will be called the *end-sum* of M and N associated to c_1 and c_2, and it is denoted by $M \natural N = (M \setminus \mathring{V}_1) \cup_\partial (N \setminus \mathring{V}_2)$. In the case where N and M have exactly one end and are both homeomorphic to $S^3 \times \mathbb{R}^+$, the paths c_1 and c_2 are unique up to ambient isotopy and thus the smooth structure of $M \natural N$ does not depend on the choice of paths. End-sum was the first technique which made it possible to find infinitely many exotic structures on $\mathbb{R}^4$ [17] and it is an important tool for dealing with the problem of generating infinitely many smooth structures on open 4–manifolds [3; 14].

Lemma 1.8 [33, Lemma 5.2] *If $\boldsymbol{R}$ and $\boldsymbol{S}$ are exotic $\mathbb{R}^4$'s, then the inequality $\gamma(\boldsymbol{R} \natural \boldsymbol{S}) \le \gamma(\boldsymbol{R}) + \gamma(\boldsymbol{S})$ holds.*

We now give a version of Taubes' theorem [32, Theorem 1.4] sufficient for our purposes.

Definition 1.9 (periodic end) Let M be an open smooth manifold with an end homeomorphic to the end of $S^3 \times [0,\infty)$. We say this end is *smoothly periodic* if there exists an open neighborhood $V \subset M$ of the end that is homeomorphic to $S^3 \times (0,\infty)$

and a diffeomorphism $h\colon V \to h(V) \subset V$ such that $h^n(V)$ defines the given end (ie $\{h^n(V)\}$ is a neighborhood base for the end). If M has exactly one end then it is said that M is *smoothly periodic*.

Note that this notion of smoothly periodic end is a particular case of the admissible periodic ends considered in [32, Definition 1.3].

Theorem 1.10 (Taubes [32]) *Suppose that M is an open smooth simply connected 4–manifold with a definite intersection form and exactly one end. If there exists an open neighborhood of the end of M which is homeomorphic to $S^3 \times (0,\infty)$ and smoothly periodic, then the intersection form is isomorphic to a diagonal form.*

1b Our exotic models

Definition 1.11 Throughout this work $\mathcal{M}_-$ (resp. $\mathcal{M}_+$) will denote the family of smoothings of closed topological 4–manifolds M with exactly one puncture such that there exists a positive integer s such that $\natural_{i=1}^{s} M$ is end-diffeomorphic to a smoothing of a once-punctured closed topological simply connected negative (resp. positive) definite but not diagonal 4–manifold. Set $\mathcal{M} = \mathcal{M}_- \cup \mathcal{M}_+$. These manifolds and their ends will be called *Taubes-like*.

The set $\mathcal{S}$ will denote the family of all exotic $\mathbb{R}^4$'s $\boldsymbol{R}$ for which there exist two integers $s,k > 0$ such that the s–fold end-sum $\natural_{i=1}^{s} \boldsymbol{R}$ is end-diffeomorphic to a smoothing of a once-punctured closed simply connected spin $-k(E_8 \oplus E_8)$ manifold.[1] The subfamily $\mathcal{R}$ will be formed by the exotica in $\mathcal{S}$ with finite Taylor index.

Remark 1.12 Of course, $\mathcal{R} \subset \mathcal{S} \subset \mathcal{M}_-$. Observe that $(\mathcal{M}_\pm,\natural)$, $(\mathcal{S},\natural)$ and $(\mathcal{R},\natural)$ are semigroups; this comes from the fact that the sum of nondiagonal definite forms of the same sign are still definite and nondiagonal by the Eichler–Kneser theorem (see eg [15, Theorem 9.24]).

Proposition 1.13 (see [33, Theorem 5.3]) *Let $\boldsymbol{R} \in \mathcal{S}$, and take $s,k \in \mathbb{N}$ so that the s–fold end-sum $\natural^s \boldsymbol{R}$ is end-diffeomorphic to a spin simply connected once-punctured closed $-k(E_8 \oplus E_8)$ manifold. Then $0 < 2k/s < \gamma(\boldsymbol{R})$. Therefore $\gamma\big(\natural^n \boldsymbol{R}\big)$ tends to ∞ as $n \to \infty$, and so $\gamma\big(\natural_{i=1}^{\infty} \boldsymbol{R}\big) = \infty$.*

Lemma 1.14 *Let M and N be two smooth once-punctured closed 4–manifolds end-homeomorphic to $\mathbb{R}^4$ such that $M \subset N$ and $N \setminus M$ is homeomorphic to $S^3 \times [0,\infty)$ with topologically bicollared boundary. If $M \in \mathcal{M}$ then $N \in \mathcal{M}$.*

[1] These are called (s,k)–*simple-semidefinite* in [33].

Proof Since $M \in \mathcal{M}$, there exists $s > 0$ such that $\natural^s M$ is end-diffeomorphic to a once-punctured closed simply connected smooth manifold having a definite but not diagonal intersection form, which we denote by W. By the hypothesis, it is clear that there exists a smooth embedding $j\colon \natural^s M \to \natural^s N$ such that $\natural^s N \setminus j(\natural^s M)$, with the induced smooth structure, is still homeomorphic to $S^3 \times [0,\infty)$ with a bicollared boundary. Let $W' \subset W$ be an open set large enough that $W \setminus W'$ is diffeomorphic to a neighborhood of the end of $\natural^s M$. It follows that $\natural^s N$ is end-diffeomorphic to $W' \cup_\partial (W \setminus W') \cup_j \natural^s N \setminus j(\natural^s M)$, which is homeomorphic to a once-punctured closed simply connected manifold having the same intersection form as W, so $N \in \mathcal{M}$. □

Corollary 1.15 *The semigroup $(\mathcal{S}, \natural)$ is closed under countably infinite end-sums. Moreover, if $\boldsymbol{R} \in \mathcal{S}$ and $\boldsymbol{E}$ is any smoothing of $\mathbb{R}^4$ then $\boldsymbol{R} \natural \boldsymbol{E} \in \mathcal{S}$.*

Proof Since $\boldsymbol{R} \in \mathcal{S}$ there exist $s, k > 0$ such that $\natural^s \boldsymbol{R}$ is end-diffeomorphic to a once-punctured closed simply connected smooth $-k(E_8 \oplus E_8)$ manifold, which we denote by W. Thus, there exists a (topological) disk $\boldsymbol{D} \subset \boldsymbol{R}$ sufficiently large that $\natural^s \boldsymbol{R} \setminus \natural^s \mathring{\boldsymbol{D}}$ is diffeomorphic to a neighborhood of the end of W with topologically bicollared boundary. Thus $\natural^s \mathring{\boldsymbol{D}}$ is end-diffeomorphic to $W \setminus (\natural^s \boldsymbol{R} \setminus \natural^s \mathring{\boldsymbol{D}})$, which is still a $-k(E_8 \oplus E_8)$ manifold. Therefore $\mathring{\boldsymbol{D}} \in \mathcal{S} \subset \mathcal{M}$. Observe that $\boldsymbol{D} \subset \boldsymbol{R} \subset \boldsymbol{R} \natural \boldsymbol{E}$, so $\mathring{\boldsymbol{D}}$ and $\boldsymbol{R} \natural \boldsymbol{E}$ satisfy the conditions of Lemma 1.14, so $\boldsymbol{R} \natural \boldsymbol{E} \in \mathcal{M}$ and therefore it belongs to $\mathcal{S}$ since it is an exotic $\mathbb{R}^4$.

The first affirmation follows by choosing $\boldsymbol{E}$ as any infinite end-sum of elements in $\mathcal{S}$. □

Remark 1.16 The above Corollary 1.15 does not hold for the semigroup $(\mathcal{R}, \natural)$, by Proposition 1.13; of course it is also false for $\mathcal{M}_\pm$ just by purely topological reasons.

Remark 1.17 (see also [33, Theorem 5.4]) No Taubes-like manifold is smoothly periodic: If $M \in \mathcal{M}$ is smoothly periodic then $\natural^s M$ will be also smoothly periodic for all $s \in \mathbb{N}$. But for some s this manifold would be end-diffeomorphic to a definite nondiagonal simply connected 4–manifold (by definition of $\mathcal{M}$), but this is not possible by Taubes' theorem. In particular, Taubes-like ends are exotic, ie they are not diffeomorphic to the ends of the standard $S^3 \times \mathbb{R}$, which are obviously smoothly periodic.

Notation 1.18 Let $\boldsymbol{R}$ be an exotic $\mathbb{R}^4$ and let $\psi_{\boldsymbol{R}}\colon \mathbb{R}^4 \to \boldsymbol{R}$ be a homeomorphism. Let us denote $\boldsymbol{K}_t^{\psi_{\boldsymbol{R}}} = \psi_{\boldsymbol{R}}(D(0,t))$, where $D(0,t)$ is the standard closed disk of radius t, and consider the smooth structure induced on $\mathring{\boldsymbol{K}}_t^{\psi_{\boldsymbol{R}}}$ by $\boldsymbol{R}$. For future reference, we choose the homeomorphism $\psi_{\boldsymbol{R}}$ so that for each $t > 0$ the boundary of the topological disk $\boldsymbol{K}_t^{\psi_{\boldsymbol{R}}}$ is smooth in a neighborhood of some point. The existence of such $\psi_{\boldsymbol{R}}$'s is clear; in fact, they can be chosen to be smooth in a neighborhood of one of the axes [29]. By an abuse of notation, we shall use the notation $\boldsymbol{K}_t$ instead of $\boldsymbol{K}_t^{\psi_{\boldsymbol{R}}}$ whenever the underlying exotic $\mathbb{R}^4$ is clear from the context and $\psi_{\boldsymbol{R}}$ is any homeomorphism as above.

Corollary 1.19 [33, Theorem 5.4] *Let $\boldsymbol{R} \in \mathcal{S}$ and let $\psi\colon \mathbb{R}^4 \to \boldsymbol{R}$ be a homeomorphism. Then there exists $r > 0$ such that $\mathring{\boldsymbol{K}}_t$ is not diffeomorphic to $\mathring{\boldsymbol{K}}_s$ for any $t > s > r$.*

In the following, whenever $\boldsymbol{R} \in \mathcal{S}$ (resp. $\mathcal{R}$), we will assume $r_{\psi_{\boldsymbol{R}}}$ is large enough that $\mathring{\boldsymbol{K}}_t \in \mathcal{S}$ (resp. $\mathcal{R}$) for all $t > r_{\psi_{\boldsymbol{R}}}$.

If N_1 and N_2 are oriented manifolds with connected boundaries, where each boundary is assumed to be smooth in a neighborhood of some point, then the *boundary connected sum* $M = N_1 \#_\partial N_2$ is obtained by identifying embedded smooth closed disks in ∂N_1 and ∂N_2 by an orientation-reversing diffeomorphism (so that the resulting manifold is oriented) and smoothing the result. Then the interior of the boundary connected sum is the end-sum of the interiors: $\mathring{M} = \mathring{N}_1 \natural \mathring{N}_2$. If N_1 and N_2 are disjoint connected oriented codimension-zero submanifolds with connected boundaries embedded in a connected oriented manifold M so as to respect the orientations, then $N_1 \#_\partial N_2$ can also be embedded in M using a standard cylinder to join smooth standard disks in the boundaries of N_1 and N_2.

Proposition 1.20 [33, Proposition 2.2] *Let $e_i \in \mathcal{E}(E)$, where $i = 1, \dots, k$, be pairwise disjoint closed topological disks in E (respecting the orientations). Then there exists $e \in \mathcal{E}(E)$ such that $e(\mathring{D}^4)$ is diffeomorphic to $\natural_{i=1}^k e_i(\mathring{D}^4)$.*

By the choice of $\psi_{\boldsymbol{R}}$, it follows that $\boldsymbol{K}_t \in \mathcal{E}(\boldsymbol{R})$ (ie $\boldsymbol{K}_t$ is a topologically embedded ball with the properties indicated in Definition 1.6). The next proposition can be seen as a corollary of [33, Theorem 7.1].

Proposition 1.21 [33, Theorem 7.1] *No $\boldsymbol{R} \in \mathcal{R}$ can be a smooth covering space of a smooth compact 4–manifold.*

Proof If this were the case then there would exist a properly discontinuous smooth $\mathbb{Z}$–action on $\boldsymbol{R}$. It follows that $\boldsymbol{R}$ would contain infinitely many pairwise disjoint copies of sets diffeomorphic to $\boldsymbol{K}_t$ for any $t > r_{\psi_{\boldsymbol{R}}}$. By Proposition 1.20, it follows that $\natural^\infty \mathring{\boldsymbol{K}}_t$ could be embedded in $\boldsymbol{R}$. Since $r_{\psi_{\boldsymbol{R}}}$ was chosen so that $\boldsymbol{K}_t$ belongs to $\mathcal{S}$, it follows that $\gamma(\boldsymbol{R}) = \infty$ by Proposition 1.13. But $\gamma(\boldsymbol{R})$ must be finite since $\boldsymbol{R} \in \mathcal{R}$. □

On the other hand, $\natural_{i=1}^{\infty} \boldsymbol{R}$ can be a nontrivial covering space of an open manifold. In fact it admits several free actions (see eg [18; 19]); for example, this exotic is diffeomorphic to the end-sum $\natural_{i\in\mathbb{Z}} \boldsymbol{R}$, which admits an obvious free action of $\mathbb{Z}$ whose quotient is an exotic $\mathbb{R}^3 \times S^1$.

Remark 1.22 Let $\boldsymbol{R} \in \mathcal{S}$. Given a strictly increasing sequence $\{t_k\}$ tending to infinity with every $t_k > r_{\psi_{\boldsymbol{R}}}$, let us set $\boldsymbol{C}_k^{\psi_{\boldsymbol{R}}} = \boldsymbol{K}_{t_k} \setminus \mathring{\boldsymbol{K}}_{t_{k-1}}$, each of which is homeomorphic to $S^3 \times [0,1]$. We use the notation $\boldsymbol{C}_k$ instead of $\boldsymbol{C}_k^{\psi_{\boldsymbol{R}}}$ whenever $\boldsymbol{R}$ and $\psi_{\boldsymbol{R}}$ are clear from the context.

Of course, $\boldsymbol{C}_k$ also depends on the sequence $\{t_k\}$, but these data are inessential and will be omitted for the sake of simplicity. A similar notation can be adapted to the end of any Taubes-like manifold M. Let X be a cylindrical neighborhood of the end of $M \in \mathcal{M}$ and let $\psi_X \colon X \to S^3 \times [0,\infty)$ be a homeomorphism. Given an increasing sequence of positive numbers t_k going to infinity, we can set $\boldsymbol{C}_k = \psi_X^{-1}(S^3 \times [t_{k-1}, t_k])$ with the induced smooth structure as a subset of M. Again there is a dependence on the sequence, the neighborhood X, and the homeomorphism ψ_X which will also be omitted.

By means of the above construction, $M \setminus \mathring{\boldsymbol{C}}_k$ has two components, one of them compact, say K_k, and the other unbounded, say X_k. Let us denote the (topological) boundary component of $\boldsymbol{C}_k$ which bounds K_k by $\partial^- \boldsymbol{C}_k$ and the boundary component which bounds X_k by $\partial^+ \boldsymbol{C}_k$. We claim that $\mathring{K}_k$ is also a Taubes-like manifold for every sufficiently large k: There exists s such that $\natural^s M$ is end-diffeomorphic to a definite nondiagonal open 4–manifold, say N. Let Y be a neighborhood of the end of N which is homeomorphic to $S^3 \times [0,\infty)$ and diffeomorphic (preserving orientation) to a neighborhood Y' of the end of $\natural^s M$. For all sufficiently large k the set $\natural^s \mathring{K}_k \cap Y'$ is also homeomorphic to $S^3 \times [0,\infty)$. Then $\natural^s \mathring{K}_k$ is end-diffeomorphic to $(N \setminus Y) \cup_\partial \left(\natural^s \mathring{K}_k \cap Y'\right)$, which is also a simply connected definite nondiagonal manifold (homeomorphic to N), so $\mathring{K}_k$ is in fact Taubes-like.

Remark 1.23 All the above results given for $\mathcal{R}$ and $\mathcal{S}$ can be applied verbatim to $\overline{\mathcal{R}}$ and $\overline{\mathcal{S}}$, which denote the same manifolds but with the orientation reversed (so they are elements in $\mathcal{M}_+$). This follows from the fact that changing orientations does not affect Furuta's theorem (Theorem 1.3) or the Taylor index.

1c Examples

Example 1.24 The existence of exotica in $\mathcal{R}$ and $\mathcal{S}$ with these properties is well known (see eg [14; 19; 33]). Let M_0 be the K3 Kummer surface. It is known that the intersection form of M_0 can be written as $-2E_8 \oplus 3H$, where the six elements in $H_2(M_0, \mathbb{Z})$ spanning the summand $3H$ can be represented by six Casson handles C_i attached to a closed 4–dimensional ball B^4 inside M_0. Let $U = \operatorname{int}\big(B^4 \cup \bigcup_{i=1}^6 C_i\big)$, which is clearly homeomorphic to a once-punctured $\#^3 S^2 \times S^2$ by Freedman's theorem (Theorem 1.1). Let S be the union of the cores of the Casson handles, which we consider to be inside $\#^3 S^2 \times S^2$. By Theorem 1.4 the manifold $\boldsymbol{P} = \big(\#^3 S^2 \times S^2\big) \setminus S$ is homeomorphic to $\mathbb{R}^4$. If this $\boldsymbol{P}$ were standard then we could smoothly replace the $3H$ part in the intersection form of M_0 by a standard ball, so the resulting smooth closed manifold would have intersection form $-(E_8 \oplus E_8)$, in contradiction to Donaldson's theorem (Theorem 1.2), since $(-E_8 \oplus E_8)$ is not isomorphic to a diagonal form. Since $\boldsymbol{P}$ is contained in $\#^3 S^2 \times S^2$ it follows that $\gamma(\boldsymbol{P}) \leq 3$ and therefore $\boldsymbol{P} \in \mathcal{R}$.

Example 1.25 With some care in the above construction (see eg [33, Example 5.6]) the six Casson handles can be arranged as three diffeomorphic pairs that can each be embedded in $S^2 \times S^2$. As above, the complement of a neighborhood of each core in $S^2 \times S^2$ is an exotic $\mathbb{R}^4$, say $\boldsymbol{T}$, such that $\natural^3 \boldsymbol{T} = \boldsymbol{P}$. It is clear that $\boldsymbol{T}$ cannot be end-diffeomorphic to any spin once-punctured closed mE_8 manifold with $m \in \mathbb{Z}$ and $m \neq 0$, for otherwise we would obtain a smoothable closed spin simply connected $(mE_8) \oplus H$ manifold: just use the end of $\boldsymbol{T}$ to attach $S^2 \times S^2 \setminus \boldsymbol{T}$ to that punctured manifold. But this is impossible by Furuta's and Rokhlin's theorems (depending on whether m is even or odd). This is another large exotic $\mathbb{R}^4$ which belongs to $\mathcal{R}$.

Example 1.26 Another interesting exotic $\mathbb{R}^4$ is given as follows. Take the indefinite form $-(E_8 \oplus E_8) \oplus \langle 1 \rangle$. It follows by Freedman's theorem (Theorem 1.1) that there exists a closed simply connected manifold with such an intersection form that is homeomorphic to $\#^{16} \overline{\mathbb{C}P^2} \# \mathbb{C}P^2$ (where $\overline{\mathbb{C}P^2}$ denotes the complex projective plane with the opposite orientation). Thus the former manifold is smoothable but it is impossible to represent this manifold as a connected sum of a $-(E_8 \oplus E_8)$ manifold with $\mathbb{C}P^2$ using a smooth 3–sphere. But the homology generator of the $\mathbb{C}P^2$ summand can be represented by a Casson handle. As above, that Casson handle can be embedded in $\mathbb{C}P^2$ representing its homology. Removing a suitable neighborhood of its core we get an exotic $\mathbb{R}^4$, which will be called $\boldsymbol{E}$, that is end-diffeomorphic to a once-punctured closed spin simply connected $-(E_8 \oplus E_8)$ manifold (so $\boldsymbol{E} \in \mathcal{S}$) and can be embedded in $\mathbb{C}P^2$ (see details in [33, Example 5.10]). It is unknown whether its Taylor index is finite. In particular it is unknown if $\boldsymbol{E}$ is diffeomorphic to $\boldsymbol{P}$.

Example 1.27 The family $\mathcal{S}$ includes the universal exotic $\mathbb{R}^4$ presented in [12] and any other possible universal smoothing of $\mathbb{R}^4$. Let $\boldsymbol{U}$ be a universal exotic $\mathbb{R}^4$. By means of universality, there exists a smooth embedding of any element $\boldsymbol{R} \in \mathcal{S}$ in $\boldsymbol{U}$. Let $\boldsymbol{D} \subset \boldsymbol{R} \subset \boldsymbol{U}$ be a disk (with topologically bicollared boundary) sufficiently large that $\mathring{\boldsymbol{D}}$ is still an element of $\mathcal{S}$ (see the proof of Corollary 1.15 for details). Thus $\mathring{\boldsymbol{D}}$ and $\boldsymbol{U}$ satisfy the conditions of Lemma 1.14 and therefore $\boldsymbol{U} \in \mathcal{S}$.

Example 1.28 Finally, a smoothing of a once-punctured closed $\pm E_8$ manifold is trivially a Taubes-like manifold, so its end is not standard. But it is unknown whether this manifold is end-diffeomorphic to an exotic $\mathbb{R}^4$.

Remark 1.29 In [33, Example 5.10], uncountably many nondiffeomorphic smooth structures on $\mathbb{R}^4$ with infinite Taylor index are exhibited. Consider the element $\boldsymbol{E} \in \mathcal{S}$ presented in Example 1.26. It is shown that the manifolds $\mathring{\boldsymbol{K}}_t^{\psi_{\boldsymbol{E}}} \natural (\natural_{i=1}^{\infty} \boldsymbol{E})$ are pairwise nondiffeomorphic for all $t > r_{\psi_{\boldsymbol{E}}}$. In fact, their ends are pairwise not end-diffeomorphic. Note also that, although $\natural_{i=1}^{\infty} \boldsymbol{E}$ cannot be embedded in any spin closed manifold with hyperbolic intersection form, it can be embedded in $\mathbb{C}P^2$.

2 Exotic simply connected smooth 4–manifolds and foliations

2a The family of exotica

Here we define certain exotic simply connected smooth manifolds that we shall show cannot be diffeomorphic to leaves of a C^2 codimension-one foliation in a compact smooth manifold. As mentioned in the introduction, we are interested in the set of open 4–manifolds which are obtained by removing a finite nonzero number of points from a closed, connected, simply connected topological 4–manifold.

Definition 2.1 Let $\mathcal{Y}$ be the set of open smooth 4–manifolds Y (up to diffeomorphism) that are homeomorphic to simply connected closed topological 4–manifolds with finitely many punctures, such that Y satisfies one of the following two conditions (see Definition 1.11 and Remark 1.23):

(1) $Y \in \mathcal{R} \cup \overline{\mathcal{R}}$ (and so Y is homeomorphic to $\mathbb{R}^4$);

(2) Y is not homeomorphic to $\mathbb{R}^4$ and at least one (exotic) end is diffeomorphic (preserving orientation) to the end of some element in $\mathcal{M}$ (ie it is a Taubes-like end).

Observe that $\mathcal{Y}$ is the (nondisjoint) union of two families: $\mathcal{Y}_f$, where at least one Taubes-like end is diffeomorphic to the end of an exotic $\mathbb{R}^4$ with finite Taylor index, and $\mathcal{Y}_\infty$, where at least one Taubes-like end is not diffeomorphic to the end of an exotic $\mathbb{R}^4$ with finite Taylor index. By definition, $\mathcal{R}\cup\overline{\mathcal{R}}\subset\mathcal{Y}_f$, hence only $\mathcal{Y}_f$ contains exotic $\mathbb{R}^4$'s.

Example 2.2 If Z is a simply connected smooth closed 4–manifold that is not homeomorphic to S^4, then $Z\,\#\,\boldsymbol{R}$ belongs to $\mathcal{Y}$ for all $\boldsymbol{R}\in\mathcal{S}\cup\overline{\mathcal{S}}$. All these manifolds are homeomorphic but not diffeomorphic to the standard $Z\setminus\{*\}$ by Taubes' work.

If Z is an arbitrary simply connected but nonsmoothable closed 4–manifold, after removing a point it becomes smoothable (every open 4–manifold is smoothable; see eg [11]). In the proof of [18, Theorem 2.1], it is shown that $Z\setminus\{*\}$ admits a smoothing which is end-diffeomorphic to some simply connected punctured spin $-kE_8$ manifold (in fact, we can take $k=2$ if the Kirby–Siebenmann invariant $\mathrm{ks}(Z)$ is trivial and $k=3$ otherwise). So in both cases we get smooth Taubes-like manifolds in $\mathcal{Y}$ homeomorphic to $Z\setminus\{*\}$. Moreover, when $\mathrm{ks}(Z)=0$ that end is shown to be end-diffeomorphic to an exotic $\mathbb{R}^4$ with finite Taylor index (it can be embedded in $\#^n S^2\times S^2$ for some sufficiently large n), so that the smoothing belongs to $\mathcal{Y}_f$. In any case, after forming an infinite end-sum with (possibly distinct) elements in $\mathcal{S}$ we also obtain smooth manifolds in $\mathcal{Y}_\infty$ homeomorphic to $Z\setminus\{*\}$.

Of course, we can add more punctures to these smoothings and we still get elements in $\mathcal{Y}$, so any topological simply connected 4–manifold obtained by removing finitely many punctures from a closed manifold is homeomorphic to some element in $\mathcal{Y}$.

Remark 2.3 Taubes' theorem (Theorem 1.10, Remark 1.17 and Corollary 1.19) shows that for any $Y\in\mathcal{Y}$ there exists an uncountable family of smooth manifolds in $\mathcal{Y}$ which are homeomorphic but nondiffeomorphic to Y. The same argument works for elements in $\mathcal{Y}_f$, ie for any element in $\mathcal{Y}_f$ there is a continuum of elements in $\mathcal{Y}_f$ which are homeomorphic but not diffeomorphic to it.

Although Taubes' theorem also applies to elements in $\mathcal{Y}_\infty$ we cannot derive so easily a continuum of smoothings of $Y\in\mathcal{Y}_\infty$ which belong to $\mathcal{Y}_\infty$. For instance, consider $\natural^\infty\boldsymbol{P}$ (see Example 1.24 for the definition of $\boldsymbol{P}$); it has infinite Taylor index by Proposition 1.13 but any $\mathring{\boldsymbol{K}}_t$ has finite Taylor index since it is contained in a finite end-sum of copies of $\boldsymbol{P}$, which has finite Taylor index by Lemma 1.8.

However Taylor's results (see Remark 1.29) show that for all $Y\in\mathcal{Y}_\infty$ which are end-diffeomorphic to $\natural^\infty\boldsymbol{E}$ there is a continuum of manifolds in $\mathcal{Y}_\infty$ which are homeomorphic but nondiffeomorphic to Y (this is also true if the orientation is reversed). Let M be a topological (nonsmoothable) simply connected compact manifold. If $\mathrm{ks}(M)=0$

then $M \# \mathbb{C}P^2$ or $M \# \overline{\mathbb{C}P^2}$ smoothable by Freedman's theorem (Theorem 1.1), since the intersection form becomes indefinite. By surgering the $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$ summand by means of a suitable neighborhood of the core of a Casson handle (diffeomorphic to the one given in Example 1.26, since any two Casson handles have a common refinement) we can obtain a smoothing of $M \setminus \{*\}$ end-diffeomorphic to $\boldsymbol{E}$ or $\overline{\boldsymbol{E}}$. It follows that every simply connected topological 4–manifold with finite punctures and trivial modulo 2 Kirby–Siebenmann invariant admits a continuum of smoothings in $\mathcal{Y}_\infty$. It is not clear for us how to solve this question when $\mathrm{ks}(M) \neq 0$ and there is a single puncture; for two or more punctures, just take a smoothing where one end is standard and then form an end-sum with $\natural^\infty \boldsymbol{E}$.

2b The proof of Proposition 2

Recall that Proposition 2 states that if a leaf of a $C^{1,0}$ codimension-one foliation of a compact smooth manifold is diffeomorphic to $Y \in \mathcal{Y}$, then the leaf is proper and each connected component of the union of all leaves diffeomorphic to Y fibers over the circle with the leaves as fibers. Let us recall the meaning of $C^{1,0}$–regularity.

Definition 2.4 A codimension-one foliation is said to have regularity $C^{1,0}$ if its leaves are tangent to a continuous hyperplane distribution of codimension one.[2]

Remark 2.5 It follows that holonomy maps of a $C^{1,0}$ foliation are only continuous maps and projections from one plaque to another in a foliated chart are C^1–diffeomorphisms. Under this hypothesis it is possible to show the existence of a foliated atlas such that the leaves are C^∞ manifolds (see [4, Example 1.2.25]).

Definition 2.6 A foliated map (ie leaf-preserving map) $h\colon (M, \mathcal{F}) \to (N, \mathcal{G})$ between $C^{1,0}$ foliations is said to be of class $C^{1,0}$ if it is continuous. its restriction to each leaf is C^1, and every first partial derivative is also continuous.

In proving Proposition 2 we use the basic theory of codimension-one foliations of smooth compact manifolds presented as integrable plane fields. Note that in this general situation there exists a smooth transverse 1–dimensional foliation $\mathcal{N}$ and a biregular foliated atlas, ie one in which each coordinate neighborhood is foliated simultaneously as a product by $\mathcal{F}$ and $\mathcal{N}$. The transverse coordinate changes are only assumed to be continuous but the leaves can be taken to be smooth manifolds and the local projection along $\mathcal{N}$ of one plaque onto another plaque in the same chart

[2] This regularity is denoted by $C^{1,0+}$ in [4, Definition 1.2.24].

is a diffeomorphism. Our basic tools are Dippolito's octopus decomposition and his semistability theorem [4; 8] as well as the trivialization lemma of Hector [22]. We assume that our foliation is transversely oriented, which is not a real restriction since every manifold we are considering to be a leaf is simply connected and therefore, by passing to the transversely oriented double cover, a transversely oriented foliation with a leaf diffeomorphic to it is obtained.

For a saturated open set U of $(M, \mathcal{F})$, let $\widehat{U}$ be the completion of U for the restriction of a Riemannian metric on M to U. The inclusion $i\colon U \to M$ clearly extends to an immersion $i\colon \widehat{U} \to M$, which is at most 2-to-1 on the boundary leaves of $\widehat{U}$. We shall use ∂^τ and $\partial^\pitchfork$ to denote the tangential and transverse boundaries, respectively.

Theorem 2.7 (octopus decomposition [4, Proposition 5.2.14; 8, Theorem 1]) *Let U be a connected saturated open set of a codimension-one transversely orientable $C^{1,0}$ foliation $\mathcal{F}$ with a transverse 1–dimensional foliation $\mathcal{N}$ on a compact manifold M. Then there is a compact submanifold K (the nucleus) with boundary and corners such that:*

1. $\partial^\tau K \subset \partial^\tau \widehat{U}$.
2. $\partial^\pitchfork K$ *is saturated for* $i^*\mathcal{N}$.
3. *The set $\widehat{U} \setminus K$ is the union of finitely many noncompact connected components $B_1, \dots, B_m$ (the arms) with boundary, where each B_i is diffeomorphic to a product $S_i \times [0,1]$ by a $C^{1,0}$ diffeomorphism $\phi_i\colon S_i \times [0,1] \to B_i$ such that the leaves of $i^*\mathcal{N}$ exactly match the fibers $\phi_i(\{*\} \times [0,1])$.*
4. *The foliation $i^*\mathcal{F}$ in each B_i is defined as the suspension of a homomorphism from $\pi_1(S_i)$ to the group of homeomorphisms of $[0,1]$. Thus the holonomy in each arm of this decomposition is completely described by the action of $\pi_1(S_i)$ on a common complete transversal.*

Observe that this decomposition is far from being canonical, for the compact set K can be extended in many ways yielding other decompositions. We do not consider the transverse boundary of B_i to be a part of B_i; in particular, the leaves of $i^*\mathcal{F}|_{B_i}$ are open sets in leaves of $i^*\mathcal{F}$. Observe that each set B_i has boundary; we assume that each $C^{1,0}$ diffeomorphism ϕ_i can be extended over the boundary.

Lemma 2.8 [22, page 154, trivialization lemma] *Let J be an arc in a leaf of $\mathcal{N}$. Assume that each leaf meets J in at most one point. Then the saturation of J is $C^{1,0}$ diffeomorphic to $L \times J$, where L is a leaf of $\mathcal{F}$, and the diffeomorphism carries the bifoliation $\mathcal{F}$ and $\mathcal{N}$ to the product bifoliation of $L \times J$ (with leaves $L \times \{*\}$ and $\{*\} \times J$).*

Theorem 2.9 (Dippolito semistability theorem [4, Theorem 5.3.4; 8, Theorem 3]) *Let L be a semiproper leaf which is semistable on the proper side; ie there exists a sequence of points fixed by all the holonomy maps of L converging to L on the proper side. Then there exists a sequence of leaves L_n converging to L on the proper side and projecting diffeomorphically onto L along the fibration defined by $\mathcal{N}$.*

Notation 2.10 Let X be a closed neighborhood of the ends of $Y \in \mathcal{Y}$ identified (topologically) with $\bigsqcup_{i=1}^n S^3 \times [0,\infty)$ such that the boundaries $\bigsqcup_{i=1}^n S^3 \times \{0\}$ are (topologically) bicollared in Y. Then we have the decomposition

$$Y = K_Y \cup X,$$

where K_Y is $Y \setminus \mathring{X}$, so it is compact with boundary, and, in the case that Y is not homeomorphic to $\mathbb{R}^4$ with finite punctures, it has nontrivial second homology by Freedman's theorem (Theorem 1.1), since removing a finite number of points does not change the second homology. Since the boundary is a disjoint union of (topological) 3–spheres, it also follows that $H_2(\partial K_Y) = 0$.

Now we have enough information to begin to follow the line of reasoning of Ghys [16]. For the rest of this section we assume that $Y \in \mathcal{Y}$ is diffeomorphic to a leaf, and we shall find some constraints.

Definition 2.11 We say that a leaf $L \in \mathcal{F}$ *contains a lacunary vanishing cycle* if there exists a topologically bicollared connected embedded closed oriented 3–manifold $\Sigma \subset L$ and a family of connected 3–manifolds $\{\Sigma(n) \mid n \in \mathbb{N}\}$ embedded in the same leaf L that are null-homologous on L and converge to Σ along leaves of the transverse foliation $\mathcal{N}$. It is a *trivial lacunary vanishing cycle* if Σ is null-homologous on L.

Lemma 2.12 *Let L be a simply connected leaf with an end $\boldsymbol{e}$ homeomorphic to $S^3 \times (0,\infty)$. If $L \subset \lim_{\boldsymbol{e}}(L)$ then L does not contain any nontrivial lacunary vanishing cycle homeomorphic to S^3.*

To prove this, we shall use a special case of a weak generalization of Novikov's theorem on the existence of Reeb components [30, Theorem 4]. Recall that a (generalized) Reeb component with connected boundary is a compact $(k+1)$–manifold with a codimension-one foliation such that the boundary is a leaf and the interior fibers over the circle with the leaves as fibers.

Suppose we are given a compact $(k+1)$–manifold M with a transversely oriented codimension-one foliation $\mathcal{F}$, a transverse 1–dimensional foliation $\mathcal{N}$, a closed connected $(k-1)$–manifold B and also a bifoliated map $h\colon B\times[a,b]\to M$, where $[a,b]$ is an interval in the real line, such that $h(B\times\{t\})$ is contained in a leaf L_t of $\mathcal{F}$ for every $t\in[a,b]$ and $h_a\colon B\to L_a$ is an embedding with bicollared image, where $h_t(x)=h(x,t)$. Since h is bifoliated, it follows that h_t is an embedding for all t.

Theorem 2.13 (see [30, Theorem 2.13(2)]) *If $B_t=h_t(B)$ bounds a compact connected region in L_t for every $t\in(a,b]$, but B_a does not bound on L_a, then L_a is the boundary of a Reeb component whose interior leaves are the leaves L_t for $t\in(a,b]$.*

Proof of Lemma 2.12 Suppose that $L\subset\lim_{\boldsymbol{e}}(L)$ and that $\Sigma\subset L$ is a nontrivial lacunary vanishing cycle on L that is homeomorphic to S^3. Since $\mathcal{F}$ is assumed to be transversely oriented, the transverse foliation $\mathcal{N}$ defines a map

$$\Phi\colon S^3\times\mathbb{R}\to M$$

that takes each set $S^3\times\{t\}$ into a leaf of $\mathcal{F}$ and such that $\Phi(S^3\times\{0\})=\Sigma$. Let $\Phi^*(\mathcal{F})$ be the pullback foliation on $S^3\times\mathbb{R}$ (recall that this is only a C^0 foliation) and set $\Sigma_t=\Phi_t(S^3)$. Since the cycle is a lacunary vanishing cycle contained in L and $L\subset\lim_{\boldsymbol{e}}(L)$, it follows (possibly after reversing the sign of $\mathbb{R}$) that there exists a decreasing sequence $t_n\to 0$, $n\in\mathbb{N}$, such that each Σ_{t_n} is contained in L and bounds a manifold $C_n\subset L$. Now C_n must be simply connected by van Kampen's theorem, since both S^3 and L are, so the manifolds C_n lift to nearby leaves. By continuation from C_1, using Reeb stability, there exists a minimal $a\in\mathbb{R}\cup\{-\infty\}$ such that Σ_t bounds a manifold homeomorphic to C_1 for all $a<t\le t_1$.

If $a<0$ the lacunary vanishing cycle would be trivial, so $0\le a<t_1$. Then by the preceding theorem with $B=S^3$ and $k=4$, we know that Σ_a will be contained in the compact boundary leaf of a generalized Reeb component and L is an interior leaf of that component, so L cannot meet Σ_t for $t\le a$, contradicting the hypothesis that $\Sigma_0\subset L$. □

Proposition 2.14 *Suppose that $\mathcal{F}$ is a codimension-one $C^{1,0}$ foliation in a compact 5–manifold M. If there exists a leaf L of $\mathcal{F}$ diffeomorphic to $Y\in\mathcal{Y}$, then L is a proper leaf without holonomy.*

Proof Since L is simply connected, it is a leaf without holonomy. We also observe that L has a saturated neighborhood not meeting any compact leaves, since a limit leaf of compact leaves is compact (see [21] or [4, Theorem 6.1.1]).

First consider the case that $H_2(Y) \neq 0$. Let $K_L \subset L$ be a compact connected submanifold diffeomorphic to K_Y (see Notation 2.10). By Reeb stability there exists a neighborhood U of K_L bifoliated $C^{1,0}$–diffeomorphically as a product (see eg [4, Section 2.4]). If L meets U in more than one connected component then there exists a compact subset $B \subset L$ homeomorphic to K_L (via the transverse projection in U) and disjoint from K_L. This is impossible since the inclusion $i_0\colon K_L \hookrightarrow L$ induces an isomorphism $i_{0*}\colon H_2(K_L) \to H_2(L)$ and the Mayer–Vietoris sequence applied to $(K_L \cup B) \cup (L \setminus (\mathring{K}_L \cup \mathring{B})$ shows that B would give an additional nontrivial summand in $H_2(L)$. So in this case L is a proper leaf.

Let us consider now the case where $Y \in \mathcal{R} \cup \overline{\mathcal{R}}$ is an exotic $\mathbb{R}^4$ (with finite Taylor index). Let $t > r_{\psi_{\boldsymbol{R}}}$, let $g\colon Y \to L$ be a diffeomorphism, and set $D = g(\boldsymbol{K}_t)$. Since $\mathcal{F}$ is $C^{1,0}$, by Reeb stability D lifts to disks on nearby leaves such that their interiors are diffeomorphic to $\mathring{\boldsymbol{K}}_t$. If L is nonproper then Reeb stability will produce infinitely many pairwise disjoint smooth embeddings of $\boldsymbol{K}_t$ in Y, so $\gamma(Y) = \infty$ by Propositions 1.20 and 1.13, which gives a contradiction.

Finally consider the case that $H_2(Y) = 0$ and Y has at least two ends (so it is homeomorphic to $\mathbb{R}^4$ with at least one puncture). Let $\boldsymbol{e}$ be an end of Y diffeomorphic to the end of some $M \in \mathcal{M}$ and recall that for all $n \in \mathbb{N}$, we can express M as $M = K_n \cup \boldsymbol{C}_n \cup X_n$ (see Remark 1.22), where X_n is a neighborhood of the end of M, K_n is a compact region and $\boldsymbol{C}_n$ is a topological compact cylinder bounding both manifolds. If L is not proper, since the number of ends of Y is finite, there exists an end $\boldsymbol{e}'$ such that $L \subset \lim_{\boldsymbol{e}'}(L)$. By hypothesis, for some sufficiently large k, there exists a smooth embedding $g\colon \mathring{X}_{k-2} \to L$ such that $g(\mathring{X}_{k-2})$ is a neighborhood of the end $\boldsymbol{e}$; therefore the simply connected compact set $C = g(\boldsymbol{C}_k)$ is contained in a neighborhood of $\boldsymbol{e}$ and separates the end $\boldsymbol{e}$ from the other ends of L. In addition we can assume that $\mathring{K}_k \in \mathcal{M}$ (see Remark 1.22). By Reeb stability there exists a neighborhood of C bifoliated as a product. Thus there exists a $C^{1,0}$ smooth $C^{1,0}$ bifoliated embedding $j\colon C \times (-1,1) \to (M, \mathcal{F})$ such that $j(C \times \{0\}) = C$. As above, the projection of a tangential leaf to another in this neighborhood is a C^1 diffeomorphism. Thus $L \cap j(C \times (-1,1))$ contains a nontrivial sequence of tangential fibers $j(C \times \{s_m\})$, with s_m tending to 0, all of them contained in a neighborhood $X_{\boldsymbol{e}'}$ of the recurrent end $\boldsymbol{e}'$.

By Lemma 2.12, for all sufficiently large m, we have that $j(C \times \{s_m\})$ disconnects $\boldsymbol{e}'$ from the other ends. Otherwise infinitely many $j(C \times \{s_m\})$ would bound disks and L would contain a nontrivial lacunary vanishing cycle homeomorphic to S^3.

Note that C has two boundary components, corresponding to $\partial^-\boldsymbol{C}_k$ and $\partial^+\boldsymbol{C}_k$. We say that $j(C\times\{s_m\})$ is *positively oriented* if $\boldsymbol{e}'$ is an end of the connected component of $L\setminus j(\mathring{C}\times\{s_m\})$ which contains the component of ∂C corresponding to $\partial^+\boldsymbol{C}_k$; otherwise we say that it is *negatively oriented*.

Let s_1 and s_2 be distinct values in the sequence $\{s_m\}_{m\in\mathbb{N}}$ (possibly changing the subscripts), sufficiently close to 0 that both $j(C\times\{s_1\})$ and $j(C\times\{s_2\})$ disconnect $\boldsymbol{e}'$ from the other ends and such that $j(C\times\{s_1\})$ and $j(C\times\{s_2\})$ have the same orientation. Let $N=j(\mathring{C}\times\{s_1\})\cup P\cup j(\mathring{C}\times\{s_2\})$, where P is the connected component of $L\setminus\big(j(\mathring{C}\times\{s_1\})\cup j(\mathring{C}\times\{s_2\})\big)$ that meets both $C\times\{s_1\}$ and $C\times\{s_2\}$. Thus N is an exotic relatively compact cylinder in $X_{\boldsymbol{e}'}$ that contains both $j(\mathring{C}\times\{s_1\})$ and $j(\mathring{C}\times\{s_2\})$ as its extremities.

We can assume that $j(C\times\{s_1\})$ contains the *negative* boundary of N, ie the boundary component that corresponds to $\partial^-\boldsymbol{C}_k$ and $j(C\times\{s_2\})$ contains the *positive* boundary of N, ie the boundary component that corresponds to $\partial^+\boldsymbol{C}_k$. Let us consider the manifold

$$W=(K_k\cup\mathring{\boldsymbol{C}}_k)\cup_{i_-}N\cup_i N\cup_i\cdots,$$

where i_- is an orientation-preserving diffeomorphism which maps $\mathring{\boldsymbol{C}}_k$ to $j(\mathring{C}\times\{s_1\})$ and i is another orientation-preserving diffeomorphism from $j(\mathring{C}\times\{s_2\})$ to $j(\mathring{C}\times\{s_1\})$. By Lemma 1.14, $W\in\mathcal{M}$ since $\mathring{K}_k\in\mathcal{M}$. But this is in contradiction with Taubes' theorem, which implies that no element in $\mathcal{M}$ is smoothly periodic (Remark 1.17). □

Proposition 2.15 *If there exists a leaf L diffeomorphic to Y, then there exists an open $\mathcal{F}$–saturated neighborhood U of L which is diffeomorphic to $L\times(-1,1)$ by a diffeomorphism which carries the bifoliation $\mathcal{F}$ and $\mathcal{N}$ to the product bifoliation. In particular, all the leaves of $\mathcal{F}|_U$ are diffeomorphic to Y.*

Proof Since L is a proper leaf, there exists a path, $c\colon[0,1)\to M$, transverse to $\mathcal{F}$, with positive orientation and such that $L\cap c([0,1))=\{c(0)\}$. Let U be the saturation of $c((0,1))$, which is a connected saturated open set, and consider the octopus decomposition of $\widehat{U}$ as described in Theorem 2.7. Clearly one of the boundary leaves of $\widehat{U}$ is diffeomorphic to L because it is proper without holonomy and $c(0)\in L$. We identify this boundary leaf with L and extend the nucleus K so that the set $K'=\partial^\tau K\cap L$ is homeomorphic to K_Y. By Reeb stability, there exists a neighborhood of K' foliated as a product by $K_Y\times\{*\}$. Since $L\subset\partial\widehat{U}$ has an end, there is an arm B_1 that meets L.

The corresponding S_1 is diffeomorphic to a neighborhood of an end and homeomorphic to $S^3 \times (0, \infty)$, thus B_1 is foliated as a product (ie the suspension must be trivial). The union of a smaller product neighborhood of $L \cap B_1$ and the product neighborhood of K_Y meeting L gives a product neighborhood on the positive side of $S_1 \cup K_Y$. We can proceed in the same way for all the ends (which are finitely many), thus obtaining a product neighborhood on the positive side of $L \equiv K_Y \cup S_1 \cup \cdots \cup S_k$.

Proceeding in the same way on the negative side of L we can find the desired product neighborhood of L. Each leaf is clearly diffeomorphic to Y since the projection to L along leaves of $\mathcal{N}$ is a local diffeomorphism and bijective by the product structure. □

Let Ω be the union of leaves diffeomorphic to Y. By the previous proposition this is an open set on which the restriction $\mathcal{F}|_\Omega$ is defined by a locally trivial fibration, so its leaf space is homeomorphic to a (possibly disconnected) 1–dimensional manifold. Let Ω_1 be one connected component of Ω.

Lemma 2.16 *The completed manifold $\widehat{\Omega}_1$ is not compact.*

Proof First we note that $\partial\widehat{\Omega}_1$ cannot be empty, for otherwise all the leaves would be diffeomorphic to Y, hence proper and noncompact. It is a well-known fact (see eg [4]) that a foliation in a compact manifold with all leaves proper must have a compact leaf, for every minimal set of such a foliation is a compact leaf.

Now suppose that $\widehat{\Omega}_1$ is compact and let L be a leaf diffeomorphic to Y. Take an exotic end $\boldsymbol{e}$ of L that is end-diffeomorphic to an element in $\mathcal{M}$. Then the limit set of $\boldsymbol{e}$ of L contains a minimal set, which must be contained in the boundary of $\widehat{\Omega}_1$ and must be a compact leaf. The holonomy of the leaf F is not the identity and has no fixed points (otherwise it would produce nontrivial holonomy on an interior leaf). Since all the orbits are proper, the holonomy group of each boundary leaf must be isomorphic to $\mathbb{Z}$.

The contracting map that generates the holonomy of F extends to a $C^{1,0}$ bifoliated map $h'\colon X \to X$ that preserves each leaf of both $\mathcal{F}$ and $\mathcal{N}$ on a neighborhood X of F in $\widehat{\Omega}_1$ (just by following the flow $\mathcal{N}$ in the direction towards F). Since the holonomy is cyclic, each connected component of $L \cap X$ is end-diffeomorphic to an end of a cyclic covering space of F, so there exists an open neighborhood V of $\boldsymbol{e}$ in L where h' is defined so that $V \subset X$ and $h = h'|_V$ is an embedding $h\colon V \to h(V) \subset V$ such that $\{h^n(V)\}_{n \geq 0}$ is a neighborhood base of the end $\boldsymbol{e}$. But this contradicts Taubes' theorem (Theorem 1.10) and Remark 1.17, which implies that e cannot be smoothly periodic. □

Following the approach of Ghys in [16], we have a dichotomy: the leaf space of $\mathcal{F}|_{\Omega_1}$, which is a connected 1–dimensional manifold, must be either $\mathbb{R}$ or S^1.

Proposition 2.17 *The leaf space of $\mathcal{F}|_{\Omega_1}$ cannot be $\mathbb{R}$.*

Proof Since $\widehat{\Omega}_1$ is not compact there exists at least one arm for its octopus decomposition. Let B_1 be such an arm that is $C^{1,0}$ diffeomorphic to $S_1 \times [0,1]$ via a $C^{1,0}$ diffeomorphism ϕ_1 carrying the vertical foliation to $i^*\mathcal{N}$. If the leaf space is $\mathbb{R}$, then $\phi_1(\{*\} \times (0,1))$ must meet each leaf in at most one point. Then the trivialization lemma (Lemma 2.8) shows that the saturation of $\phi_1(\{*\} \times (0,1))$ is $C^{1,0}$ diffeomorphic to a product $L \times (0,1)$. Then the process of completing Ω_1 to $\widehat{\Omega}_1$ shows that the product $L \times (0,1)$ extends to a product $L \times [0,1)$, so the boundary leaf corresponding to $L \times \{0\}$ must be diffeomorphic to Y, but this is a contradiction since leaves diffeomorphic to Y have to be interior leaves of Ω. □

Since Ω_1 cannot fiber over the line, it must fiber over the circle, but this is just the conclusion of Proposition 2, so its proof is complete.

2c Proof of Theorem 1

The rest of this section is devoted to proving Theorem 1. It will be a quick corollary of Proposition 2 but first we have to introduce some terminology.

Let U be a saturated open set of a $C^{1,0}$ foliation $\mathcal{F}$ on a compact manifold and let $L \subset U$ be a leaf. For a Dippolito decomposition (Theorem 2.7) $\widehat{U} = K \cup B_1 \cup \cdots \cup B_n$, let us choose a transverse fiber of each arm B_i and denote it by T_i.

Definition 2.18 Let U be a saturated open set of a $C^{1,0}$ foliation $\mathcal{F}$ on a compact manifold and let $L \subset U$ be a leaf. Then L is said to be *trivial at infinity* for U if there exists a Dippolito decomposition $\widehat{U} = K \cup B_1 \cup \cdots \cup B_n$, and total transversals T_i of each B_i, such that for each i $L \cap T_i$ consists of fixed points for every element of the total holonomy group associated to that arm.

The next theorem is a consequence of the so-called *generalized Kopell lemma* for foliations, which can be found in Cantwell and Conlon [5] and in [4, proof of Theorem 8.1.26]. We thank the first referee for suggesting this argument.

Theorem 2.19 [5] *Let $\mathcal{F}$ be a transversely oriented codimension-one C^2 foliation on a compact manifold. If L is a proper leaf then it is trivial at infinity for every saturated open set containing L.*

From this theorem we can deduce the following.

Lemma 2.20 *Suppose $\mathcal{F}$ is a transversely oriented codimension-one C^2 foliation on a compact manifold. Let U be an open saturated set. If $\widehat{U}$ is noncompact and $L \subset U$ is a proper leaf such that L meets the transverse fiber associated to the arm of a Dippolito decomposition in infinitely many points, then L has infinitely many ends.*

Proof Let $\widehat{U} = K \cup B_1 \cup \cdots \cup B_n$ be a Dippolito decomposition and let $T_1, \ldots, T_n$ be the transverse fibers associated to each arm, since $\widehat{U}$ is noncompact there exists at least one arm in any decomposition. Assume that $L \cap T_i$ consists of infinitely many points for some $1 \leq i \leq n$. By Theorem 2.19, L is trivial at infinity, so there exists another decomposition $\widehat{U} = K' \cup B'_1 \cup \cdots \cup B'_m$ such that all the points of $L \cap T'_j$ are fixed points for the total holonomy group of each arm B'_j, for $1 \leq j \leq m$, where T'_j denotes a transverse fiber of the arm B'_j. Without loss of generality we can assume that $K \subset K'$ and therefore every arm B'_j is contained in some arm B_k. Consider an arm $B'_r \subset B_i$. Clearly there exists a holonomy map $h_{ri}\colon T_i \to T'_r$, and therefore $L \cap T'_r$ also consists of infinitely many points. Since all of these points are fixed by the total holonomy group of B'_r, it follows that $L \cap B'_r$ is a disjoint union of infinitely many copies of the base manifold of that arm. Each one of these copies defines a different end for the leaf L. □

Proof of Theorem 1 Let L be a leaf diffeomorphic to some $Y \in \mathcal{Y}$ and suppose that $\mathcal{F}$ is C^2 and transversely oriented (otherwise pass to the transversely oriented double cover). By Proposition 2, L is proper and the set Ω_1 (the connected component of the union of the leaves diffeomorphic to L that contains L) is an open, connected and saturated set fibering over the circle. Moreover, Lemma 2.16 implies that $\widehat{\Omega}_1$ is noncompact. Since Ω_1 fibers over the circle we can associate to every leaf $L \in \mathcal{F}|_{\Omega_1}$ a diffeomorphism $h\colon L \to L$ (the monodromy of the fibration) which associates to each point $x \in L$ the next element in the negative direction of the transverse foliation $\mathcal{N}$ which belongs to L. Let T be the transverse fiber of any arm of a Dippolito decomposition of $\widehat{\Omega}_1$ and let L be a leaf which meets T in a point x. Since, by construction, T is contained in a flow line of $\mathcal{N}$, it follows that every $h^n(x)$, where $n \in \mathbb{Z}$, also belongs to T and all of them are distinct points since $\mathcal{N}$ has no closed orbits contained in any arm. Therefore L satisfies the hypothesis of Lemma 2.20

and thus it must have infinitely many ends, in contradiction to the fact that every manifold in $\mathcal{Y}$ has finitely many ends. □

3 Final comments

It is possible to enlarge the family $\mathcal{Y}$ if, instead of working with ends homeomorphic to $S^3 \times [0,\infty)$, we allow admissible ends (in the sense of [32, Definition 1.3]) that are simply connected. We avoid working with this generality in the interests of readability.

Recall that Proposition 2 says that if $Y \in \mathcal{Y}$ is diffeomorphic to a leaf, then it is a proper leaf without holonomy contained in an open saturated set Ω_1 which consists of leaves diffeomorphic to Y and fibers over the circle. As we saw in the proof of Theorem 1 (see also [16, Section 5]), for any $L \in \mathcal{F}|_{\Omega_1}$, the monodromy map $h\colon L \to L$ induces an orientation-preserving automorphism of our exotic manifold Y. We saw in the description of exotic structures that self-diffeomorphisms of $\boldsymbol{R} \in \mathcal{R} \cup \overline{\mathcal{R}}$ are in some sense rigid; this allows us to say what kind of monodromy is admissible, in the $C^{1,0}$ category, at the present state of the art. For instance, the compact set $\boldsymbol{K}_t \subset \boldsymbol{R}$ with $t > r_{\psi_{\boldsymbol{R}}}$ must meet the nucleus of any octopus decomposition of Ω_1 and it seems likely that the monodromy should have fixed points.

As far as we know, this work gives the first insight into the problem of realizing exotic structures on open 4–manifolds as leaves of a foliation in a compact manifold. We express our hopes in the following conjecture, which we are far from proving, since it includes the higher-codimension case and lower regularity assumptions, which are not treated in this paper. It is a goal for future research.

Conjecture 3.1 No open 4–manifold with an isolated Taubes-like end is diffeomorphic to a leaf of a $C^{1,0}$ foliation of arbitrary codimension in a compact manifold.

The corresponding conjecture is in fact open for all the known families (of ends) of exotic $\mathbb{R}^4$'s but, as a consequence of this work, those which are Taubes-like are the best candidates. Another interesting question is what can be said if we allow infinitely many ends, since in this case Lemma 2.20 will not work. For instance S^4 minus a Cantor set is diffeomorphic to a leaf of a C^ω codimension-one compact manifold; observe that it is the universal covering space of a compact smooth 4–manifold with fundamental group isomorphic to $\mathbb{Z} * \mathbb{Z}$, so any suspension of an analytic action of that group on S^1 does the job (see [27] for many such actions). It would be interesting to exhibit smooth structures on this manifold which are not diffeomorphic to leaves. This question will be treated in a forthcoming work of the authors.

Let us say something about small exotica (those which embed as open sets in the standard $\mathbb{R}^4$). Small exotica are more interesting from a physical point of view since they support Stein structures (see eg [19]). There is a Taubes-type theorem for them based on the work of De Michelis and Freedman [7], and with more generality in [34], implying the existence of uncountably many small exotica. By means of a cardinality argument, there should exist small exotica that are not smoothly periodic, but it is difficult to tell whether or not a given small exotic $\mathbb{R}^4$ is smoothly periodic. Therefore it seems possible to prove a version of Lemma 2.16 for certain families of small exotica. Unfortunately, there is no known "Taylor index" invariant and therefore the first part of our arguments, which shows that the leaf must be a proper leaf, fails for small exotica, although it works for punctured simply connected 4–manifolds obtained by removing finitely many points from closed manifolds not homeomorphic to S^4, since for these manifolds the argument is purely topological.

It is also worth noting that if the smooth 4–dimensional Poincaré conjecture is false then it is easy to produce exotic $\mathbb{R}^4$'s which are leaves of a transversely analytic (in particular C^2) foliation. Consider $S^4 \times S^1$ with the product foliation, where S^4 has an exotic smooth structure, and insert a Reeb component along a transverse curve, for example $\{*\} \times S^1$. This can easily be done so as to preserve the transverse analyticity. The leaves would be exotic $\mathbb{R}^4$'s with a standard smooth structure at the end.

Finally we include a last remark. Recent work of J Álvarez López and R Barral Lijó [26] states that every Riemannian manifold with bounded geometry can be realized isometrically as a leaf in a compact foliated space. It is known [20] that every smooth manifold supports such a geometry, so it follows as a corollary that every smooth manifold is diffeomorphic to a leaf in a compact foliated space. In particular this holds for any exotic $\mathbb{R}^4$. However the transverse topology of this foliated space would in general be far from being a manifold. Anyway, this gives us some hope of finding an explicit description of exotic structures by using finite data: the tangential change of coordinates of a finite foliated atlas.

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Departamento de Análise, Instituto de Matemática e Estatística
Universidade Federal Fluminense
Niterói, Rio de Janeiro-RJ, Brazil

Departamento de Matemática, Pontificia Universidade Católica do Rio de Janeiro
Gávea, Rio de Janeiro-RJ, Brazil

carlos.meninho@gmail.com, paul37sj@gmail.com

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