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Moduli stacks of semistable sheaves and representations of Ext–quivers

YUKINOBU TODA

We show that the moduli stacks of semistable sheaves on smooth projective varieties are analytic locally on their coarse moduli spaces described in terms of representations of the associated Ext–quivers with convergent relations. When the underlying variety is a Calabi–Yau 3–fold, our result describes the above moduli stacks as critical loci analytic locally on the coarse moduli spaces. The results in this paper will be applied to the wall-crossing formula of Gopakumar–Vafa invariants defined by Maulik and the author.

14D22, 14D23; 14D15

1 Introduction

1.1 Motivation

The purpose of this paper is to give descriptions of moduli stacks of semistable sheaves on smooth projective varieties in terms of quivers with (formal but convergent) relations, analytic locally on their coarse moduli spaces. The relevant quiver is the Ext–quiver associated to the simple collection of coherent sheaves, determined by a polystable sheaf corresponding to a point of the coarse moduli space. Probably the main results have been folklore for experts of moduli of sheaves (at least on formal neighborhoods at closed points of the coarse moduli space), but we cannot find any reference and our purpose is to give precise statements and details of the proofs. The main results in this paper will be used in the companion paper [33] in the proof of the wall-crossing formula of Gopakumar–Vafa invariants introduced by Maulik and the author [25].

1.2 Results

Let X be a smooth projective variety over $\mathbb{C}$ and ω an ample divisor on it. Let $\mathcal{M}_\omega$ be the moduli stack of ω–Gieseker semistable sheaves on X, and M_ω the coarse moduli space of their S–equivalence classes. There is a natural morphism

$$p_M\colon \mathcal{M}_\omega \to M_\omega$$

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sending a semistable sheaf to its S–equivalence class. A closed point of M_ω corresponds to a polystable sheaf, ie a direct sum

$$E = \bigoplus_{i=1}^{k} V_i \otimes E_i, \tag{1-1}$$

where each of the $E_1, \dots, E_k$ are mutually nonisomorphic ω–Gieseker stable sheaves with the same reduced Hilbert polynomials.

The *Ext–quiver* Q associated to the collection $(E_1, \dots, E_k)$ is defined by the quiver whose vertex is $\{1, \dots, k\}$ and the number of arrows from i to j is the dimension of $\operatorname{Ext}^1(E_i, E_j)$. We denote by $\mathcal{M}_Q$ the moduli stack of finite-dimensional Q–representations with dimension vector $(\dim V_i)_{1 \le i \le k}$, and M_Q the coarse moduli space of semisimple Q–representations with dimension vector as above. We have the natural morphism

$$p_Q \colon \mathcal{M}_Q \to M_Q$$

sending a Q–representation to its semisimplification. There is a point $0 \in M_Q$ represented by the semisimple Q–representation $\bigoplus_{i=1}^k V_i \otimes S_i$, where S_i is a simple Q–representation corresponding to the vertex i. The following is our main result.

Theorem 1.1 (Theorem 3.2) *For $p \in M_\omega$ represented by a polystable sheaf* (1-1)*, let Q be the* Ext*–quiver associated to $(E_1, \dots, E_k)$. Then there exist analytic open neighborhoods $p \in U \subset M_\omega$ and $0 \in V \subset M_Q$, a closed analytic substack $\mathcal{Z} \subset p_Q^{-1}(V)$ with the natural morphism to its coarse moduli space $p_Q \colon \mathcal{Z} \to Z$ and the commutative isomorphisms*

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\cong} & p_M^{-1}(U) \\ {\scriptstyle p_Q}\downarrow & & \downarrow{\scriptstyle p_M} \\ Z & \xrightarrow{\cong} & U \end{array}$$

Indeed we can define the (formal but convergent) relation I of the Ext–quiver Q using the minimal A_∞–structure of the dg-category generated by $(E_1, \dots, E_k)$. The convergence of I will be proved by generalizing the gauge theory arguments of Fukaya [11] and Tu [34] for deformations of vector bundles to the case of resolutions of coherent sheaves by complexes of vector bundles. The substack $\mathcal{Z} \subset p_Q^{-1}(V)$ is then defined to be the stack of Q–representations satisfying the relation I.

When X is a smooth projective Calabi–Yau (CY) 3–fold, we can take the relation I to be the derivation of a convergent superpotential of the quiver Q. So we have:

Corollary 1.2 (Corollary 5.7) *In the situation of Theorem 1.1, suppose that X is a smooth projective CY 3–fold. Then there is a morphism of complex analytic stacks $W\colon p_Q^{-1}(V) \to \mathbb{C}$ such that*

$$\mathcal{Z} = \{dW=0\} \xrightarrow{\cong} p_M^{-1}(U).$$

A similar result was already proved by Ben-Bassat–Brav–Bussi–Joyce [3] and Joyce–Song [17], where the stack $\mathcal{M}_\omega$ is described as a critical locus locally on $\mathcal{M}_\omega$. Our description is more global, as we describe the stack $\mathcal{M}_\omega$ as a critical locus on the preimage of an open subset of the coarse moduli space M_ω. The result of Corollary 5.7 is also compatible with the d–critical structure introduced by Joyce [16]. By Pantev–Toën–Vaquié–Vezzosi [28], the stack $\mathcal{M}_\omega$ is a truncation of a derived scheme with a (-1)–shifted symplectic structure. Using this fact, it is proved in [3] that the stack $\mathcal{M}_\omega$ has a canonical d–critical structure. From the construction of W in Corollary 1.2, the data $(p_M^{-1}(U), p_Q^{-1}(V), W)$ is shown to give a d–critical chart of the d–critical stack $\mathcal{M}_\omega$; see [33, Appendix A].

In the case of moduli spaces of one-dimensional sheaves, we also investigate the wall-crossing phenomena of these moduli spaces with respect to the twisted stability. Let $A(X)_{\mathbb{C}}$ be the complexified ample cone of X, and take an element

$$\sigma = B + i\omega \in A(X)_{\mathbb{C}}.$$

Let M_σ be the coarse moduli space of one-dimensional B–twisted ω–semistable sheaves on X. We will see that the result of Theorem 1.1 is also applied for the moduli space M_σ of twisted semistable sheaves. If we take $\sigma^+ \in A(X)_{\mathbb{C}}$ to be sufficiently close to σ, we have the natural projective morphism

$$q_M\colon M_{\sigma^+} \to M_\sigma. \tag{1-2}$$

Theorem 1.3 (Theorem 7.7) *For $p \in M_\sigma$, let an open subset $p \in U \subset M_\sigma$, a quiver Q, and an analytic space Z be as in Theorem 1.1. Then there is a stability condition ξ on the category of Q–representations such that we have the commutative diagram of isomorphisms*

$$\begin{array}{ccc} Z_\xi & \xrightarrow{\cong} & q_M^{-1}(U) \\ \downarrow & & \downarrow q_M \\ Z & \xrightarrow{\cong} & U \end{array} \tag{1-3}$$

Here Z_ξ is the coarse moduli space of ξ–semistable Q–representations satisfying the relation I.

When X is a K3 surface, the morphism (1-2) was studied by Arbarello–Saccà [2]. In this case, they showed that the morphism (1-2) is analytic locally on M_{σ} described as a symplectic resolution of singularities of Nakajima quiver varieties via variation of stability conditions of representations of quivers. One can check that the result of Theorem 1.3 gives the same description of the morphism (1-2) as in [2], if we know the formality of the dg-algebra $\mathbf{R}\mathrm{Hom}(E, E)$ for a polystable sheaf $[E] \in M_{\sigma}$.

The results of Corollary 1.2 and Theorem 1.3 will be used in [33] to show the wall-crossing formula of (the generalization of) Gopakumar–Vafa (GV) invariants introduced by Maulik and the author [25]. Roughly speaking, the idea is as follows. In [33], we construct some perverse sheaves $\phi_{M_{\sigma+}}$, $\phi_{M_{\sigma}}$ on the moduli spaces $M_{\sigma+}$, M_{σ} in (1-2), respectively, following the analogy of BPS sheaves introduced by Davison–Meinhardt [7]. It turns out that there is a natural morphism

$$\phi_{M_{\sigma}} \to \mathbf{R}q_{M*}\phi_{M_{\sigma+}}, \tag{1-4}$$

and we want to show that this is an isomorphism. The results of Corollary 1.2 and Theorem 1.3 enable us to reduce to the case of quivers with convergent superpotentials. In the case of quivers with superpotentials, the similar question was addressed and solved in [7], and we can use the results and arguments therein to show that (1-4) is an isomorphism.

In a similar way, using the result of Corollary 1.2, it should be possible to reduce several problems in Donaldson–Thomas (DT) theory on CY 3–folds to the case of representations of quivers with convergent superpotentials, which is easier in many cases. For example it was recently announced by Davison–Meinhardt that the integrality conjecture of generalized DT invariants (see Joyce–Song [17] and Kontsevich–Soibelman [21]) on CY 3–folds can be proved using the result of Corollary 1.2.

1.3 Plan of the paper

The organization of this paper is as follows. In Section 2, we introduce the notion of quivers with convergent relations and construct the moduli spaces of their representations. In Section 3, we fix some notation on the moduli spaces of semistable sheaves and state the precise form of Theorem 1.1. In Section 4, we describe deformation theory of coherent sheaves in terms of minimal A_{∞}–structures. In Section 5, we complete the proof of Theorem 1.1. In Section 6, we recall NC deformation theory and relate it with the result of Theorem 1.1. In Section 7, we prove Theorem 1.3.

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2 Quivers with convergent relations

In this section, we recall some basic notions on quivers, their representations and moduli spaces. We also introduce the concept of convergent relations of quivers, and moduli spaces of quiver representations satisfying such relations.

2.1 Representations of quivers

Recall that a *quiver* Q consists data

$$Q = (V(Q), E(Q), s, t),$$

where $V(Q)$ and $E(Q)$ are finite sets, and s and t are maps

$$s, t\colon E(Q) \to V(Q).$$

The set $V(Q)$ is the set of vertices and $E(Q)$ is the set of edges. For $e \in E(Q)$, the vertex $s(e)$ is the source and $t(e)$ is the target. For $i, j \in V(Q)$, we use the notation

$$E_{i,j} := \{e \in E(Q) : s(e) = i,\ t(e) = j\}; \tag{2-1}$$

ie $E_{i,j}$ is the set of edges from i to j.

A *Q–representation* consists of the data

$$\mathbb{V} = \{(V_i, u_e) :\ i \in V(Q),\ e \in E(Q),\ u_e\colon V_{s(e)} \to V_{t(e)}\}, \tag{2-2}$$

where V_i is a finite-dimensional $\mathbb{C}$–vector space and u_e is a linear map. For a Q–representation (2-2), the vector

$$\vec{m} = (m_i)_{i \in V(Q)}, \quad m_i = \dim V_i, \tag{2-3}$$

is called the *dimension vector*.

Given a dimension vector (2-3), let V_i be a $\mathbb{C}$–vector space with dimension m_i. Set

$$G := \prod_{i \in Q(V)} \mathrm{GL}(V_i) \quad \text{and} \quad \mathrm{Rep}_Q(\vec{m}) := \prod_{e \in E(V)} \mathrm{Hom}(V_{s(e)}, V_{t(e)}).$$

The algebraic group G acts on $\mathrm{Rep}_Q(\vec{m})$ by

$$g \cdot u = \{g_{t(e)}^{-1} \circ u_e \circ g_{s(e)}\}_{e \in E(Q)} \tag{2-4}$$

for $g = (g_i)_{i \in V(Q)} \in G$ and $u = (u_e)_{e \in E(Q)}$. A Q–representation with dimension vector $\vec{m}$ is determined by a point in $\mathrm{Rep}_Q(\vec{m})$ up to G–action. The moduli stack of Q–representations with dimension vector $\vec{m}$ is given by the quotient stack

$$\mathcal{M}_Q(\vec{m}) := [\mathrm{Rep}_Q(\vec{m})/G].$$

It has the coarse moduli space given by

$$p_Q \colon \mathcal{M}_Q(\vec{m}) \to M_Q(\vec{m}) := \mathrm{Rep}_Q(\vec{m}) /\!\!/ G. \tag{2-5}$$

Here in general, if a reductive algebraic group G acts on an affine scheme $Y = \mathrm{Spec}\, R$, then its affine GIT quotient is given by

$$Y /\!\!/ G := \mathrm{Spec}\, R^G.$$

For two points in $x_1, x_2 \in Y$, they are mapped to the same point in $Y /\!\!/ G$ if and only if their G–orbit closures intersect, ie

$$\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \varnothing.$$

In the case of G–action on $\mathrm{Rep}_Q(\vec{m})$, the above condition is also equivalent to that the corresponding Q–representations have the isomorphic semisimplifications. The quotient space $M_Q(\vec{m})$ parametrizes semisimple Q–representations with dimension vector $\vec{m}$, and the map (2-5) sends a Q–representation to its semisimplification; see [26, Section 5; 19, Section 3] for details.

For $i \in V(Q)$, let S_i be the simple Q–representation corresponding to the vertex i; ie it is the unique Q–representation with dimension vector $m_i = 1$ and $m_j = 0$ for $j \neq i$. The point $0 \in \mathrm{Rep}_Q(\vec{m})$ and its image $0 \in M_Q(\vec{m})$ by the map (2-5) correspond to semisimple Q–representation $\bigoplus_{i \in V(Q)} V_i \otimes S_i$. A Q–representation (2-2) is called *nilpotent* if any sufficiently large number of compositions of the linear maps u_e becomes zero. It is easy to see that a Q–representation is nilpotent if and only if it is an iterated extensions of simple objects $\{S_i\}_{i \in V(Q)}$. In particular, the fiber

$$p_Q^{-1}(0) \subset \mathcal{M}_Q(\vec{m})$$

for the morphism (2-5) consists of nilpotent Q–representations with dimension vector $\vec{m}$.

2.2 Quivers with convergent relations

Recall that a *path* of a quiver Q is a composition of edges in Q:

$$e_1 e_2 \cdots e_n, \quad \text{with } e_i \in E(Q) \text{ and } t(e_i) = s(e_{i+1}).$$

The number n above is called the *length* of the path. The *path algebra* of a quiver Q is a $\mathbb{C}$–vector space spanned by paths in Q:

$$\mathbb{C}[Q] := \bigoplus_{n \geq 0} \bigoplus_{\substack{e_1, \dots, e_n \in E(Q) \\ t(e_i) = s(e_{i+1})}} \mathbb{C} \cdot e_1 e_2 \cdots e_n.$$

Here a path of length zero is a trivial path at each vertex of Q, and the product on $\mathbb{C}[Q]$ is defined by composition of paths. By taking the completion of $\mathbb{C}[Q]$ with respect to the length of the path, we obtain the formal path algebra

$$\mathbb{C}[\![Q]\!] := \prod_{n \geq 0} \bigoplus_{\substack{e_1, \dots, e_n \in E(Q) \\ t(e_i) = s(e_{i+1})}} \mathbb{C} \cdot e_1 e_2 \cdots e_n.$$

Note that an element $f \in \mathbb{C}[\![Q]\!]$ is written as

$$f = \sum_{\substack{n \geq 0 \\ \psi : \{1, \dots, n+1\} \to V(Q)}} \sum_{e_i \in E_{\psi(i), \psi(i+1)}} a_{\psi, e_\bullet} \cdot e_1 e_2 \cdots e_n. \tag{2-6}$$

Here $a_{\psi, e_\bullet} \in \mathbb{C}$, $e_\bullet = (e_1, \dots, e_n)$ and $E_{\psi(i), \psi(i+1)}$ is defined as in (2-1). The above element f lies in $\mathbb{C}[Q]$ if and only if $a_{\psi, e_\bullet} = 0$ for $n \gg 0$.

Definition 2.1 We define the subalgebra

$$\mathbb{C}\{Q\} \subset \mathbb{C}[\![Q]\!]$$

to be elements (2-6) such that $|a_{\psi, e_\bullet}| < C^n$ for some constant $C > 0$ which is independent of n.

Note that $\mathbb{C}\{Q\}$ contains $\mathbb{C}[Q]$ as a subalgebra. For an element $f \in \mathbb{C}\{Q\}$, we write it as (2-6) and consider the following $\mathrm{Hom}(V_a, V_b)$–valued formal function of $u = (u_e)_{e \in E(Q)} \in \mathrm{Rep}_Q(\vec{m})$:

$$f(a,b,\vec{m}) := \sum_{\substack{n \geq 0 \\ \psi\colon \{1,\ldots,n+1\} \to V(Q) \\ \psi(1)=a,\ \psi(n+1)=b}} \sum_{e_i \in E_{\psi(i),\psi(i+1)}} a_{\psi,e_\bullet} \cdot u_{e_n} \circ \cdots \circ u_{e_2} \circ u_{e_1}. \tag{2-7}$$

By the definition of $\mathbb{C}\{Q\}$, the above $\mathrm{Hom}(V_a,V_b)$–valued formal function on $\mathrm{Rep}_Q(\vec{m})$ has a convergent radius. So there is an analytic open neighborhood

$$0 \in \mathcal{U} \subset \mathrm{Rep}_Q(\vec{m}) \tag{2-8}$$

such that the function (2-7) absolutely converges on it and determines the complex analytic map

$$f(a,b,\vec{m})\colon \mathcal{U} \to \mathrm{Hom}(V_a, V_b).$$

In particular, the equations $f(a,b,\vec{m}) = 0$ for all $a, b \in V(Q)$ determines the closed complex analytic subspace of $\mathcal{U}$.

2.3 Saturated open subsets

We will extend the arguments in the previous subsection to a preimage of an open subset in $\mathrm{Rep}_Q(\vec{m}) /\!\!/ G$. Before doing this, we prepare some general definitions and lemmas for the action of a reductive algebraic group on affine schemes or analytic spaces.

Definition 2.2 Let G be a reductive group acting on an affine algebraic $\mathbb{C}$–scheme Y. Then an analytic open set $U \subset Y$ is called saturated if for any $x \in U$, the orbit closure $\overline{G \cdot x} \subset Y$ is contained in U.

Note that a saturated open subset is in particular G–invariant. Let

$$\pi_Y\colon Y \to Y /\!\!/ G \tag{2-9}$$

be the quotient map and $V \subset Y /\!\!/ G$ an analytic open subset. Then $\pi_Y^{-1}(V)$ is obviously saturated. Indeed, the converse is also true. In order to see this, we recall the following fact on the topology of affine GIT quotient $Y /\!\!/ G$.

Theorem 2.3 [27; 31] *In the situation of Definition 2.2, let $K \subset G$ be a maximal compact subgroup of G. Then there is a K–invariant closed subset $S \subset Y$ in analytic topology, called a* ***Kempf–Ness set****, satisfying the following: for any $x \in S$ the G–orbit $G \cdot x$ is closed in Y and the inclusion $S \subset Y$ induces the homeomorphism*

$$\iota\colon S/K \xrightarrow{\cong} Y /\!\!/ G. \tag{2-10}$$

Here the topology of S/K is a quotient topology induced from the analytic topology of S, and that of $Y /\!\!/ G$ is the analytic topology. In particular, the analytic topology of $Y /\!\!/ G$ is the quotient topology induced from the analytic topology of Y.

The following lemma follows from the above theorem:

Lemma 2.4 *In the situation of Definition 2.2, an analytic open subset $U \subset Y$ is saturated if and only if there is an analytic open set $V \subset Y /\!\!/ G$ such that $U = \pi_Y^{-1}(V)$, where $\pi_Y\colon Y \to Y /\!\!/ G$ is the quotient morphism.*

Proof For $x \in U$ and $y \in Y$, suppose that $\pi_Y(x) = \pi_Y(y)$, ie that $\overline{G \cdot x}$ and $\overline{G \cdot y}$ intersect. Since U is saturated, we have $\overline{G \cdot x} \subset U$. Then we have $\overline{G \cdot y} \cap U \neq \varnothing$, and since U is open there is $g \in G$ such that $g \cdot y \in U$. Therefore we have $y \in U$. This implies that there is a subset $V \subset Y /\!\!/ G$ such that $U = \pi_Y^{-1}(V)$. By Theorem 2.3, the subset V is analytic open, hence the lemma holds. □

We also have the following lemma.

Lemma 2.5 *In the situation of Definition 2.2, let $y \in Y$ be a G–fixed point and $U \subset Y$ a G–invariant analytic open subset with $y \in U$. Then there is an analytic open subset $U' \subset Y$ which is saturated and satisfies $0 \in y \in U' \subset U$.*

Proof Let $S \subset Y$ be the Kempf–Ness set as in Theorem 2.3. Since $y \in Y$ is G–fixed, we have $y \in S$ by the homeomorphism (2-10). Then we have $y \in S \cap U$, and $S \cap U$ is a K–invariant open subset in S. Therefore we have $S \cap U = \pi_S^{-1}(V)$ for some open subset $V \subset S/K$, where $\pi_S\colon S \to S/K$ is the quotient map. Since the map τ in (2-10) is a homeomorphism, the subset $\iota(V) \subset Y /\!\!/ G$ is open. We set a saturated open subset $U' \subset Y$ to be $U' = \pi_Y^{-1}(\iota(V))$ for the quotient map (2-9). Since $\pi_S(y) \in V$, we have $y \in U'$. It is enough to check that $U' \subset U$. By the construction of U', for $x \in U'$ there is $z \in S \cap U$ such that $\pi_Y(x) = \pi_Y(z)$, ie the closures of $G \cdot x$ and $G \cdot z$ intersect. Since $G \cdot z$ is closed, we have $z \in \overline{G \cdot x}$. Therefore there is $g \in G$ such that $g \cdot x \in U$. Since U is G–invariant, we have $x \in U$, hence the lemma is proved. □

2.4 Analytic Hilbert quotients

Later we will take GIT-type quotients for nonalgebraic complex analytic spaces. Here we recall the basic notions for such quotients. The following definition appears in [14; 12] for reduced complex analytic spaces.

Definition 2.6 Let G be a reductive algebraic group acting on a complex analytic space Z. Then a complex analytic space $Z /\!\!/ G$ together with a morphism

$$\pi_Z\colon Z \to Z /\!\!/ G \tag{2-11}$$

is called an *analytic Hilbert quotient* if the following conditions hold:

(1) π_Z is a locally Stein map; ie there is an open cover $Z /\!\!/ G = \bigcup_\lambda \mathcal{U}_\lambda$ by Stein open subsets $\mathcal{U}_\lambda$ such that $\pi_Z^{-1}(\mathcal{U}_\lambda)$ is Stein.

(2) We have $(\pi_{Z*}\mathcal{O}_Z)^G = \mathcal{O}_{Z/\!\!/G}$.

An analytic Hilbert quotient is known to exist when Z is a reduced Stein space, which is unique up to isomorphism [13]. In [14; 12], analytic Hilbert quotients are discussed under the assumption that Z is reduced. It seems that such quotients for nonreduced analytic spaces are not available in the literature. We don't develop generality of such quotients for nonreduced analytic spaces, but show the existence of such quotients in some special cases discussed below, and their universality.

We show the following lemma on the existence of analytic Hilbert quotients, which may be well known, but we include it here as we cannot find a reference.

Lemma 2.7 *Let Y be an affine algebraic $\mathbb{C}$–scheme with G–action. Then for the affine GIT quotient $\pi_Y\colon Y \to Y /\!\!/ G$, its analytification*

$$\pi_Y^{\mathrm{an}}\colon Y^{\mathrm{an}} \to (Y /\!\!/ G)^{\mathrm{an}}$$

is an analytic Hilbert quotient.

Proof The condition Definition 2.6(1) is obvious as Y^{an} and $(Y /\!\!/ G)^{\mathrm{an}}$ are Stein, so we only prove (2). First suppose that $Y = \mathbb{C}^n$ and the G–action on it is linear. In this case, Definition 2.6(2) is proved in [24]. In general, there is a G–invariant closed embedding $Y \subset \mathbb{C}^n$, where G acts on $\mathbb{C}^n$ linearly, and the commutative diagram

$$\begin{array}{ccc} Y & \hookrightarrow & \mathbb{C}^n \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_{\mathbb{C}^n}} \\ Y /\!\!/ G & \hookrightarrow & \mathbb{C}^n /\!\!/ G \end{array} \tag{2-12}$$

Since G is reductive, the functor $(-)^G$ sending a G–representation to its G–invariant part is exact. So the natural map $\Gamma(\mathcal{O}_{\mathbb{C}^n})^G \to \Gamma(\mathcal{O}_Y)^G$ is surjective, so the bottom arrow of (2-12) is a closed embedding.

By taking the analytification of (2-12), we obtain the commutative diagram of analytic sheaves on $(\mathbb{C}^n /\!\!/ G)^{\mathrm{an}}$:

$$\begin{array}{ccc} \mathcal{O}_{(\mathbb{C}^n/\!\!/G)^{\mathrm{an}}} & \xrightarrow{\cong} & (\pi^{\mathrm{an}}_{\mathbb{C}^n *}\mathcal{O}_{(\mathbb{C}^n)^{\mathrm{an}}})^G \\ \downarrow & & \downarrow \\ \mathcal{O}_{(Y/\!\!/G)^{\mathrm{an}}} & \longrightarrow & (\pi^{\mathrm{an}}_{Y*}\mathcal{O}_{Y^{\mathrm{an}}})^G \end{array} \tag{2-13}$$

Since $\pi^{\mathrm{an}}_{\mathbb{C}^n}$ is locally Stein, and the functor $(-)^G$ is exact, the vertical arrows of (2-13) are surjections. Therefore the bottom arrow of (2-13) is surjective. Also as $\mathcal{O}_{Y/\!\!/G} = (\pi_{Y*}\mathcal{O}_Y)^G$ for Zariski sheaves, we have an injection $\mathcal{O}_{Y/\!\!/G} \hookrightarrow \pi_{Y*}\mathcal{O}_Y$, which is also injective after taking completions at each closed point of $\mathcal{O}_{Y/\!\!/G}$. Hence the bottom arrow of (2-13) is also injective, so it is an isomorphism; ie π^{an}_Y satisfies Definition 2.6(2). □

By Lemma 2.7, for an analytic open subset $U \subset Y /\!\!/ G$ the map

$$\pi_Y \colon \pi_Y^{-1}(U) \to U \tag{2-14}$$

is an analytic Hilbert quotient of $\pi_Y^{-1}(U)$. We also have the following lemma:

Lemma 2.8 *Let $Z \subset \pi_Y^{-1}(U)$ be a G–invariant closed analytic subspace. Then there is a closed analytic subspace $Z /\!\!/ G \hookrightarrow U$ and an analytic Hilbert quotient $\pi_Z \colon Z \to Z /\!\!/ G$.*

Proof Since (2-14) is an analytic Hilbert quotient and the functor $(-)^G$ is exact, we have the surjection

$$\mathcal{O}_U = (\pi_{Y*}\mathcal{O}_{\pi_Y^{-1}(U)})^G \twoheadrightarrow (\pi_{Y*}\mathcal{O}_Z)^G.$$

Thus by setting $Z /\!\!/ G$ to be the complex analytic subspace of U defined by the ideal of the above kernel, we obtain the analytic Hilbert quotient $\pi_Z = \pi_Y|_Z \colon Z \to Z /\!\!/ G$. □

By gluing the above construction, we have the following lemma:

Lemma 2.9 *Let Y be an algebraic $\mathbb{C}$–scheme with G–action and $\pi_Y\colon Y \to Y'$ a G–invariant morphism of algebraic $\mathbb{C}$–schemes, where G acts on Y' trivially. Suppose that $Y' = \bigcup_{i \in I} V_i'$ is an affine open cover such that $V_i = \pi_Y^{-1}(V_i')$ is affine and $\pi|_{V_i}\colon V_i \to V_i'$ is isomorphic to $V_i \to V_i /\!\!/ G$. Then for an analytic open subset $U \subset Y'$ and a G–invariant closed analytic subspace $Z \subset \pi_Y^{-1}(U)$, the analytic Hilbert quotient $Z /\!\!/ G$ exists as a closed analytic subspace of U.*

Proof Let $U_i = U \cap V_i'$ and $Z_i = Z \cap V_i$. Applying Lemma 2.8 to $Z_i \subset \pi_Y^{-1}(U_i) \subset V_i$, we obtain the analytic Hilbert quotient $Z_i /\!\!/ G \subset U_i$. By the construction, they glue to give a desired analytic Hilbert quotient $Z /\!\!/ G \subset U$. □

Remark 2.10 The situation of Lemma 2.9 happens for a GIT quotient of semistable locus with respect to a G–linearization on a quasiprojective scheme.

We next discuss the universality of analytic Hilbert quotients:

Definition 2.11 An analytic Hilbert quotient (2-11) satisfies the universality if for any G–invariant analytic map $h\colon Z \to Z'$ to a complex analytic space Z', there is a unique factorization

$$h\colon Z \xrightarrow{\pi_Z} Z /\!\!/ G \to Z'. \tag{2-15}$$

The above universality is proved in [13, Corollary 4] when Z is a reduced Stein space and $Z' = \mathbb{C}^n$. Below we show the universality for the analytic Hilbert quotients given in Lemma 2.9. We prepare for this with the following lemma:

Lemma 2.12 *Let $\pi_Z\colon Z \to Z/\!\!/G$ be the analytic Hilbert quotient given in Lemma 2.9. Then for any family of G–invariant closed (not necessary analytic) subsets $\{W_\lambda\}_{\lambda \in \Lambda}$ in Z, the image $\pi_Z(W_\lambda)$ is closed in $Z /\!\!/ G$, and we have the identity*

$$\pi_Z\Big(\bigcap_{\lambda \in \Lambda} W_\lambda\Big) = \bigcap_{\lambda \in \Lambda} \pi_Z(W_\lambda). \tag{2-16}$$

Proof The question is local on $Z /\!\!/ G$, so we may assume that Y is affine and $Y' = Y /\!\!/ G$. Since Z and $Z /\!\!/ G$ are closed in $\pi_Y^{-1}(U)$ and U, we may also assume that $Z = \pi_Y^{-1}(U)$ and $Z/\!\!/G = U$. Let $S \subset Y$ be a Kempf–Ness set as in Theorem 2.3. Then for $S' := \pi_Y^{-1}(U) \cap S$, we have the homeomorphism $S'/K \xrightarrow{\cong} U$. Therefore for $W_\lambda' := S' \cap W_\lambda$, we have $\pi_Z(W_\lambda') = \pi_Z(W_\lambda)$. Since each W_λ' is a K–invariant closed subset of S', its image $\pi_Z(W_\lambda')$ is a closed subset of U and the identity (2-16) holds. □

The desired universality is proved in the following lemma:

Lemma 2.13 *The analytic Hilbert quotient $\pi_Z\colon Z \to Z /\!/ G$ in Lemma 2.9 satisfies the universality in Definition 2.11.*

Proof Let $h\colon Z \to Z'$ be a G–invariant analytic map to a complex analytic space Z'. We take an open cover $Z' = \bigcup_{\lambda\in\Lambda} \mathcal{U}'_\lambda$ such that $\mathcal{U}'_\lambda$ is a closed analytic subspace of an open subset in $\mathbb{C}^n$. Let $W'_\lambda := Z' \setminus \mathcal{U}'_\lambda$ and $W_\lambda := h^{-1}(W'_\lambda)$. Then each W_λ is a G–invariant closed subset of Z. By Lemma 2.12, the image $\pi_Z(W_\lambda) \subset Z /\!/ G$ is closed, and

$$\bigcap_{\lambda\in\Lambda} \pi_Z(W_\lambda) = \pi_Z\Big(\bigcap_{\lambda\in\Lambda} W_\lambda\Big) = \pi_Z \circ h^{-1}\Big(\bigcap_{\lambda\in\Lambda} W'_\lambda\Big) = \varnothing.$$

Here the last identity follows because $\{\mathcal{U}'_\lambda\}_{\lambda\in\Lambda}$ is an open cover of Z'. It follows that by setting $\mathcal{U}_\lambda := (Z /\!/ G) \setminus \pi_Z(W_\lambda)$, we have an open cover $Z /\!/ G = \bigcup_{\lambda\in\Lambda} \mathcal{U}_\lambda$ and the diagram

$$\begin{array}{ccccc}
\pi_Z^{-1}(\mathcal{U}_\lambda) & \hookrightarrow & h^{-1}(\mathcal{U}'_\lambda) & & \\
\downarrow{\scriptstyle \pi_Z} & & \downarrow{\scriptstyle h} & & \\
\mathcal{U}_\lambda & \dashrightarrow & \mathcal{U}'_\lambda & \hookrightarrow & \mathbb{C}^n
\end{array}$$

Here the top horizontal arrow is an open immersion, and the right horizontal arrow is a locally closed embedding. By the property Definition 2.6(2), there is a unique analytic map $\mathcal{U}_\lambda \to \mathcal{U}'_\lambda$ which makes the above diagram commutes. By the uniqueness, they glue to give a desired factorization (2-15). □

2.5 Moduli spaces of representations of quivers with convergent relations

We return to the situation of Section 2.2.

Definition 2.14 A convergent relation I of a quiver Q is a collection of finite number of elements

$$I = (f_1, \dots, f_l), \quad f_i \in \mathbb{C}\{Q\}.$$

Using the lemmas in the previous subsection, we have the following:

Lemma 2.15 *Given a convergent relation $I = (f_1, \dots, f_l)$ of a quiver Q and its dimension vector $\vec{m}$, there is an analytic open neighborhood of* 0

$$0 \in V \subset M_Q(\vec{m})$$

such that each $\mathrm{Hom}(V_a, V_b)$*–valued formal function* $f_i(a,b,\vec{m})$ *defined by* (2-7) *for* $f = f_i$ *absolutely converges on* $\pi_Q^{-1}(V)$*. Here* π_Q *is the quotient map*

$$\pi_Q\colon \mathrm{Rep}_Q(\vec{m}) \to M_Q(\vec{m}).$$

Proof Let $\mathcal{U}$ be an open neighborhood of $0 \in \mathrm{Rep}_Q(\vec{m})$ as in (2-8), where each $f_i(a,b,\vec{m})$ absolutely converges on $\mathcal{U}$. Since for $g = (g_i)_{i \in V(Q)} \in G$ and $u = (u_e)_{e \in E(Q)}$, we have

$$f_i(a,b,\vec{m})(g \cdot u) = g_b^{-1} \circ f_i(a,b,\vec{m})(u) \circ g_a.$$

The $\mathrm{Hom}(V_a, V_b)$–valued function $f_i(a,b,\vec{m})$ absolutely converges on $G \cdot \mathcal{U}$. By Lemma 2.5, there is a saturated open subset $0 \in \mathcal{V} \subset G \cdot \mathcal{U}$. Then by Lemma 2.4, $\mathcal{V} = \pi_Q^{-1}(V)$ for an open subset $0 \in V \subset M_Q(\vec{m})$. □

For a quiver Q with a convergent relation $I = (f_1, \ldots, f_l)$, let $\vec{m}$ be its dimension vector and take an open subset $V \subset M_Q(\vec{m})$ as in Lemma 2.15. By that lemma, we have the G–invariant closed analytic subspace of $\pi_Q^{-1}(V)$

$$\mathrm{Rep}_{(Q,I)}(\vec{m})|_V \subset \pi_Q^{-1}(V) \tag{2-17}$$

whose structure sheaf is given by

$$\mathcal{O}_{\mathrm{Rep}_{(Q,I)}(\vec{m})|_V} = \mathcal{O}_{\pi_Q^{-1}(V)}/(f_i(a,b,\vec{m})_{jk}, \quad a,b \in V(Q)).$$

Here $f_i(a,b,\vec{m})_{jk}$ is the matrix component of the analytic map

$$f_i(a,b,\vec{m})\colon \pi_Q^{-1}(V) \to \mathrm{Hom}(V_a, V_b).$$

By taking the quotient by G, we have the following definition:

Definition 2.16 Let Q be a quiver with a convergent relation I, and $\vec{m}$ its dimension vector. Then for a sufficiently small analytic open neighborhood $0 \in V \subset M_Q(\vec{m})$, we define the complex analytic stack $\mathcal{M}_{(Q,I)}(\vec{m})|_V$ and complex analytic space $M_{(Q,I)}(\vec{m})|_V$ by

$$\mathcal{M}_{(Q,I)}(\vec{m})|_V := [\mathrm{Rep}_{(Q,I)}(\vec{m})|_V/G],$$
$$M_{(Q,I)}(\vec{m})|_V := \mathrm{Rep}_{(Q,I)}(\vec{m})|_V /\!\!/ G.$$

Here $\mathrm{Rep}_{(Q,I)}(\vec{m})|_V /\!\!/ G$ is the analytic Hilbert quotient of $\mathrm{Rep}_{(Q,I)}(\vec{m})|_V$, given in Lemma 2.8.

2.6 Convergent superpotential

For a quiver Q, its convergent superpotential is defined as follows.

Definition 2.17 A convergent superpotential of a quiver Q is an element

$$W \in \mathbb{C}\{Q\}/[\mathbb{C}\{Q\}, \mathbb{C}\{Q\}].$$

A convergent superpotential W of Q is represented by a formal sum

$$W = \sum_{n\geq 1} \sum_{\substack{\psi: \{1,\ldots,n+1\}\to V(Q) \\ \psi(n+1)=\psi(1)}} \sum_{e_i \in E_{\psi(i),\psi(i+1)}} a_{\psi,e_\bullet} \cdot e_1 e_2 \cdots e_n$$

with $|a_{\psi,e_\bullet}| < C^n$ for a constant $C > 0$.

For $i, j \in V(Q)$, let $\boldsymbol{E}_{i,j}$ be the $\mathbb{C}$–vector space spanned by $E_{i,j}$. We set

$$E_{i,j}^{\vee} := \{e^{\vee} : e \in E_{i,j}\} \subset \boldsymbol{E}_{i,j}^{\vee}. \tag{2-18}$$

Here for $e \in E_{i,j}$, the element $e^{\vee} \in \boldsymbol{E}_{i,j}^{\vee}$ is defined by the condition $e^{\vee}(e) = 1$ and $e^{\vee}(e') = 0$ for any $e \neq e' \in E_{i,j}$; ie $E_{i,j}^{\vee}$ is the dual basis of $E_{i,j}$. For a map $\psi\colon \{1, \ldots, n+1\} \to V(Q)$ with $\psi(1) = \psi(n+1)$ and elements $e_i \in E_{\psi(i),\psi(i+1)}$ and $e \in E(Q)$, we set

$$\partial_{e^{\vee}}(e_1 \cdots e_n) = \sum_{a=1}^{n} e^{\vee}(e_a) e_{a+1} \cdots e_n e_1 \cdots e_{a-1}.$$

Here $e^{\vee}(e_a) = 0$ if $(s(e_a), t(e_a)) \neq (s(e), t(e))$. The above partial differential extends to a linear map

$$\partial_{e^{\vee}}\colon \mathbb{C}\{Q\}/[\mathbb{C}\{Q\}, \mathbb{C}\{Q\}] \to \mathbb{C}\{Q\}.$$

For a convergent superpotential W, the set of elements in $\mathbb{C}\{Q\}$

$$\partial W := \{\partial_{e^{\vee}} W : e \in E(Q)\}$$

is a convergent relation of Q.

For a dimension vector $\vec{m}$ of Q, let $\operatorname{tr} W$ be the formal function of $u = (u_e)_{e\in E(Q)} \in \operatorname{Rep}_Q(\vec{m})$ defined by

$$\operatorname{tr} W(u) := \sum_{n\geq 1} \sum_{\substack{\psi: \{1,\ldots,n+1\}\to V(Q) \\ \psi(n+1)=\psi(1)}} \sum_{e_i \in E_{\psi(i),\psi(i+1)}} a_{\psi,e_\bullet} \cdot \operatorname{tr}(u_{e_n} \circ u_{e_{n-1}} \circ \cdots \circ u_{e_1}).$$

This formal function on $\mathrm{Rep}_Q(\vec{m})$ is G–invariant. By the argument of Lemma 2.15, there is an analytic open neighborhood $0 \in V \subset M_Q(\vec{m})$ such that the formal function $\mathrm{tr}\, W$ absolutely converges on $\pi_Q^{-1}(V)$ to give a G–invariant holomorphic function

$$\mathrm{tr}\, W \colon \pi_Q^{-1}(V) \to \mathbb{C}.$$

Then for the relation $I = \partial W$, it is easy to see (and well known when W is a typical superpotential of Q) that the analytic subspace (2-17) equals to the critical locus of $\mathrm{tr}\, W$ in $\pi_Q^{-1}(V)$:

$$\mathrm{Rep}_{(Q,\partial W)}(\vec{m})|_V = \{d(\mathrm{tr}\, W)=0\}.$$

In particular, we have

$$\mathcal{M}_{(Q,\partial W)}(\vec{m})|_V = [\{d(\mathrm{tr}\, W)=0\}/G].$$

3 Moduli stacks of semistable sheaves

In this section, we recall some basic notions and facts on moduli spaces of semistable sheaves, whose details are available in [15]. Then we state the precise form of Theorem 1.1 in Theorem 3.2. In what follows, we always assume that the varieties or schemes are defined over $\mathbb{C}$.

3.1 Gieseker semistable sheaves

Let

$$(X, \mathcal{O}_X(1))$$

be a polarized smooth projective variety with $\omega = c_1(\mathcal{O}_X(1))$. For a coherent sheaf E on X, its *Hilbert polynomial* is defined by

$$\chi(E \otimes \mathcal{O}_X(m)) = a_d m^d + a_{d-1} m^{d-1} + \cdots,$$

where $d = \dim \mathrm{Supp}(E)$ and a_d is a positive rational number. The *reduced Hilbert polynomial* is defined by

$$\overline{\chi}(E, m) := \frac{\chi(E \otimes \mathcal{O}_X(m))}{a_d} \in \mathbb{Q}[m].$$

For polynomials $p_i(m) \in \mathbb{Q}[m]$ with $i = 1, 2$, we write $p_1(m) \succ p_2(m)$ if $\deg p_1 < \deg p_2$, or if $\deg p_1 = \deg p_2$ and $p_1(m) > p_2(m)$ for $m \gg 0$. Then $(\mathbb{Q}[m], \succ)$ is an ordered set.

By definition, a coherent sheaf E on X is said to be *ω–Gieseker (semi)stable* if for any nonzero subsheaf $E' \subsetneq E$, we have the inequality

$$\overline{\chi}(E', m) \prec (\preceq)\, \overline{\chi}(E, m).$$

For any Gieseker semistable sheaf E on X, it has a filtration (called *Jordan–Hölder (JH) filtration*)

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_k = E$$

such that each F_i/F_{i-1} is ω–Gieseker stable whose reduced Hilbert polynomial coincides with $\overline{\chi}(E, m)$. The JH filtration is not necessary unique, but its subquotient

$$\mathrm{gr}(E) := \bigoplus_{i=1}^{k} F_i/F_{i-1}$$

is uniquely determined up to isomorphism. For two ω–Gieseker semistable sheaves E and E' on X, they are called *S-equivalent* if $\mathrm{gr}(E)$ and $\mathrm{gr}(E')$ are isomorphic.

3.2 Moduli spaces of semistable sheaves

Let $\mathcal{M}$ be the 2–functor

$$\mathcal{M}\colon \mathcal{S}\mathrm{ch}/\mathbb{C} \to \mathcal{G}\mathrm{roupoid} \tag{3-1}$$

which sends a $\mathbb{C}$–scheme S to the groupoid of S–flat coherent sheaves on $X \times S$. The stack $\mathcal{M}$ is an algebraic stack locally of finite type over $\mathbb{C}$. Let Γ be the image of the Chern character map

$$\Gamma := \mathrm{Im}(\mathrm{ch}\colon K(X) \to H^*(X, \mathbb{Q})).$$

For each $v \in \Gamma$, we have an open substack of finite type

$$\mathcal{M}_\omega(v) \subset \mathcal{M}$$

consisting of flat families of ω–Gieseker semistable sheaves with Chern character v.

The stack $\mathcal{M}_\omega(v)$ is constructed as a global quotient stack of a quasiprojective scheme. For $[E] \in \mathcal{M}_\omega(v)$, we take $m \gg 0$ and a vector space V satisfying

$$\dim V = \chi(E(m)) = \dim H^0(E(m)).$$

The above condition depends only on v, and independent of E for $m \gg 0$. Let $\mathrm{Quot}(V, v)$ be the Grothendieck Quot scheme parametrizing quotients

$$s\colon V \otimes \mathcal{O}_X(-m) \twoheadrightarrow E \tag{3-2}$$

in $\mathrm{Coh}(X)$ with $\mathrm{ch}(E) = v$. Then there is an open subscheme

$$\mathrm{Quot}^{\circ}(V, v) \subset \mathrm{Quot}(V, v)$$

parametrizing quotients (3-2) such that E is ω–Gieseker semistable and the induced linear map $V \to H^0(E(m))$ is an isomorphism. The algebraic group $\mathrm{GL}(V)$ acts on $\mathrm{Quot}^{\circ}(V, v)$ by

$$g \cdot (V \otimes \mathcal{O}_X(-m) \stackrel{s}{\twoheadrightarrow} E) = (V \otimes \mathcal{O}_X(-m) \stackrel{s \circ g}{\twoheadrightarrow} E),$$

and the stack $\mathcal{M}_{\omega}(v)$ is described as

$$\mathcal{M}_{\omega}(v) = [\mathrm{Quot}^{\circ}(V, v) / \mathrm{GL}(V)].$$

The above construction is compatible with the geometric invariant theory (GIT). If we take the closure of $\mathrm{Quot}^{\circ}(V, v)$

$$\overline{\mathrm{Quot}}^{\circ}(V, v) \subset \mathrm{Quot}(V, v),$$

then there is a $\mathrm{GL}(V)$–linearized polarization on $\overline{\mathrm{Quot}}^{\circ}(V, v)$ such that its open locus $\mathrm{Quot}^{\circ}(V, v)$ is the GIT semistable locus with respect to the above $\mathrm{GL}(V)$–linearized polarization. In particular, we have the good quotient morphism (which is in particular a good moduli space in the sense of [1])

$$p_M \colon \mathcal{M}_{\omega}(v) \to M_{\omega}(v) := \mathrm{Quot}^{\circ}(V, v) /\!\!/ \mathrm{GL}(V).$$

Namely, there is a $\mathrm{GL}(V)$–invariant affine open cover

$$\mathrm{Quot}^{\circ}(V, v) = \bigcup_i U_i, \quad U_i = \mathrm{Spec}\, R_i$$

such that $M_{\omega}(v)$ has the following affine open cover

$$M_{\omega}(v) = \bigcup_i U_i /\!\!/ \mathrm{GL}(V), \quad U_i /\!\!/ \mathrm{GL}(V) = \mathrm{Spec}\, R_i^{\mathrm{GL}(V)}.$$

By the GIT construction of $M_{\omega}(v)$, two points $x_1, x_2 \in \mathrm{Quot}^{\circ}(V)$ are mapped to the same point by p_M if and only if their orbit closures intersect, ie

$$\overline{\mathrm{GL}(V) \cdot x_1} \cap \overline{\mathrm{GL}(V) \cdot x_2} \neq \varnothing.$$

It is also known that the above condition is equivalent to that, if x_i corresponds to a ω–Gieseker semistable sheaf E_i, then E_1 and E_2 are S–equivalent. In fact, the projective scheme $M_{\omega}(v)$ is the coarse moduli space of S–equivalence classes of ω–Gieseker semistable sheaves with Chern character v. So every point $p \in M_{\omega}(v)$ is represented by a direct sum of ω–Gieseker stable sheaves E (called a *polystable sheaf*), written as

$$E = \bigoplus_{i=1}^{k} V_i \otimes E_i. \tag{3-3}$$

Here each V_i is a finite-dimensional vector space, E_i is a ω–Gieseker stable sheaf with $\overline{\chi}(E_i, m) = \overline{\chi}(E, m)$ for all i.

3.3 Ext–quiver

Suppose that $E \in \mathrm{Coh}(X)$ is of the form (3-3). Then the collection of the sheaves $(E_1, \dots, E_k)$ forms a simple collection, defined as follows:

Definition 3.1 A collection of coherent sheaves $(E_1, \dots, E_k)$ is called a simple collection if $\mathrm{Hom}(E_i, E_j) = \mathbb{C} \cdot \delta_{ij}$.

Let $E_\bullet = (E_1, \dots, E_k)$ be a simple collection of coherent sheaves on X. For each $1 \le i, j \le k$, we fix a finite subset

$$E_{i,j} \subset \mathrm{Ext}^1(E_i, E_j)^\vee \tag{3-4}$$

giving a basis of $\mathrm{Ext}^1(E_i, E_j)^\vee$. We define the quiver $Q_{E_\bullet}$ as follows. The set of vertices and edges are given by

$$V(Q_{E_\bullet}) = \{1, 2, \dots, k\}, \quad E(Q_{E_\bullet}) = \coprod_{1 \le i,j \le k} E_{i,j}.$$

The maps $s, t \colon E(Q_{E_\bullet}) \to V(Q_{E_\bullet})$ are given by

$$s|_{E_{i,j}} = i, \quad t|_{E_{i,j}} = j.$$

The resulting quiver $Q_{E_\bullet}$ is called the *Ext–quiver* of $E_\bullet$.

We can now give the precise statement of Theorem 1.1:

Theorem 3.2 *Let X be a smooth projective variety, and let $\mathcal{M}_\omega(v)$ be the moduli stack of ω–Gieseker semistable sheaves on X with Chern character v. We have the natural morphism to its coarse moduli space*

$$p_M \colon \mathcal{M}_\omega(v) \to M_\omega(v).$$

For $p \in M_\omega(v)$, it is represented by a sheaf E of the form

$$E = \bigoplus_{i=1}^{k} V_i \otimes E_i,$$

where $E_\bullet = (E_1, \dots, E_k)$ *is a simple collection. Let* $Q_{E_\bullet}$ *be the corresponding Ext–quiver and* $\vec{m}$ *its dimension vector given by* $\vec{m} = (m_1, \dots, m_k)$*, where* $m_i = \dim V_i$*. Then there is a convergent relation* $I_{E_\bullet}$ *of* $Q_{E_\bullet}$*, analytic open neighborhoods* $p \in U \subset M_\omega(v)$ *and* $0 \in V \subset M_{Q_{E_\bullet}}(\vec{m})$*, and commuting isomorphisms*

$$\begin{array}{ccc} \mathcal{M}_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V & \xrightarrow{\cong} & p_M^{-1}(U) \\ \big\downarrow p_Q & & \big\downarrow p_M \\ M_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V & \xrightarrow{\cong} & U \end{array} \tag{3-5}$$

Here the bottom arrow sends 0 *to* p*.*

The proof of Theorem 3.2 will be completed in Proposition 5.4 below.

4 Deformations of coherent sheaves

In this section, we describe deformation theory of coherent sheaves via dg-algebras and their minimal A_∞–models. The arguments are already known for vector bundles [11; 34], and we apply similar arguments for resolutions of coherent sheaves by vector bundles.

The above description will give local atlas of the moduli stack $\mathcal{M}$ in Section 3.2 via finite-dimensional A_∞–algebras. More precisely for a given coherent sheaf E on a smooth projective variety X, we compare the following three descriptions of the deformation space of E:

(1) An open neighborhood of the algebraic stack $\mathcal{M}$ given in Section 3.2 at the point $[E] \in \mathcal{M}$.

(2) The Maurer–Cartan locus associated with the infinite-dimensional dg-algebra $\mathbf{R}\mathrm{Hom}(E, E)$.

(3) The Maurer–Cartan locus associated with the finite-dimensional minimal A_∞–algebra $\mathrm{Ext}^*(E, E)$.

We will compare the above descriptions by first constructing the map $(3) \Longrightarrow (2)$ in Lemma 4.2. Then we will construct a map $(2) \Longrightarrow (1)$, and then composing we get the desired atlas $(3) \Longrightarrow (1)$ in Proposition 4.3.

4.1 Deformations of vector bundles

We recall some basic facts on the deformation theory of vector bundles via gauge theory, and fix some notation; see [11] for details. For a holomorphic vector bundle $\mathcal{E} \to X$ on a smooth projective variety X, we denote by $\mathcal{A}^{p,q}(\mathcal{E})$ the sheaf of $\mathcal{E}$–valued (p,q)–forms on X, and we set

$$A^{p,q}(\mathcal{E}) := \Gamma(X, \mathcal{A}^{p,q}(\mathcal{E})).$$

The holomorphic structure on $\mathcal{E}$ is given by the Dolbeault connection

$$\bar{\partial}_{\mathcal{E}}\colon \mathcal{A}^{0,0}(\mathcal{E}) \to \mathcal{A}^{0,1}(\mathcal{E}).$$

The Dolbeault connection extends to the Dolbeault complex

$$0 \to \mathcal{A}^{0,0}(\mathcal{E}) \to \mathcal{A}^{0,1}(\mathcal{E}) \to \cdots \to \mathcal{A}^{0,i}(\mathcal{E}) \to \mathcal{A}^{0,i+1}(\mathcal{E}) \to \cdots$$

giving a resolution of $\mathcal{E}$. The complex $\mathcal{A}^{0,*}(\mathcal{E})$ is an elliptic complex (see [35, Chapter IV, Section 5]), whose global section computes $H^*(X,\mathcal{E})$, ie

$$H^k(X,\mathcal{E}) = \mathcal{H}^k(A^{0,*}(\mathcal{E})).$$

Any other holomorphic structure on $\mathcal{E}$ is given by the Dolbeault connection of the form

$$\bar{\partial}_{\mathcal{E}} + A\colon \mathcal{A}^{0,0}(\mathcal{E}) \to \mathcal{A}^{0,1}(\mathcal{E})$$

for some $A \in A^{0,1}(\mathcal{E}\mathrm{nd}(\mathcal{E}))$. Conversely given $A \in A^{0,1}(\mathcal{E}\mathrm{nd}(\mathcal{E}))$, the connection $\bar{\partial}_{\mathcal{E}} + A$ gives a holomorphic structure on $\mathcal{E}$ if and only if its square is zero, ie

$$\mathrm{ad}(\bar{\partial}_{\mathcal{E}})(A) + A \circ A = 0.$$

The above equation is the Maurer–Cartan (MC) equation of the dg-algebra

$$\mathfrak{g}^*_{\mathcal{E}} := A^{0,*}(\mathcal{E}\mathrm{nd}(\mathcal{E})). \tag{4-1}$$

The quotient of the solution space of the MC equation of $\mathfrak{g}^*_{\mathcal{E}}$ by the gauge group of $\mathcal{C}^\infty$–automorphisms of $\mathcal{E}$ describes the deformation space of $\mathcal{E}$ as holomorphic vector bundles.

4.2 Deformations of complexes

We have a similar deformation theory for complexes of vector bundles. Let

$$\mathcal{E}^\bullet = (\cdots \to 0 \to \mathcal{E}^i \xrightarrow{d^i} \mathcal{E}^{i+1} \to \cdots \to \mathcal{E}^j \to 0 \to \cdots) \tag{4-2}$$

be a bounded complex of holomorphic vector bundles on X. By taking the Dolbeault complex $\mathcal{A}^{0,*}(\mathcal{E}^i)$ for each $\mathcal{E}^i$, we obtain the double complex $\mathcal{A}^{0,*}(\mathcal{E}^\bullet)$. Let $\mathrm{Tot}(-)$ means the total complex of the double complex. We set

$$A^{0,*}(\mathcal{E}^\bullet) := \mathrm{Tot}(\Gamma(X, \mathcal{A}^{0,*}(\mathcal{E}^\bullet))). \tag{4-3}$$

Similarly to the vector bundle case, the complex $\mathrm{Tot}(\mathcal{A}^{0,*}(\mathcal{E}^\bullet))$ is elliptic, and its global section computes the hypercohomology of $\mathcal{E}^\bullet$

$$\mathcal{H}^k(\mathbf{R}\Gamma(X, \mathcal{E}^\bullet)) = \mathcal{H}^k(A^{0,*}(\mathcal{E}^\bullet)). \tag{4-4}$$

Applying the construction (4-3) to the inner $\mathcal{H}\mathrm{om}$ complex $\mathcal{H}\mathrm{om}^*(\mathcal{E}^\bullet, \mathcal{E}^\bullet)$, we obtain the complex

$$\mathfrak{g}^*_{\mathcal{E}^\bullet} := A^{0,*}(\mathcal{H}\mathrm{om}^*(\mathcal{E}^\bullet, \mathcal{E}^\bullet)).$$

Its degree-k part is given by

$$\mathfrak{g}^k_{\mathcal{E}^\bullet} = \bigoplus_{p+q=k} \prod_i A^{0,q}(\mathcal{H}\mathrm{om}(\mathcal{E}^i, \mathcal{E}^{i+p})), \tag{4-5}$$

and the differential $d_{\mathfrak{g}}$ is induced by the Dolbeault connections $\bar{\partial}_{\mathcal{E}_i}$ on each $\mathcal{E}_i$ together with the differentials d^* in (4-2). Also the composition

$$A^{0,q}(\mathcal{H}\mathrm{om}(\mathcal{E}^i, \mathcal{E}^{i+p})) \times A^{0,q'}(\mathcal{H}\mathrm{om}(\mathcal{E}^{i+p}, \mathcal{E}^{i+p+p'})) \to A^{0,q+q'}(\mathcal{H}\mathrm{om}(\mathcal{E}^i, \mathcal{E}^{i+p+p'}))$$

defines the product structure $\cdot$ on $\mathfrak{g}^*_{\mathcal{E}^\bullet}$. Then it is straightforward to check that the data

$$(\mathfrak{g}^*_{\mathcal{E}^\bullet}, d_{\mathfrak{g}}, \cdot) \tag{4-6}$$

is a dg-algebra.

Let $\mathfrak{mc}$ be the map defined by

$$\mathfrak{mc}\colon \mathfrak{g}^1_{\mathcal{E}^\bullet} \to \mathfrak{g}^2_{\mathcal{E}^\bullet}, \quad \alpha \mapsto d_{\mathfrak{g}}(\alpha) + \alpha \cdot \alpha.$$

Its zero set

$$\mathrm{MC}(\mathfrak{g}^*_{\mathcal{E}^\bullet}) = \{\alpha \in \mathfrak{g}^1_{\mathcal{E}^\bullet} : \mathfrak{mc}(\alpha) = 0\} \tag{4-7}$$

is the solution of the Maurer–Cartan equation of the dg-algebra $\mathfrak{g}^*_{\mathcal{E}^\bullet}$. Note that an element $\alpha \in \mathfrak{g}^1_{\mathcal{E}^\bullet}$ satisfies the MC equation if and only if

$$(d_{\mathcal{A}^{0,*}(\mathcal{E}^\bullet)} + \alpha)^2 = 0$$

on $\mathcal{A}^{0,*}(\mathcal{E}^\bullet)$. In this case, the data

$$(\mathcal{A}^{0,*}(\mathcal{E}^\bullet), d_{\mathcal{A}^{0,*}(\mathcal{E}^\bullet)} + \alpha) \tag{4-8}$$

determines a dg-$\mathcal{A}^{0,*}(\mathcal{O}_X)$–module. Then (4-8) is a bounded complex of $\mathcal{O}_X$–modules whose cohomologies are coherent (see [4, Lemma 4.1.5]), giving a deformation of the complex (4-2) in the derived category.

More explicitly, by (4-5) an element $\alpha \in \mathfrak{g}^1_{\mathcal{E}^\bullet}$ consists of data

$$\alpha = (\alpha^i_0, \alpha^i_1, \alpha^i_2, \dots), \quad \alpha^i_j \in A^{0,j}(\mathcal{H}\mathrm{om}(\mathcal{E}^i, \mathcal{E}^{i-j+1})). \tag{4-9}$$

Suppose that the above α satisfies the MC equation $\mathfrak{mc}(\alpha) = 0$. Then the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathcal{A}^{0,0}(\mathcal{E}^{i-1}) & \longrightarrow & \mathcal{A}^{0,0}(\mathcal{E}^{i}) & \xrightarrow{d^i+\alpha^i_0} & \mathcal{A}^{0,0}(\mathcal{E}^{i+1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \bar{\partial}_{\mathcal{E}^i}+\alpha^i_1} & & \downarrow & & \\
\cdots & \longrightarrow & \mathcal{A}^{0,1}(\mathcal{E}^{i-1}) & \longrightarrow & \mathcal{A}^{0,1}(\mathcal{E}^{i}) & \longrightarrow & \mathcal{A}^{0,1}(\mathcal{E}^{i+1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \bar{\partial}_{\mathcal{E}^i}+\alpha^i_1} & & \downarrow & & \\
\cdots & \longrightarrow & \mathcal{A}^{0,2}(\mathcal{E}^{i-1}) & \longrightarrow & \mathcal{A}^{0,2}(\mathcal{E}^{i}) & \longrightarrow & \mathcal{A}^{0,2}(\mathcal{E}^{i+1}) & \longrightarrow & \cdots
\end{array}$$

is a complex in the horizontal direction, each square is commutative, and the compositions of vertical arrows are homotopic to zero with homotopy given by α^i_2.

In particular if $\alpha^i_j = 0$ for $j \geq 2$, then the above diagram extends to a double complex. In this case,

$$\mathcal{E}^i_\alpha = (\mathcal{A}^{0,0}(\mathcal{E}^i), \bar{\partial}_{\mathcal{E}^i} + \alpha^i_1)$$

is a holomorphic structure on $\mathcal{E}^i$. By setting

$$d^i_\alpha = d^i + \alpha^i_0 \colon \mathcal{A}^{0,0}(\mathcal{E}^i) \to \mathcal{A}^{0,0}(\mathcal{E}^{i+1}),$$

we have the bounded complex of holomorphic vector bundles on X

$$\cdots \to 0 \to \mathcal{E}^{-n}_\alpha \xrightarrow{d^{-n}_\alpha} \cdots \to \mathcal{E}^{-1}_\alpha \xrightarrow{d^{-1}_\alpha} \mathcal{E}^0_\alpha \to 0 \to \cdots \tag{4-10}$$

giving a deformation of $\mathcal{E}^\bullet$ as complexes. Conversely given a deformation of $\mathcal{E}^\bullet$ as a complex, it gives rise to the solution of MC equation of the form $\alpha = (\alpha^i_0, \alpha^i_1, 0, \dots)$.

For $\alpha, \alpha' \in \mathrm{MC}(\mathfrak{g}^*_{\mathcal{E}^\bullet})$, α and α' are called *gauge-equivalent* if there exist

$$\gamma = \{(\gamma^i_0, \gamma^i_1, \gamma^i_2, \dots)\}_i \in \mathfrak{g}^0_{\mathcal{E}^\bullet}, \quad \gamma^i_j \in A^{0,j}(\mathcal{H}\mathrm{om}(\mathcal{E}^i, \mathcal{E}^{i-j})),$$

where γ_0^i gives an isomorphism $\mathcal{E}^i \xrightarrow{\cong} \mathcal{E}^i$ as C^∞–vector bundles, such that we have

$$\gamma \circ (d_{\mathcal{A}^{0,*}(\mathcal{E}^\bullet)} + \alpha) \circ \gamma^{-1} = d_{\mathcal{A}^{0,*}(\mathcal{E}^\bullet)} + \alpha'. \tag{4-11}$$

In this case, we have the isomorphism of the dg-$\mathcal{A}^{0,*}(\mathcal{O}_X)$–modules

$$\gamma\colon (\mathcal{A}^{0,*}(\mathcal{E}^\bullet), d_{\mathcal{A}^{0,*}(\mathcal{E}^\bullet)} + \alpha) \xrightarrow{\cong} (\mathcal{A}^{0,*}(\mathcal{E}^\bullet), d_{\mathcal{A}^{0,*}(\mathcal{E}^\bullet)} + \alpha')$$

giving isomorphic deformations of (4-2) in the derived category.

Suppose that the complex (4-2) is quasi-isomorphic to a coherent sheaf E. Let Def_E be the deformation functor

$$\mathrm{Def}_E\colon \mathcal{A}\mathrm{rt} \to \mathcal{S}\mathrm{et}$$

sending a finite-dimensional commutative local $\mathbb{C}$–algebra $(R, \boldsymbol{m})$ to the set of isomorphism classes of R–flat deformations of E to $X \times \operatorname{Spec} R$. Then it is shown in [10, Section 8] that we have the functorial isomorphism

$$\mathrm{MC}(\mathfrak{g}^*_{\mathcal{E}^\bullet} \otimes \boldsymbol{m})/(\text{gauge equivalence}) \xrightarrow{\cong} \mathrm{Def}_E(R)$$

by sending a solution of the MC equation to the cohomology of the corresponding deformation (4-8).

4.3 Resolutions of coherent sheaves

For a smooth projective variety X, we consider the deformation theory of a sheaf

$$E \in \mathrm{Coh}(X)$$

in terms of the dg-algebra. As we recalled in Section 4.1, when E is a vector bundle its deformation theory is described in terms of the dg-algebra (4-1). In general, we take a resolution of E by vector bundles and consider the associated dg-algebra (4-6).

We first fix a resolution of E by vector bundles in the following way. Let $\mathcal{O}_X(1)$ be an ample line bundle on X. Then for $m_0 \gg 0$ we have the surjection

$$H^0(E(m_0)) \otimes \mathcal{O}_X(-m_0) \twoheadrightarrow E.$$

Applying this construction to the kernel of the above morphism and repeating, we obtain the resolution of E of the form

$$\cdots \to W^i \otimes \mathcal{O}_X(-m_i) \xrightarrow{d^i} W^{i+1} \otimes \mathcal{O}_X(-m_{i+1}) \to \cdots \to W^0 \otimes \mathcal{O}_X(-m_0) \to E \to 0$$

for finite-dimensional vector spaces W^i. Since X is smooth, the kernel of d^i for $i=-N$ with $N \gg 0$ is a vector bundle on X. Therefore we obtain the bounded resolution of E

$$0 \to \mathcal{E}^{-N} \xrightarrow{d^{-N}} \cdots \to \mathcal{E}^{-1} \xrightarrow{d^{-1}} \mathcal{E}^0 \to E \to 0, \tag{4-12}$$

where $\mathcal{E}^{-N} = \mathrm{Ker}(d^{-N})$ and $\mathcal{E}^i = W^i \otimes \mathcal{O}_X(-m_i)$ for $-N < i \le 0$.

By replacing m_i and n if necessary, the above construction can be extended to local universal family of deformations of E. Let $\mathcal{M}$ be the stack (3-1), and take its local atlas

$$(A, p) \to (\mathcal{M}, [E]) \tag{4-13}$$

at $[E] \in \mathcal{M}$ such that A is a finite-type affine scheme and a point $p \in A$ is sent to $[E]$. Let

$$E_A \in \mathrm{Coh}(X \times A)$$

be the universal family. Let $\mathcal{O}_{X\times A}(1)$ be the pull-back of $\mathcal{O}_X(1)$ to $X \times A$. For $m_0 \gg 0$, the $\mathcal{O}_A$–module $H^0(E_A(-m_0))$ is locally free of finite rank and we have the surjection

$$H^0(E_A(m_0)) \otimes_{\mathcal{O}_A} \mathcal{O}_{X\times A}(-m_0) \twoheadrightarrow E_A.$$

Similarly as above, we obtain the resolution of E_A of the form

$$\cdots \to \mathcal{W}^i \otimes_{\mathcal{O}_A} \mathcal{O}_{X\times A}(-m_i) \to \mathcal{W}^{i+1} \otimes_{\mathcal{O}_A} \mathcal{O}_{X\times A}(-m_{i+1}) \to \cdots$$
$$\to W^0 \otimes_{\mathcal{O}_A} \mathcal{O}_{X\times A}(-m_0) \to E_A \to 0$$

for locally free $\mathcal{O}_A$–modules $\mathcal{W}^i$ of finite rank. By taking the kernel at $i=-N$ for $N \gg 0$, we obtain the resolution of E_A

$$0 \to \mathcal{E}_A^{-N} \to \cdots \to \mathcal{E}_A^{-1} \to \mathcal{E}_A^0 \to E_A \to 0. \tag{4-14}$$

For $N \gg 0$, each $\mathcal{E}_A^i$ is a vector bundle on $X \times A$, since E_A is a A–flat perfect object. By restricting it to $X \times \{p\}$, we obtain the resolution (4-12).

4.4 Minimal A_∞–algebras

For a coherent sheaf E on X, we fix a resolution $\mathcal{E}^\bullet$ as in (4-12) and consider the dg-algebra (4-6)

$$\mathfrak{g}_E^* := \mathfrak{g}_{\mathcal{E}^\bullet}^*. \tag{4-15}$$

When E is a vector bundle, we just take the dg-algebra (4-1) in the argument below. By (4-4) we have

$$\mathrm{Ext}^k(E,E) = \mathcal{H}^k(\mathfrak{g}_E^*).$$

By the homological transfer theorem, there exists a minimal A_∞–algebra structure $\{m_n\}_{n\geq 2}$ on $\mathrm{Ext}^*(E,E)$, and a quasi-isomorphism

$$I\colon (\mathrm{Ext}^*(E,E), \{m_n\}_{n\geq 2}) \to (\mathfrak{g}_E^*, d_{\mathfrak{g}}, \cdot) \tag{4-16}$$

as A_∞–algebras. Here the A_∞–structure on $\mathrm{Ext}^*(E,E)$ consists of linear maps

$$m_n\colon \mathrm{Ext}^*(E,E) \to \mathrm{Ext}^{*+2-n}(E,E), \quad n \geq 2, \tag{4-17}$$

and the quasi-isomorphism (4-16) is a collection of linear maps

$$I_n\colon \mathrm{Ext}^*(E,E)^{\otimes n} \to \mathfrak{g}_E^{*+1-n}.$$

Both of m_n and I_n satisfy the A_∞–constraints. The maps m_n and I_n are explicitly described in terms of Kontsevich–Soibelman's tree formula [20] given as follows.

Let us choose a Kähler metric on X, Hermitian metrics on vector bundles $\mathcal{E}^i$, and fix them. A standard argument in Hodge theory for elliptic complexes (for example, see [35]) yields a linear embedding

$$i\colon \mathrm{Ext}^*(E,E) \hookrightarrow \mathfrak{g}_E^*$$

which identifies $\mathrm{Ext}^*(E,E)$ with $\Delta = 0$, where Δ is the Laplacian operator

$$\Delta = d_{\mathfrak{g}} d_{\mathfrak{g}}^* + d_{\mathfrak{g}}^* d_{\mathfrak{g}}\colon \mathfrak{g}_E^* \to \mathfrak{g}_E^*.$$

Here $d_{\mathfrak{g}}^*$ is the adjoint map of $d_{\mathfrak{g}}$ with respect to the above chosen Kähler metric on X and Hermitian metrics on $\mathcal{E}^i$. Moreover we have linear operators

$$p\colon \mathfrak{g}_E^* \twoheadrightarrow \mathrm{Ext}^*(E,E), \quad h\colon \mathfrak{g}_E^* \to \mathfrak{g}_E^{*-1} \tag{4-18}$$

satisfying the relations

$$p \circ i = \mathrm{id}, \quad i \circ p = \mathrm{id} + d_{\mathfrak{g}} \circ h + h \circ d_{\mathfrak{g}}. \tag{4-19}$$

The homotopy operator h is given by

$$h = -d_{\mathfrak{g}}^* \circ G, \tag{4-20}$$

where G is the Green's operator, which is an operator of order -2 (see [35, Chapter IV]), hence h is of order -1.

The A_∞–product (4-17) is described by Kontsevich–Soibelman's tree formula as

$$m_n = \sum_{T \in \mathcal{O}(n)} \pm m_{n,T}, \tag{4-21}$$

where $\mathcal{O}(n)$ is the set of isomorphism classes of binary rooted trees with n leaves. Here $m_{n,T}$ is the operation given by the composition associated to T, by putting i on leaves, the product map $\cdot$ of $\mathfrak{g}_E^*$ on internal vertices, the homotopy h on internal edges, and the projection p on the root of T. For example, m_3 is given by

$$m_3(x_1, x_2, x_3) = \pm p\big(h(i(x_1) \cdot i(x_2)) \cdot i(x_3)\big) \pm p\big(i(x_1) \cdot h(i(x_1) \cdot i(x_2))\big).$$

The operation I_n is similarly given by

$$I_n = \sum_{T \in \mathcal{O}(n)} \pm I_{n,T}, \tag{4-22}$$

where $I_{n,T}$ is defined by replacing p by h in the construction of $m_{n,T}$. For example, I_3 is given by

$$I_3(x_1, x_2, x_3) = \pm h\big(h(i(x_1) \cdot i(x_2)) \cdot i(x_3)\big) \pm h\big(i(x_1) \cdot h(i(x_1) \cdot i(x_2))\big).$$

By [34, Appendix A], there exists another A_∞–homomorphism

$$P\colon (\mathfrak{g}_E^*, d_{\mathfrak{g}}, \cdot) \to (\operatorname{Ext}^*(E, E), \{m_n\}_{n \geq 2}) \tag{4-23}$$

which is a homotopy inverse of I, ie

$$P \circ I = \mathrm{id}, \quad I \circ P \overset{\text{homotopic}}{\sim} \mathrm{id}.$$

Here two A_∞–morphisms $f_1, f_2\colon A_1 \to A_2$ between A_∞–algebras A_1, A_2 are called *homotopic* if there is an A_∞–homomorphism

$$H\colon A_1 \to A_2 \otimes \Omega^*_{[0,1]}$$

such that $H(0) = f_1$ and $H(1) = f_2$, where $\Omega^*_{[0,1]}$ is the dg-algebra of $\mathcal{C}^\infty$–differential forms on the interval $[0,1]$. The A_∞–homomorphism P consists of linear maps

$$P_n\colon (\mathfrak{g}_E^*)^{\otimes n} \to \operatorname{Ext}^{*+1-n}(E, E)$$

which are also described in terms of the tree formula, whose details we omit (see [34, Appendix A] for details).

Later we will use some boundedness properties of linear maps m_n, I_n and P_n. Let us take an even number $l \gg 0$, eg $l > 2 \dim X$, and consider the Sobolev $(l,2)$–norm $\|-\|_l$

on $\mathfrak{g}_E^*$. It also induces a norm $\|-\|_l$ on $\mathrm{Ext}^*(E,E)$ by the embedding i in (4-18). We denote by

$$\mathfrak{g}_E^* \subset \widehat{\mathfrak{g}}_{E,l}^*$$

the completion of $\mathfrak{g}_E^*$ with respect to the Sobolev norm $\|-\|_l$.

Lemma 4.1 *There is a constant $C>0$ independent of n such that*

$$\|m_n\|_l < C^n, \quad \|I_n\|_l < C^n, \quad \|P_n\|_l < C^n.$$

Here $\|-\|_l$ for linear maps mean the operator norm with respect to the norm $\|-\|_l$ on $\mathfrak{g}_E^*$ or $\mathrm{Ext}^*(E,E)$.

Proof When E is a vector bundle, the lemma is proved in [11, Proposition 2.3.2] and [34, Lemmas A.1.1, A.1.2 and A.1.5]. The key ingredient of the proof is that the maps m_n, I_n and P_n are constructed as in (4-21) using rooted trees, whose cardinality is bounded as

$$\#\mathcal{O}(n) = \frac{(2n-2)!}{(n-1)!n!} < 4^{n-1},$$

and the fact that the homotopy operator h, the product map on $\mathfrak{g}_E^*$ are extended to bounded operators

$$\widehat{\mathfrak{g}}_{E,l}^* \xrightarrow{h} \widehat{\mathfrak{g}}_{E,l}^*, \quad \widehat{\mathfrak{g}}_{E,l}^* \times \widehat{\mathfrak{g}}_{E,l}^* \xrightarrow{\cdot} \widehat{\mathfrak{g}}_{E,l}^*.$$

When E is a coherent sheaf which is not necessary a vector bundle, the above property still hold for the complex (4-12) without any modification: the boundedness of h is a general fact for elliptic complexes (see [35, Theorem 4.12]), as it is an operator of degree -1 given by (4-20), and that of the product $\cdot$ follows from our choice of $l \gg 0$ and a standard result of Sobolev spaces (for example, see [36, Theorem 25]). Therefore the same argument for the vector bundle case proves the lemma. □

4.5 Deformations by A_∞–algebras

For $x \in \mathrm{Ext}^1(E,E)$, we consider the infinite series

(4-24) $$\kappa(x) := \sum_{n \geq 2} m_n(x,\dots,x)$$

where each term $m_n(x,\dots,x)$ is an element of $\mathrm{Ext}^2(E,E)$. By Lemma 4.1, there is an analytic open neighborhood

(4-25) $$0 \in \mathcal{U} \subset \mathrm{Ext}^1(E,E)$$

such that the series (4-24) absolutely converges on $\mathcal{U}$ to give a complex analytic morphism

$$\kappa\colon \mathcal{U} \to \mathrm{Ext}^2(E, E). \tag{4-26}$$

The equation $\kappa(x) = 0$ is the Mauler–Cartan equation for the A_∞–algebra (4-17). We set T to be

$$T := \kappa^{-1}(0) \subset \mathcal{U}; \tag{4-27}$$

ie T is the closed complex analytic subspace defined by the ideal of zero of the map (4-26).

On the other hand, for $x \in \mathrm{Ext}^1(E, E)$ we also consider the infinite series

$$I_*(x) := \sum_{n \geq 1} I_n(x, \dots, x), \tag{4-28}$$

where each term $I_n(x, \dots, x)$ is an element of $\mathfrak{g}_E^1$. By Lemma 4.1, for a sufficiently small open subset (4-25) the series (4-28) absolutely converges on $\mathcal{U}$ to give a morphism of Banach analytic spaces

$$I_*\colon \mathcal{U} \to \hat{\mathfrak{g}}^1_{E,l}. \tag{4-29}$$

Lemma 4.2 *The morphism* (4-29) *restricts to the morphism of Banach analytic spaces*

$$I_*\colon T \to \mathrm{MC}(\mathfrak{g}_E^*). \tag{4-30}$$

Here $\mathrm{MC}(\mathfrak{g}_E^*)$ *is the solution of the Maurer–Cartan equation* (4-7) *of the dg-algebra* $\mathfrak{g}_E^*$.

Proof The result is proved in [34, Section 2.2, Lemma A.1.3] when E is a vector bundle, and the same argument applies for the complex (4-12). Since I_* is an A_∞–homomorphism, it preserves the MC locus, so it sends T to $\mathrm{MC}(\hat{\mathfrak{g}}^*_{E,l})$. For $x \in T$, the smoothness of $I_*(x)$ follows along with the argument of [34, Lemma A.1.3], by replacing $\bar{\partial}$ in [loc. cit.] by the differential $d_{\mathfrak{g}}$ of $\hat{\mathfrak{g}}^*_{E,l}$. Therefore we obtain the morphism (4-30). □

Let $\mathcal{M}$ be the moduli stack of coherent sheaves on X, and we regard it as a complex analytic stack. The above lemma implies the following proposition:

Proposition 4.3 *By shrinking* $\mathcal{U}$ *if necessary, the morphism* (4-29) *induces the morphism of complex analytic stacks*

$$I_*\colon T \to \mathcal{M}. \tag{4-31}$$

Proof The map in Lemma 4.2 corresponds to the element

$$\alpha \in \mathfrak{g}_E^1 \otimes \Gamma(\mathcal{O}_T)$$

satisfying the MC equation of the dg-algebra $\mathfrak{g}_E^* \otimes \Gamma(\mathcal{O}_T)$. Then we obtain the dg-$\mathcal{A}^{0,*}(\mathcal{O}_X) \otimes \mathcal{O}_T$–module

$$(\mathcal{A}^{0,*}(\mathcal{E}^\bullet) \otimes \mathcal{O}_T, d_{\mathcal{A}^{0,*}(\mathcal{E}^\bullet)\otimes\mathcal{O}_T} + \alpha). \tag{4-32}$$

Here $\mathcal{E}^\bullet$ is the complex (4-12).

The dg-module (4-32) is a bounded complex of $\mathcal{O}_{X\times T}$–modules. We can show that each cohomology of (4-32) is a coherent $\mathcal{O}_{X\times T}$–module as in [4, Lemma 4.1.5], which essentially follows the argument in [8, pages 51–52]. Indeed for each $t \in T$ and $x \in X$, by the proof of [4, Lemma 4.1.5] there is an open neighborhood $x \in U$ such that there is a degree-zero C^∞–isomorphism

$$\phi_t \colon \mathcal{A}^{0,*}(\mathcal{E}^\bullet)|_U \xrightarrow{\cong} \mathcal{A}^{0,*}(\mathcal{E}^\bullet)|_U \tag{4-33}$$

satisfying

$$\phi_t^{-1} \circ (d_{\mathcal{A}^{0,*}(\mathcal{E}^\bullet)} + \alpha_t) \circ \phi_t = d_{\mathcal{A}^{0,*}(\mathcal{E}^\bullet)} + \beta_t.$$

Here in the notation (4-9), β_t is of the form

$$\beta_t = ((\beta_0^i)_t, 0, 0, \dots), \quad (\beta_0^i)_t \in \operatorname{Hom}(\mathcal{E}^i|_U, \mathcal{E}^{i+1}|_U).$$

This implies that the dg-module (4-32) restricted to $U \times \{t\}$ is gauge-equivalent to a complex which is quasi-isomorphic to a bounded complex of holomorphic vector bundles on U. The isomorphism (4-33) can be found by solving a certain differential equation, as in [8, pages 51–52]. As remarked in [8, page 52], the solution ϕ_t is analytic in $t \in T$ as α_t is. Therefore by shrinking U and T if necessary we see that (4-32) restricted to $U \times T$ is gauge-equivalent to a complex which is quasi-isomorphic to a bounded complex of analytic vector bundles on $U \times T$. In particular, each cohomology of (4-32) is coherent.

Therefore (4-32) determines an object

$$\mathcal{E}_T^\bullet \in D^b_{\mathrm{Coh}(X\times T)}(\mathrm{Mod}\,\mathcal{O}_{X\times T}).$$

We show that by shrinking $\mathcal{U}$ if necessary, the object $\mathcal{E}_T^\bullet$ is quasi-isomorphic to a T–flat sheaf

$$E_T := \mathcal{H}^0(\mathcal{E}_T^\bullet) \in \mathrm{Coh}(X \times T).$$

By the construction of $\mathcal{E}_T^\bullet$, at $t=0$ we have $\mathcal{E}_T^\bullet \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_T} \mathcal{O}_{\{0\}} \cong E$. We have the spectral sequence

$$E_2^{p,q} = \mathcal{T}\mathrm{or}_{-p}^{\mathcal{O}_{X\times T}}(\mathcal{H}^q(\mathcal{E}_T^\bullet), \mathcal{O}_{X\times\{0\}}) \Rightarrow \mathcal{H}^{p+q}(E).$$

Let q_0 be the maximal $q\in\mathbb{Z}$ such that $\mathcal{H}^q(\mathcal{E}_T^\bullet)\neq 0$. If $q_0>0$, then by the above spectral sequence we have $\mathcal{H}^{q_0}(\mathcal{E}_T^\bullet)|_{t=0}=0$. Therefore by shrinking $\mathcal{U}$, we have $q_0\leq 0$, and as $E\neq 0$ it follows that $q_0=0$ by the above spectral sequence. Moreover we have $E_2^{-1,0}=0$, which implies that E_T is flat at $t=0$, hence $E_2^{p,0}=0$ for any $p<0$. Then by the above spectral sequence again, we have $E_2^{0,-1}=0$, hence we may assume $\mathcal{H}^{-1}(\mathcal{E}_T^\bullet)=0$. Inductively, by shrinking $\mathcal{U}$ we see that $\mathcal{H}^q(\mathcal{E}_T^\bullet)=0$ for any $q<0$. Therefore the above claim holds.

By the universal property of $\mathcal{M}$, the sheaf E_T defines the morphism (4-31). □

Proposition 4.4 *The morphism of complex analytic stacks $I_*\colon T\to\mathcal{M}$ in* (4-31) *is smooth of relative dimension* $\dim\mathrm{Aut}(E)$.

Proof We first show that $I_*\colon T\to\mathcal{M}$ is smooth. Let (S,s) be a complex analytic space and $(S,s)\to(\mathcal{M},[E])$ a morphism of complex analytic stacks which sends s to $[E]$. It is enough to show that, after replacing S by its open neighborhood at $s\in S$ if necessary, we have the factorization

$$(S,s)\to(T,0)\xrightarrow{I_*}(\mathcal{M},[E]). \tag{4-34}$$

By shrinking S if necessary, we may assume that $S\to\mathcal{M}$ factors through

$$(S,s)\xrightarrow{f_1}(A,p)\to(\mathcal{M},[E]),$$

where the right morphism is the local atlas in (4-13). Let $\mathcal{E}_A^\bullet$ be the complex on $X\times A$ constructed in (4-14). By pulling $\mathcal{E}_A^\bullet$ back by f_1^*, we obtain the complex

$$\mathcal{E}_S^\bullet = f_1^*\mathcal{E}_A^\bullet.$$

Then as described in Section 4.1, the complex structures of each term of $\mathcal{E}_S^\bullet$ and their differentials give rise to the solution of the MC equation of the dg-algebra $\mathfrak{g}_E^*\otimes\mathcal{O}_S(S)$. Thus we obtain a map of Banach analytic spaces

$$f_2\colon (S,s)\to(\mathrm{MC}(\mathfrak{g}_E^*),0).$$

We are left to prove the existence of the morphism $f_3\colon (S,s)\to(T,0)$ such that the composition

$$(S,s)\xrightarrow{f_3}(T,0)\xrightarrow{I_*}(\mathrm{MC}(\mathfrak{g}_E^*),0)$$

differs from f_2 only up to gauge equivalence. The existence of such f_3 is proved in [34, Theorem 2.2.2] when E is a vector bundle, and the same argument applies for the complex of vector bundles (4-12). Below we give an outline of the proof.

For $y \in \mathfrak{g}_E^1$, consider the series

$$P_*(y) := \sum_{n \geq 1} P_n(y, \dots, y),$$

where P is the homotopy inverse of I in (4-23). By Lemma 4.1, there is an open neighborhood $0 \in \mathcal{U}' \subset \mathfrak{g}_E^1$ in $\|-\|_l$–norm such that P_* gives the analytic map

$$P_*\colon \mathcal{U}' \to \operatorname{Ext}^1(E, E).$$

Since P is an A_∞–homomorphism, after shrinking $\mathcal{U}'$ if necessary the above map induces the morphism of Banach analytic spaces

$$P_*\colon \operatorname{MC}(\mathfrak{g}_E^*) \cap \mathcal{U}' \to T.$$

Therefore by shrinking S if necessary so that $f_2(S) \subset \mathcal{U}'$, we have the analytic map

$$f_3 = P_* \circ f_2\colon (S, s) \to (T, 0).$$

It remains to show that two maps

$$I_* \circ f_3 = I_* \circ P_* \circ f_2, \quad f_2\colon (S, s) \to (\operatorname{MC}(\mathfrak{g}_E^*), 0)$$

are gauge-equivalent. Since P is a homotopy inverse of I, we have

$$H\colon \mathfrak{g}_E^* \to \mathfrak{g}_E^* \otimes \Omega_{[0,1]}^*,$$

an A_∞–homomorphism with $H(0) = \mathrm{id}$ and $H(1) = I \circ P$. Then H also satisfies the boundedness property as in Lemma 4.1 (see [34, Corollary A.2.7]), so that after shrinking $\mathcal{U}'$ if necessary the A_∞–homomorphism H induces the analytic map

$$H_*\colon \operatorname{MC}(\mathfrak{g}_E^*) \cap \mathcal{U}' \to \operatorname{MC}(\mathfrak{g}_E^* \otimes \Omega_{[0,1]}^*).$$

Then the analytic map

$$H_* \circ f_2\colon S \to \operatorname{MC}(\mathfrak{g}_E^* \otimes \Omega_{[0,1]}^*)$$

satisfies

$$H_* \circ f_2(0) = f_2, \quad H_* \circ f_2(1) = I_* \circ P_* \circ f_2.$$

This implies that f_2 and $I_* \circ P_* \circ f_2$ are gauge-equivalent in the sense of [11, Definition 2.2.2]. As proved in [11, Lemma 2.2.2], this notion of gauge equivalence coincides with the gauge equivalence in (4-11). Therefore the smoothness of I_* follows.

Finally, the relative dimension of $I_*\colon T \to \mathcal{M}$ is $\dim \operatorname{Aut}(E)$ since the dimension of the tangent space of T at 0 is $\dim \operatorname{Ext}^1(E,E)$, and that of $\mathcal{M}$ at $[E]$ is $\dim \operatorname{Ext}^1(E,E) - \dim \operatorname{Aut}(E)$. □

5 Local descriptions of moduli stacks of semistable sheaves

In this section, we use the results in the previous sections to prove Theorem 3.2. By applying the arguments to the CY 3–fold case, we also obtain Corollary 5.7.

5.1 Convergent relation of the Ext–quiver

For a smooth projective variety X, let

$$E_\bullet = (E_1, \dots, E_k)$$

be a simple collection of coherent sheaves on X, and $Q_{E_\bullet}$ the associated Ext–quiver; see Section 3.3. Here we construct a convergent relation of $Q_{E_\bullet}$ from the minimal A_∞–structure on the derived category of coherent sheaves on X.

Let us consider the sheaf on X of the form

$$E = \bigoplus_{i=1}^{k} V_i \otimes E_i \tag{5-1}$$

for vector spaces V_i, and set $m_i = \dim V_i$. Note that we have the decomposition

$$\operatorname{Ext}^*(E,E) = \bigoplus_{1 \le a,b \le k} \operatorname{Hom}(V_a, V_b) \otimes \operatorname{Ext}^*(E_a, E_b). \tag{5-2}$$

Let us take a resolution $\mathcal{E}^\bullet \to E$ as in (4-12). From its construction, it naturally decomposes into the direct sum of resolutions of E_i. Namely, let

$$0 \to \mathcal{E}_i^{-N} \xrightarrow{d_i^{-N}} \cdots \to \mathcal{E}_i^{-1} \xrightarrow{d_i^{-1}} \mathcal{E}_i^0 \to E_i \to 0$$

be the resolution (4-12) applied for E_i. By taking $N \gg 0$, we may assume that N is independent of i. Then the complex $\mathcal{E}^\bullet$ in (4-12) is

$$\mathcal{E}^\bullet = \bigoplus_{i=1}^{k} V_i \otimes \mathcal{E}_i^\bullet.$$

Therefore we have the decompositions

$$\mathfrak{g}_E^* = \bigoplus_{1\le a,b\le k} \mathrm{Hom}(V_a, V_b) \otimes A^{0,*}(\mathcal{H}\mathrm{om}^*(\mathcal{E}_a^\bullet, \mathcal{E}_b^\bullet)). \tag{5-3}$$

Here $\mathfrak{g}_E^*$ is the dg-algebra (4-15), defined via the above complex $\mathcal{E}^\bullet$. The decomposition of $\mathfrak{g}_E^*$ is compatible with the Laplacian operator Δ. Indeed each complex $\mathcal{A}^{0,*}(\mathcal{H}\mathrm{om}^*(\mathcal{E}_a^\bullet, \mathcal{E}_b^\bullet))$ is elliptic and hence we have linear operators

$$\begin{aligned}
i_{a,b}&\colon \mathrm{Ext}^*(E_a, E_b) \hookrightarrow A^{0,*}(\mathcal{H}\mathrm{om}^*(\mathcal{E}_a^\bullet, \mathcal{E}_b^\bullet)),\\
p_{a,b}&\colon A^{0,*}(\mathcal{H}\mathrm{om}^*(\mathcal{E}_a^\bullet, \mathcal{E}_b^\bullet)) \twoheadrightarrow \mathrm{Ext}^*(E_a, E_b),\\
h_{a,b}&\colon A^{0,*}(\mathcal{H}\mathrm{om}^*(\mathcal{E}_a^\bullet, \mathcal{E}_b^\bullet)) \to A^{0,*-1}(\mathcal{H}\mathrm{om}^*(\mathcal{E}_a^\bullet, \mathcal{E}_b^\bullet))
\end{aligned}$$

satisfying the same relations as (4-19) and

$$\star = \bigoplus_{1\le a,b\le k} \mathrm{id}_{\mathrm{Hom}(V_a,V_b)} \otimes \star_{a,b}, \tag{5-4}$$

where $\star$ is either i or p or h as given in Section 4.4.

Let $\bar{E}$ be the coherent sheaf on X defined by

$$\bar{E} := \bigoplus_{i=1}^{k} E_i \tag{5-5}$$

and consider the A_∞–product

$$m_n\colon \mathrm{Ext}^1(\bar{E}, \bar{E})^{\otimes n} \to \mathrm{Ext}^2(\bar{E}, \bar{E}). \tag{5-6}$$

By the relation (5-4) and the explicit formula (4-17) of the A_∞–product, the map (5-6) only consists of the direct sum factors of the form

$$\begin{aligned}
m_n\colon \mathrm{Ext}^1(E_{\psi(1)}, E_{\psi(2)}) \otimes \mathrm{Ext}^1(E_{\psi(2)}, E_{\psi(3)}) \otimes \cdots \otimes \mathrm{Ext}^1(E_{\psi(n)}, E_{\psi(n+1)})\\
\to \mathrm{Ext}^2(E_{\psi(1)}, E_{\psi(n+1)})
\end{aligned} \tag{5-7}$$

for maps $\psi\colon \{1,\dots,n+1\} \to \{1,\dots,k\}$, which give a minimal A_∞–category structure on the dg-category generated by $(E_1,\dots,E_k)$. By taking the dual and the products of (5-7) for all $n \ge 2$, we obtain the linear map

$$\begin{aligned}
\boldsymbol{m}^\vee := \prod_{n\ge 2} m_n^\vee\colon \mathrm{Ext}^2(\bar{E}, \bar{E})^\vee\\
\to \prod_{n\ge 2} \bigoplus_{\substack{\psi\colon \{1,\dots,n+1\}\\ \to\{1,\dots,k\}}} \mathrm{Ext}^1(E_{\psi(1)}, E_{\psi(2)})^\vee \otimes \cdots \otimes \mathrm{Ext}^1(E_{\psi(n)}, E_{\psi(n+1)})^\vee.
\end{aligned}$$

Note that an element of the right-hand side is an element of $\mathbb{C}[\![Q_{E_\bullet}]\!]$ by (2-6). Let $\{\boldsymbol{o}_1, \dots, \boldsymbol{o}_l\}$ be a basis of $\mathrm{Ext}^2(\bar{E}, \bar{E})^\vee$ and set

$$f_i = \boldsymbol{m}^\vee(\boldsymbol{o}_i) \in \mathbb{C}[\![Q_{E_\bullet}]\!].$$

Then by Lemma 4.1, we have $f_i \in \mathbb{C}\{Q_{E_\bullet}\}$. We obtain the convergent relation of $Q_{E_\bullet}$

$$I_{E_\bullet} := (f_1, \dots, f_l). \tag{5-8}$$

5.2 Deformations of direct sums of simple collections

We consider the deformations of sheaves of the form (5-1). By the decomposition (5-2), the space $\mathrm{Ext}^1(E, E)$ is identified with the space of $Q_{E_\bullet}$–representations

$$\mathrm{Ext}^1(E, E) = \mathrm{Rep}_{Q_{E_\bullet}}(\vec{m}). \tag{5-9}$$

Here $\vec{m}$ is the dimension vector of $Q_{E_\bullet}$ given by $m_i = \dim V_i$. We also have

$$G = \mathrm{Aut}(E) = \prod_{i=1}^{k} \mathrm{GL}(V_i), \tag{5-10}$$

and the adjoint action of $\mathrm{Aut}(E)$ on $\mathrm{Ext}^1(E, E)$ coincides with the action (2-4) under the identification (5-9). Recall that in (4-26) and (4-29), we constructed analytic maps

$$\kappa\colon \mathcal{U} \to \mathrm{Ext}^2(E, E), \quad I_*\colon \mathcal{U} \to \hat{\mathfrak{g}}^*_{E,l} \tag{5-11}$$

for a sufficiently small analytic open subset $0 \in \mathcal{U} \subset \mathrm{Ext}^1(E, E)$. Explicitly under the identification (5-9), for a $Q_{E_\bullet}$–representation

$$u = (u_e)_{e \in E(Q_{E_\bullet})} \in \mathcal{U}, \quad u_e\colon V_{s(e)} \to V_{t(e)},$$

we have the following identities by the decompositions (5-2), (5-3) and (5-4):

$$\begin{aligned} \kappa(u) &= \sum_{\substack{n \geq 2,\\ \psi\colon\{1,\dots,n+1\} \to \{1,\dots,k\}}} \sum_{e_i \in E_{\psi(i),\psi(i+1)}} m_n(e_1^\vee, \dots, e_n^\vee) \cdot u_{e_n} \circ \cdots \circ u_{e_2} \circ u_{e_1}, \\ I_*(u) &= \sum_{\substack{n \geq 2,\\ \psi\colon\{1,\dots,n+1\} \to \{1,\dots,k\}}} \sum_{e_i \in E_{\psi(i),\psi(i+1)}} I_n(e_1^\vee, \dots, e_n^\vee) \cdot u_{e_n} \circ \cdots \circ u_{e_2} \circ u_{e_1}. \end{aligned} \tag{5-12}$$

Here for $e \in E_{i,j}$, the element $e^\vee \in \mathrm{Ext}^1(E_i, E_j)$ is defined as in (2-18).

Lemma 5.1 *There is a saturated open subset $\mathcal{V}$ in $\operatorname{Ext}^1(E,E)$ with respect to the G–action on $\operatorname{Ext}^1(E,E)$, satisfying*

$$0 \in \mathcal{V} \subset G\cdot\mathcal{U} \subset \operatorname{Ext}^1(E,E)$$

such that the maps in (5-11) induce G–equivariant analytic maps

$$\kappa\colon \mathcal{V} \to \operatorname{Ext}^2(E,E), \quad I_*\colon \mathcal{V} \to \hat{\mathfrak{g}}^*_{E,l}.$$

Here G acts on $\operatorname{Ext}^2(E,E)$ and $\hat{\mathfrak{g}}^*_{E,l}$ by adjoint.

Proof The formal series κ and I_* in (5-12) are obviously G–equivariant. Therefore for a choice of $\mathcal{U}$ in (4-26) and (4-29), the maps κ and I_* can be extended to analytic maps

$$\kappa\colon G\cdot\mathcal{U} \to \operatorname{Ext}^2(E,E), \quad I_*\colon G\cdot\mathcal{U} \to \hat{\mathfrak{g}}^*_{E,l}.$$

By Lemma 2.5, there is a saturated analytic open subset $\mathcal{V} \subset G\cdot\mathcal{U}$ which contains $0 \in \operatorname{Ext}^1(E,E)$, so the lemma follows. □

Let $\mathcal{V} \subset \operatorname{Ext}^1(E,E)$ be as in Lemma 5.1. By Lemma 2.4, it is written as

$$\mathcal{V} = \pi^{-1}_{Q_{E_\bullet}}(V)$$

for some analytic open subset $0 \in V \subset M_{Q_{E_\bullet}}(\vec{m})$, where $\pi_{Q_{E_\bullet}}$ is the quotient map

$$\pi_{Q_{E_\bullet}}\colon \operatorname{Rep}_{Q_{E_\bullet}}(\vec{m}) \to M_{Q_{E_\bullet}}(\vec{m}).$$

Let $R \subset \mathcal{V}$ be the closed analytic subspace given by

$$R := \kappa^{-1}(0) \subset \mathcal{V} \subset \operatorname{Ext}^1(E,E).$$

By the definition of $I_{E_\bullet}$ in (5-8), under the identification (5-9) we have

$$R = \operatorname{Rep}_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V.$$

Here we have used the notation (2-17) for the right-hand side. Therefore in the notation of Definition 2.16, we have

$$\mathcal{M}_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V = [R/G].$$

Lemma 5.2 *By shrinking V if necessary, the map I_* in Lemma 5.1 induces the smooth morphism of relative dimension zero*

$$I_*\colon \mathcal{M}_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V \to \mathcal{M}. \tag{5-13}$$

Here $\mathcal{M}$ is the moduli stack of coherent sheaves on X.

Proof By Lemma 4.2 and Proposition 4.4, the map I_* in Lemma 5.1 gives the analytic maps

$$I_*\colon R \cap \mathcal{U} \to \mathrm{MC}(\mathfrak{g}_E^*), \quad I_*\colon R \cap \mathcal{U} \to \mathcal{M}.$$

Then by the G–equivalence of I_* and the property $\mathcal{V} \subset G \cdot \mathcal{U}$ in Lemma 5.1, the above maps extend to the G–equivariant analytic maps

$$I_*\colon R \to \mathrm{MC}(\mathfrak{g}_E^*), \quad I_*\colon R \to \mathcal{M}. \tag{5-14}$$

Here the right map is induced by the left map as in the proof of Proposition 4.4. By the G–equivalence of I_*, the right map of (5-14) descends to the quotient by G to induce (5-13), which is of relative dimension zero by Lemma 5.1. □

5.3 Functoriality of I_*

In this subsection, by the explicit description (5-12) of the map I_* in Proposition 5.4, we see that it has some functorial property. In particular, it implies that I_* sends subsheaves to subrepresentations of Ext–quivers. This fact will not be used in the rest of this section, but will be used in the proof of Theorem 6.8, which will be used in Theorem 7.7 to compare stability conditions of sheaves and quiver representations.

For each $i \in V(Q_{E_\bullet}) = \{1, 2, \dots, k\}$, let V_i and V_i' be vector spaces with dimensions m_i and m_i', and set

$$E = \bigoplus_{i=1}^{k} V_i \otimes E_i, \quad E' = \bigoplus_{i=1}^{k} V_i' \otimes E_i.$$

Let us take

$$u = (u_e)_{e \in E(Q_{E_\bullet})}, \quad u' = (u_e')_{e \in E(Q_{E_\bullet})}, \tag{5-15}$$

where u_e and u_e' are linear maps

$$u_e\colon V_{s(e)} \to V_{t(e)}, \quad u_e'\colon V_{s(e)}' \to V_{t(e)}'$$

whose operator norms are sufficiently small that they give $Q_{E_\bullet}$–representations satisfying the relation $I_{E_\bullet}$. Let $\phi_i\colon V_i \to V_i'$ be linear maps for $1 \le i \le k$ such that the following diagram commutes for each $e \in E(Q_{E_\bullet})$:

$$\begin{array}{ccc} V_{s(e)} & \xrightarrow{u_e} & V_{t(e)} \\ {\scriptstyle \phi_{s(e)}}\downarrow & & \downarrow{\scriptstyle \phi_{t(e)}} \\ V_{s(e)}' & \xrightarrow[u_e']{} & V_{t(e)}' \end{array}$$

Then each term of

$$I_*(u) \in \mathrm{MC}(\mathfrak{g}_E^*), \quad I_*(u') \in \mathrm{MC}(\mathfrak{g}_{E'}^*) \tag{5-16}$$

in (5-12) satisfies

$$I_n(e_1^\vee, \dots, e_n^\vee) \cdot \phi_{t(e_n)} \circ u_{e_n} \circ \cdots \circ u_{e_1} = I_n(e_1^\vee, \dots, e_n^\vee) \cdot u'_{e_n} \circ \cdots \circ u'_{e_1} \circ \phi_{s(e_1)}.$$

This implies that the map

$$\bigoplus_{i=1}^k \phi_i \otimes \mathrm{id}\colon \Big(\mathcal{A}^{0,*}\Big(\bigoplus_{i=1}^k V_i \otimes \mathcal{E}_i^\bullet\Big), d_{\mathcal{A}^{0,*}(\bigoplus_{i=1}^k V_i \otimes \mathcal{E}_i^\bullet)} + I_*(u)\Big) \\ \to \Big(\mathcal{A}^{0,*}\Big(\bigoplus_{i=1}^k V'_i \otimes \mathcal{E}_i^\bullet\Big), d_{\mathcal{A}^{0,*}(\bigoplus_{i=1}^k V'_i \otimes \mathcal{E}_i^\bullet)} + I_*(u')\Big)$$

is a map of dg-$\mathcal{A}^{0,*}(\mathcal{O}_X)$–modules. By taking the cohomology of the above map, we obtain the morphism of coherent sheaves

$$\mathcal{H}^0\Big(\bigoplus_{i=1}^k \phi_i \otimes \mathrm{id}\Big)\colon E_u \to E_{u'}. \tag{5-17}$$

Here E_u and $E_{u'}$ are coherent sheaves corresponding to u and u' under the map in Proposition 4.3, respectively.

Remark 5.3 In the above argument, we assumed that the operator norms of u and u' are small enough that I_* is defined. We can relax this condition in the following cases. First suppose that each ϕ_i is injective or surjective. Then the operator norm of u is bounded by that of u', so if the operator norm of u' is enough small then so is u, and $I_*(u)$ is defined. Next if u and u' correspond to nilpotent $Q_{E_\bullet}$–representations, then whatever the operator norms of u and u', the infinite sums $I_*(u)$ and $I_*(u')$ in (5-12) are finite sums. So in the above cases, E_u, $E_{u'}$ and the morphism (5-17) are well defined.

5.4 Étale slice

Below we use the notation in Section 3.2. Let $\mathcal{M}_\omega(v)$ be the moduli stack of ω–Gieseker semistable sheaves on X with Chern character v, $M_\omega(v)$ its coarse moduli space. Let E be a polystable sheaf of the form (3-3), and take closed points

$$p = [E] \in M_\omega(v), \quad p' = [E] \in \mathcal{M}_\omega(v).$$

For $m \gg 0$, let $\boldsymbol{V}$ be the vector space given by

$$\boldsymbol{V} = H^0(E(m)) = \bigoplus_{i=1}^{k} V_i \otimes H^0(E_i(m)).$$

Let $q \in \mathrm{Quot}^\circ(\boldsymbol{V}, v)$ be a point which is mapped to p' under the quotient morphism $\mathrm{Quot}^\circ(\boldsymbol{V}, v) \to \mathcal{M}_\omega(v)$. Then we have

$$\mathrm{Stab}_{\mathrm{GL}(\boldsymbol{V})}(q) = G \subset \mathrm{GL}(\boldsymbol{V}),$$

where G is given as in (5-10). By Luna's étale slice theorem [23], there is an affine locally closed G–invariant subscheme

$$q \in Z \subset \mathrm{Quot}^\circ(\boldsymbol{V}, v)$$

such that the natural $\mathrm{GL}(\boldsymbol{V})$–equivariant morphism

$$\mathrm{GL}(\boldsymbol{V}) \times_G Z \to \mathrm{Quot}^\circ(\boldsymbol{V}, v)$$

is étale. Moreover by taking the quotients by $\mathrm{GL}(\boldsymbol{V})$, we obtain the Cartesian diagram

(5-18)
$$\begin{array}{ccc} [Z/G] & \longrightarrow & \mathcal{M}_\omega(v) \\ {\scriptstyle p_Z}\downarrow & \square & \downarrow{\scriptstyle p_M} \\ Z /\!\!/ G & \longrightarrow & M_\omega(v) \end{array}$$

such that each horizontal arrows are étale. Therefore there is a saturated analytic open subset $\mathcal{W} \subset Z$ (with respect to the G–action on Z) which contains q and the Cartesian diagram of complex analytic stacks

$$\begin{array}{ccc} [\mathcal{W}/G] & \longrightarrow & \mathcal{M}_\omega(v) \\ {\scriptstyle p_{\mathcal{W}}}\downarrow & \square & \downarrow{\scriptstyle p_M} \\ \mathcal{W} /\!\!/ G & \longrightarrow & M_\omega(v) \end{array}$$

such that each horizontal arrows are analytic open immersions.

On the other hand, let us consider the morphism I_* in Lemma 5.2 applied for the above polystable sheaf $p' = [E] \in \mathcal{M}_\omega(v)$. By the openness of stability, by shrinking $\mathcal{U}$ in Lemma 5.1 if necessary, the map I_* in Lemma 5.2 factors through the open substack $\mathcal{M}_\omega(v) \subset \mathcal{M}$:

(5-19)
$$I_*\colon \mathcal{M}_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V \to \mathcal{M}_\omega(v).$$

Now the following proposition completes the proof of Theorem 3.2.

Proposition 5.4 *By shrinking $\mathcal{V}$ in Lemma 5.1 and $\mathcal{W}$ if necessary (while keeping the condition to be saturated in* $\mathrm{Ext}^1(E,E)$, Z *respectively) the map* (5-19) *induces the commuting isomorphisms*

$$\begin{array}{ccc} [R/G] = \mathcal{M}_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V & \xrightarrow[\cong]{I_*} & [\mathcal{W}/G] \\ \downarrow{\scriptstyle p_Q} & & \downarrow{\scriptstyle p_{\mathcal{W}}} \\ R /\!\!/ G = M_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V & \xrightarrow[\cong]{} & \mathcal{W} /\!\!/ G \end{array} \tag{5-20}$$

Proof The map (5-19) induces the analytic map $R/\!\!/G \to M_\omega(v)$. So by shrinking $0 \in V \subset M_{Q_{E_\bullet}}(\vec{m})$ if necessary, we may assume that the above map factors through $R/\!\!/G \to \mathcal{W}/\!\!/G$. Then we have the commutative diagram

$$\begin{array}{ccc} [R/G] & \xrightarrow{I_*} & [\mathcal{W}/G] \\ \downarrow{\scriptstyle p_Q} & & \downarrow{\scriptstyle p_{\mathcal{W}}} \\ R/\!\!/G & \longrightarrow & \mathcal{W}/\!\!/G \end{array}$$

Let $K \subset G$ be a maximal compact subgroup, and take a sufficiently small K–invariant analytic open subset $q \in \mathcal{W}_1 \subset \mathcal{W}$. Then as in the proof of Proposition 4.4, the composition

$$\mathcal{W}_1 \to \mathcal{W} \to [\mathcal{W}/G] \subset \mathcal{M}_\omega(v)$$

admits a lift $\phi \colon \mathcal{W}_1 \to R$ using the homotopy inverse P of I. Moreover, that proof immediately implies that ϕ can taken to be K–equivariant. (Indeed if the map f_2 in that proof is K–equivariant, then so is f_3 as P_* is K–equivariant.) So we have the commutative diagram

$$\begin{array}{ccc} R & \xleftarrow{\phi} & \mathcal{W}_1 \\ \downarrow & & \downarrow \\ [R/G] & \xrightarrow{I_*} & [\mathcal{W}/G] \end{array} \tag{5-21}$$

Note that the bottom arrow is a smooth morphism of relative dimension zero by Lemma 5.2. Let $0 \in R_1 \subset R$ be a sufficiently small K–invariant analytic open neighborhood. Since both R_1 and $\mathcal{W}_1$ are the bases of versal families of flat deformations of E with tangent space $\operatorname{Ext}^1(E, E)$, and ϕ is an isomorphism at the tangent by the diagram (5-21), the K–equivariant map ϕ gives an isomorphism $\psi\colon \mathcal{W}_1 \xrightarrow{\cong} R_1$ for some suitable choices of $\mathcal{W}_1$ and R_1. By setting $\psi = \phi^{-1}$, we obtain the commutative diagram

$$\begin{array}{ccc} R_1 & \xrightarrow[\cong]{\psi} & \mathcal{W}_1 \\ \downarrow & & \downarrow \\ [R/G] & \xrightarrow{I_*} & [\mathcal{W}/G] \end{array} \tag{5-22}$$

By Lemma 5.5 below, after shrinking R_1 if necessary we can extend the K–equivariant isomorphism $\psi\colon R_1 \xrightarrow{\cong} \mathcal{W}_1$ to a G–equivariant isomorphism between G–invariant open subsets in R and $\mathcal{W}$

$$\widetilde{\psi}\colon R_2 \xrightarrow{\cong} \mathcal{W}_2, \quad \text{where } R_2 := G \cdot R_1 \text{ and } \mathcal{W}_2 := G \cdot \mathcal{W}_1, \tag{5-23}$$

by sending $g \cdot x$ to $g \cdot \psi(x)$ for $g \in G$ and $x \in R_1$. Then by Lemma 5.6 below, the isomorphism (5-23) restricts to the isomorphism of saturated open subsets. By taking the quotients of G–actions, we obtain the desired isomorphisms (5-20). □

In the proof of the above proposition, we postponed the following two lemmas:

Lemma 5.5 *The map* (5-23) *is well defined and an isomorphism.*

Proof The lemma is essentially proved in the proof of [17, Theorem 5.5]. In order to show that (5-23) is well defined, it is enough to show that if $g_1 R_1 \cap g_2 R_1 \neq \varnothing$ for $g_1, g_2 \in G$, then we have the identity $g_1 \psi g_1^{-1} = g_2 \psi g_2^{-1}$ on $g_1 R_1 \cap g_2 R_1$. By applying g_2^{-1}, we may assume that $g_2 = 1$. Let $G' \subset G$ be the open subset given by

$$G' := \{g \in G : gR_1 \cap R_1 \neq \varnothing\}.$$

If we define

$$G'' := \{g \in G' : g\psi g^{-1} = \psi \text{ on } gR_1 \cap R_1\},$$

then G'' is a closed analytic subset of G' which contains K. Therefore if $(G')^\circ$ and $(G'')^\circ$ are the connected components of G' and G'' which contain K, then we have $(G')^\circ = (G'')^\circ$. Then we take a sufficiently small K–invariant open subset $0 \in R_1' \subset R_1$ satisfying the following: for any $x_1, x_2 \in R_1'$ with $G \cdot x_1 = G \cdot x_2$, the connected component of $(G \cdot x_1) \cap R_1$ containing x_1 should contain x_2. The above choice of R_1'

implies that

$$G''' := \{g \in G : gR'_1 \cap R'_1 \neq \varnothing\} \subset (G')^\circ.$$

Therefore as $(G')^\circ = (G'')^\circ$, for $g \in G'''$ we have $g\psi g^{-1} = \psi$ on $gR'_1 \cap R'_1 \neq \varnothing$. By replacing R_1 with R'_1, we see that (5-23) is well defined. Applying the above argument for the inverse of $\psi\colon R_1 \xrightarrow{\cong} \mathcal{W}_1$, we have the inverse of (5-23), showing that (5-23) is an isomorphism. □

Lemma 5.6 *There exist saturated open subsets $\widetilde{\mathcal{V}} \subset \mathrm{Ext}^1(E,E)$ and $\widetilde{\mathcal{W}} \subset Z$ satisfying $0 \in R \cap \widetilde{\mathcal{V}} \subset R_2$ and $q \in \widetilde{\mathcal{W}} \subset \mathcal{W}_2$ such that the isomorphism* (5-23) *restricts to the isomorphism*

$$\widetilde{\psi}\colon R \cap \widetilde{\mathcal{V}} \xrightarrow{\cong} \widetilde{\mathcal{W}}.$$

Proof Let $\mathcal{W}_3 \subset Z$ be a saturated open subset in Z satisfying $q \in \mathcal{W}_3 \subset \mathcal{W}_2$, which exists by Lemma 2.5, and set $R_3 := \widetilde{\psi}^{-1}(\mathcal{W}_3) \subset R_2$. Then R_3 is written as $R_3 = R \cap \mathcal{V}'$ for some G–invariant open subset $0 \in \mathcal{V}' \subset \mathcal{V}$. Let $\mathcal{V}'' \subset \mathrm{Ext}^1(E,E)$ be a saturated open subset satisfying $0 \in \mathcal{V}'' \subset \mathcal{V}'$, which again exists by Lemma 2.5, and set $R_4 := R \cap \mathcal{V}'' \subset R_3$. Let $\mathcal{W}_4 := \widetilde{\psi}(R_4)$. We show that $\mathcal{W}_4$ is a saturated open subset in Z. Indeed for $x \in \mathcal{W}_4$, the orbit closure $\overline{G \cdot x}$ in Z is contained in $\mathcal{W}_3$ since $\mathcal{W}_3$ is saturated. Take $y \in \overline{G \cdot x}$ and consider $\widetilde{\psi}^{-1}(y) \in R_3$. Then since $\mathcal{V}''$ is saturated, we have $\widetilde{\psi}^{-1}(y) \in R_4$, hence $y \in \mathcal{W}_4$ as desired. Now $\mathcal{V}''$ and $\mathcal{W}_4$ are saturated in $\mathrm{Ext}^1(E,E)$ and Z. By setting $\widetilde{\mathcal{V}} = \mathcal{V}''$ and $\widetilde{\mathcal{W}} = \mathcal{W}_4$, we obtain the lemma. □

5.5 Calabi–Yau 3–fold case

We keep the situation as in the previous subsections. Suppose furthermore that X is a smooth projective CY 3–fold; ie

$$\dim X = 3, \quad \mathcal{O}_X(K_X) \cong \mathcal{O}_X.$$

In this case, the A_∞–structure (5-6) is cyclic (see [29]); ie for a map

$$\psi\colon \{1, \ldots, n+1\} \to \{1, \ldots, k\}, \quad \psi(1) = \psi(n+1)$$

and elements

$$a_i \in \mathrm{Ext}^1(E_{\psi(i)}, E_{\psi(i+1)}), \quad 1 \le i \le n,$$

we have the relation

$$(m_{n-1}(a_1, \ldots, a_{n-1}), a_n) = (m_{n-1}(a_2, \ldots, a_n), a_1). \tag{5-24}$$

Here m_n is the A_∞–product (5-7), $(-,-)$ is the Serre duality pairing

$$(-,-)\colon \mathrm{Ext}^j(E_a,E_b)\times \mathrm{Ext}^{3-j}(E_b,E_a)\to \mathrm{Ext}^3(E_a,E_a)\xrightarrow{\int_X \mathrm{tr}} \mathbb{C}. \tag{5-25}$$

Let $W_{E_\bullet}\in \mathbb{C}[\![Q_{E_\bullet}]\!]$ be defined by

$$W_{E_\bullet} := \sum_{n\geq 3}\ \sum_{\substack{\psi\colon\{1,\dots,n+1\}\to\{1,\dots,k\}\\ \psi(1)=\psi(n+1)}}\ \sum_{e_i\in E_{\psi(i),\psi(i+1)}} a_{\psi,e_\bullet}\cdot e_1e_2\cdots e_n.$$

Here the coefficient $a_{\psi,e_\bullet}$ is given by

$$a_{\psi,e_\bullet} = \frac{1}{n}\big(m_{n-1}(e_1^\vee,e_2^\vee,\dots,e_{n-1}^\vee),e_n^\vee\big). \tag{5-26}$$

Then by Lemma 4.1, we have

$$W_{E_\bullet}\in \mathbb{C}\{Q_{E_\bullet}\}\subset \mathbb{C}[\![Q_{E_\bullet}]\!].$$

Therefore $W_{E_\bullet}$ determines a convergent superpotential of $Q_{E_\bullet}$ (see Definition 2.17).

Let $\bar{E}$ be the object given by (5-5). By the Serre duality, $\mathrm{Ext}^2(\bar{E},\bar{E})^\vee$ is identified with $\mathrm{Ext}^1(\bar{E},\bar{E})$. Thus

$$\{e^\vee : e\in E(Q_{E_\bullet})\}\subset \mathrm{Ext}^1(\bar{E},\bar{E}) \tag{5-27}$$

gives a basis of $\mathrm{Ext}^2(\bar{E},\bar{E})^\vee$. Using this basis, the relation $I_{E_\bullet}$ defined in (5-8) satisfies

$$I_{E_\bullet} = \{\boldsymbol{m}^\vee(e^\vee) : e\in E(Q_{E_\bullet})\} = \partial W_{E_\bullet}.$$

Here the first identity is due to the definition of $I_{E_\bullet}$ via the basis (5-27), and the second identity follows from the construction of $W_{E_\bullet}$ and the cyclic condition (5-24). As a corollary of Theorem 3.2, we obtain the following:

Corollary 5.7 *In the situation of Theorem 3.2, suppose furthermore that X is a smooth projective CY 3–fold. Then there is a convergent superpotential $W_{E_\bullet}$ of $Q_{E_\bullet}$, analytic open neighborhoods $p\in U\subset M_\omega(v)$, $0\in V\subset M_{Q_{E_\bullet}}(\vec{m})$ and commuting isomorphisms*

$$\begin{array}{ccccccc} p_M^{-1}(U) & \xleftarrow[I_*]{\cong} & \mathcal{M}_{(Q_{E_\bullet},\partial W_{E_\bullet})}(\vec{m})|_V & = & [\{d(\mathrm{tr}\,W_{E_\bullet})=0\}/G] & \hookrightarrow & [\pi_Q^{-1}(V)/G] \\ \downarrow{\scriptstyle p_M} & & \downarrow{\scriptstyle p_Q} & & & & \downarrow{\scriptstyle \mathrm{tr}\,W_{E_\bullet}} \\ U & \xleftarrow{\cong} & M_{(Q_{E_\bullet},\partial W_{E_\bullet})}(\vec{m})|_V & & & & \mathbb{C} \end{array} \tag{5-28}$$

Here the bottom arrow sends 0 *to* p, *the map* $\pi_Q\colon \operatorname{Rep}_{Q_{E_\bullet}}(\vec{m}) \to M_{Q_{E_\bullet}}(\vec{m})$ *is the quotient morphism, and* $\operatorname{tr} W_{E_\bullet}$ *is the* G*–invariant analytic function on the smooth analytic space* $\pi_Q^{-1}(V)$ (see Section 2.6).

6 Noncommutative deformation theory

Note that the diagram (3-5) in Theorem 3.2 in particular implies the isomorphism

$$I_*\colon p_Q^{-1}(0) \xrightarrow{\cong} p_M^{-1}(p). \tag{6-1}$$

In this section, we recall the NC deformation theory associated to a simple collection of sheaves, and explain its relationship to the isomorphism (6-1).

More precisely in Theorem 6.8, using NC deformation theory we show that the map I_* gives an equivalence of categories between the category of nilpotent representations of the Ext–quiver and the subcategory of coherent sheaves on X generated by the given simple collection. Theorem 6.8 immediately implies the isomorphism (6-1), so giving an interpretation of (6-1) via NC deformation theory. The result of Theorem 6.8 will be only used in the proof of Lemma 7.8 in the next section, but seems to be an interesting result in its own right as it gives intrinsic understanding of the isomorphism (6-1).

6.1 NC deformation functors

Let X be a smooth projective variety, on which we take a simple collection of coherent sheaves

$$E_\bullet = (E_1, E_2, \dots, E_k). \tag{6-2}$$

The NC deformation theory associated to the simple collection (6-2) is formulated for such a collection [22; 9; 18; 5]. The following convention is due to Kawamata [18].

By definition, a *k–pointed $\mathbb{C}$–algebra* is an associative ring R with $\mathbb{C}$–algebra homomorphisms

$$\mathbb{C}^k \xrightarrow{p} R \xrightarrow{q} \mathbb{C}^k$$

whose composition is the identity. Then R decomposes as

$$R = \mathbb{C}^k \oplus \boldsymbol{m}, \quad \boldsymbol{m} := \operatorname{Ker} q.$$

For $1 \leq i \leq k$, let $\boldsymbol{m}_i$ be the kernel of the composition

$$R \xrightarrow{q} \mathbb{C}^k \to \mathbb{C},$$

where the second map is the i^{th} projection. Note that $\boldsymbol{m} = \bigcap_{i=1}^k \boldsymbol{m}_i$. We define $\mathcal{A}\mathrm{rt}_k$ to be the category of finite-dimensional k–pointed $\mathbb{C}$–algebras $R = \mathbb{C}^k \oplus \boldsymbol{m}$ such that $\boldsymbol{m}$ is nilpotent.

For a simple collection (6-2), we have the NC deformation functor

$$\mathrm{Def}^{\mathrm{nc}}_{E_\bullet} \colon \mathcal{A}\mathrm{rt}_k \to \mathcal{S}\mathrm{et}. \tag{6-3}$$

The above functor is defined by sending $R = \mathbb{C}^k \oplus \boldsymbol{m}$ to the set of isomorphism classes of pairs

$$(\mathcal{E}, \psi), \quad \mathcal{E} \in \mathrm{Coh}(R \otimes_{\mathbb{C}} \mathcal{O}_X),$$

where $\mathcal{E}$ is a coherent left $R \otimes_{\mathbb{C}} \mathcal{O}_X$–module which is flat over R, and ψ is an isomorphism $R/\boldsymbol{m} \otimes_R \mathcal{E} \xrightarrow{\cong} \bigoplus_i E_i$ which induces isomorphisms

$$R/\boldsymbol{m}_i \otimes_R \mathcal{E} \xrightarrow{\cong} E_i, \quad 1 \leq i \leq k.$$

6.2 Prorepresentable hull

Let $\widehat{\mathcal{A}\mathrm{rt}}_k$ be the category whose objects consist of $\mathbb{C}^k$–algebras given by inverse limits of objects in $\mathcal{A}\mathrm{rt}_k$. An object $A \in \widehat{\mathcal{A}\mathrm{rt}}_k$ is called a *prorepresentable hull* of the functor $\mathrm{Def}^{\mathrm{nc}}_{E_\bullet}$ if there is a formally smooth morphism

$$\mathrm{Hom}_{\widehat{\mathcal{A}\mathrm{rt}}_k}(A, -) \to \mathrm{Def}^{\mathrm{nc}}_{E_\bullet}(-)$$

which are isomorphisms in first orders. A prorepresentable hull is, if it exists, unique up to noncanonical isomorphisms; see [30].

A prorepresentable hull of the functor $\mathrm{Def}^{\mathrm{nc}}_{E_\bullet}$ is known to exist by [22; 9]. By [18], it is explicitly constructed by taking the iterated universal extensions of sheaves E_i, which we review here. We first set $E_i^{(0)} = E_i$ for $1 \leq i \leq k$. Suppose that $E_i^{(n)}$ is constructed for some $n \geq 0$ and all $1 \leq i \leq k$. Then $E_i^{(n+1)}$ is constructed as the universal extension

$$0 \to \bigoplus_{j=1}^k \mathrm{Ext}^1(E_i^{(n)}, E_j)^\vee \otimes E_j \to E_i^{(n+1)} \to E_i^{(n)} \to 0. \tag{6-4}$$

Let us set

$$E^{(n)} := \bigoplus_{i=1}^{n} E_i^{(n)}, \quad R^{(n)} := \mathrm{Hom}(E^{(n)}, E^{(n)}).$$

Then by [18, Theorem 4.8], $R^{(n)}$ is an object of $\mathcal{A}\mathrm{rt}_k$, and $E^{(n)}$ is an element of $\mathrm{Def}^{\mathrm{nc}}_{E_\bullet}(R^{(n)})$. Moreover by [18, Lemma 4.3, Corollary 4.6, Theorem 4.8], there exist natural surjections $R^{(n+1)} \twoheadrightarrow R^{(n)}$ such that the inverse limit

$$R^{\mathrm{nc}}_{E_\bullet} = \varprojlim R^{(n)} \in \widehat{\mathcal{A}\mathrm{rt}}_k$$

is a prorepresentable hull of (6-3). Moreover the surjection $E^{(n+1)} \twoheadrightarrow E^{(n)}$ induces the isomorphism

$$R^{(n)} \otimes_{R^{(n+1)}} E^{(n+1)} \xrightarrow{\cong} E^{(n)}. \tag{6-5}$$

By the surjection $R^{(n+1)} \twoheadrightarrow R^{(n)}$, we have the fully faithful embedding

$$\mathrm{mod}\, R^{(n)} \hookrightarrow \mathrm{mod}\, R^{(n+1)}. \tag{6-6}$$

Then the category $\mathrm{mod}_{\mathrm{nil}}\, R^{\mathrm{nc}}_{E_\bullet}$ is defined by

$$\mathrm{mod}_{\mathrm{nil}}\, R^{\mathrm{nc}}_{E_\bullet} := \varinjlim(\mathrm{mod}\, R^{(n)}). \tag{6-7}$$

The above category is identified with the abelian category of nilpotent finite-dimensional right $R^{\mathrm{nc}}_{E_\bullet}$–modules.

6.3 Equivalence of categories via NC deformations

In what follows, we show that the category (6-7) is equivalent to the subcategory of $\mathrm{Coh}(X)$

$$\langle E_1, E_2, \ldots, E_k \rangle \subset \mathrm{Coh}(X)$$

given by the extension closure of $E_1, \ldots, E_k$, ie the smallest extension closed subcategory of $\mathrm{Coh}(X)$ which contains $E_1, \ldots, E_k$.

Lemma 6.1 *For $T \in \mathrm{mod}\, R^{(n)}$, we have*

$$\Phi(T) := T \otimes_{R^{(n)}} E^{(n)} \in \langle E_1, \ldots, E_k \rangle. \tag{6-8}$$

Proof Since $R^{(n)} \in \mathcal{A}\mathrm{rt}_k$, it decomposes as $R^{(n)} = \mathbb{C}^k \oplus \boldsymbol{m}^{(n)}$. We take the following filtration in $\mathrm{mod}\, R^{(n)}$

$$\cdots \subset T(\boldsymbol{m}^{(n)})^j \subset T(\boldsymbol{m}^{(n)})^{j-1} \subset \cdots \subset T\boldsymbol{m}^{(n)} \subset T.$$

Then the subquotient

$$T^{(j)} := T(\boldsymbol{m}^{(n)})^j / T(\boldsymbol{m}^{(n)})^{j+1}$$

is a $\mathbb{C}^k$–module, which is zero for $j \gg 0$. Since $E^{(n)}$ is an NC deformation of $E_\bullet$ to $R^{(n)}$, it follows that $T^{(j)} \otimes_{R^{(n)}} E^{(n)}$ is a direct sum of objects in $(E_1, \dots, E_k)$. Since T is given by iterated extensions of $T^{(j)}$, the lemma follows. □

The functor

$$\Phi\colon \operatorname{mod} R^{(n)} \to \langle E_1, \dots, E_k \rangle$$

given by Lemma 6.1 commutes with the embedding (6-6) by the isomorphism (6-5). Hence we obtain the functor

$$\Phi\colon \operatorname{mod}_{\mathrm{nil}} R^{\mathrm{nc}}_{E_\bullet} \to \langle E_1, \dots, E_k \rangle. \tag{6-9}$$

We will show that the functor (6-9) is an equivalence of categories, for which we now prepare some lemmas.

Lemma 6.2 *We have* $\operatorname{Hom}(E_i^{(n)}, E_j) = \mathbb{C}^{\delta_{ij}}$*, and the natural map*

$$\operatorname{Ext}^1(E_i^{(n)}, E_j) \to \operatorname{Ext}^1(E_i^{(n+1)}, E_j)$$

is a zero map.

Proof The lemma follows from the exact sequence

$$0 \to \operatorname{Hom}(E_i^{(n)}, E_j) \to \operatorname{Hom}(E_i^{(n+1)}, E_j) \to \operatorname{Ext}^1(E_i^{(n)}, E_j) \xrightarrow{\mathrm{id}} \operatorname{Ext}^1(E_i^{(n)}, E_j) \to \operatorname{Ext}^1(E_i^{(n+1)}, E_j)$$

obtained by applying $\operatorname{Hom}(-, E_j)$ to the exact sequence (6-4). □

Lemma 6.3 *For any* $U \in \langle E_1, \dots, E_k \rangle$ *and* $n \geq 0$*, the natural map*

$$\operatorname{Ext}^1(E_i^{(n)}, U) \to \operatorname{Ext}^1(E_i^{(n+l)}, U) \tag{6-10}$$

is a zero map for $l \gg 0$*.*

Proof If $U = E_j$ for some $1 \leq j \leq k$, the lemma follows from Lemma 6.2. Otherwise there is an exact sequence

$$0 \to U' \to U \to U'' \to 0, \quad U', U'' \in \langle E_1, \dots, E_k \rangle \setminus \{0\}.$$

Suppose that the lemma holds for U' and U''. For $l' \gg 0$ and $l'' \gg 0$, We have the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^1(E^{(n)}, U') & \longrightarrow & \mathrm{Ext}^1(E^{(n)}, U) & \longrightarrow & \mathrm{Ext}^1(E^{(n)}, U'') \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 0} \\
\mathrm{Ext}^1(E^{(n+l'')}, U') & \longrightarrow & \mathrm{Ext}^1(E^{(n+l'')}, U) & \longrightarrow & \mathrm{Ext}^1(E^{(n+l'')}, U'') \\
\downarrow{\scriptstyle 0} & & \downarrow & & \downarrow \\
\mathrm{Ext}^1(E^{(n+l'+l'')}, U') & \longrightarrow & \mathrm{Ext}^1(E^{(n+l'+l'')}, U) & \longrightarrow & \mathrm{Ext}^1(E^{(n+l+l'')}, U'')
\end{array}$$

Here the horizontal arrows are exact sequences. The map (6-10) for $l = l' + l''$ is the composition of middle vertical arrows, which is zero by a diagram chasing. Therefore the lemma follows by the induction on the number of iterated extensions of U by $E_1, \dots, E_k$. □

Lemma 6.4 *For any $U \in \langle E_1, \dots, E_k \rangle$, the sequence*

$$\mathrm{Hom}(E^{(0)}, U) \subset \mathrm{Hom}(E^{(1)}, U) \subset \cdots \subset \mathrm{Hom}(E^{(n)}, U) \subset \cdots \tag{6-11}$$

terminates for $n \gg 0$.

Proof The lemma can be proved by the induction on the number of iterated extensions of U by $E_1, \dots, E_k$. If $U = E_i$ for some i, then the sequence (6-11) terminates by Lemma 6.2. Otherwise there is an exact sequence

$$0 \to E_i \to U \to U' \to 0$$

for some $1 \le i \le k$ and $U' \in \langle E_1, \dots, E_k \rangle$. By applying $\mathrm{Hom}(E^{(n)}, -)$, we obtain the exact sequence

$$0 \to \mathrm{Hom}(E^{(n)}, E_i) \to \mathrm{Hom}(E^{(n)}, U) \to \mathrm{Hom}(E^{(n)}, U').$$

By Lemma 6.2, it follows that

$$\mathrm{Hom}(E^{(n)}, U) \le \mathrm{Hom}(E^{(n)}, U') + 1.$$

By the induction hypothesis, $\mathrm{Hom}(E^{(n)}, U')$ is bounded above by a number which is independent of n. Therefore $\mathrm{Hom}(E^{(n)}, U)$ is also bounded above. □

By Lemma 6.4, we have the functor

$$\Psi\colon \langle E_1, \dots, E_k \rangle \to \mathrm{mod}_{\mathrm{nil}}\, R^{\mathrm{nc}}_{E_\bullet} \tag{6-12}$$

sending U to $\mathrm{Hom}(E^{(n)}, U)$ for $n \gg 0$.

Lemma 6.5 *The functor* (6-12) *is exact.*

Proof It is enough to show that (6-12) is right exact. Let $0 \to U' \to U \to U'' \to 0$ be an exact sequence in $\langle E_1, \dots, E_k \rangle$. For $n \gg 0$ and $l \gg 0$, we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(E^{(n)}, U) & \longrightarrow & \mathrm{Hom}(E^{(n)}, U'') & \longrightarrow & \mathrm{Ext}^1(E^{(n)}, U') \\ \downarrow \cong & & \downarrow \cong & & \downarrow 0 \\ \mathrm{Hom}(E^{(n+l)}, U) & \longrightarrow & \mathrm{Hom}(E^{(n+l)}, U'') & \longrightarrow & \mathrm{Ext}^1(E^{(n+l)}, U') \end{array}$$

Here the isomorphisms of the left and middle vertical arrows follow from Lemma 6.4 and the right vertical arrow is a zero map by Lemma 6.3. Therefore the right bottom horizontal arrow is a zero map, which shows that $\mathrm{Hom}(E^{(n)}, U) \to \mathrm{Hom}(E^{(n)}, U'')$ is surjective for $n \gg 0$. Therefore the functor (6-12) is exact. □

Proposition 6.6 *The functor* (6-9) *is an equivalence of categories.*

Proof The functor (6-12) is a right adjoint functor of Φ, so there exist canonical natural transformations

$$\mathrm{id} \to \Psi \circ \Phi(-), \quad \Phi \circ \Psi(-) \to \mathrm{id}.$$

It is enough to show that both of them are isomorphisms of functors.

As $E^{(n)}$ is flat over $R^{(n)}$, the functor Φ is exact. The functor Ψ is also exact by Lemma 6.5, so the compositions $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are also exact. Therefore by the induction on the number of iterated extensions by simple objects and the five lemma, it is enough to check the isomorphisms

$$S_i \xrightarrow{\cong} \Psi \circ \Phi(S_i), \quad \Phi \circ \Psi(E_i) \xrightarrow{\cong} E_i.$$

Here $S_1, \dots, S_k$ are simple $R^{(0)} = \mathbb{C}^k$–modules. Since $\Phi(S_i) = E_i$ and $\Psi(E_i) = S_i$, the above isomorphisms are obvious. □

6.4 Maurer–Cartan formalism of NC deformations

We can interpret the NC deformation functor (6-3) in terms of Maurer–Cartan formalism. The argument below is also available in [32].

For $R \in \mathcal{A}\mathrm{rt}_k$ with the decomposition $R = \mathbb{C}^k \oplus \boldsymbol{m}$, an argument similar to Section 4.2 shows that

$$\text{(6-13)} \qquad \mathrm{Def}^{\mathrm{nc}}_{E_\bullet}(R) \cong \mathrm{MC}\left(A^{0,*}\left(\mathcal{H}\mathrm{om}^*\left(\bigoplus_{i=1}^{k} \mathcal{E}_i^\bullet, \bigoplus_{i=1}^{k} \mathcal{E}_i^\bullet\right) \underline{\otimes}\, \boldsymbol{m}\right)\right)\Big/\sim$$
$$= \mathrm{MC}\left(\bigoplus_{i,j} A^{0,*}(\mathcal{H}\mathrm{om}^*(\mathcal{E}_i^\bullet, \mathcal{E}_j^\bullet)) \otimes_{\mathbb{C}} \boldsymbol{m}_{ij}\right)\Big/\sim.$$

Here $\sim$ means gauge equivalence, $\underline{\otimes}$ is the tensor product of k–pointed $\mathbb{C}$–algebras (see [32, Section 1.3]), and $\boldsymbol{m}_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{m} \cdot \boldsymbol{e}_j$ for the idempotents $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k\}$ of R. Then using the A_∞–operation $\{I_n\}_{n \geq 1}$ in Section 4.4, we have the map

$$\text{(6-14)} \quad I_*\colon \mathrm{MC}\left(\bigoplus_{i,j} \mathrm{Ext}^*(E_i, E_j) \otimes_{\mathbb{C}} \boldsymbol{m}_{ij}\right) \to \mathrm{MC}\left(\bigoplus_{i,j} A^{0,*}(\mathcal{H}\mathrm{om}(\mathcal{E}_i^\bullet, \mathcal{E}_j^\bullet)) \otimes_{\mathbb{C}} \boldsymbol{m}_{ij}\right),$$

which is an isomorphism after taking the quotients by gauge equivalence. Here the left-hand side is the solution of the MC equation of the A_∞–algebra

$$\bigoplus_{i,j} \mathrm{Ext}^*(E_i, E_j) \otimes_{\mathbb{C}} \boldsymbol{m}_{ij}$$

whose A_∞–product is given by (5-7), and the map I_* is constructed as in (4-28).

Let A be the $\mathbb{C}^k$–algebra defined by

$$\text{(6-15)} \qquad A := \mathbb{C}[\![Q_{E_\bullet}]\!]/(f_1, \dots, f_l),$$

where $(f_1, \dots, f_l)$ is the convergent relation of $Q_{E_\bullet}$ given in (5-8). We have the tautological identification

$$\text{(6-16)} \qquad \mathrm{MC}\left(\bigoplus_{i,j} \mathrm{Ext}^*(E_i, E_j) \otimes_{\mathbb{C}} \boldsymbol{m}_{ij}\right) = \mathrm{Hom}_{\widehat{\mathcal{A}\mathrm{rt}}_k}(A, R).$$

Here $(e_{i,j} \otimes r_{i,j})$ in the left-hand side corresponds to $A \to R$ given by

$$z \mapsto e_{i,j}(z) \cdot r_{i,j} \quad \text{for } z \in E_{i,j} \subset \mathrm{Ext}^1(E_i, E_j)^\vee.$$

As proved in [32, Proposition 2.13], under the above identification the gauge equivalence in the left-hand side corresponds to the conjugation by an element in $1 + \bigoplus_i \boldsymbol{m}_{ii}$ in the right-hand side.

Thus we see that A is a prorepresentable hull of $\mathrm{Def}^{\mathrm{nc}}_{E_\bullet}$. By the uniqueness of prorepresentable hull, we have an isomorphism

$$R^{\mathrm{nc}}_{E_\bullet} \cong A$$

which commutes with maps to $\mathrm{Def}^{\mathrm{nc}}_{E_\bullet}$. Combined with Proposition 6.6, we have the following corollary:

Corollary 6.7 *We have an equivalence of categories*

$$\Phi\colon \mathrm{mod}_{\mathrm{nil}}\, A \xrightarrow{\sim} \langle E_1, E_2, \dots, E_k\rangle. \tag{6-17}$$

Here A is the $\mathbb{C}^k$–algebra (6-15).

6.5 Equivalence of categories via I_*

Let us take a nilpotent $Q_{E_\bullet}$–representation

$$u = (u_e)_{e\in E(Q_{E_\bullet})}, \quad u_e\colon V_{s(e)} \to V_{t(e)}. \tag{6-18}$$

By the argument in Section 5.3 and Remark 5.3, the correspondence $u \mapsto I_*(u)$ forms a functor

$$I_*\colon \mathrm{mod}_{\mathrm{nil}}(A) \to \mathrm{Coh}(X). \tag{6-19}$$

We compare the above functor with the equivalence (6-17) in the following proposition:

Theorem 6.8 *The functor* (6-19) *is isomorphic to the functor Φ in* (6-9). *In particular, the functor I_* in* (6-19) *is an equivalence of categories*

$$I_*\colon \mathrm{mod}_{\mathrm{nil}}(A) \xrightarrow{\sim} \langle E_1, E_2, \dots, E_k\rangle \subset \mathrm{Coh}(X).$$

Proof Let $A = \mathbb{C}^k \oplus \boldsymbol{m}$ be the decomposition and $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k\}$ the idempotents of A, and set $A^{(n)} := A/\boldsymbol{m}^{n+1}$ and $\boldsymbol{m}^{(n)} := \boldsymbol{m}/\boldsymbol{m}^{n+1}$. Then for an element u as in (6-18), the compositions of u_e for $e \in E(Q_{E_\bullet})$ along with the path in $Q_{E_\bullet}$ defines the linear map

$$\boldsymbol{u}\colon \boldsymbol{m}^{(n)}_{ij} \to \mathrm{Hom}(V_i, V_j), \quad \boldsymbol{m}^{(n)}_{ij} := \boldsymbol{e}_i \cdot \boldsymbol{m}^{(n)} \cdot \boldsymbol{e}_j.$$

On the other hand, let

$$c^{(n)} \in \mathrm{MC}\Big(\bigoplus_{i,j} \mathrm{Ext}^*(E_i, E_j) \otimes_{\mathbb{C}} \boldsymbol{m}^{(n)}_{ij}\Big)$$

be the canonical element corresponding to the surjection $A \twoheadrightarrow A^{(n)}$ under the tautological identity (6-16). Applying the map (6-14), we obtain

$$I_*(c^{(n)}) \in \mathrm{MC}\Big(\bigoplus_{i,j} A^{0,*}(\mathcal{H}om^*(\mathcal{E}^\bullet_i, \mathcal{E}^\bullet_j)) \otimes_{\mathbb{C}} \boldsymbol{m}^{(n)}_{ij}\Big). \tag{6-20}$$

Then for $n \gg 0$, we have the identity

$$I_*(u) = \boldsymbol{u} \circ I_*(c^{(n)}) \in \mathrm{MC}(\mathfrak{g}_E^*). \tag{6-21}$$

Let $\mathcal{F}^{(n)} \in \mathrm{Def}^{\mathrm{nc}}_{E_\bullet}(A^{(n)})$ the NC deformation of $E_\bullet$ over $A^{(n)}$ corresponding to (6-20) under the isomorphism (6-13). Note that $\mathcal{F}^{(n)}$ is the universal NC deformation over A pulled back by the surjection $A \twoheadrightarrow A^{(n)}$. Let $T \in \mathrm{mod}_{\mathrm{nil}}(A)$ be the object given by the $Q_{E_\bullet}$–representation u. Then the identity (6-21) implies that

$$I_*(T) \cong T \otimes_{A^{(n)}} \mathcal{F}^{(n)}.$$

By the construction of Φ in (6-17), which goes back to the construction in Lemma 6.1, and the universality of $\mathcal{F}^{(n)}$, we have $\Phi(T) = T \otimes_{A^{(n)}} \mathcal{F}^{(n)}$. Therefore, the proposition holds. □

In the diagram (3-5), note that $p_Q^{-1}(0)$ consists of nilpotent A–modules and $p_M^{-1}(p)$ consists of objects in the extension closure $\langle E_1, \dots, E_k \rangle$. The above proposition implies that the isomorphism (6-1) is induced by the universal family over NC deformations.

7 Moduli spaces of one-dimensional semistable sheaves

In this section, we focus on the case of moduli spaces of one-dimensional semistable sheaves, and prove Theorem 1.3.

7.1 Twisted semistable sheaves

Let X be a smooth projective variety, and $A(X)_{\mathbb{C}}$ its complexified ample cone

$$A(X)_{\mathbb{C}} := \{B + i\omega \in \mathrm{NS}(X)_{\mathbb{C}} : \omega \text{ is ample}\}.$$

Let

$$\mathrm{Coh}_{\le 1}(X) \subset \mathrm{Coh}(X)$$

be the abelian subcategory of coherent sheaves whose supports have dimensions less than or equal to one. For an object $E \in \mathrm{Coh}_{\le 1}(X)$ and $B + i\omega \in A(X)_{\mathbb{C}}$, the *$B$–twisted ω–slope* $\mu_{B,\omega}(E)$ is defined by

$$\mu_{B,\omega}(E) := \frac{\chi(E) - B \cdot \mathrm{ch}_{d-1}(E)}{\omega \cdot \mathrm{ch}_{d-1}(E)} \in \mathbb{R} \cup \{\infty\}.$$

Here $d = \dim X$, and we set $\mu_{B,\omega}(E) = \infty$ if $\omega \cdot \mathrm{ch}_{d-1}(E) = 0$, ie if E is a zero-dimensional sheaf.

Definition 7.1 An object $E \in \mathrm{Coh}_{\leq 1}(X)$ is (B, ω)–(semi)stable if for any nonzero subsheaf $F \subsetneq E$, we have the inequality

$$\mu_{B,\omega}(F) < (\leq)\, \mu_{B,\omega}(E).$$

Remark 7.2 If $B = 0$, then $E \in \mathrm{Coh}_{\leq 1}(X)$ is $(0, \omega)$–(semi)stable if and only if it is ω–Gieseker (semi)stable sheaf.

Remark 7.3 For any integer $k \geq 1$ and a line bundle $\mathcal{L}$ on X, we have

$$\mu_{B,\omega}(E) = \mu_{kB,k\omega}(E) = \mu_{kB+c_1(\mathcal{L}),k\omega}(E \otimes \mathcal{L}).$$

In particular if B and ω are elements of $\mathrm{NS}(X)_{\mathbb{Q}}$ such that kB and $k\omega$ are integral, then for a line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = -kB$, a sheaf $E \in \mathrm{Coh}_{\leq 1}(X)$ is (B, ω)–semistable if and only if $E \otimes \mathcal{L}$ is a ω–Gieseker semistable sheaf.

The (B, ω)–stability condition is interpreted in terms of Bridgeland stability conditions [6] as follows. Let $N_1(X) \subset H_2(X, \mathbb{Z})$ be the group of numerical classes of algebraic one cycles on X and set

$$\Gamma_X := N_1(X) \oplus \mathbb{Z}.$$

Let cl be the group homomorphism defined by

$$\mathrm{cl}\colon K(\mathrm{Coh}_{\leq 1}(X)) \to \Gamma_X, \quad E \mapsto ([E], \chi(E)), \tag{7-1}$$

where $[E]$ is the fundamental one cycle associated to E. By definition, a *Bridgeland stability condition* on $D^b(\mathrm{Coh}_{\leq 1}(X))$ with respect to the group homomorphism map (7-1) consists of the data

$$\sigma = (Z, \mathcal{A}), \quad Z\colon \Gamma_X \to \mathbb{C}, \quad \mathcal{A} \subset D^b(\mathrm{Coh}_{\leq 1}(X)), \tag{7-2}$$

where Z is a group homomorphism and $\mathcal{A}$ is the heart of a bounded t-structure satisfying some axioms; see [6; 21] for details. It determines the set of σ–*(semi)stable objects*: $E \in D^b(\mathrm{Coh}_{\leq 1}(X))$ is σ–(semi)stable if $E[k] \in \mathcal{A}$ for some $k \in \mathbb{Z}$, and for any nonzero subobject $0 \neq F \subsetneq E[k]$ in $\mathcal{A}$, we have the inequality in $(0, \pi]$

$$\arg Z(\mathrm{cl}(F)) < (\leq) \arg Z(\mathrm{cl}(E[k])).$$

The set of Bridgeland stability conditions (7-2) forms a complex manifold, which we denote by $\mathrm{Stab}_{\leq 1}(X)$. The forgetting map $(Z, \mathcal{A}) \mapsto Z$ gives a local homeomorphism

$$\mathrm{Stab}_{\leq 1}(X) \to (\Gamma_X)^{\vee}_{\mathbb{C}}.$$

For a given element $B+i\omega \in A(X)_{\mathbb{C}}$, let $Z_{B,\omega}$ be the group homomorphism $\Gamma_X \to \mathbb{C}$ defined by

(7-3) $$Z_{B,\omega}(\beta, m) := -m + (B+i\omega)\beta.$$

Then the pair

(7-4) $$\sigma_{B,\omega} := (Z_{B,\omega}, \mathrm{Coh}_{\le 1}(X))$$

determines a point in $\mathrm{Stab}_{\le 1}(X)$.

It is obvious that an object in $\mathrm{Coh}_{\le 1}(X)$ is (B,ω)–(semi)stable if and only if it is Bridgeland $\sigma_{B,\omega}$–(semi)stable. We also call (B,ω)–(semi)stable sheaves as $\sigma_{B,\omega}$–*(semi)stable objects*. Moreover the map

$$A(X)_{\mathbb{C}} \to \mathrm{Stab}_{\le 1}(X), \quad (B,\omega) \mapsto \sigma_{B,\omega},$$

is a continuous injective map, whose image is denoted by

$$U(X) \subset \mathrm{Stab}_{\le 1}(X).$$

7.2 Moduli stacks of twisted semistable sheaves

For $\sigma = \sigma_{B,\omega} \in U(X)$ and $v \in \Gamma_X$, let

$$\mathcal{M}_\sigma(v) \subset \mathcal{M}$$

be the moduli stack of σ–semistable $E \in \mathrm{Coh}_{\le 1}(X)$ with $\mathrm{cl}(E) = v$. As in the case of Gieseker stability, we have the following:

Lemma 7.4 *The stack $\mathcal{M}_\sigma(v)$ is an algebraic stack of finite type with a projective coarse moduli space $M_\sigma(v)$. So we have the natural morphism*

$$p_M\colon \mathcal{M}_\sigma(v) \to M_\sigma(v).$$

Moreover for each closed point $p \in M_\sigma(v)$, the same conclusion of Theorem 3.2 holds.

Proof If B and ω are rational, then we can reduce the lemma in the case of $B=0$ and ω is integral by Remark 7.3. In that case, the lemma follows from Theorem 3.2. In general by wall-chamber structure on the space of Bridgeland stability conditions, there is a collection of real codimension one submanifolds $\{\mathcal{W}_j\}_{j\in J}$ in $A(X)_{\mathbb{C}}$ called *walls* such that $\mathcal{M}_\sigma(v)$ is constant if σ is contained in a stratum

(7-5) $$\bigcap_{j\in J'} \mathcal{W}_j \setminus \bigcup_{j\notin J'} \mathcal{W}_j$$

for some subset $J' \subset J$. Each wall is given by $\mu_{B,\omega}(\beta, n) = \mu_{B,\omega}(\beta', n')$ for other $(\beta', n') \in \Gamma_X$ which is not proportional to (β, n), ie

$$(n'\beta - n\beta')\omega = B\beta' \cdot \omega\beta - B\beta \cdot \omega\beta'.$$

The above equation determines a hypersurface in $A(X)_{\mathbb{C}}$ which contains dense rational points. Therefore if (B, ω) is not rational, then we can perturb it in the strata (7-5) and can assume that (B, ω) is rational. □

7.3 Moduli stacks of semistable Ext–quiver representations

For $v \in \Gamma_X$ and $\sigma = \sigma_{B,\omega} \in U(X)$, take a point $p \in M_\sigma(v)$. Suppose that p is represented by a (B, ω)–polystable sheaf E of the form

$$E = \bigoplus_{i=1}^{k} V_i \otimes E_i, \tag{7-6}$$

where $E_i \in \mathrm{Coh}_{\le 1}(X)$ is (B, ω)–stable with $\mu_{B,\omega}(E_i) = \mu_{B,\omega}(E)$. Then we have the Ext–quiver $Q_{E_\bullet}$ associated to the simple collection

$$E_\bullet = (E_1, \dots, E_k),$$

together with a convergent relation $I_{E_\bullet}$ as in (5-8). For $i \in V(Q_{E_\bullet}) = \{1, 2, \dots, k\}$, let S_i be the one-dimensional $Q_{E_\bullet}$–representation corresponding to the vertex i. We denote by $K(Q_{E_\bullet})$ the Grothendieck group of finite-dimensional $Q_{E_\bullet}$–representations, and take the group homomorphism

$$\mathbf{dim}\colon K(Q_{E_\bullet}) \to \Gamma_Q := \bigoplus_{i=1}^{k} \mathbb{Z} \cdot \mathbf{dim}(S_i)$$

by taking the dimension vectors.

Let us take another stability condition

$$\sigma^+ = \sigma_{B^+,\omega^+} = (Z_{B^+,\omega^+}, \mathrm{Coh}_{\le 1}(X)) \in U(X). \tag{7-7}$$

Then we have the group homomorphism

$$Z_Q^+\colon K(Q_{E_\bullet}) \xrightarrow{\mathbf{dim}} \Gamma_Q \to \mathbb{C}, \quad [S_i] \mapsto Z_{B^+,\omega^+}(E_i).$$

The above group homomorphism determines a Bridgeland stability condition on the category of $Q_{E_\bullet}$–representations, and the associated (semi)stable representations. They are described in terms of slope stability condition as in Definition 7.1. Let μ_Q^+ be the slope function on the category of $Q_{E_\bullet}$–representations defined by

$$\mu_Q^+(-) := -\frac{\operatorname{Re} Z_Q^+(-)}{\operatorname{Im} Z_Q^+(-)}.$$

Note that if $\mathbb{V}$ is a $Q_{E_\bullet}$–representation with dimension vector

$$\vec{m} = (m_i)_{1 \le i \le k}, \quad m_i = \dim V_i, \tag{7-8}$$

then we have the identity

$$\mu_Q^+(\mathbb{V}) = \mu_{B^+,\omega^+}(E), \tag{7-9}$$

where E is given by (7-6). We have the following definition:

Definition 7.5 A $Q_{E_\bullet}$–representation $\mathbb{V}$ is μ_Q^+–(semi)stable if for any $Q_{E_\bullet}$–subrepresentation $0 \neq \mathbb{V}' \subsetneq \mathbb{V}$, we have the inequality

$$\mu_Q^+(\mathbb{V}') < (\le)\, \mu_Q^+(\mathbb{V}).$$

For the dimension vector (7-8), let

$$\operatorname{Rep}_{Q_{E_\bullet}}^+(\vec{m}) \subset \operatorname{Rep}_{Q_{E_\bullet}}(\vec{m})$$

be the (Zariski) open subset consisting of μ_Q^+–semistable $Q_{E_\bullet}$–representations. The above open subset is a GIT semistable locus with respect to a certain character of G; see [19, Section 3]. The quotients by G

$$\mathcal{M}_{Q_{E_\bullet}}^+(\vec{m}) = [\operatorname{Rep}_{Q_{E_\bullet}}^+(\vec{m})/G], \quad M_{Q_{E_\bullet}}^+(\vec{m}) = \operatorname{Rep}_{Q_{E_\bullet}}^+(\vec{m}) /\!\!/ G$$

are the moduli stack of μ_Q^+–semistable $Q_{E_\bullet}$–representations with dimension vector $\vec{m}$, and its coarse moduli space, respectively. We have the commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{Q_{E_\bullet}}^+(\vec{m}) & \hookrightarrow & \mathcal{M}_{Q_{E_\bullet}}(\vec{m}) \\ \downarrow{\scriptstyle p_Q^+} & & \downarrow{\scriptstyle p_Q} \\ M_{Q_{E_\bullet}}^+(\vec{m}) & \xrightarrow[q_Q]{} & M_{Q_{E_\bullet}}(\vec{m}) \end{array}$$

Here the vertical arrows are natural morphisms to the coarse moduli spaces, the top horizontal arrow is an open immersion and the bottom horizontal arrow q_Q is induced by the universality of the GIT quotients. Note that q_Q is projective due to a general argument of affine GIT quotients; see [26, Section 6].

Let $0 \in V \subset M_{Q_{E_\bullet}}(\vec{m})$ be a sufficiently small analytic open subset as in Definition 2.16. Let

$$\mathrm{Rep}^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V \subset \mathrm{Rep}_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V$$

be the open locus consisting of μ_Q^+–semistable representations, where the right-hand side is defined as in (2-17). Then we set

$$\mathcal{M}^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V := [\mathrm{Rep}^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V / G],$$
$$M^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V := \mathrm{Rep}^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V /\!\!/ G.$$

Here $M^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V$ is the analytic Hilbert quotient given in Lemma 2.9, which is a closed analytic subspace of $V^+ = q_Q^{-1}(V)$. We have the commutative diagram

(7-10)
$$\begin{array}{ccc} \mathcal{M}^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V & \hookrightarrow & \mathcal{M}_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V \\ {\scriptstyle p^+_{(Q,I)}}\downarrow & \searrow^{r_{(Q,I)}} & \downarrow{\scriptstyle p_{(Q,I)}} \\ M^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V & \xrightarrow[q_{(Q,I)}]{} & M_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V \end{array}$$

Here the vertical arrows are natural morphisms to the coarse moduli spaces, the top horizontal arrow is an open immersion and the bottom horizontal arrow $q_{(Q,I)}$ is induced by the universality of analytic Hilbert quotients (see Lemma 2.13).

Lemma 7.6 *The morphism $q_{(Q,I)}$ in diagram* (7-10) *is projective.*

Proof We have the commutative diagram

$$\begin{array}{ccccc} M^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V & \hookrightarrow & V^+ & \hookrightarrow & M^+_{Q_{E_\bullet}}(\vec{m}) \\ {\scriptstyle q_{(Q,I)}}\downarrow & & \downarrow & \square & \downarrow{\scriptstyle q_Q} \\ M_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V & \hookrightarrow & V & \hookrightarrow & M_{Q_{E_\bullet}}(\vec{m}) \end{array}$$

Here the right diagram is a Cartesian square whose horizontal arrows are open immersions, and the horizontal arrows in the left diagram are closed immersions. Since q_Q is projective, the morphism $q_{(Q,I)}$ is projective by the above diagram. $\square$

7.4 Moduli stacks of semistable sheaves under the change of stability

Let us take σ^+ in (7-7) sufficiently close to σ. Then by wall-chamber structure on $U(X)$, any σ^+–semistable object E with $\mathrm{cl}(E) = v$ is σ–semistable. Then we have the commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{\sigma^+}(v) & \hookrightarrow & \mathcal{M}_{\sigma}(v) \\ {\scriptstyle p_M^+}\downarrow & \searrow^{r_M} & \downarrow{\scriptstyle p_M} \\ M_{\sigma^+}(v) & \xrightarrow[q_M]{} & M_{\sigma}(v) \end{array} \tag{7-11}$$

Here the vertical arrows are natural morphisms to the coarse moduli spaces, the top arrow is an open immersion and the bottom arrow is induced by the universality of coarse moduli spaces. The following is the main result in this section.

Theorem 7.7 *For a closed point $p \in M_\sigma(v)$ represented by a polystable sheaf* (7-6), *there are analytic open neighborhoods $p \in U \subset M_\sigma(v)$ and $0 \in V \subset M_{Q_{E_\bullet}}(\vec{m})$, where $Q_{E_\bullet}$ is the Ext–quiver associated to p with convergent relation $I_{E_\bullet}$, and the dimension vector $\vec{m}$ is given by* (7-8), *such that the diagram* (7-11) *pulled back to U*

$$\begin{array}{ccc} r_M^{-1}(U) & \hookrightarrow & p_M^{-1}(U) \\ {\scriptstyle p_M^+}\downarrow & & \downarrow{\scriptstyle p_M} \\ q_M^{-1}(U) & \xrightarrow[q_M]{} & U \end{array}$$

is isomorphic to the diagram (7-10).

Proof We take $U = \mathcal{W}/\!\!/G$, $V \subset M_{Q_{E_\bullet}}(\vec{m})$ and the isomorphism

$$I_*\colon \mathcal{M}_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V \xrightarrow{\cong} p_M^{-1}(U) \tag{7-12}$$

as in Proposition 5.4. It is enough to show that the isomorphism (7-12) restricts to the isomorphism

$$I_*\colon \mathcal{M}^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V \xrightarrow{\cong} r_M^{-1}(U). \tag{7-13}$$

For a $\mathbb{C}$–valued point $x \in \mathcal{M}_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V$, let $\mathbb{V}_x$ be the corresponding $Q_{E_\bullet}$–representation, and $E_x \in \mathrm{Coh}_{\leq 1}(X)$ the (B, ω)–semistable sheaf corresponding to $I_*(x) \in p_M^{-1}(U)$. Let $\mathcal{Z} \subset \mathcal{M}^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V$ be the closed substack given by

$$\mathcal{Z} := \left\{ x \in \mathcal{M}^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V : I_*(x) \notin r_M^{-1}(U) \right\}.$$

Namely $x \in \mathcal{M}_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V$ is a $\mathbb{C}$–valued point of $\mathcal{Z}$ if and only if $\mathbb{V}_x$ is μ_Q^+–semistable but E_x is not (B^+, ω^+)–semistable. Below we use the notation in the diagram (7-10). By Lemma 7.8 below, we have

$$\mathcal{Z} \cap (r_{(Q,I)})^{-1}(0) = \varnothing. \tag{7-14}$$

On the other hand, by Lemma 2.12 the subset

$$p^+_{(Q,I)}(\mathcal{Z}) \subset M^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V$$

is closed. Together with Lemma 7.6, we see that

$$r_{(Q,I)}(\mathcal{Z}) = q_{(Q,I)} \circ p^+_{(Q,I)}(\mathcal{Z}) \subset M_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V$$

is a closed subset. By (7-14), the above closed subset does not contain 0. Therefore by shrinking V if necessary, we may assume that $\mathcal{Z} = \varnothing$, ie (7-12) takes $\mathcal{M}^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V$ to $r_M^{-1}(U)$.

Next for $x \in \mathcal{M}_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V$, suppose that E_x is (B^+, ω^+)–semistable, ie $I_*(x) \in r_M^{-1}(U)$. Note that by (7-9), we have

$$\mu_Q^+(\mathbb{V}_x) = \mu_{B^+, \omega^+}(E_x).$$

By the functoriality of I_* in Section 5.3 and the above equality, if a $Q_{E_\bullet}$–subrepresentation $\mathbb{V}' \subset \mathbb{V}_x$ destabilizes $\mathbb{V}_x$ in μ_Q^+–stability, then by applying I_* and noting Remark 5.3 we obtain the subsheaf $E' \subset E_x$ which destabilizes E_x in (B^+, ω^+)–stability. This is a contradiction, so $\mathbb{V}_x$ is μ_Q^+–semistable; ie $x \in \mathcal{M}^+_{(Q_{E_\bullet}, I_{E_\bullet})}(\vec{m})|_V$. Therefore we obtain the desired isomorphism (7-13). □

We have used the following lemma:

Lemma 7.8 *Under the equivalence I_* in Theorem 6.8, an object $\mathbb{V} \in \mathrm{mod}_{\mathrm{nil}}(A)$ with $\dim \mathbb{V} = \vec{m}$ is μ_Q^+–semistable if and only if $F = I_*(\mathbb{V})$ is (B^+, ω^+)–semistable in $\mathrm{Coh}_{\le 1}(X)$.*

Proof The if direction is proved in the first part of the proof of Theorem 7.7, so we only prove the only if direction. Suppose by contradiction that $\mathbb{V}$ is μ_Q^+–semistable but F is not (B^+, ω^+)–semistable. Then there is a nonzero subsheaf $F' \subsetneq F$ such that $\mu_{B^+, \omega^+}(F') > \mu_{B^+, \omega^+}(F)$. On the other hand, as σ^+ is sufficiently close to σ we may assume that there is no wall between σ and σ^+ with respect to the numerical class $\mathrm{cl}(F)$. So we have $\mu_{B,\omega}(F') \ge \mu_{B,\omega}(F)$. Since $F \in \langle E_1, \dots, E_k \rangle$ and each E_i

is (B,ω)–stable with the same slope, the sheaf F is (B,ω)–semistable. Therefore we have $\mu_{B,\omega}(F') \leq \mu_{B,\omega}(F)$, thus $\mu_{B,\omega}(F') = \mu_{B,\omega}(F)$ and F' is also (B,ω)–semistable. By the uniqueness of JH factors of (B,ω)–semistable sheaves, we have $F' \in \langle E_1, \dots, E_k \rangle$. Then by the equivalence I_* in Theorem 6.8, we find a subobject $\mathbb{V}' \subset \mathbb{V}$ in $\mathrm{mod}_{\mathrm{nil}}(A)$ with $I_*(\mathbb{V}') \cong F'$. By the identity (7-9), the subobject $\mathbb{V}'$ destabilizes $\mathbb{V}$, hence a contradiction. □

References

[1] **J Alper**, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) 63 (2013) 2349–2402 MR

[2] **E Arbarello**, **G Saccà**, *Singularities of moduli spaces of sheaves on K3 surfaces and Nakajima quiver varieties*, preprint (2015) arXiv

[3] **O Ben-Bassat**, **C Brav**, **V Bussi**, **D Joyce**, *A "Darboux theorem" for shifted symplectic structures on derived Artin stacks, with applications*, Geom. Topol. 19 (2015) 1287–1359 MR

[4] **J Block**, *Duality and equivalence of module categories in noncommutative geometry*, from "A celebration of the mathematical legacy of Raoul Bott" (P R Kotiuga, editor), CRM Proc. Lecture Notes 50, Amer. Math. Soc., Providence, RI (2010) 311–339 MR

[5] **A Bodzenta**, **A Bondal**, *Flops and spherical functors*, preprint (2015) arXiv

[6] **T Bridgeland**, *Stability conditions on triangulated categories*, Ann. of Math. 166 (2007) 317–345 MR

[7] **B Davison**, **S Meinhardt**, *Cohomological Donaldson–Thomas theory of a quiver with potential and quantum enveloping algebras*, preprint (2016) arXiv

[8] **S K Donaldson**, **P B Kronheimer**, *The geometry of four-manifolds*, Clarendon, New York (1990) MR

[9] **E Eriksen**, *Computing noncommutative deformations of presheaves and sheaves of modules*, Canad. J. Math. 62 (2010) 520–542 MR

[10] **D Fiorenza**, **D Iacono**, **E Martinengo**, *Differential graded Lie algebras controlling infinitesimal deformations of coherent sheaves*, J. Eur. Math. Soc. 14 (2012) 521–540 MR

[11] **K Fukaya**, *Deformation theory, homological algebra and mirror symmetry*, from "Geometry and physics of branes" (U Bruzzo, V Gorini, U Moschella, editors), IOP, Bristol (2003) 121–209 MR

[12] **D Greb**, *Complex-analytic quotients of algebraic G–varieties*, Math. Ann. 363 (2015) 77–100 MR

[13] **P Heinzner**, *Geometric invariant theory on Stein spaces*, Math. Ann. 289 (1991) 631–662 MR

[14] **P Heinzner**, **L Migliorini**, **M Polito**, *Semistable quotients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 26 (1998) 233–248 MR

[15] **D Huybrechts**, **M Lehn**, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics 31, Vieweg, Braunschweig (1997) MR

[16] **D Joyce**, *A classical model for derived critical loci*, J. Differential Geom. 101 (2015) 289–367 MR

[17] **D Joyce**, **Y Song**, *A theory of generalized Donaldson–Thomas invariants*, Mem. Amer. Math. Soc. 1020, Amer. Math. Soc., Providence, RI (2012) MR

[18] **Y Kawamata**, *On multi-pointed non-commutative deformations and Calabi–Yau threefolds*, preprint (2015) arXiv

[19] **A D King**, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser. 45 (1994) 515–530 MR

[20] **M Kontsevich**, **Y Soibelman**, *Homological mirror symmetry and torus fibrations*, from "Symplectic geometry and mirror symmetry" (K Fukaya, Y-G Oh, K Ono, G Tian, editors), World Scientific, River Edge, NJ (2001) 203–263 MR

[21] **M Kontsevich**, **Y Soibelman**, *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations*, preprint (2008) arXiv

[22] **O A Laudal**, *Noncommutative deformations of modules*, Homology Homotopy Appl. 4 (2002) 357–396 MR The Roos Festschrift

[23] **D Luna**, *Slices étales*, from "Sur les groupes algébriques", Mém. Soc. Math. France 33, Soc. Math. France, Paris (1973) 81–10 MR

[24] **D Luna**, *Fonctions différentiables invariantes sous l'opération d'un groupe réductif*, Ann. Inst. Fourier (Grenoble) 26 (1976) 33–49 MR

[25] **D Maulik**, **Y Toda**, *Gopakumar–Vafa invariants via vanishing cycles*, preprint (2016) arXiv

[26] **S Mukai**, *An introduction to invariants and moduli*, Cambridge Stud. Adv. Math. 81, Cambridge Univ. Press (2003) MR

[27] **A Neeman**, *The topology of quotient varieties*, Ann. of Math. 122 (1985) 419–459 MR

[28] **T Pantev**, **B Toën**, **M Vaquié**, **G Vezzosi**, *Shifted symplectic structures*, Publ. Math. Inst. Hautes Études Sci. 117 (2013) 271–328 MR

[29] **A Polishchuk**, *Homological mirror symmetry with higher products*, from "Winter school on mirror symmetry, vector bundles and Lagrangian submanifolds" (C Vafa, S-T Yau, editors), AMS/IP Stud. Adv. Math. 23, Amer. Math. Soc., Providence, RI (2001) 247–259 MR

[30] **M Schlessinger**, *Functors of Artin rings*, Trans. Amer. Math. Soc. 130 (1968) 208–222 MR

[31] **G W Schwarz**, *The topology of algebraic quotients*, from "Topological methods in algebraic transformation groups" (H Kraft, T Petrie, G W Schwarz, editors), Progr. Math. 80, Birkhäuser, Boston (1989) 135–151 MR

[32] **E Segal**, *The A_∞ deformation theory of a point and the derived categories of local Calabi–Yaus*, J. Algebra 320 (2008) 3232–3268 MR

[33] **Y Toda**, *Gopakumar–Vafa invariants and wall-crossing*, preprint (2017) arXiv

[34] **J Tu**, *Homotopy L-infinity spaces*, preprint (2014) arXiv

[35] **R O Wells, Jr**, *Differential analysis on complex manifolds*, Prentice-Hall, Englewood Cliffs, NJ (1973) MR

[36] **W W-Y Wong**, *Lecture notes on Sobolev spaces for CCA*, lecture notes (2010) Available at `http://sma.epfl.ch/~wwywong/papers/sobolevnotes.pdf`

Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo
Kashiwa, Japan

yukinobu.toda@ipmu.jp

Proposed: Richard Thomas Received: 10 October 2017
Seconded: Jim Bryan, Dan Abramovich Revised: 14 December 2017