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Phase tropical hypersurfaces

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We prove a conjecture of Viro (Tr. Mat. Inst. Steklova 273 (2011) 271–303) that a smooth complex hypersurface in $(\mathbb{C}^*)^n$ is homeomorphic to the corresponding phase tropical hypersurface.

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1 Introduction

Consider a hypersurface $H_f \subset (\mathbb{C}^*)^n$ defined by a Laurent polynomial

$$f = \sum_{a \in A} c_a z^a,$$

where $A \subset \mathbb{Z}^n$ is the set of monomials. Let Q be its Newton polytope, that is, the convex hull of A. Given a function $\eta\colon A \to \mathbb{R}$ whose upper graph induces a triangulation of Q, one considers the associated phase tropical hypersurface $\mathcal{TH}_\eta \subset \mathbb{R}^n \times \mathbb{T}^n \cong (\mathbb{C}^*)^n$. This is a polyhedral object which surjects onto the tropical hypersurface $\mathcal{H}_\eta \subset \mathbb{R}^n$. Over the relative interior of a face of $\mathcal{H}_\eta$ dual to a simplex Q' in the triangulation, the fiber of this surjection is the coamoeba of the hypersurface

$$\sum_{a \in \operatorname{vert} Q'} c_a z^a = 0.$$

Our main result, Theorem 21, states that for a generic polynomial f, the complex hypersurface H_f is homeomorphic to the phase tropical hypersurface $\mathcal{TH}_\eta$, which was a conjecture of Viro [13].

In Section 3 we reduce the case of a general hypersurface to finite abelian coverings of a pair of pants using Viro's patchworking [12] and a nonunimodular version of Mikhalkin's pair-of-pants decomposition [9]. Thus, the core of the proof is the case of a pair of pants. The closures $\overline{P}^{n-1}$ and $\overline{\mathcal{T}}\mathcal{P}^{n-1}$ of the corresponding pairs of pants carry induced stratifications from a certain compactification of $(\mathbb{C}^*)^n$. The key technical result of Section 2 is that the closed strata are balls. A homeomorphism

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$\overline{P}^{n-1} \approx \overline{\mathcal{T}}\mathcal{P}^{n-1}$ then follows from an isomorphism between the two regular CW–complexes, one for $\overline{P}^{n-1}$ and the other for $\overline{\mathcal{T}}\mathcal{P}^{n-1}$.

In the final stages of writing the paper we were made aware of an announcement by Kim and Nisse of similar results in Theorem 1.1 and Proposition 5.2 of [5].

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2 Pair of pants

The main result of this section is a homeomorphism between the complex pair of pants and the phase tropical pair of pants; see Theorem 15. The idea is to endow both spaces with structures of regular CW–complexes which are isomorphic.

2.1 Notation

Throughout the paper we identify $\mathbb{C}^*$ with $\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$. In particular, we will identify $(\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ with

$$(\mathbb{R}^{n+1}/\mathbb{R}) \times ((\mathbb{R}/2\pi\mathbb{Z})^{n+1}/(\mathbb{R}/2\pi\mathbb{Z})),$$

where both $\mathbb{R}$ and $\mathbb{R}/2\pi\mathbb{Z}$ act diagonally. We denote the second factor by

$$\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^{n+1}/(\mathbb{R}/2\pi\mathbb{Z}) \cong \mathbb{R}^n/2\pi\mathbb{Z}^n.$$

We will use homogeneous (additive for the last two cases) coordinates

$$\begin{aligned} &[z_0, \dots, z_n] \quad \text{in } (\mathbb{C}^*)^{n+1}/\mathbb{C}^*, \\ &[x_0, \dots, x_n] \quad \text{in } \mathbb{R}^{n+1}/\mathbb{R}, \\ &[\theta_0, \dots, \theta_n] \quad \text{in } \mathbb{T}^n. \end{aligned} \tag{1}$$

An element in $\mathbb{T}^n$ can be thought of as a configuration of marked points $\theta_0, \theta_1, \dots, \theta_n$ on the unit circle up to simultaneous rotation.

Let $\hat{n}$ denote the set $\{0, \dots, n\}$. For any subset $I \subseteq \hat{n}$ we denote by I^c its complement. We denote by $\pi_I = [\theta_0, \dots, \theta_n]$ the point in $\mathbb{T}^n$ with coordinates

$$\theta_i = \begin{cases} \pi, & i \in I, \\ 0, & i \notin I. \end{cases} \tag{2}$$

The points π_I and π_{I^c} coincide. The origin $[0, \dots, 0]$ is denoted by 0.

Let

$$\Delta := \left\{(y_0, \dots, y_n) \in \mathbb{R}^{n+1} : y_i \geq 0, \sum y_i = 1\right\}$$

be the standard n–simplex. For a nonempty subset $J \subseteq \hat{n}$ the face Δ_J of Δ is defined by $y_i = 0$ for $i \in J^c$. We will identify $\mathbb{R}^{n+1}/\mathbb{R}$ with the interior of Δ via the map

$$[x_0, \dots, x_n] \mapsto \left(\frac{e^{x_0}}{e^{x_0} + \dots + e^{x_n}}, \dots, \frac{e^{x_n}}{e^{x_0} + \dots + e^{x_n}} \right). \tag{3}$$

Multiplying by the factor $\mathbb{T}^n$ leads to a compactification of $(\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ to the space $\Delta \times \mathbb{T}^n$. For any subset $Y \subset (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ we define its *compactified* version $\overline{Y}$ to be the closure of Y in $\Delta \times \mathbb{T}^n$ via the map (3) above.

2.2 The face lattice $\mathcal{W}$ of the future CW–complex

We say that $\sigma = \langle I_1, \dots, I_k \rangle$ is a *cyclic partition* of the set $\hat{n} = \{0, \dots, n\}$ if $\hat{n}$ is a disjoint union of the sets $I_1, \dots, I_k$ and the sets $I_1, \dots, I_k$ are cyclically ordered. The elements within each I_s are not ordered. If all I_s are 1–element sets then we simply write $\sigma = \langle i_0, \dots, i_n \rangle$. Our main source of cyclic partitions of $\hat{n}$ will be configurations of marked points $\theta_0, \theta_1, \dots, \theta_n$ on the oriented circle.

The set of hyperplanes $\theta_i = \theta_j$ for $i, j \in \hat{n}$ stratifies the torus $\mathbb{T}^n$ with strata $\mathbb{T}^n_\sigma$ labeled by cyclic partitions σ. On the other hand the simplex Δ has a natural stratification by its faces Δ_J. The product of the two stratifications induces a stratification on any closed subset $\overline{Y} \subseteq \Delta \times \mathbb{T}^n$. The strata $Y_{\sigma,J}$ of $\overline{Y}$ are labeled by the pairs (σ, J), where $\sigma = \langle I_1, \dots, I_k \rangle$ is a cyclic partition of $\hat{n}$ and $J \subseteq \hat{n}$. The inclusion of the strata closures $\overline{Y}_{\sigma',J'} \subseteq \overline{Y}_{\sigma,J}$ gives a partial order among the pairs: $(\sigma', J') \preceq (\sigma, J)$ if σ is a refinement of σ' (we write $\sigma' \preceq \sigma$) and $J' \subseteq J$.

To simplify notation we will often drop the index J from the subscript if $J = \hat{n}$. For any nonempty subset $J \subseteq \hat{n}$ a cyclic partition $\sigma = \langle I_1, \dots, I_k \rangle$ of $\hat{n}$ induces a cyclic partition $\sigma_J = \langle J_1, \dots, J_r\rangle$ of J by intersecting each I_s with J. We will drop the empty intersections and shift the indices; in this case r will be smaller than k.

Our main focus will be on the poset $\mathcal{W}$ which consists of pairs (σ, J) such that J contains elements in at least two of the subsets $I_1, \dots, I_k$ of $\sigma = \langle I_1, \dots, I_k\rangle$. In this case we say that σ *divides* J and write $\sigma \mid J$. This, in particular, means that $k \geq 2$ and $|J| \geq 2$. We set the rank function to be

$$\mathrm{rk}(\sigma, J) := k + |J| - 4.$$

The poset $\mathcal{W}$ will be the face lattice of our regular CW–complex and $\mathrm{rk}(\sigma, J)$ will be the dimension of the (σ, J)–cell.

Conjecture 1 *For each element $(\sigma, J) \in \mathcal{W}$, its lower interval*

$$\mathcal{W}_{\preceq(\sigma,J)} := \{(\sigma', J') \in \mathcal{W} : (\sigma', J') \preceq (\sigma, J)\}$$

is isomorphic to the face lattice of a simple polytope.

It is clear that for any pair $(\sigma', J') \preceq (\sigma, J)$ the interval $[(\sigma', J'), (\sigma, J)]$ is Boolean, which means that the polytope would have to be simple. The conjecture is manifest for $n = 2$: maximal faces $\mathcal{W}_{\preceq\sigma}$ are hexagons. For $n = 3$ each maximal face $\mathcal{W}_{\preceq\sigma}$ is the 4–dimensional polytope with 20 vertices and 8 facets, dual to P^8_{35}, one of the 37 simplicial polytopes on 8 vertices classified by Grünbaum and Sreedharan [4]. The next problem would be to realize $\mathcal{W}_{\preceq\sigma}$ inside a linear space, which is already interesting for $n = 2$ and 3.

2.3 Complex pair of pants as a CW–complex

The $(n-1)$–dimensional *pair of pants* P^{n-1} is the complement of $n+1$ generic hyperplanes in $\mathbb{CP}^{n-1}$. By an appropriate choice of coordinates we can identify P^{n-1} with the affine hypersurface in $(\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ given by the homogenous equation

$$z_0 + z_1 + \dots + z_n = 0.$$

We define the *compactified pair of pants* $\bar{P}^{n-1}$ to be the closure of P^{n-1} in $\Delta \times \mathbb{T}^n$ via the map (3). This is a manifold with corners, and it can be thought of as a real oriented blow-up of $\mathbb{CP}^{n-1}$ along its intersections with the coordinate hyperplanes in $\mathbb{CP}^n$.

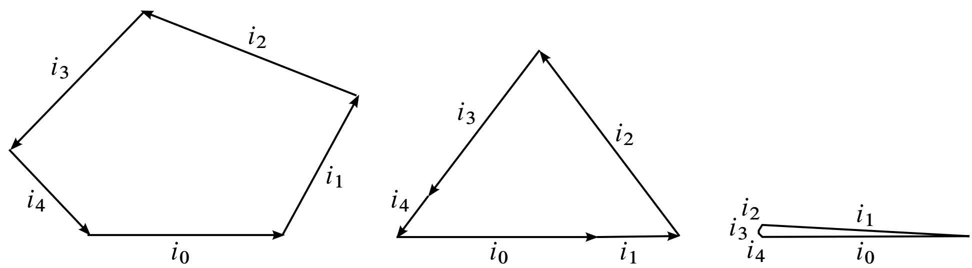

Figure 1: Polygons represent points in $\Phi_{\langle i_0,\dots,i_4\rangle}$, $\Phi_{\langle\{i_0,i_1\},i_2,\{i_3,i_4\}\rangle}$ and $\Phi_{\langle i_0,\dots,i_4\rangle,J=\{i_0,i_1\}}$

We can view points in P^{n-1} as closed oriented broken lines with $n+1$ marked segments in the plane defined up to rigid motions and scaling. The segments represent the complex numbers $z_0,\dots,z_n$. In the compactification $\overline{P}^{n-1}$ the broken lines may have sides of zero length but with directions still recorded.

Recall that the (σ,J)–stratification on $\Delta\times\mathbb{T}^n$ induces a stratification on any closed subset of $\Delta\times\mathbb{T}^n$, in particular on $\overline{P}^{n-1}$. We denote by $\Phi_{\sigma,J}$ the corresponding stratum of $\overline{P}^{n-1}$ and by $\overline{\Phi}_{\sigma,J}$ its closure in $\overline{P}^{n-1}$.

For $[z_0,\dots,z_n]$, a point in a stratum $\Phi_{\sigma,J}$ of $\overline{P}^{n-1}$, one can rearrange the variables so that their arguments are (partially) ordered counterclockwise on the circle. The order of the z_i is defined up to permutations within the subsets I_s in $\sigma=\langle I_1,\dots,I_k\rangle$. Then the circuit of vectors $z_{i_0},\dots,z_{i_n}$ forms a convex (possibly degenerate) polygon $\mathcal{D}$ in the plane (see Figure 1). The vertices of $\mathcal{D}$ separate the subsets I_s in σ.

One can deform a polygon $\mathcal{D}$ representing a point in $\Phi_{\sigma,J}$ by "bending" its edges within each I_s (which refines σ) and introducing small lengths for zero edges (which increases J). Thus, we have the following observation:

Proposition 2 *A closed stratum $\overline{\Phi}_{\sigma,J}$ contains $\Phi_{\sigma',J'}$ if and only if $(\sigma',J')\preceq(\sigma,J)$.*

Next we argue that the (σ,J)–stratification defines a CW–structure on $\overline{P}^{n-1}$.

Lemma 3 *$\Phi_{\sigma,J}$ is homeomorphic to $\mathbb{R}^{\mathrm{rk}(\sigma,J)}$ if $(\sigma,J)\in\mathcal{W}$ and is empty if $(\sigma,J)\notin\mathcal{W}$.*

Proof If $(\sigma,J)\notin\mathcal{W}$ then the set of nonzero edges J falls in a single subset I_s of $\sigma=\langle I_1,\dots,I_k\rangle$. But it is impossible to build a closed circuit with just one nonzero side.

Now let $(\sigma,J)\in\mathcal{W}$ be a maximal stratum, that is, $\sigma=\langle i_0,\dots,i_n\rangle$ and $J=\widehat{n}$ (remember we drop the subscript J from $\Phi_{\sigma,J}$ in this case). We set $z_{i_0}=1$. That fixes the rotational and scaling ambiguity and we can think of Φ_σ as a subset in $(\mathbb{C}^*)^n$.

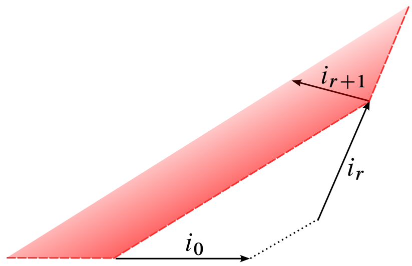

Figure 2: Linear inequalities for $z_{i_{r+1}}$ defining the fiber

Denote by $\Phi_\sigma^{(r)} \subset (\mathbb{C}^*)^r$ the image of Φ_σ under the projection onto the first r coordinates $z_{i_1}, \dots, z_{i_r}$. Notice that $\Phi_\sigma^{(1)}$ is the upper half-plane. For $0 < r < n-1$ the fiber of the projection $\Phi_\sigma^{(r+1)} \to \Phi_\sigma^{(r)}$ over a point $(z_{i_1}, \dots, z_{i_r})$ is an open polyhedral domain in the plane defined by 3 linear inequalities (the red region in Figure 2). Finally, for $r = n-1$ the fiber is a point: the last vector z_{i_n} has to close the circuit. By induction this shows that the Φ_σ is homeomorphic to $\mathbb{R}^{2n-2}$.

For general σ and J one can first replace the vectors in each part J_s of the induced cyclic partition $\sigma_J = \langle J_1, \dots, J_l \rangle$ by their sum, thus reducing the number of edges to l. This projects $\Phi_{\sigma,J}$ to a lower-dimensional maximal case, which is $\mathbb{R}^{2l-4}$ by the previous argument. A fiber of this projection consists of possible splittings of the edge vectors into several parallel nonzero vectors from the same J_s, which gives $\mathbb{R}^{|J|-l}$, plus choosing the arguments θ_{I_s} for the subsets I_s with $I_s \cap J = \varnothing$, according to their order in σ, which gives another $\mathbb{R}^{k-l}$. Thus, we conclude that the total space of the fibration is homeomorphic to $\mathbb{R}^{k+|J|-4}$. □

Lemma 4 *Each closed stratum $\overline{\Phi}_{\sigma,J}$ is a topological manifold with boundary.*

Proof Let $\mathcal{D}$ be a k–gon which represents some point in a stratum $\Phi_{\sigma',J'}$ in $\overline{\Phi}_{\sigma,J}$. Here $\sigma' = \langle I'_1, \dots, I'_{k'} \rangle$ is a coarsening of $\sigma = \langle I_1, \dots, I_k \rangle$ and $J' \subseteq J$. We describe a coordinate system in a neighborhood of $\mathcal{D}$ in $\overline{\Phi}_{\sigma,J}$ which maps it to a neighborhood of a corner point in

$$\mathbb{R}_{\geq 0}^{|J|-|J'|} \times \mathbb{R}^{|J'|-2} \times \mathbb{R}_{\geq 0}^{k-k'} \times \mathbb{R}^{k'-2}. \tag{4}$$

We choose $\langle I_-, I_+ \rangle$, a cyclic 2–partition coarsening of σ', and two elements $j_\pm \in J'_\pm := I_\pm \cap J'$ (remember σ' divides J'). Set $J_\pm := I_\pm \cap J$. Let $\mathcal{V}_\pm$ be the sets of vertices of $\mathcal{D}$ which separate subsets of σ' in $I_\pm$, respectively. Together there are $k'-2$ vertices in $\mathcal{V}_-$ and $\mathcal{V}_+$. Let $\mathcal{V}'$ be the set of vertices of $\mathcal{D}$ which separate subsets of σ inside the subsets of σ'. There are $k-k'$ vertices in $\mathcal{V}'$.

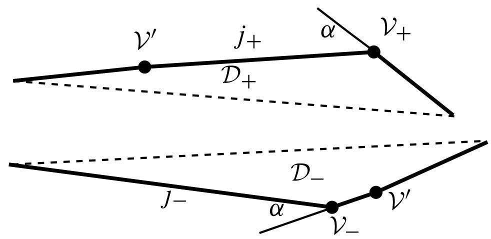

Figure 3: Gluing polygon from its two deformed halves

The first $|J|-2$ coordinates are given by the lengths of edges in J_- and J_+ relative the lengths of j_- and j_+, respectively. Namely we set $x_j := |z_j|/|z_{j_\pm}|$ for $j \in J_\pm \setminus j_\pm$. Note that $x_j = 0$ for $j \in J \setminus J'$ at $\mathcal{D}$ and they can deform only positively. The coordinates x_j give the first two factors in (4). The last two factors in (4) are formed by the exterior angles α_r at the vertices $\mathcal{V}_\pm$ and $\mathcal{V}'$ of $\mathcal{D}$. The angles at $\mathcal{V}'$ are zero at $\mathcal{D}$ and can only deform positively to maintain convexity of nearby polygons in $\overline{\Phi}_{\sigma,J}$.

Any small variation of the x_i and the α_r from the original values at $\mathcal{D}$ will independently deform the two halves $\mathcal{D}_\pm$, which correspond to $I_\pm$ (see Figure 3). Then one uniquely reconstructs a polygon $\mathcal{D}'$ by rescaling (the values of the x_i and the α_r are not changed) and gluing the deformed halves $\mathcal{D}'_\pm$ at the ends. □

Remark The above argument shows that $\overline{\Phi}_{\sigma,J}$ is, in fact, a manifold with corners.

Proposition 5 *Each closed stratum $\overline{\Phi}_{\sigma,J}$ is homeomorphic to a closed ball.*

Proof Lemmas 3 and 4 show that $\overline{\Phi}_{\sigma,J}$ is a compact topological manifold with boundary whose interior is homeomorphic to the Euclidean space. In particular its boundary $\partial\overline{\Phi}_{\sigma,J}$ is simply connected (unless it is of dimension 1). We can remove a topological ball from the interior and use a collar of the boundary to get an h–cobordism between the boundary and the standard sphere which has to be trivial, at least in dimension > 4 [6]. Gluing the ball back in we conclude that $\overline{\Phi}_{\sigma,J}$ has to be a closed ball. In dimensions ≤ 4 one can give an explicit homeomorphism of $\overline{\Phi}_{\sigma,J}$ with a simple polytope (see the remark after Conjecture 1). □

Combining Propositions 2 and 5 we arrive at the desired CW–decomposition of the pair of pants (see eg [7] for details about regular CW–complexes).

Proposition 6 $\overline{P}^{n-1} = \bigcup_{(\sigma,J)\in\mathcal{W}} \Phi_{\sigma,J}$ *is a regular CW–complex.*

2.4 The coamoeba and its decompositions

Consider the argument map

$$\mathrm{Arg}\colon (\mathbb{C}^*)^{n+1}/\mathbb{C}^* \to \mathbb{T}^n, \quad [z_0,\dots,z_n] \mapsto [\arg(z_0),\dots,\arg(z_n)].$$

The closure of the image $\mathrm{Arg}(P^{n-1})$ in $\mathbb{T}^n$ is the *coamoeba* $\mathcal{C}^n$ of the pair of pants. The argument map extends to a continuous surjective map $\mathrm{Arg}\colon \overline{P}^{n-1} \to \mathcal{C}^n$ via the projection from $\Delta \times \mathbb{T}^n$ onto the second factor.

We can think of points in $\mathcal{C}^n$ as allowed configurations of $n+1$ marked points $\theta_0,\dots,\theta_n$ on the circle. A configuration is *allowed* if not all points lie on an open half-circle. Any allowed configuration is realized by a point in $\overline{P}^{n-1}$: we circumscribe a polygon $\mathcal{D}$ around the circle with edges tangent at the θ_i. Excluding *disallowed* configurations leads to a well-known description of $\mathcal{C}^n$ as the complement of the interior of the zonotope (see eg [11, Proposition 2.1])

$$Z = \sum_{i=0}^{n} [0,\pi_i],$$

where an interval $[0,\pi_i] \subset \mathbb{T}^n$ is defined by $\theta_0 = \cdots \widehat{\theta_i} \cdots = \theta_n$ and $\theta_i - \theta_j \in [0,\pi]$ for any $j \neq i$. The facets of Z are given by hyperplanes $\theta_i - \theta_j = \pi$. Among all boundary points of $\mathcal{C}^n$ only the vertices π_I for $I \neq \varnothing$ or $\hat{n}$ are in the image $\mathrm{Arg}(P^{n-1})$.

For any subset $J \subseteq \hat{n}$ we define the partial coamoeba $\mathcal{C}_J$ to be the closure of $\mathrm{Arg}(P_J^{n-1})$ in $\mathbb{T}^n$, where $P_J^{n-1} \subset (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ is the hypersurface given by

$$\sum_{j \in J} z_j = 0.$$

An allowed configuration of points on the circle remains allowed if we add more points to it. This shows that $\mathcal{C}_I \subseteq \mathcal{C}_J$ for $I \subseteq J$. In particular, all $\mathcal{C}_J$ are closed subsets of $\mathcal{C}^n$. Note that $\mathcal{C}_J$ is empty unless $|J| \geq 2$.

We call a subset in $\mathbb{T}^n$ a *polytope* if it is a bijective image of a convex polytope in the universal cover $\mathbb{R}^{n+1}/\mathbb{R}$. We will often define a polytope by a set of inequalities in $\mathbb{R}^{n+1}/\mathbb{R}$ which depends on a cyclic partition σ along with a choice of an initial subset in σ. However, the image polytope in $\mathcal{C}^n$ will be independent of that choice. We give two polytopal decompositions of $\mathcal{C}^n$. The second is a refinement of the first.

The *octahedral* decomposition (the name comes from the case $n = 3$; see Figure 4) is the restriction to $\mathcal{C}^n$ of the stratification $\mathbb{T}^n_\sigma$ of $\mathbb{T}^n$ by the cyclic partitions of $\hat{n}$. For

$\sigma = \langle I_1, \dots, I_k \rangle$ the octahedron $\mathcal{O}_\sigma := \mathcal{C}^n \cap \overline{\mathbb{T}}^n_\sigma$ is given (in $\mathbb{R}^{n+1}/\mathbb{R}$) by

$$\begin{aligned} &\theta_i = \theta_{i'} =: \theta_{I_s} \quad \text{for } i, i' \in I_s, \\ &\theta_{I_s} \le \theta_{I_{s+1}} \le \theta_{I_s} + \pi \quad \text{for } s = 1, \dots, k-1, \\ &\theta_{I_k} \le \theta_{I_1} + 2\pi \le \theta_{I_k} + \pi. \end{aligned} \tag{5}$$

Left inequalities reflect the order of σ. Right inequalities define the boundary of $\mathcal{C}^n$: they exclude disallowed configurations. Changing the initial subset from I_1 to I_r in σ would amount to shifting $\theta_1, \dots, \theta_{r-1}$ by 2π. Note that two distinct lifts to $\mathbb{R}^{n+1}/\mathbb{R}$ of a point in $\mathbb{T}^n$ cannot both satisfy (5). In particular, this means that $\mathcal{O}_\sigma$ is a polytope in $\mathbb{T}^n$.

The full-dimensional octahedra correspond to maximal cyclic partitions. In general, the dimension of $\mathcal{O}_\sigma$ is $k-1$, where k is the number of sets in σ. The vertices are exceptions from this rule; they correspond to cyclic 2–partitions. And there are no 1–dimensional octahedra.

Remark The octahedral decomposition is not a polyhedral complex. The faces of octahedra on the boundary of $\mathcal{C}^n$, except vertices, are not octahedra themselves. In particular, the edges always lie on the boundary.

Example 7 For $n = 2$ there are 2 maximal octahedra (triangles in this case). For $n = 3$ there are 6 maximal octahedra. In Figure 4 (we set $\theta_0 = 0$ and $0 \le \theta_i \le 2\pi$) the red octahedron is $\mathcal{O}_{0213}$ and the blue one is $\mathcal{O}_{0321}$ (we dropped commas and brackets from the subscripts). The triangle face common to the red and green octahedra is $\mathcal{O}_{02\{13\}}$. For $n \ge 3$ a maximal octahedron has $n+1$ pairs of facets (corresponding to $n+1$ pairs of inequalities (5)). The $n+1$ facets in the interior of $\mathcal{C}^n$ are the $(n-1)$–maximal octahedra. Opposite to each such octahedron lies an $(n-1)$–simplex, which is on the boundary of $\mathcal{C}^n$.

Let us look at the induced decompositions of partial coamoebas $\mathcal{C}_J$. Let $\sigma = \langle I_1, \dots, I_k \rangle$ and $\sigma_J = \langle J_1, \dots, J_r \rangle$. The intersection $\mathcal{O}_{\sigma,J} := \mathcal{C}_J \cap \mathcal{O}_\sigma$ is not an octahedron anymore, though it is still a polytope in $\mathbb{T}^n$. Namely, it is cut out by the inequalities (in $\mathbb{R}^{n+1}/\mathbb{R}$)

$$\begin{aligned} &\theta_i = \theta_{i'} =: \theta_{I_s} \quad \text{for } i, i' \in I_s, \\ &\theta_{I_1} \le \dots \le \theta_{I_k} \le \theta_{I_1} + 2\pi, \\ &\theta_{j_{s+1}} \le \theta_{j_s} + \pi \quad \text{and} \quad \theta_{j_1} + \pi \le \theta_{j_r}, \quad \text{where } j_s \in J_s. \end{aligned} \tag{6}$$

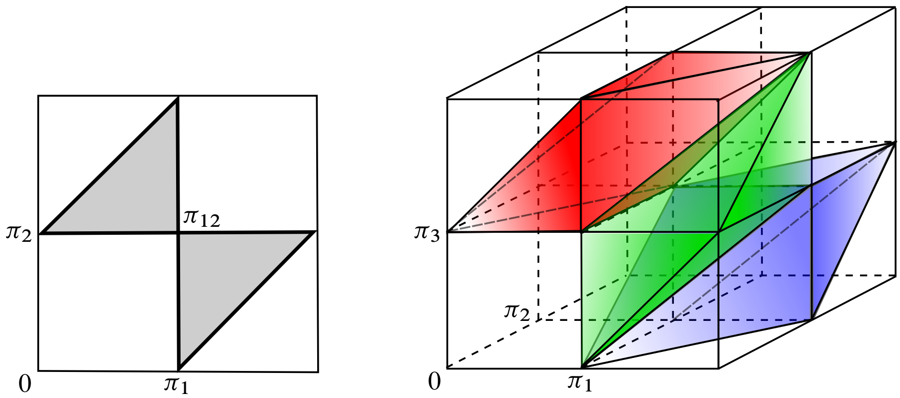

Figure 4: Octahedral subdivisions: the two triangles for $n=2$ and three of the six octahedra for $n=3$

Recall that we drop the empty sets $I_s \cap J$ from σ_J and shift the indexing. Thus, the inequalities in the third line of (6), which define the boundary of the partial coamoeba, are generally stronger than the ones in (5).

The *alcove* decomposition of $\mathcal{C}^n$ (the name comes from the affine root system $\widehat{A}_n$) is the restriction of the triangulation of $\mathbb{T}^n$ induced from the decomposition of $\mathbb{R}^{n+1}/\mathbb{R}$ by the hyperplanes

$$\theta_i - \theta_j \in \pi\mathbb{Z} \quad \text{for all pairs } i, j \in \hat{n}. \tag{7}$$

The octahedra and their intersections with partial coamoebas are cut out by hyperplanes of the same form which means that all $\mathcal{O}_{\sigma,J}$ are triangulated by alcoves.

The hyperplanes (7) break $\mathbb{T}^n$ into $n!\cdot 2^n$ maximal simplices. The $(n+1)!$ of them are incident to 0; they form the zonotope Z and are not part of the coamoeba. Thus, each of the $n!$ maximal octahedra in $\mathcal{C}^n$ consists of $2^n - n - 1$ maximal alcoves. For example, for $n=3$ each of the maximal octahedra is broken into 4 alcoves according to the relative directions of the opposite pairs of edges in the representing polygons

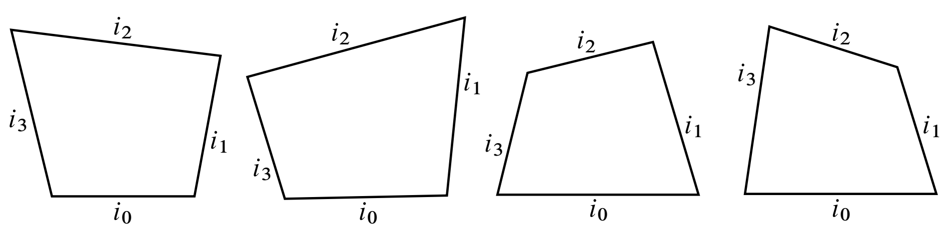

Figure 5: The four shapes of generic quadrilaterals

(see Figure 5). The zero-dimensional octahedra are also the zero-dimensional alcoves, and they are the vertices of the coamoeba.

To label alcoves we introduce certain combinatorial objects τ which refine cyclic partitions σ. Think of σ as coming from a configuration of points $\theta_0, \dots, \theta_n$ on the circle. Identifying the opposite points of the circle gives a new configuration of points on the quotient circle, that is, another cyclic partition $\widetilde{\sigma}$. The purpose of τ is to encode both σ and $\widetilde{\sigma}$.

Given a cyclic partition $\sigma = \langle I_1, \dots, I_k \rangle$ we mark k distinct points on the boundary of a disk (which we will call vertices) and label the k boundary arcs between the vertices by the sets I_s in the order given by σ. We say that a nonempty collection of chords in the disk with endpoints at the marked vertices is a *net* τ if any two chords intersect (possibly at the endpoints). If some of the vertices on the circle are not used by any of the chords in τ we can join the nonseparated arcs together, thus getting a coarsening of σ, which we denote by $\sigma(\tau)$. Instead of the original σ we rather let the cyclic partition $\sigma(\tau)$ be a part of the intrinsic information in τ.

One can think of nets as the Möbius band decompositions as follows. We put the disk in $\mathbb{RP}^2$ and extend chords in τ to lines in $\mathbb{RP}^2$. The complement of the disc is the Möbius band with boundary broken into arcs also labeled by $\sigma(\tau)$. Any two chords intersect inside the disk, which means that their complements, which we call *intervals*, do not intersect. That is, τ can be thought of a decomposition of the Möbius band by intervals into triangles and trapezoids. Maximal decompositions with fixed $\sigma = \sigma(\tau)$ are triangulations (no trapezoids) and they use k intervals. Minimal decompositions with fixed $\sigma = \sigma(\tau)$ consist of trapezoids (plus one triangle if k is odd). Thus, the number l of chords in a net can be any integer between $\frac{k}{2}$ and k.

The midcircle in the Möbius band (the "horizon" in $\mathbb{RP}^2$) oriented along its boundary defines a new cyclic partition $\widetilde{\sigma}(\tau) = \langle K_1, \dots, K_l \rangle$ of $\hat{n}$, which we call the *shuffle* of τ. Opposite sides of trapezoids in τ are combined into single subsets in $\widetilde{\sigma}(\tau)$. The number of subsets in $\widetilde{\sigma}(\tau)$ equals the number of chords in τ.

Example 8 Figure 6, left, is an example of a 4–chord net τ with $\sigma(\tau) = \langle i_0, \dots, i_4 \rangle$. The corresponding Möbius band on the right is glued by identifying left and right blue intervals (turning one of the them upside down). We can always cut the Möbius band along an interval in τ and picture the subdivision like that. This helps to see the shuffle order, which in this case is $\widetilde{\sigma}(\tau) = \langle i_0, i_3, \{i_1, i_4\}, i_2 \rangle$.

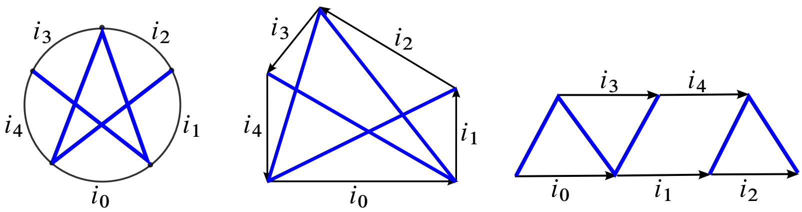

Figure 6: Net of chords, diameters in $\mathcal{D}$ (sides i_1 and i_4 are parallel) and a subdivision of the Möbius band

Given a net τ we define the alcove $\mathcal{A}_\tau \subset \mathbb{T}^n$ as the image of a simplex in $\mathbb{R}^{n+1}/\mathbb{R}$. As before we choose an initial set in $\sigma(\tau) = \langle I_1, \dots, I_k \rangle$. First, for $i, i' \in I_s$ we set

$$\theta_i = \theta_{i'} =: \theta_{I_s}. \tag{8}$$

Next, if I_s and I_r are opposite sides of a trapezoid in τ, we set

$$\theta_{I_r} + \pi = \theta_{I_s} \quad \text{if } r < s. \tag{9}$$

Finally, we describe the inequalities, one for each chord (or interval) in τ. Let $I_s \subseteq K_{s'}$ follow right after $I_r \subseteq K_{r'}$ in the shuffle order (that is, $s' = r' + 1$, or $r' = l$ and $s' = 1$). If I_s also follows right after I_r in the $\sigma(\tau)$–order (that is $s = r + 1$, or $r = k$ and $s = 1$) we set

$$\begin{aligned} \theta_{I_r} &\le \theta_{I_s} && \text{if } 1 \le r < k, \\ \theta_{I_r} &\le \theta_{I_s} + 2\pi && \text{if } r = k \text{ and } s = 1. \end{aligned} \tag{10}$$

If I_s does not follow I_r in the $\sigma(\tau)$–order, we set

$$\begin{aligned} \theta_{I_r} + \pi &\le \theta_{I_s} && \text{if } r < s, \\ \theta_{I_r} &\le \theta_{I_s} + \pi && \text{if } r > s. \end{aligned} \tag{11}$$

If $K_{r'}$ or $K_{s'}$ (or both) contains more than one (ie two) subsets from $\sigma(\tau)$ then, given (9), any choice of a pair $(I_r, I_s) \subseteq (K_{r'}, K_{s'})$ gives rise to the same inequality.

Altogether, the inequalities (8), (9), (10) and (11) define an $(l-1)$–dimensional simplex in $\mathbb{R}^{n+1}/\mathbb{R}$ which descends to a simplex in $\mathbb{T}^n$. This is the alcove $\mathcal{A}_\tau$. Its relative interior is defined by replacing (10) and (11) with strict inequalities. In Example 8 (see

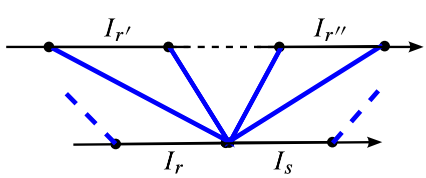

Figure 7: A fragment of the Möbius band

Figure 6) the alcove $\mathcal{A}_\tau$ is defined by

$$\theta_{i_4} = \theta_{i_1} + \pi, \quad \theta_{i_0} + \pi \le \theta_{i_3}, \quad \theta_{i_3} \le \theta_{i_4}, \quad \theta_{i_1} \le \theta_{i_2}, \quad \theta_{i_2} \le \theta_{i_0} + \pi.$$

There is another, nonminimal but more intuitive, set of inequalities which defines $\mathcal{A}_\tau$ directly in $\mathbb{T}^n$. It keeps track of relative positions of *all* pairs of points θ_i and θ_j on the circle, ie which half of the circle the differences $\theta_j - \theta_i$ belong to. If i and j are elements in different subsets in $\sigma(\tau)$ then any chord in τ which does *not* divides $\{i, j\}$ defines an order between i and j (going counterclockwise). All such chords in τ give the same order (otherwise they would not intersect). We write $i \to_\tau j$ if i comes first in this order. Two elements $i, j \in \hat{n}$ may be divided by

(1) no chords in τ, that is, i and j belongs to the same I_s in $\sigma(\tau)$;

(2) all chords in τ, that is, i and j lie in opposite sides of a trapezoid; or

(3) some but not all chords in τ, that is, i and j belong to different cells in the Möbius band decomposition.

We set

$$\begin{aligned} \theta_i &= \theta_j && \text{in case (1)}, \\ \theta_i - \theta_j &= \pi \bmod 2\pi && \text{in case (2)}, \\ \theta_j - \theta_i &\in [0, \pi] \bmod 2\pi && \text{in case (3) with } i \to_\tau j. \end{aligned} \tag{12}$$

Lemma 9 *The alcove $\mathcal{A}_\tau$ is defined by* (12).

Proof In the lift to $\mathbb{R}^{n+1}/\mathbb{R}$ associated with the initial subset I_1 in $\sigma(\tau)$ the first two equations in (12) are the same as (8) and (9). The inequalities (10) and (11) form a subset of the third line in (12) for pairs i and j next to each other in the shuffle order.

In the opposite direction, let us deduce, say, $\theta_{I_r} \le \theta_{I_s}$ for $I_r \to_\tau I_s$ and $r < s$. The other inequalities in (12) are similar. Choose a chord which does not divide I_r and I_s and cut the Möbius band along it. Then the Möbius band unfolds into a strip with I_r and I_s on one side. By induction, we may assume that I_r and I_s are neighbors in the $\sigma(\tau)$–order, that is, $s = r+1$, but there may be several subsets $I_{r'}, \dots, I_{r''}$ "shuffled" in-between in the shuffle order; see Figure 7.

There are several cases for the order among the subscripts r, r' and r''. We consider eg the case $r'' < r < s = r+1 < r'$, the others being similar. Then (10) and (11) will read

$$\begin{aligned} &\pi + \theta_{I_r} \le \theta_{I_{r'}} \le \dots \le \theta_{I_k} \le \theta_{I_1} + 2\pi \le \dots \le \theta_{I_{r''}} + 2\pi, \\ &\theta_{I_{r''}} + \pi \le \theta_{I_s}, \end{aligned} \tag{13}$$

which imply $\theta_{I_r} \le \theta_{I_s}$. □

All nets, or equivalently all subdivisions of the Möbius band, form a poset under refinements. We set $\operatorname{rk}\tau := l-1$, where l is the number of chords in τ. This is the dimension of the alcove $\mathcal{A}_\tau$. Clearly, $\tau \preceq \tau' \Longrightarrow \sigma(\tau) \preceq \sigma(\tau')$. For any $J \subseteq \hat{n}$ we say a chord in τ *divides* J if J does not lie on one side of it. We say τ *divides* J (and write $\tau \mid J$) if all of its chords do. In Example 8 (see Figure 6) the net τ divides only $J = \{i_1, i_4\}$ among all two-element subsets, but it divides any $J \subseteq \hat{n}$ with $|J| \geq 3$.

Proposition 10 *$\mathcal{A}_\tau \subseteq \mathcal{O}_{\sigma,J}$ if and only if $\sigma(\tau) \preceq \sigma$ and τ divides J.*

Proof The "if" part follows directly from Lemma 9. Indeed, the inequalities (6), which define $\mathcal{O}_{\sigma,J}$, are special cases of (12) for i and j belonging to neighboring subsets in $\sigma(\tau)$.

For the converse, to a polygon $\mathcal{D}$ representing a point in $\mathcal{O}_{\sigma,J}$ we associate a net τ with $\sigma(\tau) \preceq \sigma$ and $\tau \mid J$ as follows. Given a line through a vertex v of $\mathcal{D}$ we say that $\mathcal{D}$ lies *strictly* on one side of the line if both adjacent edge vectors from v lie in the same *open* half-plane. Two vertices are connected by a *diameter* if $\mathcal{D}$ lies strictly between two parallel lines through these two vertices. Any polygon $\mathcal{D}$ has at least one diameter, a geometric one, and clearly any two diameters intersect. That is, the set of diameters forms a net τ on a disk with boundary arcs labeled by the sides of $\mathcal{D}$ (see Figure 6). Moreover any diameter must have nonzero edges on both sides of it, that is, τ divides J.

A set of diameters in a polygon $\mathcal{D}$ recovers all pairwise relations among the directions θ_i of its sides. It is easy to see that these relations are given by (12) for the corresponding net τ. Thus, the point in $\mathcal{O}_{\sigma,J}$ represented by $\mathcal{D}$ falls into the alcove $\mathcal{A}_\tau$ in $\mathcal{C}^n$. □

2.5 Phase tropical pair of pants as a CW–complex

Consider

$$F(x) = \max\{x_0, x_1, \ldots, x_n\},$$

a convex PL function on $\mathbb{R}^{n+1}$. Its corner locus is invariant under the diagonal translation by $\mathbb{R}$, hence it descends to an $(n-1)$–dimensional polyhedral fan in $\mathbb{R}^{n+1}/\mathbb{R}$, which is known as the *tropical hyperplane* $\mathcal{P}^{n-1}$. The cones $\mathcal{P}_I$ in $\mathcal{P}^{n-1}$ are indexed by subsets $I \subseteq \hat{n}$ of size $|I| \geq 2$: the cone $\mathcal{P}_I$ is defined by

$$x_i = x_j \geq x_k \quad \text{for all } i, j \in I \text{ and } k \notin I.$$

The vertex $\mathcal{P}_{\hat{n}}$ of $\mathcal{P}^{n-1}$ is at the origin in $\mathbb{R}^{n+1}/\mathbb{R}$.

The closure $\overline{\mathcal{P}}^{n-1}$ of $\mathcal{P}^{n-1}$ in Δ is a polyhedral complex: the map (3) takes each cone $\mathcal{P}_I$ to a linear subspace in the interior of Δ and the linearity extends to the closure. Also note that the face $\Delta_{I'}$ of the simplex Δ intersects the closure of $\mathcal{P}_I$ only if the subset I' contains I, which gives additional labeling to the boundary faces of $\overline{\mathcal{P}}^{n-1}$ by subsets $I' \supseteq I$. Each face $\mathcal{P}_{I,I'}$ of $\overline{\mathcal{P}}^{n-1}$ is a polytope of dimension $|I'|-|I|$.

The *phase tropical pair of pants* $\mathcal{TP}^{n-1} \subset (\mathbb{C}^*)^{n+1}/\mathbb{C}^* = (\mathbb{R}^{n+1}/\mathbb{R}) \times \mathbb{T}^n$ is the union

$$\mathcal{TP}^{n-1} := \bigcup_{I \subseteq \hat{n}} (\mathcal{P}_I \times \mathcal{C}_I).$$

The compactified version $\mathcal{T}\overline{\mathcal{P}}^{n-1}$ is the closure of $\mathcal{TP}^{n-1}$ in $\Delta \times \mathbb{T}^n$. The (σ, J)–stratification on $\Delta \times \mathbb{T}^n$ induces a stratification of $\mathcal{T}\overline{\mathcal{P}}^{n-1}$. We denote by $\overline{\Psi}_{\sigma,J}$ the corresponding closed stratum of $\mathcal{T}\overline{\mathcal{P}}^{n-1}$.

Proposition 10 says that each $\mathcal{O}_{\sigma,I}$ is triangulated into alcoves $\mathcal{A}_\tau$ with $\sigma(\tau) \preceq \sigma$ and $\tau \mid I$. This makes $\overline{\Psi}_{\sigma,J}$ into a polyhedral complex in $\Delta \times \mathbb{T}^n$,

$$\overline{\Psi}_{\sigma,J} = \bigcup_{(I,I',\tau)} \mathcal{P}_{I,I'} \times \mathcal{A}_\tau,$$

where the triples (I, I', τ) satisfy $I \subseteq I' \subseteq J$, $\sigma(\tau) \preceq \sigma$ and $\tau \mid I$. The face order between legitimate triples is $(I, I', \tau) \preceq (\tilde{I}, \tilde{I}', \tilde{\tau})$ if $I \supseteq \tilde{I}$, $I' \subseteq \tilde{I}'$ and $\tau \preceq \tilde{\tau}$.

Proposition 11 *The decomposition of $\mathcal{T}\overline{\mathcal{P}}^{n-1}$ into $\overline{\Psi}_{\sigma,J}$ is a regular CW–complex.*

To prove the proposition we will show (Lemmas 12 and 13) that each $\overline{\Psi}_{\sigma,J}$ is a collapsible PL manifold with boundary of dimension $\operatorname{rk}(\sigma, J)$. We begin with the collapsibility. Recall the collapsing operation on a polyhedral complex X. Let F be a face of X and let G be a facet of F, such that G is not a subface of any other face in X. Then we can remove both F and G and call this an elementary collapse. We say that a polyhedral complex is *collapsible* if it can be reduced to a vertex by a sequence of elementary collapses.

Lemma 12 *$\overline{\Psi}_{\sigma,J}$ is a collapsible polyhedral complex of pure dimension* $\operatorname{rk}(\sigma, J)$.

Proof The maximal faces (I, I', τ) in $\overline{\Psi}_{\sigma,J}$ are of two types:

Type I τ is a maximal net with $\sigma(\tau) = \sigma$ and I has three elements (a maximal τ cannot divide a set of two elements).

Type II τ contains a single trapezoid (and $k-2$ triangles) and $|I|=2$; its two elements belong to the opposite sides of the trapezoid.

In both cases, $I'=J$ and

$$\dim(\mathcal{P}_{I,J}\times\mathcal{A}_{I(\tau)})=|J|-|I|+(\#\{\text{chords in }\tau\}-1)=|J|+k-4=\mathrm{rk}(\sigma,J).$$

Any face (I,I',τ) is a subface of (I,J,τ). Adding chords to τ and/or removing elements from I, one can see that any face is a subface of a maximal face. That is, the complex is indeed of pure dimension $\mathrm{rk}(\sigma,J)$.

To collapse $\overline{\Psi}_{\sigma,J}$ we look at its face lattice. For a given pair (I,τ) the interval between (I,I,τ) and (I,J,τ) consists of all subsets I' between I and J. In particular, it is Boolean, hence possesses a matching unless $I=J$. Note that if τ is maximal, elements in the interval $[(I,I,\tau),(I,J,\tau)]$ are not subfaces of anything outside the interval. Then we can remove the entire interval. The same can be said about intervals with $|I|=2$. Proceeding by alternating induction on the number of chords in τ and number of elements in I, we remove all faces of $\overline{\Psi}_{\sigma,J}$ except those with $I=J$.

Thus, it remains to collapse the fiber over the vertex $\mathcal{P}_{J,J}$. This fiber is the polytope $\mathcal{O}_{\sigma,J}$ triangulated into alcoves, which is clearly collapsible. □

Lemma 13 *$\overline{\Psi}_{\sigma,J}$ is a topological manifold with boundary.*

Before proving Lemma 13 we discuss a certain general property of convex cones. Let V be an n–dimensional vector space over $\mathbb{R}$ and write V^* for its dual. Let $\mathcal{R}\subset V$ and $\mathcal{R}^\vee\subset V^*$ be a dual pair of convex cones. That is,

$$\mathcal{R}=\{v\in V:\lambda(v)\geq 0,\ \lambda\in\mathcal{R}^\vee\},\tag{14}$$

and vice versa. We assume both $\mathcal{R}$ and $\mathcal{R}^\vee$ have nonempty interiors. For any $v\in\mathcal{R}$ we define the *supporting tangent cone* $T_v\mathcal{R}$ to be the set of vectors in V lying in supporting hyperplanes to $\mathcal{R}$ at v.

We define the *total supporting tangent space* of $\mathcal{R}$ to be the union

$$T\mathcal{R}:=\bigcup_{v\in\mathcal{R}}(\{v\}\times T_v\mathcal{R})=\bigcup_{\lambda\in\mathcal{R}^\vee\setminus\{0\}}((\ker\lambda\cap\mathcal{R})\times\ker\lambda)\subset V\times V.\tag{15}$$

The first presentation shows that $T\mathcal{R}$ is a fibration over $\mathcal{R}$ supported on its boundary ∂R. An example of a two-dimensional polyhedral cone $\mathcal{R}$ is illustrated in Figure 8.

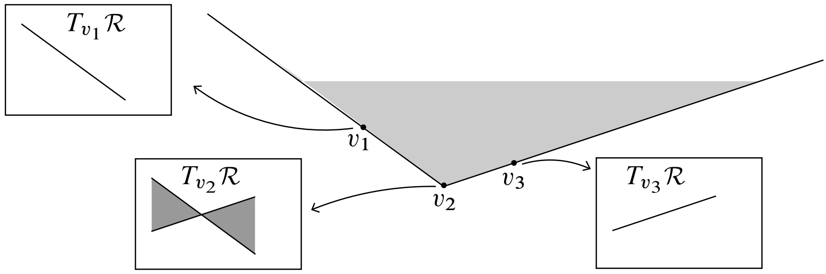

Figure 8: Two-dimensional cone and fibers of the total supporting tangent space of its boundary

We fix a vector $\widetilde{v}$ in the interior of $\mathcal{R}$ and a vector $\widetilde{\lambda}$ in the interior of $\mathcal{R}^\vee$ such that $\widetilde{\lambda}(\widetilde{v}) = 1$. Then in the second presentation of $T\mathcal{R}$ in (15) it is enough to take the union over $\{\lambda \in \mathcal{R}^\vee : \lambda(\widetilde{v}) = 1\}$.

Denote by W the quotient space $V/(\mathbb{R}\widetilde{v})$, and we write $\pi\colon V \to W$ for the projection. Consider the map $\phi\colon T\mathcal{R} \to W$ given by $\phi(v,u) = \pi(u)$.

Lemma 14 *The total supporting tangent space $T\mathcal{R}$ is homeomorphic to $\mathbb{R}^{2n-2}$. Moreover, the map $\phi\colon T\mathcal{R} \to W$ is a trivial fiber bundle with fiber homeomorphic to $W \cong \mathbb{R}^{n-1}$.*

Proof We will show that the map $\psi\colon T\mathcal{R} \to W \times W$ given by

$$\psi(v,u) = (\pi(v) + \widetilde{\lambda}(u) \cdot \pi(u), \pi(u)) \tag{16}$$

is a homeomorphism. The geometric meaning of the map ψ is to "stretch out" the fibers $\phi^{-1}(w)$ into $\mathbb{R}^{n-1}$; see Figure 9. Then $\phi\colon T\mathcal{R} \to W$ is the composition of ψ with the projection onto the second factor and, hence, is a topologically trivial fiber bundle with fiber homeomorphic to W.

The map ψ is clearly continuous. Surjectivity follows from the intermediate value theorem applied to each fiber $\phi^{-1}(w)$ for a given $w \in W$.

Injectivity Let $\psi(v_1,u_1) = \psi(v_2,u_2)$. Then according to (16) we must have

$$\begin{aligned} v_1 - v_2 + \widetilde{\lambda}(u_1 - u_2) \cdot u_1 &= 0 \mod \widetilde{v}, \\ v_1 - v_2 + \widetilde{\lambda}(u_1 - u_2) \cdot u_2 &= 0 \mod \widetilde{v}. \end{aligned} \tag{17}$$

Say $v_1, u_1 \in \ker \lambda_1$ and $v_2, u_2 \in \ker \lambda_2$ for some $\lambda_{1,2} \in \mathcal{R}^\vee$ with $\lambda_{1,2}(\widetilde{v}) = 1$. Note that $\lambda_1(v_2) \ge 0$ and $\lambda_2(v_1) \ge 0$. Applying λ_1 and λ_2 to the respective equations (17),

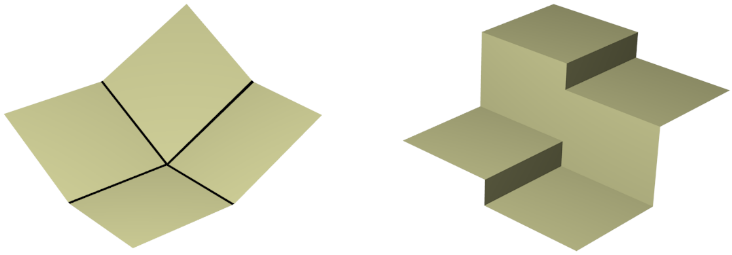

Figure 9: On the left is the boundary of $\partial\mathcal{R}$, and on the right is an example of the fiber $\phi^{-1}(w)$ for some $w \in W$, which is homeomorphic to $\mathbb{R}^2$.

we get

$$v_1 - v_2 + \widetilde{\lambda}(u_1 - u_2)\cdot u_1 = -\lambda_1(v_2)\cdot\widetilde{v},$$
$$v_1 - v_2 + \widetilde{\lambda}(u_1 - u_2)\cdot u_2 = \lambda_2(v_1)\cdot\widetilde{v}.$$

Subtracting one equation from another we get

$$\widetilde{\lambda}(u_1 - u_2)\cdot(u_1 - u_2) = -(\lambda_1(v_2) + \lambda_2(v_1))\cdot\widetilde{v}. \tag{18}$$

Finally, applying $\widetilde{\lambda}$ we arrive at $\widetilde{\lambda}(u_1 - u_2)^2 \le 0$, which, combined with $\pi(u_1) = \pi(u_2)$, gives $u_1 = u_2$. That implies that $\pi(v_1) = \pi(v_2)$, and since both $v_1, v_2 \in \partial\mathcal{R}$, we have $v_1 = v_2$.

Continuity of the inverse Fix a Euclidean metric on $\ker\widetilde{\lambda} \cong W$ and extend it to a Euclidean metric on V. Then notice that the linear functionals $\lambda \in \mathcal{R}^\vee$ with $\lambda(\widetilde{v}) = 1$ are uniformly bounded, and so are the ratios $|u|/|\pi(u)|$ and $|v|/|\pi(v)|$ for all $(v,u) \in T\mathcal{R}$. Suppose $|\psi(v_1,u_1) - \psi(v_2,u_2)| < \epsilon$. Then, following the injectivity arguments above, we deduce that $|v_1 - v_2| < C\epsilon$ and $|u_1 - u_2| < C\epsilon$ for some universal constant C. □

Remark The lemma is a consequence of the convexity property of $\mathcal{R}$. It holds for $\mathcal{R}$ replaced by the upper graph of any convex function $f\colon W \to \mathbb{R}$.

For the proof of Lemma 13 we will apply the above lemma to the following convex cone. Let $\sigma_0 = \langle I_-, I_+\rangle$ be a cyclic 2–partition of $\hat{n}$ together with a choice of the initial subset. Define a convex polyhedral cone $\mathcal{R} \subset \mathbb{R}^{n+1}/\mathbb{R}$ by the set of inequalities

$$x_{i_-} \le x_{i_+}, \quad i_- \in I_-,\ i_+ \in I_+. \tag{19}$$

Its boundary $\partial\mathcal{R}$ is a polyhedral fan, whose cones $\mathcal{R}_I$ are labeled by subsets $I \subseteq \hat{n}$ which are divided by σ_0, ie both $I_- \cap I$ and $I_+ \cap I$ are nonempty. That is, the face lattice of $\partial\mathcal{R}$ is dual to the face lattice of the product of simplices $\Delta_{I_-} \times \Delta_{I_+}$. The cone $\mathcal{R}_I$ is defined by

$$\begin{aligned} x_i &= x_{i'} \quad \text{for } i, i' \in I, \\ x_{i_-} &\le x_{i_+} \quad \text{for } i_\pm \in I_\pm. \end{aligned} \tag{20}$$

We identify the tangent space of $\mathbb{R}^{n+1}/\mathbb{R}$ at any point with $\mathbb{R}^{n+1}/\mathbb{R}$ and consider the total supporting tangent space

$$T\mathcal{R} \subset (\mathbb{R}^{n+1}/\mathbb{R}) \times (\mathbb{R}^{n+1}/\mathbb{R}).$$

Let $[u_0, \dots, u_n]$ be the homogeneous coordinates in the second (tangent) factor which are parallel to $[x_0, \dots, x_n]$. Following the notation introduced just before Lemma 14, we let $\widetilde{v} \in \mathbb{R}^{n+1}/\mathbb{R}$ be the vector with coordinates

$$\begin{cases} u_{i_-} = 0, & i_- \in I_-, \\ u_{i_+} = 1, & i_+ \in I_+, \end{cases} \tag{21}$$

and denote by W the quotient $(\mathbb{R}^{n+1}/\mathbb{R})/(\mathbb{R}\widetilde{v})$. Let $\phi\colon T\mathcal{R} \to W$ be the projection onto W in the tangent factor.

Now let

$$\sigma = \langle \underbrace{I_1, \dots, I_r}_{I_-}, \underbrace{I_{r+1}, \dots, I_k}_{I_+} \rangle,$$

be a refinement of σ_0. The set of inequalities

$$\begin{aligned} u_i = u_{i'} =: u_{I_s} \quad &\text{for } i, i' \in I_s, \\ u_{I_1} \le \dots \le u_{I_r}, \quad &u_{I_{r+1}} \le \dots \le u_{I_k}, \end{aligned} \tag{22}$$

defines a cone $C_{\sigma_0,\sigma}$ in $W = (\mathbb{R}^{n+1}/\mathbb{R})/(\mathbb{R}\widetilde{v})$. Then, by Lemma 14, $\phi^{-1}(C_{\sigma_0,\sigma})$ is homeomorphic to $W \times C_{\sigma_0,\sigma}$, which is a manifold with boundary.

Next we describe the fibers of the projection of $\phi^{-1}(C_{\sigma_0,\sigma}) \subset \partial\mathcal{R} \times (\mathbb{R}^{n+1}/\mathbb{R})$ onto the first factor. Let x be a point in the relative interior of a face $\mathcal{R}_I$ of $\partial\mathcal{R}$. A vector $u \in \mathbb{R}^{n+1}/\mathbb{R}$ is in the supporting tangent cone $T_x\mathcal{R}$ if and only if it is in the kernel of some nonzero linear functional $\lambda \in \mathcal{R}^\vee$ with $\lambda(x) = 0$. But $\mathcal{R}^\vee \cap (\ker x)$ is positively spanned by the vectors $e_{i_+} - e_{i_-}$ for $i_\pm \in I_\pm \cap I$, where $\{e_i\}$ is the dual basis to the coordinates $[x_0, \dots, x_n]$. It means that there are two pairs $i_{1,2}^- \in I_- \cap I$ and $i_{1,2}^+ \in I_+ \cap I$ such that

$$u_{i_1^-} - u_{i_1^+} \le 0 \quad \text{and} \quad u_{i_2^-} - u_{i_2^+} \ge 0. \tag{23}$$

Consider the partition of I induced by σ,

$$\sigma_I = \langle \underbrace{I^-_{\min}, \dots, I^-_{\max}}_{I_- \cap I}, \underbrace{I^+_{\min}, \dots, I^+_{\max}}_{I_+ \cap I} \rangle. \tag{24}$$

Then, given the inequalities (22) for the cone $C_{\sigma_0,\sigma}$, the existence of pairs $i^-_{1,2} \in I_- \cap I$ and $i^+_{1,2} \in I_+ \cap I$ satisfying (23) becomes equivalent to

$$u_{i^-_{\max}} \geq u_{i^+_{\min}} \quad \text{and} \quad u_{i^+_{\max}} \geq u_{i^-_{\min}}, \quad \text{where } i^{\pm}_{\min,\max} \in I^{\pm}_{\min,\max}. \tag{25}$$

Thus, the fiber of $\phi^{-1}(C_{\sigma_0,\sigma})$ over the relative interior of a face $\mathcal{R}_I$ is cut out by the set of inequalities

$$\begin{gathered} u_i = u_{i'} =: u_{I_s} \quad \text{for } i, i' \in I_s, \\ u_{I_1} \leq \dots \leq u_{I_r}, \quad u_{I_{r+1}} \leq \dots \leq u_{I_k}, \\ u_{i^-_{\max}} \geq u_{i^+_{\min}} \quad \text{and} \quad u_{i^+_{\max}} \geq u_{i^-_{\min}}, \quad \text{where } i^{\pm}_{\min,\max} \in I^{\pm}_{\min,\max}. \end{gathered} \tag{26}$$

Finally, we remark that up to an isomorphism the space $\phi^{-1}(C_{\sigma_0,\sigma})$ does not depend on the choice of the initial subset in σ_0. Had we chosen I_+ instead of I_-, the cone $\mathcal{R}$ would have changed to its opposite. The fibers over the corresponding cones in $\partial\mathcal{R}$ would remain the same.

Let us return to $\mathcal{T}\overline{\mathcal{P}}^{n-1}$. First, recall the notion of the local fan in a polyhedral complex. If v is a vertex in a face F of a polyhedral complex $X \subset \mathbb{R}^n$, then the *local cone* $\Sigma_v F$ of F at v is the set of vectors

$$\{w \in T_v\mathbb{R}^n : v + \epsilon w \in F \text{ for some } \epsilon > 0\} \tag{27}$$

in the tangent space $T_v\mathbb{R}^n \cong \mathbb{R}^n$. The *local fan* $\Sigma_v X$ of X at v is the union of local cones over all faces F of X containing v.

Proof of Lemma 13 We will show that the local fan $\Sigma_v \overline{\Psi}_{\sigma,J}$ at any vertex v of $\overline{\Psi}_{\sigma,J}$ is homeomorphic to the $(k+|J|-4)$–dimensional half-space. First, we consider a maximal case: $J = \hat{n}$ and v is a vertex of $\overline{\Psi}_\sigma$ which lies over the vertex of $\mathcal{P}^{n-1}$. That is, $v = \mathcal{P}_{\overline{n}} \times \{a\}$, where $a = \pi_{I_-} = \pi_{I_+}$ (see Section 2.1), is a vertex of $\mathcal{O}_\sigma$. The corresponding cyclic 2–partition

$$\sigma_a = \langle I_-, I_+ \rangle = \langle I_1 \cup \dots \cup I_r, I_{r+1} \cup \dots \cup I_k \rangle$$

is a coarsening of σ.

The local fan $\Sigma_v \overline{\Psi}_\sigma$ maps to the tropical hyperplane $\mathcal{P}^{n-1}$. The fiber over the relative interior of a cone $\mathcal{P}_I$ is nonempty if and only if $a \in \mathcal{O}_{\sigma,I}$, that is, if both sets $I \cap I_\pm$

are nonempty. Thus, the collection of faces $\mathcal{P}_I$ with nonempty fibers forms a subfan $\mathcal{P}(a)$ of $\mathcal{P}^{n-1}$ whose face lattice is dual to the face lattice of $\Delta_{I_-} \times \Delta_{I_+}$. In particular, the fan $\mathcal{P}(a)$ is isomorphic to the boundary fan $\partial\mathcal{R}$ of the polyhedral cone $\mathcal{R}$ defined in (19).

Next we describe the fiber over the relative interior of $\mathcal{P}_I$. It is the relative cone of the polytope $\mathcal{O}_{\sigma,I}$ at its vertex a. Let u_i be the homogeneous coordinates on the tangent space $T_v\mathbb{T}^n = \mathbb{R}^{n+1}/\mathbb{R}$ which are parallel to the coordinates θ_i on $\mathbb{T}^n$, and let

$$\sigma_I = \langle \underbrace{I^-_{\min}, \dots, I^-_{\max}}_{I\cap I_-}, \underbrace{I^+_{\min}, \dots, I^+_{\max}}_{I\cap I_-} \rangle$$

be the cyclic partition of I induced by σ. Then the subset of the defining inequalities (6) for the polytope $\mathcal{O}_{\sigma,I}$ at a is identical to (26). Thus, $\Sigma_v\Psi_\sigma$ is homeomorphic to $\phi^{-1}(C_{\sigma_0,\sigma})$, which by Lemma 14 is homeomorphic to the $(n+k-3)$–dimensional half-space.

Now we allow J to be a proper subset of $\hat{n}$, but still consider a vertex v of $\overline{\Psi}_{\sigma,J}$ which lies over the vertex of the corresponding lower-dimensional tropical hyperplane $\mathcal{P}^{|J|-2} \subset \Delta_J$. That is, $v = \mathcal{P}_{J,J} \times \{a\}$, where a is a vertex of $\mathcal{O}_{\sigma,J}$. Locally near v the space $(\mathcal{T}\overline{\mathcal{P}}^{n-1}) \cap (\Delta_J \times \mathbb{T}^n)$ is the product $\mathcal{T}\mathcal{P}^{|J|-2} \times \mathbb{T}^{\hat{n}\setminus J}$. Moreover, by choosing the splitting $\mathbb{R}^{n+1}/\mathbb{R} = \mathbb{R}^J/\mathbb{R} \times \mathbb{R}^{\hat{n}\setminus J}$ so that the vector $\tilde{v}$ (see (21)) lies in $\mathbb{R}^J/\mathbb{R}$ the product structure can be made compatible with the projection by $\tilde{v}$. Then the projection $\phi\colon \phi^{-1}(C_{\sigma_a,\sigma}) \to W$ is again a trivial fiber bundle with fibers homeomorphic to $(\mathbb{R}^J/\mathbb{R})/(\mathbb{R}\tilde{v})$, as in the maximal case for a lower-dimensional pair of pants.

Finally, for a noncentral vertex $v = \mathcal{P}_{I,I} \times \{a\}$ of $\Psi_{\sigma,J}$, where $I \subsetneq J$, the local fan $\Sigma_v\Psi_{\sigma,J}$ is just the product $\Sigma_v\Psi_{\sigma,I} \times \mathbb{R}^{J\setminus I}_{\geq 0}$. $\square$

Remark Although the polyhedral fans $\partial\mathcal{R}$ and $\mathcal{P}(v)$ are isomorphic, they are really different fans in $\mathbb{R}^{n+1}/\mathbb{R}$. The former bounds a convex cone; the latter generally does not.

For any vertex a of the coamoeba $\mathcal{C}^n$ one can identify the local fan of $\mathcal{T}\mathcal{P}^{n-1}$ at $v = \mathcal{P}_{\overline{n}} \times \{a\}$ with the total supporting tangent space $T\mathcal{R}$ of the cone $\mathcal{R}$ associated with the corresponding 2–partition $\sigma_a = \langle I_-, I_+ \rangle$, which is homeomorphic to a $(2n-2)$–ball. That directly proves a conjecture of Viro [13, Section 5.10] that $\mathcal{T}\mathcal{P}^{n-1}$ is a topological manifold.

Proof of Proposition 11 Lemmas 12 and 13 show that each $\overline{\Psi}_{\sigma,J}$ is a collapsible PL manifold with boundary of dimension $\mathrm{rk}(\sigma, J)$. Then a version of the regular neighborhood theorem (see eg [2, Theorem 1.6]) implies that $\overline{\Psi}_{\sigma,J}$ is homeomorphic to the closed ball of dimension $\mathrm{rk}(\sigma, J)$. □

Theorem 15 *$\mathcal{T}\overline{\mathcal{P}}^{n-1}$ is homeomorphic to $\overline{P}^{n-1}$ and $\mathcal{T}\mathcal{P}^{n-1}$ is homeomorphic to P^{n-1}.*

Proof Two isomorphic regular CW–complexes are homeomorphic (see eg [1]). A homeomorphism between $\mathcal{T}\overline{\mathcal{P}}^{n-1}$ and $\overline{P}^{n-1}$ can be chosen to respect the stratification. In particular, it restricts to a homeomorphism between $\mathcal{T}\mathcal{P}^{n-1}$ and P^{n-1}. □

Remark It may be desirable to extend the result to a homeomorphism of pairs

$$(P^{n-1}, (\mathbb{C}^*)^{n+1}/\mathbb{C}) \approx (\mathcal{T}\mathcal{P}^{n-1}, (\mathbb{C}^*)^{n+1}/\mathbb{C}).$$

Indeed, the respective complements are the higher-dimensional pairs of pants themselves. The problem, however, is that even the local homeomorphism $\Sigma_v \mathcal{T}\mathcal{P}^{n-1} \approx \mathbb{R}^{2n-2}$ given by (16) is fairly complicated and it does not seem to have an obvious extension to a tubular neighborhood.

3 Phase tropical varieties of hypersurfaces in $(\mathbb{C}^*)^n$

3.1 Phase tropical varieties

In this section, we will give the necessary background to state Theorem 21, our main result. We begin with some preliminary definitions which may be found in [3].

Let $N \cong \mathbb{Z}^n$, $M = \mathrm{Hom}(N, \mathbb{Z})$ and, for any abelian group $\mathbb{K}$, write $M_{\mathbb{K}}$ for $M \otimes_{\mathbb{Z}} \mathbb{K}$ and similarly for N. Let A be a finite subset of M and Q its convex hull in $M_{\mathbb{R}}$. We call (Q, A) a *marked polytope*. Another marked polytope (Q', A') will be called a *face* of (Q, A) if Q' is a face of Q and $A' = A \cap Q'$. If A is affinely independent, we say it is a *marked simplex*. A subdivision $\mathcal{S} = \{(Q_\gamma, A_\gamma) : \gamma \in \Gamma\}$ of (Q, A) is a collection of marked polytopes satisfying

(1) for any $\gamma \in \Gamma$, every face of (Q_γ, A_γ) is in $\mathcal{S}$,

(2) for any $\gamma, \tilde{\gamma} \in \Gamma$, the intersection $(Q_\gamma \cap Q_{\tilde{\gamma}}, A_\gamma \cap A_{\tilde{\gamma}})$ is in $\mathcal{S}$ and is a face of both (Q_γ, A_γ) and $(Q_{\tilde{\gamma}}, A_{\tilde{\gamma}})$,

(3) the union $\bigcup_{\gamma\in\Gamma} Q_\gamma$ equals Q.

The poset Γ is the face lattice of the subdivision. Note that it is not necessarily the case that $\bigcup_{\gamma\in\Gamma} A_\gamma = A$.

We will be particularly interested in subdivisions that are *coherent*. The basic ingredient in this construction is a function $\eta\colon A \to \mathbb{R}$. From η, we define a polyhedron in $M_\mathbb{R}\times\mathbb{R}$ as the convex hull

$$\overline{Q}_\eta := \mathrm{Conv}\{(\alpha, r) \in A\times\mathbb{R} : r \geq \eta(\alpha)\}. \tag{28}$$

Let $\overline{A}_{\overline{F}}$ be the set of vertices over A of any compact face $\overline{F}$ of $\overline{Q}_\eta$, and take A_F and F to be their projections onto $M_\mathbb{R}$. Define the subdivision

$$\mathcal{S}_\eta = \{(F, A_F) : \overline{F} \text{ a compact face of } \overline{Q}_\eta\}.$$

When each (F, A_F) is a marked simplex, we say that η induces a *coherent triangulation* of (Q, A) (see [3, Chapter 7]).

Consider the piecewise linear function $F_\eta\colon N_\mathbb{R}\to\mathbb{R}$ (or tropical polynomial) defined by

$$F_\eta(\boldsymbol{x}) = \max\{\alpha(\boldsymbol{x}) - \eta(\alpha) : \alpha\in A\}. \tag{29}$$

The *tropical hypersurface* $\mathcal{H}_\eta \subset N_\mathbb{R}$ is the corner locus of F_η; see eg [8] for details. Note that $\mathcal{H}_\eta$ is a polyhedral complex. For any $k\in\mathbb{Z}$, let $\mathcal{S}_\eta^{\geq k}$ be the set of faces (Q_γ, A_γ) for which $\dim Q_\gamma \geq k$. There is an order-reversing bijection

$$\mathcal{S}_\eta^{\geq 1} \xrightarrow{\Phi} \{\text{faces of } \mathcal{H}_\eta\}, \tag{30}$$

where $\Phi(Q_\gamma, A_\gamma) = \{\boldsymbol{x}\in N_\mathbb{R} : \alpha(\boldsymbol{x}) - \eta(\alpha) = F_\eta(\boldsymbol{x}) \text{ for all } \alpha\in A_\gamma\}$. For any $(Q_\gamma, A_\gamma)\in\mathcal{S}_\eta^{\geq 1}$, write $\mathcal{H}_{\eta,\gamma}$ for the relative interior of $\Phi(Q_\gamma, A_\gamma)$, so that

$$\mathcal{H}_\eta = \bigcup_{\gamma\in\Gamma} \mathcal{H}_{\eta,\gamma}. \tag{31}$$

Example 16 Consider an example of $A = \{(0,0),(1,0),(0,1),(2,3)\}$, with Q its convex hull. Take $\eta\colon A\to\mathbb{R}$ equal to 0 except at $(2,3)$, where it equals 1. That gives the coherent triangulation and tropical hypersurface illustrated in Figure 10. The correspondence Φ is indicated by the coloring of the simplices in the triangulation and the faces of the tropical hypersurface.

Turning to complex polynomials, given a Laurent polynomial $f = \sum_{m\in\mathcal{M}} c_m z^m$, where $\mathcal{M}$ is a finite set in M and $c_m\neq 0$ for all $m\in\mathcal{M}$, we say that $(\mathrm{Conv}(\mathcal{M}), \mathcal{M})$

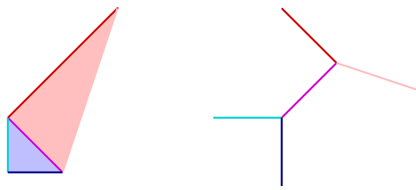

Figure 10: A triangulation and tropical hypersurface induced by η

is the *marked Newton polytope* of f. For any $A \subseteq \mathcal{M}$ we consider the truncated polynomial

$$f_A = \sum_{a \in A} c_a z^a. \tag{32}$$

Given a Laurent polynomial f, its zero locus H_f lives in the complex torus $N_{\mathbb{C}^*}$. In the notation from Section 2.4 we have the argument map $\operatorname{Arg}\colon N_{\mathbb{C}^*} \to N_{\mathbb{T}}$ to the real n–torus $N_{\mathbb{T}}$. For any polynomial f, we define its *coamoeba* $\mathcal{C}_f \subset N_{\mathbb{T}}$ to be the closure of the image of H_f under the argument map.

Example 17 As will be shown in the next subsection, the coamoeba of a simplex is a finite cover of the product of the coamoeba of a pair of pants and a torus. Take f to be a generic polynomial with marked Newton polytope (Q, A) from Example 16. Figure 11 illustrates the coamoebas, in the cover $N_{\mathbb{R}}$ of $N_{\mathbb{T}}$, of f truncated to the three indicated simplices.

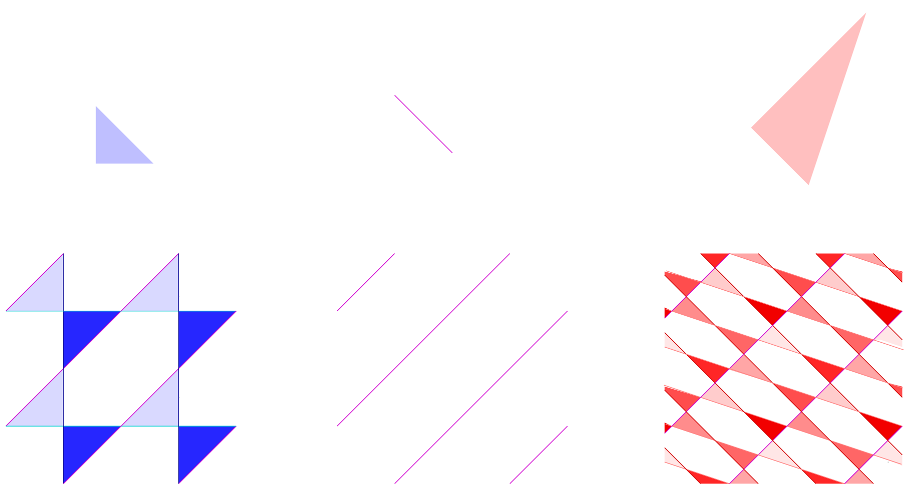

Figure 11: Coamoebas associated to simplices

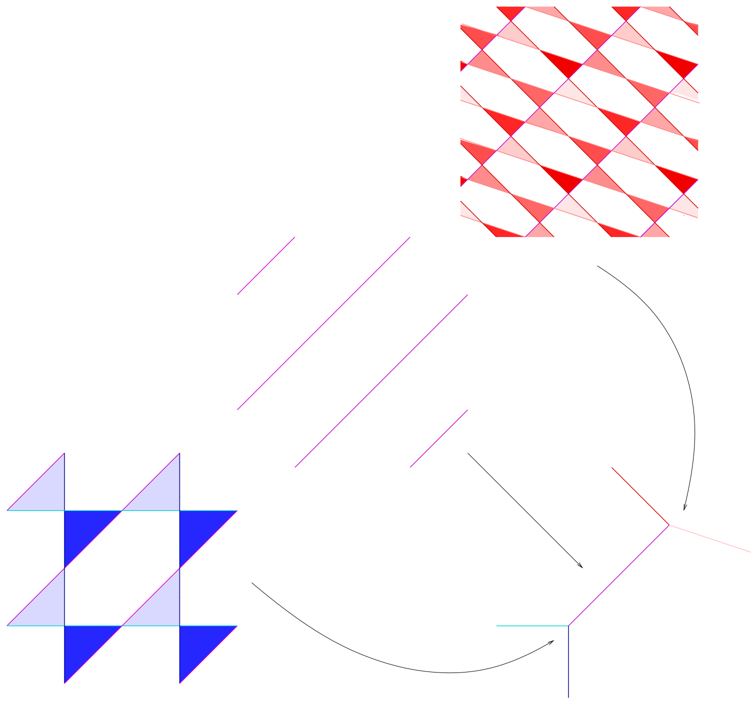

Figure 12: The phase tropical hypersurface

Definition 18 Let $f \in \mathbb{C}[M]$ and (Q, A) be its marked Newton polytope. Given a coherent triangulation $\mathcal{S}_\eta = \{(Q_\gamma, A_\gamma) : \gamma \in \Gamma\}$ induced by $\eta\colon A \to \mathbb{R}$, the *phase tropical hypersurface* of f defined by η is

$$\mathcal{TH}_{\eta,f} := \bigcup_{\gamma \in \Gamma} \mathcal{H}_{\eta,\gamma} \times \mathcal{C}_{f_{A_\gamma}} \subset N_\mathbb{R} \times N_\mathbb{T} = N_{\mathbb{C}^*}.$$

Example 19 Combining Examples 16 and 17 we obtain a partial picture of a phase tropical hypersurface $\mathcal{TH}_{\eta,f}$, illustrated in Figure 12. Here we have not illustrated the coamoebas over the noncompact faces of the tropical hypersurface.

We will also consider a compactified version of the phase tropical hypersurface. For this, we identify $N_\mathbb{R} \times N_\mathbb{T}$ with $N_{\mathbb{C}^*}$ via the exponential map and consider the algebraic moment map $\mu\colon N_{\mathbb{C}^*} \to M_\mathbb{R}$ defined as

$$\mu(z) = \frac{1}{\sum_{a \in A} |z^a|} \sum_{a \in A} |z^a| a.$$

It is clear that μ is independent of the $N_{\mathbb{T}}$ factor and, for any $\theta \in N_{\mathbb{T}}$, we denote the restriction of μ to $N_{\mathbb{R}} \times \{\theta\}$ by $\mu_{\mathbb{R}}\colon N_{\mathbb{R}} \to M_{\mathbb{R}}$. It is also easy to observe that the image of μ is the relative interior of Q. If we wish to reference the marked polytope in the notation, we write μ^A for μ and $\mu_{\mathbb{R}}^A$ for $\mu_{\mathbb{R}}$.

Definition 20 Let Q be an n–dimensional polytope. The *compactified phase tropical hypersurface* $\mathcal{T}\overline{\mathcal{H}}_{\eta,f}$ is the closure

$$\overline{(\mu_{\mathbb{R}} \times \mathrm{Id})(\mathcal{T}\mathcal{H}_{\eta,f})} \subset Q \times N_{\mathbb{T}}$$

of the phase tropical hypersurface in $Q \times N_{\mathbb{T}}$.

Of course, we may also compactify the complex hypersurface $H_f \subset N_{\mathbb{C}^*}$ by taking its closure under the image of the moment map. When the Newton polytope of f is n–dimensional, we call

$$\overline{H}_f = \overline{\mu(H_f)} \subset Q \times N_{\mathbb{T}} \tag{33}$$

the compactification of H_f.

The boundaries of $\overline{H}_f$ and $\mathcal{T}\overline{\mathcal{H}}_{\eta,f}$ both have stratifications indexed by the face lattice of Q. It will be helpful later on to describe these boundary strata intrinsically. Suppose (Q, A) is a marked polytope in M, not necessarily full-dimensional, f is a polynomial whose marked Newton polytope contains (Q, A) and η is any function on A. Let

$$N(A) := \{\boldsymbol{x} \in N : a(\boldsymbol{x}) = a'(\boldsymbol{x}) \text{ for all } a, a' \in A\}$$

be the sublattice orthogonal to the affine span of A, and $N_{\mathbb{K}}(A) = N(A) \otimes_{\mathbb{Z}} \mathbb{K}$. Then it is easy to see that the tropical hypersurface $\mathcal{H}_\eta \subset N_{\mathbb{R}}$ is invariant under translations by $N_{\mathbb{R}}(A)$. Define the space

$$\mathcal{T}\mathcal{H}_{\eta,f,A} = \left\{ (\boldsymbol{x} + N_{\mathbb{R}}(A), \boldsymbol{\theta}) \in \frac{N_{\mathbb{R}}}{N_{\mathbb{R}}(A)} \times N_{\mathbb{T}} : (\boldsymbol{x}, \boldsymbol{\theta}) \in \mathcal{T}\mathcal{H}_{\eta,f_A} \right\}. \tag{34}$$

We may compactify $\mathcal{T}\mathcal{H}_{\eta,f,A}$ by using the moment map μ^A associated to A. More explicitly, let $\iota_{\mathbb{R}}\colon N_{\mathbb{R}}/N_{\mathbb{R}}(A) \to N_{\mathbb{R}}$ be any section of the quotient map. Then define

$$\tilde{\mu}_{\mathbb{R}}^A\colon \frac{N_{\mathbb{R}}}{N_{\mathbb{R}}(A)} \to Q \tag{35}$$

to be the composition $\mu_{\mathbb{R}}^A \circ \iota_{\mathbb{R}}$. As $\mu_{\mathbb{R}}^A$ is invariant with respect to translations by $N(A)$, it is clear that $\tilde{\mu}_{\mathbb{R}}^A$ is independent of the choice of ι. Note that the image of $\tilde{\mu}_{\mathbb{R}}^A$ is the relative interior of Q. For the compactification of $\mathcal{TH}_{\eta,f,A}$ we take

$$\mathcal{T}\overline{\mathcal{H}}_{\eta,f,A} = \overline{(\tilde{\mu}_{\mathbb{R}}^{A} \times \mathrm{Id})(\mathcal{TH}_{\eta,f,A})}. \tag{36}$$

One may relate $\mathcal{T}\overline{\mathcal{H}}_{\eta,f,A}$ to the lower-dimensional phase tropical hypersurface associated to f. To do this, let $M(A)$ be the saturation of the affine lattice spanned by A, $(\widetilde{Q}, \widetilde{A})$ the image of A in $M(A)$ and $\tilde{\eta}\colon \widetilde{A} \to \mathbb{R}$ the function equal to η; then $\mathcal{T}\overline{\mathcal{H}}_{\eta,f,A}$ is homeomorphic to $\mathcal{T}\overline{\mathcal{H}}_{\tilde{\eta},f} \times N_{\mathbb{T}}(A)$. To define a homeomorphism, one can use a lift $\iota_{\mathbb{R}}\colon N_{\mathbb{R}}/N_{\mathbb{R}}(A) \to N_{\mathbb{R}}$ to equate the argument factor of $\mathcal{TH}_{\eta,f,A}$ with $N_{\mathbb{T}}/N_{\mathbb{T}}(A) \times N_{\mathbb{T}}(A)$.

If (Q', A') is a face of (Q, A) where the marked Newton polytope of f contains (Q, A), then there is a natural inclusion

$$i_{A',A}\colon \mathcal{T}\overline{\mathcal{H}}_{\eta,f,A'} \to \mathcal{T}\overline{\mathcal{H}}_{\eta,f,A} \tag{37}$$

whose image is the inverse image of Q' in $Q \times N_{\mathbb{T}}$ under the moment map. To define this map, it suffices to consider the noncompact strata. But there is already a map $(\tilde{\mu}_{\mathbb{R}}^{A'} \times \mathrm{Id})\colon \mathcal{TH}_{\eta,f,A'} \to Q' \times N_{\mathbb{T}} \subset Q \times N_{\mathbb{T}}$ and this maps bijectively onto $\mathcal{T}\overline{\mathcal{H}}_{\eta,f,A}$ over the relative interior of Q'.

Thus we obtain a stratification of the compactified phase tropical hypersurface

$$\mathcal{T}\overline{\mathcal{H}}_{\eta,f} = \bigcup_{(Q',A') \text{ a face of } (Q,A)} \mathcal{TH}_{\eta,f,A'} \tag{38}$$

whose strata are in bijective correspondence with the positive-dimensional faces of Q.

Turning to the geometry of complex hypersurfaces, we note that their tropical compactifications also carry a stratification by the boundary faces of Q. Here, assume f is a Laurent polynomial with marked Newton polytope containing the face (Q, A). We denote by $H_{f,A}$ the quotient of the hypersurface H_{f_A} in $N_{\mathbb{C}^*} = N_{\mathbb{R}} \times N_{\mathbb{T}}$ by $N_{\mathbb{R}}(A)$ (note that f_A is homogeneous with respect to this action, so that the quotient is well defined). The space $H_{f,A}$ is not compact, but again we may compactify by taking its closure under the restricted moment map

$$\overline{H}_{f,A} = \overline{(\tilde{\mu}_{\mathbb{R}}^{A} \times \mathrm{Id})(H_{f,A})}. \tag{39}$$

As in the phase tropical setting, for a face (Q', A') of (Q, A), and marked Newton polytope of f containing (Q, A), there are natural maps

$$j_{A',A}\colon \overline{H}_{f,A'} \to \overline{H}_{f,A} \tag{40}$$

defined analogously to those in (37). The associated stratification is then

$$\overline{H}_f = \bigcup_{(Q',A') \text{ a face of } (Q,A)} H_{f,A'}. \tag{41}$$

Even for smooth hypersurfaces H_f, the tropical compactification may contain unwanted singularities on the boundary strata. To prevent such singularities, we call a Laurent polynomial f *nondegenerate* if 0 is a regular value of $f_{A'}$ for every face (Q', A') of (Q, A).

Theorem 21 *Given a nondegenerate polynomial f with marked Newton polytope (Q, A) and $\eta\colon A \to \mathbb{R}$ defining a coherent triangulation, there are homeomorphisms ψ and $\overline{\psi}$ for which the diagram*

$$\begin{tikzcd} H_f \arrow[r, "\psi"] \arrow[d, hook] & \mathcal{TH}_{\eta,f} \arrow[d, hook] \\ \overline{H}_f \arrow[r, "\overline{\psi}"'] & \mathcal{T}\overline{\mathcal{H}}_{\eta,f} \end{tikzcd} \tag{42}$$

commutes.

3.2 Simple hypersurfaces

Often when considering pair-of-pants decompositions induced by a coherent triangulation $\mathcal{S} = \{(Q_\gamma, A_\gamma) : \gamma \in \Gamma\}$ of a marked polytope (Q, A) (eg as in [9]), the simplices (Q_γ, A_γ) are required to have volume $\frac{1}{n!}$ (or normalized volume 1). Such triangulations are referred to as maximal or unimodular. In practice, unimodular triangulations are comparatively rare and there will usually be several simplices in any given triangulation with larger volume. Indeed, there are many cases of marked polytopes without a single unimodular triangulation. For a simplex of volume greater than $\frac{1}{n!}$, the associated phase tropical hypersurface is no longer a pair of pants, but rather a finite cover of the pair of pants, called a *simple hypersurface* in [10]. Following loc. cit., we take a moment to consider this cover and that of the associated hypersurface H_f.

First, let us establish some notation. Let (Q, B) be a marked simplex in $M \cong \mathbb{Z}^n$ for which B affinely spans $M_\mathbb{R}$ and

$$f = \sum_{b \in B} c_b z^b \in \mathbb{C}[M]$$

with $c_b \neq 0$ for every $b \in B$. Fix an ordering of $B = \{b_0, \dots, b_n\}$, write c_i for c_{b_i} and identify any subset $I \subseteq \hat{n}$ with its corresponding subset $\{b_i : i \in I\} \subseteq B$. Consider the map $\phi_B\colon N_{\mathbb{C}^*} \to (\mathbb{C}^*)^{n+1}/\mathbb{C}^* \subset \mathbb{P}^n$ defined by

$$\phi_B(z) := [c_0 z^{b_0}, \dots, c_n z^{b_n}]. \tag{43}$$

One notes that ϕ_B extends via the inverse of the moment map to $\overline{\phi}_B\colon Q \times N_{\mathbb{T}} \to \Delta \times \mathbb{T}^n$, where Δ is the standard simplex.

Write $\Xi_B \subseteq M$ for the sublattice which is the $\mathbb{Z}$–span of $\{b_i - b_j : b_i, b_j \in B\}$. Then there are containments $N \subseteq \Xi_B^\vee \subset N_{\mathbb{R}}$ and we write Λ_B for the image of $\Xi_B^\vee$ in the quotient $N_{\mathbb{T}} = N_{\mathbb{R}}/N$. Then, using notation from Section 2.4, we have the following basic result:

Lemma 22 *The maps ϕ_B and $\overline{\phi}_B$ are quotient maps by Λ_B. Furthermore, for any subset $I \subseteq \{0, \dots, n\}$ with $|I| \geq 2$, the restriction of ϕ_B to H_{f_I} is a covering map to the complex pair of pants P_I^{n-1}.*

Proof To verify the statement that ϕ_B is the quotient map, first observe that

$$\phi_B(z) = [c_0 z^{b_0}, \dots, c_n z^{b_n}] = [c_0, c_1 z^{b_1 - b_0}, \dots, c_n z^{b_n - b_0}].$$

This implies that ϕ_B factors through the homomorphism $N_{\mathbb{C}^*} \to (\mathbb{C}^*)^n$ given by $z \mapsto (z^{b_1 - b_0}, \dots, z^{b_n - b_0})$, which has kernel Λ_B. The extension to $\overline{\phi}_B$ follows immediately since $\Lambda_B \subset N_{\mathbb{T}}$ acts only on the $N_{\mathbb{T}}$ factor.

The fact that ϕ_B and $\overline{\phi}_B$ restrict to give covering maps from the hypersurface H_{f_I} to the pair of pants follows from the definition of $P_I^{n-1} \subset (\mathbb{C}^*)^{n+1}/\mathbb{C}^* \subset \mathbb{P}^n$ as the zero locus of $\sum_{i \in I} z_i = 0$. In particular, $f_I = \phi_B^*\bigl(\sum_{i \in I} z_i\bigr)$, so that ϕ_B restricts to H_{f_I} to give a well-defined and an onto map. □

We now turn our attention to the phase tropical hypersurface of a marked simplex (Q, B) with the function $\eta\colon B \to \mathbb{R}$ and a polynomial f. We first make an observation that the tropical hypersurface $\mathcal{H}_\eta$ depends on η only up to a translation in $N_{\mathbb{R}}$. In particular, since η is defined on a simplex, it is the restriction of an affine function $n_\eta + c$ on $M_{\mathbb{R}}$ to B, where $n_\eta \in N_{\mathbb{R}}$ and c is a constant. In this instance, one observes that $\mathcal{H}_\eta = \mathcal{H}_{\mathbf{0}} + n_\eta$. The coamoeba $\mathcal{C}_{f_I}$ is independent of the function η and only depends on I and the arguments of the coefficients $\{c_i : i \in I\}$ of f. Thus, there is an elementary homeomorphism $\mathcal{TH}_{\eta, f} \cong \mathcal{TH}_{\mathbf{0}, f}$ given by translating by n_η in the $N_{\mathbb{R}}$ factor of $N_{\mathbb{R}} \times N_{\mathbb{T}}$. Consequently, the topology of the phase tropical hypersurface of a simplex is independent of the function η. For convenience, we choose $\eta_f\colon B \to \mathbb{R}$ to be the function $\eta_f(b_i) = -\log|c_i|$.

Lemma 23 *The restriction of $\overline{\phi}_B$ to $\mathcal{T}\overline{\mathcal{H}}_{\eta_f,f}$ is a covering map onto $\mathcal{T}\overline{\mathcal{P}}^{n-1}$.*

Proof View the map ϕ_B as a map from $N_{\mathbb{R}} \times N_{\mathbb{T}}$ to $(\mathbb{C}^*)^{n+1}/\mathbb{C}^* \subset \mathbb{P}^n$. Consider the linear map $\xi\colon \mathbb{R}^{n+1} \to M_{\mathbb{R}}$ defined by $\xi(e_i) = b_i$ and let $\boldsymbol{c} = \sum_i \log|c_i| e_i \in \mathbb{R}^{n+1}$. It is then clear that the diagram

$$\begin{array}{ccc} N_{\mathbb{R}} \times N_{\mathbb{T}} & \xrightarrow{\phi_B} & (\mathbb{C}^*)^{n+1}/\mathbb{C}^* \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \mathrm{Log}} \\ N_{\mathbb{R}} & \xrightarrow[\xi^\vee + \boldsymbol{c}]{} & \mathbb{R}^{n+1}/\mathbb{R} \end{array} \tag{44}$$

commutes, where π_1 is projection to the first factor. Furthermore, the composition of the affine map $\xi^\vee + \boldsymbol{c}\colon N_{\mathbb{R}} \to \mathbb{R}^{n+1}$ with every dual basis vector $e_i^\vee\colon \mathbb{R}^{n+1} \to \mathbb{R}$ yields the map $b_i - \eta_f(b_i)$. Thus, the tropical polynomial for the standard pair of pants pulls back to F_{η_f} and $\xi^\vee + \boldsymbol{c}$ maps the tropical hypersurface $\mathcal{H}_f$ to the tropical pair of pants $\mathcal{P}^{n-1}$.

For $I \subseteq \{0, \dots, n\}$ with $|I| \geq 2$, denote by Q_I the convex hull of the corresponding set in B. Then, over the face $\Phi(Q_I, I)$ of the tropical hypersurface, in the phase tropical hypersurface $\mathcal{T}\mathcal{H}_{f,\eta_f}$, lies the coamoeba $\mathcal{C}_{f_I}$. Utilizing Lemma 22 and a commutative diagram analogous to (44) with argument maps, we have that $\mathcal{C}_{f_I}$ maps to the coamoeba $\mathcal{C}_I$. This implies the result. $\square$

Combining Lemmas 22 and 23, we obtain an extension of Theorem 15 to the simple hypersurface case.

Theorem 24 *Given a marked simplex (Q, B) which affinely spans $M_{\mathbb{R}}$ and any $\eta\colon B \to \mathbb{R}$, there is a homeomorphism $\overline{\psi}\colon \overline{H}_f \to \mathcal{T}\overline{\mathcal{H}}_{\eta,f}$.*

Proof It suffices to prove this theorem in the noncompact case. We write $\phi\colon P^{n-1} \to \mathcal{T}\mathcal{P}^{n-1}$ for the homeomorphism in Theorem 15. Note that both inclusions $P^{n-1} \hookrightarrow (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ and $\mathcal{T}\mathcal{P}^{n-1} \hookrightarrow (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$ induce isomorphisms on first homology (and on the fundamental group when $n > 2$). This follows by looking at the coamoeba (see eg [11]) for the complex and phase tropical pair of pants and its covering in $\mathbb{R}^{n+1}$, which is simply connected for $n \geq 3$ and is the universal abelian cover for $n = 2$. Moreover, it is evident from the construction of ϕ that the diagram

$$\begin{tikzcd} H_1(P^{n-1};\mathbb{Z}) \arrow[rr, "H(\phi)"] \arrow[dr, "\cong"'] & & H_1(\mathcal{TP}^{n-1};\mathbb{Z}) \arrow[dl, "\cong"] \\ & H_1((\mathbb{C}^*)^{n+1}/\mathbb{C}^*;\mathbb{Z}) \cong \mathbb{Z}^n & \end{tikzcd} \tag{45}$$

commutes.

By Lemmas 22 and 23, H_f and $\mathcal{TH}_{\eta,f}$ are covers of P^{n-1} and $\mathcal{TP}^{n-1}$ obtained by pulling back the subspaces along the cover $\phi_B\colon N_{\mathbb{C}^*} \to (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$. This cover corresponds to a sublattice of $H_1((\mathbb{C}^*)^{n+1}/\mathbb{C}^*;\mathbb{Z}) \cong \pi_1((\mathbb{C}^*)^{n+1}/\mathbb{C}^*)$. The commutativity of (45), pulled back along the Hurewicz homomorphism, then implies that $\pi_1(\phi)$ takes the normal subgroup associated to $\phi_B|_{H_f}$ to that of $\phi_B|_{\mathcal{TH}_{\eta,f}}$ implying the result. □

The arguments given in Lemmas 22 and 23 easily extend to the strata $\overline{H}_{f,B'}$ and $\mathcal{T}\overline{\mathcal{H}}_{\eta,f,B'}$. We record the stratified version of Theorem 24 as a corollary.

Corollary 25 *Given a marked simplex (Q,B) contained in the marked Newton polytope of f, $\eta\colon B \to \mathbb{R}$ and any subsimplex (Q',B'), there are homeomorphisms $\overline{\psi}_{B'}$ and $\overline{\psi}_B$ for which*

$$\begin{tikzcd} \overline{H}_{f,B'} \arrow[r, "\overline{\psi}_{B'}"] \arrow[d, hook, "j_{B',B}"'] & \mathcal{T}\overline{\mathcal{H}}_{\eta,f,B'} \arrow[d, hook, "i_{B',B}"] \\ \overline{H}_{f,B} \arrow[r, "\overline{\psi}_B"'] & \mathcal{T}\overline{\mathcal{H}}_{\eta,f,B} \end{tikzcd} \tag{46}$$

commutes.

3.3 Proof of Theorem 21

Having extended Theorem 15 to simple hypersurfaces and their stratified compactifications, we now apply the results of Milkalkin [9] based on Viro's patchworking method [12] to obtain Theorem 21 for general hypersurfaces.

Proof of Theorem 21 One first observes that, since f is assumed to be nondegenerate, H_f is diffeomorphic to H_g for any other nondegenerate polynomial g with marked Newton polytope (Q, A). To see this, one can compactify $N_{\mathbb{C}^*}$ to the toric variety Y_Q and resolve all singularities of the toric boundary to obtain X_Q with a normal crossing divisor D. Then Laurent polynomials with Newton polytope Q may be identified with a dense open subset of sections of the line bundle $\mathcal{O}(1)$ determined by Q. Moreover, the condition of nondegeneracy implies that the zero locus Z_f, which is an analytic compactification of H_f, transversely intersects the divisor D. Taking a family $\xi\colon [0,1] \to \Gamma(X_Q, \mathcal{O}(1))$ of such nondegenerate sections, one may consider the incidence variety $\mathcal{Y} = \{(t, z) : z \in Z_{\xi(t)}\}$ along with the function $\pi\colon \mathcal{Y} \to [0,1]$ induced by projection. Let $\mathcal{D} = \{(t, z) \in \mathcal{Y} : z \in D\}$. By the openness of the transversality condition, π and $\pi|_D$ are trivial families and equipping $\mathcal{Y}$ with a connection for which $\mathcal{D}$ is horizontal and taking parallel transport gives a diffeomorphism of the pair $(Z_{\xi(0)}, Z_{\xi(0)} \cap D)$ with $(Z_{\xi(1)}, Z_{\xi(1)} \cap D)$. Excising the respective subspaces then produces the diffeomorphism.

Next we note that, for $t \in \mathbb{R}_{>0}$ small, the polynomial $f_t = \sum_{a\in A} c_a t^{\eta(a)} z^a$ is nondegenerate, regardless of the coefficients $\boldsymbol{c}_{\mathbb{C}} := (c_a) \in \mathbb{C}^A$ (see [3]). In particular, an alternative definition of a nondegenerate polynomial f is that the principal A–determinant $E_A(f)$ is nonzero. By [3, Theorem 10.1.4], E_A is a polynomial in the coefficients (c_a) whose Newton polytope is the secondary polytope $\Sigma(A)$. It can be shown that the Log of the coefficients $\boldsymbol{c}_{\mathbb{C}} t^{\eta}$ of f_t lie in the interior of the cone dual to the triangulation defined by η for sufficiently small t. This implies that, for such f_t, the point $\mathrm{Log}(\boldsymbol{c}_{\mathbb{C}})$ lies outside the amoeba of E_A implying $E_A(\boldsymbol{c}_{\mathbb{C}}) \neq 0$.

To complete the proof we apply the reconstruction result [9, Theorem 4], adapted to the nonunimodular case. In this modified form, it asserts that for η inducing the coherent triangulation $\mathcal{S} = \{(Q_\gamma, A_\gamma) : \gamma \in \Gamma\}$, there is homeomorphism between $\overline{H}_f$ and the topological direct limit (ie the colimit in the category of topological spaces). Thus, we obtain

$$\overline{H}_f \approx \varinjlim \overline{H}_{f, A_\gamma}. \tag{47}$$

Here we mean that one may consider the face lattice Γ of $\mathcal{S}$ given by inclusions as a category and $\overline{H}_{f,-}$ as a functor from Γ to topological spaces. Then the limit of this functor is achieved by gluing simple hypersurfaces along common boundary strata.

Using the dual subdivision of $N_{\mathbb{R}}$ to that given by the tropical hypersurface, one obtains a decomposition $\overline{\mathcal{H}}_\eta = \bigcup_{\gamma\in\Gamma} Y_\gamma$. We then have that $\mathcal{T}\overline{\mathcal{H}}_{\eta,f} = \bigcup_{\gamma\in\Gamma} \mathcal{T}Y_\gamma$, where $\mathcal{T}Y_\gamma$ is the part of the phase tropical hypersurface lying over Y_γ. Each $\mathcal{T}Y_\gamma$ can be identified with a partially contracted $\mathcal{T}\overline{\mathcal{H}}_{\eta,f,A_\gamma}$ which is clearly homeomorphic to the original $\mathcal{T}\overline{\mathcal{H}}_{\eta,f,A_\gamma}$. These identifications are compatible with inclusion maps $i_{A_\gamma,A_{\tilde{\gamma}}}$, when (Q_γ, A_γ) if a face $(Q_{\tilde{\gamma}}, A_{\tilde{\gamma}})$. In particular, we have that

$$\mathcal{T}\overline{\mathcal{H}}_{\eta,f} \approx \varinjlim \mathcal{T}\overline{\mathcal{H}}_{\eta,f,A_\gamma}. \tag{48}$$

Here again we regard this as a limit of the functor from the poset category Γ to topological spaces taking γ to $\mathcal{T}\overline{\mathcal{H}}_{\eta,f,A_\gamma}$ and inclusions of faces (Q_γ, A_γ) of $(Q_{\tilde{\gamma}}, A_{\tilde{\gamma}})$ to $i_{A_\gamma,A_{\tilde{\gamma}}}$.

By Corollary 25, we have a natural isomorphism of functors from $\overline{H}_{f,-}$ to $\mathcal{T}\overline{\mathcal{H}}_{\eta,f,-}$, implying their limits are homeomorphic. Equations (47) and (48) then give that $\overline{H}_f \approx \mathcal{T}\overline{\mathcal{H}}_{\eta,f}$. Removing the boundary strata on both sides of the equation gives the open case. □

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