Geometry & Topology 22 (2018) 3321–3394

On the Farrell–Jones conjecture for Waldhausen's A–theory

NILS-EDVIN ENKELMANN
WOLFGANG LÜCK
MALTE PIEPER
MARK ULLMANN
CHRISTOPH WINGES

We prove the Farrell–Jones conjecture for (nonconnective) A–theory with coefficients and finite wreath products for hyperbolic groups, CAT(0)–groups, cocompact lattices in almost connected Lie groups and fundamental groups of manifolds of dimension less or equal to three. Moreover, we prove inheritance properties such as passing to subgroups, colimits of direct systems of groups, finite direct products and finite free products. These results hold also for Whitehead spectra and spectra of stable pseudoisotopies in the topological, piecewise linear and smooth categories.

19D10; 57Q10, 57Q60

1 Introduction

We investigate the Farrell–Jones conjecture for Waldhausen's A–theory. Our main result is:

Published: 23 September 2018 DOI: 10.2140/gt.2018.22.3321

Theorem 1.1 (main result) *Let $\mathcal{FJ}_A$ be the class of groups for which the Farrell–Jones conjecture 2.15 for (nonconnective) A–theory with coefficients and finite wreath products holds.*

(i) *The class $\mathcal{FJ}_A$ contains the following groups:*
- *Hyperbolic groups.*
- $\mathrm{CAT}(0)$*–groups.*
- *Virtually polycyclic groups.*
- *Cocompact lattices in almost connected Lie groups.*
- *Fundamental groups of (not necessarily compact) d–dimensional manifolds (possibly with boundary) for $d \leq 3$.*

(ii) *The class $\mathcal{FJ}_A$ has the following inheritance properties:*
- *If G_1 and G_2 belong to $\mathcal{FJ}_A$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}_A$.*
- *If H is a subgroup of G and $G \in \mathcal{FJ}_A$, then $H \in \mathcal{FJ}_A$.*
- *Let $1 \to K \to G \xrightarrow{p} Q \to 1$ be an extension of groups. Suppose that K, Q and $p^{-1}(C)$ for every infinite cyclic subgroup $C \subseteq Q$ belong to $\mathcal{FJ}_A$. Then G belongs to $\mathcal{FJ}_A$.*
- *If $H \subseteq G$ is a subgroup of G with $[G:H] < \infty$ and $H \in \mathcal{FJ}_A$, then $G \in \mathcal{FJ}_A$.*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}_A$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}_A$.*

The Farrell–Jones conjecture for A–theory aims at the computation of the homotopy groups of $\boldsymbol{A}(B\mathrm{G})$ for a group G, where $\boldsymbol{A}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ sends a space X to the nonconnective A–theory spectrum $\boldsymbol{A}(X)$ modeling Waldhausen's A–theory space $A(X)$. More precisely, it predicts the bijectivity of the assembly map

$$H_n^G(E_{\mathcal{VCY}}(G); \boldsymbol{A}^B) \xrightarrow{\cong} H_n^G(G/G; \boldsymbol{A}^B) = \pi_n(\boldsymbol{A}(B\mathrm{G}))$$

induced by the projection of the classifying space $E_{\mathcal{VCY}}(G)$ for the family of virtually cyclic subgroups of G to G/G. It essentially reduces the computation of $\pi_n(\boldsymbol{A}(B\mathrm{G}))$ to the computation of the system $\{\pi_n(\boldsymbol{A}(BV))\}$, where V ranges over the virtually cyclic subgroups V of G. Following the setup of Davis and Lück [13], we give the precise formulations of the various versions of the Farrell–Jones conjecture in Section 2. The equivalent original formulation of their conjecture can be found in Farrell and Jones [21].

Section 3 relates the Farrell–Jones conjecture for A–theory to the corresponding conjectures for other functors. In particular, we discuss the equivalence of the conjectures for $\boldsymbol{A}$, $\mathbf{Wh}^{\mathrm{CAT}}$ and $\boldsymbol{P}^{\mathrm{CAT}}$, where the latter denote the nonconnective spectra modeling the Whitehead space and the space of stable pseudoisotopies, with CAT being TOP, PL or DIFF.

As an illustration of the impact of the Farrell–Jones conjecture for A–theory, we discuss applications to the automorphism groups of aspherical closed manifolds in Section 4.1, where also the proof of the following Theorem 1.3 is given.

Let $\boldsymbol{NA}(\{\bullet\})$ be the Nil–term occurring in the Bass–Heller–Swan-isomorphisms for nonconnective A–theory; see Hüttemann, Klein, Vogell, Waldhausen and Williams [30; 31]. We have

$$\textbf{(1.2)}\quad \pi_n(\boldsymbol{A}(S^1)) = \pi_n(\boldsymbol{A}(\{\bullet\})) \oplus \pi_{n-1}(\boldsymbol{A}(\{\bullet\})) \oplus \pi_n(\boldsymbol{NA}(\{\bullet\})) \oplus \pi_n(\boldsymbol{NA}(\{\bullet\})).$$

We conclude $\pi_n(\boldsymbol{NA}(\{\bullet\})) = \{0\}$ for $n \le 1$ and $\pi_n(\boldsymbol{NA}(\{\bullet\})) \otimes_{\mathbb{Z}} \mathbb{Q} = \{0\}$ for $n \in \mathbb{Z}$ from Theorem 3.7 and Lück and Steimle [38, Theorem 0.3]. On the other hand, $\pi_n(\boldsymbol{NA}(\{\bullet\}))$ for $n = 2, 3$ is an infinite-dimensional $\mathbb{F}_2$–vector space. For more information about $\pi_n(\boldsymbol{NA}(\{\bullet\}))$ we refer to Grunewald, Klein and Macko [28] and Hesselholt [29]. The next result is already explained in the special case of closed manifolds with negative sectional curvature by Weiss and Williams [59, Section 6.3], based on the work of Farrell and Jones [17; 18; 19; 20; 21], and we can extend it to torsionfree hyperbolic groups.

Theorem 1.3 (i) *Let G be a torsionfree hyperbolic group. Then we get an equivalence*

$$\mathbf{Wh}^{\mathrm{TOP}}(BG) \simeq \bigvee_C \mathbf{Wh}^{\mathrm{TOP}}(BC) \simeq \bigvee_C \boldsymbol{NA}(\{\bullet\}) \vee \boldsymbol{NA}(\{\bullet\}),$$

where C ranges over the conjugacy classes of maximal infinite cyclic subgroups of G.
In particular, $\mathbf{Wh}^{\mathrm{TOP}}(BG)$ is connective.

(ii) *Let M be a smoothable aspherical closed manifold of dimension ≥ 10 whose fundamental group π is hyperbolic.*
Then there is a $\mathbb{Z}/2$–action on $\mathbf{Wh}^{\mathrm{TOP}}(B\pi)$ such that we obtain, for $1 \le n \le \min\{\frac{1}{2}(\dim M - 7), \frac{1}{3}(\dim M - 4)\}$, isomorphisms

$$\pi_n(\mathrm{TOP}(M)) \cong \pi_{n+2}\big(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \big(\bigvee_C \mathbf{Wh}^{\mathrm{TOP}}(BC)\big)\big)$$

and an exact sequence

$$1 \to \pi_2\big(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \big(\bigvee_C \mathbf{Wh}^{\mathrm{TOP}}(BC)\big)\big) \to \pi_0(\mathrm{TOP}(M)) \to \mathrm{Out}(\pi) \to 1,$$

where C ranges over the conjugacy classes of maximal infinite cyclic subgroups of π.

Remark 1.4 The $\mathbb{Z}/2$–action we refer to in Theorem 1.3 is induced by the one given Weiss and Williams [57]. Vogell described in [48] another $\mathbb{Z}/2$–action on $\mathbf{Wh}(B\pi)$ which depends on the choice of a spherical fibration over $B\pi$. It was shown by Hüttemann et al [31] that for a certain fibration this action corresponds under the Bass–Heller–Swan decomposition to switching the Nil–terms via a homeomorphism. If the arguments presented in [31] carry over to other spherical fibrations and the two actions on $\mathbf{Wh}^{\mathrm{TOP}}(B\pi)$ agree for a suitably chosen fibration, then the homotopy orbits appearing in Theorem 1.3 can be identified as

$$E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \big(\bigvee_C \mathbf{Wh}^{\mathrm{TOP}}(BC)\big) \simeq \bigvee_C \boldsymbol{NA}(\{\bullet\}).$$

These issues will be discussed in Pieper's forthcoming PhD thesis [42].

The rest of the paper is devoted to the proof of Theorem 1.1. The main technical part of this paper concerns the proof for hyperbolic groups and CAT(0)–groups. It is given in Sections 6 and 7, and is motivated by the proof of the K–theoretic Farrell–Jones conjecture for CAT(0)–groups given by Wegner [55] based on the method of Bartels and Lück [5]. Our approach, which is based on work of Ullmann and Winges [47], requires us to define an analog of the transfer on geometric modules which works on Waldhausen categories of controlled retractive spaces. Virtually polycyclic groups have already been treated in [47].

In conjunction with the inheritance properties in Theorem 1.1(ii), the case of a cocompact lattice in an almost-connected Lie group or a fundamental group of a (not necessarily compact) d–dimensional manifold (possibly with boundary) for $d \le 3$ follows via the argument presented in Bartels, Farrell and Lück [3]. The inheritance properties for the A–theoretic conjecture are taken care of in Section 5.

Remark 1.5 (solvable groups) The class $\mathcal{FJ}_A$ has also been shown to contain all virtually solvable groups by Kasprowski, Ullmann, Wegner and Winges [33]. Therefore, $\mathcal{FJ}$ also contains any (not necessarily cocompact) lattice in a second countable, locally compact Hausdorff group with finitely many path components, the groups $\mathrm{GL}_n(\mathbb{Q})$

and $\mathrm{GL}_n(F(t))$ for $F(t)$ the function field over a finite field F, and all S–arithmetic groups. The arguments of Kammeyer, Lück and Rüping [32] and Rüping [46] carry over to show the prerequisites of Theorem 1.1 and Corollary 6.20, respectively.

Acknowledgments

The idea to use the categories of "cellwise 0–controlled morphisms" in Section 7 is due to Arthur Bartels and Paul Bubenzer. The paper is financially supported by the Leibniz-Preis of Lück granted by the DFG, the ERC Advanced Grant *KL2MG-interactions* (no. 662400) of Lück granted by the European Research Council, the Cluster of Excellence *Hausdorff Center for Mathematics* at Bonn, the SFB 647 *Space–Time–Matter* at Berlin, the SFB 878 *Groups, Geometry & Actions* at Münster, and the Junior Hausdorff Trimester Program *Topology* at the Hausdorff Research Institute for Mathematics (HIM).

We thank Xiaolei Wu for comments on an earlier version of this article.

2 The isomorphism conjecture

In this section we state various versions of the isomorphism conjectures we want to consider. We assume familiarity with the notion of a G–equivariant homology theory from [13] and the notion of an equivariant homology theory from [36]. As usual, we use a convenient category of compactly generated spaces.

2.1 The metaisomorphism conjecture for functors from spaces to spectra

Let $\boldsymbol{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be a covariant functor. Throughout this section we will assume that $\boldsymbol{S}$ *respects weak equivalences and disjoint unions*, ie a weak homotopy equivalence of spaces $f\colon X \to Y$ is sent to a weak homotopy equivalence of spectra $\boldsymbol{S}(f)\colon \boldsymbol{S}(X) \to \boldsymbol{S}(Y)$ and, for a collection of spaces $\{X_i \mid i \in I\}$ for an arbitrary index set I, the canonical map

$$\bigvee_{i\in I} \boldsymbol{S}(X_i) \to \boldsymbol{S}\big(\textstyle\coprod_{i\in I} X_i\big) \tag{2.1}$$

is a weak homotopy equivalence of spectra. Weak equivalences of spectra are understood to be the stable equivalences, ie the maps which induce isomorphisms on all stable homotopy groups. We obtain a covariant functor

$$\boldsymbol{S}^B\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}, \quad \mathcal{G} \mapsto \boldsymbol{S}(B\mathcal{G}), \tag{2.2}$$

where $B\mathcal{G}$ is the classifying space of the groupoid $\mathcal{G}$ which is the geometric realization of the simplicial set given by its nerve and denoted by $B^{\mathrm{bar}}\mathcal{G}$ in [13, page 227]. Let $H_n^?(-;\boldsymbol{S}^B)$ be the equivariant homology theory in the sense of [36, Section 1] which is associated to $\boldsymbol{S}^B$ by the construction in [37, Proposition 157 on page 796]. Equivariant homology theory essentially means that we get for every group G a G–homology theory $H_n^G(-;\boldsymbol{S}^B)$ satisfying the disjoint union axiom and for every group homomorphism $\alpha\colon H \to G$ and H–CW–pair (X,A) we get natural maps compatible with boundary homomorphisms of pairs $H_*^H(X,A;\boldsymbol{S}^B) \to H_*^G(\alpha_*(X,A);\boldsymbol{S}^B)$, which are bijective if the kernel of α acts freely on X. Moreover, for any group G, subgroup $H \subseteq G$ and $n \in \mathbb{Z}$ we have canonical identifications

$$H_n^G(G/H;\boldsymbol{S}^B) \cong H_n^H(H/H;\boldsymbol{S}^B) \cong \pi_n(\boldsymbol{S}(BH)).$$

Conjecture 2.3 (metaisomorphism conjecture for functors from spaces to spectra) *Let $\boldsymbol{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be a covariant functor which respects weak equivalences and disjoint unions. The group G satisfies the* ***metaisomorphism conjecture for $\boldsymbol{S}$*** *with respect to the family $\mathcal{F}$ of subgroups of G if the assembly map induced by the projection* pr: $E_{\mathcal{F}}(G) \to G/G$,

$$H_n^G(\mathrm{pr};\boldsymbol{S}^B)\colon H_n^G(E_{\mathcal{F}}(G);\boldsymbol{S}^B) \to H_n^G(G/G;\boldsymbol{S}^B) \cong \pi_n(\boldsymbol{S}(B\mathrm{G})),$$

is bijective for all $n \in \mathbb{Z}$.

Example 2.4 (the K– and L–theoretic Farrell–Jones conjectures) Let R be a ring (with involution). There are covariant functors [37, Theorem 158]

$$\boldsymbol{K}_R\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA},$$
$$\boldsymbol{L}_R^{\langle -\infty\rangle}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$$

such that, for every group G, which we can consider as a groupoid $\underline{G}$ with precisely one object and G as its group of automorphisms, and $n \in \mathbb{Z}$, we have

$$K_n(RG) = \pi_n(\boldsymbol{K}_R(\underline{G})),$$
$$L_n^{\langle -\infty\rangle}(RG) = \pi_n(\boldsymbol{L}_R^{\langle -\infty\rangle}(\underline{G})).$$

Then the K–theoretic and L–theoretic Farrell–Jones conjectures, which were originally formulated in [21, Conjecture 1.6 on page 257], are equivalent to the statement that the covariant functors $\boldsymbol{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$, given by the composition of $\boldsymbol{K}_R$

and $\boldsymbol{L}_R$ with the functor sending a space to its fundamental groupoid, satisfy the Metaisomorphism conjecture 2.3 for the family $\mathcal{VCY}$ of virtually cyclic subgroups of G.

Our main example is the following case. Let $\boldsymbol{A}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be the functor sending a space X to the spectrum $\boldsymbol{A}(X)$ given by the nonconnective version of Waldhausen's algebraic K–theory of spaces in the sense of [47].

Lemma 2.5 (i) *The functor $\boldsymbol{A}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ respects weak equivalences and disjoint unions.*

(ii) *For any directed systems of spaces $\{X_i \mid i \in I\}$ indexed over an arbitrary directed set I, the canonical map*

$$\operatorname*{hocolim}_{i \in I} \boldsymbol{A}(X_i) \to \boldsymbol{A}\big(\operatorname*{hocolim}_{i \in I} X_i\big)$$

is a weak homotopy equivalence.

Proof In the connective case, Waldhausen proved in [53, Proposition 2.1.7] that A–theory preserves weak equivalences. The other two properties follow upon inspection of the explicit model as finite retractive CW–complexes. Since the homotopy groups of connective and nonconnective A–theory agree in positive degrees and the indexing category I is assumed to be directed in (ii), this proves the claim for π_n for $n \geq 1$.

Note that the algebraic K–theory functor which sends X to $K(\mathbb{Z}\Pi(X))$ enjoys the properties claimed for $\boldsymbol{A}$. Since Vogell showed that the linearization map $\boldsymbol{L}\colon \boldsymbol{A} \to \boldsymbol{K}$ induces an isomorphism on all nonpositive homotopy groups [50], the general case follows.

It will be shown in [42] that the nonconnective deloopings described by Ullmann and Winges in [47] and Vogell in [49] are equivalent. □

Conjecture 2.6 (the Farrell–Jones conjecture for A–theory) *A group G satisfies the Farrell–Jones conjecture for A–theory if the Metaisomorphism conjecture 2.3 holds for $\boldsymbol{A}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ and the family $\mathcal{VCY}$, ie for every $n \in \mathbb{Z}$ the projection $E_{\mathcal{VCY}}(G) \to G/G$ induces an isomorphism*

$$H_n^G(\mathrm{pr}; \boldsymbol{A}^B)\colon H_n^G(E_{\mathcal{VCY}}(G); \boldsymbol{A}^B) \to H_n^G(G/G; \boldsymbol{A}^B) = \pi_n(\boldsymbol{A}(B\mathrm{G})).$$

2.2 The metaisomorphism conjecture for functors from spaces to spectra with coefficients

Let G be a group and Z be a G–CW–complex. Define a covariant $\mathsf{Or}(G)$–spectrum

$$\textbf{(2.7)} \qquad \boldsymbol{S}_Z^G\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}, \quad G/H \mapsto \boldsymbol{S}(G/H \times_G Z),$$

where $G/H \times_G Z$ is the orbit space of the diagonal G–action on $G/H \times Z$. Notice that there is an obvious homeomorphism $G/H \times_G Z \xrightarrow{\cong} Z/H$. Denote by $H_n^G(-;\boldsymbol{S}_Z^G)$ the G–homology theory in the sense of [36, Section 1] which is associated to $\boldsymbol{S}_Z^G$ by the construction of [37, Proposition 156 on page 795] and satisfies $H_n^G(G/H;\boldsymbol{S}_Z^G) \cong \pi_n(\boldsymbol{S}_Z^G(G/H)) = \pi_n(\boldsymbol{S}(Z/H))$ for any homogeneous G–space G/H and $n \in \mathbb{Z}$.

Conjecture 2.8 (metaisomorphism conjecture for functors from spaces to spectra with coefficients) *Let $\boldsymbol{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be a covariant functor which respects weak equivalences and disjoint unions. The group G satisfies the* ***metaisomorphism conjecture for $\boldsymbol{S}$ with coefficients*** *with respect to the family $\mathcal{F}$ of subgroups of G if for any free G–CW–complex Z the assembly map*

$$H_n^G(\mathrm{pr};\boldsymbol{S}_Z^G)\colon H_n^G(E_{\mathcal{F}}(G);\boldsymbol{S}_Z^G) \to H_n^G(G/G;\boldsymbol{S}_Z^G) = \pi_n(\boldsymbol{S}(Z/G)),$$

induced by the projection pr: $E_{\mathcal{F}}(G) \to G/G$, *is bijective for all $n \in \mathbb{Z}$.*

Example 2.9 ($Z = EG$) If we take $Z = EG$ in Conjecture 2.8, then Conjecture 2.8 reduces to Conjecture 2.3. Namely, for a G–set S let $\mathcal{T}^G(S)$ be its *transport groupoid* whose set of objects is S, the set of morphisms from s_1 to s_2 is the set $\{g \in G \mid s_2 = gs_1\}$ and composition comes from the multiplication in G. There is a homotopy equivalence $B\mathcal{T}^G(G/H) \xrightarrow{\simeq} G/H \times_G EG$ which is natural in G/H. Hence, we get a weak homotopy equivalence of $\mathsf{Or}(G)$–spectra $\boldsymbol{S}^B(\mathcal{T}^G(G/?)) \xrightarrow{\simeq} \boldsymbol{S}_{EG}^G$. It induces an isomorphism of G–homology theories — see [13, Lemma 4.6] —

$$H_*^G(-;\boldsymbol{S}^B) \xrightarrow{\cong} H_*^G(-;\boldsymbol{S}_{EG}^G).$$

Remark 2.10 (relation to the original formulation) In [21, Section 1.7 on page 262], Farrell and Jones formulate a fibered version of their conjectures for a covariant functor $\boldsymbol{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ for every (Serre) fibration $\xi\colon Y \to X$ over a connected CW–complex X. In our setup this corresponds to choosing Z to be the total space of the fibration obtained from $Y \to X$ by pulling back along the universal covering $\widetilde{X} \to X$. This space Z is a free G–CW–complex for $G = \pi_1(X)$. Note that every free G–CW–complex Z can always be obtained in this fashion from the fiber bundle $EG \times_G Z \to B\mathrm{G}$ up to G–homotopy; compare [21, Corollary 2.2.1 on page 263].

We sketch the proof of this identification. Let A be a G–CW–complex. Let $f\colon \mathcal{E}(X) \to X$ be the map obtained by taking the quotient of the $G = \pi_1(X)$–action on the G–map $A \times \widetilde{X} \to \widetilde{X}$ given by the projection. Denote by $\widehat{p}\colon \mathcal{E}(\xi) \to \mathcal{E}(X)$ the pullback of ξ with f. Let $q\colon \mathcal{E}(\xi) \to A/G$ be the composite of $\widehat{p}$ with the map $\mathcal{E}(X) \to A/G$ induced by the projection $A \times \widetilde{X} \to A$. This is a stratified fibration and one can consider the spectrum $\mathbb{H}(A/G; \mathcal{S}(q))$ in the sense of Quinn [43, Section 8]. Put

$$H_n^G(A;\xi) := \pi_n(\mathbb{H}(A/G;\mathcal{S}(q))).$$

The projection pr: $A \to G/G$ induces a map

$$a(A)\colon \mathbb{H}(A/G;\mathcal{S}(q)) \to \mathbb{H}(G/G;\mathcal{S}(Y \to G/G)) = \boldsymbol{S}(Y), \tag{2.11}$$

which is the assembly map in [21, Section 1.7 on page 262] if we take $A = E_{\mathcal{VCY}}(G)$. The construction of $H_n^G(A;\xi) := \mathbb{H}(A/G;\mathcal{S}(q))$ is very complicated, but, fortunately, for us only two facts are relevant. We obtain a G–homology theory $H_n^G(-;\xi)$ and for every $H \subseteq G$ we get a natural identification $H_n^G(G/H;\xi) = \boldsymbol{S}_Z^G(G/H)$. Hence, the functor G–CW–$\mathsf{COMPLEXES} \to \mathsf{SPECTRA}$ given by $A \mapsto \mathbb{H}(A/G;\mathcal{S}(q))$ is weakly excisive and its restriction to $\mathsf{Or}(G)$ is the functor $\boldsymbol{S}_Z^G$. We conclude from [13, Theorem 6.3] that the map (2.11) can be identified with the map induced by the projection $A \to G/G$,

$$H_n^G(A;\boldsymbol{S}_Z^G) \to H_n^G(G/G;\boldsymbol{S}_Z^G) = \pi_n(\boldsymbol{S}(Z/G)) = \pi_n(\boldsymbol{S}(Y)),$$

which appears in Metaisomorphism conjecture 2.8 for functors from spaces to spectra with coefficients.

Remark 2.12 (the condition free is necessary in Conjecture 2.8) Conjecture 2.8 is only true very rarely if we drop the condition that Z is free. Take for instance $Z = G/G$. Then Conjecture 2.8 predicts that the projection $E_{\mathcal{F}}(G)/G \to G/G$ induces for all $n \in \mathbb{Z}$ an isomorphism

$$H_n(\mathrm{pr};\boldsymbol{S}(\{\bullet\}))\colon H_n(E_{\mathcal{F}}(G)/G;\boldsymbol{S}(\{\bullet\})) \to H_n(\{\bullet\},\boldsymbol{S}(\{\bullet\})),$$

where $H_*(-;\boldsymbol{S}(\{\bullet\}))$ is the (nonequivariant) homology theory associated to the spectrum $\boldsymbol{S}(\{\bullet\})$. This statement is in general wrong, except in extreme cases such as $\mathcal{F} = \mathcal{ALL}$.

Conjecture 2.13 (the Farrell–Jones conjecture for A–theory with coefficients) *A group G satisfies the **Farrell–Jones conjecture for A–theory with coefficients** if the Metaisomorphism conjecture 2.8 with coefficients holds for $\boldsymbol{A}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ and the family $\mathcal{VCY}$, ie for every $n \in \mathbb{Z}$ and free G–CW–complex Z the projection $E_{\mathcal{VCY}}(G) \to G/G$ induces an isomorphism*

$$H_n^G(\mathrm{pr}; \boldsymbol{A}_Z^G)\colon H_n^G(E_{\mathcal{VCY}}(G); \boldsymbol{A}_Z^G) \to H_n^G(G/G; \boldsymbol{A}_Z^G) = \pi_n(\boldsymbol{A}(Z/G)).$$

2.3 The metaisomorphism conjecture for functors from spaces to spectra with coefficients and finite wreath products

There are also versions with finite wreath products. Recall that for groups G and F their *wreath product* $G \wr F$ is defined to be the semidirect product $\bigl(\prod_F G\bigr) \rtimes F$, where F acts on $\prod_F G$ by permuting the factors. Fix a class of groups $\mathcal{C}$ which is closed under isomorphisms, taking subgroups and taking quotients. Examples are the classes $\mathcal{FIN}$ and $\mathcal{VCY}$ of finite and of virtually cyclic groups. For a group G define the family of subgroups $\mathcal{C}(G) := \{K \subseteq G \mid K \in \mathcal{C}\}$.

Conjecture 2.14 (the metaisomorphism conjecture for functors from spaces to spectra with coefficients and finite wreath products) *Let $\boldsymbol{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be a covariant functor which respects weak equivalences and disjoint unions. The group G satisfies the **metaisomorphism conjecture with coefficients and finite wreath products** for the functor $\boldsymbol{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ with respect to the class $\mathcal{C}$ of groups if, for any finite group F, the wreath product $G \wr F$ satisfies the Metaisomorphism conjecture 2.8 with coefficients for the functor $\boldsymbol{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ with respect to the family $\mathcal{C}(G \wr F)$ of subgroups of G.*

Conjecture 2.15 (the Farrell–Jones conjecture for A–theory with coefficients and finite wreath products) *A group G satisfies the **Farrell–Jones conjecture for A–theory with coefficients and finite wreath products** if the Metaisomorphism conjecture 2.14 with coefficients and finite wreath products holds for $\boldsymbol{A}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ and the class $\mathcal{VCY}$ of virtually cyclic groups.*

The next two lemmas will be needed later.

Lemma 2.16 *Let $\boldsymbol{E}$ be a spectrum such that $\boldsymbol{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ is given by $Y \mapsto Y_+ \wedge \boldsymbol{E}$.*

(i) *Then, for any group G, any G–CW–complex X which is contractible (after forgetting the G–action) and any free G–CW–complex Z, the projection $X \to G/G$ induces for all $n \in \mathbb{Z}$ an isomorphism*

$$H_n^G(X; \mathbf{S}_Z^G) \xrightarrow{\cong} H_n^G(G/G; \mathbf{S}_Z^G).$$

(ii) *Conjectures 2.3, 2.8 and 2.14 hold for such an $\mathbf{S}$ for every group G and every family $\mathcal{F}$ of subgroups of G.*

An $\mathbf{S}$ given by $Y \mapsto Y_+ \wedge \mathbf{E}$ is a homology theory, and thus the lemma states that the conjectures hold for homology theories.

Proof (i) There are natural isomorphisms of spectra

$$\begin{aligned}&\mathrm{map}_G((G/?), X)_+ \wedge_{\mathrm{Or}(G)} ((G/? \times_G Z)_+ \wedge \mathbf{E}) \\ &\qquad \xrightarrow{\cong} \big(\big(\mathrm{map}_G((G/?), X) \times_{\mathrm{Or}(G)} G/?\big) \times_G Z\big)_+ \wedge \mathbf{E} \xrightarrow{\cong} (X \times_G Z)_+ \wedge \mathbf{E},\end{aligned}$$

where the second isomorphism comes from the G–homeomorphism

$$\mathrm{map}_G((G/?), X) \times_{\mathrm{Or}(G)} G/? \xrightarrow{\cong} X$$

of [13, Theorem 7.4(1)]. Since Z is a free G–CW–complex and X is contractible (after forgetting the group action), the projection $X \times_G Z \to G/G \times_G Z$ is a homotopy equivalence and hence induces a weak homotopy equivalence

$$(X \times_G Z)_+ \wedge \mathbf{E} \xrightarrow{\simeq} (G/G \times_G Z)_+ \wedge \mathbf{E}.$$

Thus, we get a weak homotopy equivalence

$$\mathrm{map}_G((G/?), X)_+ \wedge_{\mathrm{Or}(G)} ((G/? \times_G Z)_+ \wedge \mathbf{E}) \to (G/G \times_G Z)_+ \wedge \mathbf{E}.$$

Under the identifications coming from the definitions

$$\begin{aligned}H_n^G(X; \mathbf{S}_Z^G) &= \pi_n\big(\mathrm{map}_G((G/?), X)_+ \wedge_{\mathrm{Or}(G)} ((G/? \times_G Z)_+ \wedge \mathbf{E})\big),\\ H_n^G(G/G; \mathbf{S}_Z^G) &= \pi_n((G/G \times_G Z)_+ \wedge \mathbf{E}),\end{aligned}$$

this weak homotopy equivalence induces on homotopy groups the isomorphism

$$H_n^G(X; \mathbf{S}_Z^G) \to H_n^G(G/G; \mathbf{S}_Z^G).$$

(ii) This follows from assertion (i). □

Lemma 2.17 *Let $\mathbf{S}, \mathbf{T}, \mathbf{U}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be covariant functors which respect weak equivalences and disjoint unions. Let $\mathbf{i}\colon \mathbf{S} \to \mathbf{T}$ and $\mathbf{p}\colon \mathbf{T} \to \mathbf{U}$ be natural transformations such that for any space Y the sequence of spectra $\mathbf{S}(Y) \xrightarrow{\mathbf{i}(Y)} \mathbf{T}(Y) \xrightarrow{\mathbf{p}(Y)} \mathbf{U}(Y)$ is up to weak homotopy equivalence a cofiber sequence of spectra.*

(i) *Then we obtain for every group G and all G–CW–complexes X and Z a natural long exact sequence*

$$\cdots \to H_n^G(X; \mathbf{S}_Z^G) \to H_n^G(X; \mathbf{T}_Z^G) \to H_n^G(X; \mathbf{U}_Z^G) \to H_{n-1}^G(X; \mathbf{S}_Z^G) \to H_{n-1}^G(X; \mathbf{T}_Z^G) \to H_{n-1}^G(X; \mathbf{U}_Z^G) \to \cdots .$$

(ii) *Let G be a group and $\mathcal{F}$ be a family of subgroups of G. Then the Metaisomorphism conjecture 2.3 for functors from spaces to spectra holds for all three functors $\mathbf{S}$, $\mathbf{T}$ and $\mathbf{U}$ for $(G, \mathcal{F})$ if it holds for two of the functors $\mathbf{S}$, $\mathbf{T}$ and $\mathbf{U}$ for $(G, \mathcal{F})$.*

The analogous statement is true for the Metaisomorphism conjecture 2.8 for functors from spaces to spectra with coefficients and for the Metaisomorphism conjecture 2.14 for functors from spaces to spectra with coefficients and finite wreath products.

Proof (i) The version for spectra of [13, Theorem 3.11] implies that we obtain, up to weak homotopy equivalence, a cofiber sequence of spectra

$$\operatorname{map}_G(G/?, X)_+ \wedge_{\mathrm{Or}(G)} \mathbf{S}(G/? \times_G Z) \to \operatorname{map}_G(G/?, X)_+ \wedge_{\mathrm{Or}(G)} \mathbf{T}(G/? \times_G Z) \to \operatorname{map}_G(G/?, X)_+ \wedge_{\mathrm{Or}(G)} \mathbf{U}(G/? \times_G Z).$$

and passing to its associated long exact sequence of homotopy groups yields the result.

(ii) This follows from assertion (i) and the five lemma. □

3 Relations between the conjectures for various theories

There are other prominent covariant functors $\mathsf{SPACES} \to \mathsf{SPECTRA}$ which respect weak homotopy equivalences and disjoint unions. Notice in the sequel that we are always considering the nonconnective versions. We are thinking of the stable pseudoisotopy spectrum $\mathbf{P}^{\mathrm{CAT}}$ and the Whitehead spectrum $\mathbf{Wh}^{\mathrm{CAT}}$, where CAT can be the topological category TOP, the PL–category PL or the smooth category DIFF. For the definition of $\mathbf{P}^{\mathrm{CAT}}$ we refer to [14; 42; 57].

Usually, the Whitehead spectrum is defined as a connective spectrum — see [54] — and see [58, Section 2.2] for a definition of the classical assembly map. We make the obvious generalization.

Definition 3.1 The *topological nonconnective Whitehead spectrum* $\mathbf{Wh}^{\mathrm{TOP}}(X)$ is the homotopy cofiber of the classical assembly map in nonconnective A–theory:

$$X_+ \wedge \boldsymbol{A}(\{\bullet\}) \to \boldsymbol{A}(X) \to \mathbf{Wh}^{\mathrm{TOP}}(X).$$

The *piecewise-linear nonconnective Whitehead spectrum* is, by definition,

$$\mathbf{Wh}^{\mathrm{PL}}(X) := \mathbf{Wh}^{\mathrm{TOP}}(X).$$

Further, we define the *smooth nonconnective Whitehead spectrum* $\mathbf{Wh}^{\mathrm{DIFF}}(X)$ as the homotopy cofiber of the sequence

$$\Sigma^\infty X_+ \to \boldsymbol{A}(X) \to \mathbf{Wh}^{\mathrm{DIFF}}(X),$$

where $\Sigma^\infty X_+ \to \boldsymbol{A}(X)$ factors as the unit map $\Sigma^\infty X_+ = X_+ \wedge \mathbb{S} \to X_+ \wedge \boldsymbol{A}(\{\bullet\})$ and assembly.

Theorem 3.2 (relations between the various functors) (i) *There is a zigzag of natural equivalences,*

$$\boldsymbol{P}^{\mathrm{CAT}} \xleftrightarrow{\simeq} \Omega^2 \mathbf{Wh}^{\mathrm{CAT}},$$

where CAT *can be taken to be* TOP, PL *or* DIFF.

(ii) *The canonical map*

$$\boldsymbol{P}^{\mathrm{PL}} \xrightarrow{\simeq} \boldsymbol{P}^{\mathrm{TOP}}$$

is a natural equivalence.

Proof The connective, objectwise case of (i) follows from the equivalence $\boldsymbol{P}(M) \simeq \Omega^2 \mathbf{Wh}^{\mathrm{PL}}(M)$, which was originally stated in [52] and fully proved in [54, Theorem 0.2].

There are some issues concerning the full functoriality of pseudoisotopy, which will be clarified in [14; 42]. The full statement will be established in [42].

The objectwise version of (ii) has been shown in [11; 12]. An argument for the full statement will be given in [14]. □

Lemma 3.3 *If the Metaisomorphism conjecture 2.3 for functors from spaces to spectra holds for the group G and the family $\mathcal{F}$ for one of the functors $\boldsymbol{A}$, $\mathbf{Wh}^{\mathrm{TOP}}$, $\mathbf{Wh}^{\mathrm{PL}}$, $\mathbf{Wh}^{\mathrm{DIFF}}$, $\boldsymbol{P}^{\mathrm{TOP}}$, $\boldsymbol{P}^{\mathrm{PL}}$ and $\boldsymbol{P}^{\mathrm{DIFF}}$, then it holds for all of them.*

The analogous statement holds for the Metaisomorphism conjecture 2.8 for functors from spaces to spectra with coefficients and for the Metaisomorphism conjecture 2.14 for functors from spaces to spectra with coefficients and finite wreath products.

Proof This follows from Lemmas 2.16 and 2.17. □

Remark 3.4 (the nonconnective spectrum of stable h–cobordisms) There is also the nonconnective stable h–cobordism spectrum $\boldsymbol{H}^{\mathrm{CAT}}(M)$ of a compact manifold (possibly with boundary) M. Note that h–cobordisms are (usually) only defined as a functor in codimension-zero embeddings. As such, they are related to the previous functors. For every compact manifold M (possibly with boundary), there are natural weak homotopy equivalences

$$\boldsymbol{H}^{\mathrm{CAT}}(M) \xrightarrow{\simeq} \Omega\mathbf{Wh}^{\mathrm{CAT}}(M)$$

and

$$\boldsymbol{P}^{\mathrm{CAT}}(M) \xrightarrow{\simeq} \Omega\boldsymbol{H}^{\mathrm{CAT}}(M).$$

For the proof and more information we refer to [54].

Finally, we explain the relationship between A–theory and algebraic K–theory of integral group rings.

For a space X, denote its fundamental groupoid by $\Pi(X)$. There is a so-called *linearization map*, natural in X,

$$\boldsymbol{L}(X)\colon \boldsymbol{A}(X) \to \boldsymbol{K}(\mathbb{Z}\Pi_1(X)) \tag{3.5}$$

The next result follows combining [50, Section 4] and [52, Propositions 2.2 and 2.3].

Theorem 3.6 (connectivity of the linearization map) *Let X be a CW–complex. Then:*

(i) *The linearization map $\boldsymbol{L}(X)$ of (3.5) is 2–connected, ie the map*

$$L_n(X) := \pi_n(\boldsymbol{L}(X))\colon A_n(X) \to K_n(\mathbb{Z}\Pi(X))$$

is bijective for $n \le 1$ and surjective for $n = 2$.

(ii) *The map L_n is rationally bijective for all $n \in \mathbb{Z}$ provided that each component of X is aspherical.*

This implies that the K–theoretic Farrell–Jones conjecture for $\mathbb{Z}G$ and the A–theoretic Farrell–Jones conjecture for $\boldsymbol{A}(BG)$ are equivalent in degree ≤ 1 and rationally equivalent in all degrees. More precisely, we have:

Theorem 3.7 (relating A–theory to algebraic K–theory) *Consider a group G and a family $\mathcal{F}$ of subgroups of G. The linearization map* (3.5) *and the projection $E_{\mathcal{F}}(G) \to G/G$ yield a commutative diagram*

$$\begin{array}{ccc} H_n^G(E_{\mathcal{F}}(G); \boldsymbol{A}^{\boldsymbol{B}}) & \longrightarrow & H_n^G(G/G; \boldsymbol{A}^{\boldsymbol{B}}) = A_n(B\mathrm{G}) \\ \downarrow & & \downarrow \\ H_n^G(E_{\mathcal{F}}(G); \boldsymbol{K}_{\mathbb{Z}}) & \longrightarrow & H_n^G(G/G; \boldsymbol{K}_{\mathbb{Z}}) = K_n(\mathbb{Z}G) \end{array}$$

where $\boldsymbol{K}_{\mathbb{Z}}$: $\mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ has been recalled in Example 2.4. The vertical arrows are bijective for $n \le 1$ and surjective for $n = 2$. They are rationally bijective for all $n \in \mathbb{Z}$.

4 Some applications to automorphism groups of aspherical closed manifolds

Before we begin with the proof of Theorem 1.1, we want to illustrate the impact of the Farrell–Jones conjecture by discussing automorphism groups of aspherical closed manifolds. For rational computations the Farrell–Jones conjecture for K–theory and L–theory suffices. For potential integral computations one needs the Farrell–Jones conjecture for A–theory and for L–theory. More details about automorphism groups of closed manifolds can be found in [59].

4.1 Topological automorphism groups of aspherical closed manifolds

Let $\mathrm{TOP}(M)$ be the topological group of self-homeomorphisms of the closed manifold M. Denote by $\mathrm{G}(M)$ the monoid of self-homotopy equivalences $M \to M$. Let $\widetilde{\mathrm{TOP}}(M)$ and $\widetilde{\mathrm{G}}(M)$ be the block versions; see [59, page 168] for a survey and further references. There are natural maps making the diagram

$$\begin{array}{ccc} \mathrm{TOP}(M) & \longrightarrow & \widetilde{\mathrm{TOP}}(M) \\ \downarrow & & \downarrow \\ \mathrm{G}(M) & \longrightarrow & \widetilde{\mathrm{G}}(M) \end{array}$$

commute.

Define $\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M)$, $\widetilde{\mathrm{G}}(M)/\mathrm{TOP}(M)$ and $\mathrm{G}(M)/\mathrm{TOP}(M)$ to be the homotopy fibers of the maps $B\mathrm{TOP}(M) \to B\widetilde{\mathrm{TOP}}(M)$, $B\mathrm{TOP}(M) \to B\widetilde{\mathrm{G}}(M)$ and

$B\mathrm{TOP}(M) \to B\mathrm{G}(M)$. We obtain a commutative diagram with horizontal fiber sequences

$$\begin{tikzcd}
\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \arrow[r] \arrow[d] & B\mathrm{TOP}(M) \arrow[r] \arrow[d, "\mathrm{id}"] & B\widetilde{\mathrm{TOP}}(M) \arrow[d] \\
\widetilde{\mathrm{G}}(M)/\mathrm{TOP}(M) \arrow[r] & B\mathrm{TOP}(M) \arrow[r] & B\widetilde{\mathrm{G}}(M) \\
\mathrm{G}(M)/\mathrm{TOP}(M) \arrow[r] \arrow[u] & B\mathrm{TOP}(M) \arrow[r] \arrow[u, "\mathrm{id}"] & B\mathrm{G}(M). \arrow[u]
\end{tikzcd}$$

According to [44, Theorem 5.8], there is no real difference between self-homotopy equivalences and their block version.

Lemma 4.1 *The map $\mathrm{G}(M) \to \widetilde{\mathrm{G}}(M)$ and hence the map $B\mathrm{G}(M) \to B\widetilde{\mathrm{G}}(M)$ are weak homotopy equivalences.*

The relative homotopy groups of the map $\widetilde{\mathrm{TOP}}(M) \to \widetilde{\mathrm{G}}(M)$ can be identified with the groups $\mathcal{S}^s(M \times D^n, \partial)$ as explained in [15, page 285]. The next lemma follows in combination with [5, Proposition 0.3]. Recall that a space X is *aspherical* if $\pi_i(X) = 0$ for $i \neq 1$.

Lemma 4.2 *Suppose that M is an aspherical closed manifold of dimension ≥ 5 and both the K– and L–theoretic Farrell–Jones conjectures hold for $\mathbb{Z}\pi_1(M)$.*

Then $\mathcal{S}^s(M \times D^n, \partial)$ is trivial for $n \geq 0$ and the map

$$\widetilde{\mathrm{TOP}}(M) \to \widetilde{\mathrm{G}}(M)$$

is a weak homotopy equivalence.

For aspherical spaces X, the homotopy groups of $\mathrm{G}(X)$ can be computed from the long exact sequence of homotopy groups associated to the evaluation map $\mathrm{G}(X) \xrightarrow{\mathrm{ev}_{x_0}} X$ for some basepoint $x_0 \in X$:

Lemma 4.3 *Let X be an aspherical CW–complex. Then*

$$\pi_n(\mathrm{G}(X)) \cong \begin{cases} \mathrm{Out}(\pi_1(X)) & \text{if } n = 0, \\ \mathrm{center}(\pi_1(X)) & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

We conclude from Lemmas 4.1, 4.2 and 4.3:

Corollary 4.4 *If M is an aspherical closed manifold of dimension ≥ 5 with fundamental group π, and both the K–theoretic and L–theoretic Farrell–Jones conjectures hold for $\mathbb{Z}\pi$, then there are natural zigzags of homotopy equivalences*

$$\widetilde{\mathrm{TOP}}(M) \simeq \mathrm{G}(M)$$

and

$$B\widetilde{\mathrm{TOP}}(M) \simeq B\mathrm{G}(M)$$

and we get

$$\pi_n(\widetilde{\mathrm{TOP}}(M)) \cong \begin{cases} \mathrm{Out}(\pi) & \text{if } n = 0, \\ \mathrm{center}(\pi) & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Theorem 4.5 *There is a map*

$$\widetilde{\mathrm{TOP}}(M)/\,\mathrm{TOP}(M) \to \Omega^\infty(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \Omega\mathbf{Whs}^{\mathrm{TOP}}(M))$$

which is $(k+1)$–connected if k is in the topological concordance stable range for M. Here $\mathbf{Whs}^{\mathrm{TOP}}(M)$ denotes the connective cover of the Whitehead spectrum $\mathbf{Wh}^{\mathrm{TOP}}(M)$.

Proof It suffices to show that the spectrum denoted by $\mathbf{Wh}^{\mathrm{TOP}}(M)$ in [57] is a model for the homotopy cofiber of the assembly map; see Definition 3.1. This follows from combining [57, Theorem A], the equivalence $\boldsymbol{P}^{\mathrm{TOP}}(M) \simeq \Omega\boldsymbol{H}^{\mathrm{TOP}}(M)$ and [54, Theorem 0.2]. □

We conclude from Theorem 3.2(ii), Corollary 4.4, Theorem 4.5 and the lower bound on the topological concordance stable range given in [54, Corollary 1.4.2]:

Theorem 4.6 *Let M be a smoothable aspherical closed manifold of dimension ≥ 10 with fundamental group π. Suppose that the Farrell–Jones conjecture for A–theory for $B\pi$ and the Farrell–Jones conjecture for L–theory for $\mathbb{Z}\pi$ hold.*

Then we obtain for $2 \leq n \leq \min\{\frac{1}{2}(\dim M - 7), \frac{1}{3}(\dim M - 4)\}$ isomorphisms

$$\pi_n(\mathrm{TOP}(M)) \xrightarrow{\cong} \pi_{n+2}(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \mathbf{Wh}^{\mathrm{TOP}}(B\pi)),$$

and an exact sequence

$$\begin{aligned} 1 \to \pi_3(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \mathbf{Wh}^{\mathrm{TOP}}(B\pi)) \to \pi_1(\mathrm{TOP}(M)) \to \mathrm{center}(\pi) \\ \to \pi_2(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \mathbf{Wh}^{\mathrm{TOP}}(B\pi)) \to \pi_0(\mathrm{TOP}(M)) \to \mathrm{Out}(\pi) \to 1. \end{aligned}$$

Next we give the proof of Theorem 1.3.

Proof of Theorem 1.3 Let G be a torsionfree hyperbolic group. Then the Farrell–Jones conjecture for A–theory for BG and the Farrell–Jones conjectures for algebraic K–theory and for L–theory for $\mathbb{Z}G$ hold by Theorem 1.1 and [5; 6].

Since G is torsionfree, we have $K_{-i}(\mathbb{Z}G) = 0$ for all $i \geq 1$ and $\widetilde{K}_0(\mathbb{Z}G) = 0$, and thus the spectra under consideration are connective by Theorem 3.6(i). It follows from Lemma 3.3 that there is a weak homotopy equivalence

$$\mathcal{H}^G(E_{\mathcal{VCY}}(G); (\mathbf{Wh}^{\mathrm{TOP}})^B) \xrightarrow{\simeq} \mathbf{Wh}^{\mathrm{TOP}}(B\mathrm{G}).$$

The arguments in [38, Section 10] based on [39, Corollary 2.8 and Example 3.6] for algebraic K–theory carry over to $\mathbf{Wh}^{\mathrm{TOP}}$ and imply that there is a weak homotopy equivalence induced by the various inclusions $C \to G$ of representatives of the conjugacy classes of maximal cyclic subgroups of G,

$$\textstyle\bigvee_C \mathbf{Wh}^{\mathrm{TOP}}(BC) \xrightarrow{\simeq} \mathbf{Wh}^{\mathrm{TOP}}(B\mathrm{G}).$$

From the Bass–Heller–Swan decomposition (1.2) we obtain a weak homotopy equivalence

$$\boldsymbol{NA}(\{\bullet\}) \vee \boldsymbol{NA}(\{\bullet\}) \xrightarrow{\simeq} \mathbf{Wh}^{\mathrm{TOP}}(BC).$$

This proves part (i) of Theorem 1.3.

Part (ii) follows from part (i) and Theorem 4.6 together with the fact that the center of a hyperbolic group which is torsionfree and not cyclic is trivial. □

Theorems 3.7 and 4.6 imply:

Theorem 4.7 (rational homotopy groups of $\mathrm{TOP}(M)$ for an aspherical closed manifold) *Let M be a smoothable aspherical closed manifold of dimension ≥ 10 with fundamental group π. Suppose that the Farrell–Jones conjectures for K–theory and for L–theory for $\mathbb{Z}\pi$ hold.*

Then for $1 \leq n \leq \min\left\{\frac{1}{2}(\dim M - 7), \frac{1}{3}(\dim M - 4)\right\}$ we have

$$\pi_n(\mathrm{TOP}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathrm{center}(\pi) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } n = 1, \\ \{0\} & \text{if } n \geq 2. \end{cases}$$

4.2 Smooth automorphism groups of aspherical closed smooth manifolds

Taking the computation of $K_i(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ of Borel [10] into account, we get from Theorems 3.2 and 3.7 and [38, Theorem 0.3]:

Theorem 4.8 *Let M be an aspherical closed smooth manifold of dimension ≥ 10 with fundamental group π. Suppose that the Farrell–Jones conjectures for K–theory and for L–theory for $\mathbb{Z}\pi$ hold.*

Then we get, for all $n \in \mathbb{Z}$,

$$\pi_n(\mathbf{Wh}^{\mathrm{DIFF}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{k=1}^{\infty} H_{n-4k-1}(M;\mathbb{Q}),$$

$$\pi_n(\boldsymbol{P}^{\mathrm{DIFF}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{k=1}^{\infty} H_{n-4k+1}(M;\mathbb{Q}).$$

For the proof of the next result, which does involve the involutions on higher algebraic K–theory, we refer to [15, Lecture 5], [16] or [19, Section 2].

Theorem 4.9 (rational homotopy groups of $\mathrm{DIFF}(M)$ for an aspherical closed smooth manifold) *Let M be an aspherical closed smooth manifold of dimension ≥ 10 with fundamental group π. Suppose that the Farrell–Jones conjectures for K–theory and for L–theory for $\mathbb{Z}\pi$ hold.*

Then for $1 \leq n \leq \min\{\frac{1}{2}(\dim M - 7), \frac{1}{3}(\dim M - 4)\}$ we have

$$\pi_n(\mathrm{DIFF}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathrm{center}(\pi) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } n = 1, \\ \bigoplus_{j=1}^{\infty} H_{(n+1)-4j}(M;\mathbb{Q}) & \text{if } n \geq 2,\ \dim M \text{ odd}, \\ \{0\} & \text{if } n \geq 2,\ \dim M \text{ even}. \end{cases}$$

Remark 4.10 (surfaces and simply connected manifolds) There are very interesting computations of the cohomology of $B\mathrm{DIFF}(M)$ in a range and under stabilization with taking the connected sum with $S^n \times S^n$ for 2–dimensional manifolds or simply connected high-dimensional manifolds by Berglund, Galatius, Madsen, Randal-Williams, Weiss and others; see for instance [8; 9; 23; 24; 25; 26; 27; 40; 41]. The methods used in these papers are quite different. Notice that taking the connected sum with $S^n \times S^n$ will destroy asphericity except for $n = 1$, so that it is not clear what stabilization could mean in the context of aspherical manifolds in high dimensions.

5 Inheritance properties of the isomorphism conjectures

The main result of this section is:

Theorem 5.1 (inheritance properties of the metaconjecture with coefficients) *Let $S\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be a covariant functor which respects weak equivalences and disjoint unions. Let $\mathcal{C}$ be a class of groups which is closed under isomorphisms, taking subgroups and taking quotients.*

(i) *Suppose that the Metaisomorphism conjecture 2.8 with coefficients holds for $(G, \mathcal{C}(G))$, ie it holds for G with respect to the family of subgroups $\mathcal{C}(G) = \{H \subseteq G \mid H \in \mathcal{C}\}$ of G. Let $H \subseteq G$ be a subgroup. Then Conjecture 2.8 holds for $(H, \mathcal{C}(H))$.*

(ii) *Let $1 \to K \to G \xrightarrow{p} Q \to 1$ be an extension of groups. Suppose that $(Q, \mathcal{C}(Q))$ and $(p^{-1}(H), \mathcal{C}(p^{-1}(H))$ for every $H \in \mathcal{C}(Q)$ satisfy Conjecture 2.8. Then $(G, \mathcal{C}(G))$ satisfies Conjecture 2.8.*

(iii) *Suppose that Conjecture 2.8 is true for $(H_1 \times H_2, \mathcal{C}(H_1 \times H_2))$ for every $H_1, H_2 \in \mathcal{C}$. Then, for two groups G_1 and G_2, Conjecture 2.8 is true for the direct product $G_1 \times G_2$ and the family $\mathcal{C}(G_1 \times G_2)$ if and only if it is true for $(G_k, \mathcal{C}(G_k))$ for $k = 1, 2$.*

(iv) *Suppose that for any directed systems of spaces $\{X_i \mid i \in I\}$ indexed over an arbitrary directed set I the canonical map*

$$\operatorname*{hocolim}_{i \in I} S(X_i) \to S\big(\operatorname*{hocolim}_{i \in I} X_i\big)$$

is a weak homotopy equivalence. Let $\{G_i \mid i \in I\}$ be a directed system of groups over a directed set I (with arbitrary structure maps). Put $G = \operatorname{colim}_{i \in I} G_i$. Suppose that Conjecture 2.8 holds for $(G_i, \mathcal{C}(G_i))$ for every $i \in I$. Then Conjecture 2.8 holds for $(G, \mathcal{C}(G))$.

(v) *The analogs of assertions* (i), (ii), (iii) *and* (iv) *hold for the Metaisomorphism conjecture 2.14 with coefficients and finite wreath products. Moreover, if G is a group and $H \subseteq G$ is a subgroup of finite index, then Conjecture 2.14 holds for $(G, \mathcal{C}(G))$ if and only if Conjecture 2.14 holds for $(H, \mathcal{C}(H))$.*

Let us remark that the case of free products is missing in Theorem 5.1. It will be treated in Section 5.6 below.

5.1 The fibered metaisomorphism conjecture for equivariant homology theories

Next we introduce the metaconjecture and its fibered version in terms of G–homology theories. In this setting the analog of Theorem 5.1 has already been proved and we want to reduce the case coming from a functor from spaces to spectra to this situation.

Conjecture 5.2 (metaisomorphism conjecture) *The group G satisfies the **metaisomorphism conjecture** with respect to the G–homology theory $\mathcal{H}_*^G$ and the family $\mathcal{F}$ of subgroups of G if the assembly map*

$$\mathcal{H}_n^G(\mathrm{pr})\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(G/G)$$

induced by the projection pr: $E_{\mathcal{F}}(G) \to G/G$ *is bijective for all $n \in \mathbb{Z}$.*

Let X be a G–CW–complex. Let $\alpha\colon H \to G$ be a group homomorphism. Denote by $\alpha^* X$ the H–CW–complex obtained from X by *restriction with α*. Given an H–CW–complex Y, we denote the G–CW–complex given by *induction* by $\alpha_* Y$.

Fix a group Γ. An equivariant homology theory $H_*^?$ over Γ in the sense of [2, Definition 2.3] assigns to a group (G, ψ) over Γ, ie a group G together with a homomorphism $\psi\colon G \to \Gamma$, a G–homology theory $H_n^{G,\psi}$, sometimes denoted just by H_*^G. For two groups (G, ψ) and (G', ψ') over Γ and a morphism φ between them, ie a group homomorphism $\varphi\colon G \to G'$ with $\psi' \circ \alpha = \psi$, one obtains homomorphisms $\mathrm{ind}_\alpha\colon H_*^G(X, A) \to H_*^{G'}(\alpha_*(X, A))$ for every G–CW–pair (X, A), which are bijective if the kernel of α acts freely on (X, A) and compatible with the boundary homomorphisms associated to pairs. If Γ is trivial, this is just an equivariant homology theory.

Conjecture 5.3 (fibered metaisomorphism conjecture) *A group (G, ψ) over Γ satisfies the **fibered metaisomorphism conjecture with respect to $\mathcal{H}_*^?$ and the family $\mathcal{F}$ of subgroups of G** if for each group homomorphism $\varphi\colon K \to G$ the group K satisfies the Metaisomorphism conjecture 5.2 with respect to the K–homology theory $\mathcal{H}_*^{K,\psi\circ\varphi}$ and the family $\varphi^*\mathcal{F} = \{H \subseteq G \mid \varphi(H) \in \mathcal{F}\}$ of subgroups of K.*

Lemma 5.4 *Let (G, ψ) be a group over Γ and $\varphi\colon K \to G$ be a group homomorphism. If (G, ψ) satisfies the Fibered metaisomorphism conjecture 5.3 with respect to the family $\mathcal{F}$ of subgroups of G, then the group $(K, \psi \circ \varphi)$ over Γ satisfies the Fibered metaisomorphism conjecture 5.3 with respect to the family $\varphi^*\mathcal{F}$.*

Proof If $\vartheta\colon L \to K$ is a group homomorphism, then $\vartheta^*(\varphi^*\mathcal{F}) = (\varphi \circ \vartheta)^*\mathcal{F}$. □

5.2 Some adjunctions

Let $\boldsymbol{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be a covariant functor. Throughout this section we will assume that it respects weak equivalences and disjoint unions.

Lemma 5.5 *Let $\psi\colon K_1 \to K_2$ be a group homomorphism.*

(i) *If Z is a K_1–CW–complex and X is a K_2–CW–complex, then there is a natural isomorphism*

$$H_n^{K_1}(\psi^* X; \boldsymbol{S}_Z^{K_1}) \xrightarrow{\cong} H_n^{K_2}(X; \boldsymbol{S}_{\psi_* Z}^{K_2}).$$

(ii) *If Z is a K_2–CW–complex and X is a K_1–CW–complex, then there is a natural isomorphism*

$$H_n^{K_1}(X; \boldsymbol{S}_{\psi^* Z}^{K_1}) \xrightarrow{\cong} H_n^{K_2}(\psi_* X; \boldsymbol{S}_Z^{K_2}).$$

Proof (i) The fourth isomorphism appearing in [13, Lemma 1.9], together with [13, Lemma 4.6], applied levelwise implies that it suffices to construct a natural weak homotopy equivalence of $\mathsf{Or}(K_2)$–spectra

$$u(\psi, Z)\colon \psi_* \boldsymbol{S}_Z^{K_1} \xrightarrow{\simeq} \boldsymbol{S}_{\psi_* Z}^{K_2},$$

where $\psi_* \boldsymbol{S}_Z^{K_1}$ is the $\mathsf{Or}(K_2)$–spectrum obtained by induction in the sense of [13, Definition 1.8] with the functor $\mathsf{Or}(\psi)\colon \mathsf{Or}(K_1) \to \mathsf{Or}(K_2)$, $K_1/H_1 \mapsto \psi_*(K_1/H_1)$, applied to the $\mathsf{Or}(K_1)$–spectrum $\boldsymbol{S}_Z^{K_1}$. For a homogeneous space K_2/H we define $u(\psi, Z)(K_2/H)$ to be the composite

$$\begin{aligned}
\psi_* \boldsymbol{S}_Z^{K_1}(K_2/H) = \operatorname{map}_{K_2}&(\psi_*(K_1/?), K_2/H)_+ \wedge_{\mathsf{Or}(K_1)} \boldsymbol{S}(K_1/? \times_{K_1} Z) \\
&\xrightarrow{\cong} \operatorname{map}_{K_1}((K_1/?), \psi^*(K_2/H))_+ \wedge_{\mathsf{Or}(K_1)} \boldsymbol{S}(K_1/? \times_{K_1} Z) \\
&\xrightarrow{\simeq} \boldsymbol{S}(\psi^*(K_2/H) \times_{K_1} Z) \\
&\xrightarrow{\cong} \boldsymbol{S}(K_2/H \times_{K_2} \psi_* Z) =: \boldsymbol{S}_{\psi_* Z}^{K_2}(K_2/H).
\end{aligned}$$

Here the first map comes from the adjunction isomorphism

$$\operatorname{map}_{K_2}(\psi_*(K_1/?), K_2/H) \xrightarrow{\cong} \operatorname{map}_{K_1}(K_1/?), \psi^*(K_2/H)),$$

and the third map comes from the canonical homeomorphism

$$\psi^*(K_2/H) \times_{K_1} Z \xrightarrow{\cong} K_2/H \times_{K_2} \psi_* Z.$$

The second map is the special case $T = \psi^* K_2/?$ of the natural weak homotopy equivalence defined for any K_1–set T,

$$\kappa(T)\colon \operatorname{map}_{K_1}((K_1/?), T)_+ \wedge_{\mathsf{Or}(K_1)} \boldsymbol{S}(K_1/? \times_{K_1} Z) \xrightarrow{\simeq} \boldsymbol{S}(T \times_{K_1} Z),$$

which is given by $(u\colon K_1/? \to T) \times s \mapsto \boldsymbol{S}(u \times_{K_1} \mathrm{id}_Z)(s)$. If T is a transitive K_1–set, then $\kappa(T)$ is even an isomorphism by the Yoneda lemma. The left-hand side is compatible with disjoint unions in T; the right-hand side is compatible with disjoint unions in T up to homotopy, where we use that $\boldsymbol{S}$ respects disjoint unions. As every K_1–set is the disjoint union of homogeneous K_1–sets, $\kappa(T)$ is a weak homotopy equivalence for every K_1–set T.

(ii) The third isomorphism appearing in [13, Lemma 1.9] together with [13, Lemma 4.6] implies that it suffices to construct a natural weak homotopy equivalence of $\mathsf{Or}(K_1)$–spectra

$$v(\psi, Z)\colon \psi^* \boldsymbol{S}_Z^{K_2} \xrightarrow{\simeq} \boldsymbol{S}_{\psi^* Z}^{K_1},$$

where $\psi^* \boldsymbol{S}_Z^{K_2}$ is the $\mathsf{Or}(K_1)$–spectrum obtained by restriction in the sense of [13, Definition 1.8] with the functor $\mathsf{Or}(\psi)\colon \mathsf{Or}(K_1) \to \mathsf{Or}(K_2)$, $K_1/H \mapsto \psi_*(K_1/H)$, applied to the $\mathsf{Or}(K_2)$–spectrum $\boldsymbol{S}_Z^{K_2}$. Actually, we obtain even an isomorphism $v(\psi, Z)$ using the adjunction

$$\psi_*(K_1/H) \times_{K_2} Z \cong K_1/H \times_{K_1} \psi^* Z$$

for any subgroup $H \subseteq K_1$. $\square$

5.3 The fibered metaisomorphism conjecture with coefficients for functors from spaces to spectra

Notice that for a homomorphism $\varphi\colon H \to G$ the restriction $\varphi^* Z$ of a free G–CW–complex Z is free again if and only if φ is injective. We have already explained in Remark 2.12 that the assumption that Z is free is needed in Conjecture 2.8. In the Fibered metaisomorphism conjecture 5.3 it is crucial not to require that $\varphi\colon H \to G$ is injective since we want to have good inheritance properties. Therefore, we have to blow up Z everywhere by passing to $EG \times Z$, as explained below.

Let G be a group and Z be a G–CW–complex. Recall that $\underline{G}$ denotes the groupoid with precisely one object, which has G as its automorphism group. Let $\mathsf{GROUPOIDS} \downarrow G$ be the category of groupoids over $\underline{G}$. Objects are groupoids $\mathcal{G}$ together with a functor $P\colon \mathcal{G} \to \underline{G}$. A morphism from $P\colon \mathcal{G} \to \underline{G}$ to $P'\colon \mathcal{G}' \to \underline{G}$ is a covariant functor

$F\colon \mathcal{G} \to \mathcal{G}'$ satisfying $P' \circ F = P$. Given a groupoid $\mathcal{G}$, we obtain a contravariant functor $E(? \downarrow \mathcal{G})\colon \mathcal{G} \to \mathsf{SPACES}$ by sending an object x to the classifying space of the category $x \downarrow \mathcal{G}$ of objects in $\mathcal{G}$ under x. We get from Z, by restriction along P, a covariant functor $P^*Z\colon \mathcal{G} \to \mathsf{SPACES}$, where we think of the left G–space Z as a covariant functor $\underline{G} \to \mathsf{SPACES}$. The tensor product over $\mathcal{G}$ — see [13, Section 1] — yields a space $E(? \downarrow \mathcal{G}) \times_{\mathcal{G}} P^*Z(?)$. Thus, we obtain a covariant functor

$$\begin{aligned} \boldsymbol{S}_Z^{\downarrow G}\colon \mathsf{GROUPOIDS} \downarrow G &\to \mathsf{SPECTRA}, \\ P\colon (\mathcal{G} \to \underline{G}) &\mapsto \boldsymbol{S}(E(? \downarrow \mathcal{G}) \times_{\mathcal{G}} P^*Z(?)). \end{aligned} \tag{5.6}$$

It yields an equivariant homology theory $H_n^?(-; \boldsymbol{S}_Z^{\downarrow G})$ over G; see [2, Lemma 7.1]. Given a homomorphism $\psi\colon K \to G$ we get an identification of K–homology theories

$$H_*^{K,\psi}(-; \boldsymbol{S}_Z^{\downarrow G}) \cong H_*^K(-; \boldsymbol{S}^K_{EK \times \psi^* Z}), \tag{5.7}$$

which is induced by a homotopy equivalence, natural in K/H,

$$E(? \downarrow \mathcal{T}^K(K/H)) \times_{\mathcal{T}^K(K/H)} \psi^* Z(?) \xrightarrow{\simeq} K/H \times_K (EK \times \psi^* Z)$$

and [13, Lemma 4.6], where $\mathcal{T}$ denotes the transport groupoid from Example 2.9 and ψ also denotes its induced map $\mathcal{T}^K(K/H) \to \underline{G}$. For any group $\psi\colon K \to G$ over G, inclusion $i\colon H \to K$ of a subgroup H of K, and $n \in \mathbb{Z}$, we have canonical identifications

$$H_n^{K,\psi}(K/H; \boldsymbol{S}_Z^{\downarrow G}) \xrightarrow{\cong} H_n^{H,\psi\circ i}(H/H; \boldsymbol{S}_Z^{\downarrow G}) \cong \pi_n(\boldsymbol{S}(EH \times_H (\psi \circ i)^* Z)).$$

Lemma 5.8 *Let $\varphi\colon H \to K$ and $\psi\colon K \to G$ be group homomorphisms.*

(i) *Let X be a G–CW–complex and let Z be a K–CW–complex. Then we obtain a natural isomorphism*

$$H_n^{H,\varphi}(\varphi^*\psi^*X; \boldsymbol{S}_Z^{\downarrow K}) \xrightarrow{\cong} H_n^G(X; \boldsymbol{S}^G_{(\psi\circ\varphi)_*(EH \times \varphi^* Z)}).$$

(ii) *Let X be an H–CW–complex and let Z be a G–CW–complex. Then we obtain a natural isomorphism*

$$H_n^{H,\varphi}(X; \boldsymbol{S}_{\psi^*Z}^{\downarrow K}) \xrightarrow{\cong} H_n^{H,\psi\circ\varphi}(X; \boldsymbol{S}_Z^{\downarrow G}).$$

Proof (i) We get, from (5.7),

$$H_n^{H,\varphi}(\varphi^*\psi^*X; \boldsymbol{S}_Z^{\downarrow K}) := H_n^H(\varphi^*\psi^*X; \boldsymbol{S}^H_{EH \times \varphi^* Z}).$$

Now apply Lemma 5.5(i).

(ii) We get, from (5.7),

$$\begin{aligned} H_n^{H,\varphi}(X; \mathbf{S}^{\downarrow K}_{\psi^* Z}) &:= H_n^H(X; \mathbf{S}^H_{EH\times\varphi^*\psi^* Z}) \\ &= H_n^H(X; \mathbf{S}^H_{EH\times(\psi\circ\varphi)^* Z}) =: H_n^{H,\psi\circ\varphi}(X; \mathbf{S}^{\downarrow G}_Z). \end{aligned}$$ $\square$

Conjecture 5.9 (fibered metaisomorphism conjecture for a functor from spaces to spectra with coefficients) *Let $\mathbf{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$, as before, respect weak equivalences and disjoint unions. We say that $\mathbf{S}$ satisfies the fibered metaisomorphism conjecture for a functor from spaces to spectra with coefficients for the group G and the family of subgroups $\mathcal{F}$ of G if the following holds: For any G–CW–complex Z, the equivariant homology theory $H_*^?(-; \mathbf{S}^{\downarrow G}_Z)$ over G satisfies the Fibered metaisomorphism conjecture 5.3 for the group (G, id_G) over G and the family $\mathcal{F}$.*

Note that Conjecture 2.8 deals with the G–homology theory $H_*^G(-; \mathbf{S}^G)$, whereas Conjecture 5.9 deals with the equivariant homology theory $H_*^?(-; \mathbf{S}^{\downarrow G})$ over G. Moreover, Conjecture 5.9 is unchanged if we additionally require that the G–CW–complex Z is free. Namely, for any G–CW–complex Z, the G–CW–complex $EG\times Z$ is free and the projection $EG\times Z \to Z$ induces an isomorphism $H_*^?(-; \mathbf{S}^{\downarrow G}_{EG\times Z}) \xrightarrow{\cong} H_*^?(-; \mathbf{S}^{\downarrow G}_Z)$ of equivariant homology theories over G because of (5.7) and [13, Lemma 4.6].

For the rest of this section, we abbreviate the different conjectures as follows:

- C2.8 is the Metaisomorphism conjecture 2.8 for functors from spaces to spectra with coefficients. This is the conjecture we want to know about in the end.
- MIC5.2 and FMIC5.3 denote the Metaisomorphism conjecture 5.2, and the Fibered metaisomorphism conjecture 5.3. These are statements about a (G–)equivariant homology theory.
- S5.9 denotes the Fibered metaisomorphism conjecture 5.9 for a functor from spaces to spectra with coefficients. This takes as input a functor $\mathbf{S}$ and is the most general version of a conjecture we are interested it.

Lemma 5.10 *Let $\psi\colon K \to G$ be a group homomorphism.*

(i) *Suppose that C2.8 holds for the group G and the family $\mathcal{F}$. Then S5.9 holds for the group K and the family $\psi^*\mathcal{F}$.*

(ii) *If S5.9 holds for the group G and the family $\mathcal{F}$, then C2.8 holds for the group G and the family $\mathcal{F}$.*

(iii) *Suppose that S5.9 holds for the group K and the family $\mathcal{F}$. Then, for every G–CW–complex Z, FMIC5.3 holds for the equivariant homology theory $H_n(-; \mathbf{S}_Z^{\downarrow G})$ over G for the group (K, ψ) over G and the family $\mathcal{F}$ of subgroups of K.*

Proof (i) This follows from Lemma 5.8(i), since in the notation used there we have $\varphi^*\psi^* E_{\mathcal{F}}(G) = \varphi^* E_{\psi^*\mathcal{F}}(K)$ and $\varphi^*\psi^* G/G = H/H$, and $(\psi \circ \varphi)_*(EH \times \varphi^* Z)$ is a free G–CW–complex.

(ii) This follows from applying Conjecture 5.9 to the special case $\psi = \mathrm{id}_G$ and the fact that for a free G–CW–complex Z the projection $EG \times Z \to Z$ is a G–homotopy equivalence and hence we get, from (5.7) and [13, Lemma 4.6], natural isomorphisms

$$H_n^{G,\mathrm{id}_G}(X; \mathbf{S}_Z^{\downarrow G}) \cong H_n^G(X; \mathbf{S}_{EG\times Z}^G) \cong H_n^G(X; \mathbf{S}_Z^G)$$

for every G–CW–complex X and $n \in \mathbb{Z}$.

(iii) This follows from Lemma 5.8(ii). □

5.4 Strongly continuous equivariant homology theories over a group

Fix a group Γ and an equivariant homology theory $\mathcal{H}_*^?$ over Γ.

Let X be a G–CW–complex and let $\alpha\colon H \to G$ be a group homomorphism. The functors $\alpha_*\colon H\text{–CW} \rightleftarrows G\text{–CW} :\alpha^*$ are adjoint to one another. In particular, the adjoint of the identity on $\alpha^* X$ is a natural G–map

(5.11) $$f(X,\alpha)\colon \alpha_*\alpha^* X \to X, \quad (g,x) \mapsto gx.$$

Consider a map $\alpha\colon (H,\xi) \to (G,\mu)$ of groups over Γ. Define the Λ–map

$$a_n = a_n(X,\alpha)\colon \mathcal{H}_n^H(\alpha^* X) \xrightarrow{\mathrm{ind}_\alpha} \mathcal{H}_n^G(\alpha_*\alpha^* X) \xrightarrow{\mathcal{H}_n^G(f(X,\alpha))} \mathcal{H}_n^G(X).$$

If $\beta\colon (G,\mu) \to (K,\nu)$ is another morphism of groups over Γ, then by the axioms of an induction structure, see [36], the composite

$$\mathcal{H}_n^H(\alpha^*\beta^* X) \xrightarrow{a_n(\beta^* X,\alpha)} \mathcal{H}_n^G(\beta^* X) \xrightarrow{a_n(X,\beta)} \mathcal{H}_n^K(X)$$

agrees with $a_n(X, \beta\circ\alpha)\colon \mathcal{H}_n^H(\alpha^*\beta^* X) = \mathcal{H}_n^H((\beta\circ\alpha)^* X) \to \mathcal{H}_n^K(X)$ for a K–CW–complex X.

Consider a directed system of groups $\{G_i \mid i \in I\}$ with $G = \mathrm{colim}_{i\in I} G_i$ and structure maps $\psi_i\colon G_i \to G$ for $i \in I$ and $\varphi_{i,j}\colon G_i \to G_j$ for $i, j \in I$ with $i \le j$. We obtain for every G–CW–complex X a system $a_n(\psi_j^* X, \varphi_{i,j})\colon \mathcal{H}^{G_i}(\psi_i^* X) \to \mathcal{H}^{G_j}(\psi_j^* X)$. We get a map

$$t_n^G(X) := \operatorname*{colim}_{i\in I} a_n(X,\psi_i)\colon \operatorname*{colim}_{i\in I} \mathcal{H}_n^{G_i}(\psi_i^*(X)) \to \mathcal{H}_n^G(X). \tag{5.12}$$

The next definition is taken from [2, Definition 3.3].

Definition 5.13 (strongly continuous equivariant homology theory over a group) An equivariant homology theory $\mathcal{H}_*^?$ over the group Γ is called *strongly continuous* if, for every group (G,ξ) over Γ and every directed system of groups $\{G_i \mid i\in I\}$ with $G = \operatorname{colim}_{i\in I} G_i$ and structure maps $\psi_i\colon G_i \to G$ for $i\in I$, the map

$$t_n^G(\{\bullet\})\colon \operatorname*{colim}_{i\in I} \mathcal{H}_n^{G_i}(\{\bullet\}) \to \mathcal{H}_n^G(\{\bullet\})$$

is an isomorphism for every $n\in\mathbb{Z}$.

Lemma 5.14 *Suppose that, for any directed system of spaces $\{X_i \mid i\in I\}$ indexed over an arbitrary directed set I, the canonical map*

$$\operatorname*{hocolim}_{i\in I} \boldsymbol{S}(X_i) \to \boldsymbol{S}\big(\operatorname*{hocolim}_{i\in I} X_i\big)$$

is a weak homotopy equivalence.

Then, for every group Γ and Γ–CW–complex Z, the equivariant homology theory over Γ given by $H_*^?(-;\boldsymbol{S}_Z^{\downarrow\Gamma})$ is strongly continuous.

Proof We only treat the case $\Gamma = G$ and $\psi = \mathrm{id}_G$; the general case of a group $\psi\colon G\to\Gamma$ over Γ is completely analogous. Consider a directed system of groups $\{G_i \mid i\in I\}$ with $G = \operatorname{colim}_{i\in I} G_i$. Let $\psi_i\colon G_i\to G$ be the structure map for $i\in I$.

As I is directed, the canonical map

$$\operatorname*{hocolim}_{i\in I} \boldsymbol{S}(EG_i \times_{G_i} \psi_i^* Z) \to \boldsymbol{S}\big(\operatorname*{hocolim}_{i\in I}(EG_i \times_{G_i} \psi_i^* Z)\big) \tag{5.15}$$

is by assumption a weak homotopy equivalence. We have the homeomorphisms

$$EG_i \times_{G_i} \psi_i^* Z \xrightarrow{\cong} (\psi_i)_* EG_i \times_G Z,$$
$$\big(\operatorname*{hocolim}_{i\in I}(\psi_i)_* EG_i\big) \times_G Z \xrightarrow{\cong} \operatorname*{hocolim}_{i\in I}((\psi_i)_* EG_i \times_G Z).$$

They induce a homeomorphism

$$\boldsymbol{S}(\operatorname*{hocolim}_{i\in I}(EG_i \times_{G_i} \psi_i^* Z)) \xrightarrow{\cong} \boldsymbol{S}\big(\big(\operatorname*{hocolim}_{i\in I}(\psi_i)_* EG_i\big) \times_G Z\big). \tag{5.16}$$

The canonical map

$$\operatorname*{hocolim}_{i \in I} (\psi_i)_* EG_i \to EG$$

is a G–homotopy equivalence. The proof of this fact is a special case of the argument appearing in the proof of [39, Theorem 4.3 on page 516]. It induces a weak homotopy equivalence

(5.17)
$$\boldsymbol{S}\big(\big(\operatorname*{hocolim}_{i \in I} (\psi_i)_* EG_i\big) \times_G Z\big) \to \boldsymbol{S}(EG \times_G Z).$$

Hence, we get, by taking the composite of the maps (5.15), (5.16) and (5.17), a weak homotopy equivalence

$$\operatorname*{hocolim}_{i \in I} \boldsymbol{S}(EG_i \times_{G_i} \psi_i^* Z) \to \boldsymbol{S}(EG \times_G Z).$$

As I is directed, it induces, after taking homotopy groups for every $n \in \mathbb{Z}$, an isomorphism

$$\operatorname*{colim}_{i \in I} \pi_n(\boldsymbol{S}(EG_i \times_{G_i} \psi_i^* Z)) \to \pi_n(\boldsymbol{S}(EG \times_G Z)),$$

which can be identified using (5.7) with the canonical map

$$t_n^G(\{\bullet\}) \colon \operatorname*{colim}_{i \in I} H_n^{G_i}(\{\bullet\}; \boldsymbol{S}_Z^{\downarrow G}) \to H_n^G(\{\bullet\}; \boldsymbol{S}_Z^{\downarrow G}).$$

This finishes the proof of Lemma 5.14. □

5.5 Proof of Theorem 5.1

In this section we give the proof of Theorem 5.1. We use the notation from there.

Proof (i) Consider a free H–CW–complex Z. Let $i \colon H \to G$ be the inclusion. Then $i_* Z$ is a free G–CW–complex, $i^* E_{\mathcal{C}(G)}(G)$ is a model for $E_{\mathcal{C}(H)}(H)$ and $i^* G/G = H/H$. From Lemma 5.5(i), we obtain a commutative diagram with isomorphisms as vertical maps

$$\begin{tikzcd}
H_n^H(E_{\mathcal{C}(H)}(H); \boldsymbol{S}_Z^H) \arrow[r] \arrow[d, "\cong"'] & H_n^H(H/H; \boldsymbol{S}_Z^G) \arrow[d, "\cong"] \\
H_n^G(E_{\mathcal{C}(G)}(G); \boldsymbol{S}_{i_* Z}^G) \arrow[r] & H_n^G(G/G; \boldsymbol{S}_{i_* Z}^G)
\end{tikzcd}$$

where the horizontal maps are induced by the projections. The lower map is bijective by assumption. Hence, the upper map is bijective as well.

(ii) As C2.8 holds for $(Q, \mathcal{Q})$, by Lemma 5.10(i), S5.9 holds for $(G, p^*\mathcal{C}(Q))$. By Lemma 5.10(i) again, for every $H \in \mathcal{C}(Q)$, C2.8 holds for $\big(p^{-1}(H), \mathcal{C}(p^{-1}(H))\big)$. Naturally, $p^{-1}(H) \subseteq G$ is a group over G for which, by Lemma 5.10(iii), FMIC5.3 holds for $H_n^?(-; \boldsymbol{S}_Z^{\downarrow G})$ for any G–CW–complex Z and the family $\mathcal{C}(p^{-1}(H)) = \mathcal{C}(G)|_{p^{-1}(H)}$. Let $L \in p^*\mathcal{C}(Q)$. Then, using Lemma 5.4 for the map $L \to p^{-1}(p(L))$, FMIC5.3 holds for $(L, \mathcal{C}|_L)$ and $H_n^?(-; \boldsymbol{S}_Z^{\downarrow G})$. As FMIC5.3 holds for $(G, p^*C(Q))$ and for $(L, \mathcal{C}|_L)$ for every $L \in p^*C(Q)$, the transitivity principle — see Theorem 4.3 of [2] — implies that FMIC5.3 holds for $(G, \mathcal{C})$. By Lemma 5.10(ii), then also C2.8 holds for $(G, \mathcal{C})$.

(iii) If C2.8 holds for $(G_1 \times G_2, \mathcal{C}(G_1 \times G_2))$, it holds for G_k and the family $\mathcal{C}(G_k) = \mathcal{C}(G_1 \times G_2)|_{G_k}$ for $k = 1, 2$ by assertion (i).

Suppose that C2.8 holds for $(G_k, \mathcal{C}(G_k))$ for $k = 1, 2$. By assertion (ii) applied to the split exact sequence

$$1 \to H_2 \to G_1 \times H_2 \to G_1 \to 1,$$

C2.8 holds for $(G_1 \times H_2, \mathcal{C}(G_1 \times H_2))$ for every $H_2 \in \mathcal{C}(G_2)$. By assertion (ii) applied to the split exact sequence $1 \to G_1 \to G_1 \times G_2 \to G_2 \to 1$, C2.8 holds for $(G_1 \times G_2, \mathcal{C}(G_1 \times G_2))$.

(iv) Since C2.8 holds for G_i and $\mathcal{C}(G_i)$ for every $i \in I$ by assumption, we conclude from Lemma 5.10(i) that S5.9 holds for the group G_i and the family $\mathcal{C}(G_i)$ for every $i \in I$. Lemma 5.10(iii) implies that, for every $i \in I$ and G–CW–complex Z, FMIC5.3 holds for the equivariant homology theory $H_n(-; \boldsymbol{S}_Z^{\downarrow G})$ over G for the group $\psi_i \colon G_i \to G$ over G and the family $\mathcal{C}(G_i)$. We conclude from [2, Theorem 5.2] and Lemma 5.14 that, for every G–CW–complex Z, FMIC5.3 holds for the equivariant homology theory $H_*^?(-; \boldsymbol{S}_Z^{\downarrow G})$ over G for the group (G, id_G) over G and the family $\mathcal{C}(G)$. In other words, S5.9 holds for the group G and the family $\mathcal{C}(G)$. Lemma 5.10(ii) implies that C2.8 holds for the group G and the family $\mathcal{C}(G)$.

(v) The analogs of (i), (ii), (iii) and (iv) hold for the Metaisomorphism conjecture 2.14 with coefficients and finite wreath products by [34, Lemmas 3.2, 3.15 and 3.16 and Satz 3.5].

For a group G and two finite groups F_1 and F_2, we have $(H \wr F_1) \wr F_2 \subseteq H \wr (F_1 \wr F_2)$ and $F_1 \wr F_2$ is finite. In particular, if G satisfies Conjecture 2.14 with wreath products, then the same is true for any wreath product $G \wr F$ with F finite. If $H \subseteq G$ is a subgroup of finite index, then G can be embedded in $H \wr F$ for some finite group F; see [56, Proof of Proposition 2.17]. Hence, the Theorem 5.1. $\square$

5.6 Proof of Theorem 1.1(ii)

By Lemma 2.5, the functor $\boldsymbol{A}$ satisfies all assumptions of Theorem 5.1. The claim of the inheritance properties appearing in Theorem 1.1(ii) follows immediately from Theorem 5.1 except for the statements about extensions, direct products and free products. For extensions, it follows from the inheritance under finite-index supergroups. For direct products, note that the product of two virtually cyclic groups is virtually abelian; hence, by [47] it satisfies the conjecture.

For free products, note that due to the inheritance under filtered colimits, we can assume our groups are finitely generated, so in particular countable. For $G_1, G_2 \in \mathcal{FJ}_A$ consider the canonical map $p\colon G_1 * G_2 \to G_1 \times G_2$. We already know $G_1 \times G_2 \in \mathcal{FJ}_A$ and hence that it suffices to prove $p^{-1}(C) \in \mathcal{FJ}_A$, where C is the trivial or any infinite cyclic subgroup of $G_1 \times G_2$. By [45, Lemma 5.2], all such $p^{-1}(C)$ are free and hence hyperbolic, as $G_1 \times G_2$ is countable.

6 Proof of the Farrell–Jones conjecture for hyperbolic and CAT(0)–groups

Thanks to the framework established in [47], we can proceed similarly to the linear case as in [55] and reduce the proof to the construction of a transfer map. This reduction is carried out in this section, while the construction of the transfer occupies Section 7.

6.1 Homotopy coherent actions and homotopy transfer reducibility

The geometric criterion we use to prove the conjecture relies on the notion of a homotopy coherent diagram, which goes back to Vogt [51]. For applications to the Farrell–Jones conjecture, it is enough to consider the case of a homotopy coherent diagram of shape G, regarding G as a one-object groupoid. In this special case, Vogt's definition was rediscovered by Wegner [55, Definition 2.1], who called it "strong homotopy action".

Definition 6.1 A *homotopy coherent G–action* of a group G on a topological space X is a continuous map

$$\Gamma\colon \coprod_{j=0}^{\infty} ((G \times [0,1])^j \times G \times X) \to X$$

with the following properties:

$$\Gamma(\gamma_k, t_k, \dots, \gamma_1, t_1, \gamma_0, x) = \begin{cases} \Gamma(\dots, \gamma_j, \Gamma(\gamma_{j-1}, \dots, x)) & \text{if } t_j = 0, \\ \Gamma(\dots, \gamma_j \gamma_{j-1}, \dots, x) & \text{if } t_j = 1, \\ \Gamma(\gamma_k, \dots, \gamma_2, t_2, \gamma_1, x) & \text{if } \gamma_0 = e \text{ and } 0 < k, \\ \Gamma(\gamma_k, \dots, t_{j+1} t_j, \dots, \gamma_0, x) & \text{if } \gamma_j = e \text{ and } 1 \le j < k, \\ \Gamma(\gamma_{k-1}, t_{k-1}, \dots, t_1, \gamma_0, x) & \text{if } \gamma_k = e \text{ and } 0 < k, \\ x & \text{if } \gamma_0 = e \text{ and } k = 0. \end{cases}$$

The following definition is adapted from the conditions given in [1, Theorem B], which does not use coherence conditions. We explain some notation below.

Definition 6.2 Let G be a discrete group. Let $\mathcal{F}$ be a family of subgroups of G.

Then G is *homotopy transfer reducible over* $\mathcal{F}$ if there exists a finite, symmetric generating set $S \subseteq G$ of G which contains the trivial element, as well as $N \in \mathbb{N}$ such that there are, for every $n \in \mathbb{N}$:

(i) A compact, contractible metric space (X, d_X) such that for every $\varepsilon > 0$ there is an ε–controlled domination of X by an at most N–dimensional, finite simplicial complex.

(ii) A homotopy coherent G–action Γ on X.

(iii) A G–simplicial complex Σ of dimension at most N whose isotropy is contained in $\mathcal{F}$.

(iv) A continuous map $f\colon X \to \Sigma$ which is (S, n)–*equivariant* in the sense that

- For all $x \in X$ and $s \in S^n$,

$$d^{\ell^1}(f(\Gamma(s, x)), s \cdot f(x)) \le \frac{1}{n}.$$

- For all $x \in X$ and $s_0, \dots, s_n \in S^n$,

$$\operatorname{diam}\{f(\Gamma(s_n, t_n, \dots, s_0, x)) \mid (t_1, \dots, t_n) \in [0, 1]^n\} \le \frac{2}{n}.$$

Notation 6.3 Let us briefly recall some notation used in Definition 6.2.

(i) Recall from [5, Definition 1.5] that an ε–controlled domination of a metric space (X, d) by a finite simplicial complex K consists of maps $i\colon X \to K$ and $p\colon K \to X$ together with a homotopy H from $p \circ i$ to id_X such that for every $x \in X$ the diameter of $\{H(x, t) \mid t \in [0, 1]\}$ is at most ε.

(ii) The ℓ^1–metric d^{ℓ^1} on a simplicial complex is defined in [6, Section 4.2].

(iii) If S is a finite generating set of G, we denote by $S^n \subseteq G$ the set

$$\{s_1 s_2 \dots s_n \in G \mid s_i \in S\}.$$

We always equip G with the word metric d_G with respect to S. Equivalently, S^n is the n–ball around the trivial element with respect to d_G.

We will show in Section 6.7 that a group satisfying Definition 6.2 satisfies the Farrell-Jones conjecture with coefficients in A–theory with respect to the family $\mathcal{F}$, and we show in Section 6.10 that hyperbolic and CAT(0)–groups satisfy Definition 6.2, thus proving Theorem 1.1(i).

6.2 Controlled CW–complexes

Let G be a discrete group and let $\mathcal{F}$ be a family of subgroups of G. In [47] it was shown that Theorem 6.14 below holds for G if and only if a certain spectrum $\mathbb{F}(G, W, E_{\mathcal{F}}(G))$ is weakly contractible for every free G–CW–complex W. The spectrum $\mathbb{F}(G, W, E_{\mathcal{F}}(G))$ is the algebraic K–theory of a Waldhausen category of controlled retractive G–CW–complexes, similar in spirit to the obstruction category for the isomorphism conjecture in algebraic K–theory; cf [4; 6, Section 3]. Let us recall the relevant definitions from [47] in this and the next section.

A *coarse structure* is a triple $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ such that Z is a Hausdorff G–space, $\mathfrak{C}$ is a collection of reflexive, symmetric and G–invariant relations on Z which is closed under taking finite unions and compositions — see [47, Definition 2.1] — and $\mathfrak{S}$ is a collection of G–invariant subsets of Z which is closed under taking finite unions. See [47, Definition 3.23] for the notion of a *morphism of coarse structures*.

Fix a coarse structure $\mathfrak{Z}$.

For a G–CW–complex Y relative W, denote by $\diamond Y$ the (discrete) set of relative cells of Y and by $\diamond_k Y$ the subset of all relative k–cells in Y. A *labeled G–CW–complex relative W* — see [47, Definition 2.3] — is a pair (Y, κ), where Y is a free G–CW–complex relative W together with a G–equivariant function $\kappa\colon \diamond Y \to Z$.

A *$\mathfrak{Z}$–controlled map* $f\colon (Y_1, \kappa_1) \to (Y_2, \kappa_2)$ is a G–equivariant, cellular map $f\colon Y_1 \to Y_2$ relative W such that for all $k \in \mathbb{N}$ there is some $C \in \mathfrak{C}$ for which

$$(\kappa_2, \kappa_1)\big(\{(e_2, e_1) \mid e_1 \in \diamond_k Y_1, e_2 \in \diamond Y_2, \langle f(e_1)\rangle \cap e_2 \neq \varnothing\}\big) \subseteq C$$

holds, where $\langle f(e_1)\rangle$ denotes the smallest nonequivariant subcomplex of Y_2 which contains $f(e_1)$.

A *$\mathfrak{Z}$–controlled G–CW–complex relative W* is a labeled G–CW–complex (Y,κ) relative W such that the identity is a $\mathfrak{Z}$–controlled map and for all $k \in \mathbb{N}$ there is some $S \in \mathfrak{S}$ such that

$$\kappa(\diamond_k Y) \subseteq S.$$

A *$\mathfrak{Z}$–controlled retractive space relative W* is a $\mathfrak{Z}$–controlled G–CW–complex (Y,κ) relative W together with a G–equivariant retraction $r\colon Y \to W$, ie a left inverse to the structural inclusion $W \hookrightarrow Y$. The $\mathfrak{Z}$–controlled retractive spaces relative W form a category $\mathcal{R}^G(W,\mathfrak{Z})$ in which *morphisms* are $\mathfrak{Z}$–controlled maps which additionally respect the chosen retractions.

The category of controlled G–CW–complexes (relative W) and controlled maps admits a notion of *controlled homotopies* — see [47, Definition 2.5] — via the objects $(Y \leftthreetimes [0,1], \kappa \circ \mathrm{pr}_Y)$, where $Y \leftthreetimes [0,1]$ denotes the reduced product which identifies $W \times [0,1] \subseteq Y \times [0,1]$ to a single copy of W and $\mathrm{pr}_Y\colon \diamond(Y \leftthreetimes [0,1]) \to \diamond Y$ is the canonical projection. In particular, we obtain a notion of *controlled homotopy equivalence* (or *h–equivalence*).

A $\mathfrak{Z}$–controlled retractive space (Y,κ) is called *finite* if it is finite-dimensional, the image of $Y\setminus W$ under the retraction meets the orbits of only finitely many path components of W and for each $z \in Z$ there is some open neighborhood U of z such that $\kappa^{-1}(U)$ is finite; see [47, Definition 3.3].

A $\mathfrak{Z}$–controlled retractive space (Y,κ) is called *finitely dominated* if there are a finite $\mathfrak{Z}$–controlled, retractive space D, a morphism $p\colon D \to Y$ and a $\mathfrak{Z}$–controlled map $i\colon Y \to D$ such that $p \circ i$ is controlled homotopic to id_Y.

The finite and finitely dominated $\mathfrak{Z}$–controlled retractive spaces form full subcategories $\mathcal{R}^G_{\mathrm{f}}(W,\mathfrak{Z}) \subseteq \mathcal{R}^G_{\mathrm{fd}}(W,\mathfrak{Z}) \subseteq \mathcal{R}^G(W,\mathfrak{Z})$. All three of these categories support a Waldhausen category structure in which inclusions of G–invariant subcomplexes up to isomorphism are the cofibrations and controlled homotopy equivalences are the weak equivalences; see [47, Corollary 3.22]. We denote this class of weak equivalences by h.

Note that a controlled homotopy equivalence is a *morphism*, but only admits a controlled homotopy inverse *map*, which does not need to be compatible with the retractions to W. This is similar to the classical situation [53, Section 2.1].

6.3 The obstruction category

Let M be a metric space with free, isometric G–action. Define the *bounded morphism control condition on* M, $\mathfrak{C}_{\mathrm{bdd}}(M)$, to be the collection of all subsets $C \subseteq M \times M$ which are of the form

$$C = \{(m, m') \in M \times M \mid d(m, m') \leq \alpha\}$$

for some $\alpha \geq 0$.

Let X be a G–CW–complex. Define further the *G–continuous control condition* $\mathfrak{C}_{G\text{-cc}}(X)$ to be the collection of all $C \subseteq (X \times [1, \infty[) \times (X \times [1, \infty[)$ which satisfy the following:

(i) For every $x \in X$ and every G_x–invariant open neighborhood U of (x, ∞) in $X \times [1, \infty]$, there exists a G_x–invariant open neighborhood $V \subseteq U$ of (x, ∞) such that $\big(((X \times [1, \infty[) \smallsetminus U) \times V\big) \cap C = \varnothing$.

(ii) Let $p_{[1,\infty[}\colon X \times [1, \infty[\to [1, \infty[$ be the projection map. Equip $[1, \infty[$ with the Euclidean metric. Then there exists some $B \in \mathfrak{C}_{\mathrm{bdd}}([1, \infty[)$ such that $C \subseteq p_{[1,\infty[}^{-1}(B)$.

(iii) C is symmetric, G–invariant and contains the diagonal.

We can combine the two morphism control conditions into one set of conditions on $M \times X \times [1, \infty[$: Let $p_M\colon M \times X \times [1, \infty[\to M$ and $p_{X \times [1,\infty[}\colon M \times X \times [1, \infty[\to X \times [1, \infty[$ denote the projection maps. Then $\mathfrak{C}(M, X)$ is the collection of all subsets $C \subseteq (M \times X \times [1, \infty[)^2$ which are of the form

$$C = p_M^{-1}(B) \cap p_{X \times [1,\infty[}^{-1}(C')$$

for some $B \in \mathfrak{C}_{\mathrm{bdd}}(M)$ and $C' \in \mathfrak{C}_{G\text{-cc}}(X)$.

Finally, define $\mathfrak{S}(M, X)$ to be the collection of all subsets $S \subseteq M \times X \times [1, \infty[$ which are of the form $S = K \times [1, \infty[$ for some G–compact subset $K \subseteq M \times X$.

Recall that $E_{\mathcal{F}}(G)$ denotes the classifying space of G with respect to $\mathcal{F}$. We also consider G as a metric space with the word metric induced by a generating set S.

Definition 6.4 With the above definitions we obtain a coarse structure

$$\mathbb{J}(M, X) := (M \times X \times [1, \infty[, \mathfrak{C}(M, X), \mathfrak{S}(M, X)).$$

Define the "obstruction category" as the category of finite controlled CW–complexes relative W, ie as

$$\big(\mathcal{R}_{\mathrm{f}}^G\big(W, \mathbb{J}(G, E_{\mathcal{F}}(G))\big), h\big);$$

see [47, Example 2.2 and Definition 6.1]. The spectrum $\mathbb{F}(G, W, E_{\mathcal{F}}(G))$ alluded to before is the nonconnective K–theory spectrum of $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(G, E_{\mathcal{F}}(G)))$ with respect to the h–equivalences; see [47, Section 5] and Definition 6.13 below. If $M = G$, we often abbreviate $\mathbb{J}(G, X)$ as $\mathbb{J}(X)$.

By [47, Corollary 6.11], a group G satisfies the Farrell–Jones conjecture 2.13 with coefficients in A–theory with respect to $\mathcal{F}$ if and only if $\mathbb{F}(G, W, E_{\mathcal{F}}(G))$ is weakly contractible for every free G–CW–complex W.

6.4 The target of the transfer

Suppose that G is homotopy transfer reducible in the sense of Definition 6.2. The key step in proving the weak contractibility of $\mathbb{F}(G, W, E_{\mathcal{F}}(G))$ will be the construction of a "transfer map". We need a generalization of the coarse structure $\mathbb{J}(M, X)$ to define the target of the transfer.

Suppose that $(M_n)_n$ is a sequence of metric spaces with a free, isometric G–action. Let X be a G–CW–complex. Following [47, Section 7], define the coarse structure

$$\mathbb{J}((M_n)_n, X) := \Big(\coprod_n M_n \times X \times [1, \infty[, \mathfrak{C}((M_n)_n, X), \mathfrak{S}((M_n)_n, X)\Big)$$

as follows: Members of $\mathfrak{C}((M_n)_n, X)$ are of the form $C = \coprod_n C_n$ with $C_n \in \mathfrak{C}(M_n, X)$, and we additionally require that C satisfies the *uniform metric control condition*: There is some $\alpha > 0$, independent of n, such that for all $((m, x, t), (m', x', t')) \in C$ we have $d(m, m') < \alpha$. Members of $\mathfrak{S}((M_n)_n, X)$ are sets of the form $S = \coprod_n S_n$ with $S_n \in \mathfrak{S}(M_n, X)$. The resulting category $\mathcal{R}^G(W, \mathbb{J}((M_n)_n, X))$ has a canonical faithful functor into the product category $\prod_n \mathcal{R}^G(W, \mathbb{J}(M_n, X))$.

Fix a symmetric, finite generating set S of G. Let d_G denote the word metric on G with respect to S. Since G is homotopy transfer reducible by assumption, there exists a natural number $N \in \mathbb{N}$ such that we can choose, for each $n \in \mathbb{N}$,

(i) a compact, contractible metric space (X_n, d_{X_n}) such that for every $\varepsilon > 0$ there is an ε–controlled domination of X_n by an at most N–dimensional, finite simplicial complex;

(ii) a homotopy coherent G–action Γ_n on X_n;

(iii) a G–simplicial complex Σ_n of dimension at most N whose isotropy is contained in $\mathcal{F}$;

(iv) a map $f_n\colon X \to \Sigma_n$ which is (S,n)–equivariant, ie

(a) for all $x \in X_n$ and $s \in S^n$,

$$d^{\ell^1}(f(\Gamma_n(s,x)), s \cdot f_n(x)) \le \frac{1}{n}; \tag{6.5}$$

(b) for all $x \in X_n$ and $s_0, \dots, s_n \in S^n$,

$$\operatorname{diam}\{f_n(\Gamma_n(s_n, t_n, \dots, s_0, x)) \mid (t_1, \dots, t_n) \in [0,1]^n\} \le \frac{2}{n}. \tag{6.6}$$

Definition 6.7 We equip $\Sigma_n \times G$ with the metric $n \cdot d^{\ell^1}(x,y) + d_G(g,h)$.

Recall that an extended metric satisfies the usual axioms of a metric, but it is allowed to take the value ∞. The following definition will be used to produce a metric on $X_n \times G$ for each $n \in \mathbb{N}$.

Definition 6.8 Let (X, d_X) be a metric space, Γ a homotopy-coherent G–action on X, and $S \subseteq G$ a finite subset containing the trivial element. Let $k \in \mathbb{N}$ and $\Lambda > 0$. Define on $X \times G$ the extended metric

$$d_{S,k,\Lambda}((x,g),(y,h)) \in [0,\infty]$$

to be the infimum over the numbers

$$l + \sum_{i=0}^{l} \Lambda \cdot d_X(x_i, z_i),$$

where the infimum is taken over all $l \in \mathbb{N}$, $x_0, \dots, x_l$, $z_0, \dots, z_l \in X$ and $a_1, \dots, a_l$, $b_1, \dots, b_l \in S$ such that

(i) $x_0 = x$ and $z_l = y$;

(ii) $g a_1^{-1} b_1 \dots a_l^{-1} b_l = h$;

(iii) for each $1 \le i \le l$ there are elements $r_0, \dots, r_k, s_0, \dots, s_k \in S$ such that $a_i = r_k \dots r_0$, $b_i = s_k \dots s_0$ and $\Gamma(r_k, t_k, \dots, r_0, z_{i-1}) = \Gamma(s_k, u_k, \dots, s_0, x_i)$ for some $t_1, \dots, t_k, u_1, \dots, u_k \in [0,1]$.

If no such data exist, take the infimum to be ∞.

This definition is analogous to [5, Definition 3.4; 55, Definition 2.3]. Since we only consider the coherent G–action Γ_n on X_n, we drop Γ_n from the notation of [55]. The proof of the next lemma is analogous to the one given in [5, Lemma 3.5].

Lemma 6.9 *Let $k \in \mathbb{N}$.*

(i) *For all $\Lambda > 0$, the function $d_{S,k,\Lambda}$ is an extended metric on $X \times G$ which is G–invariant if we let G act on $X \times G$ by $\gamma \cdot (x, g) = (x, \gamma g)$. It is a metric if and only if S generates G.*

(ii) *We have $d_{S,k,\Lambda}((x, g), (y, h)) < 1$ if and only if $g = h$ and $\Lambda \cdot d_X(x, y) < 1$ holds, in which case we have $d_{S,k,\Lambda}((x, g), (y, h)) = \Lambda \cdot d_X(x, y)$. In particular, the topology induced by $d_{S,k,\Lambda}$ is the product topology.* □

6.5 The actual target of the transfer

We now specialize the construction of Section 6.4 to our needs. Assume that G is homotopy transfer reducible, ie it satisfies Definition 6.2. That definition provides us for every n with a metric space X_n, as well as Γ_n, f_n and Σ_n. From Definition 6.8 and Lemma 6.9 we obtain for any sequence $(\Lambda_n)_n$ a sequence of metric spaces $(X_n \times G, d_{S^n,n,\Lambda_n})_n$. Although we do not need to restrict to a specific choice of $(\Lambda_n)_n$ until a little later, we wish to avoid spreading our choices throughout the whole proof. Therefore, we will now fix a specific sequence $(\Lambda_n)_n$.

Since each X_n is compact, f_n is uniformly continuous. Hence, there exists for each $n \in \mathbb{N}$ some $\delta_n > 0$ such that for all $x, y \in X_n$ with $d_X(x, y) < \delta_n$ we have $d^{\ell^1}(f_n(x), f_n(y)) < \frac{1}{n}$.

Definition 6.10 Choose such δ_n for all n and set

$$\Lambda_n := \frac{n+1}{\delta_n}.$$

Define a metric d_n on $X_n \times G$ by

$$d_n((x, g), (y, h)) := d_{S^n,n,\Lambda_n}((x, g), (y, h)) + d_G(g, h).$$

Then $X_n \times G$ carries a free and isometric G–action if we let G act on the right factor. If we make no explicit mention of a metric, we will view $X_n \times G$ as a metric space with respect to d_n in what follows. Similarly, $\Sigma_n \times G$ carries a diagonal G–action and will always be understood as a metric space with respect to the metric $n \cdot d^{\ell^1} + d_G$ from Definition 6.7. Abbreviate $E := E_{\mathcal{F}}(G)$.

The category $\mathcal{R}^G\big(W, \mathbb{J}((X_n \times G)_n, E)\big)$ will be the target of the "transfer". However, we need to equip it with another class of weak equivalences. These h^{fin}–equivalences were introduced in the proof of [47, Theorem 10.1]. Basically, they ignore the behavior of an object on finitely many factors and behave like h–equivalences otherwise.

Definition 6.11 Let $(M_n)_n$ be a sequence of metric spaces with free, isometric G–action (eg $M_n = X_n \times G$).

Let $(Y_n)_n$ be an object of $\mathcal{R}^G\big(W, \mathbb{J}((M_n)_n, E)\big)$. For $\nu \in \mathbb{N}$, we denote by $(-)_{n>\nu}$ the endofunctor which sends $(Y_n)_n$ to the sequence $(\widetilde{Y}_n)_n$ with $\widetilde{Y}_n$ equal to the zero object, ie W for $n \le \nu$ and $\widetilde{Y}_n = Y_n$ for $n > \nu$.

A morphism $(f_n)_n\colon (Y_n)_n \to (Y'_n)_n$ is an h^{fin}*–equivalence* if there is some $\nu \in \mathbb{N}$ such that $(f_n)_{n>\nu}\colon (Y_n)_{n>\nu} \to (Y'_n)_{n>\nu}$ is an h–equivalence.

Lemma 6.12 *Let $(M_n)_n$ and $(N_n)_n$ be sequences of metric spaces with free, isometric G–action. Let $(g_n)_n\colon (M_n)_n \to (N_n)_n$ be a uniformly expanding sequence of G–equivariant maps, ie for every $\alpha > 0$ there is some $\beta > 0$ such that for all $n \in \mathbb{N}$ and $x, y \in M_n$ we have $d(g_n(x), g_n(y)) < \beta$ whenever $d(x, y) < \alpha$.*

Then $(g_n)_n$ induces a map $\mathcal{R}^G(W, \mathbb{J}((M_n)_n, E)) \to \mathcal{R}^G(W, \mathbb{J}((N_n)_n, E))$ which also respects h– and h^{fin}–equivalences, as well as finiteness conditions.

Proof As $(g_n)_n$ induces a map on $\mathbb{J}((M_n)_n, E)$ which respects the control conditions, it also respects the h–equivalences. As it maps M_n to N_n, it also respects the h^{fin}–equivalences. □

We will discuss the difference between h– and the h^{fin}–equivalences in Section 6.8.

6.6 Nonconnective algebraic K–theory of controlled CW–complexes

Before we turn to the main theorem, we need to briefly recall the definition of algebraic K–theory in our setting. Let $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ be a coarse structure. Then $\mathcal{R}^G_{\mathrm{f}}(W, \mathfrak{Z})$ and its variants are Waldhausen categories, hence their algebraic K–theory is defined by [53]. However, we need the nonconnective delooping from [47, Section 5], which we briefly recall for completeness.

Definition 6.13 Let $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ be a coarse structure. For $n \in \mathbb{N}$ define the coarse structure $\mathfrak{Z}(n) = (\mathbb{R}^n \times Z, \mathfrak{C}(n), \mathfrak{S}(n))$ as follows: A set $C \subseteq (\mathbb{R}^n \times Z)^2$ is in $\mathfrak{C}(n)$ if and only if:

(i) C is symmetric, G–invariant and contains the diagonal.

(ii) $C \subseteq p_n^{-1}(C')$ for some $C' \in \mathfrak{C}_{\mathrm{bdd}}(\mathbb{R}^n)$, where $p_n\colon \mathbb{R}^n \times Z \to \mathbb{R}^n$ is the projection map.

(iii) For all $K \subseteq \mathbb{R}^n$ compact, there is a $C' \in \mathfrak{C}$ such that

$$C \cap ((K \times Z) \times (K \times Z)) \subseteq p_Z^{-1}(C'),$$

where $p_Z\colon \mathbb{R}^n \times Z \to Z$ is the projection map.

Let $\mathfrak{S}(n)$ be the collection of all $S \subseteq \mathbb{R}^n \times Z$ such that $S = p_Z^{-1}(S')$ for some $S' \in \mathfrak{S}(Z)$.

Consider for all n also the restricted coarse structures

$$\begin{aligned}\mathfrak{Z}(n+1)^+ &:= \mathfrak{Z}(n+1) \cap (\mathbb{R}^n \times \mathbb{R}_{\geq 0} \times Z),\\ \mathfrak{Z}(n+1)^- &:= \mathfrak{Z}(n+1) \cap (\mathbb{R}^n \times \mathbb{R}_{\leq 0} \times Z).\end{aligned}$$

Note that $\mathfrak{Z}(n+1) \cap (\mathbb{R}^n \times \{0\} \times Z) = \mathfrak{Z}(n)$. The inclusion maps give rise to a commutative square

$$\begin{tikzcd} hS_\bullet\mathcal{R}_{\mathrm{f}}^G(W,\mathfrak{Z}(n)) \arrow[r] \arrow[d] & hS_\bullet\mathcal{R}_{\mathrm{f}}^G(W,\mathfrak{Z}(n+1)^+) \arrow[d] \\ hS_\bullet\mathcal{R}_{\mathrm{f}}^G(W,\mathfrak{Z}(n+1)^-) \arrow[r] & hS_\bullet\mathcal{R}_{\mathrm{f}}^G(W,\mathfrak{Z}(n+1)) \end{tikzcd}$$

By an Eilenberg swindle, the top-right and bottom-left corners of this square are contractible. This provides us with structure maps (uniquely determined up to contractible choice) for a spectrum

$$\mathbb{K}^{-\infty}(\mathcal{R}_{\mathrm{f}}^G(W,\mathfrak{Z}),h)_n := K\big(\mathcal{R}_{\mathrm{f}}^G(W,\mathfrak{Z}(n)),h\big),$$

which we call the *nonconnective algebraic K–theory spectrum* of $\mathcal{R}_{\mathrm{f}}^G(W,\mathfrak{Z})$. This construction can be made functorial in $\mathfrak{Z}$; see [47, Section 5].

All the arguments which will follow do not interact with a possible $\mathbb{R}^n$–coordinate, hence can also be carried out for $n > 0$, similar to [47, Section 9]. From the next section onwards, our proofs will only treat the case $n = 0$.

6.7 The main theorem

In this part we show the following result:

Theorem 6.14 *Let G be a discrete group and let $\mathcal{F}$ be a family of subgroups of G. If G is homotopy transfer reducible over $\mathcal{F}$, then G satisfies the Farrell–Jones conjecture 2.13 with coefficients in A–theory with respect to $\mathcal{F}$.*

Theorem 1.1(i) follows from Theorem 6.14 in conjunction with the inheritance properties established in Section 5; see the introduction. We derive the validity of Theorem 1.1(i) for hyperbolic and CAT(0)–groups in Corollary 6.20 below.

We follow the strategy of [55, Section 5]. We construct a commutative diagram of Waldhausen categories and exact functors

$$\begin{tikzcd}
\big(\mathcal{R}^G_{\mathrm{fd}}\big(W, \mathbb{J}((X_n \times G)_n, E)\big), h^{\mathrm{fin}}\big) \arrow[r, "F"] \arrow[drr, "p_{X_n\times G\to G}"'] & & \big(\mathcal{R}^G_{\mathrm{fd}}\big(W, \mathbb{J}((\Sigma_n \times G)_n, E)\big), h^{\mathrm{fin}}\big) \arrow[d, "p_{\Sigma_n\times G\to G}"] \\
& & \big(\mathcal{R}^G_{\mathrm{fd}}\big(W, \mathbb{J}((G)_n, E)\big), h^{\mathrm{fin}}\big) \\
\big(\mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E)), h\big) \arrow[uu, dashed, "\mathrm{trans}"] \arrow[rr, "\mathrm{incl}"] & & \big(\mathcal{R}^G_{\mathrm{fd}}(W, \mathbb{J}(E)), h\big) \arrow[u, "\Delta"']
\end{tikzcd}$$

We define the maps trans, Δ and F below and show the following:

Proposition 6.15 (i) *The arrow* trans *exists after applying nonconnective algebraic K–theory. It will be induced by a map of spectra whose domain is weakly equivalent to $\mathbb{K}^{-\infty}(\mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E)), h)$. The square formed by $p_{X_n\times G\to G} \circ \mathrm{trans}$ and $\Delta \circ \mathrm{incl}$ commutes up to levelwise weak equivalence of spectra.*

(ii) *The functor F, defined below, is well defined.*

(iii) *The algebraic K–theory of $\mathcal{R}^G_{\mathrm{fd}}(W, \mathbb{J}((\Sigma_n \times G)_n, E), h^{\mathrm{fin}})$ vanishes.*

(iv) *The map $\Delta \circ \mathrm{incl}$ is injective on nonconnective algebraic K–theory groups.*

Given all of this, the proof can be finished as in [55, Section 5]. It is a diagram chase on the level of homotopy groups.

Before proving Proposition 6.15, let us define the maps of the diagram. The maps p are induced by the indicated projections on control spaces, "incl" is the inclusion of the finite into the finitely dominated objects. The functor Δ is induced by the diagonal map into $\prod_n \mathcal{R}^G_{\mathrm{fd}}(W, \mathbb{J}(G_n, X))$, which factors over $\big(\mathcal{R}^G_{\mathrm{fd}}\big(W, \mathbb{J}((G)_n, E)\big), h^{\mathrm{fin}}\big)$ because its image consists of uniformly controlled objects and maps. Also note that every h–equivalence is an h^{fin}–equivalence.

The functor F is defined using the maps f_n from Section 6.4. It is induced by the maps $F_n\colon X_n \times G \to \Sigma_n \times G,\ (x,g) \mapsto (gf_n(x), g)$. We show in Section 6.8 that with our choices these are uniformly bounded. The map "trans" is constructed in Section 7.

6.8 Squeezing

All claims made in Proposition 6.15 except part (i) admit fairly short proofs, which we give in this section. Part (i) will be shown in Section 7.10

Proposition 6.15(iii) follows from the "squeezing theorem" [47, Theorem 10.1] and the fact that nonconnective K–theory does not distinguish between finite and finitely dominated objects [47, Remark 5.5]. Indeed, in the proof of Theorem 10.1 of [47], in equation (21), a homotopy fiber sequence

$$\textbf{(6.16)}\quad \mathbb{K}^{-\infty}\Big(\operatorname*{colim}_n \prod_{k=1}^{n} \mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(M_k, E)), h\Big) \\ \to \mathbb{K}^{-\infty}\big(\mathcal{R}^G_{\mathrm{f}}\big(W, \mathbb{J}((M_n)_n, E)\big), h\big) \to \mathbb{K}^{-\infty}\big(\mathcal{R}^G_{\mathrm{f}}\big(W, \mathbb{J}((M_n)_n, E)\big), h^{\mathrm{fin}}\big)$$

is established for any sequence of metric spaces $(M_n)_n$. Then it is shown there that, under the assumptions from Section 6.4 on $(\Sigma_n \times G)_n$, the first map is a weak equivalence by proving that the last object is weakly contractible.

Let us discuss Proposition 6.15(ii) next. It suffices to show the following:

Lemma 6.17 *The map $(F_n)_n\colon \mathbb{J}((X_n \times G)_n, E) \to \mathbb{J}((\Sigma_n \times G)_n, E)$ is a morphism of coarse structures.*

Proof By Lemma 6.12, it suffices to check that $(F_n)_n$ is a uniformly expanding sequence. Since the proof is fairly lengthy (though still straightforward), we give the details.

Let us recall the definitions. The metric d_n on $X_n \times G$ was defined in Definition 6.10; the metric $n \cdot d^{\ell^1} + d_G$ on $\Sigma_n \times G$ was chosen in Definition 6.7. Let $\alpha > 0$.

Let $n \in \mathbb{N}$ and $(x,g),(y,h) \in X_n \times G$. Suppose that $d_n((x,g),(y,h)) < \alpha$. To prove that F_n is uniformly expanding we have to show that

$$n \cdot d^{\ell^1}(gf_n(x), hf_n(y)) + d_G(g,h) \le \beta$$

for some $\beta > 0$ which is independent of n.

In fact, it suffices to show this for $n \geq \alpha$. Then we have, by Definition 6.10,

$$\frac{\alpha}{\Lambda_n} = \frac{\alpha\delta_n}{n+1} \leq \frac{n\delta_n}{n+1} < \delta_n.$$

By definition of d_n, we have $d_G(g,h) < \alpha$ and $d_{S^n,n,\Lambda_n}((x,g),(y,h)) < \alpha$. Hence, there exist

- $l \in \mathbb{N}$,
- $x_0,\dots,x_l$, $z_0,\dots,z_l \in X_n$,
- $a_1,\dots,a_l$, $b_1,\dots,b_l \in S^n$

such that

(i) $x_0 = x$ and $z_l = y$;

(ii) $ga_1^{-1}b_1 \dots a_l^{-1}b_l = h$;

(iii) for each $1 \leq i \leq l$ there are elements $r_0,\dots,r_n,s_0,\dots,s_n \in S^n$ such that $a_i = r_n \dots r_0$, $b_i = s_n \dots s_0$ and, for some $t_1,\dots,t_n,u_1,\dots,u_n \in [0,1]$, $\Gamma_n(r_n,t_n,\dots,t_0,z_{i-1}) = \Gamma_n(s_n,u_n,\dots,s_0,x_i)$ holds;

(iv) $l + \sum_{i=0}^{l} \Lambda_n \cdot d_{X_n}(x_i,z_i) < \alpha$.

This implies $l < \alpha$ and $d_{X_n}(x_i,z_i) < \alpha/\Lambda_n < \delta_n$. By Definition 6.10 of δ_n, this implies $d^{\ell^1}(f_n(x_i), f_n(z_i)) < \frac{1}{n}$.

We proceed by induction on l. For $l = 0$, we have $g = h$ and $d^{\ell^1}(f_n(x), f_n(y)) < \frac{1}{n}$. For the induction step, with $a_1 = r_n \dots r_0$ and $b_1 = s_n \dots s_0$ we have

$$\begin{aligned}
&d^{\ell^1}(gf_n(x), hf_n(y)) \\
&\quad = d^{\ell^1}(gf_n(x), ga_1^{-1}b_1 \dots a_l^{-1}b_l f_n(y)) \\
&\quad = d^{\ell^1}(f_n(x), a_1^{-1}b_1 \dots a_l^{-1}b_l f_n(y)) \\
&\quad \leq d^{\ell^1}(f_n(x_0), f_n(z_0)) + d^{\ell^1}\big(f_n(z_0), a_1^{-1} f_n(\Gamma_n(r_n,1,\dots,1,r_0,z_0))\big) \\
&\qquad\qquad + d^{\ell^1}\big(f_n(\Gamma_n(r_n,1,\dots,1,r_0,z_0)), f_n(\Gamma_n(r_n,t_n,\dots,t_1,r_0,z_0))\big) \\
&\qquad\qquad + d^{\ell^1}\big(f_n(\Gamma_n(s_n,u_n,\dots,u_1,s_0,x_1)), f_n(\Gamma_n(s_n,1,\dots,1,s_0,x_1))\big) \\
&\qquad\qquad + d^{\ell^1}(f_n(\Gamma_n(s_n,1,\dots,1,s_0,x_1)), b_1 f_n(x_1)) \\
&\qquad\qquad + d^{\ell^1}(f_n(x_1), f_n(z_1)) \\
&\qquad\qquad + d^{\ell^1}(f_n(z_1), a_2^{-1}b_2 \dots a_l^{-1}b_l f_n(y)).
\end{aligned}$$

We give an estimate for each summand. We already know

$$d^{\ell^1}(f_n(x_0), f_n(z_0)) < \frac{1}{n}, \quad d^{\ell^1}(f_n(x_1), f_n(z_1)) < \frac{1}{n}.$$

For the second summand, we have, by (6.5),

$$\begin{aligned} d^{\ell^1}(f_n(z_0), a_1^{-1} f_n(\Gamma_n(r_n, 1, \dots, 1, r_0, z_0))) &= d^{\ell^1}\big(a_1 \cdot f_n(z_0), f_n(\Gamma_n(a_1, z_0))\big) \\ &\leq \frac{1}{n}, \end{aligned}$$

and similarly for $d^{\ell^1}\big(f(\Gamma_n(s_n, 1, \dots, 1, s_0, x_1)), b_1 f(x_1)\big)$. Furthermore, we have

$$\begin{aligned} d^{\ell^1}\big(f_n(\Gamma_n(r_n, 1, \dots, 1, r_0, z_0)), f_n(\Gamma_n(r_n, t_n, \dots, t_1, r_0, z_0))\big) &\leq \frac{2}{n}, \\ d^{\ell^1}(f_n(\Gamma_n(s_n, u_n, \dots, u_1, s_0, x_1)), f_n(\Gamma_n(s_n, 1, \dots, 1, s_0, x_1))) &\leq \frac{2}{n}, \end{aligned}$$

by (6.6). Finally, we choose the induction hypothesis to be

$$d^{\ell^1}(f_n(z_1), a_2^{-1} b_2 \dots a_l^{-1} b_l f_n(y)) < \frac{8(l-1)+1}{n}.$$

Thus, we obtain

$$d^{\ell^1}(g f_n(x), h f_n(y)) < \frac{8l+1}{n}.$$

Since we also have $d_G(g, h) < \alpha$, we conclude that

$$n \cdot d^{\ell^1}(g f_n(x), h f_n(y)) + d_G(g, h) < 9\alpha + 1. \qquad \square$$

6.9 Injectivity of the Δ–map

Now we show Proposition 6.15(iv). Namely, we have to show that Δ induces an injective map on algebraic K–theory. Our argument is a straightforward adaptation of the argument used in [55, Section 5]. As usual, we abbreviate $\pi_m(\mathbb{K}^{-\infty}(\dots))$ by $K_m(\dots)$.

Lemma 6.18 *The map*

$$K_m(\Delta) \circ K_m(\mathrm{incl})\colon K_m\big(\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E)), h\big) \to K_m\big(\mathcal{R}_{\mathrm{fd}}^G\big(W, \mathbb{J}((G)_n, E)\big), h^{\mathrm{fin}}\big)$$

is injective for each $m \geq 0$.

Proof The map $K_m(\mathrm{incl})\colon K_m\big(\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E)), h\big) \to K_m\big(\mathcal{R}_{\mathrm{fd}}^G(W, \mathbb{J}(E)), h\big)$ is an isomorphism by [47, Remark 5.5]. Hence, we only have to show that $K_m(\Delta)$ is injective. To increase readability, we shorten $\mathcal{R}^G_{\mathrm{fd}}(W,\dots)$ to $\mathcal{R}(\dots)$ in the following commutative diagram:

$$\begin{tikzcd}
 & K_m\bigl(\mathcal{R}(\mathbb{J}(E)),h\bigr) \arrow[d, "\Delta_*"] \arrow[dr, "\Delta_*"] & \\
K_m\bigl(\prod^{\mathrm{fin}}\mathcal{R}(\mathbb{J}(E)),h\bigr) \arrow[r, "\mathrm{incl}"] \arrow[d, "\cong"] & K_m\bigl(\mathcal{R}\bigl(\mathbb{J}((G)_n,E)\bigr),h\bigr) \arrow[r, "\mathrm{id}_*"'] \arrow[d, "\prod_{n\in\mathbb{N}} p_n"] & K_m\bigl(\mathcal{R}\bigl(\mathbb{J}((G)_n,E)\bigr),h^{\mathrm{fin}}\bigr) \\
\bigoplus_{n\in\mathbb{N}} K_m\bigl(\mathcal{R}(\mathbb{J}(E)),h\bigr) \arrow[r, "\mathrm{incl}"] & \prod_{n\in\mathbb{N}} K_m\bigl(\mathcal{R}(\mathbb{J}(E)),h\bigr) &
\end{tikzcd}$$

The middle row is exact due to the homotopy fiber sequence (6.16), where we abbreviated $\operatorname{colim}_n \prod_{k=1}^n$ as $\prod^{\mathrm{fin}}$. The left vertical map is an isomorphism, because algebraic K–theory commutes with directed colimits and is compatible with finite products. The map is defined using the projections onto the factors of the product. Note that after projection on any n, the middle column is the identity. A diagram chase finishes the proof. □

6.10 Homotopy transfer reducible follows from strongly transfer reducible

We can now prove Theorem 1.1(i) for hyperbolic and CAT(0)–groups.

In [55, Definition 3.1], Wegner defined when a group G is *strongly transfer reducible over a family $\mathcal{F}$*. As we will not need the precise definition here, we refer to loc. cit. for the definition. We will use the definitions from Section 6.1.

Theorem 6.19 *Let G be strongly transfer reducible over $\mathcal{F}$. Then G is homotopy transfer reducible over $\mathcal{F}$ and the Farrell–Jones conjecture 2.13 for A–theory with coefficients holds for G relative to $\mathcal{F}$.*

Proof Assume that G is strongly transfer reducible. We show it is homotopy transfer reducible and apply Theorem 6.14.

According to [55, Proposition 3.6], there exists $N\in\mathbb{N}$ such that there are, for every $n\in\mathbb{N}$,

(i) a compact, contractible metric space (X,d_X) such that for every $\varepsilon>0$ there is an ε–controlled domination of X by an at most N–dimensional, finite simplicial complex;

(ii) a homotopy coherent G–action Γ on X;

(iii) a G–simplicial complex Σ of dimension at most N whose isotropy is contained in $\mathcal{F}$;

(iv) a positive real number Λ;

(v) a G–equivariant map $\varphi\colon G \times X \to \Sigma$ such that

$$n \cdot d^{\ell^1}(\varphi(g,x), \varphi(h,y)) \le d_{S^n,n,\Lambda}((g,x),(h,y))$$

holds for all $(g,x),(h,y) \in G \times X$, where G acts on the G–factor.

Fix $n \in \mathbb{N}$ and choose X, Γ, Σ, Λ and φ as above. Define $f := \varphi|_{\{e\}\times X}\colon X \to \Sigma$. Then we have, for all $x \in X$ and $s \in S^n$,

$$\begin{aligned} n \cdot d^{\ell^1}\big(f(\Gamma(s,x)), s \cdot f(x)\big) &= n \cdot d^{\ell^1}\big(\varphi(e,\Gamma(s,x)), \varphi(s,x)\big) \\ &\le d_{S^n,n,\Lambda}\big((e,\Gamma(s,x)),(s,x)\big) \\ &\le 1. \end{aligned}$$

Similarly, we find, for all $x \in X$, $s_0,\dots,s_n \in S^n$ and $t_1,\dots,t_n,u_1,\dots,u_n \in I^n$,

$$\begin{aligned} &n \cdot d^{\ell^1}\big(f(\Gamma(s_n,t_n,\dots,s_0,x)), f(\Gamma(s_n,u_n,\dots,s_0,x))\big) \\ &\qquad = n \cdot d^{\ell^1}\big(\varphi(e,\Gamma(s_n,t_n,\dots,s_0,x)), \varphi(e,\Gamma(s_n,u_n,\dots,s_0,x))\big) \\ &\qquad \le d_{S^n,n,\Lambda}\big((e,\Gamma(s_n,t_n,\dots,s_0,x)),(e,\Gamma(s_n,u_n,\dots,s_0,x))\big) \\ &\qquad \le 2. \end{aligned}$$

Hence, G is homotopy transfer reducible over $\mathcal{F}$ and we can apply Theorem 6.14. □

Corollary 6.20 *The Farrell–Jones conjecture 2.15 for A–theory with coefficients and finite wreath products is true for hyperbolic and* CAT(0)*–groups.*

Proof By [55, Example 3.2 and Theorem 3.4], finitely generated hyperbolic groups as well as CAT(0)–groups are strongly transfer reducible with respect to the family of virtually cyclic subgroups. Thus, these groups are homotopy transfer reducible over the same family by Theorem 6.19. If a group is homotopy transfer reducible with respect to $\mathcal{F}$, then the wreath product $G \wr F$ with a finite group F is homotopy transfer reducible over $\mathcal{F} \wr F$. This follows as in [7, Section 5]; basically, one takes the F–fold product of X and Σ and uses eg [47, Lemma 11.14] for the estimate. As the A–theoretic Farrell–Jones conjecture with coefficients holds for virtually finitely generated abelian groups [47, Proposition 11.9], the A–theoretic Farrell–Jones conjecture with coefficients and finite wreath products holds for hyperbolic and CAT(0)–groups by the transitivity principle [47, Proposition 11.2]. □

7 The transfer: final part of the proof

We turn now to the construction of the transfer map whose existence was claimed in the first part of Proposition 6.15(i). In the uncontrolled setting, these transfers are well known and are induced by an appropriate pullback construction; see [35; 5, Appendix A] for a description in the linear case, and [60, Section 2.4] for retractive spaces. Let X be as in Definition 6.2, but suppose for simplicity that X carries a strict G–action. The transfer we are considering is along the trivial bundle $G \times X \to G$, and therefore the transfer amounts to taking the product with the fiber X. Since X is not a CW–complex and we require retractive spaces to have a CW–structure, we replace X by the geometric realization $\|S_\bullet(X)\|$ of its singular set.

Of course, this construction does not yet make sense in the controlled setting. A controlled refinement of the linear transfer has been employed in [5; 6; 55]. Translated to the world of CW–complexes, the main idea is to replace the singular set of X by the subsimplicial set $S^\delta_\bullet(X)$ of singular simplices whose diameter is bounded by a sufficiently small number δ. It is a consequence of excision that this substitution does not alter the homotopy type.

However, since G does not act isometrically on X, the simplicial set $S^\delta_\bullet(X)$ does not carry a G–action. This is seemingly remedied by the fact that we chose a new metric on $G \times X$ (Definition 6.8 and Lemma 6.9) and let G act only on the first coordinate of $G \times X$. Unfortunately, the problem has only been moved somewhere else. Unwinding the definitions, one observes that if a cell c is attached to a lower-dimensional cell c' in a retractive space Y, we have little control over the distance of the cells $c \times \sigma$ and $c' \times \sigma$ in the product $Y \times \|S^\delta_\bullet(X)\|$. The best we can show is that this distance grows in a uniform fashion which only depends on the amount of bounded control Y satisfies with respect to the G–coordinate of the control space; see Lemma 7.8 and Corollary 7.9. This observation suggests the following solution: If we fix a bound on both the dimension and the amount of bounded control of Y in terms of G, a controlled version of $Y \times \|S^\delta_\bullet(X)\|$ can be constructed in which the diameter of singular simplices varies with the dimension of cells in Y. Since this forces us to take a product with a different version of $\|S^\delta_\bullet(X)\|$ for each cell in Y and glue these products accordingly, this construction is best expressed in the form of a coend ("balanced product") over the poset of cells of Y. We do this in Sections 7.2, 7.4 and 7.5.

The transfer construction sketched so far leaves us with two problems, which turn out to have the same solution. First, the collection of controlled retractive spaces with a fixed bound on the amount of bounded control in terms of G is not closed under pushouts, and hence does not form a Waldhausen subcategory of all controlled retractive spaces. Second, while there is a candidate definition for extending the transfer construction from objects to morphisms (see the proof of Proposition 7.20, where we do need this construction), it is not functorial, essentially for the same reason that we could not take the product of Y with one fixed $\|S^{\delta}_{\bullet}(X)\|$.

We avoid these problems by implementing an idea due to Arthur Bartels and Paul Bubenzer: If we only consider morphisms which are *cellwise* 0*–controlled* (which essentially means that individual cells are not moved in terms of the G–coordinate; see Definition 7.1 for the precise definition), pushouts along cofibrations preserve the bounded control of retractive spaces (see Section 7.1), and the transfer construction on morphisms becomes functorial (see Section 7.6). Moreover, restricting from the category of all morphisms to the subcategory of cellwise 0–controlled morphisms makes no difference in K–theory (Proposition 7.2).

In order to verify that the functor we have constructed preserves weak equivalences, we rely on the more general transfer construction on morphisms hinted at earlier (Proposition 7.20). The resulting map on K–theory is essentially independent of the choices we had to make (Proposition 7.19).

Finally, to ensure that the target of the transfer is not trivial due to an Eilenberg swindle, we prove in Section 7.9 that the transfer preserves finiteness conditions imposed on the objects; this is essentially guaranteed by the existence of arbitrarily controlled finite dominations of X.

The final Section 7.10 summarizes the preceding discussion to show that the desired transfer map exists.

7.1 The domain of the transfer

We now define the appropriate subcategories of cellwise 0–controlled morphisms, which will serve as the source of the transfer.

Let M be a metric space with a free, isometric G–action, and consider the category $\mathcal{R}^G(W, \mathbb{J}(M, E))$. For $(Y, \kappa) \in \mathcal{R}^G(W, \mathbb{J}(M, E))$, let κ_M denote the composition of the control map κ with the projection map $M \times E \times [1, \infty[\to M$.

Definition 7.1 Let $f\colon (Y_1, \kappa_1) \to (Y_2, \kappa_2)$ be a morphism in $\mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(M, E))$. We say that f is *regular* if the image of each open cell in Y_1 is either equal to an open cell in Y_2 or completely contained in W. That is, either $f(\operatorname{int} e) = \operatorname{int} e'$ or $f(\operatorname{int} e) \subseteq W$.

We say that f is *cellwise* 0*–controlled over* M if f is regular and satisfies the property that $\kappa_{1,M}(e) = \kappa_{2,M}(f(e))$ for all cells $e \in \diamond Y_1$.

The composition of two morphisms which are cellwise 0–controlled over M is again cellwise 0–controlled over M, so we can consider the subcategory

$$\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_0 \subseteq \mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))$$

which has the same objects as $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))$, but contains only those morphisms which are cellwise 0–controlled over M. The category $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_0$ inherits its cofibrations and weak equivalences from $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))$. It is a Waldhausen subcategory of $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))$.

For $\alpha > 0$, we may further restrict to the full subcategory

$$\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_\alpha \subseteq \mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_0$$

consisting only of those objects which are *α–controlled over* M, ie those (Y, κ) such that $\kappa_M(\diamond\langle e\rangle) \subseteq B_\alpha(\kappa_M(e))$ for every cell $e \in \diamond Y$. (Recall that $\langle e\rangle$ denotes the smallest subcomplex of Y containing e.) The category $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_\alpha$ also inherits the structure of a Waldhausen category, as the pushout of α–controlled complexes along cellwise 0–controlled morphisms is again α–controlled.

Finally, we can filter $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_\alpha$ by

$$\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_{\alpha,0} \subseteq \mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_{\alpha,1} \subseteq \cdots \subseteq \mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_\alpha,$$

where $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_{\alpha,d}$ denotes the full subcategory of $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_\alpha$ containing those objects whose dimension is at most d. Note that

$$\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_\alpha = \operatorname*{colim}_d \mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_{\alpha,d},$$

as each object in $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))$ is finite-dimensional.

Proposition 7.2 *There is a natural weak equivalence*

$$\operatorname*{hocolim}_{\alpha,d} K\big(\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_{\alpha,d}\big) \xrightarrow{\sim} K\big(\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))\big).$$

Proof We have $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_0 = \operatorname{colim}_{\alpha,d} \mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_{\alpha,d}$. Since K–theory commutes with directed colimits, we obtain a natural weak equivalence

$$\operatorname*{hocolim}_{\alpha,d} K\big(\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_{\alpha,d}\big) \xrightarrow{\sim} K\big(\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M, E))_0\big).$$

Now consider the inclusion functor $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M,E))_0 \hookrightarrow \mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M,E))$. We show that Waldhausen's approximation theorem [53, Theorem 1.6.7] applies.

The cylinder functor on $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M,E))$ constructed in [47, Lemma 3.14] restricts to a cylinder functor on $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M,E))_0$; in particular, the inclusion of the source is always cellwise 0–controlled. By definition, the inclusion functor satisfies the first part of the approximation property. To verify the second part of the approximation property, let $f\colon Y_1 \to Y_2$ be an arbitrary morphism in $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(M,E))$. Then the factorization of f via the cylinder functor $Y_1 \rightarrowtail Mf \xrightarrow{\sim} Y_2$ decomposes f into a cellwise 0–controlled morphism and a weak equivalence. So the approximation theorem implies that the inclusion functor induces an equivalence on algebraic K–theory. $\square$

Remark 7.3 The upshot of Proposition 7.2 is that we do not have to define a "global" transfer functor on $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E))$. Instead, it suffices to define a transfer functor $\mathrm{trans}^{\alpha,d}\colon \mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E))_{\alpha,d} \to \big(\mathcal{R}_{\mathrm{fd}}^G\big(W, \mathbb{J}((X_n \times G)_n, E)\big), h^{\mathrm{fin}}\big)$ on each subcategory such that the induced diagrams on K–theory

$$\begin{tikzcd}
K\big(\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E))_{\alpha,d}, h\big) \arrow[r, "K(\mathrm{trans}^{\alpha,d})"] \arrow[d] & K\big(\mathcal{R}_{\mathrm{fd}}^G\big(W, \mathbb{J}((X_n \times G)_n, E)\big), h^{\mathrm{fin}}\big) \\
K\big(\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E))_{\alpha+1,d+1}, h\big) \arrow[ur, "K(\mathrm{trans}^{\alpha+1,d+1})"'] &
\end{tikzcd}$$

are homotopy-commutative.

7.2 Balanced products of CW–complexes

We introduce a slight generalization of the balanced products discussed in [13] as a means to define the transfer in Section 7.5.

Let W be a topological space and $\mathcal{C}$ a small category. A *$\mathcal{C}$–CW–complex relative* W is a functor Y from $\mathcal{C}$ to topological spaces such that $Y(c)$ is a CW–complex relative W and the morphisms in $\mathcal{C}$ are mapped to cellular maps relative W. A (relative) free $\mathcal{C}$–n–cell based at c, with $c \in \mathcal{C}$, is a pair $(\eta, \partial\eta)$ of $\mathcal{C}$–CW–complexes relative W, where $\eta = \mathcal{C}(c,-) \times D^n \amalg W$ and $\partial\eta = \mathcal{C}(c,-) \times S^{n-1} \amalg W$. Attaching a free $\mathcal{C}$–cell η to Y means taking the pushout along a map $\partial\eta \to Y$. Note that W itself defines a (constant) covariant $\mathcal{C}$–CW–complex relative W.

We say that Y is a *free* $\mathcal{C}$–CW–complex relative W if it comes equipped with a filtration $W = \mathrm{sk}_{-1}(Y) \subseteq \mathrm{sk}_0(Y) \subseteq \mathrm{sk}_1(Y) \subseteq \cdots$ such that $Y = \mathrm{colim}_n\, \mathrm{sk}_n(Y)$ and for every $n \geq 0$ there exists a pushout in the category of $\mathcal{C}$–CW–complexes relative W

$$\begin{array}{ccc} \Big(\coprod\limits_{i\in I_n} \mathcal{C}(c_i,-)\times S^{n-1}\Big) \amalg W & \longrightarrow & \mathrm{sk}_{n-1}(Y)(-) \\ \big\downarrow & & \big\downarrow \\ \Big(\coprod\limits_{i\in I_n} \mathcal{C}(c_i,-)\times D^{n}\Big) \amalg W & \longrightarrow & \mathrm{sk}_{n}(Y)(-) \end{array}$$

Hence, a free $\mathcal{C}$–CW–complex arises by attaching free $\mathcal{C}$–cells. The set of free $\mathcal{C}$–n–cells is in bijection with I_n. Note that the attaching map of a $\mathcal{C}$–n–cell based at c is the same as a map $S^{n-1} \to \mathrm{sk}_{n-1}(Y)(c)$, hence we can consider η as a map $D^n \to Y(c)$.

Let Y be a covariant $\mathcal{C}$–CW–complex relative W and $X\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{CW\text{–}COMPLEXES}$ be a contravariant $\mathcal{C}$–CW–complex. Define the *reduced balanced product* $X \curlywedge_{\mathcal{C}} Y$ as the pushout

$$\begin{array}{ccc} X \times_{\mathcal{C}} W & \longrightarrow & X \times_{\mathcal{C}} Y \\ \big\downarrow & & \big\downarrow \\ * \times_{\mathcal{C}} W & \longrightarrow & X \curlywedge_{\mathcal{C}} Y \end{array}$$

where $\times_{\mathcal{C}}$ denotes the balanced product from [13]. If $\mathcal{C}$ is connected, we have $* \times_{\mathcal{C}} W \cong W$.

Proposition 7.4 *Let X be a contravariant $\mathcal{C}$–space, Y a covariant $\mathcal{C}$–CW–complex relative W and Z a space relative $* \times_{\mathcal{C}} W$. There is a natural homeomorphism*

$$\hom^{*\times_{\mathcal{C}} W}(X \curlywedge_{\mathcal{C}} Y, Z) \cong \hom_{\mathcal{C}}^{*\times_{\mathcal{C}} W}(Y, \hom(X,Z)).$$

Here, $\hom(X,Z)$ is a covariant $\mathcal{C}$–space relative $* \times_{\mathcal{C}} W$ via the inclusion that sends a point $w \in * \times_{\mathcal{C}} W$ based at $c \in \mathcal{C}$ to the constant map $X(c) \to \{w\} \subseteq Z$, $\hom_{\mathcal{C}}^{*\times_{\mathcal{C}} W}$ denotes the natural transformations which are relative $* \times_{\mathcal{C}} W$, and $\hom^{*\times_{\mathcal{C}} W}$ denotes just the set of maps relative $* \times_{\mathcal{C}} W$.

Proof By definition, a map $X \curlywedge_{\mathcal{C}} Y \to Z$ is the same as three compatible maps from $* \times_{\mathcal{C}} W \leftarrow X \times_{\mathcal{C}} W \to X \times_{\mathcal{C}} Y$ to Z. Using that $\hom(X \times_{\mathcal{C}} Y, Z)$ is isomorphic to $\hom_{\mathcal{C}}(Y, \hom(X,Z))$, the result is easy to deduce. □

It follows that $X \curlywedge_{\mathcal{C}} Y$ commutes with colimits in the "Y"–variable. We can therefore determine the cell structure of $X \curlywedge_{\mathcal{C}} Y$. The attachment of a free $\mathcal{C}$–n–cell η to Y

gives a pushout $X \curlywedge_{\mathcal{C}} \eta(-) \cup_{X \curlywedge_{\mathcal{C}} \partial\eta} Y$. Now

$$X \curlywedge_{\mathcal{C}} \eta \cong \big((X \times_{\mathcal{C}} \mathcal{C}(c,-)) \times D^n\big) \amalg (* \times_{\mathcal{C}} W) \cong (X(c) \times D^n) \amalg (* \times_{\mathcal{C}} W)$$

and similarly for $\partial\eta$. First, this gives a filtration on $X \curlywedge_{\mathcal{C}} Y$, namely

$$\cdots \subseteq X \curlywedge_{\mathcal{C}} \mathrm{sk}_{n-1}(Y) \subseteq X \curlywedge_{\mathcal{C}} \mathrm{sk}_n(Y) \subseteq \cdots . \tag{7.5}$$

Second, as $X(c)$ is a CW–complex, we can now read off the cell structure of $X \curlywedge_{\mathcal{C}} Y$:

Proposition 7.6 (cf [13, Lemma 3.19(2)]) *Let Y be a free covariant $\mathcal{C}$–CW–complex relative W and X a contravariant $\mathcal{C}$–CW–complex.*

Then $X \curlywedge_{\mathcal{C}} Y$ is a CW–complex relative $* \times_{\mathcal{C}} W$, and there is a canonical identification

$$\diamond(X \curlywedge_{\mathcal{C}} Y) \cong \{(\xi,\eta) \mid \eta \text{ is a free } \mathcal{C}\text{–cell based at } c \text{ and } \xi \in \diamond X(c)\}.$$

Let $(\xi,\eta) \in \diamond(X \curlywedge_{\mathcal{C}} Y)$. If $\Phi\colon D^p \to X(c)$ and $\Psi\colon \mathcal{C}(c,-) \times D^q \to Y(-)$ are characteristic maps for ξ and η, respectively, then

$$D^p \times D^q \to X \curlywedge_{\mathcal{C}} Y, \quad (a,b) \mapsto [\Phi(a), \Psi(\mathrm{id}_c, b)],$$

is a characteristic map for (ξ,η).

Let $(\xi,\eta), (\xi',\eta') \in \diamond(X \curlywedge_{\mathcal{C}} Y)$ be two cells, with η based at c and η' based at c'. Then $(\xi,\eta) \subseteq \langle(\xi',\eta')\rangle$ if and only if there exists a morphism $\gamma\colon c \to c'$ such that $\gamma_*\eta \subseteq \langle\eta'\rangle \subseteq Y(c')$ and $\xi \subseteq \langle\gamma^*\xi'\rangle \subseteq X(c)$.

In greater generality, (7.5) gives a filtration for an inclusion $Y_1 \hookrightarrow Y_2$ of $\mathcal{C}$–spaces in which Y_2 is obtained from Y_1 by the attachment of free $\mathcal{C}$–cells. This observation allows us to translate the constructions for geometric modules to CW–complexes.

Let us conclude this section with a short remark about functoriality of the balanced product construction. In addition to the obvious functoriality properties, we have the following: Let X be a contravariant and Y be a covariant $\mathcal{C}$–space. Let $F\colon \mathcal{D} \to \mathcal{C}$ be a functor. Then there is an induced map

$$\iota_{\mathrm{f}}\colon F^*X \times_{\mathcal{D}} F^*Y \to X \times_{\mathcal{C}} Y, \quad [d,x,y] \mapsto [F(d),x,y].$$

This map is functorial in the sense that $\iota_{F_2}\iota_{F_1} = \iota_{F_2F_1}$ for any two composable functors F_1 and F_2. In particular, if $F\colon \mathcal{C} \xrightarrow{\cong} \mathcal{C}$ is an automorphism of the indexing category, then ι_{f} is an isomorphism.

7.3 Conventions

For the following sections, fix the following data:

(i) Natural numbers $\alpha, d \in \mathbb{N}$ and a natural number $n > \max\{d+1, \alpha\}$.

(ii) A natural number $N \in \mathbb{N}$.

(iii) A compact and contractible metric space (X, d_X) such that for every $\varepsilon > 0$ there is an ε–controlled domination of X by an at most N–dimensional, finite simplicial complex.

(iv) A homotopy coherent G–action Γ on X.

(v) A positive real number Λ.

As before, we consider $X \times G$ equipped with the metric $d_{S^n,n,\Lambda} + d_G$.

7.4 Y as a $\diamond_+ Y$–CW–complex and a $(\diamond_+ Y)^{\mathrm{op}}$–CW–complex from X

Let $(Y, \kappa) \in \mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))_{\alpha,d}$. If $c \in \diamond Y$ is a cell of Y, we will frequently need to refer to the G–component of $\kappa(c)$; we denote this by $\kappa_G(c)$.

Define a relation $\leq$ on the set of cells $\diamond Y$ by saying that $c \leq c'$ if and only if $c \subseteq \langle c' \rangle$. Then $\diamond Y$ forms a poset under the relation $\leq$. We define $\diamond_+ Y$ as the category given by this poset, where we add an additional initial object (which corresponds to W). The complex Y itself gives rise to a covariant $\diamond_+ Y$–CW–complex (relative W) $\mathcal{C}_Y$ by setting

$$\mathcal{C}_Y(c) := \langle c \rangle$$

and sending a morphism $c \leq c'$ to the obvious inclusion $\langle c \rangle \hookrightarrow \langle c' \rangle$. Observe that $\mathcal{C}_Y$ is a free $\diamond_+ Y$–CW–complex; the set of free $\diamond_+ Y$–cells of $\mathcal{C}_Y$ is in canonical bijection with the cells of Y. Note that a cellwise 0–controlled map $Y \to Y'$ gives rise to a functor $\diamond_+ Y \to \diamond_+ Y'$. Last, each cell in $\diamond_+ Y$ has a dimension $|c|$, where we assign the initial object the dimension -1.

The metric space X gives rise to a contravariant $\diamond_+ Y$–CW–complex, but the construction is more involved. We mimic the construction used in [55], but do not pass to the cellular chain complex. Instead, we simply stick with the space of controlled simplices.

In the first step, we pass from the homotopy-coherent G–action Γ on X to an honest G–action on a closely related space. This is accomplished by strictifying the homotopy-coherent diagram Γ; see [51, proof of Proposition 5.4]. Define $M\Gamma$ to be the space

$$M\Gamma := \Big(\coprod_{k\geq 0} G^{k+1} \times [0,1]^k \times X \Big) \Big/ \sim,$$

where $\sim$ is the equivalence relation generated by

$$(\gamma_{k+1}, t_k, \gamma_k, \dots, \gamma_1, x) \sim \begin{cases} (\gamma_{k+1}, t_k, \dots, \gamma_2, x) & \text{if } \gamma_1 = e, \\ (\gamma_{k+1}, \dots, t_i t_{i-1}, \dots, \gamma_1, x) & \text{if } \gamma_i = e, 2 \leq i \leq k, \\ (\gamma_{k+1}, \dots, \gamma_{i+1}\gamma_i, \dots, \gamma_1, x) & \text{if } t_i = 1, 1 \leq i \leq k, \\ (\gamma_{k+1}, \dots, \gamma_{i+1}, \Gamma(\gamma_i, \dots, \gamma_1, x)) & \text{if } t_i = 0, 1 \leq i \leq k. \end{cases}$$

Then G acts on $M\Gamma$ by

$$g \cdot [\gamma, t_k, \gamma_k, \dots, \gamma_1, x] := [g\gamma, t_k, \gamma_k, \dots, \gamma_1, x].$$

We have a map $X \to M\Gamma$ via $x \mapsto [e, x]$. Let $R\colon M\Gamma \to X$ be the retraction induced by Γ; explicitly, $R([\gamma, t_k, \gamma_k, \dots, \gamma_1, x]) = \Gamma(\gamma, t_k, \gamma_k, \dots, \gamma_1, x)$. Using the axioms of a homotopy-coherent action from Definition 6.1, one checks this is a well-defined map. The homotopy

$$\begin{aligned} &H\colon M\Gamma \times [0,1] \to M\Gamma, \\ &([\gamma, t_k, \gamma_k, \dots, \gamma_1, x], u) \mapsto [e, u, \gamma, t_k, \gamma_k, \dots, \gamma_1, x], \end{aligned} \tag{7.7}$$

then shows that X is a strong deformation retract of $M\Gamma$.

The space $M\Gamma$ comes with a filtration by subspaces $M\Gamma^{l,r}$, where we set

$$M\Gamma^{l,r} := \{[e, t_k, \gamma_k, \dots, \gamma_1, x] \in M\Gamma \mid k \leq l,\ \gamma_i \in B_r(e)\}.$$

For $\delta > 0$, define $S_\bullet^\delta(M\Gamma^{l,r})$ to be the subsimplicial set of the singular simplicial set $S_\bullet(M\Gamma^{l,r})$ containing those singular simplices $\sigma\colon \Delta^{|\sigma|} \to M\Gamma^{l,r}$ which fulfill

$$\operatorname{diam}_{X\times G}((R\circ\sigma)(\Delta^{|\sigma|}) \times \{e\}) \leq \delta,$$

where diameters in $X \times G$ are taken with respect to the metric $d_{S^n,n,\Lambda}$. Note that we could replace e by any other group element without changing the diameter, as the metric is G–invariant.

Finally, we can define the contravariant $\diamond_+ Y$–CW–complex $\mathcal{S}^{\alpha,d}_{X,Y}$: Let

$$\delta_k^d := 4(d+1-k) \quad \text{and} \quad l_k^d := d+1-k.$$

Typically, we will omit d from the notation. On objects, we set

$$\mathcal{S}^{\alpha,d}_{X,Y}(c) := \|S^{\delta_{|c|}}_{\bullet}(M\Gamma^{l_{|c|},\alpha})\|,$$

where $|c|$ denotes the dimension of the cell c and $\|-\|$ is fat geometric realization, ie the realization after forgetting the degeneracies. Note that we have the canonical inclusion $\iota_{c'}$ of $\mathcal{S}^{\alpha,d}_{X,Y}(c')$ into $\|S_{\bullet}(M\Gamma^{l})\|$. The latter has an honest G–action. For a morphism $c' \to c = c \leq c'$ in $(\diamond_{+} Y)^{\mathrm{op}}$ define $\mathcal{S}^{\alpha,d}_{X,Y}(c' \to c)$ as the factorization of

$$\mathcal{S}^{\alpha,d}_{X,Y}(c' \to c) := \kappa_G(c)^{-1}\kappa_G(c') \cdot \iota_{c'}(-)$$

over ι_c. We have to check that it is well defined, ie that it actually factors. We require the following observation:

Lemma 7.8 *Let $[e, t_b, \dots, \gamma_1, x] \in M\Gamma^{l,\alpha}$. Suppose that $l < n$ and $\alpha \leq n$. Let $h \in B_\alpha(e)$. Then*

$$d_{S^n,n,\Lambda}\big(R([h, t_b, \dots, \gamma_1, x], g), (R([e, t_b, \dots, \gamma_1, x]), gh)\big) \leq 2$$

for all $g \in G$.

Proof Note that $b < n$. We use the definition of the metric. Let $x_0 = z_0 = \Gamma(h, t_b, \gamma_b, \dots, \gamma_1, x)$, $x_1 = z_1 = x$ and $x_2 = z_2 = \Gamma(e, t_b, \gamma_b, \dots, \gamma_1, x)$. Furthermore, we set $a_1 = e$, $b_1 = h\gamma_b \dots \gamma_1$, $a_2 = \gamma_b \dots \gamma_1$ and $b_2 = e$. Now we can estimate

$$\begin{aligned}
&d_{S^n,n,\Lambda}\big((\Gamma(h, t_b, \gamma_b, \dots, \gamma_1, x), g), (\Gamma(e, t_b, \gamma_b, \dots, \gamma_1, x), gh)\big)\\
&\quad \leq 2 + \Lambda \cdot d_X(\Gamma(h, t_b, \gamma_b, \dots, \gamma_1, x), \Gamma(h, t_b, \gamma_b, \dots, \gamma_1, x))\\
&\qquad\quad + \Lambda \cdot d_X(x, x) + \Lambda \cdot d_X(\Gamma(e, t_b, \gamma_b, \dots, \gamma_1, x), \Gamma(e, t_b, \gamma_b, \dots, \gamma_1, x))\\
&\quad = 2.
\end{aligned}$$

□

Corollary 7.9 *Assume $\alpha \leq n$ and $d+1 < n$. Then the functor $\mathcal{S}^{\alpha,d}_{X,Y}\colon (\diamond_{+} Y)^{\mathrm{op}} \to \mathrm{CW}$ is well defined.*

Proof Since $M\Gamma$ carries an honest G–action, functoriality will be clear as soon as we have convinced ourselves that $\mathcal{S}^{\alpha,d}_{X,Y}$ is well defined on morphisms. Let $c' \to c$ be a morphism in $(\diamond\, Y)^{\mathrm{op}}$ and $\sigma \in S^{\delta_{|c'|}}_{\bullet}(M\Gamma^{l_{|c'|},\alpha})$. We only need to check nonidentity morphisms. Hence, we assume $|c'| \geq |c| + 1$. Let $[e, t_b, \gamma_b, \dots, \gamma_1, x]$ be a point in the image of σ. By definition, we have $b \leq l_{|c'|}$ and $\gamma_i \in B_\alpha(e)$ for all i. Set

$\gamma_{c,c'} := \kappa_G(c)^{-1}\kappa_G(c')$. Note that $\gamma_{c,c'} \in B_\alpha(e)$ since Y is α–controlled over G. Then we obtain

$$\begin{aligned}\gamma_{c,c'} \cdot [e, t_b, \gamma_b, \dots, \gamma_1, x] &= [\gamma_{c,c'}, t_b, \gamma_b, \dots, \gamma_1, x] \\ &= [e, 1, \gamma_{c,c'}, t_b, \gamma_b, \dots, \gamma_1, x] \in M\Gamma^{l_{|c'|+1},\alpha} \subseteq M\Gamma^{l_{|c|},\alpha}.\end{aligned}$$

Hence, $\gamma_{c,c'} \cdot \sigma$ is a singular simplex in $M\Gamma^{l_{|c|},\alpha}$.

Let $[e, t'_{b'}, \gamma'_{b'}, \dots, \gamma'_1, x']$ be another point in the image of σ. Then

$$R(\gamma_{c,c'} \cdot [e, t_b, \gamma_b, \dots, \gamma_1, x]) = \Gamma(\gamma_{c,c'}, t_b, \gamma_b, \dots, \gamma_1, x),$$

and similarly for $[e, t'_{b'}, \gamma'_{b'}, \dots, \gamma'_1, x']$. We calculate

$$\begin{aligned}
& d_{S^n,n,\Lambda}\big((\Gamma(\gamma_{c,c'}, t_b, \gamma_b, \dots, \gamma_1, x), \kappa_G(c)), (\Gamma(\gamma_{c,c'}, t'_{b'}, \gamma'_{b'}, \dots, \gamma'_1, x'), \kappa_G(c))\big) \\
&\quad \le d_{S^n,n,\Lambda}\big((\Gamma(\gamma_{c,c'}, t_b, \gamma_b, \dots, \gamma_1, x), \kappa_G(c)), (\Gamma(e, t_b, \gamma_b, \dots, \gamma_1, x), \kappa_G(c'))\big) \\
&\qquad + d_{S^n,n,\Lambda}\big((\Gamma(e, t_b, \gamma_b, \dots, \gamma_1, x), \kappa_G(c')), (\Gamma(e, t'_{b'}, \gamma'_{b'}, \dots, \gamma'_1, x'), \kappa_G(c'))\big) \\
&\qquad + d_{S^n,n,\Lambda}\big((\Gamma(e, t'_{b'}, \gamma'_{b'}, \dots, \gamma'_1, x'), \kappa_G(c')), (\Gamma(\gamma_{c,c'}, t'_{b'}, \gamma'_{b'}, \dots, \gamma'_1, x'), \kappa_G(c))\big) \\
&\quad \le 2 + \delta_{|c'|} + 2 \\
&\quad = 4(d + 1 - (|c'| - 1)) \\
&\quad \le 4(d + 1 - |c|) = \delta_{|c|},
\end{aligned}$$

where we used Lemma 7.8 for the second inequality. This shows that multiplication by $\gamma_{c,c'}$ indeed defines a map

$$\gamma_{c,c'} \cdot - \colon S_\bullet^{\delta_{|c'|}}(M\Gamma^{l_{|c'|},\alpha}) \to S_\bullet^{\delta_{|c|}}(M\Gamma^{l_{|c|},\alpha}).$$

So the functor $\mathcal{S}^{\alpha,d}_{X,Y}$ is well defined. □

7.5 The transfer on objects

Recall that by our assumptions in Section 7.3 we have $\alpha < n$ and $d + 1 < n$.

Definition 7.10 The *transfer* $\mathrm{trans}^{\alpha,d}_X(Y)$ *of* Y *with respect to* X is defined to be

$$\mathrm{trans}^{\alpha,d}_X(Y) := \mathcal{S}^{\alpha,d}_{X,Y} \curlywedge_{\diamond_+ Y} \mathcal{C}_Y.$$

If α, d or both of them are understood, we abbreviate $\mathrm{trans}^{\alpha,d}_X(Y)$ to $\mathrm{trans}^{d}_X(Y)$, $\mathrm{trans}^{\alpha}_X(Y)$ or $\mathrm{trans}_X(Y)$, respectively.

Since $\mathcal{C}_Y$ is a free $\diamond_+ Y$–CW–complex, the space $\mathrm{trans}_X(Y)$ is a CW–complex relative W by Proposition 7.6. The natural transformation $\mathcal{S}^{\alpha,d}_{X,Y} \to *$ to the constant functor with value the one-point space induces a map $\mathcal{S}^{\alpha,d}_{X,Y} \curlywedge_{\diamond_+ Y} \mathcal{C}_Y \to * \curlywedge_{\diamond_+ Y} \mathcal{C}_Y \cong Y$ of CW–complexes relative W. We regard $\mathcal{S}^{\alpha,d}_{X,Y} \curlywedge_{\diamond_+ Y} \mathcal{C}_Y$ as a retractive space via this map.

We equip $\mathrm{trans}_X(Y)$ with a G–action as follows. Observe that G acts on the indexing category $\diamond_+ Y$; let $\mu_g\colon \diamond_+ Y \to \diamond_+ Y$ denote the functor induced by the action of $g \in G$. The action of g on Y induces a natural isomorphism $\mathcal{C}_Y \xrightarrow{\cong} \mathcal{C}_Y \circ \mu_g$, and hence a cellular homeomorphism

$$\tau_g\colon \mathcal{S}^{\alpha,d}_{X,Y} \curlywedge_{\diamond_+ Y} \mathcal{C}_Y \xrightarrow{\cong} \mathcal{S}^{\alpha,d}_{X,Y} \curlywedge_{\diamond_+ Y} (\mathcal{C}_Y \circ \mu_g).$$

Observing that $\mathcal{S}^{\alpha,d}_{X,Y} \circ \mu_g = \mathcal{S}^{\alpha,d}_{X,Y}$, we obtain from the functoriality of $\curlywedge_{\diamond_+ Y}$ in Section 7.2 a cellular homeomorphism

$$\iota_{\mu_g}\colon \mathcal{S}^{\alpha,d}_{X,Y} \curlywedge_{\diamond_+ Y} (\mathcal{C}_Y \circ \mu_g) = (\mathcal{S}^{\alpha,d}_{X,Y} \circ \mu_g) \curlywedge_{\diamond_+ Y} (\mathcal{C}_Y \circ \mu_g) \xrightarrow{\cong} \mathcal{S}^{\alpha,d}_{X,Y} \curlywedge_{\diamond_+ Y} \mathcal{C}_Y.$$

Define the action map of $g \in G$ as the composition

$$g \cdot - := \iota_{\mu_g} \circ \tau_g\colon \mathrm{trans}_X(Y) \xrightarrow{\cong} \mathrm{trans}_X(Y).$$

Explicitly, this map is given by $g \cdot [c, x, y] \mapsto [gc, x, gy]$, and defines a group action by cellular homeomorphisms.

Again by Proposition 7.6, we have a canonical identification

$$\diamond\, \mathrm{trans}_X(Y) \cong \{(\sigma, c) \mid c \in \diamond\, Y,\ \sigma \in S^{\delta_{|c|}}_\bullet(M\Gamma^{l_{|c|},\alpha})\},$$

which translates the G–action on the set of cells of $\mathrm{trans}_X(Y)$ to $g \cdot (\sigma, c) = (\sigma, gc)$. Hence, $\mathrm{trans}_X(Y)$ is a free G–CW–complex.

Continuing to use the above identification of $\diamond\, \mathrm{trans}_X(Y)$, we define a control map for $\mathrm{trans}_X(Y)$: Let β_p denote the barycenter of the standard p–simplex. Then set

$$\mathrm{trans}_X(\kappa)\colon \diamond\, \mathrm{trans}_X(Y) \to X \times G \times E \times [1, \infty[, \quad (\sigma, c) \mapsto ((R \circ \sigma)(\beta_{|\sigma|}), \kappa(c)).$$

Lemma 7.11 *The pair* $(\mathrm{trans}_X(Y), \mathrm{trans}_X(\kappa))$ *is an object in* $\mathcal{R}^G(W, \mathbb{J}(X \times G, E))$ *which is* $(\alpha+\delta_0+2)$*–controlled over* $X \times G$.

Proof By construction, the labeled G–CW–complex $(\mathrm{trans}_X(Y), \mathrm{trans}_X(\kappa))$ satisfies the G–continuous control condition. It also has the correct support, since X is compact.

So it is only necessary to check that it satisfies bounded control over $X \times G$. Let (σ, c) and (σ', c') be cells such that $(\sigma, c) \subseteq \langle(\sigma', c')\rangle$. By Proposition 7.6, this is equivalent to the conditions $c \subseteq \langle c'\rangle$ and $\sigma \subseteq \langle \kappa_G(c)^{-1}\kappa_G(c')\sigma'\rangle \subseteq \|S_\bullet^{\delta_{|c|}}(M\Gamma^{l_{|c|},\alpha})\|$. Set $\gamma_{c,c'} := \kappa_G(c)^{-1}\kappa_G(c')$. Then we have

$$\begin{aligned}
&d_{S^n,n,\Lambda}((\mathrm{trans}_X(\kappa)(\sigma, c), \mathrm{trans}_X(\kappa)(\sigma', c'))\\
&\quad = d_{S^n,n,\Lambda}\big(((R\circ\sigma)(\beta_{|\sigma|}), \kappa_G(c)), ((R\circ\sigma')(\beta_{|\sigma'|}), \kappa_G(c'))\big)\\
&\quad \le d_{S^n,n,\Lambda}\big(((R\circ\sigma)(\beta_{|\sigma|}), \kappa_G(c)), ((R\circ\gamma_{c,c'}\sigma')(\beta_{|\sigma'|}), \kappa_G(c))\big)\\
&\qquad\qquad + d_{S^n,n,\Lambda}\big(((R\circ\gamma_{c,c'}\sigma')(\beta_{|\sigma'|}), \kappa_G(c)), ((R\circ\sigma')(\beta_{|\sigma'|}), \kappa_G(c'))\big)\\
&\quad \le \delta_{|c|} + 2 \le \delta_0 + 2,
\end{aligned}$$

where the last inequality follows from our assumption and Lemma 7.8. Using

$$d_G(\kappa_G(c), \kappa_G(c')) \le \alpha,$$

we conclude that $(\mathrm{trans}_X(Y), \mathrm{trans}_X(\kappa))$ is $(\alpha+\delta_0+2)$–controlled over $X \times G$. $\square$

7.6 The transfer on cellwise 0–controlled morphisms

In the next step, we extend the assignment $(Y, \kappa) \mapsto (\mathrm{trans}_X(Y), \mathrm{trans}_X(\kappa))$ to a functor $\mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))_{\alpha,d} \to \mathcal{R}^G(W, \mathbb{J}(X\times G, E))$. Let $f\colon (Y_1, \kappa_1) \to (Y_2, \kappa_2)$ be a morphism in $\mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))_{\alpha,d}$. Since f is a regular map, we have an induced functor $\diamond_+ f\colon \diamond_+ Y_1 \to \diamond_+ Y_2$, which is compatible with the G–actions. (G acts trivially on the initial object.) Define a natural transformation $\mathcal{C}_{\mathrm{f}}\colon \mathcal{C}_{Y_1} \to \mathcal{C}_{Y_2} \circ \diamond_+ f$ by

$$\mathcal{C}_{f,c}\colon \mathcal{C}_{Y_1}(c) = \langle c\rangle \xrightarrow{f} f(\langle c\rangle) = \langle f(c)\rangle = (\mathcal{C}_{Y_2}\circ\diamond_+ f)(c).$$

Define a natural transformation $\mathcal{S}_f\colon \mathcal{S}^{\alpha,d}_{X,Y_1} \to \mathcal{S}^{\alpha,d}_{X,Y_2}\circ\diamond_+ f$ by

$$\mathcal{S}_{f,c}\colon \|S^{\delta_{|c|}}(M\Gamma^{l_{|c|},\alpha})\| \subseteq \|S_\bullet^{\delta_{|f(c)|}}(M\Gamma^{l_{|f(c)|},\alpha})\|.$$

Then $\mathrm{trans}_X(f)$ is defined as the composition

$$\begin{aligned}
\mathcal{S}^{\alpha,d}_{X,Y_1} \curlywedge_{\diamond_+ Y_1} \mathcal{C}_{Y_1} &\xrightarrow{\mathcal{S}_f \curlywedge_{\diamond_+ Y_1} \mathcal{C}_{\mathrm{f}}} (\mathcal{S}^{\alpha,d}_{X,Y_2}\circ(\diamond_+ f)) \curlywedge_{\diamond_+ Y_1} \mathcal{C}_{Y_2}\circ(\diamond_+ f))\\
&\xrightarrow{\iota(\diamond_+ f)} \mathcal{S}^{\alpha,d}_{X,Y_2} \curlywedge_{\diamond_+ Y_2} \mathcal{C}_{Y_2}.
\end{aligned}$$

Lemma 7.12 *This defines a functor*

$$\mathrm{trans}_X\colon \mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))_{\alpha,d} \to \mathcal{R}^G(W, \mathbb{J}(X\times G, E))_0.$$

Proof To see that the construction of $\mathrm{trans}_X(f)$ is functorial, it is best to translate the above formalism again into an explicit mapping rule. Concretely, $\mathrm{trans}_X(f)$ is given by $[c,x,y] \mapsto [f(c),x,f(y)]$, and functoriality becomes obvious.

We also need to check that $\mathrm{trans}_X(f)$ is a controlled map. It suffices to consider bounded control over $X \times G$. Note that $\mathrm{trans}_X(f)$ is regular as f and $\mathcal{S}_{f,c}$ are regular. Hence, it is enough to compute, for $(\sigma, c) \in \diamond\, \mathrm{trans}_X(Y_1)$,

$$\begin{aligned} d_{S^n,n,\Lambda}\big(&\mathrm{trans}_X(\kappa_1)(\sigma,c), \mathrm{trans}_X(\kappa_2)(\mathrm{trans}_X(f)(\sigma,c))\big) + d_G\big(\kappa_{1,G}(c), \kappa_{2,G}(f(c))\big) \\ &= d_{S^n,n,\Lambda}\big(((R\circ\sigma)(\beta_{|\sigma|}), \kappa_{1,G}(c)), \big((R\circ\sigma)(\beta_{|\sigma|}), \kappa_{2,G}(f(c))\big)\big) + 0 \\ &= 0. \end{aligned}$$

So $\mathrm{trans}_X(f)$ is in fact cellwise 0–controlled over $X \times G$. Hence, we have defined a functor $\mathrm{trans}_X \colon \mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))_{\alpha,d} \to \mathcal{R}^G(W, \mathbb{J}(X \times G, E))_0$. □

Remark 7.13 One can adapt the constructions presented in this paper to chain complexes over geometric modules to obtain a linear transfer. The linearization map, which assigns to a CW–complex its cellular chain complex, translates our transfer functor into its linear counterpart.

Moreover, the natural inclusion of geometric modules into chain complexes makes these constructions compatible with the transfers defined in [5; 6; 55]. Thus, the transfer for geometric $\mathbb{Z}[G]$–modules corresponds to our construction restricted to 0–dimensional CW–complexes.

7.7 Transferring cofibrations

Our next aim is to show that the transfer is a functor of categories with cofibrations. Since the cofibrations under consideration are essentially inclusions of CW–subcomplexes, it comes as no surprise that this result relies on an analysis of the CW–structure of $\mathrm{trans}_X(Y)$.

Let $(Y,\kappa) \in \mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))$ as before. It defines a $\diamond_+ Y$–CW–complex $\mathcal{C}_Y$ relative W. Let $B \subseteq Y$ be a subcomplex. We get a $\diamond_+ Y$–CW–complex $\mathcal{C}^B_Y$ relative W by setting $\mathcal{C}^B_Y(c) := \langle c \rangle \cap B$. As before, $\mathrm{colim}_{\diamond_+ Y}\, \mathcal{C}^B_Y \cong B$.

Let $A \subseteq B \subseteq Y$ be subcomplexes. Assume that B arises from A by attaching cells η_i for $i \in I$. From Section 7.2 we get a pushout diagram

$$\begin{tikzcd}
\coprod^W_{i\in I} \partial\eta_i \arrow[r] \arrow[d] & \mathcal{C}^A_Y \arrow[d] \\
\coprod^W_{i\in I} \eta_i \arrow[r] & \mathcal{C}^B_Y
\end{tikzcd}$$

in $\diamond_+ Y$–CW–complexes relative W. This becomes a pushout diagram in retractive spaces if we equip everything with the retractions into W arising from Y. Now, $\mathcal{S}^{\alpha,d}_{X,Y} \leftthreetimes_{\diamond_+ Y} (-)$ commutes with pushouts, so we get the following result:

Lemma 7.14 *There is a pushout diagram*

$$\begin{tikzcd}
\mathcal{S}^{\alpha,d}_{X,Y} \leftthreetimes_{\diamond_+ Y} \left(\coprod^W_{i\in I} \partial\eta_i\right) \arrow[r] \arrow[d] & \mathcal{S}^{\alpha,d}_{X,Y} \leftthreetimes_{\diamond_+ Y} \mathcal{C}^A_Y \arrow[d] \\
\mathcal{S}^{\alpha,d}_{X,Y} \leftthreetimes_{\diamond_+ Y} \left(\coprod^W_{i\in I} \eta_i\right) \arrow[r] & \mathcal{S}^{\alpha,d}_{X,Y} \leftthreetimes_{\diamond_+ Y} \mathcal{C}^B_Y
\end{tikzcd}$$

in $\mathcal{R}^G(W, \mathbb{J}(X \times G, E))$. Here the coproducts on the left are disjoint unions over W. As these are cells, the space on the lower left is isomorphic to

$$\coprod_{i\in I} (\| S_\bullet^{\delta^d_{|c_i|}} (M\Gamma^{l^d_{|c_i|},\alpha}) \| \times D^{|c_i|}) \amalg W,$$

when η_i is a cell based at c_i, and similarly for the upper left. □

This enables us to do inductive arguments over the cells in Y. Note that if $A \subseteq Y$ is a subcomplex, we can interpret A as a $\diamond_+ A$–CW–complex, or as $\diamond_+ Y$–CW–complex. We can define the transfer also for A as a $\diamond_+ Y$–CW–complex, and it is canonically isomorphic to Definition 7.10.

Lemma 7.15 *The functor* trans_X *preserves the zero object, cofibrations and admissible pushout diagrams, ie it is a functor of categories with cofibrations.*

Proof Let $f\colon (Y_1, \kappa_1) \rightarrowtail (Y_2, \kappa_2)$ be a cofibration in $\mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))_{\alpha,d}$. Without loss of generality, we can assume that Y_2 is obtained from Y_1 by attaching free G–cells, and that f is the inclusion of the subcomplex Y_1 into Y_2. Then it follows from Lemma 7.14 and interpreting Y_1 as a $\diamond_+ Y_2$–CW–complex that $\mathrm{trans}_X(f)$ is also a cofibration. For the same reason, trans_X preserves all relevant pushout squares. Last, it maps the zero object W to W. □

7.8 The transfer on general morphisms and weak equivalences

Next, we construct natural transformations between our transfer functors for various indices. Once we have shown that they are weak equivalences, it follows that the diagram in Remark 7.3 is homotopy-commutative. In addition, these enter the proof that $\mathrm{trans}^{\alpha,d}$ preserves weak equivalences.

Definition 7.16 Let $\alpha' > \alpha$ and $d' > d$ satisfying $n > \max\{d'+1, \alpha'\}$. Then both $\mathcal{S}^{\alpha,d}_{X,Y}$ and $\mathcal{S}^{\alpha',d'}_{X,Y}$ are defined and give rise to transfer functors $\mathrm{trans}^{\alpha,d}_X$ and $\mathrm{trans}^{\alpha',d'}_X$. For every $(Y,\kappa) \in \mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))_{\alpha,d}$, there is a natural transformation $\mathcal{S}^{\alpha,d}_{X,Y} \to \mathcal{S}^{\alpha',d'}_{X,Y}$ which is given at $c \in \diamond_+ Y$ by the obvious inclusion

$$\|S_\bullet^{\delta^d_{|c|}}(M\Gamma^{l^d_{|c|},\alpha})\| \subseteq \|S_\bullet^{\delta^{d'}_{|c|}}(M\Gamma^{l^{d'}_{|c|},\alpha'})\|.$$

Hence, we obtain an induced natural morphism

$$\rho^{\alpha,\alpha',d,d'}_Y\colon \mathrm{trans}^{\alpha,d}_X(Y) \to \mathrm{trans}^{\alpha',d'}_X(Y).$$

Lemma 7.17 *Let $\delta > 0$. Consider X as a subspace of $M\Gamma^{l,s}$ via the embedding $x \mapsto [e,x]$. There exists a δ–controlled strong deformation retraction*

$$H\colon \|S^\delta_\bullet(M\Gamma^{l,s})\| \times [0,1] \to \|S^\delta_\bullet(M\Gamma^{l,s})\|$$

onto $\|S^\delta_\bullet(X)\|$.

Proof There is a (topological) inclusion

$$i\colon \|S^\delta_\bullet(M\Gamma^{l,s})\| \times [0,1] \to \|S^\delta_\bullet(M\Gamma^{l,s} \times [0,1])\|$$

which maps each prism $\Delta^p \times [0,1]$ to its canonical triangulation. The strong deformation retraction from (7.7) restricts to a strong deformation retraction

$$H'\colon M\Gamma^{l,s} \times [0,1] \to M\Gamma^{l,s}$$

of $M\Gamma^{l,s}$ onto X, which is given by

$$H'([e, t_k, \gamma_k, \dots, \gamma_1, x], u) := [e, u\cdot t_k, \gamma_k, \dots, \gamma_1, x].$$

It has the property that $R \circ H'(m,u) = R(m)$, so

$$\mathrm{diam}_{X\times G}\{(R\circ H'(m,u), g) \mid u \in [0,1]\} = 0$$

for all $m \in M\Gamma^{l,s}$ and $g \in G$. Hence, H' induces a map

$$H'_*\colon \|S^\delta_\bullet(M\Gamma^{l,s} \times [0,1])\| \to \|S^\delta_\bullet(M\Gamma^{l,s})\|.$$

Then $H := H'_* \circ i$ is a strong deformation retraction onto $\|S^{\delta}_{\bullet}(X)\|$. As the target is δ–controlled, H is δ–controlled. □

Corollary 7.18 *Let $\delta > 0$, $l \le l'$ and $s \le s'$. Then the canonical inclusion map $\|S^{\delta}_{\bullet}(M\Gamma^{l,s})\| \hookrightarrow \|S^{\delta}_{\bullet}(M\Gamma^{l',s'})\|$ is a 2δ–controlled homotopy equivalence (with respect to $d_{S^n,n,\Lambda}$).*

Proof Lemma 7.17 yields homotopy equivalences $\|S^{\delta}_{\bullet}(X)\| \hookrightarrow \|S^{\delta}_{\bullet}(M\Gamma^{l,s})\|$ and $\|S^{\delta}_{\bullet}(X)\| \hookrightarrow \|S^{\delta}_{\bullet}(M\Gamma^{l',s'})\|$ which are δ–controlled. Since the triangle

$$\begin{tikzcd}
\|\mathcal{S}^{\delta}_{\bullet}(X)\| \arrow[r] \arrow[dr] & \|\mathcal{S}^{\delta}_{\bullet}(M\Gamma^{l,s})\| \arrow[d] \\
 & \|\mathcal{S}^{\delta}_{\bullet}(M\Gamma^{l',s'})\|
\end{tikzcd}$$

commutes, the result follows. □

Proposition 7.19 *The morphisms $\rho_Y^{\alpha,\alpha',d,d'}$ are weak equivalences.*

Proof We prove that $\rho_Y^{\alpha,\alpha',d,d'}$ is a weak equivalence for all $Y \in \mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))_{\alpha,d}$ by induction over the dimension of Y. For (-1)–dimensional objects, which is the start of the induction, the claim is trivial. For the induction step, we apply Lemma 7.14 to see that the inclusion of the p–skeleton into the $(p+1)$–skeleton induces a pushout

$$\begin{tikzcd}
\mathcal{S}^{\alpha,d}_{X,Y} \curlywedge_{\diamond_+ Y} \left(\coprod^W_{c\in\diamond_{p+1} Y} \partial c\right) \arrow[r] \arrow[d] & \mathcal{S}^{\alpha,d}_{X,Y} \curlywedge_{\diamond_+ Y} \mathrm{sk}_p(\mathcal{C}_Y) \arrow[d] \\
\mathcal{S}^{\alpha,d}_{X,Y} \curlywedge_{\diamond_+ Y} \left(\coprod^W_{c\in\diamond_{p+1} Y} c\right) \arrow[r] & \mathcal{S}^{\alpha,d}_{X,Y} \curlywedge_{\diamond_+ Y} \mathrm{sk}_{p+1}(\mathcal{C}_Y)
\end{tikzcd}$$

in $\mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(X \times G, E))_0$. The lower-left corner is, again by Lemma 7.14, identified as

$$\coprod_{c\in\diamond_{p+1} Y} (\|S^{\delta^d_{|c|}}_{\bullet}(M\Gamma^{l^d_{|c|},\alpha})\| \times D^{|c|}) \amalg W,$$

and similarly for the upper corner, with $D^{|c|}$ replaced with $\partial D^{|c|}$. There is an analogous pushout square with α and d replaced by α' and d', respectively. Moreover, the former square maps to the latter via the transformation $\rho_Y^{\alpha,\alpha',d,d'}$. On the left-hand side this is identified with the canonical inclusion maps. This transformation is a weak equivalence on the top-right corner of the diagram by the induction hypothesis, and it is a $2\delta_{p+1}$–controlled homotopy equivalence on the top-left and bottom-left corners combining

Corollary 7.18 and Lemma 7.21 below. Hence, the gluing lemma implies that it is also a weak equivalence on the bottom-right corner. This finishes the induction step and finite-dimensionality of Y proves the claim. □

In order to show exactness, we will need that the transfer maps h–equivalences to h^{fin}–equivalences later. The following proposition implies this.

Proposition 7.20 *Let f be a weak equivalence in $\mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))_{\alpha,d}$ which is an α'–controlled homotopy equivalence over G. Suppose that $n > \max\{d+2, \alpha, \alpha'\}$.*

Then $\mathrm{trans}_X(f)$ is a controlled homotopy equivalence.

Proof To show the proposition, we exploit the fact that maps which are not cellwise 0–controlled over G can also be transferred, but in a less functorial fashion. Let (Y_1, κ_1) and (Y_2, κ_2) be objects of $\mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))_{\alpha,d}$, and $f\colon Y_1 \to Y_2$ be an arbitrary map in $\mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))$. Choose $\alpha' > 0$ such that f is α'–controlled over G. We construct an induced map

$$\mathrm{trans}_{\alpha,\alpha'}(f)\colon \mathrm{trans}_X^{\alpha,d}(Y_1) \to \mathrm{trans}_X^{\max\{\alpha,\alpha'\},d+1}(Y_2).$$

To define $\mathrm{trans}_{\alpha,\alpha'}(f)$, consider first a single cell $c \in \diamond_+ Y_1$; denote by η_c the corresponding $\diamond_+ Y_1$–n–cell of $\mathcal{C}_{Y_2}$. We define the function

$$\begin{aligned} t_c\colon \|S_\bullet^{\delta_{|c|}}(M\Gamma^{l_{|c|},\alpha})\| \curlywedge_{\diamond_+ Y_1} \eta_c &\to \mathcal{S}^{\max\{\alpha,\alpha'\},d+1}_{X,Y_2} \curlywedge_{\diamond_+ Y_2} \mathcal{C}_{Y_2}, \\ (x,y) &\mapsto [\mathrm{supp}(f(y)), \gamma^c_y \cdot x, f(y)], \end{aligned}$$

where $\mathrm{supp}(f(y))$ denotes the support of $f(y)$, ie the unique open cell $\mathrm{supp}(f(y))$ of Y_2 such that $f(y) \in \mathrm{supp}(f(y))$, and $\gamma^c_y := \kappa_{2,G}(\mathrm{supp}(f(y)))^{-1}\kappa_{1,G}(c)$. We will glue the different t_c together to get the transfer for f.

Let us check that the target space is large enough that $t_c(x,y)$ is contained in it: Recall that $\gamma^c_y \cdot x$ is defined via the G–action which $\|S_\bullet(M\Gamma)\|$ inherits from $M\Gamma$. Since f is α'–controlled, we have $\gamma^c_y \in B_{\alpha'}(e)$. Therefore, we can regard multiplication with γ^c_y as a map $M\Gamma^{l_{|c|},\alpha} \to M\Gamma^{l_{|c|}+1,\max\{\alpha,\alpha'\}}$. In addition, $M\Gamma^{l_{|c|}+1,\max\{\alpha,\alpha'\}}$ is contained in $M\Gamma^{l_{|\mathrm{supp}(f(y))|}+1,\max\{\alpha,\alpha'\}}$, so $[\mathrm{supp}(f(y)), \gamma^c_y \cdot x, f(y)]$ defines a point in the target space.

We need to check that t_c is continuous. It suffices to show continuity on finite subcomplexes. These are metrizable, so it is enough to show that t_c is sequentially continuous. Let $(x_l, y_l)_l$ be a convergent sequence in $\|S_\bullet^{\delta_{|c|}}(M\Gamma^{l_{|c|},\alpha})\| \times \eta_c$ with limit point (x,y). As f is continuous, $f(y_l)$ converges against $f(y)$. Hence, $S :=$

$\{\mathrm{supp}(f(y_l)) \mid l \in \mathbb{N}\}$ is a finite set, and we can assume that for each $s \in S$ there are infinitely many l such that $\mathrm{supp}(f(y_l)) = s$. We treat the s individually and restrict to the corresponding subsequence. If s happens to be equal to $\mathrm{supp}(f(y))$, then $\gamma_y^c = \gamma_{y_l}^c$ and continuity follows. Otherwise, $f(y)$ must still lie in the closure of the cell s, ie $\mathrm{supp}(f(y)) \subseteq \langle s\rangle$. Hence,

$$\begin{aligned}
&[\mathrm{supp}(f(y)), \gamma_y^c \cdot x, f(y)] \\
&\qquad = [\mathrm{supp}(f(y)), \kappa_{2,G}(\mathrm{supp}(f(y)))^{-1}\kappa_{2,G}(s)\kappa_{2,G}(s)^{-1}\kappa_{1,G}(c)x, f(y)] \\
&\qquad = [s, \kappa_{2,G}(s)^{-1}\kappa_{1,G}(c)x, f(y)],
\end{aligned}$$

and continuity becomes obvious.

Suppose now that $c \le c'$ in $\diamond Y_1$. For $y \in \langle c\rangle$ and $x \in \|S_\bullet^{\delta_{|c'|}}(M\Gamma^{l_{|c'|},\alpha})\|$, we obtain

$$\begin{aligned}
t_{c'}(x, y) &= [\mathrm{supp}(f(y)), \gamma_y^{c'} x, f(y)] \\
&= [\mathrm{supp}(f(y)), \kappa_{2,G}(\mathrm{supp}(f(y)))^{-1}\kappa_{1,G}(c)\kappa_{1,G}(c)^{-1}\kappa_{1,G}(c')x, f(y)] \\
&= t_c(\kappa_{1,G}(c)^{-1}\kappa_{1,G}(c')x, y).
\end{aligned}$$

Therefore, the collection $\{t_c\}_{c\in\diamond Y_1}$ induces a continuous, cellular map relative W,

$$\mathrm{trans}_{\alpha,\alpha'}(f)\colon \mathrm{trans}_X^{\alpha,d}(Y_1) \to \mathrm{trans}_X^{\max\{\alpha,\alpha'\},d+1}(Y_2).$$

Using Lemmas 7.8 and 7.11, it is not hard to show that the map $\mathrm{trans}_{\alpha,\alpha'}(f)$ is $(\max\{\alpha,\alpha'\}+\alpha'+\delta_0+4)$–controlled over $X \times G$, as $n > \alpha'$. Note that for cellwise 0–controlled maps, $\mathrm{trans}_{\alpha,\alpha'}(f)$ agrees with the previous defined transfer from Section 7.6. The only reason we increased d is the argument which follows; it was not needed in the construction so far.

Suppose now that $f\colon (Y_1,\kappa_1) \to (Y_2,\kappa_2)$ is a weak equivalence in $\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E))_{\alpha,d}$ which is α'–controlled over G as a homotopy equivalence, ie its inverses and the homotopies are α'–controlled over G. Then there exists some α'–controlled map $\bar f\colon (Y_2,\kappa_2) \to (Y_1,\kappa_1)$ such that $\bar f f$ and $f\bar f$ are α'–controlled homotopic to the identity. Consider the diagram

$$\begin{tikzcd}
\mathrm{trans}_X^{\alpha,d}(Y_1) \arrow[r, "\mathrm{trans}_X^{\alpha,d}(f)"] \arrow[d, tail, "\rho_{Y_1}^{\alpha,\max\{\alpha,\alpha'\},d,d+1}"'] & \mathrm{trans}_X^{\alpha,d}(Y_2) \arrow[d, tail, "\rho_{Y_2}^{\alpha,\max\{\alpha,\alpha'\},d,d+1}"] \arrow[dl, dashed, "\mathrm{trans}_{\alpha,\alpha'}(\bar f)"'] \\
\mathrm{trans}_X^{\max\{\alpha,\alpha'\},d+1}(Y_1) \arrow[r, "\mathrm{trans}_X^{\max\{\alpha,\alpha'\},d+1}(f)"'] & \mathrm{trans}_X^{\max\{\alpha,\alpha'\},d+1}(Y_2)
\end{tikzcd}$$

in which the outer square commutes. The vertical maps $\rho_{Y_i}^{\alpha,\max\{\alpha,\alpha'\},d,d+1}$ for $i = 1, 2$ are weak equivalences by Proposition 7.19. We claim that the two triangles involving the dashed diagonal map $\mathrm{trans}_{\alpha,\alpha'}(\bar{f})$ commute up to controlled homotopy. If this is true, it follows that $\mathrm{trans}_X^{\alpha,d}(f)$ is a weak equivalence.

Let $h\colon Y_1 \curlywedge [0,1] \to Y_1$ be an α'–controlled homotopy from $\bar{f} f$ to id_{Y_1}. Note that $\mathrm{trans}_{\alpha,\alpha'}(\bar{f}) \circ \mathrm{trans}_X^{\alpha,d}(f) = \mathrm{trans}_{\alpha,\alpha'}(\bar{f} f)$. Since $n > d+2$, we can apply $\mathrm{trans}_X^{\alpha,d}$ also to $Y_1 \curlywedge [0,1]$ and consider the controlled map

$$\mathrm{trans}_{\alpha,\alpha'}(h)\colon \mathrm{trans}_X^{\alpha,d}(Y_1 \curlywedge [0,1]) \to \mathrm{trans}_X^{\max\{\alpha,\alpha'\},d+1}(Y_1).$$

The domain of this map is not equal to $\mathrm{trans}_X^{\alpha,d}(Y_1) \curlywedge [0,1]$, but it is contained in $\mathrm{trans}_X^{\alpha,d}(Y_1) \curlywedge [0,1]$ as a controlled strong deformation retract. This follows by an induction argument similar to Proposition 7.19. Essentially, we can construct both objects as the balanced products over $\diamond_+(Y_1 \curlywedge [0,1])$ and use that the inclusion $\|S_\bullet^{\delta_{|c|}}(M\Gamma^{l_{|c|},\alpha})\| \to \|S_\bullet^{\delta_{|c|+1}}(M\Gamma^{l_{|c|+1},\alpha})\|$ is a controlled deformation retraction by Corollary 7.18 and Lemma 7.21. The retraction induces the required controlled homotopy.

The argument for the second triangle is analogous. $\square$

7.9 Restricting the target category

Now we show that the transfer functor factors over the full subcategory of finitely dominated objects. The following result was already used in the proof of Proposition 7.20.

Lemma 7.21 *Let (M, d) be a metric space.*

- (i) *Let $\delta > 0$. The natural inclusion map $\|S_\bullet^\delta(M)\| \to \|S_\bullet(M)\|$ is a homotopy equivalence.*
- (ii) *Let $0 < \delta \le \delta'$. Then the inclusion map $\|S_\bullet^\delta(M)\| \to \|S_\bullet^{\delta'}(M)\|$ is a δ'–controlled homotopy equivalence (with respect to the metric on M, labeling simplices by the image of their barycenter).*
- (iii) *Suppose $|K|$ is the realization of an ordered (abstract) simplicial complex K and suppose that $p\colon |K| \to M$ is a continuous map. Let $\kappa\colon \diamond K \to M$ be the labeling sending a cell (ie simplex) to the image of its barycenter under p. Let $\delta > 0$. Let $S_\bullet^\delta(|K|, p)$ denote the (semi)simplicial set of all singular simplices σ in $|K|$ such that the diameter of $p \circ \sigma$ is at most δ.*

If the characteristic maps of all simplices of K lie in $S^{\delta}_{\bullet}(|K|, p)$, then the canonical map $|K| \to \|S^{\delta}_{\bullet}(|K|, p)\|$ is a δ–controlled homotopy equivalence (measuring control in M via p).

Proof The proof proceeds in analogy to [6, Lemma 6.7]. The first part follows directly from an appropriate formulation of excision, eg [22, Theorem 4.6.9].

For the second part, let $\mathcal{A}$ be the poset of closed subsets of X, considered as a category. Then the $\mathcal{A}$–CW–complex $\mathcal{S}^{\delta}_{\mathcal{A}}$ given by

$$\mathcal{S}^{\delta}_{\mathcal{A}}(A) := \|S^{\delta}_{\bullet}(A)\|$$

is a free $\mathcal{A}$–CW–complex, whose free cells are of the form $\hom_{\mathcal{A}}(\sigma(\Delta^{|\sigma|}), -) \times D^{|\sigma|}$ since $\diamond\|S^{\delta}_{\bullet}(A)\| \cong \coprod_{\sigma \in S^{\delta}_{\bullet}(X)} \hom_{\mathcal{A}}(\sigma(\Delta^{|\sigma|}), A)$. Since both $\|S^{\delta}_{\bullet}(A)\| \hookrightarrow \|S_{\bullet}(A)\|$ and $\|S^{\delta'}_{\bullet}(A)\| \hookrightarrow \|S_{\bullet}(A)\|$ are homotopy equivalences for every $A \in \mathcal{A}$, so is the inclusion $\|S^{\delta}_{\bullet}(A)\| \hookrightarrow \|S^{\delta'}_{\bullet}(A)\|$. Hence, the natural transformation $\mathcal{S}^{\delta}_{\mathcal{A}} \to \mathcal{S}^{\delta'}_{\mathcal{A}}$ is a homotopy equivalence of (free) $\mathcal{A}$–CW–complexes by [13, Corollary 3.5]. This in particular means that there is an inverse map, compatible with the structure map, as well as compatible homotopies. It is easy to check that such a map has the right control.

The third claim follows by similar reasoning, substituting the poset of subcomplexes for the poset of closed subsets. □

Lemma 7.22 *Suppose that (X, d_X) admits a finite ε–domination. Then $\|S^{\delta}_{\bullet}(M\Gamma^{l,\alpha})\|$ is $4\delta + 6\Lambda\varepsilon$–dominated over $X \times G$ (with respect to the metric $d_{S^n,n,\Lambda}$).*

Proof By Lemma 7.17, the complex $\|S^{\delta}_{\bullet}(X)\|$ is a δ–controlled strong deformation retract of $\|S^{\delta}_{\bullet}(M\Gamma^{l,\alpha})\|$. Choose an appropriate controlled retraction r.

Pick an ε–domination of X by a finite simplicial complex $|K|$, ie a sequence of maps $X \xrightarrow{\iota} |K| \xrightarrow{\pi} X$ together with a homotopy $h\colon \pi \circ \iota \simeq \mathrm{id}_X$, and such that the diameter, measured with respect to the original metric d_X on X, of $h(x, [0, 1])$ is at most ε for every $x \in X$. Then the given domination induces maps

$$\|S^{\delta}_{\bullet}(X)\| \xrightarrow{\iota_*} \|S^{\delta+2\Lambda\varepsilon}_{\bullet}(|K|, \pi)\| \xrightarrow{\pi_*} \|S^{\delta+2\Lambda\varepsilon}_{\bullet}(X)\|,$$

where similarly to Section 7.4 we measure distances of points in $|K|$ via π. These are maps of labeled complexes over $X \times G$: Pick an arbitrary group element $g \in G$. Then we label simplices σ in $S^{\delta}_{\bullet}(X)$ or $S^{\delta+2\Lambda\varepsilon}_{\bullet}(X)$ by $(\sigma(\beta_{|\sigma|}), g)$ and simplices σ in $S^{\delta+2\Lambda\varepsilon}_{\bullet}(|K|, \pi)$ by $(\pi(\sigma(\beta_{|\sigma|})), g)$ (see Lemma 7.17 and Corollary 7.18).

Choose an iterated barycentric subdivision K' of K such that the characteristic map of each simplex of K' is a simplex in $S_\bullet^{\delta+2\Lambda\varepsilon}(|K|,\pi)$; note that K' is ordered if we subdivide at least once. Since $|K'|$ is then naturally a subcomplex of $S_\bullet^{\delta+2\Lambda\varepsilon}(|K|,\pi)$, we endow it with the induced control map. The canonical inclusion $i\colon |K'| \to \|S_\bullet^{\delta+2\Lambda\varepsilon}(|K|,\pi)\|$ is a $(\delta+2\Lambda\varepsilon)$–controlled homotopy equivalence over $X \times G$ by Lemma 7.21. Choose an appropriate controlled homotopy inverse p.

Finally, the inclusion $\|S_\bullet^{\delta}(X)\| \hookrightarrow \|S_\bullet^{\delta+2\Lambda\varepsilon}(X)\|$ is a $(\delta+2\Lambda\varepsilon)$–controlled homotopy equivalence by Lemma 7.21; let f be an appropriate controlled homotopy inverse.

Then

$$\|S_\bullet^{\delta}(M\Gamma^{l,\alpha})\| \xrightarrow{r} \|S_\bullet^{\delta}(X)\| \xrightarrow{\iota_*} \|S_\bullet^{\delta+2\Lambda\varepsilon}(|K|,\pi)\| \xrightarrow{p} |K'|$$

and

$$|K'| \xrightarrow{i} \|S_\bullet^{\delta+2\Lambda\varepsilon}(|K|,\pi)\| \xrightarrow{\pi_*} \|S_\bullet^{\delta+2\Lambda\varepsilon}(X)\| \xrightarrow{f} \|S_\bullet^{\delta}(X)\| \hookrightarrow \|S_\bullet^{\delta}(M\Gamma^{l,\alpha})\|$$

yield the desired domination of $\|S_\bullet^{\delta}(X)\|$; from the previous control estimates we see that there is a $(4\delta+6\Lambda\varepsilon)$–controlled homotopy between the composition of these two maps and the identity on $\|S_\bullet^{\delta}(M\Gamma^{l,\alpha})\|$. $\square$

Proposition 7.23 *Suppose that (X, d_X) admits a finite ε–domination for every ε. Let $(Y,\kappa) \in \mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))_{\alpha,d}$. Then $\mathrm{trans}_X(Y,\kappa)$ is controlled finitely dominated, ie it defines an object in $\mathcal{R}^G_{\mathrm{fd}}(W, \mathbb{J}(X \times G, E))$. We can choose the control estimate to be independent of the constants Λ and n from the metric.*

Proof We prove the claim by induction on the dimension of Y. The case of a (-1)–dimensional object is trivial and provides the start of the induction.

For the induction step, we use Lemma 7.14 to obtain a pushout square

$$\begin{tikzcd}
\coprod\limits_{c\in\diamond_{p+1}Y} (\|S_\bullet^{\delta^d_{p+1}}(M\Gamma^{l^d_{p+1},\alpha})\| \times \partial D^{|c|}) \amalg W \arrow[r] \arrow[d] & \mathcal{S}^{\alpha,d}_{X,Y} \curlywedge_{\diamond_+ Y} \mathrm{sk}_p(\mathcal{C}_Y) \arrow[d] \\
\coprod\limits_{c\in\diamond_{p+1}Y} (\|S_\bullet^{\delta^d_{p+1}}(M\Gamma^{l^d_{p+1},\alpha})\| \times D^{|c|}) \amalg W \arrow[r] & \mathcal{S}^{\alpha,d}_{X,Y} \curlywedge_{\diamond_+ Y} \mathrm{sk}_{p+1}(\mathcal{C}_Y)
\end{tikzcd}$$

in $\mathcal{R}^G(W, \mathbb{J}(X \times G, E))$.

By the induction hypothesis, the object at the top-right corner of this square is finitely dominated. Thus, we only need to find a controlled finite domination for

$\|S_\bullet^{\delta_{p+1}}(M\Gamma^{l_{p+1},\alpha})\|$, as Y itself is (locally) finite. By Lemma 7.22, such a domination indeed exists.

Note that the same bound works if we increase n, and we can choose ε to be $\frac{1}{\Lambda}$. Then the estimate of the metric does not depend on n and Λ, which finishes the proof. □

Finally, we show that, after forgetting the labeling in X, the transfer does not alter the homotopy type of a given object.

Proposition 7.24 *Let $P\colon \mathcal{R}^G_{\mathrm{fd}}(W, \mathbb{J}(X \times G, E)) \to \mathcal{R}^G_{\mathrm{fd}}(W, \mathbb{J}(E))$ denote the functor induced by the projection map $X \times G \to G$. Let $(Y, \kappa) \in \mathcal{R}^G_{\mathrm{f}}(W, \mathbb{J}(E))_{\alpha,d}$.*

Then there is an α–controlled natural weak equivalence

$$P(\mathrm{trans}_X(Y)) \xrightarrow{\sim} Y.$$

Proof The relevant map is induced by the projection map $M\Gamma \to *$. As in the proofs of Propositions 7.20 and 7.23, the claim follows by another induction along the skeleta of Y, using Lemmas 7.14 and 7.21 together with the fact that the projection map $\|S_\bullet(X)\| \to *$ is a homotopy equivalence. Since the bounded control is only over (G, d_G), it is not hard to check that the weak equivalence is α–controlled. □

7.10 The transfer map

We combine all of the results established so far to show Proposition 6.15(i).

Let $N \in \mathbb{N}$. Suppose that we have chosen, for every $n \in \mathbb{N}$,

(i) a compact, contractible metric space (X_n, d_{X_n}) such that for every $\varepsilon > 0$ there is an ε–controlled domination of X_n by an at most N–dimensional, finite simplicial complex;

(ii) a homotopy coherent G–action Γ_n on X_n;

(iii) a positive real number Λ_n.

We equip $X_n \times G$ with the metric $d_{S^n,n,\Lambda_n} + d_G$. As in Section 7.3, we set

$$\delta_k := 4(d + 1 - k), \quad l_k := d + 1 - k.$$

Proposition 7.25 *Let $\alpha, d \in \mathbb{N}$. The assignment*

$$(Y,\kappa) \mapsto \mathrm{trans}^{\alpha,d}(Y,\kappa) := (\mathrm{trans}^{\alpha,d}_{X_n}(Y,\kappa))_{n>\max\{d+1,\alpha\}},$$
$$f \mapsto (\mathrm{trans}^{\alpha,d}_{X_n}(f))_{n>\max\{d+1,\alpha\}},$$

defines an exact functor

$$\mathrm{trans}^{\alpha,d}\colon \big(\mathcal{R}^G_{\mathrm{f}}(W,\mathbb{J}(E)),h\big)_{\alpha,d} \to \big(\mathcal{R}^G_{\mathrm{fd}}\big(W,\mathbb{J}((X_n\times G)_n,E)\big),h^{\mathrm{fin}}\big).$$

Proof According to Lemmas 7.11, 7.12 and 7.15 the assignment yields a functor of categories with cofibrations

$$\mathrm{trans}^{\alpha,d}\colon \mathcal{R}^G_{\mathrm{f}}(W,\mathbb{J}(E))_{\alpha,d} \to \prod_{n\in\mathbb{N}} \mathcal{R}^G_{\mathrm{fd}}(W,\mathbb{J}(X_n\times G,E)).$$

We have to show that it factors over the subcategory $\mathcal{R}^G_{\mathrm{fd}}\big(W,\mathbb{J}((X_n\times G)_n,E)\big)$. This is the case if all objects and morphisms in the image of $\mathrm{trans}^{\alpha,d}$ are uniformly boundedly controlled over $X_n\times G$. Essentially, we have to see that all of the necessary control estimates are independent of $n\in\mathbb{N}$.

For this, recall that the map

$$\mathcal{R}^G_{\mathrm{fd}}\big(W,\mathbb{J}((X_n\times G)_n,E)\big) \to \prod_{n\in\mathbb{N}} \mathcal{R}^G_{\mathrm{fd}}(W,\mathbb{J}(X_n\times G,E))$$

works as follows. Essentially, an object in the source is a CW–complex relative W, where we have a partition of its cells into $\mathbb{N}$–many sets and no boundary and no map is allowed to hit a cell which is in a different set. Hence, we can write the object as the coproduct (over W), indexed by $\mathbb{N}$, of CW–complexes relative W. The collection of summands defines an element in the target. If the transfer satisfies a uniform metric control condition, it factors over this map. Hence, we need to check that the previous results of this section give uniform bounds for all n.

Since $\mathrm{trans}^{\alpha,d}_{X_n}(Y,\kappa)$ is $(\alpha+\delta_0+2)$–controlled over $X_n\times G$ for every n by Lemma 7.11 and $\mathrm{trans}^{\alpha,d}_{X_n}(f)$ is cellwise 0–controlled by Lemma 7.12, all objects and morphisms are uniformly bounded, as desired. Proposition 7.23 shows that each component $\mathrm{trans}^{\alpha,d}(Y,\kappa)$ is finitely dominated, but we need it uniformly. For this, note that the proof of Proposition 7.23 can actually be done with $(X_n)_n$ replacing X. Roughly, we would get an extra coproduct over $\mathbb{N}$ everywhere, and everything else would need to get an extra index, which is why Proposition 7.23 is not stated that way. However, the control estimations come from applications of the gluing lemma and an induction over the cells of Y. But the gluing lemma preserves the property of everything being uniformly controlled, and we start the induction with uniform control arising from f and the δ_k, so we can do the same induction.

Proposition 7.20 tells us that $\mathrm{trans}_{X_n}^{\alpha,d}$ sends h–equivalences to h^{fin}–equivalences since it applies for sufficiently large n. Again, the proof can be done for $(X_n)_n$ instead of X, and the control estimates come from an induction over the cells of Y and the gluing lemma, so they will be uniform. □

Proposition 7.26 *Let $\alpha, d \in \mathbb{N}$ and let*

$$i_{\alpha,d}\colon \mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E))_{\alpha,d} \hookrightarrow \mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E))_{\alpha+1,d+1}$$

be the obvious inclusion functor. Then there is a natural h^{fin}–equivalence

$$\mathrm{trans}^{\alpha,d} \xrightarrow{\simeq} \mathrm{trans}^{\alpha+1,d+1} \circ i_{\alpha,d}.$$

Proof There is a natural transformation $\mathrm{trans}^{\alpha,d} \to \mathrm{trans}^{\alpha+1,d+1} \circ i_{\alpha,d}$ given by the sequence $(\rho^{\alpha,\alpha+1,d,d+1})_{n>\max\{d+2,\alpha+1\}}$ from Definition 7.16. These are homotopy equivalences by Proposition 7.19, and the control estimates in the proof of Proposition 7.19 show that they are also uniformly boundedly controlled homotopy equivalences. □

To obtain the transfer map whose existence was claimed in Proposition 6.15, we proceed as follows. Let $k \in \mathbb{N}$. Consider the inclusion $j_k\colon \mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E))_{k,k} \hookrightarrow \mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E))_{k+1,k+1}$. By Proposition 7.26, there is a natural weak equivalence

$$\rho_k\colon \mathrm{trans}^{k,k} \xrightarrow{\simeq} \mathrm{trans}^{k+1,k+1} \circ j_k.$$

Hence, we obtain an induced homotopy

$$K(\mathrm{trans}^{k,k}) \simeq K(\mathrm{trans}^{k+1,k+1}) \circ K(j_k).$$

Thinking of $\mathrm{hocolim}_k\, K(\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E))_{k,k}, h)$ as the mapping telescope of

$$K\big(\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E))_{1,1}, h\big) \xrightarrow{K(j_1)} K\big(\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E))_{2,2}, h\big) \xrightarrow{K(j_2)} \cdots,$$

these homotopies serve to define a map

$$\mathrm{trans}\colon \operatorname*{hocolim}_k K\big(\mathcal{R}_{\mathrm{f}}^G(W, \mathbb{J}(E))_{k,k}, h\big) \to K\big(\mathcal{R}_{\mathrm{fd}}^G\big(W, \mathbb{J}((X_n \times G)_n, E)\big), h^{\mathrm{fin}}\big).$$

Proposition 7.27 *The map* trans *satisfies Proposition 6.15(i).*

Proof That trans is the required map and that the diagram commutes up to homotopy is immediate from Propositions 7.2 and 7.24, noting again that the latter proof can be done uniformly. □

References

[1] **A Bartels**, *On proofs of the Farrell–Jones conjecture*, from "Topology and geometric group theory" (M W Davis, J Fowler, J-F Lafont, I J Leary, editors), Springer Proc. Math. Stat. 184, Springer (2016) 1–31 MR

[2] **A Bartels**, **S Echterhoff**, **W Lück**, *Inheritance of isomorphism conjectures under colimits*, from "K–theory and noncommutative geometry" (G Cortiñas, J Cuntz, M Karoubi, R Nest, C A Weibel, editors), Eur. Math. Soc., Zürich (2008) 41–70 MR

[3] **A Bartels**, **F T Farrell**, **W Lück**, *The Farrell–Jones conjecture for cocompact lattices in virtually connected Lie groups*, J. Amer. Math. Soc. 27 (2014) 339–388 MR

[4] **A Bartels**, **T Farrell**, **L Jones**, **H Reich**, *On the isomorphism conjecture in algebraic K–theory*, Topology 43 (2004) 157–213 MR

[5] **A Bartels**, **W Lück**, *The Borel conjecture for hyperbolic and* $\mathrm{CAT}(0)$*–groups*, Ann. of Math. 175 (2012) 631–689 MR

[6] **A Bartels**, **W Lück**, **H Reich**, *The K–theoretic Farrell–Jones conjecture for hyperbolic groups*, Invent. Math. 172 (2008) 29–70 MR

[7] **A Bartels**, **W Lück**, **H Reich**, **H Rüping**, *K- and L–theory of group rings over* $\mathrm{GL}_n(\mathbf{Z})$, Publ. Math. Inst. Hautes Études Sci. 119 (2014) 97–125 MR

[8] **A Berglund**, **I Madsen**, *Homological stability of diffeomorphism groups*, Pure Appl. Math. Q. 9 (2013) 1–48 MR

[9] **A Berglund**, **I Madsen**, *Rational homotopy theory of automorphisms of manifolds*, preprint (2014) arXiv

[10] **A Borel**, *Cohomologie réelle stable de groupes S–arithmétiques classiques*, C. R. Acad. Sci. Paris Sér. A-B 274 (1972) A1700–A1702 MR

[11] **D Burghelea**, **R Lashof**, *The homotopy type of the space of diffeomorphisms, I*, Trans. Amer. Math. Soc. 196 (1974) 1–36 MR

[12] **D Burghelea**, **R Lashof**, *The homotopy type of the space of diffeomorphisms, II*, Trans. Amer. Math. Soc. 196 (1974) 37–50 MR

[13] **J F Davis**, **W Lück**, *Spaces over a category and assembly maps in isomorphism conjectures in K– and L–theory*, K–Theory 15 (1998) 201–252 MR

[14] **N-E Enkelmann**, *Pseudoisotopy and automorphism groups of manifolds*, PhD Thesis, in preparation, Rheinische Wilhelms-Universität Bonn

[15] **F T Farrell**, *The Borel conjecture*, from "Topology of high-dimensional manifolds, I" (F T Farrell, L Göttsche, W Lück, editors), ICTP Lect. Notes 9, Abdus Salam Int. Cent. Theoret. Phys., Trieste (2002) 225–298 MR

[16] **F T Farrell**, **W C Hsiang**, *On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds*, from "Algebraic and geometric topology, I" (R J Milgram, editor), Proc. Sympos. Pure Math. XXXII, Amer. Math. Soc., Providence, RI (1978) 325–337 MR

[17] **F T Farrell**, **L E Jones**, *Classical aspherical manifolds*, CBMS Regional Conference Series in Mathematics 75, Amer. Math. Soc., Providence, RI (1990) MR

[18] **F T Farrell**, **L E Jones**, *Computations of stable pseudoisotopy spaces for aspherical manifolds*, from "Algebraic topology" (S Jackowski, B Oliver, K Pawałowski, editors), Lecture Notes in Math. 1474, Springer (1991) 59–74 MR

[19] **F T Farrell**, **L E Jones**, *Rigidity in geometry and topology*, from "Proceedings of the International Congress of Mathematicians" (I Satake, editor), volume I, Math. Soc. Japan, Tokyo (1991) 653–663 MR

[20] **F T Farrell**, **L E Jones**, *Stable pseudoisotopy spaces of compact non-positively curved manifolds*, J. Differential Geom. 34 (1991) 769–834 MR

[21] **F T Farrell**, **L E Jones**, *Isomorphism conjectures in algebraic K–theory*, J. Amer. Math. Soc. 6 (1993) 249–297 MR

[22] **R Fritsch**, **R A Piccinini**, *Cellular structures in topology*, Cambridge Studies in Advanced Mathematics 19, Cambridge Univ. Press (1990) MR

[23] **S Galatius**, **O Randal-Williams**, *Detecting and realising characteristic classes of manifold bundles*, from "Algebraic topology: applications and new directions" (U Tillmann, S Galatius, D Sinha, editors), Contemp. Math. 620, Amer. Math. Soc., Providence, RI (2014) 99–110 MR

[24] **S Galatius**, **O Randal-Williams**, *Stable moduli spaces of high-dimensional manifolds*, Acta Math. 212 (2014) 257–377 MR

[25] **S Galatius**, **O Randal-Williams**, *Abelian quotients of mapping class groups of highly connected manifolds*, Math. Ann. 365 (2016) 857–879 MR

[26] **S Galatius**, **O Randal-Williams**, *Homological stability for moduli spaces of high dimensional manifolds, II*, Ann. of Math. 186 (2017) 127–204 MR

[27] **S Galatius**, **O Randal-Williams**, *Homological stability for moduli spaces of high dimensional manifolds, I*, J. Amer. Math. Soc. 31 (2018) 215–264 MR

[28] **J Grunewald**, **J R Klein**, **T Macko**, *Operations on the A-theoretic nil-terms*, J. Topol. 1 (2008) 317–341 MR

[29] **L Hesselholt**, *On the Whitehead spectrum of the circle*, from "Algebraic topology" (N A Baas, E M Friedlander, B Jahren, P A Østvær, editors), Abel Symp. 4, Springer (2009) 131–184 MR

[30] **T Hüttemann**, **J R Klein**, **W Vogell**, **F Waldhausen**, **B Williams**, *The "fundamental theorem" for the algebraic K–theory of spaces, I*, J. Pure Appl. Algebra 160 (2001) 21–52 MR

[31] **T Hüttemann**, **J R Klein**, **W Vogell**, **F Waldhausen**, **B Williams**, *The "fundamental theorem" for the algebraic K–theory of spaces, II: The canonical involution*, J. Pure Appl. Algebra 167 (2002) 53–82 MR

[32] **H Kammeyer**, **W Lück**, **H Rüping**, *The Farrell–Jones conjecture for arbitrary lattices in virtually connected Lie groups*, Geom. Topol. 20 (2016) 1275–1287 MR

[33] **D Kasprowski**, **M Ullmann**, **C Wegner**, **C Winges**, *The A–theoretic Farrell–Jones conjecture for virtually solvable groups*, Bull. Lond. Math. Soc. 50 (2018) 219–228

[34] **P Kühl**, *Isomorphismusvermutungen und 3–Mannigfaltigkeiten*, preprint (2009) arXiv

[35] **W Lück**, *The transfer maps induced in the algebraic K_0– and K_1–groups by a fibration, I*, Math. Scand. 59 (1986) 93–121 MR

[36] **W Lück**, *Chern characters for proper equivariant homology theories and applications to K- and L–theory*, J. Reine Angew. Math. 543 (2002) 193–234 MR

[37] **W Lück**, **H Reich**, *The Baum–Connes and the Farrell–Jones conjectures in K– and L–theory*, from "Handbook of K–theory" (E M Friedlander, D R Grayson, editors), volume I, Springer (2005) 703–842 MR

[38] **W Lück**, **W Steimle**, *Splitting the relative assembly map, nil-terms and involutions*, Ann. K–Theory 1 (2016) 339–377 MR

[39] **W Lück**, **M Weiermann**, *On the classifying space of the family of virtually cyclic subgroups*, Pure Appl. Math. Q. 8 (2012) 497–555 MR

[40] **I Madsen**, **M Weiss**, *The stable mapping class group and stable homotopy theory*, from "European Congress of Mathematics" (A Laptev, editor), Eur. Math. Soc., Zürich (2005) 283–307 MR

[41] **I Madsen**, **M Weiss**, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. 165 (2007) 843–941 MR

[42] **M Pieper**, *Assembly maps and pseudoisotopy functors*, PhD thesis, in preparation, Rheinische Wilhelms-Universität Bonn

[43] **F Quinn**, *Ends of maps, I*, Ann. of Math. 110 (1979) 275–331 MR

[44] **C P Rourke**, **B J Sanderson**, *Block bundles, I*, Ann. of Math. 87 (1968) 1–28 MR

[45] **S K Roushon**, *The Farrell–Jones isomorphism conjecture for 3–manifold groups*, J. K–Theory 1 (2008) 49–82 MR

[46] **H Rüping**, *The Farrell–Jones conjecture for S–arithmetic groups*, J. Topol. 9 (2016) 51–90 MR

[47] **M Ullmann**, **C Winges**, *On the Farrell–Jones conjecture for algebraic K–theory of spaces: the Farrell–Hsiang method*, preprint (2015) arXiv

[48] **W Vogell**, *The canonical involution on the algebraic K–theory of spaces*, from "Algebraic topology, Aarhus 1982" (I Madsen, B Oliver, editors), Lecture Notes in Math. 1051, Springer (1984) 156–172 MR

[49] **W Vogell**, *Algebraic K–theory of spaces, with bounded control*, Acta Math. 165 (1990) 161–187 MR

[50] **W Vogell**, *Boundedly controlled algebraic K–theory of spaces and its linear counterparts*, J. Pure Appl. Algebra 76 (1991) 193–224 MR

[51] **R M Vogt**, *Homotopy limits and colimits*, Math. Z. 134 (1973) 11–52 MR

[52] **F Waldhausen**, *Algebraic K–theory of topological spaces, I*, from "Algebraic and geometric topology, I" (R J Milgram, editor), Proc. Sympos. Pure Math. XXXII, Amer. Math. Soc., Providence, RI (1978) 35–60 MR

[53] **F Waldhausen**, *Algebraic K–theory of spaces*, from "Algebraic and geometric topology" (A Ranicki, N Levitt, F Quinn, editors), Lecture Notes in Math. 1126, Springer (1985) 318–419 MR

[54] **F Waldhausen**, **B Jahren**, **J Rognes**, *Spaces of PL manifolds and categories of simple maps*, Annals of Mathematics Studies 186, Princeton Univ. Press (2013) MR

[55] **C Wegner**, *The K–theoretic Farrell–Jones conjecture for* CAT(0)*–groups*, Proc. Amer. Math. Soc. 140 (2012) 779–793 MR

[56] **C Wegner**, *The Farrell–Jones conjecture for virtually solvable groups*, J. Topol. 8 (2015) 975–1016 MR

[57] **M Weiss**, **B Williams**, *Automorphisms of manifolds and algebraic K–theory, I*, K–Theory 1 (1988) 575–626 MR

[58] **M Weiss**, **B Williams**, *Assembly*, from "Novikov conjectures, index theorems and rigidity, II" (S C Ferry, A Ranicki, J Rosenberg, editors), London Math. Soc. Lecture Note Ser. 227, Cambridge Univ. Press (1995) 332–352 MR

[59] **M Weiss**, **B Williams**, *Automorphisms of manifolds*, from "Surveys on surgery theory, II" (S Cappell, A Ranicki, J Rosenberg, editors), Ann. of Math. Stud. 149, Princeton Univ. Press (2001) 165–220 MR

[60] **B Williams**, *Bivariant Riemann Roch theorems*, from "Geometry and topology" (K Grove, I H Madsen, E K Pedersen, editors), Contemp. Math. 258, Amer. Math. Soc., Providence, RI (2000) 377–393 MR

Mathematisches Institut, Rheinische Wilhelms-Universität Bonn
Bonn, Germany

Mathematisches Institut, Rheinische Wilhelms-Universität Bonn
Bonn, Germany

Mathematisches Institut, Rheinische Wilhelms-Universität Bonn
Bonn, Germany

Institut für Mathematik, FU Berlin
Berlin, Germany

Max-Planck-Institut für Mathematik
Bonn, Germany

nils.edvin@gmail.com, wolfgang.lueck@him.uni-bonn.de,
mpieper@uni-bonn.de, mark.ullmann.math@gmail.com,
winges@mpim-bonn.mpg.de

https://www.hcm.uni-bonn.de/people/profile/nils-edvin-enkelmann/,
https://www.him.uni-bonn.de/lueck/, https://www.hcm.uni-bonn.de/
people/profile/malte-pieper/, http://people.mpim-bonn.mpg.de/winges/

Proposed: Steve Ferry Received: 26 November 2016
Seconded: Martin Bridson, Benson Farb Accepted: 4 March 2018