

Generators for a complex hyperbolic braid group

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We give generators for a certain complex hyperbolic braid group. That is, we remove a hyperplane arrangement from complex hyperbolic \mathbb{B}^3 -space, take the quotient of the remaining space by a discrete group, and find generators for the orbifold fundamental group of the quotient space. These generators have the most natural form: loops corresponding to the hyperplanes which come nearest the basepoint. Our results support the conjecture that motivated this study, the “monstrous proposal”, which posits a relationship between this braid group and the monster finite simple group.

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1 Introduction

We are interested in finding generators and relations for complex hyperbolic braid groups. Complex hyperbolic space \mathbb{B}^n is a complex manifold, diffeomorphic to the unit ball in \mathbb{C}^n , and closely analogous to real hyperbolic space. Both are negatively curved manifolds with transitive isometry groups. For us the main conceptual difference is that real hyperbolic space gets cut into pieces by hyperplanes, while hyperplanes in \mathbb{B}^n have complex codimension 1, so their removal leaves a connected space. (In fact, \mathbb{B}^n has no totally geodesic submanifolds of real codimension 1, which causes great difficulty in constructing fundamental domains for discrete groups.)

Our braid groups, which we called braid-like in [5], arise by removing a locally finite arrangement of complex hyperplanes, quotienting the remaining space by the action of a discrete group, and then taking the fundamental group. This is like a well-known construction of the classical braid group (see Fox and Neuwirth [21]): remove the hyperplanes $x_i = x_j$ from \mathbb{C}^n , and quotient what's left by the symmetric group S_n . A similar construction, with S_n replaced by any other Coxeter group, yields the Artin groups; see Brieskorn [15] and van der Lek [24]. In a different direction, replacing S_n by a finite complex reflection group leads to the complex braid groups, whose presentations and key properties were worked out only recently; see Bessis [13]. One can generalize the classical braid group in both these ways simultaneously, and our paper studies a case of this double generalization.

Our main result, [Theorem 1.2](#), gives a natural set of generators for a particular complex hyperbolic braid group coming from \mathbb{B}^{13} . Finding enough relations that give a presentation of the fundamental group appears to be rather harder. We will explain why this example is interesting, but to set the scene we begin with our broader motivations. In a sense these are detours, since we do not develop them further in this paper.

The first motivation involves singularity theory. The Artin groups of types A_n , D_n and E_n appear in nature as the fundamental groups of the discriminant complements of the simple singularities, which are also called A_n , D_n and E_n . Under a technical assumption that often holds, this has the following consequence. Suppose we are given a family of complex varieties over a base variety B . Write $B_0 \subseteq B$ for the set of smooth fibers, and suppose the fiber over $b \in B$ has some simple singularities but no worse singularities. Then these singularities have types A_n , D_n and E_n for some choices of subscripts, and there is a neighborhood U of b such that $\pi_1(B_0 \cap U)$ is the direct product of the corresponding Artin groups. Informally: the A_n , D_n and E_n Artin groups appear “locally” in the fundamental groups of “most” families of algebraic varieties. For example, by considering families of Riemann surfaces, one automatically expects suitable elements of mapping class groups to satisfy the braid relation; an expectation fulfilled by Dehn twists around curves that meet just once (and transversely). See Libgober [25], Looijenga [29] and Allcock, Carlson and Toledo [6, (3.5)–(3.7); 7, Lemma 1.5 and Theorem 7.1] for examples and applications of these ideas; the needed technical assumption is that the family B should provide a simultaneous versal deformation of all the singularities.

Now, singularities well beyond the simple ones have been classified, and at the next level of complexity are the *affine* singularities \tilde{A}_n , \tilde{D}_n , \tilde{E}_n . From the notation one

naturally expects that the fundamental groups of their discriminant complements are the corresponding affine Artin groups. But they are not quite the same; see van der Lek [24]. And more-singular singularities have discriminant complements whose fundamental groups are even further from the Artin groups. For example, Lönne [26, Theorem 4.3] gives presentations in the case of the Brieskorn–Pham singularities $x_1^{d_1} + \cdots + x_n^{d_n} = 0$, where each d_i is at least 2. These presentations include Artin relations but also additional relations. But in many cases, for example in Looijenga [27] and [28, Section 10], and Laza [23], these fundamental groups are braid groups in our sense, with \mathbb{B}^n replaced by the symmetric space $\mathrm{SO}(2, n)/S(\mathrm{O}(2) \times \mathrm{O}(n))$. The reasoning we used for simple singularities generalizes to any singularity, so these new braid groups appear automatically when considering families of complex varieties. In particular, they appear in nature in a way that the infinite-type Artin groups do not. Brieskorn [16] has asked for presentations for more discriminant complements, and we hope our methods will contribute to this. We regard braid groups coming from hyperplane arrangements in $\mathbb{B}^n \simeq \mathrm{SU}(n, 1)/S(\mathrm{U}(n) \times \mathrm{U}(1))$ as an easier analogue of the $\mathrm{SO}(2, n)/S(\mathrm{O}(2) \times \mathrm{O}(n))$ case, hence a test-bed for our ideas.

Our second motivation is the braid groups of the finite complex reflection groups. The known presentations (see Bessis [13]) are obtained from Lefschetz pencils, and we suspect that “better” presentations might be obtained directly from the arrangement of the hyperplanes. For example, in the case of Artin groups, the standard basepoint lies in the interior of the Weyl chamber, and the standard generators are the following paths, called *meridians*. Each starts at the basepoint, moves directly toward a facet F of the chamber until close to it, then travels along a semicircle around the complexification of F , and then moves directly to the image of the basepoint under the reflection across F . Although this is not a loop, it becomes one after quotienting by the Weyl group. In this way the standard generators correspond to the mirrors nearest the basepoint. We hope that there are analogous good generating sets for the complex braid groups. The Weyl chamber is not available in this context, but there may still be natural basepoints (possibly the *Weyl vectors* of Basak [10]), and generators like those for Artin groups, coming from the mirrors nearest the basepoint.

Now we discuss the braid group that is the subject of this paper. It comes from a group $P\Gamma$ acting on \mathbb{B}^{13} with finite covolume, generated by triflections (complex reflections of order 3). The hyperplane arrangement \mathcal{H} is the union of the mirrors of the triflections. In a sense this is the hardest example available, because $n = 13$ is the highest dimension for which there is a known finite-covolume complex hyperbolic reflection group on \mathbb{B}^n .

Our main result, [Theorem 1.2](#), concerns a particular basepoint $\tau \in \mathbb{B}^{13}$ (defined in [Section 2.6](#)) and the meridians based there. Meridians are defined below, and more generally in [Section 3](#), but they are similar to the Artin group case. The main difference is that the circular-arc portion of the path is only one third of a circle, not half, because the complex reflections in $P\Gamma$ have order 3, not 2. [Theorem 1.2](#) says that the braid group $G_\tau = \pi_1^{\text{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, \tau)$ is generated by the meridians associated to the 26 mirrors nearest τ . The notation π_1^{orb} indicates the orbifold fundamental group, which is needed because $P\Gamma$ does not act freely on $\mathbb{B}^{13} - \mathcal{H}$. See [Section 2.9](#) for our conventions about orbifold fundamental groups.

The motivation to study this particular example is the first author's *monstrous proposal*, namely the following conjecture concerning the sporadic finite simple group M known as the monster; see Allcock [\[2\]](#) for background.

Conjecture 1.1 (monstrous proposal) *The quotient of G_τ by the subgroup N normally generated by the squares of the meridians is isomorphic to the **bimonster** $B = (M \times M) \rtimes \mathbb{Z}/2$, where M is the monster finite simple group and $\mathbb{Z}/2$ acts by exchanging the factors in the obvious way.*

A known presentation of B has 26 generators of order 2, corresponding to the thirteen points and thirteen lines of the projective plane $P^2\mathbb{F}_3$ over \mathbb{F}_3 . Two of these generators braid or commute according to whether the corresponding points/lines are incident in the usual sense of projective geometry. There is one additional relation, called *deflation*; see Conway and Simons [\[19\]](#). The amazing coincidence is that the 26 generators of G_τ may be indexed by the points and lines of $P^2\mathbb{F}_3$ in the same way, and they satisfy exactly the same commutation and braid relations; see Basak [\[11, Theorem 4.7\]](#). And although it has not yet been verified, there are good geometric grounds to expect that our 26 generators for G_τ also satisfy the deflation relation, or one equivalent to it modulo the (meridian)² = 1 relations. Checking this relation is difficult, because it amounts to finding a disk in the hyperplane complement, bounding a particular loop. If the deflation relation holds, then G_τ/N is a quotient of B , hence is isomorphic to B or $\mathbb{Z}/2$ or the trivial group. To prove [Conjecture 1.1](#) one would need to rule out the last two cases. So we regard [Theorem 1.2](#) as significant progress toward [Conjecture 1.1](#).

Of course, ultimately we would like to find relations for G_τ , not just generators. Then one could check whether the relations in G_τ hold in B . If they do then the conjecture holds. We remark that the natural surjection $G_\tau \rightarrow P\Gamma$ (see [Section 2.9](#)) can be

described by imposing the relations (meridian)³ = 1. This is essentially the same as the fact that killing the squares of the standard generators of the usual braid group yields the symmetric group. (The exponent 3 in the $P\Gamma$ case comes from the generating reflections having order 3 rather than 2.) We do not know yet how to find defining relators for G_τ or even $P\Gamma$. We also don't know whether $\mathbb{B}^{13} - \mathcal{H}$ has contractible universal cover, which would be analogous to the main results of Bessis [13] for finite complex reflection groups. (A local condition on \mathcal{H} from Allcock [4] would imply this, but we don't know how to check it.)

Now we develop just enough background to make precise the objects we have discussed. For additional background see Section 2, in particular Sections 2.4 and 2.6. We write \mathcal{E} for the ring $\mathbb{Z}[e^{2\pi i/3}]$ of Eisenstein integers. The central player in this paper is a particular hermitian \mathcal{E} -lattice L , namely the unique one that has signature (13, 1) and equals $\sqrt{-3}$ times its dual lattice. A concrete model for L appears in Section 2.4; here is a summary. It is described in terms of the projective plane over the field of order 3. An old-fashioned approach to $P^2\mathbb{F}_3$ is enough for this paper: it has 13 points and 13 lines, each point lies on 4 lines, each line contains 4 points, any two points lie on a unique line, and any two lines meet at a unique point. Often we speak of a point and line being *incident*, rather than the line containing the point, because of symmetry: one can exchange the set of points with the set of lines in a manner which preserves incidence. The projective linear group $\text{PGL}_3\mathbb{F}_3$, of order 5616, permutes the points among themselves and the lines among themselves, also preserving incidence.

We will build L as a lattice in \mathbb{C}^{14} . We choose the standard hermitian form of signature $- + \cdots +$, and index the last 13 coordinates by the points of $P^2\mathbb{F}_3$. In Section 2.6 we define the *point-roots* as the 13 vectors $(0; \sqrt{-3}, 0, \dots, 0)$, where the $\sqrt{-3}$ may appear in any of the last 13 positions. And for each of the 13 lines, we define the corresponding *line-root* as the vector $(1; (0 \text{ or } 1), \dots, (0 \text{ or } 1))$, where each of the last 13 entries is 1 or 0 according to whether the corresponding point of $P^2\mathbb{F}_3$ lies on the line or not. We can now define L : it is the \mathcal{E} -span of the point- and line-roots.

The complex ball \mathbb{B}^{13} means the set of complex lines of negative norm in \mathbb{C}^{14} . A root means a lattice vector of norm 3, for example a point- or line-root. If r is any root then we write R_r for the triflection in r , meaning the isometry of L which multiplies r by $e^{2\pi i/3}$ and fixes r^\perp pointwise. The mirror of this complex reflection means the fixed-point set in \mathbb{B}^{13} . The hyperplane arrangement \mathcal{H} is the union of the mirrors, and $P\Gamma$ is the subgroup of $\text{Aut } \mathbb{B}^{13}$ generated by the triflections. The braid group we will

study in this paper is $\pi_1^{\text{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma)$, but before stating our main result we must choose a basepoint.

The mirrors of the 13 point-roots meet orthogonally at a point of \mathbb{B}^{13} , and similarly for the 13 line-roots. The basepoint τ is the midpoint of the segment joining these two points, and the mirrors nearest τ are exactly these 26 mirrors; see [Lemma A.5](#). For any point- or line-root r , the corresponding meridian in $G_\tau = \pi_1^{\text{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, \tau)$ is represented by the following three-part path. Let p be the projection of τ to the mirror r^\perp , and let U be an open ball centered at p , small enough so that the only mirror it meets is r^\perp . Let d be a point of the geodesic $\overline{\tau p}$, lying in U and different from p . The first part of the meridian is the geodesic $\overline{\tau d}$. The second part is the circular arc having angle $\frac{2\pi}{3}$ from d to $R_r(d)$, centered at p and positively oriented in the complex line containing $\overline{\tau p}$. The third part of the meridian is the geodesic $\overline{R_r(d)R_r(\tau)}$. We call these elements of G_τ the point- and line-meridians. Now our main theorem has precise meaning:

Theorem 1.2 (main theorem) *The 13 point-meridians and the 13 line-meridians generate the orbifold fundamental group of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$, based at τ .*

Our starting point for the proof is Theorem 1.5 of our paper [\[5\]](#), which gives a specific infinite generating set for this orbifold fundamental group, but based “at” a certain cusp ρ of $P\Gamma$. This generating set consists of the meridians based “at” ρ and corresponding to the infinitely many mirrors which come closest to it. (See [Theorem 4.1](#) for a precise statement, and [Section 3](#) for what we mean by taking the basepoint at a cusp.)

Beyond this result, we will require only one fleeting reference to [\[5\]](#): the proof of [Lemma 5.2](#) uses Theorem 1.2 from that paper to give explicit generators for the braid group of a certain finite complex reflection group. That theorem gives such a result for any hyperplane arrangement and basepoint that satisfy a certain condition, which we will verify for this finite complex reflection group. (Sadly, that condition fails badly for \mathcal{H} and τ , which is why the additional work in this paper is necessary to prove [Theorem 1.2](#).) Other references to [\[5\]](#) are meant to be informative but are not dependencies.

Starting from Theorem 1.5 from [\[5\]](#), our first step is to reduce the infinite generating set to a finite subset. This occupies [Section 4](#), and the key argument concerns generators for the $P\Gamma$ -stabilizer of ρ . Our second step is to show that the meridians corresponding to the point- and line-roots, but based at ρ rather than τ , are also a generating set. This

is Section 5, and the method is to show that the subgroup of the orbifold fundamental group they generate contains all the generators from Section 4. Finally, in Section 6 we show that moving the basepoint from ρ to τ along the geodesic $\overline{\rho\tau}$ identifies the point- and line-meridians based at ρ with those based at τ , in the obvious way. This implies Theorem 1.2. At heart, all of our arguments involve concrete homotopies between various paths in $\mathbb{B}^{13} - \mathcal{H}$. Besides setting up our general definition of meridians, Section 3 contains several theorems saying that such homotopies exist, provided that certain totally geodesic triangles in \mathbb{B}^{13} miss \mathcal{H} . The paper rests on the verification of this property for a total of 10 triangles, in the appendices.

Part of this verification relies on computer calculation; we also used the computer to verify the paper's many hand calculations involving vectors in \mathbb{C}^{14} . These calculations are involved enough that a reader skimming over them might imagine that the main theorem is a numerical accident. In fact, behind most of these calculations lurk special properties of the Leech lattice, such as inequalities that are exactly what is needed to complete a proof. So we will emphasize these properties when they arise; they lend the calculations a certain sense of inevitability.

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2 Background, conventions, notation

2.1 Eisenstein lattices

Let $\omega = e^{2\pi i/3}$ and $\theta = \omega - \bar{\omega} = \sqrt{-3}$. Let \mathcal{E} be the ring $\mathbb{Z}[\omega]$ of Eisenstein integers. An *Eisenstein lattice* K means an hermitian \mathcal{E} -lattice, ie a free \mathcal{E} -module with an hermitian form $\langle \cdot | \cdot \rangle: K \times K \rightarrow \mathbb{Q}(\omega)$, linear in the first variable and antilinear in the second. We abbreviate $K \otimes_{\mathcal{E}} \mathbb{C}$ to $K \otimes \mathbb{C}$. If K is nondegenerate then its *dual lattice* is defined as $K^* = \{x \in K \otimes \mathbb{C} : \langle x | k \rangle \in \mathcal{E} \text{ for all } k \in K\}$. The norm v^2 of $v \in K$ means $\langle v | v \rangle$. If X is a subset of a lattice then we write X^\perp for the set of lattice vectors orthogonal to it. If x, y, \dots lie in an \mathcal{E} -lattice, then $\langle x, y, \dots \rangle$ means their \mathcal{E} -span.

In the appendices, starting with Lemma A.8, the Eisenstein integer $\psi = 1 - 3\bar{\omega}$ plays an important role.

2.2 Complex hyperbolic space

We call an Eisenstein lattice K *Lorentzian* if it has signature $(n, 1)$. In that case we let $\mathbb{B}(K) \subseteq P(K \otimes \mathbb{C})$ denote the set of complex lines of negative norm in $K \otimes \mathbb{C}$. Topologically it is a complex ball of dimension n . It has a natural Riemannian metric with negative sectional curvature called the Bergman metric, and is sometimes called *complex hyperbolic space*. For background on negative curvature geometry and complex hyperbolic space, see [14] and [22]. If it is clear what lattice we mean then we sometimes write \mathbb{B}^n in place of $\mathbb{B}(K)$. In particular, \mathbb{B}^{13} will always mean $\mathbb{B}(L)$ for the lattice L defined in Section 2.4 below. An inclusion of Lorentzian lattices induces an inclusion of their complex balls. If this inclusion has codimension 1 then we call the smaller ball a hyperplane.

Any vector v of negative norm in $K \otimes \mathbb{C}$ determines a point $\mathbb{C}v$ in $\mathbb{B}(K)$. Often we use the same symbol for the point and the vector. The distance between two points of $\mathbb{B}(K)$ is given by

$$(2-1) \quad d(v, w) = \cosh^{-1} \sqrt{\frac{|\langle v | w \rangle|^2}{v^2 w^2}},$$

where v and w are two negative-norm vectors of $K \otimes \mathbb{C}$. Similarly, if $v, s \in K \otimes \mathbb{C}$ have negative and positive norm respectively, then

$$(2-2) \quad d(v, \mathbb{B}(s^\perp)) = \sinh^{-1} \sqrt{-\frac{|\langle v | s \rangle|^2}{v^2 s^2}}.$$

When s is a root (see Section 2.5 below) we usually write s^\perp in place of $\mathbb{B}(s^\perp)$ when it is clear that we mean this rather than the orthogonal complement in the lattice. Formulas (2-1) and (2-2) differ from those in [22] by an unimportant factor of 2.

The boundary $\partial\mathbb{B}(K)$ of $\mathbb{B}(K)$ in $P(K \otimes \mathbb{C})$ is a real $(2n-1)$ -sphere. Its points are the projectivizations of null vectors, meaning nonzero vectors in $K \otimes \mathbb{C}$ of norm 0. Given such a vector ρ , we define a sort of distance-to- ρ function on $\mathbb{B}(K)$, called the *height*:

$$(2-3) \quad \text{ht}_\rho(v) = -\frac{|\langle v | \rho \rangle|^2}{v^2}.$$

This is invariant under scaling v , so it descends to a function on $\mathbb{B}(K)$. We will say that one point of $\mathbb{B}(K)$ is closer to ρ than another point is, if its value of ht_ρ is smaller. The *horosphere* centered at ρ , of height h , means the set of $v \in \mathbb{B}(K)$ with $\text{ht}_\rho(v) = h$. We define open and closed *horoballs* the same way, replacing $=$

by $<$ and \leq . Scaling ρ by $\lambda \in \mathbb{C}$ scales the function ht_ρ by $|\lambda|^2$. But for us there will be a canonical normalization, because we will always take ρ to be a primitive lattice vector. Under this condition, the only allowed scaling of ρ is by sixth roots of unity, which does not affect the height function. We call the point of $\partial\mathbb{B}(K)$ represented by a primitive lattice vector a *cuspl*, and for convenience we also call that vector itself a *cuspl*. We write $\overline{\mathbb{B}(K)}$ for the topological closure $\mathbb{B}(K) \cup \partial\mathbb{B}(K)$ of $\mathbb{B}(K)$ in $P(K \otimes \mathbb{C})$.

2.3 Geodesics and totally geodesic triangles

Continuing to take K as above, suppose $v, w \in K \otimes \mathbb{C}$ represent points of $\overline{\mathbb{B}(K)}$. Then the geodesic joining them may be described as follows. First we rescale one or both of v, w so that $\langle v | w \rangle \in (-\infty, 0]$; this inner product will be negative unless v and w both represent the same point of $\partial\mathbb{B}(K)$. After this rescaling, the geodesic is the image in projective space of the real line segment in $K \otimes \mathbb{C}$ joining the vectors. If we have in mind a third point of $\overline{\mathbb{B}(K)}$, represented by $x \in K \otimes \mathbb{C}$, then there may or may not be a totally geodesic surface in $\overline{\mathbb{B}(K)}$ whose closure contains all three points. If there is one then we write Δvwx for their convex hull. There are two cases in which we use this notation. One is when v, w, x all lie in some complex 2-space in $K \otimes \mathbb{C}$. Then they lie in (the closure of) some \mathbb{B}^1 . In this case we call Δvwx a complex triangle.

The other case occurs when it is possible to scale v, w, x so that their pairwise inner products lie in $(-\infty, 0]$. Suppose this has been accomplished. The image in $\mathbb{B}(K)$ of the set of negative-norm vectors in the real span of v, w, x is a totally geodesic copy of the real hyperbolic plane. In this case the projectivization of the convex hull of v, w, x in $K \otimes \mathbb{C}$ is the convex hull of their images in $\overline{\mathbb{B}(K)}$. In this case we call Δvwx a totally real triangle, and don't usually distinguish between the triangle in $K \otimes \mathbb{C}$ and its image in $\overline{\mathbb{B}(K)}$.

For us this situation occurs as follows. Let b be a point of $\overline{\mathbb{B}(K)}$, H be a hyperplane in $\mathbb{B}(K)$, p be the point of H closest to b , and q be another point of H . We claim that Δbpq is totally real. We use the same letters for vectors representing these points, and s for a positive-norm vector orthogonal to H . We first scale b so that its inner product with s is real, and then scale q so that its inner product with b lies in $(-\infty, 0]$. Then $p = b - \langle b | s \rangle s / s^2$ has inner product $b^2 - \langle b | s \rangle^2 / s^2 \leq 0$ with b , and by $s \perp q$ we have $\langle p | q \rangle = \langle b | q \rangle \leq 0$. After these scalings, the vectors b, p and q have norms and inner products in $(-\infty, 0]$, so the triangle is totally real.

2.4 The $P^2\mathbb{F}_3$ model of the lattice L

Now we set up an explicit model for the \mathcal{E} -lattice L that governs everything in this paper, called the $P^2\mathbb{F}_3$ model. It was implicit in [9, Equation 25 in proof of Proposition 6.1] and was defined explicitly in [3]. Most of our computations will be done in this coordinate system. See Section 2.7 for an alternative model of L , the Leech model.

We write elements of \mathbb{C}^{14} as vectors $x = (x_0; x_1, \dots, x_{13})$ and use the standard hermitian form of signature $(13, 1)$ on \mathbb{C}^{14} , namely

$$\langle x | y \rangle = -x_0\bar{y}_0 + x_1\bar{y}_1 + \dots + x_{13}\bar{y}_{13}.$$

We index the last 13 coordinates by the points of $P^2\mathbb{F}_3$. L consists of all vectors $x \in \mathcal{E}^{14}$ such that $x_0 \cong x_1 + \dots + x_{13} \pmod{\theta}$ and that (x_1, \dots, x_{13}) , modulo θ , is an element of the *line code*, meaning the 7-dimensional subspace of \mathbb{F}_3^{13} spanned by the characteristic functions of the lines of $P^2\mathbb{F}_3$. The elements of the line code appear in Tables 2 and 3 of [3]; the explicit list is not needed in this paper. We write Γ for $\text{Aut } L$. It contains the simple group $L_3(3) = \text{PGL}_3(\mathbb{F}_3)$, acting by permuting the points of $P^2\mathbb{F}_3$ in the obvious way. It also contains the group $3^{13} = (\mathbb{Z}/3)^{13}$, acting by multiplying the last 13 coordinates by cube roots of unity.

An important property of L is that it equals $\theta \cdot L^*$. The proof amounts to checking that all inner products are divisible by θ and that $\det L = -3^7$. The first part is easy, using the point- and line-roots from Section 2.6. The second part follows from the fact that $L/(\theta\mathcal{E})^{14} \cong \mathbb{F}_3^7$.

2.5 Roots, mirrors and the hyperplane arrangement \mathcal{H}

A *root* of L means a lattice vector of norm 3. What makes roots special is that their *triflections* preserve L . That is, supposing s is a root, we define R_s as the automorphism of $L \otimes \mathbb{C}$ that fixes s^\perp pointwise and multiplies s by the cube root of unity ω . A formula is

$$(2-4) \quad R_s: x \mapsto x - (1 - \omega) \frac{\langle x | s \rangle}{s^2} s.$$

One can show that this isometry of $L \otimes \mathbb{C}$ preserves L . (The key is that all inner products in L are divisible by θ .) It is called the ω -reflection in s . Replacing ω by $\bar{\omega}$ gives the $\bar{\omega}$ -reflection, which is also R_s^{-1} . Both of these isometries are complex

reflections of order 3, sometimes called triflections. Although we don't need it, we remark that Γ is generated by the triflections in the roots of L ; see [9] or [3].

As the fixed-point set of the reflection R_s , the hyperplane $\mathbb{B}(s^\perp) \subseteq \mathbb{B}(L)$ is called the *mirror* of s . Sometimes, when there is no chance of confusion, we shall write s^\perp instead of $\mathbb{B}(s^\perp)$. In this paper we use the word mirror exclusively for hyperplanes orthogonal to roots. The union of all mirrors is called \mathcal{H} . This hyperplane arrangement is central to the paper, since our goal is to study the orbifold fundamental group of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$.

2.6 Point-roots, line-roots, 13–points and 26–points

It is possible to number the points and lines of $P^2\mathbb{F}_3$ by the numbers $1, \dots, 13$, such that the j^{th} line is the set of points $\{j, j + 1, j + 3, j + 9\}$. Here the indices should be read modulo 13. We adopt this numbering for the following important roots of L . First, the 13 point-roots p_i are the vectors

$$p_1 = (0; \theta, 0, \dots, 0), \quad \dots, \quad p_{13} = (0; 0, \dots, 0, \theta).$$

And second, for $j = 1, \dots, 13$, the line-root l_j is the vector of the form

$$(1; (0 \text{ or } 1), \dots, (0 \text{ or } 1)),$$

with entries of 1 along the j^{th} line of $P^2\mathbb{F}_3$. Explicitly,

$$\begin{aligned} l_1 &= (1; 1101000001000), \\ l_2 &= (1; 0110100000100), \quad l_8 = (1; 0001000110100), \\ l_3 &= (1; 0011010000010), \quad l_9 = (1; 0000100011010), \\ l_4 &= (1; 0001101000001), \quad l_{10} = (1; 0000010001101), \\ l_5 &= (1; 1000110100000), \quad l_{11} = (1; 1000001000110), \\ l_6 &= (1; 0100011010000), \quad l_{12} = (1; 0100000100011), \\ l_7 &= (1; 0010001101000), \quad l_{13} = (1; 1010000010001). \end{aligned}$$

It is easy to see that the point- and line-roots span L . We speak of a point-root p_i and a line-root l_j as being *incident* when the corresponding point and line of $P^2\mathbb{F}_3$ are. In particular, whether p_i and l_j are incident depends on the geometry of $P^2\mathbb{F}_3$ rather than whether the corresponding hyperplanes meet in \mathbb{B}^{13} . In fact, these hyperplanes always meet, but how they meet depends on incidence: $\langle p_i | l_j \rangle = \theta$ or 0 according to whether or not p_i and l_j are incident. Also, distinct point-roots are orthogonal, as are distinct line-roots.

There is some obvious symmetry preserving this configuration of roots: $L_3(3)$ sends point-roots to point-roots and line-roots to line-roots. But there is additional symmetry. From a correlation of $P^2\mathbb{F}_3$, ie an incidence-preserving exchange of points with lines, one can construct an isometry of L that sends the point-roots to line-roots and the line-roots to negated point-roots. (We give such an isometry explicitly in the proof of Lemma A.13.) Together with scalars and $L_3(3)$, this generates a subgroup $(6 \times L_3(3)) \cdot 2$ of Γ , whose image in $P\Gamma$ is $L_3(3):2$. Here we are using ATLAS [18] notation: a group has *structure* $A.B$ if it has a normal subgroup isomorphic to a group A , with quotient isomorphic to a group B . If the extension splits then one can indicate this by writing $A:B$ instead. If it does not split then one can write $A \cdot B$. We are also using another piece of ATLAS notation: writing n to indicate a cyclic group of order n .

We define the point- and line-mirrors to be the mirrors of the point- and line-roots. The 13 point-mirrors are mutually orthogonal and intersect at a single point of \mathbb{B}^{13} , represented by

$$p_\infty = (\bar{\theta}; 0^{13}).$$

For lack of a better name we call it a 13–point to indicate the 13 mirrors passing through it. (It is easy to see that there are no mirrors through it except the point-mirrors.) We apply the same term to its Γ –translates, such as the intersection point of the 13 line-mirrors, represented by

$$l_\infty = (4; 1^{13}).$$

We summarize the inner product information about $p_\infty, p_1, \dots, p_{13}, l_\infty, l_1, \dots, l_{13}$ as follows. First, all have norm 3 except for p_∞ and l_∞ , which have norm -3 . Second, if $i \neq j$ then $p_i \perp p_j$ and $l_i \perp l_j$. Third, $\langle p_\infty | l_\infty \rangle = 4\theta$. Finally, if i and j are not both ∞ , then $\langle p_i | l_j \rangle$ is θ or 0 according to whether or not p_i and l_j are incident. Here we are regarding p_∞ as incident to every l_j , and l_∞ as incident to every p_i .

We define the basepoint τ used in Theorem 1.2 as the midpoint of the segment joining p_∞ and l_∞ . It is represented by the vector

$$\tau = l_\infty + ip_\infty = (4 + \sqrt{3}; 1^{13})$$

of norm $-6 - 8\sqrt{3}$, and is the unique fixed point of $L_3(3):2 \subseteq P\Gamma$. It is known that the mirrors closest to τ are exactly the 26 point- and line-mirrors. (See [9, Proposition 1.2], where τ was called $\bar{\rho}$, or Lemma A.5 in this paper.) Two consequences of this are that no mirrors pass through τ , and that $L_3(3):2$ is the full $P\Gamma$ –stabilizer of τ . For lack

of a better name, we call τ a 26–point to indicate these 26 nearest mirrors. We use the same language for its Γ –translates.

2.7 The Leech model of L

By the Leech lattice we mean what might more properly be called the complex Leech lattice Λ . It is a 12–dimensional positive-definite \mathcal{E} –lattice, described in detail in [31]. At the smallest scale at which it is integral as an \mathcal{E} –lattice, it has minimal norm 6 and satisfies $\Lambda = \theta \cdot \Lambda^*$ and $\det \Lambda = 3^6$. Also see [31] for a thorough study of $\text{Aut } \Lambda$, which is the universal central extension $6 \cdot \text{Suz}$ of Suzuki’s sporadic finite simple group Suz . At one point we will need the fact that $6 \cdot \text{Suz}$ acts transitively on the lattice vectors of norm 6 and those of norm 9 (see [31, page 155] and note that these are his “type 2” and “type 3” vectors). We postpone an explicit description of Λ until the second half of Appendix A, because we won’t need it until then.

The Leech model of L is useful when one has in mind a Leech cusp (see below). It is: $L \cong \Lambda \oplus \begin{pmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{pmatrix}$. This means that we write lattice vectors as $(x; y, z)$, where $x \in \Lambda$ and $y, z \in \mathcal{E}$, with the inner product given by

$$(2-5) \quad \langle (x; y, z) | (x'; y', z') \rangle = \langle x | x' \rangle + (y z) \begin{pmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{pmatrix} \begin{pmatrix} \bar{y}' \\ \bar{z}' \end{pmatrix}.$$

This model of L was introduced in [1], and proven to be isometric to the $P^2\mathbb{F}_3$ model in [9, Lemma 2.6]. We give an explicit isomorphism in Lemma A.10. In the Leech model, the Leech cusp $\rho = (0; 0, 1)$ is distinguished, and the mirrors nearest it can be conveniently parametrized as explained in Section 2.8.

2.8 Leech cusps and Leech roots

Earlier we declared that if $v \in L$ is a primitive null vector then we call it (or the point of $\partial\mathbb{B}^{13}$ it represents) a cusp. We refine this to *Leech cusp* in the special case that $v^\perp / \langle v \rangle$ is isometric to the Leech lattice. An example which will play a key role in this paper is the primitive null vector ρ defined as $(0; 0, 1)$ in the Leech model. Although we don’t need it, for background we remark that there are five Γ –orbits on primitive null vectors $v \in L$, corresponding to the five possibilities for the isometry class of $v^\perp / \langle v \rangle$, which are the *Eisenstein Niemeier lattices*. In particular, there is a unique Γ –orbit of Leech cusps. (See Lemma 2 and Theorem 4 of [3].)

In the $P^2\mathbb{F}_3$ model it is harder to find a Leech cusp, but an example is

$$\rho = (3\omega - 1; -1, \dots, -1).$$

(Proof sketch, following the proof of [3, Theorem 1]: $\rho^\perp/\langle\rho\rangle$ is an Eisenstein Niemeier lattice, and its isometry group obviously contains $L_3(3)$. By [3, Theorem 4] the Leech lattice is the only Eisenstein Niemeier lattice whose symmetry group contains $L_3(3)$.) We hope that the simultaneous use of ρ for $(0; 0, 1)$ in the Leech model and $(3\omega - 1; -1, \dots, -1)$ in the $P^2\mathbb{F}_3$ model is helpful rather than confusing: we think of the models as two ways to describe a single lattice. In Lemma A.10 we complete the identification of the two models by explicitly writing down the point- and line-roots in the Leech model.

The following description of vectors $s \in L \otimes \mathbb{C}$ not orthogonal to ρ is very useful: every such s can be written uniquely in the form

$$(2-6) \quad s = \left(\sigma; m, \frac{\theta}{\bar{m}} \left(\frac{\sigma^2 - N}{6} + \nu \right) \right),$$

where $\sigma \in \Lambda \otimes \mathbb{C}$, $m \in \mathbb{C} - \{0\}$, N is the norm s^2 , and ν is purely imaginary. Restricting the first component to Λ and the others to \mathcal{E} gives the elements of $L - \rho^\perp$. Further restricting N to 3 gives the roots of L , and finally restricting m to 1 gives the Leech roots (see Equation (2-8) below). We remark that scaling s by $\lambda \in \mathbb{C} - \{0\}$ scales σ and m by λ , and scales N and ν by $|\lambda|^2$.

The elaborate form of the last coordinate in (2-6) allows the following interpretation of $\langle s | s' \rangle$, where s is from (2-6) and similarly s' :

$$(2-7) \quad \langle s | s' \rangle = m\bar{m}' \left[\frac{1}{2} \left(\frac{N'}{|m'|^2} + \frac{N}{|m|^2} - \left(\frac{\sigma}{m} - \frac{\sigma'}{m'} \right)^2 \right) + \text{Im} \left\langle \frac{\sigma}{m} \middle| \frac{\sigma'}{m'} \right\rangle + 3 \left(\frac{\nu'}{|m'|^2} - \frac{\nu}{|m|^2} \right) \right].$$

The content of this is that $\langle s | s' \rangle$ is governed by the relative positions of σ/m and σ'/m' in $\Lambda \otimes \mathbb{C}$, with ν and ν' influencing only the imaginary part of the bracketed term. (*Caution:* we are using the convention that the imaginary part of a complex number is imaginary; for example $\text{Im } \theta$ is θ rather than $\sqrt{3}$.) One proves the formula by writing out $\langle s | s' \rangle$, completing the square and patiently rearranging.

Because the Leech lattice has no vectors of norm 3, a Leech cusp $\rho \in \partial\mathbb{B}^{13}$ lies in (the closures of) no mirrors. The mirrors which come closest to it are called first-shell mirrors, and the second- and third-shell mirrors are defined similarly. We use the same language for their corresponding roots. In particular, s is a first-, second- or third-shell root if $|\langle \rho | s \rangle|^2$ is 3, 9 or 12 respectively. It will be useful to fix a scaling of the first-shell roots: a *Leech root* (with respect to ρ) means a root s with $\langle \rho | s \rangle = \theta$, and we call its mirror a *Leech mirror*.

In the Leech model, the Leech roots (with respect to ρ) are the vectors

$$(2-8) \quad s = \left(\sigma; 1, \theta \left(\frac{\sigma^2 - 3}{6} + \nu \right) \right)$$

with $\sigma \in \Lambda$ and $\nu \in \frac{1}{\theta}\mathbb{Z} + \frac{1}{2\theta}$ if σ^2 is divisible by 6, or $\nu \in \frac{1}{\theta}\mathbb{Z}$ otherwise. This is a specialization of (2-6). We note that if $(\sigma; 1, \alpha)$ is a Leech root, then the Leech roots of the form $(\sigma; 1, *)$ are $(\sigma; 1, \alpha + n)$ as n runs over \mathbb{Z} . In the $P^2\mathbb{F}_3$ model there is no simple formula for the Leech roots, but (with respect to ρ) the point-roots are examples of Leech roots, and the line-roots are examples of second-shell roots. For later use, we record the inner products

$$\langle \rho | p_i \rangle = \theta \quad \text{and} \quad \langle \rho | l_i \rangle = 3\bar{\omega}$$

for $i = 1, \dots, 13$.

In Section 3 we will choose a closed horoball A centered at ρ and disjoint from \mathcal{H} , and choose a point $a \in A$. We will speak of “meridians” to refer to certain elements $M_{a,A,H}$ of the orbifold fundamental group (see below) of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$, or their underlying paths $\mu_{a,A,H}$. The definition of a meridian involves a choice of a mirror H ; we will call the meridian a *Leech meridian* if H is a Leech mirror.

2.9 Meridians and the orbifold fundamental group

The orbifold fundamental group $G_b = \pi_1^{\text{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, b)$ depends on the choice of a base point $b \in \mathbb{B}^{13} - \mathcal{H}$. It is defined as the following set of equivalence classes of pairs (γ, g) , where $g \in P\Gamma$ and γ is a path in $\mathbb{B}^{13} - \mathcal{H}$ from b to $g(b)$. One such pair is equivalent to another one (γ', g') if $g = g'$ and γ and γ' are homotopic in $\mathbb{B}^{13} - \mathcal{H}$, rel endpoints. The group operation is $(\gamma, g) \cdot (\gamma', g') = (\gamma$ followed by $g \circ \gamma', gg')$. Inversion is given by $(\gamma, g)^{-1} = (g^{-1} \circ \text{reverse}(\gamma), g^{-1})$. Projection of (γ, g) to g defines a homomorphism $G_b \rightarrow P\Gamma$. It is surjective because $\mathbb{B}^{13} - \mathcal{H}$ is path-connected. The kernel is obviously $\pi_1(\mathbb{B}^{13} - \mathcal{H}, b)$, yielding the exact sequence

$$(2-9) \quad 1 \rightarrow \pi_1(\mathbb{B}^{13} - \mathcal{H}, b) \rightarrow G_b \rightarrow P\Gamma \rightarrow 1.$$

The elements of G_b we will use are the meridians $M_{b,H}$ and $M_{b,q,H}$, where H is a mirror and q is a point of H . These are defined in Section 3. Their underlying paths are written $\mu_{b,H}$ and $\mu_{b,q,H}$ and are also called meridians. The meridians associated to the point- and line-mirrors are called the point- and line-meridians. The main theorem of this paper is that G_τ is generated by these 26 meridians, when the basepoint τ is

the 26–point from Section 2.6. There are also *fat basepoint* versions of meridians: $M_{a,A,H}$, $M_{a,A,q,H}$, $\mu_{a,A,H}$ and $\mu_{a,A,q,H}$ with A a horoball centered at a Leech cusp, small enough to miss \mathcal{H} , and a a point of A . These are also defined in Section 3. Of the meridians of this sort, we mostly use those with H a Leech mirror, which we call *Leech meridians*. We also speak of *second-shell meridians*, with the corresponding meaning.

In any μ_{\dots} or M_{\dots} , we write just s instead of s^\perp in the subscript when the hyperplane is the mirror s^\perp of a root. For example, $\mu_{b,s}$ rather than μ_{b,s^\perp} . Here we are also following the convention from Section 2.5 of writing just s^\perp for $\mathbb{B}(s^\perp)$.

3 Meridians and homotopies between them

At its heart this paper consists of explicit manipulations of paths in $\mathbb{B}^{13} - \mathcal{H}$. In this section we describe the most important paths and manipulations. For concreteness we work only with our particular arrangement \mathcal{H} in \mathbb{B}^{13} , but it will be obvious that everything works equally well in the generality of [5]. (In particular, for the mirror arrangements of finite complex reflection groups.) The ideas are simple, and we hope the following summary will enable the reader to skip the technical details.

Our basic path is called a *meridian* and written $\mu_{b,q,H}$. Here b is a point of $\mathbb{B}^{13} - \mathcal{H}$, H is a mirror, and q is a point of H . The path begins at b and travels along a geodesic towards q , stopping near it. (If this geodesic meets some other mirror then detours are required; see the technical definition later in this section.) Next it travels directly toward H along the shortest geodesic segment to H , stopping very near it. (Again, detours might be necessary.) Then it travels $\frac{2\pi}{3}$ of the way around H , on a positively oriented circular arc. Finally it “returns” by retracing the paths in the first two steps of the construction (or rather their images under the ω –reflection in H). See Figure 3.1. This path $\mu_{b,q,H}$ is not a loop, but it becomes a loop upon quotienting by the $P\Gamma$ action; this is why we used quote marks around “returns”. The term meridian is from knot theory: we think of the image of \mathcal{H} in $\mathbb{B}^{13}/P\Gamma$ as a sort of knot. “Meridian” indicates that (the image of) $\mu_{b,q,H}$ is a loop, freely homotopic to a circle around (the image of) H .

We abbreviate the notation to $\mu_{b,H}$ when q is the point of H nearest b . This is also pictured in Figure 3.1. The $\mu_{b,H}$ are the meridians considered in our earlier paper [5], where we did not need the flexibility to take q to be an arbitrary point of H .

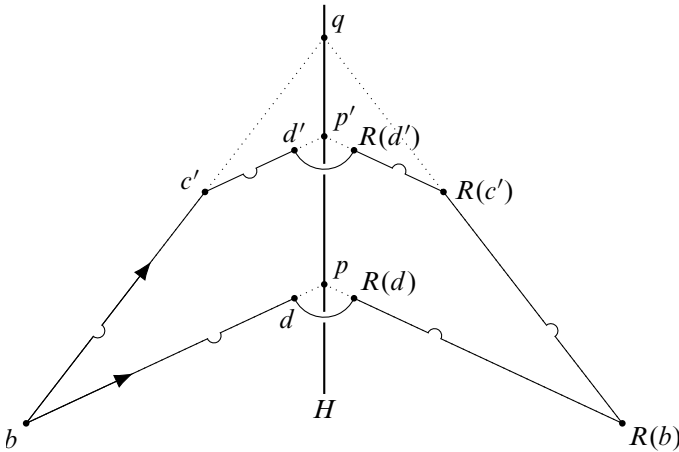


Figure 3.1: The meridians $\mu_{b,q,H}$ (top) and $\mu_{b,H}$ (bottom) go left to right from b to $R(b)$. Here b is the basepoint, H is a mirror, R is its ω -reflection, $q \in H$, and p is the point of H nearest b . The semicircular arcs in the paths indicate possible diversions around points of \mathcal{H} .

Lemmas 3.2 and 3.8 are the main results of this section. The conceptual content of Lemma 3.2 is the following. Consider a second basepoint b' , and join b to it by some path $b_t \in [0,1]$ that misses \mathcal{H} . One expects that under reasonable hypotheses, the family of paths $\mu_{b_t,q,H}$ will form a homotopy between $\mu_{b,q,H}$ and $\mu_{b',q,H}$. Lemma 3.2 gives sufficient conditions for this, and Lemma 3.4 gives stronger but simpler sufficient conditions. Example 3.3 shows that these hypotheses can't be omitted.

The conceptual content of Lemma 3.8 is the following. Consider the first turning point of $\mu_{b,q,H}$, which we recall lies near q . It is called c' in Figure 3.1. If we used a point slightly further from q in place of c' , then we would obtain a homotopic path. If we gradually moved the turning point all the way back to b , and all the intermediate paths missed \mathcal{H} , then we would have a homotopy from $\mu_{b,q,H}$ to $\mu_{b,H}$. Lemma 3.8 shows that this applies if Δbpq misses \mathcal{H} except at the obvious intersection points. Here p is the point of H closest to b .

Informally we think of the meridians $\mu_{b,q,H}$ and $\mu_{b,H}$ as elements of the orbifold fundamental group $G_b = \pi_1^{\text{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, b)$. But strictly speaking, an element of this group is an ordered pair (see Section 2.9). So we also say *meridian* for the ordered pairs $(\mu_{b,q,H}, R_H)$ and $(\mu_{b,H}, R_H)$, where R_H is the ω -reflection with mirror H . We write $M_{b,q,H}$ and $M_{b,H}$ for these ordered pairs.

In our applications the hyperplane H is always the mirror of a root s . As mentioned in Section 2.9, when writing μ_{\dots} or M_{\dots} we will write just s in the subscript rather than s^\perp , when s is a root and the hyperplane is its mirror. For example, $\mu_{b,s}$ rather than μ_{b,s^\perp} .

Our informal statements of Lemmas 3.2 and 3.8 are natural enough that the reader may wonder what else there is to say. There are three things to fuss over. First, the basepoint might not be in general position with respect to the hyperplanes. Working with basepoints in special position is essential for us, since both of our basepoints (τ and the cusp ρ) have large stabilizers, which we make essential use of. Allowing basepoints in special position forces the proper definition of $\mu_{b,q,H}$ to include detours around any hyperplanes met by the informal definition of $\mu_{b,q,H}$ given above. Then it requires work even to show that the resulting homotopy class is well-defined (Lemma 3.1). Second, we would like to use the Leech cusp ρ as a sort of basepoint, even though ρ is not a point of \mathbb{B}^{13} . We accomplish this by using the closed horoball A centered at ρ and disjoint from \mathcal{H} , and a point $a \in A$. Informally we think of ρ as the basepoint, but a is the official basepoint. In the end one can work with the resulting meridians just like the ones above. But the notation acquires an extra subscript: $\mu_{a,A,q,H}$, $\mu_{a,A,H}$, $M_{a,A,q,H}$ and $M_{a,A,H}$. Third, we must make precise the conditions under which moving the basepoint from one point b_0 to another point b_1 identifies the meridians based at b_0 with those based at b_1 . See Lemmas 3.2 and 3.4.

Now we begin the technical content of the section. For $b, c \in \overline{\mathbb{B}^{13}}$ we write \overline{bc} for the geodesic segment from b to c . Now suppose $b, c \notin \mathcal{H}$. It may happen that \overline{bc} meets \mathcal{H} , so we define a perturbation \widehat{bc} of \overline{bc} in the following obvious way. The notation may be pronounced “ b dodge c ” or “ b detour c ”. We write $\overline{bc}^{\mathbb{C}}$ for the complex line containing \overline{bc} . By the local finiteness of the mirror arrangement, $\overline{bc}^{\mathbb{C}} \cap \mathcal{H}$ is a discrete set. Consider the path obtained from \overline{bc} by using positively oriented semicircular detours in $\overline{bc}^{\mathbb{C}}$, around the points of $\overline{bc} \cap \mathcal{H}$, in place of the corresponding segments of \overline{bc} . By \widehat{bc} we mean this path, with the radius of these detours taken to be small enough. (This means: small enough for the construction to make sense and the resulting homotopy class in $\mathbb{B}^{13} - \mathcal{H}$, rel endpoints, to be radius-independent.) Note: \widehat{cb} is not the same as reverse(\widehat{bc}), because \widehat{bc} followed by \widehat{cb} encircles any mirrors that meet \overline{bc} , while \widehat{bc} followed by reverse(\widehat{bc}) is nullhomotopic.

Now we define $\mu_{b,H}$ for $b \in \mathbb{B}^{13} - \mathcal{H}$ and H a hyperplane of \mathcal{H} . See the bottom path in Figure 3.1. Write p for the point of H nearest b and R for the ω -reflection with mirror H . Choose an open ball U around p , small enough to miss all the mirrors

except those through p , and choose some $d \in U \cap (\overline{bp} - \{p\})$. (Unless otherwise stated, a ball around p always means a complex hyperbolic metric ball centered at p .) If d were to lie in \mathcal{H} , then by choice of U it would follow that d lies on a mirror H' that passes through p . This would imply that the geodesic ray joining d and p lies entirely in H' , forcing $b \in H'$, which is a contradiction. Thus, from $b \notin \mathcal{H}$ we get $d \notin \mathcal{H}$. So we define $\mu_{b,H}$ as \widehat{bd} , followed by the positively oriented circular arc in $\overline{bp}^{\mathbb{C}}$ from d to $R(d)$, centered at p , followed by $R(\text{reverse}(\widehat{bd}))$.

It is easy to see that the resulting path is independent of our choices of U and d , up to homotopy rel endpoints in $\mathbb{B}^{13} - \mathcal{H}$. As mentioned above, $M_{b,H}$ means the element $(\mu_{b,H}, R)$ of the orbifold fundamental group, and is also called a meridian. Although we don't need it, we mention for background that $M_{b,H}^3$ is the loop in $\mathbb{B}^{13} - \mathcal{H}$ that we called \widehat{bH} (read "b loop H") in [5]. It goes from b to d , encircles H in $\overline{bp}^{\mathbb{C}}$, and then returns to b .

Next we define $\mu_{b,q,H}$, where q is a point of H . See the top path in Figure 3.1. Choose an open ball U around q that is small enough to miss all the mirrors except those through q , and choose some point $c' \in U \cap (\overline{bq} - \{q\})$. From $b \notin \mathcal{H}$ we get $c' \notin \mathcal{H}$, so $\mu_{c',H}$ is defined. Define $\mu_{b,q,H}$ as $\widehat{bc'}$ followed by $\mu_{c',H}$ followed by $R(\text{reverse}(\widehat{bc'}))$. Figure 3.1 also shows primed versions of the points p and d from the previous paragraph's definition of $\mu_{c',H}$. The next lemma shows that $\mu_{b,q,H}$ is well-defined as a homotopy class. The tricky part is dealing with possible detours in the subpath $\mu_{c',H}$. (For the meridians in this paper, it happens that no such detours occur, in which case the lemma is obvious.) Just as with $M_{b,H}$, we refer to the element $M_{b,q,H} = (\mu_{b,q,H}, R)$ of the orbifold fundamental group as a meridian.

Lemma 3.1 (well-definedness of $\mu_{b,q,H}$) *The homotopy class of $\mu_{b,q,H}$, rel endpoints, is independent of the choices of U and c' .*

Proof We will refer to Figure 3.2. Suppose U is a ball around q as above, and write $c_0, c_1 \in U \cap (\overline{bq} - \{q\})$ for two candidates for c' . We choose the subscripts so that c_0 is further from q than c_1 is. We will show that the version of $\mu_{b,q,H}$ defined using $c' = c_0$ is homotopic rel endpoints to the version of $\mu_{b,q,H}$ defined using $c' = c_1$. The shaded surface in Figure 3.2 is the homotopy that we will construct. It is a homotopy from $\mu_{c_0,H}$ to $\mu_{c_1,H}$ that moves c_0 along $\overline{c_0c_1}$. This is enough to build a homotopy between the two versions of $\mu_{b,q,H}$. The only properties of U , c_0 and c_1 that we will use in this argument are that U meets no mirrors except those containing q , and that

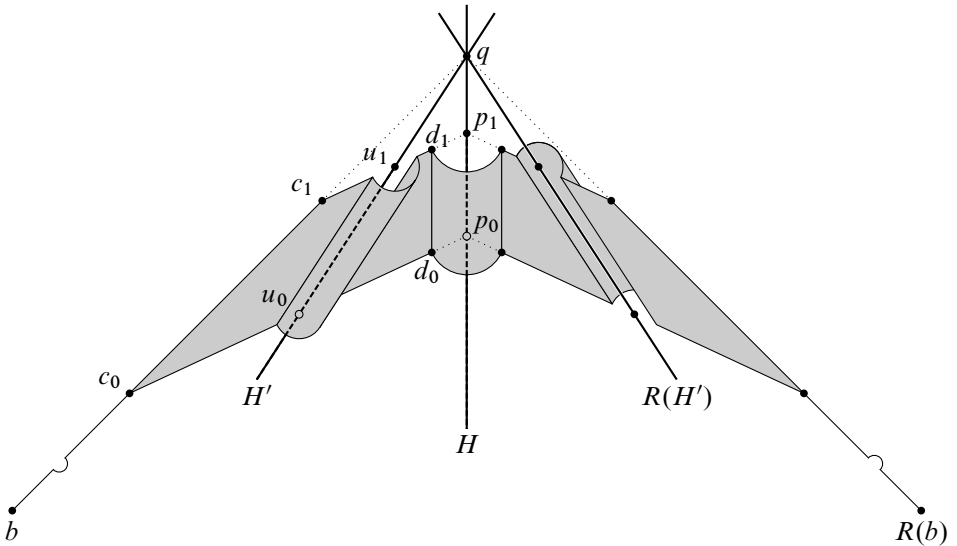


Figure 3.2: The well-definedness of $\mu_{b,q,H}$; see Lemma 3.1. The unlabeled points on the right side are the R -images of c_0, c_1, u_0, u_1, d_0 and d_1 .

$c_0, c_1 \in U \cap (\overline{bq} - \{q\})$. It follows that the homotopy class of $\mu_{b,q,H}$ is independent of the choices of U and c .

We parametrize $\overline{c_0c_1}$ by $t \mapsto c_t$ with t varying over $[0, 1]$. We write p_t for the point of H nearest c_t . The issue we must deal with is the following. If $\overline{c_t p_t} - \{p_t\}$ meets \mathcal{H} , then varying t will move the intersection points but not eliminate them. This is visible in Figure 3.2, where H' is a mirror giving rise to such intersections. The last paragraph of Section 2.3 shows that c_0, p_0 and q span a totally real triangle $\Delta c_0 p_0 q$, and also that all the p_t lie on the segment $\overline{p_0 q}$. By the nonpositive curvature of \mathbb{B}^{13} , projection to the hyperplane H is distance-nonincreasing. Therefore p_0 is closer to q than c_0 is. So $\Delta c_0 p_0 q$ lies in U . In particular, the mirrors meeting this triangle are the same as those containing q .

It follows that \mathcal{H} meets $\Delta c_0 p_0 q$ in the union of finitely many geodesic segments from its vertex q to the opposite edge $\overline{c_0 p_0}$. Now restrict this to the quadrilateral with vertices c_0, c_1, p_1 and p_0 . Its intersection with any mirror is either empty or a segment $\overline{u_0 u_1}$, which we may parametrize by $t \mapsto u_t$ with $u_t \in \overline{c_t p_t}$ as shown in Figure 3.2. If there is more than one such segment then they are disjoint from each other.

Now, by definition of U , any mirror that meets $\overline{p_0 p_1}$ contains all of $\overline{p_0 p_1}$. So we may choose $\varepsilon > 0$ such that every mirror meeting the closed ε -neighborhood of $\overline{p_0 p_1}$

contains $\overline{p_0 p_1}$. Now let D_t be the closed ε -disk in the complex geodesic $\overline{c_t p_t}^{\mathbb{C}}$, centered at p_t . The only mirrors it meets are the ones containing $\overline{p_0 p_1}$. All of these mirrors contain p_t , and thus contain no other points of D_t . So $D_t \cap \mathcal{H} = \{p_t\}$. Write d_t for the point where $\overline{c_t p_t}$ enters D_t . Because D_t misses \mathcal{H} except at p_t , we may take $\mu_{c_t, H}$ to be $\overline{c_t d_t}$, followed by the positively oriented arc in ∂D_t from d_t to $R(d_t)$, followed by $R(\text{reverse}(\overline{c_t d_t}))$. The ∂D_t portions of these paths vary continuously with t , sweeping out the $\frac{1}{3}$ -tube in the center of Figure 3.2.

The shaded surface to the left of this in Figure 3.2 is obtained by modifying the quadrilateral with edges $\overline{d_1 c_1}$, $\overline{c_1 c_0}$, $\overline{c_0 d_0}$ and $\{d_t \mid t \in [0, 1]\}$. The modification is to replace a strip around each segment $\overline{u_0 u_1}$ (notation as above and in Figure 3.2) by a semicylindrical strip that dodges \mathcal{H} . More precisely, we may take this semicylindrical strip to be a union of positively oriented semicircles S_t , where S_t lies in the complex geodesic $\overline{c_t p_t}^{\mathbb{C}}$ and has center u_t and some small constant radius. Each S_t misses \mathcal{H} by reasoning similar to the previous paragraph. Because the u_t and the $\overline{c_t p_t}^{\mathbb{C}}$ vary continuously with t , the S_t do too.

The part of the homotopy on the right side of the figure is the R -image of the part on the left. The verifications on the left-hand side carry over to the right-hand side since R is an isometry that preserves \mathcal{H} and fixes H pointwise. The left, right and middle parts fit together to give the promised homotopy from $\mu_{c_0, H}$ to $\mu_{c_1, H}$. \square

Lemma 3.2 (homotopy between meridians under change of basepoint) *Suppose H is a mirror, q is a point of H , and $b : [0, 1] \rightarrow \mathbb{B}^{13} - \mathcal{H}$ is a path, written $t \mapsto b_t$. We suppose that each geodesic $\overline{b_t q}$ meets \mathcal{H} only at q . We also assume the following hypothesis, which is expressed in terms of the point r_t of H nearest to b_t :*

$$(\star) \quad r_t \text{ is the only point of } \overline{b_t r_t} \text{ that lies on a mirror containing } q.$$

Then $M_{b_0, q, H}$ is identified with $M_{b_1, q, H}$ under the isomorphism of orbifold fundamental groups $G_{b_0} \cong G_{b_1}$ induced by b .

Example 3.3 We show how the failure of hypothesis (\star) can lead to the failure of the conclusion of Lemma 3.2; this might give more insight than the proof of the lemma. Take \mathcal{H} to be the union of the mirrors in \mathbb{C}^2 for the dihedral group of order 8, with H being the mirror $x = 0$. All of our paths will lie in the real 3-dimensional space $\text{Im } y = 0$ pictured in Figure 3.3. The direction of increasing $\text{Im } x$ is into the page, so the semicircular arcs, traversed left to right, travel positively around H . We have

not drawn the mirror $y = 0$ because it is irrelevant. But we have drawn H and the remaining two mirrors $y = \pm x$. We take q to be the origin, and $b_0 = (-3 - \varepsilon i, -2)$ and $b_1 = (-3 + \varepsilon i, -1)$, where $\varepsilon > 0$. We take b to be the segment $\overline{b_0 b_1}$, and define c_t , d_t and p_t as in the definition of $\mu_{b_t, q, H}$. The figure shows $\mu_{b_0, q, H}$ (resp. $\mu_{b_1, q, H}$) going from left to right from b_0 to $R(b_0)$ (resp. b_1 to $R(b_1)$). The key feature in the figure is that $\overline{c_0 d_0}$ (resp. $\overline{c_1 d_1}$) passes in front of (resp. behind) the mirror H' (or rather, the intersection of H' with the pictured \mathbb{R}^3). This is a consequence of the sign of the imaginary part of the x -coordinate of b_0 (resp. b_1). It is easy to see that the pictured loop is not contractible in $\mathbb{C}^2 - \mathcal{H}$ (or even $\mathbb{C}^2 - H'$). This is a way of saying that $M_{b_0, q, H}$ does *not* correspond to $M_{b_1, q, H}$ under the identification $G_{b_0} \cong G_{b_1}$ induced by b . In this example, the hypothesis (\star) fails because $\overline{b_t r_t}$ meets H' when $t = \frac{1}{2}$.

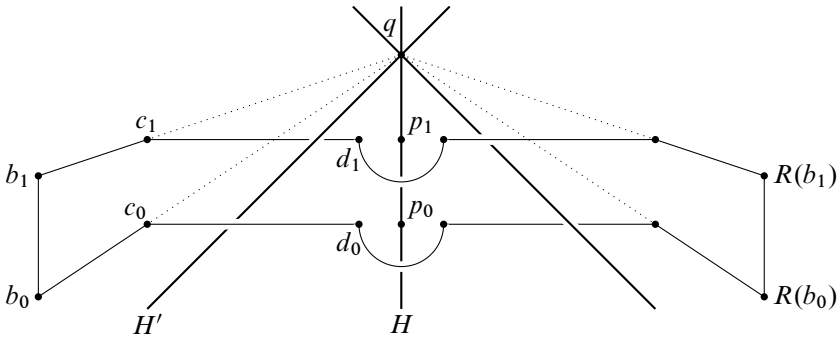


Figure 3.3: Illustration for Example 3.3. The paths $\mu_{b_0, q, H}$ and $\mu_{b_1, q, H}$ go from left to right across the bottom and top respectively. The unlabeled points on the right are the R -images of the corresponding points on the left.

Proof of Lemma 3.2 Take U to be a ball centered at q , small enough to miss all mirrors except the ones containing q . The definition of $\mu_{b_t, q, H}$ begins with the choice of a point $c_t \in U \cap (\overline{b_t q} - \{q\})$. By hypothesis, all the paths $\overline{b_t c_t}$ miss \mathcal{H} . The next step in the construction of $\mu_{b_t, q, H}$ is the definition of p_t as the point of H nearest to c_t . The essential step in the proof is this: $\overline{c_t p_t}$ meets \mathcal{H} only at p_t . Otherwise, some mirror H' would contain an interior point of $\overline{c_t p_t}$. Since $\overline{c_t p_t}$ lies in U , this H' must also pass through q . So H' contains the geodesic from q to an interior point of $\overline{c_t p_t}$. Since $\overline{c_t p_t}$ lies in the totally real triangle $\triangle b_t r_t q$, extending this geodesic shows that H' also contains an interior point of $\overline{b_t r_t}$. Which is contrary to (\star) .

Without loss of generality we may suppose that c_t varies continuously with respect to t , and it follows that p_t does also. Continuing to follow the definition of $\mu_{b_t, q, H}$, we choose a point d_t of $\overline{c_t p_t}$ very close to p_t but distinct from it. We may suppose that d_t also varies continuously with respect to t . Now, $\mu_{b_t, q, H}$ is the concatenation of five paths: first $\overline{b_t c_t}$, then $\overline{c_t d_t}$, then a circular arc from d_t to $R(d_t)$ in $\overline{d_t p_t}^C$ centered at p_t , then $R(\overline{d_t c_t})$, and finally $R(\overline{c_t b_t})$. All these paths miss \mathcal{H} , and the ingredients in their definitions vary continuously with respect to t . So we have found a homotopy in $\mathbb{B}^{13} - \mathcal{H}$ from $\mu_{b_0, q, H}$ to $\mu_{b_1, q, H}$. Since the track of b_0 is b , the proof is complete. \square

Lemma 3.4 (a variation) *Lemma 3.2 remains true if its hypothesis (\star) is replaced by: H is orthogonal to all other mirrors through q .*

Proof It suffices to show that (\star) holds, because then we can quote Lemma 3.2. We establish (\star) by deriving a contradiction from its negation. So we suppose that H is orthogonal to all other mirrors through q , and that there exists a $t \in [0, 1]$ and an interior point v of $\overline{b_t r_t}$ such that v lies in a mirror H' that contains q . We have $H' \neq H$ because otherwise H would contain the geodesic line through v and r_t , hence contain b_t . Now we use the hypothesis $H' \perp H$: it implies that for every point of H' , its projection to H also lies in H' . We apply this to v , whose projection to H is r_t because $v \in \overline{b_t r_t}$. Therefore $r_t \in H'$. Since H' contains both r_t and v , it contains the geodesic line containing them. But b_t lies on this line, which is a contradiction since $b_t \notin \mathcal{H}$. \square

Now we turn our attention to using the Leech cusp ρ as a basepoint for analyzing the orbifold fundamental group of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$. We must work around the difficulty that ρ is not a point of \mathbb{B}^{13} . To do this we follow the “fat basepoint” strategy of [5], by choosing a closed horoball A centered at ρ and small enough to miss \mathcal{H} . Such a horoball exists because no mirrors pass through ρ ; if one wants to be more concrete then one can refer to Lemma 4.3 of [5], which shows that we may take A to be the closed horoball of any height < 1 . We also choose some point a of A .

We continue to suppose that H is a mirror and R is its ω -reflection. Now we take p to be the point in H closest to ρ . (See Section 2.2 for the meaning of “closest”.) Informally, the horoball A is a sort of infinite-radius ball centered at the cusp ρ . In particular, it is convex and the geodesics orthogonal to ∂A are the same as the geodesics having ρ as a limit point. If the meridian based at ρ were defined then it would begin

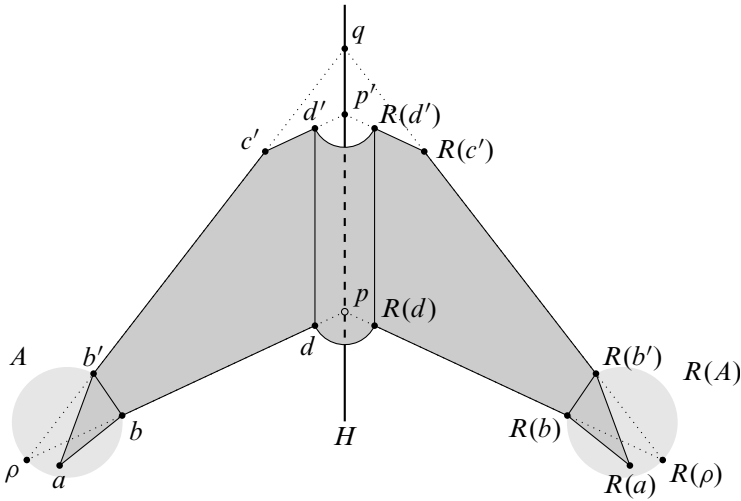


Figure 3.4: Illustration of the meridians $\mu_{a,A,H}$, left to right across the bottom, and $\mu_{a,A,q,H}$, left to right across the top. The dark region indicates the homotopy between them constructed in Lemma 3.7. The light gray balls are horoballs centered at Leech cusps ρ and $R(\rho)$.

with a geodesic (possibly with small detours) from ρ to a point near p . Because the geodesics through ρ are orthogonal to the horosphere, this path would pierce ∂A at the point of ∂A nearest to p , which we will call b . To define the corresponding path based at a we simply replace the nonexistent segment $\overline{\rho b}$ by \overline{ab} . Formally, $\mu_{a,A,H}$ is \overline{ab} followed by $\mu_{b,H}$ followed by $R(\overline{ba})$. See Figure 3.4 for a picture: $\mu_{a,A,H}$ travels from left to right across the bottom. The point marked d comes from the definition of $\mu_{b,H}$.

Similarly, if q is any point of H then we define b' as the point of ∂A nearest to q , and then define $\mu_{a,A,q,H}$ as $\overline{ab'}$ followed by $\mu_{b',q,H}$ followed by $R(\overline{b'a})$. This travels from left to right across the top of Figure 3.4. The point marked c' comes from the definition of $\mu_{b',q,H}$, and the points marked d' and p' come from the definition of the subpath $\mu_{c',H}$ of $\mu_{b',q,H}$. Just as for ordinary basepoints, we refer to the orbifold fundamental group elements $M_{a,A,H} = (\mu_{a,A,H}, R)$ and $M_{a,A,q,H} = (\mu_{a,A,q,H}, R)$ as meridians.

Remark 3.5 In several places it will clarify an argument to assume “without loss of generality” that a is some particular point of A . One should expect that the exact location of a is unimportant: if a' is another point of A then G_a and $G_{a'}$ are canonically isomorphic. One just moves the basepoint from a to a' along any path in the simply

connected subspace A . But more is true: this isomorphism identifies each meridian $M_{a,A,q,H} \in G_a$ with $M_{a',A,q,H} \in G_{a'}$. It is easy to see that this holds. In fact, arranging for it to be true was the main reason for introducing A into the definition of the meridians. This was necessary because the canonical isomorphism $G_a \cong G_{a'}$ does not identify every $M_{a,q,H}$ with the corresponding $M_{a',q,H}$.

Remark 3.6 There is a more elegant treatment of fat basepoints in [13, Appendix A]. This treats A itself as the basepoint and avoids mention of a . But this approach is not well-suited for moving the basepoint from a to the 26-point τ (a point outside A). We do this in Section 6, and it is an essential part of our proof of Theorem 1.2.

Lemma 3.7 (homotopies between meridians based “at” a cusp) *Suppose $\rho \in \partial\mathbb{B}^{13}$ is a Leech cusp, A is a closed horoball centered at ρ and disjoint from \mathcal{H} , and a is a basepoint in A . Let H be a mirror, p be the point of H closest to ρ , and q be another point of H . Our key assumption is that the totally real triangle $\Delta\rho pq$ is disjoint from \mathcal{H} except that \overline{pq} lies in H and that q might lie in additional mirrors. Then $\mu_{a,A,H}$ and $\mu_{a,A,q,H}$ are homotopic rel endpoints in $\mathbb{B}^{13} - \mathcal{H}$, and $M_{a,A,H} = M_{a,A,q,H}$ in G_a .*

Proof We hope Figure 3.4 will help the reader; we will use the notation b, d, b', c', d', p', R from the definitions given above. The paths $\mu_{a,A,q,H}$ and $\mu_{a,A,H}$ are the upper and lower paths around the dark gray region, traversed from left to right. This region indicates the homotopy between them that we will construct.

By hypothesis, for small enough $\varepsilon > 0$, the ε -neighborhood of $\overline{pp'}$ meets no mirrors except H . This neighborhood, which we call U , is convex by the nonpositive curvature of \mathbb{B}^{13} (see the remark after Proposition II.2.4 of [14]). By moving d closer to p and d' closer to p' we may suppose without loss of generality that they lie in U . The portion of the homotopy that appears in the figure as $\frac{1}{3}$ of a tube is the surface swept out by $\overline{dd'}$ under the rotations around H by angles in $[0, \frac{2\pi}{3}]$. It lies in U since $\overline{dd'}$ does. It misses \mathcal{H} because it misses H , which is the only mirror that meets U . The pentagonal surface with vertices b', c', d', d and b misses \mathcal{H} because it lies in $\Delta\rho pq - \overline{pq}$. (It makes sense to speak of this surface because all these points lie in $\Delta\rho pq$, which is totally geodesic by Section 2.3.) The triangle with vertices a, b and b' might not be totally geodesic, but it lies in A and therefore spans a disk that misses \mathcal{H} . Since R preserves \mathcal{H} , the R -images of this pentagon and triangle also miss \mathcal{H} . The $\frac{1}{3}$ -tube, two pentagons and two triangles fit together to give a homotopy between $\mu_{a,A,H}$ and $\mu_{a,A,q,H}$. □

We will also need a version of this where the basepoint is an ordinary point of $\mathbb{B}^{13} - \mathcal{H}$ rather than a Leech cusp. The picture and proof differ from Lemma 3.7 only by replacing all of ρ, A, a, b and b' by a single point b .

Lemma 3.8 (homotopies between meridians based at a point of \mathbb{B}^{13}) *Suppose $b \in \mathbb{B}^{13} - \mathcal{H}$. Let H be a mirror, p be the point of H closest to b , and q be another point of H . Suppose that the totally real triangle $\triangle bpq$ is disjoint from \mathcal{H} except that \overline{pq} lies in H and that q might lie in additional mirrors. Then $\mu_{b,H} \simeq \mu_{b,q,H}$ rel endpoints in $\mathbb{B}^{13} - \mathcal{H}$, and $M_{b,H} = M_{b,q,H}$ in G_b . \square*

4 Finitely many generators based “at” a cusp

In this section we begin the proof of our main result, Theorem 1.2. Our starting point is Theorem 1.5 from [5]:

Theorem 4.1 *Let ρ be the Leech cusp $(0; 0, 1)$ of the Eisenstein lattice L (in Leech coordinates), and let A be a closed horoball neighborhood of it, disjoint from the mirror arrangement \mathcal{H} . Choose a basepoint $a \in A$. Then the orbifold fundamental group*

$$G_a = \pi_1^{\text{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, a)$$

is generated by the set of all Leech meridians. \square

Recall from Sections 2.8–2.9 that the Leech roots are the roots r satisfying $\langle \rho | r \rangle = \theta$, and the Leech meridians are the meridians $M_{a,A,s}$ with s varying over the Leech roots. This is an infinite generating set, which we improve in Theorem 4.2 by exhibiting an explicit finite set of Leech meridians that generates G_a . The exact nature of this generating set is not so important, but it must be explicit and finite so that we will be able to further improve the generating set in Theorem 5.1.

We use $P^2\mathbb{F}_3$ coordinates in Theorem 4.2 because that is how it is used in the next section. We use Leech coordinates everywhere else in this section.

Theorem 4.2 (130 generators based “at” a cusp) *Suppose $\rho = (3\omega - 1; -1, \dots, -1)$ is the Leech cusp from Section 2.8, written in the $P^2\mathbb{F}_3$ coordinate system, A is a closed horoball centered at ρ and disjoint from \mathcal{H} , and a is an element of A . Write S for the set of 130 Leech roots*

$$pi, \quad pi - \rho, \quad \bar{\omega} pi - l_j, \quad \bar{\omega} pi - l_j - \rho,$$

with $i, j = 1, \dots, 13$ and p_i and l_j incident in the last two cases. Then G_a is generated by the Leech meridians $M_{a,A,s}$, where s varies over S .

In this section we will write G for the subgroup of G_a generated by these particular Leech meridians, and our goal is to show that G is all of G_a . Informally, the strategy is to show that the “ G -stabilizer of ρ ” acts transitively on the Leech roots. Since G contains a Leech meridian, it follows that every Leech meridian is G -conjugate into G , hence lies in G . From [Theorem 4.1](#) we know that the set of all Leech meridians generates G_a . It follows that G is all of G_a , as desired.

In order to express this idea precisely we must make sense of the “ G -stabilizer of ρ ”. The problem is that G_a is an orbifold fundamental group, not a group of isometries of \mathbb{B}^{13} , so we must say what we mean by G_a “containing” certain isometries of \mathbb{B}^{13} . We will explain this and then develop the details of the strategy just outlined. The idea is that the subgroup $P\Gamma_A$ of $P\Gamma \subseteq \text{Isom}(\mathbb{B}^{13})$ preserving A can be thought of as a subgroup of G_a . To see this, consider the obvious homomorphism

$$(4-1) \quad P\Gamma_A = \pi_1^{\text{orb}}(A/P\Gamma_A, a) \rightarrow \pi_1^{\text{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, a) = G_a.$$

The first equality holds because A is simply connected, and the map exists because A misses \mathcal{H} . This map $P\Gamma_A \rightarrow G_a$ is an embedding, because following it by the natural map $G_a \rightarrow P\Gamma$ (see [Section 2.9](#)) gives the identity on $P\Gamma_A$. In this manner, we may regard $P\Gamma_A$ as a subgroup of G_a . Concretely, if $g \in P\Gamma_A$, then the corresponding element of G_a is (γ, g) where γ is any path in A from a to $g(a)$. The choice of γ is unimportant since A is simply connected.

The elements of $P\Gamma_A$ we will need are the following translations $T_{\lambda,z}$. The language comes from their analogy with the isometries of real hyperbolic space which look like Euclidean translations in the upper half-space model. Given $\lambda \in \Lambda$ and $z \in \text{Im } \mathbb{C}$ such that $z - \frac{1}{2}\lambda^2 \in \theta\mathcal{E}$, the definition is

$$(4-2) \quad T_{\lambda,z} : \begin{cases} (l; 0, 0) \mapsto (l; 0, \bar{\theta}^{-1}\langle l|\lambda \rangle), \\ (0; 1, 0) \mapsto (\lambda; 1, \theta^{-1}(z - \frac{1}{2}\lambda^2)), \\ (0; 0, 1) \mapsto (0; 0, 1). \end{cases}$$

One can see that the translations form a Heisenberg group by checking the identities

$$(4-3) \quad T_{\lambda,z} T_{\lambda',z'} = T_{\lambda+\lambda',z+z'+\text{Im}\langle \lambda|\lambda' \rangle},$$

$$(4-4) \quad (T_{\lambda,z})^{-1} = T_{-\lambda,-z},$$

$$(4-5) \quad T_{\lambda,z} T_{\lambda',z'} T_{\lambda,z}^{-1} T_{\lambda',z'}^{-1} = T_{0,2\text{Im}(\lambda|\lambda')}.$$

Lemma 4.3 *The group of translations $T_{\lambda,z}$ preserves A and ρ and acts simply transitively on the Leech roots. Its center consists of the $T_{0,z}$ with $z \in \theta\mathbb{Z}$.*

Proof Preservation of ρ is obvious, and A is defined in terms of ρ . The simple transitivity follows by computing the image of the Leech root $(0; 1, -\omega)$ under $T_{\lambda,z}$. This turns out to be the Leech root $(\sigma; 1, \theta(\frac{1}{6}(\sigma^2 - 3) + \nu))$ from (2-8), where $\sigma = \lambda$ and $\nu = \frac{1}{2\theta} - \frac{1}{3}z$. This accounts for every Leech root exactly once. (If $2|\lambda^2$ then the conditions $z \in \text{Im } \mathbb{C}$ and $z - \frac{1}{2}\lambda^2 \in \theta\mathcal{E}$ on z are equivalent to the condition $\nu \in \frac{1}{\theta}\mathbb{Z} + \frac{1}{2\theta}$ on ν in (2-8). And similarly if $2 \nmid \lambda^2$.)

Equation (4-5) shows that $T_{\lambda,z}$ is central if and only if $\lambda \perp \Lambda$, ie if and only if $\lambda = 0$. \square

The next lemma, which is the crucial trick in this section, obtains some elements of $P\Gamma_A$ as products of Leech meridians. Then we combine these in Lemma 4.5 to obtain translations.

Lemma 4.4 *Suppose $\rho \in L$ is a Leech cusp, A is a closed horoball centered at ρ and disjoint from \mathcal{H} , and $a \in A$. Suppose s is a Leech root, and write s' for the Leech root $s - \rho$. Write H and H' for their mirrors and R and R' for their ω -reflections. Then*

$$(4-6) \quad M_{a,A,H'} \cdot M_{a,A,H} = R' \cdot R.$$

Here the product of meridians on the left is evaluated in the orbifold fundamental group G_a , and the product of reflections on the right is evaluated in $P\Gamma$, with result in $P\Gamma_A$. When stating equality we are regarding $P\Gamma_A$ as a subgroup of G_a via (4-1).

Proof We use the Leech model of L , with $\rho = (0; 0, 1)$ as usual. By the transitivity of translations on Leech roots (Lemma 4.3), we may suppose without loss of generality that $s = (0; 1, -\omega)$, so $s' = (0; 1, \bar{\omega})$. Consider the subspace $V = \{(0; x, y)\}$ of $L \otimes \mathbb{C}$. We plot the point in $P(V) \cong P^1$ represented by $(0; x, y)$ as $y/x \in \mathbb{C} \cup \{\infty\}$. Using this, we have drawn the situation in Figure 4.1. In particular, ρ corresponds to the point at infinity and $\mathbb{B}^1 = \mathbb{B}(V)$ to the upper half-plane. The orthogonal complements in V of s and s' are spanned by $(0; 1, -\bar{\omega})$ and $(0; 1, \omega)$, respectively. So $H = s^\perp$ and $H' = s'^\perp$ correspond to $-\bar{\omega}$ and ω in the upper half-plane. The reflections R' and R act by counterclockwise hyperbolic rotations by $\frac{2\pi}{3}$ around these points. It is easy to see that $R(\infty) = 0$, $R'(0) = \infty$ and $R'(\infty) = -1$. In particular, $R'R \in P\Gamma_A$,

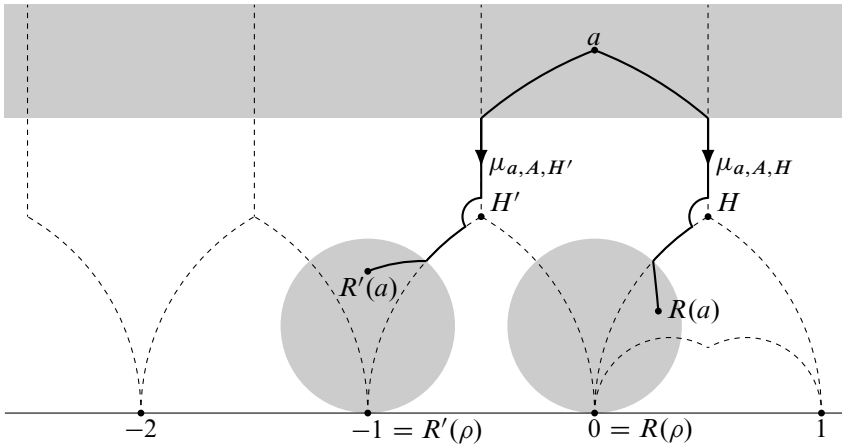


Figure 4.1: The Leech meridians appearing in (4-6). The cusp ρ is at infinity and the hyperplanes H and H' appear as the points where they meet the pictured \mathbb{B}^1 . The top gray region is the horoball A , and the regions below it are its images under R and R' .

which is part of the lemma. The closed horoball A and its images under R and R' also appear in Figure 4.1. To draw the paths we have assumed that a , a priori any point of A , actually lies in \mathbb{B}^1 and has real part 0. This is harmless, by Remark 3.5.

By definition (Section 2.9), the left side of (4-6) is equal to

$$(\mu_{a,A,H'}, R') \cdot (\mu_{a,A,H}, R) = (\mu_{a,A,H'} \text{ followed by } R' \circ \mu_{a,A,H}, R'R)$$

The path part of the right side appears in Figure 4.2. This figure is the same as Figure 4.1 except as follows. First, we have drawn $R' \circ \mu_{a,A,H}$ instead of $\mu_{a,A,H}$. Second, we have drawn the height 1 open horoballs around $\rho = \infty$ and $R'(\rho) = -1$, instead of A and its translates. It is easy to see that $\mu_{a,A,H'}$ followed by $R' \circ \mu_{a,A,H}$ lies in the union of these two horoballs. (One must check that s' and $R'(s)$ are first-shell roots with respect to $R'(\rho)$, which is almost obvious.) The union of these horoballs is contractible and misses \mathcal{H} . So our path from a to $R'R(a)$ may be homotoped (rel endpoints and missing \mathcal{H}) to a path in A . We have proven that the left side of (4-6) is equal to

$$(\text{some path in } A \text{ from } a \text{ to } R'R(a), R'R)$$

in the orbifold fundamental group G_a . This is exactly what the equality (4-6) asserts, so the proof is complete. □

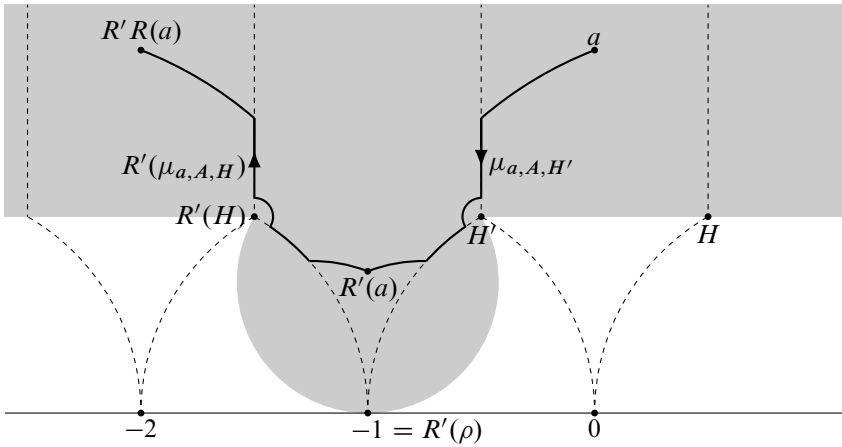


Figure 4.2: The proof of Lemma 4.4. The gray region is the union of the open height 1 horoballs around ρ and $R(\rho)$. Since these horoballs miss \mathcal{H} , the path from a to $R'R(a)$ is homotopic into A .

Remark Figures 4.1 and 4.2 bring to mind the classical action of $SL_2\mathbb{Z}$ on the upper half-plane. This is because the projective isometry group of $\begin{pmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{pmatrix}$ happens to be $PSL_2\mathbb{Z}$. The dashed lines indicate some fundamental domains for the index 2 subgroup generated by triflections.

Lemma 4.5 Suppose ρ, A and a are as in Lemma 4.4, and S_0 is a set of Leech roots whose differences span $\rho^\perp / \langle \rho \rangle$. Then the subgroup of $P\Gamma_A$ generated by the products $R'R$ of Lemma 4.4, with s varying over S_0 , contains all the translations (4-2). In particular, it acts transitively on the Leech roots.

Proof At heart this is Lemma 8 of [3]. But the statement there, and conventions and notation, are different enough that we indicate the proof. Write G for the specified subgroup of $P\Gamma_A$.

We begin by taking r to be the Leech root $(0; 1, -\omega)$ and using primes as in Lemma 4.4. So r' means $r - \rho = (0; 1, \bar{\omega})$. Write R_r, R'_r for the ω -reflections in r, r' . One checks that on the span of these roots, $R'_r R_r$ acts by $-\omega T_{0,-2\theta}$. Obviously $R'_r R_r$ acts trivially on Λ . We re-express this by saying that $R'_r R_r$ acts on L by $-\omega QT_{0,-2\theta}$, where Q scales $\Lambda \subseteq L$ by $-\bar{\omega}$ and fixes r and r' . One checks that $QT_{\lambda,z}Q^{-1} = T_{-\bar{\omega}\lambda,z}$ for every translation $T_{\lambda,z}$.

Now suppose $s \in S_0$. As a Leech root, we have $s = T_{\sigma,y}(r)$ for some translation $T_{\sigma,y}$ by Lemma 4.3. We also have $s' = T_{\sigma,y}(r')$ because $s' = s - \rho$ and $r' = r - \rho$.

Therefore,

$$R'_s R_s = T_{\sigma,y}(R'_r R_r) T_{\sigma,y}^{-1} = T_{\sigma,y} \cdot -\bar{\omega} Q T_{0,-2\theta} \cdot T_{\sigma,y}^{-1}.$$

Choosing another element t of S_0 and writing it as $T_{\tau,z}(r)$, we get a similar description of $R'_t R_t$. So

$$\begin{aligned} (R'_s R_s)(R'_t R_t)^{-1} &= T_{\sigma,y} \cdot (-\omega Q T_{0,-2\theta}) \cdot T_{\sigma,y}^{-1} \cdot T_{\tau,z} \cdot (-\bar{\omega} T_{0,2\theta} Q^{-1}) \cdot T_{\tau,z}^{-1} \\ &= T_{\sigma,y} Q (T_{-\sigma,-y} T_{\tau,z}) Q^{-1} T_{\tau,z}^{-1} \\ &= T_{\sigma,y} T_{\bar{\omega}\sigma,-y} T_{-\bar{\omega}\tau,z} T_{\tau,z}^{-1} \\ &= T_{\sigma+\bar{\omega}\sigma-\bar{\omega}\tau-\tau,?} \\ &= T_{-\omega(\sigma-\tau),?}, \end{aligned}$$

where the value of ? could be worked out but is unimportant for us.

We have shown that for each pair $s, t \in S_0$, the subgroup G contains a translation $T_{-\omega(\sigma-\tau),?}$. So it also contains a translation $(R'_s R_s) T_{-\omega(\sigma-\tau),?} (R'_s R_s)^{-1} = T_{\sigma-\tau,?}$. We have shown that for all $s, t \in S_0$, the subgroup G contains a translation $T_{\sigma-\tau,?}$ and a translation $T_{-\omega(\sigma-\tau),?}$. It follows from (4-3) that for every λ in the \mathcal{E} -span of the differences $\sigma - \tau$, the subgroup G contains a translation $T_{\lambda,?}$.

By hypothesis the differences $s - t$, with $s, t \in S_0$, projected into $\rho^\perp / \langle \rho \rangle$, span $\rho^\perp / \langle \rho \rangle$ over \mathcal{E} . Another way to say this is the \mathcal{E} -span of the corresponding differences $\sigma - \tau$ is all of Λ . So we have shown that for every $\lambda \in \Lambda$, the subgroup G contains a translation $T_{\lambda,?}$. Using these in the commutator formula (4-5), and using $\Lambda = \theta \Lambda^*$, shows that G also contains $T_{0,z}$ for every $z \in \text{Im } \mathcal{E}$. It follows that G contains all translations. □

Lemma 4.6 Suppose ρ, A, a and S_0 are as in Lemma 4.5, and define

$$S = \{s, s - \rho \mid s \in S_0\}.$$

Then G_a is generated by the Leech meridians $M_{a,A,s}$, where s varies over S .

Proof Write G for the subgroup of G_a generated by these Leech meridians. Also, fix some $s \in S$. For any other Leech root t , there is a translation $g \in P\Gamma_A$ that sends s to t . We choose a path γ in A from a to $g(a)$, and identify g with the element (γ, g) of the orbifold fundamental group G_a , via (4-1). By the previous lemma we know that (γ, g) lies in G . By drawing a picture, it is easy to see that

$$(\gamma, g) \cdot (\mu_{a,A,s}, R_s) \cdot (\gamma, g)^{-1} = (\mu_{a,A,t}, R_t)$$

in G_a , so G also contains the Leech meridian associated to t . Since t was an arbitrary Leech root, G contains all the Leech meridians. These generate G_a by [Theorem 4.1](#), completing the proof that $G = G_a$. \square

Proof of [Theorem 4.2](#) This follows from the previous lemma with S_0 equal to the set of Leech roots p_i and $\bar{\omega} p_i - l_j$, where $i, j = 1, \dots, 13$ and p_i and l_j are incident. To apply the lemma, we must show that the image of K in $\rho^\perp / \langle \rho \rangle$ is equal to $\rho^\perp / \langle \rho \rangle$, where K is the \mathcal{E} -span of $\{s - t : s, t \in S_0\}$. We show that in fact $K = \rho^\perp$.

Note that ρ has the same inner product with all elements of S_0 . So $K \subseteq \rho^\perp$. For the converse, let K_1 be the \mathcal{E} -span of $\{(p_i - p_1), (l_i - l_1) : i = 1, \dots, 13\}$. Observe that K contains $(p_1 - p_i)$. Given a line-root l_i , let p_j be the point-root incident to l_i and l_1 . Writing $(l_i - l_1) = (\bar{\omega} p_j - l_1) - (\bar{\omega} p_j - l_i)$ we find that K contains $(l_i - l_1)$ too. We have now shown that $K_1 \subseteq K$.

Now suppose $v \in \rho^\perp$; we must show that it lies in K . By adding a multiple of $(\bar{\omega} p_1 - l_1) - p_1 \in K$, which has inner product θ with p_∞ , we may suppose without loss of generality that $v \perp p_\infty$. Being orthogonal to ρ and p_∞ , the vector v is also orthogonal to l_∞ . Since L is spanned by the point- and line-roots, we may write $v = y + ap_1 + bl_1$ for some $y \in K_1$ and $a, b \in \mathcal{E}$. Taking the inner product with p_∞ and l_∞ and using the fact that $p_\infty, l_\infty \in K_1^\perp$ shows $a = b = 0$. So $v = y \in K_1 \subseteq K$. \square

5 Twenty-six generators based “at” a cusp

The goal of this section is to prove the [Theorem 5.1](#) below, which improves the 130 generators of [Theorem 4.2](#) to the 26 meridians associated to the point- and line-roots. This establishes our main theorem ([Theorem 1.2](#)), except with the wrong basepoint, which we will change to the correct basepoint in the next section. We recall that in the $P^2\mathbb{F}_3$ model of L from [Section 2.4](#), the point-roots are the 13 vectors of the form $(0; \theta, 0^{12})$ and the line-roots are the 13 vectors of the form $(1; 1^4, 0^9)$. We also recall from [Section 2.8](#) that the null vector $\rho = (3\omega - 1; -1, \dots, -1)$ represents a Leech cusp, that A indicates a closed horoball centered at ρ , small enough to miss \mathcal{H} , and that $a \in A$ indicates the basepoint. Finally, recall that if s is a point- or line-root, then we will call the corresponding meridian $M_{a,A,s} \in G_a$ a point- or line-meridian (based at a).

Theorem 5.1 (26 generators based “at” a cusp) *The 26 point- and line-meridians $M_{a,A,s}$, based at a , generate the orbifold fundamental group G_a .*

To prove it we will show that the subgroup G of G_a generated by the point- and line-meridians contains the 130 Leech meridians from [Theorem 4.2](#). That theorem says that the 130 meridians generate G_a , so [Theorem 5.1](#) follows. We remark that the point-roots are Leech roots with respect to ρ , so their corresponding meridians are Leech meridians. The line-roots are 2^{nd} -shell roots rather than Leech roots.

Here is an overview of our recipe for getting additional Leech meridians from the point- and line-meridians. Suppose x and y are Leech roots with inner product $\pm\theta$, or a point-root and an incident line-root, in which case they also have inner product $\pm\theta$. The span of two roots with inner product $\pm\theta$ is a well-known Eisenstein lattice, called $D_4^\mathcal{E}$ because its underlying \mathbb{Z} -lattice is a copy of the usual D_4 root lattice, scaled to have minimal norm 3. Up to scale, this lattice has just four roots: x , y , $R_x(y)$ and $R_y(x)$. One can work these out explicitly, and it turns out that there is exactly one Leech root that is not a scalar multiple of x or y . Call it z . We will show that the subgroup of G_a generated by $M_{a,A,x}$ and $M_{a,A,y}$ also contains $M_{a,A,z}$. “New” Leech meridians obtained this way can be combined with old ones using the same method, to obtain even more Leech meridians. Repeating the process many times enables us to prove that G contains the 130 Leech meridians of [Theorem 4.2](#).

This idea suggested itself because the braid group associated to the group generated by the triflections in $\text{Aut } D_4^\mathcal{E}$ is known to be isomorphic to the usual 3-strand braid group B_3 , with the conjugacy class of our meridians corresponding to the conjugacy class of the standard generators of B_3 . Although we don’t need it, we remark that the origin of this isomorphism is the fact that the hyperplane complement for this finite complex reflection group, modulo the group, is diffeomorphic to the corresponding space for the usual action of the symmetric group S_3 on \mathbb{C}^2 . This coincidence was first noticed by Orlik and Solomon [\[30\]](#).

So it was natural to hope that the meridians $M_{a,A,x}$, $M_{a,A,y}$ and $M_{a,A,z}$ all lie in a single copy of B_3 , with the first two of them serving as generators. So our first result is that this braid group is generated by two specific meridians, for either of two specific basepoints. It seems silly to have to prove this, since the braid groups of finite complex reflections groups are well understood [\[8; 13; 17\]](#). But we do not know of any quotable result stating generation by particular loops with particular basepoints.

Lemma 5.2 (generators for the braid group of $D_4^\mathcal{E}$) *Consider the lattice $D_4^\mathcal{E}$ in \mathbb{C}^2 , namely the \mathcal{E} -span of $\alpha = (\theta, 0)$ and $\beta = (1, \sqrt{2})$ under the usual inner product. Let*

\mathcal{H} be the union of the mirrors orthogonal to the 24 roots of $D_4^{\mathbb{C}}$, and let c be any scalar multiple of $(-1, -\bar{\omega}\sqrt{2})$ or $\frac{1}{2}\omega(\theta + 1, (\theta - 2)\sqrt{2})$. Then

$$J_c = \pi_1^{\text{orb}}((\mathbb{C}^2 - \mathcal{H})/\langle R_\alpha, R_\beta \rangle, c)$$

is generated by $M_{c,\alpha}$ and $M_{c,\beta}$.

Remark The factors of $\sqrt{2}$ can be avoided by using three coordinates with the last two being equal: $\alpha = (\theta, 0, 0)$ and $\beta = (1, 1, 1)$.

Proof We begin by remarking that $\alpha^2 = \beta^2 = 3$ and $\langle \alpha | \beta \rangle = \theta$. Let us write $\gamma = (1, \bar{\omega}\sqrt{2})$ and $\delta = (1, \omega\sqrt{2})$ for the remaining roots of $D_4^{\mathbb{C}}$, up to scale. We will begin the proof by establishing the lemma for a different basepoint $c_0 = (-2, \sqrt{2} + \sqrt{6})$. This basepoint is probably the most natural: when one plots the situation in $\mathbb{C}P^1 \cong S^2$, the four mirrors form the vertices of a regular tetrahedron and c_0 corresponds to the midpoint of the edge between α^\perp and β^\perp . We will suppress the subscript 0 until we need to refer to the other c_i .

One checks that $\langle c | \alpha \rangle$ and $\langle c | \beta \rangle$ are equal in absolute value, and smaller in absolute value than $\langle c | \gamma \rangle$ and $\langle c | \delta \rangle$. Therefore α^\perp and β^\perp are the mirrors nearest c . Write J for the subgroup of J_c generated by $M_{c,\alpha} = (\mu_{c,\alpha}, R_\alpha)$ and $M_{c,\beta} = (\mu_{c,\beta}, R_\beta)$. We are claiming that J is all of J_c . Theorem 1.2 of [5] gives the following sufficient conditions for this. First, J should surject to $\langle R_\alpha, R_\beta \rangle$, which is obvious. Second, some triflection in α or β should move c closer to c 's projection to γ^\perp , and some triflection should do the same with δ in place of γ . Direct calculation reveals that R_α does this for γ^\perp and R_α^{-1} does it for δ^\perp . This finishes the proof of the lemma for the basepoint c_0 .

A different method is required for the other basepoints. First we suppose the basepoint c is a scalar multiple of $(-1, -\bar{\omega}\sqrt{2})$. It does not matter which multiple, because scaling permutes the possibilities (and transforms their meridians accordingly). We choose c to be the particular multiple $c_1 = (\theta\bar{\omega} - \omega\sqrt{3})(-1, -\bar{\omega}\sqrt{2})$. The advantage of this scaling is that $\langle c_0 | c_1 \rangle$ is positive, so the angle between c_0 and c_1 is as small as possible. We regard this as saying that c_1 is “as close as possible” to c_0 , which is desirable because we will study the identification of J_{c_0} with J_{c_1} obtained by moving c_0 to c_1 along $\overline{c_0c_1}$. Namely, we will show that $M_{c_0,\alpha} \in J_{c_0}$ corresponds to $M_{c_1,\alpha} \in J_{c_1}$, and similarly with β in place of α . From this correspondence and the previous paragraph it follows immediately that $M_{c_1,\alpha}$ and $M_{c_1,\beta}$ generate J_{c_1} .

We will use Lemmas 3.2 and 3.8 to show that $\overline{c_0c_1}$ identifies $M_{c_0,\alpha} \in J_{c_0}$ with $M_{c_1,\alpha} \in J_{c_1}$. For concreteness we only stated these lemmas for complex hyperbolic space, but they apply equally well in complex Euclidean space. The reason is that their proofs and the definition of the meridians rely only on general concepts like geodesics, projections to hyperplanes, complex lines and totally real triangles, and the only property of the underlying space we need is nonpositive curvature.

We will write p_i for the projection of c_i to α^\perp . We begin by using Lemma 3.2 to show that $\overline{c_0c_1}$ identifies $M_{c_0,p_0,\alpha} \in J_{c_0}$ with $M_{c_1,p_0,\alpha} \in J_{c_1}$. (Note that $M_{c_0,p_0,\alpha}$ is the same as $M_{c_0,\alpha}$, since p_0 is c_0 's projection to α^\perp .) To apply this lemma we must check that the triangle $T = \Delta c_0c_1p_0$ meets \mathcal{H} only at p_0 , and that hypothesis (\star) of that lemma holds. The latter is trivial because there are no mirrors through p_0 except α^\perp . To show $\mathcal{H} \cap T = \{p_0\}$, one checks that α^\perp meets T only at p_0 and that β^\perp , γ^\perp and δ^\perp are disjoint from T . To carry out this check for α^\perp , one observes that the triangle in \mathbb{C} with vertices $\langle c_0|\alpha \rangle$, $\langle c_1|\alpha \rangle$ and $\langle p_0|\alpha \rangle$ touches the origin only at its vertex $\langle p_0|\alpha \rangle$. The corresponding triangles in \mathbb{C} for β , γ and δ do not contain the origin at all. This completes the identification of $M_{c_0,\alpha} \in J_{c_0}$ with $M_{c_1,p_0,\alpha} \in J_{c_1}$.

Next we use Lemma 3.8 to prove the equality $M_{c_1,p_0,\alpha} = M_{c_1,\alpha}$ in J_{c_1} . This requires checking $\mathcal{H} \cap \Delta c_1p_0p_1 = \overline{p_0p_1}$, which uses the same method as the check in the previous paragraph. This completes the proof that $\overline{c_0c_1}$ identifies $M_{c_0,\alpha} \in J_{c_0}$ with $M_{c_1,\alpha} \in J_{c_1}$. The same method applies with β in place of α . Since $M_{c_0,\alpha}$ and $M_{c_0,\beta}$ generate J_{c_0} , this completes the proof that $M_{c_1,\alpha}$ and $M_{c_1,\beta}$ generate J_{c_1} .

Exactly the same analysis applies to the second possibility for c . The scalar multiple of $\frac{1}{2}\omega(\theta + 1, (\theta - 2)\sqrt{2})$ that we used was

$$c_2 = -(3 + 2\sqrt{3} + \theta\sqrt{3})(\theta + 1, (\theta - 2)\sqrt{2}),$$

for the same reason as before. □

As explained above, our method starts with two Leech meridians, corresponding to Leech roots with inner product $\pm\theta$, and shows that the group they generate contains a third Leech meridian. So naturally we will need to understand such pairs of Leech roots. At the same time we will classify the pairs of Leech roots with inner product $-\frac{3}{2} \pm \frac{\theta}{2}$. We do not care about these pairs themselves. But they turn out to be key for the other half of our method: using a point-meridian and a line-meridian to generate another Leech meridian. The precise statements of our constructions of “new” Leech meridians are Lemmas 5.4(3) and 5.6.

Lemma 5.3 Under the Γ -stabilizer Γ_ρ of ρ , there are four orbits of ordered pairs (s, s') of Leech roots with $|\langle s | s' \rangle| = \sqrt{3}$. In the Leech model of L , orbit representatives are the pairs (s, s') of the form

$$s = (0; 1, -\omega) \quad \text{and} \quad s' = \begin{cases} (\lambda_6; 1, \omega) & \text{with } \langle s | s' \rangle = \theta, \\ (\lambda_9; 1, \theta) & \text{with } \langle s | s' \rangle = -\frac{3}{2} + \frac{\theta}{2} = \bar{\omega}\bar{\theta}, \end{cases}$$

and the pairs obtained from these by exchanging s, s' . Here λ_6 and λ_9 are any fixed vectors of norms 6 and 9 in Λ .

Proof Suppose (s, s') is such an ordered pair. In the Leech model, the Leech roots are parametrized by (2-8). So s has the form $(\sigma; 1, \theta(\frac{1}{6}(\sigma^2 - 3) + \nu))$, where σ lies in the Leech lattice and ν is purely imaginary and chosen so that the last coordinate lies in \mathcal{E} . And similarly for s' . By the transitivity of Γ_ρ on Leech roots (Lemma 4.3), we may take $\sigma = 0$ and $\nu = \frac{1}{2\theta}$, yielding $s = (0; 1, \theta(-\frac{1}{2} + \frac{1}{2\theta})) = (0; 1, -\omega)$. The inner product $\langle s | s' \rangle$ is best understood using (2-7), which in this case reduces to

$$\langle s | s' \rangle = \frac{1}{2}(6 - \sigma'^2) + 3(\nu' - \frac{1}{2\theta}).$$

The first term is real and the second is imaginary. Now we consider the elements of \mathcal{E} of absolute value $\sqrt{3}$. First, $\frac{3}{2} \pm \frac{\theta}{2}$ cannot occur, because σ' would have to have norm 3 in order for $\langle s | s' \rangle$ to have real part $\frac{3}{2}$. This is impossible since the Leech lattice has minimal norm 6. By the same reasoning, $\pm\theta$ can occur only when σ' has norm 6, and $-\frac{3}{2} \pm \frac{\theta}{2}$ can occur only when σ' has norm 9. Wilson proved that $\text{Aut } \Lambda = 6 \cdot \text{Suz}$ acts transitively on the norm 6 (resp. norm 9) elements of Λ ; see [31, page 155] and note that these are his “type 2” (resp. “type 3”) vectors. Thus we may suppose without loss of generality that $\sigma' = \lambda_6$ or λ_9 . Then ν' is determined by the imaginary part of $\langle s | s' \rangle$. When $\langle s | s' \rangle = \theta$ we get $3(\nu' - \frac{1}{2\theta}) = \theta$, so $\nu' = -\frac{1}{2\theta}$ and s' is as displayed. Similarly, when $\langle s | s' \rangle = -\frac{3}{2} + \frac{\theta}{2}$ one gets $\nu' = 0$. \square

The proofs of the next two lemmas depend on how certain totally real triangles in \mathbb{B}^{13} meet \mathcal{H} . Because the calculations are long, and similar computations will be needed later, we moved the verifications to Appendix A. Lemma 5.4 relies on Lemma A.3 and Lemma 5.5 relies on Lemma A.4.

Lemma 5.4 Suppose x, y are Leech roots with $\langle x | y \rangle = \theta$. Let p be the projection of ρ to x^\perp , and q the projection of ρ to $x^\perp \cap y^\perp$. Then:

- (1) Of the four mirrors of the reflection group $\langle R_x, R_y \rangle$, three are Leech mirrors (corresponding to the Leech roots x, y and $z = -\bar{\omega}x - \omega y$) and one is a second-shell mirror.

- (2) These four mirrors are the only ones containing q .
- (3) The subgroup of G_a generated by the Leech meridians $M_{a,A,x}$ and $M_{a,A,y}$ also contains the Leech meridian $M_{a,A,z}$.

Remark An example of the situation described in this lemma is given by $x = -R_{p_2}(l)$ and $y = p$, where l is a line-root and p, p_2 are two point-roots incident to l . We formulated our notion of meridian to include cases where the basepoint is not in general position. The situation in this lemma is such a case, because q is the projection of ρ to the second-shell mirror in (1), and 4 mirrors pass through q . One can prove that this is the only sort of nongenericity that occurs for our lattice L , but we shall not need this.

Proof By Lemma 5.3 we may suppose without loss of generality that $x = (0; 1, -\omega)$ and $y = (\lambda_6; 1, \omega)$. And we suppose without loss of generality that our basepoint a lies in $\overline{\rho q} \cap A$; see Remark 3.5.

(1) Using $x^2 = y^2 = 3$ and $\langle x|y \rangle = \theta$, one checks that z and $\omega x - \bar{\omega}y$ are roots. Then, using $\langle \rho|x \rangle = \langle \rho|y \rangle = \theta$, one checks that $\langle \rho|z \rangle = \theta$, so that z is also a Leech root, and that $\langle \rho|\omega x - \bar{\omega}y \rangle = 3$, so that $(\omega x - \bar{\omega}y)^\perp$ is a second-shell mirror.

(2) Lemma A.3 identifies all the mirrors that meet the totally real triangle $\Delta\rho pq$. The only ones that do are the four from (1). So they are the only ones that can contain q .

Before proving (3) we establish two preparatory results. First, there is an isometry of L that fixes ρ and permutes x, y and z cyclically. To see this, one checks that $\langle y|z \rangle = \theta$ and applies the transitivity in Lemma 5.3 to conclude that some $g \in \Gamma_\rho$ sends (x, y) to (y, z) . Now, g must permute the three Leech roots in the sublattice $\mathcal{E}x + \mathcal{E}y = \mathcal{E}y + \mathcal{E}z$. Since g sends x to y and y to z , it must also send z to x .

Second, we claim that $\mu_{a,A,x}$ is homotopic to $\mu_{a,A,q,x}$ in $\mathbb{B}^{13} - \mathcal{H}$, rel endpoints, and similarly with y or z in place of x . By the symmetry just established, the y and z cases follow from the x case. Lemma A.3 describes how the mirrors of \mathcal{H} meet $\Delta\rho pq$. Namely, x^\perp contains $\overline{\rho q}$, the other three mirrors from (1) meet the triangle at q only, and all other mirrors miss it completely. This verifies the hypothesis of Lemma 3.7, whose conclusion is that $\mu_{a,A,x} \simeq \mu_{a,A,q,x}$, as claimed.

(3) Let U be an open ball around q , small enough that the only mirrors meeting it are the ones through q . We also take it small enough to miss A . Let c be any point of $\overline{\rho q} \cap U$ other than q . Examining the definition of the meridian shows that we may suppose without loss of generality that c is the “turning point” of $\mu_{a,A,q,x}$. Formally,

$\mu_{a,A,q,x}$ equals \overline{ac} followed by $\mu_{c,x}$ followed by $R_x(\overline{ca})$. (We write \overline{ac} in place of $\overline{a\widehat{c}}$ because they are the same: $\overline{ac} \subseteq \overline{\rho q}$ misses \mathcal{H} by our choice of a at the beginning of the proof.) Another way to state the homotopy $\mu_{a,A,x} \simeq \mu_{a,A,q,x}$ of the previous paragraph is that $M_{a,A,x} = (\mu_{a,A,x}, R_x) \in G_a$ corresponds to $M_{c,x} = (\mu_{c,x}, R_x) \in G_c$ under the identification of G_a with G_c induced by \overline{ac} . And similarly with y or z in place of x , by the $x \rightarrow y \rightarrow z \rightarrow x$ symmetry established earlier. Therefore proving (3) is equivalent to proving that $M_{c,z}$ lies in the subgroup of G_c generated by $M_{c,x}$ and $M_{c,y}$.

We focus attention on U by defining a “local” analogue of G_c , namely

$$J_c = \pi_1^{\text{orb}}((U - \mathcal{H})/\langle R_x, R_y \rangle, c).$$

By considering the natural map $J_c \rightarrow G_c$, it suffices to prove that $M_{c,z}$ lies in the subgroup of J_c generated by $M_{c,x}$ and $M_{c,y}$. We convert this into a problem in the Euclidean space $T_q\mathbb{B}^{13}$ as follows. We write g_0 for the Riemannian metric on \mathbb{B}^{13} obtained by identifying \mathbb{B}^{13} with $T_q\mathbb{B}^{13}$ under the exponential map. This metric is Euclidean, so we write $\mu_{c,x}^{\text{Euc}}$ for the path defined like $\mu_{c,x}$, but using this Euclidean metric in place of the hyperbolic metric. We claim that $\mu_{c,x}^{\text{Euc}}$ and $\mu_{c,x}$ are homotopic rel endpoints in $U - \mathcal{H}$. In particular, $M_{c,x} = (\mu_{c,x}, R_x)$ and $M_{c,x}^{\text{Euc}} = (\mu_{c,x}^{\text{Euc}}, R_x)$ are the same element of J_c , and similarly with y or z in place of x . After proving this claim, it will suffice to prove that $M_{c,x}^{\text{Euc}}$ and $M_{c,y}^{\text{Euc}}$ generate J_c .

To prove the claim, we define for each $t \in (0, 1]$ a diffeomorphism S_t of \mathbb{B}^{13} , namely the one corresponding to the scaling-by- t map on $T_q\mathbb{B}^{13}$. Let g_t be the Riemannian metric obtained by pulling back the complex hyperbolic metric under S_t and multiplying it by $1/t$. It is easy to see that the g_t converge to g_0 as $t \rightarrow 0$. For any $t \in [0, 1]$ we define $\mu_{c,x}^t$ just as we did for $\mu_{c,x}$, except that we use g_t in place of the hyperbolic metric. For $t = 0$ this is just $\mu_{c,x}^{\text{Euc}}$, and for $t > 0$ this definition makes sense because (\mathbb{B}^{13}, g_t) is isometric to \mathbb{B}^{13} with its metric scaled. All the ingredients in this definition (nearest points, geodesics, etc) vary continuously with t , so the $\mu_{c,x}^t$ provide a homotopy between $\mu_{c,x}^1 = \mu_{c,x}$ and $\mu_{c,x}^0 = \mu_{c,x}^{\text{Euc}}$, as desired.

It remains to show that $M_{c,z}^{\text{Euc}}$ lies in the group generated by $M_{c,x}^{\text{Euc}}$ and $M_{c,y}^{\text{Euc}}$. Obviously we may restrict attention to the \mathbb{B}^2 orthogonal to $\langle x, y \rangle^\perp$ at q . We will identify $T_q\mathbb{B}^2$ with the \mathbb{C}^2 from Lemma 5.2, in a way which lets us quote that result. We first note that $\mathbb{B}^2 = \mathbb{B}(V)$, where V is the complex span of ρ, x and y . We write W for the subspace of V spanned by x and y , and Q for the linear projection of ρ to W^\perp .

This is a vector representing $q \in \mathbb{B}^2$. Identify \mathbb{B}^2 with a neighborhood of 0 in W , by

$$(w \in W) \leftrightarrow (\text{the image in } PV \text{ of } Q + w).$$

Up to a scaling factor, this correspondence is essentially the exponential map of $\mathbb{B}^{1,3}$ at q . This yields an identification of $T_q \mathbb{B}^2$ with $T_0 W = W$, in which the mirrors through q correspond to the mirrors of the D_4^c spanned by x and y , and this identification is equivariant under the action of the finite group generated by R_x and R_y . (We saw in (2) that no more mirrors pass through q .) Furthermore, c lies in $P(\mathbb{C}Q + \mathbb{C}\rho)$, whose corresponding complex line in W is spanned by the projection $\pi_W(\rho)$ of ρ to W . This projection can be worked out from $\langle \rho | x \rangle = \langle \rho | y \rangle = \theta$, namely $\pi_W(\rho) = \omega x - \bar{\omega} y$.

Now we identify x (resp. y) with α (resp. β) in Lemma 5.2. Under this identification, $T_q \mathbb{B}^2 - T_q \mathcal{H}$ is identified with the mirror complement of D_4^c , with c corresponding to some scalar multiple of $\omega\alpha - \bar{\omega}\beta = (-1, -\bar{\omega}\sqrt{2})$. Lemma 5.2 tells us that $M_{c,\alpha}^{\text{Euc}}$ and $M_{c,\beta}^{\text{Euc}}$ generate $\pi_1^{\text{orb}}((\mathbb{C}^2 - \mathcal{H})/\langle R_\alpha, R_\beta \rangle, c)$. Transferring this back to $U - \mathcal{H}$ finishes the proof. □

Lemma 5.5 *Suppose x, z are Leech roots with $\langle x | z \rangle = -\frac{3}{2} + \frac{\theta}{2}$, and let q be the projection of ρ to the intersection of their mirrors. Then:*

- (1) *Of the four mirrors of the reflection group $\langle R_x, R_z \rangle$, two are Leech mirrors (x^\perp and z^\perp), one is a second-shell mirror (y^\perp for $y = \bar{\omega}x - z$), and one is a third-shell mirror.*
- (2) *These mirrors are the only ones containing q .*
- (3) *The subgroup of G_a generated by the Leech meridian $M_{a,A,x}$ and the second-shell meridian $M_{a,A,y}$ also contains the Leech meridian $M_{a,A,z}$.*

Proof This is very similar to the proof of Lemma 5.4. By Lemma 5.3 we may suppose without loss of generality that $x = (0; 1, -\omega)$ and $z = (\lambda_9; 1, \theta)$, and as before we take $a \in \bar{\rho}q$.

(1) Using $x^2 = z^2 = 3$ and $\langle x | z \rangle = -\frac{3}{2} + \frac{\theta}{2}$, one checks that y and $x + z$ are roots. So the 24 roots in $\langle x, z \rangle$ are their unit multiples together with those of x and z . Using $\langle \rho | x \rangle = \langle \rho | z \rangle = \theta$, one computes $\langle \rho | y \rangle = 3\bar{\omega}$ and $\langle \rho | x + z \rangle = 2\theta$. So y is a second-shell root and $x + z$ is a third-shell root.

(2) This is just like the corresponding part of the previous lemma, except that we refer to Lemma A.4 in place of Lemma A.3.

In preparation for (3), we claim that $\mu_{a,A,x}$ is homotopic to $\mu_{a,A,q,x}$ in $\mathbb{B}^{13} - \mathcal{H}$, rel endpoints, and similarly with y or z in place of x . This is just like the corresponding part of the previous proof, except that there is no cyclic symmetry. So one has to analyze three triangles rather than just one. This is done in Lemma A.4.

(3) This is just like the corresponding part of the previous lemma. The only difference is in the very last step: now $\langle \rho | x \rangle = \theta$ and $\langle \rho | y \rangle = 3\bar{\omega}$, so $\pi_W(\rho) = -\frac{\theta}{2}\omega x + (\frac{\theta}{2} - 1)\omega y$. We still identify $T_q\mathbb{B}^2$ with \mathbb{C}^2 by taking x and y to correspond to the α and β of Lemma 5.2. This makes sense since one can check $\langle x | y \rangle = \theta$. Under this identification, $\pi_W(\rho)$ corresponds to $\frac{1}{2}\omega(\theta + 1, (\theta - 2)\sqrt{2}) \in \mathbb{C}^2$, and we can apply Lemma 5.2 just as before. □

Lemma 5.6 *Suppose x is a point-root and y is an incident line-root. Then $z = \bar{\omega}x - y$ is a Leech root, and the subgroup of G_a generated by $M_{a,A,x}$ and $M_{a,A,y}$ contains the Leech meridian $M_{a,A,z}$.*

Proof Using $x^2 = y^2 = 3$ and $\langle x | y \rangle = \theta$, one checks $z^2 = 3$, so z is a root. Using $\langle \rho | x \rangle = \theta$ and $\langle \rho | y \rangle = 3\bar{\omega}$, one checks $\langle \rho | z \rangle = \theta$, so z is a Leech root. Similarly, one checks that $\langle x | z \rangle = -\frac{3}{2} + \frac{\theta}{2}$. So we may apply Lemma 5.5 to x and z . One checks that the root called y there is the same as the one we call y here. To finish the proof we appeal to Lemma 5.5(3). □

Proof of Theorem 5.1 Let G denote the subgroup of G_a generated by the 26 point- and line-meridians $M_{a,A,s}$ based at a . We must prove $G = G_a$. By Theorem 4.2 it suffices to show that G contains the Leech meridians associated to the Leech roots $p_i, \bar{\omega}p_i - l_j, p_i - \rho$ and $\bar{\omega}p_i - l_j - \rho$, where $i, j = 1, \dots, 13$ and p_i and l_j are incident. G contains the Leech meridians associated to the p_i by definition, and those associated to the $\bar{\omega}p_i - l_j$ by Lemma 5.6.

Recall that the finite projective linear group $L_3(3)$ permutes the point- and line-roots in the same way that it permutes the points and lines of $P^2\mathbb{F}_3$. (See Section 2.6.) This group preserves ρ , hence A , and Remark 3.5 lets us suppose the basepoint $a \in A$ is also $L_3(3)$ -invariant. So $L_3(3)$ permutes the point- and line-meridians in the same way it permutes the points and lines of $P^2\mathbb{F}_3$, and therefore sends G to itself. We must prove that G contains the Leech meridians associated to the Leech roots $p_i - \rho$ and $\bar{\omega}p_i - l_j - \rho$. By using the $L_3(3)$ symmetry, it suffices to do this for $p_1 - \rho$ and $\bar{\omega}p_1 - l_1 - \rho$.

For $p_1 - \rho$, consider the sequence of Leech roots

$$\begin{array}{lll}
 s_1 = \bar{\omega} p_1 - l_1, & s_7 = -\bar{\omega} s_1 - \omega p_4, & \\
 s_2 = \bar{\omega} p_1 - l_{11}, & s_8 = -\bar{\omega} s_4 - \omega p_3, & s_{13} = -\bar{\omega} s_{12} - \omega s_{11}, \\
 s_3 = \bar{\omega} p_1 - l_{13}, & s_9 = -\bar{\omega} s_4 - \omega p_5, & s_{14} = -\bar{\omega} s_{13} - \omega p_1, \\
 s_4 = \bar{\omega} p_2 - l_2, & s_{10} = -\bar{\omega} s_5 - \omega s_8, & s_{15} = -\bar{\omega} s_3 - \omega s_{14}, \\
 s_5 = \bar{\omega} p_5 - l_5, & s_{11} = -\bar{\omega} s_6 - \omega s_9, & s_{16} = -\bar{\omega} s_{15} - \omega s_7, \\
 s_6 = \bar{\omega} p_{11} - l_{11}, & s_{12} = -\bar{\omega} s_2 - \omega s_{10}, & s_{17} = -\bar{\omega} s_{16} - \omega p_{10}.
 \end{array}$$

In each equation in the left column, the point- and line-root on the right-hand side are incident. By Lemma 5.6, G contains the Leech meridian associated to the Leech root defined by that equation. In each equation in the other two columns, the roots appearing on the right side are Leech roots and $\langle \text{the first one} | \text{the second} \rangle = \theta$. By repeated use of Lemma 5.4(3), G contains the meridians associated to the Leech roots defined by these equations. And one checks that $s_{17} = p_1 - \rho$.

For $\bar{\omega} p_1 - l_1 - \rho$ the argument is the same, defining

$$\begin{array}{lll}
 s_1 = \bar{\omega} p_3 - l_2, & s_6 = -\bar{\omega} s_1 - \omega p_5, & \\
 s_2 = \bar{\omega} p_6 - l_5, & s_7 = -\bar{\omega} s_5 - \omega p_{12}, & s_{11} = -\bar{\omega} s_8 - \omega s_{10}, \\
 s_3 = \bar{\omega} p_8 - l_5, & s_8 = -\bar{\omega} s_4 - \omega s_6, & s_{12} = -\bar{\omega} s_2 - \omega s_{11}, \\
 s_4 = \bar{\omega} p_2 - l_6, & s_9 = -\bar{\omega} s_3 - \omega s_7, & s_{13} = -\bar{\omega} s_{12} - \omega s_1, \\
 s_5 = \bar{\omega} p_2 - l_{12}, & s_{10} = -\bar{\omega} s_9 - \omega p_{13}, & s_{14} = -\bar{\omega} s_{13} - \omega p_{11},
 \end{array}$$

and checking that $s_{14} = \bar{\omega} p_1 - l_1 - \rho$. We found these sequences of Leech roots by a rather intensive computer search. But their validity can be verified easily. \square

6 Change of basepoint

In this section we prove the main theorem of the paper, Theorem 1.2: the orbifold fundamental group $G_\tau = \pi_1^{\text{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, \tau)$ is generated by the 26 point- and line-meridians $M_{\tau, H}$. Here $\tau = (4 + \sqrt{3}; 1, \dots, 1)$ is the 26–point specified in Section 2.6 and H varies over the point- and line-mirrors. The starting point of the proof is Theorem 5.1: for any basepoint a in the horoball A centered at the Leech cusp ρ , the 26 point- and line-meridians $M_{a, A, H}$ generate G_a . Here $\rho = (3\omega - 1; -1, \dots, -1)$ is the Leech cusp defined in Section 2.8 and used throughout Section 5, and H varies over the same 26 mirrors. By Remark 3.5 we may choose a to be a point of $\bar{\tau}\rho$ very

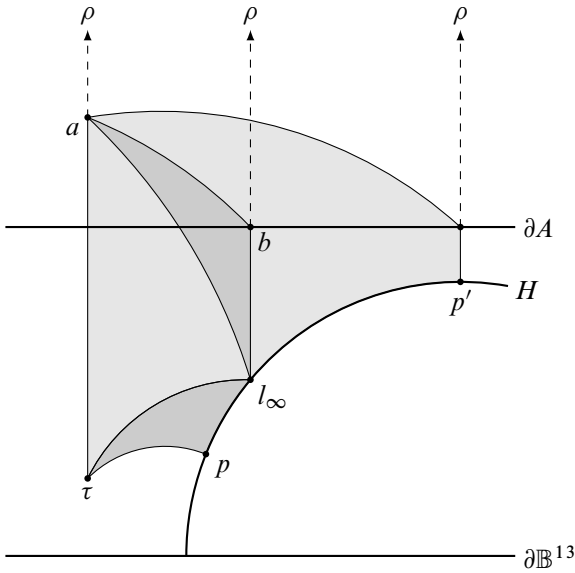


Figure 6.1: The regions correspond to the four steps in the proof of Lemma 6.1, starting from the bottom. We use the upper half-space model with the Leech cusp ρ at vertical infinity and A being the horoball centered there, whose boundary is the horizontal line. Each of the regions misses \mathcal{H} except for obvious intersection points.

close to ρ ; exactly how close will be specified later. So $\overline{\tau a}$ is a subsegment of $\overline{\tau \rho}$. In light of Theorem 5.1, Theorem 1.2 follows immediately from the next lemma.

Lemma 6.1 *Suppose H is a point- or line-mirror. Then the meridian $M_{\tau,H} \in G_\tau$ corresponds to the meridian $M_{a,A,H} \in G_a$ under the isomorphism $G_\tau \cong G_a$ induced by the path $\overline{\tau a}$.*

For the lemma to make sense, one must verify that $\overline{\tau a}$ misses \mathcal{H} . Because $\overline{\tau a} \subseteq \overline{\tau \rho}$, this follows from the stronger result (Lemma B.2) that the complex triangle $\Delta \rho \tau l_\infty$ misses \mathcal{H} except at l_∞ .

Proof We will give the proof when H is a line-mirror, and then remark on the changes needed for the point-mirror case. Recall from Section 2.6 that l_∞ is where all 13 line-mirrors intersect. We will use a 4-step homotopy. The first step corresponds to the bottom region (shaded darkly) in Figure 6.1, the second step to the region above it (lighter), the third to the next (dark again), and the fourth to the rightmost region (light again). The key fact is that these regions miss \mathcal{H} except at known points. These verifications are Lemmas A.6, A.13 and B.2 in the appendices.

Step 1 The first step is to prove that $M_{\tau,H}$ is equal to $M_{\tau,l_\infty,H}$ in G_τ . This follows from [Lemma 3.8](#) once one checks that $\Delta\tau pl_\infty$ misses \mathcal{H} , except that $\overline{pl_\infty}$ lies in H and l_∞ may lie in additional hyperplanes. Here p means the point of H closest to τ . This check is [Lemma A.6](#).

Step 2 The second step is to prove that the isomorphism $G_\tau \cong G_a$ identifies $M_{\tau,l_\infty,H}$ with $M_{a,l_\infty,H}$. This follows from [Lemma 3.4](#), once one checks that the (complex) triangle $\Delta a\tau l_\infty$ misses \mathcal{H} except at l_∞ . This check is [Lemma B.2](#): it proves the corresponding result for $\Delta\rho\tau l_\infty$, which contains $\Delta a\tau l_\infty$.

Step 3 The third step is to prove that $M_{a,l_\infty,H}$ is equal to $M_{a,A,l_\infty,H}$ in G_a . By taking a very far up in [Figure 6.1](#) this becomes almost obvious. Namely, the uniform distance between $\overline{al_\infty}$ and the concatenation of \overline{ab} and $\overline{bl_\infty}$ tends to 0 as a approaches ρ . Here b is the point of A closest to l_∞ . Therefore, by taking a close enough to ρ , we may take the uniform distance between $M_{a,l_\infty,H}$ and $M_{a,A,l_\infty,H}$ to be arbitrarily small. We take it small enough that the straight-line homotopy misses \mathcal{H} . (This uses the fact verified in [Lemma A.13](#) that the geodesic segment joining b and l_∞ misses \mathcal{H} except at l_∞ .)

Step 4 The final step is like the first: $M_{a,A,l_\infty,H}$ is equal in G_a to $M_{a,A,H}$. This follows from [Lemma 3.7](#), once one checks that $\Delta\rho l_\infty p'$ misses \mathcal{H} except that $\overline{p'l_\infty} \subseteq H$ and that l_∞ may lie in additional mirrors. Here p' is the point of \mathcal{H} nearest ρ . This check is [Lemma A.13](#).

Putting these four steps together gives

$$M_{\tau,H} = M_{\tau,l_\infty,H} \leftrightarrow M_{a,l_\infty,H} = M_{a,A,l_\infty,H} = M_{a,A,H}.$$

Here the first two terms are equal in G_τ by the first step of the homotopy. The second and third terms correspond under the isomorphism $G_\tau \cong G_a$ induced by $\overline{\tau a}$, by the second step. And the last three terms are equal in G_a by the third and fourth steps. This finishes the proof for line-meridians.

The point-meridian case is exactly the same. Now H is a point-mirror and p and p' are the projections of τ and ρ to it. And we replace l_∞ by p_∞ . In [Step 1](#), $\Delta\tau pl_\infty$ gets replaced by $\Delta\tau pp_\infty$, whose intersection with \mathcal{H} is also given by [Lemma A.6](#). In [Step 2](#), $\Delta\rho l_\infty$ gets replaced by $\Delta\rho pp_\infty$, whose intersection with \mathcal{H} is also given by [Lemma B.2](#). In [Step 4](#), $\Delta\rho p'l_\infty$ gets replaced by $\Delta\rho p'p_\infty$, whose intersection with \mathcal{H} is given by [Lemma A.11](#) rather than [Lemma A.13](#). □

Appendix A: How eight totally real triangles meet the mirrors

In this appendix we examine how certain totally real triangles in $\overline{\mathbb{B}^{13}}$ meet the mirror arrangement \mathcal{H} . This was needed in the proofs of Lemmas 5.4, 5.5 and 6.1. Recall from Section 2.3 that if $x, y, z \in L \otimes \mathbb{C} - \{0\}$ have norms and inner products in $(-\infty, 0]$, then the convex hull of the corresponding points in $\overline{\mathbb{B}^{13}}$ is the projectivization of the convex hull of x, y and z in $L \otimes \mathbb{C}$. In Section 2.3 we called such a triangle in $\overline{\mathbb{B}^{13}}$ totally real. For totally real triangles we don't usually distinguish between the triangle in $L \otimes \mathbb{C}$ and the triangle in $\overline{\mathbb{B}^{13}}$. The following lemma is trivial but crucial.

Lemma A.1 (how to intersect a totally real triangle and a hyperplane) *Suppose $x, y, z \in L \otimes \mathbb{C}$ have norms and inner products in $(-\infty, 0]$. Let T be the totally real triangle Δxyz , and let $s \in L \otimes \mathbb{C}$. Then $T \cap s^\perp$ is the preimage of the origin under the map $T \rightarrow \mathbb{C}$ given by $v \mapsto \langle v | s \rangle$.*

In particular, if the convex hull in \mathbb{C} of $\langle x | s \rangle, \langle y | s \rangle$ and $\langle z | s \rangle$ does not contain the origin, then $T \cap s^\perp = \emptyset$. □

In the situation of the lemma we will write $\langle T | s \rangle$ for the convex hull in \mathbb{C} of $\langle x | s \rangle, \langle y | s \rangle$ and $\langle z | s \rangle$. Most arguments in this appendix amount to showing that $0 \notin \langle T | s \rangle$ for various triangles T and roots s . In all our applications, the vertices of the triangle $\langle T | s \rangle$ lie in $\mathbb{Q}(\omega, \sqrt{3})$. For some of the calculations we used the computer library PARI [12]; checking whether $\langle T | s \rangle$ contains the origin can be done with exact arithmetic in this field. Some other calculations were not computerizable because we had to consider infinitely many mirrors, for example finding which Leech mirrors meet a given triangle. For these cases we used Lemma A.1 together with special properties of the Leech lattice. For example, Lemmas A.3 and A.13 use the next result. Several times it seemed miraculous that some such property was exactly what we needed.

Lemma A.2 (Leech lattice points near λ_6/θ and λ_9/θ) *Suppose λ_6 and λ_9 are vectors in Λ with norms 6 and 9. Then the nearest points of Λ to λ_6/θ are three in number, at distance $\sqrt{2}$. And the nearest points of Λ to λ_9/θ are 36 in number, at distance $\sqrt{3}$.*

Proof The essential point is that the shortest elements of Λ which are congruent mod θ to a given lattice vector of norm 6 (resp. 9) are 3 (resp. 36) in number, all of

norm 6 (resp. 9). This is the lemma that is split between pages 153 and 155 of [31] but not related to the table on page 154. Also, his “type” is one third of our norm, so his type 2 and 3 vectors are the same as our norm 6 and 9 vectors.

We abbreviate λ_6/θ to C . The lattice Λ contains the vector $C - \lambda_6/\theta$, which lies at distance $\sqrt{2}$ from C . (It lies in Λ because it is the zero vector.) Now consider any lattice point at distance $\leq \sqrt{2}$ from C , and write it as $C + x/\theta$ with $x \in \Lambda \otimes \mathbb{C}$ of norm ≤ 6 . Since Λ contains the difference between this point and the one just mentioned, it contains $(x + \lambda_6)/\theta$. That is, x lies in Λ and is congruent to $-\lambda_6 \pmod{\theta}$. By Wilson’s result, the possibilities for x are the three minimal representatives of $-\lambda_6$ ’s congruence class mod θ . This finishes the λ_6 case. The λ_9 case is similar. \square

When we have a Leech cusp ρ in mind, we will call a horosphere centered there a *critical horosphere* if it is tangent to some mirror. We use the same language for the (open or closed) horoball it bounds. Recall the definition of a horosphere of height h given in the discussion following Equation (2-3). If s is a root then it is easy to see that the horosphere tangent to its mirror is the one of height $\frac{1}{3}|\langle \rho | s \rangle|^2$. (Compute the inner product of ρ with its projection onto s^\perp .) So the heights of the first four critical horospheres are 1, 3, 4 and 7. The first-, second- and third-shell mirrors (defined in Section 2.8) are the mirrors tangent to the first, second and third critical horoballs.

The following lemma is the first of several in this appendix having the general form: prove that some particular triangle misses the mirrors except for some obvious intersection points. The general strategy is to start by showing that the triangle is covered by the union of a ball and a horoball. We enumerate the finitely many mirrors that meet the ball, and check by direct computation that they miss the triangle. Then we enumerate the mirrors that meet the horoball; there are infinitely many such mirrors, but they correspond to roots of small height and therefore they may be parametrized. Checking that these mirrors miss the triangle requires some intricate analysis rather than just direct computation.

Lemma A.3 (the triangle needed in Lemma 5.4) *In the Leech model, consider the Leech roots $x = (0; 1, -\omega)$ and $y = (\lambda_6; 1, \omega)$, where λ_6 is a norm 6 vector in Λ . Define p (resp. q) as the projection of ρ to x^\perp (resp. $\langle x, y \rangle^\perp$). Then the totally real triangle $T = \Delta \rho p q$ meets \mathcal{H} as follows: x^\perp meets T in \overline{pq} , the other three mirrors of $\langle R_x, R_y \rangle$ meet T only at q , and all other mirrors miss T .*

Proof First we introduce various important points. Recall that $\rho = (0; 0, 1)$. By definition,

$$p = \rho - \frac{\langle \rho | x \rangle}{x^2} x = \left(0; \frac{1}{\theta}, -\frac{\bar{\omega}}{\theta} \right),$$

and one can compute $p^2 = \langle \rho | p \rangle = -1$. For later calculations, we also write p in the form (2-6), writing σ_p, m_p, N_p and v_p in place of the unsubscripted letters that appear there. Obviously we have $\sigma_p = 0$ and $m_p = \frac{1}{\theta}$. And N_p is p^2 , which we just computed to be -1 . One can then solve for v_p by equating the last coordinate of p with the form given in (2-6) and obtain $v_p = -\frac{\theta}{18}$. One can verify the following formula for q by checking that it is orthogonal to x and y :

$$q = \rho - (\omega x - \bar{\omega} y) = (\bar{\omega} \lambda_6; \bar{\theta}, \theta \bar{\omega}).$$

One can check $q^2 = \langle \rho | q \rangle = \langle p | q \rangle = -3$. Since the vectors we have chosen to represent the three vertices of $\Delta \rho p q$ have inner products in $(-\infty, 0]$, we will be able to apply Lemma A.1. Also, writing q in the form (2-6), just as we did for p , gives $\sigma_q = \bar{\omega} \lambda_6, m_q = \bar{\theta}, N_q = -3, v_q = -\frac{1}{2} \theta$. The four mirrors of the group $\langle R_x, R_y \rangle$ generated by R_x and R_y obviously meet T as claimed, so it suffices to show that no other mirrors meet T . By construction, p lies on the boundary of the first critical horoball. And $\text{ht}(q) = 3$, so q lies on the boundary of the second. Therefore only one mirror not in the first shell could meet the triangle, and then only at q . This mirror corresponds to the second-shell root $\omega x - \bar{\omega} y$ in $\mathcal{E}x + \mathcal{E}y$ that we left unnamed in Lemma 5.4(1). So it suffices to show that no Leech mirrors meet the triangle except the three coming from the sublattice $\mathcal{E}x + \mathcal{E}y$.

Our strategy is to write a general Leech root s in the form (2-8), namely

$$s = \left(\sigma; 1, \theta \left(\frac{1}{6} (\sigma^2 - 3) + v \right) \right),$$

and compute its inner products with ρ, p and q , and then apply Lemma A.1 to gain control over σ and v . To find $\langle p | s \rangle$ and $\langle q | s \rangle$ we appeal to formula (2-7); being able to apply this is the reason we computed σ_p, \dots, v_q . By (2-7), we have

$$(A-1) \quad \langle p | s \rangle = \frac{\theta}{6} \sigma^2 + \left(\frac{1}{2} - \theta v \right),$$

$$(A-2) \quad \langle q | s \rangle = \frac{\theta}{2} \left(\left(\sigma + \bar{\omega} \frac{\lambda_6}{\theta} \right)^2 - 2 \right) - \left(\theta \text{Im} \left\langle \sigma \mid \bar{\omega} \frac{\lambda_6}{\theta} \right\rangle - \frac{3}{2} + 3\theta v \right).$$

In each of these, the first term is imaginary and the second is real. Our goal is to show that only three possible pairs σ, v allow 0 to lie in the triangle $\langle T | s \rangle \subseteq \mathbb{C}$ whose vertices are (A-1), (A-2) and $\langle \rho | s \rangle = \theta$.

First we consider the case $\sigma = 0$. Then (A-1) and (A-2) simplify dramatically, and $\langle T|s \rangle$ is the triangle in \mathbb{C} with vertices $\frac{1}{2} - \theta\nu$, $3(\frac{1}{2} - \theta\nu)$ and θ . Since the last vertex is above the real axis, while the first two are real and differ by a factor of 3, the triangle $\langle T|s \rangle$ can only contain 0 if both of its first two vertices are 0. This forces $\nu = \frac{1}{2\theta}$, which leads to $s = x$, whose mirror we already know meets T .

Now we suppose $\sigma \neq 0$. So (A-1) lies above the real axis, just as $\langle \rho|s \rangle$ does. If σ lies at distance $> \sqrt{2}$ from $-\bar{\omega}\lambda_6/\theta$ then (A-2) also lies above the real axis. In this case it is obvious that $0 \notin \langle T|s \rangle$. By Lemma A.2, the only other possibility is that σ is one of the three elements of Λ that lie at distance $\sqrt{2}$ from $-\bar{\omega}\lambda_6/\theta$, in which case $\langle q|s \rangle$ is real. One of these nearest neighbors is $0 \in \Lambda$, which we treated in the previous paragraph. In each of the other two cases, $\langle p|s \rangle$ and $\langle \rho|s \rangle$ are still above the x -axis and $\langle q|s \rangle$ is on the x -axis. So the only way the origin can lie in $\langle T|s \rangle$ is for it to be the vertex $\langle q|s \rangle$. Then ν is determined in terms of σ using (A-2), since $\langle q|s \rangle = 0$. We have shown that there are at most three Leech mirrors that meet the triangle. Since we know three Leech mirrors that do meet it, coming from roots in $\langle x, y \rangle$, the proof is complete. □

Lemma A.4 (three triangles needed in Lemma 5.5) *Let us consider the Leech roots $x = (0; 1, -\omega)$ and $z = (\lambda_9; 1, \theta)$, where we are using the Leech model of L , and λ_9 is some norm 9 vector in the Leech lattice. Define the second-shell root y as $\bar{\omega}x - z$. Write X, Y and Z for the projections of ρ to x^\perp, y^\perp and z^\perp respectively, and q for the projection of ρ to $\langle x, y \rangle^\perp$. Then the only mirrors of \mathcal{H} which meet any of the totally real triangles $\Delta\rho qX, \Delta\rho qY$ and $\Delta\rho qZ$ are the four mirrors of $\langle R_x, R_y \rangle$.*

Proof We begin by finding basic data about various important points. In addition to x and z given in the statement, we have

$$\rho = (0; 0, 1) \quad \text{and} \quad y = (-\lambda_9; \theta\omega, 2\bar{\omega}).$$

Recall that x and z are Leech roots, so $\langle \rho|x \rangle = \langle \rho|z \rangle = \theta$. Also, $\langle x|z \rangle = -\frac{3}{2} + \frac{\theta}{2}$ and y is a second-shell root with $\langle \rho|y \rangle = 3\bar{\omega}$. The projections X, Y, Z are

$$\begin{aligned} X &= \rho - \frac{\langle \rho|x \rangle}{x^2}x = \rho + \frac{x}{\theta} = \left(0; \frac{1}{\theta}, -\frac{\bar{\omega}}{\theta}\right), \\ Y &= \rho - \frac{\langle \rho|y \rangle}{y^2}y = \rho - \bar{\omega}y = (\bar{\omega}\lambda_9; \bar{\theta}, 1 - 2\omega), \\ Z &= \rho - \frac{\langle \rho|z \rangle}{z^2}z = \rho + \frac{z}{\theta} = \left(\frac{\lambda_9}{\theta}; \frac{1}{\theta}, 2\right). \end{aligned}$$

Using these one checks

$$\begin{aligned} X^2 = Z^2 = -1, \quad \langle \rho | X \rangle = \langle \rho | Z \rangle = -1, \quad \langle X | y \rangle = 1 + 3\bar{\omega}, \\ Y^2 = -3, \quad \langle \rho | Y \rangle = -3, \quad \langle Y | x \rangle = -\omega\theta. \end{aligned}$$

Because $X \perp x$ and $Y \perp y$, a formula for the projection q of ρ to $\langle x, y \rangle^\perp$ is

$$q = (2\omega - 1) \left(\rho - \frac{\langle \rho | x \rangle}{\langle Y | x \rangle} Y - \frac{\langle \rho | y \rangle}{\langle X | y \rangle} X \right) = ((2\bar{\omega} - \omega)\lambda_9; 3\bar{\theta}, 3 - 3\omega).$$

(The initial factor $2\omega - 1$ makes $\langle \rho | q \rangle$ negative, and also makes q a primitive lattice vector.) Using this one can check

$$q^2 = -18 \quad \text{and} \quad \langle q | \rho \rangle = \langle q | X \rangle = \langle q | Y \rangle = \langle q | Z \rangle = -9.$$

We will be able to apply [Lemma A.1](#) to $\Delta\rho qX$ because ρ, q and X have negative inner products. And similarly with Y or Z in place of X .

We first claim that $\Delta\rho qX \cup \Delta\rho qY \cup \Delta\rho qZ$ lies in the interior of the fourth critical horoball. It suffices to show that X, Y, Z and q do. By definition, Y lies on the boundary of the 2nd critical horoball and X and Z lie on the boundary of the 1st. For q , we use $q^2 = -18$ and $\langle q | \rho \rangle = -9$ to get $\text{ht}(q) = \frac{9}{2} < 7$, as desired.

Let Q_X be $\Delta\rho qX$ minus the first (open) critical horoball around ρ , and similarly for Q_Y and Q_Z . By working in the hyperbolic plane containing $\Delta\rho qX$, it is obvious that the point of Q_X furthest from q is X . By the distance formula (2-1), we have $d(q, X) = \cosh^{-1} \sqrt{9/2}$. Similarly, Z is the point of Q_Z furthest from q , and $d(q, Z)$ is also $\cosh^{-1} \sqrt{9/2}$. Finally, the furthest point of Q_Y from q is where $\bar{Y}\rho$ intersects the first critical horosphere. We can find this point by parametrizing $\bar{Y}\rho - \{\rho\}$ by $Y + t\rho$ for $t \in [0, \infty)$. The intersection point is defined by the condition $\text{ht}_\rho(Y + t\rho) = 1$. Writing this out explicitly using (2-3) and solving for t gives $t = 1$. So the point is $Y + \rho$. The distance formula gives $d(q, Y + \rho) = \cosh^{-1} \sqrt{2}$. This is smaller than $\cosh^{-1} \sqrt{9/2}$, so we conclude that $Q_X \cup Q_Y \cup Q_Z$ lies in the closed ball around q of radius $\cosh^{-1} \sqrt{9/2}$.

Now suppose s is a root whose mirror meets $\Delta\rho qX \cup \Delta\rho qY \cup \Delta\rho qZ$. Since these triangles lie in the fourth (open) critical horoball, s must be a 1st-, 2nd- or 3rd-shell root. By scaling s we may suppose $\langle \rho | s \rangle \in \{\theta, 3, 2\theta\}$. We will work with many projections of s , so we name them all at once:

- $s_q = (\langle s | q \rangle / q^2)q$, the projection of s to the span of q ;
- $s_{q^\perp} = s - s_q$, the projection to the orthogonal complement of this;

- s_{xy} , the projection to the span of x and y ;
- $s_{qxy} = s_q + s_{xy}$, the projection to the span of q , x and y ;
- $s_{qxy\perp} = s - s_{qxy}$, the projection to the orthogonal complement of this;
- $s_{\rho x}$, the projection to the span of ρ and x ;
- s_θ , the projection to the span of $\lambda_\theta \in \Lambda$;
- $s_\Lambda = s_\theta + s_{qxy\perp}$, the projection to the summand $\Lambda = \langle \rho, x \rangle^\perp$ of L .

Because all mirrors miss the first (open) critical horoball, s^\perp must meet $Q_X \cup Q_Y \cup Q_Z$. This set lies in the closed $(\cosh^{-1} \sqrt{9/2})$ -ball around q , so $d(q, s^\perp) \leq \cosh^{-1} \sqrt{9/2}$. Using the second distance formula (2-2) gives

$$\sinh^{-1} \sqrt{-\frac{|\langle q|s \rangle|^2}{q^2 s^2}} \leq \cosh^{-1} \sqrt{\frac{9}{2}}.$$

Now $s^2 = 3$, $q^2 = -18$ and $\sinh(\cosh^{-1}(\theta)) = \sqrt{\theta^2 - 1}$ give us the inequality $|\langle q|s \rangle|^2 \leq 189$. Using $q^2 = -18$ a second time shows that the most negative s_q^2 can be is $-\frac{189}{18} = -\frac{21}{2}$. Next, $3 = s^2 = s_q^2 + (s_{q\perp})^2$, so $(s_{q\perp})^2 \leq \frac{27}{2}$. Since s_{xy} is a projection of $s_{q\perp}$ inside the positive definite space q^\perp , it follows that $s_{xy}^2 \leq \frac{27}{2}$.

We recall that $\langle x, y \rangle$ is a copy of the Eisenstein lattice $D_4^\mathcal{E}$, for which one can introduce explicit coordinates, for example as in Lemma 5.2. Since all inner products in L are divisible by θ , we have $s_{xy} \in \theta(D_4^\mathcal{E})^*$. One can check that $\theta(D_4^\mathcal{E})^*$ is a copy of $D_4^\mathcal{E}$ scaled to have minimal norm $\frac{3}{2}$. An \mathcal{E} -basis consists of $\frac{1}{2}(\theta x + y)$ and $\frac{1}{2}(\theta y - x)$. Write S_{xy} for the set of vectors in this lattice with norm $\leq \frac{27}{2}$. We have shown that the projection s_{xy} of s must lie in S_{xy} . This set can be enumerated on a computer and turns out to have size 937. (PARI has a built-in function to do this. It is most natural to rescale by multiplying all inner products by $\frac{4}{3}$, to work with a copy of the standard D_4 lattice over \mathbb{Z} . Then one enumerates all lattice vectors of norm $\leq \frac{4}{3} \cdot \frac{27}{2} = 18$. The number 937 matches what one expects from [20, Table 4.8].)

Now we consider the possibilities for s_{qxy} . We claim it lies in

$$S_{qxy} = \left\{ s_{xy} + tq \mid s_{xy} \in S_{xy} \text{ and } t = \frac{(\text{one of } \bar{\theta}, 3 \text{ and } 2\bar{\theta}) - \langle s_{xy} | \rho \rangle}{-9} \right\}.$$

This is easy: s_{qxy} equals s_{xy} plus some multiple of q , and the multiple of q is determined by the condition $\langle \rho | s_{qxy} \rangle = \langle \rho | s \rangle \in \{\theta, 3, 2\theta\}$. So S_{qxy} has cardinality $937 \cdot 3 = 2811$.

We are working here under the assumption that s is orthogonal to some point of $Q_X \cup Q_Y \cup Q_Z$. Now, s_{qxy} has the same inner products as s with all elements of the span of q, x, y , which contains $Q_X \cup Q_Y \cup Q_Z$. It follows that s_{qxy}^\perp meets one of the triangles $\triangle \rho qX$, $\triangle \rho qY$ and $\triangle \rho qZ$. Of the 2811 possibilities for s_{qxy} , only 460 satisfy this condition. And the following considerations further restrict the possibilities.

Because the span of q, x and y is the same as the span of ρ, x and λ_9 , and λ_9 is orthogonal to the span of ρ and x , we have $s_{qxy} = s_9 + s_{x\rho}$ and hence $s = s_{qxy\perp} + s_9 + s_{x\rho}$. It follows that $s_\Lambda = s_{qxy\perp} + s_9$. Note that the projection s_9 of s to $\mathbb{C}\lambda_9$ coincides with the corresponding projection of s_{qxy} . Furthermore, the equalities

$$s_\Lambda^2 = (s_{qxy\perp})^2 + s_9^2 = (3 - s_{qxy}^2) + s_9^2,$$

$$\langle s_\Lambda | \lambda_9 \rangle = \langle s_{qxy} | \lambda_9 \rangle,$$

show that s_{qxy} determines s_Λ^2 and $\langle s_\Lambda | \lambda_9 \rangle$. Therefore s_{qxy} determines all inner products in the span of s_Λ and λ_9 . Also, note that s_Λ lies in Λ , not just $\Lambda \otimes \mathbb{C}$, because Λ is an orthogonal summand of L .

Of the 460 possibilities for s_{qxy} , 449 lead to $|\langle s_\Lambda | \lambda_9 \rangle|^2 > 9s_\Lambda^2$, which violates the Cauchy–Schwarz inequality. Of the remaining 11 possibilities, 4 lead to $s_\Lambda^2 = 3$, which contradicts the fact that Λ has minimal norm 6. Of the remaining 7 possibilities, 3 lead to the similar contradiction that $(\zeta s_\Lambda - \lambda_9)^2 = 3$ for some unit ζ of \mathcal{E} . The remaining 4 possibilities for s_{qxy} turn out to be $x, \bar{\omega}y, z$ and $x + z$, all of which are roots of $\langle x, y \rangle$. In particular, they all have norm 3. It follows that

$$s_{qxy\perp}^2 = s^2 - s_{qxy}^2 = 3 - 3 = 0.$$

Since $s_{qxy\perp}$ lies in the positive-definite space q^\perp , we conclude $s_{qxy\perp} = 0$. Therefore $s = s_{qxy}$, which we have already observed is a root of $\langle x, y \rangle$. □

For the rest of the results in this section we will need to understand the mirrors near the intersection point p_∞ of all 13 point-mirrors. Just as we spoke of critical horoballs around ρ , we will speak of critical balls around p_∞ . They are defined the same way: the balls that are centered at p_∞ and tangent to some mirror. The radii of the critical balls are called critical radii. Even though some mirrors pass through p_∞ , we don't regard 0 as a critical radius.

For this analysis we use the $P^2\mathbb{F}_3$ model, in which $p_\infty = (\bar{\theta}; 0, \dots, 0)$. For a given root s of L , the distance formula (2-2) gives $d(p_\infty, s^\perp)$ in terms of $p_\infty^2 = -3$,

$s^2 = 3$ and $|\langle p_\infty | s \rangle|^2$. Since the inner product lies in $\theta\mathcal{E}$ and hence has norm in $3 \cdot \{0, 1, 3, 4, 7, 9, 12, \dots\}$, one can work out the critical radii as

$$r_1, r_2, r_3, r_4, r_5, r_6, \dots = \sinh^{-1} \sqrt{\frac{1}{3}(1, 3, 4, 7, 9, 12, \dots)}.$$

This assumes that there are indeed roots in L having the appropriate inner products with p_∞ , which is easy to check in the range we care about, which is r_1, \dots, r_6 (see Lemma A.12 below). Numerically, the first six critical radii are approximately .549, .881, .987, 1.210, 1.317 and 1.444. We call the mirrors tangent to the n^{th} critical ball around p_∞ the batch n mirrors. For completeness we also refer to the point-mirrors as the batch 0 mirrors. Lemma A.12 below describes batches $0, \dots, 5$ explicitly, but until Lemma A.13 we only need batches 0, 1 and 2. These are easy to enumerate and appear in Table 1. As an example application we reprove the following result from [9]. Although this result helps motivate the main theorem of the paper (Theorem 1.2), it is not logically necessary for us.

Lemma A.5 (mirrors near a 26–point [9, Proposition 1.2]) *The mirrors of L closest to the 26–point τ are the point- and line-mirrors, at distance $\sinh^{-1}(6+8\sqrt{3})^{-\frac{1}{2}} \approx .223$.*

Proof First we use (2-1) to compute

$$d(\tau, p_\infty) = \cosh^{-1} \sqrt{\frac{|\langle \tau | p_\infty \rangle|^2}{p_\infty^2 \tau^2}} = \cosh^{-1} \sqrt{\frac{19 + 8\sqrt{3}}{6 + 8\sqrt{3}}} \approx .740.$$

That the point- and line-mirrors lie at distance $\sinh^{-1}(6 + 8\sqrt{3})^{-\frac{1}{2}} \approx .223$ from τ is a similar calculation, using (2-2) in place of (2-1). Any mirror which passes as near or nearer to τ as these do must lie at distance at most

$$d(\tau, p_\infty) + \sinh^{-1} \sqrt{\frac{1}{6 + 8\sqrt{3}}} \approx .740 + .223 = .963$$

from p_∞ . This is less than $r_3 = \sinh^{-1} \sqrt{4/3} \approx .987$, so all such mirrors occur in batches 0, 1 and 2. Our computer iterated over these batches and found that the mirrors closest to τ are just the point- and line-mirrors. □

Lemma A.6 (two triangles needed for Step 1 of Lemma 6.1) *Let s be a point-root and x be the projection of the 26–point τ to its mirror. Then the only mirrors of L that meet the totally real triangle $\Delta\tau xp_\infty$ are the point-mirrors. The intersection of $\Delta\tau xp_\infty$ with each of them is p_∞ , except that the intersection with s^\perp is $\overline{xp_\infty}$. The same results hold if s is a line-mirror, provided we replace p_∞ by l_∞ .*

distance(s^\perp, p_∞)	representative	$ 3^{13}: L_3(3)\text{-orbit} $
0	$(0; \theta, 0^{12})$	13
$\sinh^{-1} \sqrt{1/3} \approx .549$	$(1; 1^4, 0^9)$	$1053 = 13 \cdot 3^4$
$\sinh^{-1} \sqrt{3/3} \approx .881$	$(\theta; 1^3, -1^3, 0^7)$	$113724 = 156 \cdot 3^6$
	$(\theta; \theta, \bar{\theta}, 0^{11})$	$1404 = 13 \cdot 12 \cdot 3^2$
	$(\theta; \theta, \theta, 0^{11})$	$702 = \binom{13}{2} \cdot 3^2$
	$(\theta; \bar{\theta}, \bar{\theta}, 0^{11})$	702

Table 1: The $3^{13}:L_3(3)$ -orbits of batch 0, 1 and 2 roots, up to units. Their mirrors pass through or near the 13-point p_∞ . The last 13 coordinates, read modulo θ , must give an element of the line code.

Proof We prove the point-mirror case; the line-mirror case follows by applying an element of $L_3(3):2$ that swaps the point- and line-mirrors. We gave $d(p_\infty, \tau) \approx .740$ in the previous lemma, and x is closer to p_∞ than τ is. So the triangle lies within $d(p_\infty, \tau)$ of p_∞ . This is less than $r_2 = \sinh^{-1} 1 \approx .881$, so only the batch 0 and 1 mirrors might meet the triangle. The batch 0 mirrors are the point-mirrors, which obviously meet the triangle as stated. For the batch 1 roots we used our computer to check (using Lemma A.1) that none of their mirrors meet the triangle. When doing this we replaced the vector $(\bar{\theta}; 0^{13})$ representing p_∞ by $\theta p_\infty = (3; 0^{13})$. Then the vectors representing τ , x and p_∞ have negative inner products, so Lemma A.1 applies. \square

So far we have not needed a concrete description of the Leech lattice. But we need one to check that \mathcal{H} is disjoint from the homotopies in Step 4 of the proof of Lemma 6.1. We will use Wilson’s $L_3(3)$ -invariant model from [31, page 188]. Its description involves $\psi = 1 - 3\bar{\omega} \in \mathcal{E}$ from [31, page 154], and uses 13 coordinates in \mathcal{E} , indexed by the points of $P^2\mathbb{F}_3$. The line difference code means the subspace of \mathbb{F}_3^{13} generated by the differences of the characteristic functions of the 13 lines of $P^2\mathbb{F}_3$. Although we don’t need it, we mention that the elements of this code are tabulated in [3, Table 2]. Wilson’s definition is that $(x_1, \dots, x_{13}) \in \mathcal{E}^{13}$ lies in Λ just if $x_1 + \dots + x_{13} = 0$, all coordinates are congruent modulo $\bar{\psi}$, and the element of \mathbb{F}_3^{13} obtained by reducing the components mod θ is an element of the line difference code. The inner product is the usual one divided by 13.

The following Leech lattice vectors play a key role in the rest of this appendix. For $i = 1, \dots, 13$ we define P_i as the Leech lattice vector $(12\theta, \bar{\theta}^{12})$ with the 12θ in the i^{th} position. And for $j = 1, \dots, 13$ we define L_j as the Leech lattice vector

$((-9)^4, 4^9)$ with the (-9) entries in the positions corresponding to the points of $P^2\mathbb{F}_3$ that lie on the j^{th} line. Even though the P_i and L_j are vectors, we will write $P_i \in L_j$ as shorthand for “the i^{th} point and j^{th} line of $P^2\mathbb{F}_3$ are incident”. It is easy to check that the P_i and L_j satisfy Wilson’s conditions, and in the next lemma we will use them to show that Wilson’s lattice is indeed a copy of the (complex) Leech lattice. We also record the following useful data: the P_i and L_j have norm 36, and for $i \neq j$ we have $\langle P_i | P_j \rangle = \langle L_i | L_j \rangle = -3$. Also, $\langle P_i | L_j \rangle = -9\theta$ or 4θ according to whether or not $P_i \in L_j$.

Lemma A.7 *The lattice just described is isometric to the complex Leech lattice.*

Proof Write Λ for the \mathcal{E} -lattice defined above. We will show that it is isometric to the complex Leech lattice, which we have denoted Λ elsewhere in the paper. First, define $\delta_{ij} = L_i - L_j$ and $\varepsilon_{ij} = (P_i - P_j)/\psi$. The ε_{ij} have the form $(0, \dots, 0, \bar{\psi}\theta, 0, \dots, 0, -\bar{\psi}\theta, 0, \dots, 0)$, so they lie in Λ . We claim that Λ is spanned by the δ_{ij} , ε_{ij} and any one P_i , say P_1 . To see this, given $x \in \Lambda$, the reduction of its components modulo θ lies in the line difference code. By adding suitable multiples of $\delta_{ij} = L_i - L_j$, we may suppose without loss of generality that this codeword is the zero codeword. That is, all coordinates are divisible by θ . By adding a multiple of P_1 , we may suppose that the last component is 0. It follows that all components are zero mod $\bar{\psi}$. Since all coordinates are divisible by $\bar{\psi}\theta$, and the coordinate sum is zero, x may be expressed as a linear combination of the ε_{ij} .

Second, all inner products in Λ lie in $\theta\mathcal{E}$; in particular, Λ is integral. This is just a computation using the inner product data given above. Third, the determinant of Λ is 3^6 . To see this, note that the \mathcal{E} -span of the ε_{ij} is a scaled copy of $\mathcal{E} \otimes A_{12}$, with determinant $3^{12} \cdot 13$. (Recall the factor $\frac{1}{13}$ in Λ ’s inner product.) Adjoining the δ_{ij} gives a larger lattice, whose quotient by this copy of $\mathcal{E} \otimes A_{12}$ is isomorphic to the line difference code \mathbb{F}_3^6 , and whose determinant is $3^6 \cdot 13$. Finally, adjoining P_1 reduces the determinant to 3^6 . Since Λ has determinant 3^6 , and all inner products are divisible by θ , it is an Eisenstein Niemeier lattice; see [3, Section 2]. And the only Eisenstein Niemeier lattice whose isometry group contains $L_3(3)$ is the Leech lattice; see [3, Theorem 4]. \square

Correction to [3] This proof is similar to that of Lemma 6 in [3], which asserts that there is a unique $L_3(3)$ -invariant integral lattice properly containing $\theta\mathcal{E} \otimes A_{12}$. That is wrong because it neglects the 13-part of the discriminant group of $\theta\mathcal{E} \otimes A_{12}$. A

counterexample is Λ as defined above. That lemma was used only once in [3], in the proof of Lemma 9, where it is used to recognize a certain lattice as the complex Leech lattice. That result can be saved by addressing the 13-part in a manner similar to the proof above.

We will need to understand the lattice points near a particular point C of $\Lambda \otimes \mathbb{C}$ that is not in Λ itself. This will allow us to write down Leech model versions of the point-roots, line-roots and 13-points, and thereby identify the $P^2\mathbb{F}_3$ and Leech models of L . The name *centroid* in the next lemma is explained after Lemma A.10.

Lemma A.8 (the centroid C) *Define $C = -P_1/\psi$. Then the only points of Λ at distance $< \sqrt{42/13}$ from C are the points*

$$C + \frac{1}{\psi} P_i = \frac{P_i - P_1}{\psi} = \begin{cases} (0, \dots, 0) & \text{if } i = 1, \\ (-\bar{\psi}\theta, 0, \dots, 0, \bar{\psi}\theta, 0, \dots, 0) & \text{if } i = 2, \dots, 13, \end{cases}$$

all of which lie at distance $\sqrt{36/13}$.

Similarly, the only points of $\frac{1}{\theta}\Lambda$ at distance $< \sqrt{14/13}$ from C are the points

$$C - \frac{\bar{\omega}}{\theta\psi} L_j = \begin{cases} \frac{1}{\theta}(-\omega, (-3 \text{ or } -\bar{\omega}), \dots, (-3 \text{ or } -\bar{\omega})) & \text{if } P_1 \in L_j, \\ \frac{1}{\theta}(4 - 8\omega, (-3 \text{ or } -\bar{\omega}), \dots, (-3 \text{ or } -\bar{\omega})) & \text{otherwise,} \end{cases}$$

where -3 appears exactly three times among the last twelve coordinates if $P_1 \in L_j$, or four times otherwise. All of these points lie at distance $\sqrt{12/13}$ from C .

Remark A.9 The norm of $(C + (1/\psi)P_i) - (C - (\bar{\omega}/\theta\psi)L_j)$ is 3 or 4, according to whether or not $P_i \in L_j$. In particular, since $P_1 \in L_1$ and $C + P_1/\psi = 0$, we have $(C - (\bar{\omega}/\theta\psi)L_1)^2 = 3$.

Proof of Lemma A.8 To show that these vectors lie in Λ (resp. $\frac{1}{\theta}\Lambda$), one just checks the conditions in Wilson’s definition. Their distances from C are as stated because $P_i^2 = L_j^2 = 36$ for all i and j . It is easy to check that all the P_i (resp. all the L_j) are congruent mod ψ . The rest of the proof is like the proof of Lemma A.2. That is, the lemma follows from the claim: the P_i (resp. L_j) are the only norm < 42 representatives of their mod ψ congruence class. We treat the P_i case; the L_j case is the same.

Let v be a Leech vector, distinct from all the $C + (1/\psi)P_i$, such that $(v - C)^2 < \frac{42}{13}$. Let $x = \psi(v - C)$. Then $x \in \Lambda$ has norm < 42 and is distinct from all the P_i , but congruent to them mod ψ . As a nonzero member of $\psi\Lambda$, $x - P_i$ has norm at

least $6 \cdot 13 = 78$. Since $P_i^2 = 36$ and $x^2 < 42$, the angle between x and P_i is obtuse. That is, in the real Euclidean space underlying $\Lambda \otimes \mathbb{C}$, the inner product of x and P_i is negative. This holds for all i . So x has negative inner product (in this real Euclidean space) with $P_1 + \dots + P_{13} = 0$, which is absurd. \square

Caution From Section 2.7 recall that the Leech model for our lattice is $L \simeq \Lambda \oplus H$, where H has Gram matrix $\begin{pmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{pmatrix}$. Wilson’s definition of Λ gives an $L_3(3)$ action on L . The $P^2\mathbb{F}_3$ model of L gives another $L_3(3)$ action on L . These two copies of $L_3(3)$ are not conjugate in $\text{Aut}(L)$. The lattices fixed pointwise by them are H and $(\mathcal{E}p_\infty + \mathcal{E}l_\infty)$, respectively, which are not isometric.

Now we can write down the point- and line-roots in the Leech model. We define p_i to be the Leech root obtained by using $\sigma = C + \frac{1}{\psi} P_i$ and $\nu = \frac{1}{2\bar{\theta}}$ in (2-8). Explicitly,

$$p_i = \begin{cases} (0, \dots, 0; 1, -\omega) & \text{if } i = 1, \\ (-\bar{\psi}\theta, 0, \dots, 0, \bar{\psi}\theta, 0, \dots, 0; 1, -\bar{\omega}) & \text{if } i = 2, \dots, 13, \end{cases}$$

where the $\bar{\psi}\theta$ appears in the i^{th} position. Similarly, we define l_j as the root obtained from (2-6) by taking $\sigma = (C - (\bar{\omega}/\theta\psi)L_j)\theta\omega$, $m = \theta\omega$, $N = 3$, and $\nu = \bar{\theta}$ or $\frac{1}{2}\bar{\theta}$ according to whether $P_1 \in L_j$ or not. Explicitly, l_j has the form

$$l_j = \begin{cases} (-9\bar{\omega}, (-3\omega \text{ or } -1), \dots, (-3\omega \text{ or } -1); \theta\omega, 2\bar{\omega}) & \text{if } P_1 \in L_j, \\ (4\omega - 8\bar{\omega}, (-3\omega \text{ or } -1), \dots, (-3\omega \text{ or } -1); \theta\omega, \bar{\theta}) & \text{otherwise.} \end{cases}$$

Here the $-9\bar{\omega}$ or $4\omega - 8\bar{\omega}$ appears in the first position, and the (-1) entries appear in positions corresponding to points of $P^2\mathbb{F}_3$ not lying on the j^{th} line of $P^2\mathbb{F}_3$. There are nine (-1) entries if $P_1 \in L_j$ and eight otherwise. One verifies the next lemma by direct computation.

Lemma A.10 (the point- and line-roots in the Leech model) *The vectors p_i , l_j and ρ in the Leech model have the same inner products with each other as do the vectors with the same names in the $P^2\mathbb{F}_3$ model. This defines an isometry of the two models, which identifies their point-roots, line-roots and Leech cusp ρ . Under this identification, in the Leech model we have*

$$p_\infty = (\psi C; \psi, -\bar{\omega}\psi) = (12\bar{\theta}, \theta, \dots, \theta; \psi, -\bar{\omega}\psi),$$

$$l_\infty = (-\bar{\omega}\theta\psi C; -\bar{\omega}\theta\psi, 6\bar{\omega} - \omega) = (-36\bar{\omega}, 3\bar{\omega}, \dots, 3\bar{\omega}; -\bar{\omega}\theta\psi, 6\bar{\omega} - \omega).$$

Writing these vectors in the form (2-6), their values of σ and m are their entries before and immediately after the semicolon, their values of N are both -3 , and their values of ν are $-\frac{13}{6}\theta$ and $-\frac{17}{2}\theta$, respectively. \square

Remarks Here is how we found this identification. For each i we have $\langle \rho | p_i \rangle = \theta$ in the $P^2\mathbb{F}_3$ model, so their analogues in the Leech model should have the form $(\sigma_i; 1, \dots)$ for suitable $\sigma_i \in \Lambda$. The inner product formula (2-7) shows that the differences between the σ_i must be minimal vectors of Λ . So the σ_i are the vertices of a 12-simplex in \mathbb{C}^{12} . This suggested looking for their centroid, expecting it to have stabilizer $L_3(3)$ in the affine isometry group $\Lambda:6Suz$ of the Leech lattice. We were already familiar with the principle used in Lemma A.2: the lattice vectors near an element of the rational span of a lattice are related to the short lattice vectors in a suitable congruence class. This suggested looking for a congruence class in Λ with stabilizer $L_3(3)$ and exactly 13 minimal representatives. In Wilson’s model of Λ the P_i form an $L_3(3)$ -orbit of size 13. So we investigated and found that they are congruent mod ψ . This suggested that the centroid should be C from Lemma A.8, and that $\sigma_1, \dots, \sigma_{13}$ should be its nearest neighbors. This determined the p_i up to their values of ν . For p_1 we chose $\nu = \frac{1}{2\theta}$ arbitrarily, and then used the orthogonality of the p_i and (2-7) to compute the ν values of the other p_i . Since ρ and the point-roots span L up to finite index, the expressions for all the other vectors follow. Although we do not need it, we also remark that with additional work it is possible to introduce a version of the Leech model “centered at C ”, to avoid hiding some of the $L_3(3)$ symmetry.

As we mentioned after (2-7), inner products with a vector $(\sigma; m, \dots)$ in the Leech model can be expressed in terms of $\sigma/m \in \Lambda \otimes \mathbb{C}$. For p_∞ and l_∞ this point is C . And Lemma A.8 shows that for the point-roots (resp. line-roots), the corresponding points of $\Lambda \otimes \mathbb{C}$ are the nearest neighbors of C in Λ (resp. $\frac{1}{\theta}\Lambda$). This part of Lemma A.8 will be crucial in the proofs of Lemmas A.11 and A.13.

Lemma A.11 (the triangle needed for Step 4 in Lemma 6.1, point-mirror case) *Let x be the projection of ρ onto a point-mirror, and T be the totally real triangle $\Delta\rho xp_\infty$. Then T meets that point-mirror in $\overline{xp_\infty}$, meets the other point-mirrors in p_∞ only, and misses all other mirrors.*

Proof By $L_3(3)$ symmetry it suffices to treat the point-root p_1 . Using the Leech model, and the formula for p_1 given just before Lemma A.10, we find

$$x = (0; \frac{1}{\theta}, -\frac{1}{\theta}\bar{\omega}) = \theta^{-1} p_1 + \rho,$$

which has $x^2 = \langle \rho | x \rangle = -1$. The other two vertices of T are represented by p_∞ from Lemma A.10 and $\rho = (0; 0, 1)$. In calculations we will use $\bar{\theta}\bar{\psi} p_\infty$ rather than p_∞ .

because it has negative inner product (namely -39) with both ρ and x . So we will be able to apply [Lemma A.1](#).

First we claim that T lies in the union of the second critical horoball around ρ and the second critical ball around p_∞ . To see this, note that $\text{ht}_\rho(x) = 1$, so x lies on the boundary of the first critical horoball. (In fact this holds by construction.) Also, one checks $\text{ht}_\rho(p_\infty) = 13 > 3$, so p_∞ lies outside the second critical horoball. Therefore it suffices to find where the boundary of this horoball intersects $\overline{p_\infty\rho}$ and $\overline{p_\infty x}$, and check that both intersection points lie in the second critical ball around p_∞ . Since $\overline{p_\infty\rho}$ travels directly toward the horoball, its intersection point is closer to p_∞ than is the intersection point of $\overline{p_\infty x}$. So it suffices to show that the second one lies in the second critical horoball. We found this intersection point by the method from the proof of [Lemma A.4](#), obtaining $\overline{\theta\psi} p_\infty + (9\sqrt{26} - 39)x$. This can be verified by checking that it has height 3. It has norm -702 and its inner product with $\overline{\theta\psi} p_\infty$ is $1404 - 351\sqrt{26}$. So the point of \mathbb{B}^{13} it represents lies at distance

$$\cosh^{-1}\sqrt{63 - 12\sqrt{26}} \approx .810$$

from the point p_∞ . This is less than $r_2 = \sinh^{-1}(1) \approx .881$, so this point lies in the second critical ball, as desired.

Therefore the only mirrors that might meet T are the Leech mirrors (which include the batch 0 roots, namely the point-roots) and the batch 1 mirrors around p_∞ . We used computer calculations in the $P^2\mathbb{F}_3$ model, and [Lemma A.1](#), to check that the batch 1 mirrors miss T . So it remains to examine how the mirror of a Leech root s can meet T . This part of the proof is similar to the proof of [Lemma A.3](#): we will compute $\langle T | s \rangle \subseteq \mathbb{C}$ and examine whether it contains the origin. For use in the calculation we tabulate the parameters of the important vectors when they are written as in (2-6):

	σ	m	N	ν
x (a vertex of T)	0	$\frac{1}{\theta}$	-1	$-\frac{1}{18}\theta$
$\overline{\theta\psi} p_\infty$ (a vertex of T)	$-13\theta C$	-13θ	-117	$-\frac{169}{2}\theta$
s (a Leech root)	σ	1	3	ν

The inner product formula (2-7) and some simplification gives

$$\begin{aligned} \langle x | s \rangle &= \frac{1}{6}\theta\sigma^2 - \frac{1}{3}\theta(3\nu + \frac{1}{2}\theta), \\ \langle \overline{\theta\psi} p_\infty | s \rangle &= \frac{13}{2}\theta((C - \sigma)^2 - \frac{36}{13}) - 13\theta(\text{Im}(C | \sigma) + 3(\nu + \frac{1}{6}\theta)). \end{aligned}$$

The first terms are their imaginary parts. So $\langle x | s \rangle$ lies on the real axis (if $\sigma = 0$) or above it (otherwise). Also, Lemma A.8 gives exactly the inequality needed to make $\langle \bar{\theta} \bar{\psi} p_\infty | s \rangle$ lie on or above the real axis, namely that $C - \sigma$ has norm $\geq \frac{36}{13}$. It is something of a miracle that the geometry of the Leech lattice gives us exactly the bound that we need for our argument to work. Finally, $\langle \rho | s \rangle = \theta$ since s is a Leech root. So $\langle T | s \rangle$ lies in the closed upper half-plane. If $\sigma \neq 0$ then two of its vertices lie strictly above the real axis, so $\langle T | s \rangle$ misses the origin unless the third vertex coincides with the origin, ie unless $\langle \bar{\theta} \bar{\psi} p_\infty | s \rangle = 0$. But then s is a point-root, since these are the only roots orthogonal to p_∞ . So suppose $\sigma = 0$. Then the above formulas simplify to the real numbers $\langle x | s \rangle = -\theta(v + \frac{1}{6}\theta)$ and $\langle \bar{\theta} \bar{\psi} p_\infty | s \rangle = -39\theta(v + \frac{1}{6}\theta)$. (The latter uses $C^2 = \frac{36}{13}$.) Since these differ by a positive factor, the only way $\langle T | s \rangle$ can contain the origin is for both of them to vanish. In particular $\langle p_\infty | s \rangle = 0$, so again s is a point-root, indeed p_1 . □

The analogue of Lemma A.11 for a line-mirror requires a much larger tabulation of the mirrors near p_∞ than we have needed so far. The following lemma describes batches 0, . . . , 5 in a manner suitable for machine computation. Batches 0, 1 and 2 appear in Table 1. In batches 3, 4 and 5 there are 743 418, 107 953 560 and 480 961 338 mirrors. To actually construct the batches one should refer to the tabulation of the line code in Tables 2 and 3 of [3].

Lemma A.12 (mirrors near a 13–point) *Write vectors of L in the $P^2\mathbb{F}_3$ model. Then the mirrors in batches $b = 0, \dots, 5$, ie those at distance $< \sinh^{-1} 2 \approx 1.444$ from p_∞ , are the orthogonal complements of the roots $s = (s_0; s_1, \dots, s_{13})$ described in the next paragraph. For $b = 0$ this gives all roots, while for the other cases it gives one from each scalar class.*

Define the **desired norm** $N = 3, 4, 6, 7, 10, \text{ or } 12$ and the **required coordinate sum** $S = 0, 1, 0, 1, 1, \text{ or } 0 \in \mathbb{F}_3$ according to the value of $b = 0, \dots, 5$. Choose any codeword $w = (w_1, \dots, w_{13})$ in the line code whose weight (number of nonzero coordinates) is at most N and whose coordinate sum is S . Choose e_1, \dots, e_{13} with $e_i \in \{0, 3, 6\}$ if $w_i \neq 0$ and $e_i \in \{0, 3, 9, 12\}$ if $w_i = 0$, such that the sum of the e_i equals $N - \text{weight}(w)$. Choose any Eisenstein integers s_1, \dots, s_{13} such that $s_i \bmod \theta$ is w_i , and $|s_i|^2$ is either e_i or $e_i + 1$ according to whether $w_i = 0$ or $w_i \neq 0$. Prefix the coordinates (s_1, \dots, s_{13}) by a coordinate $s_0 = 0, 1, \theta, -2, 2 - (\omega \text{ or } \bar{\omega})$, or 3 according to the value of $b = 0, \dots, 5$.

Proof Suppose $s = (s_0; s_1, \dots, s_{13})$ is a root in one of the batches $0, \dots, 5$. Then it follows that $|\langle s | p_\infty \rangle|^2 < 3 \cdot 12$, which boils down to the condition $|s_0|^2 < 12$, ie $|s_0|^2 \in \{0, 1, 3, 4, 7, 9\}$. If $|s_0|^2 = 0$ then of course $s_0 = 0$. In the other cases there is a unique way to scale s by a unit so that s_0 is as described at the end of the lemma. In all cases, $s^2 = 3$ says that the vector (s_1, \dots, s_{13}) has norm N , and the definition of L requires that its reduction w modulo θ is in the line code and has coordinate sum S . For $i = 1, \dots, 13$ we take e_i to be defined as $|s_i|^2$ if $w_i = 0$, or $|s_i|^2 - 1$ if $w_i \neq 0$. If $w_i = 0$ then s_i must be divisible by θ , and $|s_i|^2 \leq 12$, so $e_i \in \{0, 3, 9, 12\}$. If $w_i \neq 0$ then $N \leq 12$ gives $|s_i|^2 \leq 9$ since w_i has weight at least 4 (so there are at least 3 other nonzero coordinates). An element of \mathcal{E} not divisible by θ , and having norm ≤ 9 , must have norm 1, 4 or 7. That is, when $w_i \neq 0$ we have proven $e_i \in \{0, 3, 6\}$. We have just established the lemma's constraints on the e_i . The constraints on the s_i in terms of the e_i are satisfied because of the construction of the e_i . Conversely, if one follows the instructions in choosing $w, e_1, \dots, e_{13}, s_1, \dots, s_{13}, s_0$ then one obtains a norm 3 vector of L in the specified batch. \square

Lemma A.13 (the triangle needed for Step 4 in Lemma 6.1, line-mirror case) *Let x be the projection of ρ onto a line-mirror, and T be the totally real triangle $\Delta \rho x l_\infty$. Then T meets that line-mirror in $\overline{x l_\infty}$, meets the other line-mirrors in l_∞ only, and misses all other mirrors.*

Proof This is similar to Lemma A.11 but some new issues arise. By symmetry we may take the line-root to be l_1 . Computation gives $x = \rho - \bar{\omega} l_1$, with $x^2 = \langle \rho | x \rangle = -3$. In computations we use $\omega \bar{\psi} l_\infty$ in place of l_∞ because it has negative inner product with ρ and x , namely -39 . This is also its norm.

We claim that T lies in the union of the sixth critical ball around l_∞ and the third critical horoball around ρ . Following the proof of Lemma A.11, it is enough to check that the point where $\overline{l_\infty x}$ pierces the boundary of the 3rd critical horoball lies in the 6th critical ball. One checks that this point is represented by the vector $\omega \bar{\psi} l_\infty + (4\sqrt{39} - 13)x$, by computing that its height is 4. Its norm is -1404 and its inner product with $\omega \bar{\psi} l_\infty$ is $468 - 156\sqrt{39}$. This lets us compute the distance from this point to l_∞ , namely

$$\cosh^{-1} \sqrt{\frac{64}{3} - \frac{8}{3}\sqrt{39}} \approx 1.407.$$

This is less than $r_6 = \sinh^{-1}(2) \approx 1.444$, as desired.

So the only mirrors that could meet T are the first- and second-shell mirrors around ρ (this includes the line-mirrors, which are the batch 0 roots around l_∞), and the mirrors

in batches $1, \dots, 5$ around l_∞ . The latter checks are done in the $P^2\mathbb{F}_3$ model. We applied an isometry F of L that exchanges the point- and line-roots up to signs, namely $p_i \mapsto l_{14-i}$ and $l_j \mapsto -p_{14-j}$. (It follows that $F(p_\infty) = l_\infty$ and $F(l_\infty) = -p_\infty$, which is useful for writing down a matrix for F .) So it suffices to check that the triangle with vertices $F(\rho)$, $F(x)$ and $F(\omega\bar{\psi}l_\infty)$ misses all batch $1, \dots, 5$ mirrors around p_∞ . For this we enumerated these mirrors in the $P^2\mathbb{F}_3$ model by using Lemma A.12, and checked that they all miss this triangle by using Lemma A.1. This was the only computer calculation in the paper that took more than a moment: sixteen hours on a laptop.

To examine how the Leech mirrors and second-shell mirrors meet T , we return to the Leech model. We begin by writing down the parameters σ , m , N and ν when we write the following important vectors in the form (2-6):

	σ	m	N	ν
x (a vertex of T)	$-(C - (\bar{\omega}/\theta\psi)L_1)\theta$	$-\theta$	-3	$-\theta$
$\omega\bar{\psi}l_\infty$ (a vertex of T)	$-13\theta C$	-13θ	-39	$-\frac{221}{2}\theta$
s (a Leech root)	σ	1	3	ν
or s (a second-shell root)	σ	θ	3	ν

Again we will use Lemma A.1 to determine whether s^\perp meets T . We begin with the case of s a second-shell root because it is simpler. The real part of the inner product formula (2-7) is

$$(A-3) \quad \text{Re}\langle x|s \rangle = \frac{3}{2} \left(\frac{\sigma}{\theta} - \left(C - \frac{\bar{\omega}}{\theta\psi} L_1 \right) \right)^2,$$

$$(A-4) \quad \text{Re}\langle \omega\bar{\psi}l_\infty|s \rangle = \frac{39}{2} \left(\left(\frac{\sigma}{\theta} - C \right)^2 - \frac{12}{13} \right).$$

We see that (A-3) is at least 0, with equality if and only if $\sigma/\theta = C - (\bar{\omega}/\theta\psi)L_1$. And (A-4) is at least 0 by Lemma A.8. Together with $\text{Re}\langle \rho|s \rangle = 3$, we see that $\langle T|s \rangle$ lies in the closed right half-plane. If $\sigma/\theta \neq C - (\bar{\omega}/\theta\psi)L_1$ then (A-3) is strictly positive, so the only way $\langle T|s \rangle$ could contain the origin is for its vertex $\langle \omega\bar{\psi}l_\infty|s \rangle$ to be the origin. That is, $s \perp l_\infty$, so s is a line-root.

The remaining possibility for σ/θ is $C - (\bar{\omega}/\theta\psi)L_1$. Then $(s - \bar{\omega}l_1)$ is a multiple of ρ (just write s and l_1 in the Leech coordinate system). Since s and l_1 both have norm 3, it quickly follows that $s = \bar{\omega}l_1 + n\theta\rho$ for some integer n . Using the known inner products between points, lines, l_∞ and ρ , we find that

$$\langle \omega\bar{\psi}l_\infty|s \rangle = 39n\theta \quad \text{and} \quad \langle x|s \rangle = 3n\theta.$$

Since these differ by a positive factor, the only way 0 could lie in $\langle T | s \rangle$ is for both of them to be 0. Then s is a line-root, indeed l_1 .

Now suppose s is a Leech root. The imaginary part of (2-7) is

$$(A-5) \quad \text{Im}\langle x | s \rangle = \frac{\theta}{2} \left[\left(\sigma - \left(C - \frac{\bar{\omega}}{\theta\psi} L_1 \right) \right)^2 - 2 \right],$$

$$(A-6) \quad \text{Im}\langle \omega\bar{\psi}l_\infty | s \rangle = \frac{13\theta}{2} \left((C - \sigma)^2 - \frac{38}{13} \right).$$

We have $\langle \rho | s \rangle = \theta$, which lies above the real axis. We claim that (A-5) does also. To see this, recall from Remark A.9 that $C - (\bar{\omega}/\theta\psi)L_1 \in \frac{1}{\theta}\Lambda$ has norm 3. So it equals a norm 9 vector of Λ , divided by θ . By Lemma A.2, the distance from $C - (\bar{\omega}/\theta\psi)L_1$ to Λ is $\sqrt{3}$. So the first term in the brackets in (A-5) is at least 3. So (A-5) lies above the real axis. Next, Lemma A.8 says that either $(C - \sigma)^2 \geq \frac{42}{13}$ or else $\sigma = C + \frac{1}{\psi}P_i$ for some $i = 1, \dots, 13$. In the first case we see that (A-6) lies above the real axis, so T does too, so it cannot contain the origin.

The remaining possibility for σ is $C + \frac{1}{\psi}P_i$. Then $s - p_i$ is a multiple of ρ . Since s and p_1 both have norm 3, it follows that $s = p_i + n\rho$ for some integer n . Using the known inner products between points, lines, l_∞ and ρ , we find that

$$(A-7) \quad \langle x | s \rangle = \begin{cases} \theta - 3n + \bar{\omega}\theta & \text{if } P_i \in L_1, \\ \theta - 3n & \text{otherwise;} \end{cases}$$

$$(A-8) \quad \langle \omega\bar{\psi}l_\infty | s \rangle = \bar{\theta} + 6 - 39n.$$

(Note that (A-7) is above the real axis and (A-8) is below it.) We want to determine whether the origin lies in the triangle $\langle T | s \rangle$ that has (A-8), (A-7) and $\langle \rho | s \rangle = \theta$ for vertices. We can find the intersection of $\langle T | s \rangle$ with \mathbb{R} by writing A (resp. B) for the convex combination of $\langle \omega\bar{\psi}l_\infty | s \rangle$ and $\langle \rho | s \rangle$ (resp. $\langle x | s \rangle$) that has no imaginary part. Then the origin lies in $\langle T | s \rangle$ just if it lies in the interval with endpoints A and B . One works out A and B , with the result

$$A = 3 - \frac{39}{2}n, \quad B = \begin{cases} 3 - 15n & \text{if } P_i \in L_1, \\ 3 - 21n & \text{otherwise.} \end{cases}$$

If $n \leq 0$ then A and B are both positive. And if $n \geq 1$ then A and B are both negative. So the origin does not lie in $\langle T | s \rangle$. □

Appendix B: How two complex triangles meet the mirrors

In this appendix we prove [Lemma B.2](#): the complex triangles $\Delta\rho\tau p_\infty$ and $\Delta\rho\tau l_\infty$ miss the mirror arrangement except at p_∞ and l_∞ respectively. This is the key fact in [Step 2](#) in the proof of [Lemma 6.1](#). Because these triangles are complex instead of totally real, we cannot use the machinery in [Appendix A](#). Instead we exploit the fact that they lie in $\mathbb{B}(F) \cong \mathbb{B}^1$, where F is the $L_3(3)$ -invariant sublattice of L . Our first step is to understand how this \mathbb{B}^1 meets the mirrors.

Lemma B.1 *Suppose $x \in \mathbb{B}^1 \cap \mathcal{H}$. Then either x is represented by a norm -3 vector of F , or is orthogonal to a norm 3 vector of F .*

Proof Write M for the sublattice of L spanned by the roots orthogonal to x . The lattice M is nonempty because $x \in \mathcal{H}$. So $1 \leq \dim M \leq 13$. As an \mathcal{E} -lattice with all inner products divisible by θ , and spanned by roots, M is a direct sum of copies of the Eisenstein root lattices $A_2^\mathcal{E}$, $D_4^\mathcal{E}$, $E_6^\mathcal{E}$ and $E_8^\mathcal{E}$; see [[3](#), Theorem 3]. There are at most 13 direct summands of M and $L_3(3)$ acts on the set of them. Since $L_3(3)$ is simple and contains an element of order 13, the smallest nontrivial permutation representations of $L_3(3)$ are on 13 objects. Therefore either (a) $L_3(3)$ preserves each direct summand of M , or (b) M is the sum of 13 copies of $A_2^\mathcal{E}$ and $L_3(3)$ permutes them transitively.

First we treat case (a). The action of $L_3(3)$ on F^\perp is known and irreducible: it is the deleted permutation representation coming from the $L_3(3)$ action on the points of $P^2\mathbb{F}_3$. So as an $L_3(3)$ representation, x^\perp decomposes into two irreducible representations: the spaces F^\perp and $x^\perp \cap F$ of dimensions 12 and 1, respectively. In case (a) we are assuming that the underlying vector space of each direct summand of M is preserved by $L_3(3)$ and so must contain one of these irreducible factors. Since each summand of M has dimension at most 4, it follows that M must be isomorphic to $A_2^\mathcal{E}$ and lie in the fixed space of $L_3(3)$. So F contains a root orthogonal to x .

Now we treat case (b). As the orthogonal complement of 13 mutually orthogonal roots, the line in $L \otimes \mathbb{C}$ corresponding to x is represented by a lattice vector; we choose a primitive one and use the same name x for it. The product of the $\bar{\omega}$ -reflections in the 13 roots, times the scalar ω , acts on L by scaling x by ω and fixing x^\perp pointwise. So it is given by the same formula ([2-4](#)) as a reflection, namely $v \mapsto v - (1 - \omega)\langle v | x \rangle x / x^2$. (This differs from a reflection in that its fixed set in \mathbb{B}^{13} is a point not a hyperplane.)

Since this preserves L , we have $(1-\omega)\langle v|x\rangle/x^2 \in \mathcal{E}$ for every $v \in L$. Since $L = \theta \cdot L^*$, there exists $v \in L$ with $\langle v|x\rangle = 1 - \bar{\omega}$. So $3/x^2 \in \mathcal{E}$, which forces $x^2 = -3$. \square

It is easy to see that F is spanned by $\rho = (-\bar{\psi}; -1, \dots, -1)$ and $p_\infty = (\bar{\theta}; 0, \dots, 0)$. These have norms 0 and -3 , and $\langle \rho | p_\infty \rangle = \theta \bar{\psi}$. We will also use a second null vector ρ' , which is defined as $-\omega$ times the image of ρ under the “ ω -reflection” in p_∞ . (As in the proof above, this fixes a single point of \mathbb{B}^{13} rather than a hyperplane.) One checks that $\rho' = \bar{\psi} p_\infty - \omega \rho = (\bar{\omega} \bar{\psi}; \omega, \dots, \omega)$ and $\langle \rho | \rho' \rangle = 13\theta$.

Everything becomes easier if we work in a certain superlattice E of F , namely the one spanned by $\rho/\bar{\psi}$ and $\rho'/\bar{\psi}$. We use the notation $[u, v]$ to mean $(u\rho' + v\rho)/\bar{\psi}$. Obviously E contains ρ , and it also contains p_∞ by our formula for ρ' in terms of p_∞ and ρ . So E does indeed contain F . The main advantage of working in E is that the inner product has the simple form

$$\langle [u, v] | [u', v'] \rangle = \begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{pmatrix} \begin{pmatrix} \bar{u}' \\ \bar{v}' \end{pmatrix},$$

which is the same as in (2-5). One can check that $p_\infty = [1, \omega]$ and $l_\infty = [\bar{\theta}\bar{\omega}, \omega - 2]$. Now we can prove the main result of this appendix:

Lemma B.2 (two triangles needed in Step 2 of Lemma 6.1) *The only point of \mathcal{H} in $\Delta\rho\tau p_\infty$ (resp. $\Delta\rho\tau l_\infty$) is p_∞ (resp. l_∞).*

Proof Because τ is the midpoint of $\overline{p_\infty l_\infty}$, the union of these two triangles is the larger triangle $T = \Delta\rho p_\infty l_\infty$. So it suffices to show that this triangle misses \mathcal{H} except at p_∞ and l_∞ . We identify the projective space of $F \otimes \mathbb{C}$ with $\mathbb{C} \cup \{\infty\}$ by plotting a vector $[u, v]$ as v/u . Then \mathbb{B}^1 corresponds to the upper half-plane, and ρ, p_∞ and l_∞ correspond to ∞, ω and $\frac{3}{2} + \frac{\theta}{6}$, respectively. So T is the hyperbolic triangle they span. The edges of T are the vertical line through ω , the vertical line through $(\frac{3}{2} + \frac{\theta}{6})$ and an arc C of the circle $\{z : |z - \frac{1}{3}|^2 = \frac{13}{9}\}$. The image of τ in the upper half-plane is $(1 + i)$, which is the (hyperbolic) midpoint of the circular arc C .

Our strategy is to find all the norm -3 vectors of F representing points of T , and all norm 3 vectors orthogonal to points of T , and then apply Lemma B.1. It is convenient to find all such vectors of E first, and then discard the ones that lie outside F . We begin by writing an arbitrary norm -3 vector of E as

$$x = \left[m, \frac{\theta}{\bar{m}} \left(\frac{0 - (-3)}{6} + v \right) \right],$$

where $m \in \mathcal{E} - \{0\}$, and $v \in \text{Im } \mathbb{C}$ is chosen so that the second component lies in \mathcal{E} . (This is just like the analysis leading to (2-6).) The corresponding point of \mathbb{B}^1 has imaginary part $\theta/2|m|^2$. Now suppose $x \in T$. The only point of T with imaginary part $\frac{\theta}{6}$ is l_∞ , and all its other points have larger imaginary part. Therefore either x is a multiple of l_∞ , or else $|m| = 1$. In the latter case we scale x so that $m = 1$, so $x = [1, \theta(\frac{1}{2} + v)]$. Since the second component lies in \mathcal{E} , we have $v = \frac{1}{2\theta} + \frac{n}{\theta}$ for some $n \in \mathbb{Z}$. Then the point of \mathbb{B}^1 represented by x is $n - \bar{\omega}$. Since every point of T has real part at least $-\frac{1}{2}$ and at most $\frac{3}{2}$, the only possibilities for n are $-1, 0$ and 1 . The case $n = -1$ yields $x = p_\infty$. The case $n = 0$ does not actually arise, because $-\bar{\omega}$ lies below the circular arc C . The case $n = 1$ yields a point of T . We have shown that the only points of T represented by norm -3 vectors of E are p_∞, l_∞ and $[1, 1 - \bar{\omega}]$. The last of these is not in F . Therefore the only points of T represented by norm -3 vectors of F are p_∞ and l_∞ .

Now consider a point $x \in T$ orthogonal to some norm 3 vector $[u, v]$ of E . The orthogonal complement of $[u, v]$ in E is generated by $[\bar{u}, \bar{v}]$, which therefore represents x . Also, $[\bar{u}, \bar{v}]$ has norm -3 , so the previous paragraph shows that x is one of p_∞, l_∞ and $[1, 1 - \bar{\omega}]$. To finish the proof we observe that $[1, 1 - \bar{\omega}]$ is orthogonal to no norm 3 vector of F , since the norm 3 vector $[1, 1 - \omega] \in E$ orthogonal to it does not lie in F . \square

More information about the lattice F can be found in Section 5 of [11]. In particular, its Γ -stabilizer acts on $\mathbb{B}(F) = \mathbb{B}^1$ as $\Gamma_0(13) \subseteq \text{PSL}_2\mathbb{Z}$.

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