

Chern–Schwartz–MacPherson classes of degeneracy loci

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The Chern–Schwartz–MacPherson class (CSM) and the Segre–Schwartz–MacPherson class (SSM) are deformations of the fundamental class of an algebraic variety. They encode finer enumerative invariants of the variety than its fundamental class. In this paper we offer three contributions to the theory of equivariant CSM/SSM classes. First, we prove an interpolation characterization for CSM classes of certain representations. This method — inspired by recent work of Maulik and Okounkov and of Gorbounov, Rimányi, Tarasov and Varchenko — does not require a resolution of singularities and often produces explicit (not sieve) formulas for CSM classes. Second, using the interpolation characterization we prove explicit formulas — including residue generating sequences — for the CSM and SSM classes of matrix Schubert varieties. Third, we suggest that a stable version of the SSM class of matrix Schubert varieties will serve as the building block of equivariant SSM theory, similarly to how the Schur functions are the building blocks of fundamental class theory. We illustrate these phenomena, and related stability and (two-step) positivity properties for some relevant representations.

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1 Introduction

1.1 Degeneracy loci, fundamental class, Schur expansion

Many interesting varieties in geometry occur as *degeneracy locus* varieties, a notion we recall now. Let $\Sigma \subset V$ be an invariant variety of a G -representation V . Let $E \rightarrow M$ be a vector bundle over a smooth variety M with fiber V and structure group G . If $\Sigma(E)$ is the union of the Σ 's in each fiber then the subvariety $X = \sigma^{-1}(\Sigma(E))$ for a section σ transversal to $\Sigma(E)$ is called a *degeneracy locus*. Areas of geometry where degeneracy loci are abundant include Schubert calculus, moduli spaces, and singularity theory.

The general strategy of studying numerical invariants of degeneracy loci is to associate a “universal” G -characteristic class to the *local* situation $\Sigma \subset V$, and to

expect that the sought numerical invariant of X is obtained by evaluating the universal characteristic class at the bundle $E \rightarrow M$. The key example of this strategy is the *fundamental class* $[X] \in H^*(M)$. One defines the G -equivariant fundamental class $[\Sigma] \in H_G^*(V) = H^*(BG)$, and it is a fact that $[X] \in H^*(M)$ can be calculated as $[\Sigma]$ evaluated at the bundle $E \rightarrow M$.

Hence, equivariant fundamental classes $[\Sigma] \in H_G^*(V)$ and their applications have been intensively studied in numerous parts of geometry. Two interesting sets of examples are (a) quiver representations (where the fundamental class is often called the quiver polynomial) and (b) singularity theory (where the fundamental class is called the Thom polynomial). In the intersection of these two sets of examples is the Giambelli–Thom–Porteous formula for the fundamental class of the orbit closures of the representation $\mathrm{GL}_k(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ acting on $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$; see Porteous [32].

Fundamental classes in both of these sets of examples above show interesting patterns, namely stabilization and positivity properties. Stabilization properties are displayed by the fact that the classes can be encoded by (generalized, so-called iterated residue) generating sequences; see eg Bérczi–Szenes [6], Kazarian [21], and Rimányi [36]. Positivity means that the coefficients in appropriate Schur expansions of the classes are nonnegative; see eg Buch [7] and Pragacz–Weber [33].

1.2 The Schwartz–MacPherson deformation of the fundamental class

The notion of fundamental class $[X] \in H^{\mathrm{codim}(X \subset M)}(M)$ has a deformation, which comes in two versions called the Chern–Schwartz–MacPherson (CSM) and Segre–Schwartz–MacPherson (SSM) classes; see MacPherson [26]. The two versions only differ by an explicit factor. (Another name for the CSM class, after homogenization by a new variable \hbar , is “characteristic cycle” class.) The CSM/SSM classes, $c^{\mathrm{sm}}(X)$ and $s^{\mathrm{sm}}(X)$, are inhomogeneous elements of $H^*(M)$. The lowest-degree part of each is the fundamental class $[X]$. The CSM/SSM classes encrypt more intrinsic information about the variety X than the fundamental class; and their applications in enumerative geometry are hard to overestimate; see eg Fulton [16, Chapter 5]. P Aluffi [1] writes “Segre classes provide a systematic framework for enumerative geometry computation; but this is of relatively little utility, as Segre classes are in general hard to compute”. More recent applications of CSM classes (under the name of “stable envelope classes”) are in Maulik–Okounkov [27], Okounkov [30], and Rimányi–Tarasov–Varchenko [38].

Ohmoto [28; 29] showed that the above-mentioned equivariant strategy works for the SSM class. One can associate a universal characteristic class $s^{\text{sm}}(\Sigma \subset V)$ to the situation $G \curvearrowright (\Sigma \subset V)$ and the SSM class of the degeneracy locus X is this characteristic class evaluated at the defining bundle; see Theorem 2.2 below. Hence, the CSM/SSM theory of degeneracy loci is reduced to finding CSM/SSM classes for invariant varieties in *representations*. In the present paper we offer three contributions to this problem, described in the next three subsections.

1.3 Interpolation characterization of CSM classes in representations

The classical approach to calculating either the fundamental class or the CSM/SSM class of $\Sigma \subset V$ (acted upon by the group G) is via resolution of singularities. Resolutions are not known for important examples of quivers or contact singularities. Even if a resolution is known, this method produces exclusion-inclusion type formulas that hide both the stabilization and positivity structures of CSM/SSM classes. In the past decades new and effective methods of calculating fundamental classes have been found. One of these new methods is interpolation: one lists a few conditions that $[\Sigma]$ and only $[\Sigma]$ satisfies; see Rimányi [35] and Buch–Rimányi [8].

In this paper we prove that an interpolation characterization exists for CSM classes of orbits of certain representations; see Theorem 2.7 below. We expect that this new interpolation method will open the way to finding CSM/SSM classes for quivers, matroids, and singularities, similarly to what has happened for the fundamental class in the past two decades.

Although formally we will not use it, let us comment on the origin of this interpolation theorem. In Maulik–Okounkov [27], Gorbounov–Rimányi–Tarasov–Varchenko [18], and Rimányi–Tarasov–Varchenko [37], two seemingly unrelated modules are identified: (i) the regular representation of the equivariant cohomology algebra of certain symmetric spaces, and (ii) the Bethe algebra of certain quantum integrable systems. On both sides of this identification there is a “given” basis and a “sought” basis. On the physics side the coordinate basis (aka spin basis) is given, and one seeks formulas for the Bethe basis. On the geometry side the basis of torus fixed points is given, and one seeks formulas for classes coming from Schubert varieties. Our identification matches the given basis on one side with the sought basis on the other side. In particular, the coordinate basis of the Bethe algebra is matched with the CSM classes of Schubert cells; see Rimányi–Varchenko [39].

Example 1.1 A small example for the Bethe algebra of a quantum integrable system is a certain commutative algebra action (not described here; see eg [18] for more details) on $(\mathbb{C}^2)_{(1,1)}^{\otimes 2} \otimes \mathbb{C}(z_1, z_2, \hbar) = \text{span}(v_1 \otimes v_2, v_2 \otimes v_1) = \text{span}(\xi_{1,2}, \xi_{2,1})$. Here v_1 and v_2 are the coordinate vectors of \mathbb{C}^2 . The element $\xi_{1,2} = v_1 \otimes v_2$ and the element $\xi_{2,1} = -\hbar/(z_1 - z_2 - \hbar)v_1 \otimes v_2 + (z_1 - z_2)/(z_1 - z_2 - \hbar)v_2 \otimes v_1$ are singled out because they are the common eigenvectors of the action. On the other hand one considers the $((\mathbb{C}^*)^2 \times \mathbb{C}^*)$ -equivariant cohomology ring of $T^*\mathbb{P}^1$, with the following two bases: $\{\gamma - z_2, \gamma - z_1\}$ (the fixed-point basis: here γ is the Chern class of the tautological line bundle, and z_1, z_2, \hbar are the equivariant Chern roots), and $\{\kappa_1 = \gamma - z_2, \kappa_2 = \hbar + \gamma - z_1\}$. It is instructive to check that the map $[f(\gamma, z_1, z_2)] \mapsto f(z_1, z_1, z_2)/(z_1 - z_2) \cdot \xi_{1,2} + f(z_2, z_1, z_2)/(z_2 - z_1) \cdot \xi_{2,1}$ is an isomorphism (after suitable localization) from cohomology to the Bethe module, and the fixed-point basis matches the ξ -basis, while the spin basis $\{v_1 \otimes v_2, v_2 \otimes v_1\}$ matches $\{\kappa_1, \kappa_2\}$. The $\{\kappa_1, \kappa_2\}$ classes (after putting $\hbar = 1$) are the CSM classes of the Schubert cells in \mathbb{P}^1 !

Our Theorem 2.7 stems from this interplay between quantum algebra and characteristic classes of singular varieties, and it is a version of the main theorem of [39] — see also Aluffi–Mihalcea–Schuermann–Su [5] — modified from Schubert calculus settings to representations with finitely many orbits satisfying the Euler condition.

1.4 CSM/SSM classes of matrix Schubert cells: weight functions, generating functions

The building blocks of the algebraic combinatorics of fundamental classes both for quivers and for singularities are the Schur functions. Schur functions are the fundamental classes of so-called matrix Schubert varieties; see Fehér–Rimányi [11] and Knutson–Miller [23]. In Sections 5–8 we calculate the CSM and SSM classes of matrix Schubert cells. Our formulas are not of inclusion-exclusion type; rather, some of our formulas are of “localization type” (inherently displaying interpolation properties), and others are iterated residue generating sequences (inherently displaying stabilization properties of their Schur expansions).

It is important to emphasize that the connections between the CSM classes of *ordinary* Schubert cells and CSM classes of matrix Schubert cells are more complex than for the corresponding fundamental classes (which form the lowest-degree part of the CSM class). In Appendix B we summarize the differences and relations between the CSM theory of Schubert cells and matrix Schubert cells.

In Section 8 we make a conjecture about the signs of the Schur expansions of the SSM classes of matrix Schubert cells. We are not aware of a direct relation between this conjecture and the positivity theorem on CSM classes of ordinary Schubert cells conjectured by Aluffi and Mihalcea [3; 4] and proved by Huh [19]; for more comments see Appendix B.

1.5 Conjectured two-step positivity of SSM classes

As mentioned, the building blocks of cohomological fundamental class theory are the Schur functions: when a fundamental class of a geometrically relevant variety is expanded in appropriate Schur functions, the coefficients are often nonnegative; see eg Buch [7] and Pragacz–Weber [33].

Schur positivity (or alternating signs) of CSM/SSM classes break down in the simplest examples. For example, according to Parusiński–Pragacz [31], a certain SSM class of the A_2 quiver representation is (for notation see Section 6)

$$(1) \quad s^{\text{sm}}(\Sigma_{n,n+1}^0) = s_0 + (-s_2) + (2s_3 + s_{21}) \\ + (-3s_4 - 3s_{31} - s_{211}) + (4s_5 + 6s_{41} + 4s_{311} + s_{2111}) \\ + (-5s_6 - 10s_{51} - 10s_{411} + s_{33} - 5s_{3111} - s_{21111}) + \cdots.$$

The signs of the term s_0 , and also of s_{33} , violate the pattern (as do many more in higher degrees).

Yet, we conjecture that there is a sign pattern. In fact we expect that the following *two-step positivity* property holds in general, for quivers, singularities, and maybe other geometrically relevant varieties:

- *The SSM classes of geometrically interesting varieties, expanded in the new building blocks, the \tilde{s}_λ functions of Section 8, have nonnegative coefficients.*
- *The coefficients of the Schur expansions of the \tilde{s}_λ functions have alternating signs.*

The second item is our Conjecture 8.4. We prove the first item in two cases, one of which is Theorem 8.6 and the other the A_2 quiver representation in Section 9. For example, according to Theorem 9.1 we have the positive \tilde{s}_λ expansion

$$(2) \quad s^{\text{sm}}(\Sigma_{n,n+1}^0) = \tilde{s}_0 + \tilde{s}_1 + \tilde{s}_{11} + \tilde{s}_{111} + \cdots,$$

and the \tilde{s}_λ functions have alternating Schur expansions

$$\begin{aligned}
 \tilde{s}_0 &= s_0 + (-s_1) + (s_2 + s_{11}) + (-s_3 - 2s_{21} - s_{111}) \\
 &\quad + (s_4 + 3s_{31} + s_{22} + 3s_{211} + s_{1111}) + \cdots, \\
 \tilde{s}_1 &= s_0 + (-s_1) + (s_2 + s_{11}) + (-s_3 - 2s_{21} - s_{111})s_1 + (-2s_2 - 2s_{11}) \\
 (3) \quad &\quad + (3s_3 + 5s_{21} + 3s_{111}) + (-4s_4 - 9s_{31} - 3s_{22} - 9s_{211} - 4s_{1111}) + \cdots, \\
 \tilde{s}_{11} &= s_0 + (-s_1) + (s_2 + s_{11}) + (-s_3 - 2s_{21} - s_{111})s_{11} + (-2s_{21} - 3s_{111}) \\
 &\quad + (3s_{31} + 2s_{22} + 7s_{211} + 6s_{1111}) + \cdots,
 \end{aligned}$$

and so on.

As claimed in (2), adding together the expressions in (3) reproduces (1)—but the signs in (2) and (3) show patterns, which disappear in the sum. Further examples illustrating this two-step positivity phenomenon will appear in Kőmúves [22] and Promtapan [34].

1.6 Conventions

We will consider varieties over \mathbb{C} , and cohomology with coefficient group \mathbb{C} . We distinguish between a “weakly decreasing sequence of nonnegative integers” and a “partition”. The first one has a fixed length, while a partition does not change by adding a 0 at the end.

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2 Classical and equivariant CSM/SSM classes

2.1 Classical CSM theory

First we recall the main notions of the classical (nonequivariant) CSM theory; for more detailed expositions see eg [2; 1; 3; 4].

For a complex algebraic variety X let $\mathcal{F}(X)$ be the abelian group of constructible functions on X , ie the group whose elements are finite sums $\sum_i n_i \mathbb{1}_{W_i}$, where the W_i are locally closed subvarieties of X , $\mathbb{1}_W$ is the characteristic function of W (1 on W , 0 outside of W), and $n_i \in \mathbb{Z}$. For a proper morphism $f: Y \rightarrow X$ define $f_*: \mathcal{F}(Y) \rightarrow$

$\mathcal{F}(X)$ by $f_*(\mathbb{1}_W)(p) = \chi(f^{-1}(p) \cap W)$ for $p \in X$. Consider also the Borel–Moore homology functor $H_*(-)$ on complex algebraic varieties and proper morphisms.

It was conjectured by Deligne and Grothendieck and proved by MacPherson [26] that there exists a unique natural transformation $C_*: \mathcal{F}(-) \rightarrow H_*(-)$, which, for a smooth X satisfies $C_*(\mathbb{1}_X) = c(TX) \cap \mu_X$, where $\mu_X \in H_{\text{top}}(X)$ is the homological fundamental class of X . Being a natural transformation C_* also satisfies $C_*(\alpha + \beta) = C_*(\alpha) + C_*(\beta)$ and $f_*C_*(\alpha) = C_*f_*(\alpha)$ for a proper morphism f .

The (homological) Chern–Schwartz–MacPherson class $c_{\text{sm}}(X) = C_*(\mathbb{1}_X) \in H_*(X)$ is hence an inhomogeneous Borel–Moore homology class, satisfying push-forward properties, and generalizing the notion of total Chern class of the tangent bundle for situations where the tangent bundle does not exist. For a closed embedding $i: X \subset M$ one can push forward $c_{\text{sm}}(X)$ to the homology of M , and if M is also smooth with Poincaré duality \mathcal{P} then one can consider the “relative” CSM classes $c_{\text{sm}}(X \subset M) = i_*(c_{\text{sm}}(X)) \in H_*(M)$ and $c^{\text{sm}}(X \subset M) = \mathcal{P}(c_{\text{sm}}(X \subset M)) \in H^*(M)$. In this paper we will mainly be concerned with the last, *cohomology* version.

Remark 2.1 MacPherson’s proof of the existence of C_* is through the important notions of Mather class and local Euler obstruction.

2.2 Ohmoto’s G –equivariant MacPherson transformation

The equivariant CSM theory was studied for torus actions in [42]. For reductive linear groups G , Ohmoto [28; 29] defined the G –equivariant version of the C_* transformation of the preceding section. Namely, the group of “ G –invariant constructible functions” $\mathcal{F}^G(X)$ is defined for a G –space X . The characteristic functions $\mathbb{1}_W$ of G –invariant subvarieties W induce elements in $\mathcal{F}^G(X)$, and the equivariant push-forward f_*^G for these elements coincides with the ordinary push-forward of the preceding section. The G –equivariant version H_*^G of the H_* functor is defined in [10].

In his work, Ohmoto proves the existence of the equivariant MacPherson transformation $C_*^G: \mathcal{F}^G(-) \rightarrow H_*^G(-)$, functorial with respect to proper equivariant push-forward, and which satisfies $C_*^G(\mathbb{1}_X) = c^G(TX) \cap \mu_X^G$ if X is a projective smooth G –variety.

Again, the version we study (just like Ohmoto) is the following: assume $i: X \subset M$ is a G –equivariant closed embedding into the smooth variety M with equivariant Poincaré duality \mathcal{P}^G . The (cohomological) G –equivariant CSM class is defined to be $c_G^{\text{sm}}(X \subset M) = \mathcal{P}^G(i_*^G(C_*^G(\mathbb{1}_X))) \in H_G^*(M)$. Recall that G –equivariant cohomology

of a G -space M is defined to be the ordinary cohomology of the Borel construction $EG \times_G M = \{(e, m)\} / \{(e, m) \sim (eg^{-1}, gm)\}$, where EG is a contractible space with a free right G -action.

In the rest of the paper we will only use equivariant cohomological CSM classes. Hence, from now on we drop the G sub- and superscripts, and also use the shorthand notation $c^{\text{sm}}(X) = c^{\text{sm}}(X \subset M) \in H_G^*(M)$ when the ambient space M is clear from the context.

The class $s^{\text{sm}}(X) = c^{\text{sm}}(X)/c(TM)$ is called the (equivariant) Segre–Schwartz–MacPherson (SSM) class. In fact the SSM class may have nonzero components in infinitely many degrees, thus it lives in the completion of $H_G^*(M)$. In our notation we will not indicate this completion.

2.3 Properties of equivariant CSM/SSM classes

For future reference we summarize some important properties of CSM/SSM classes:

- (i) For invariant constructible functions f and g and scalars $k, l \in \mathbb{C}$ we have $c^{\text{sm}}(kf + lg) = kc^{\text{sm}}(f) + lc^{\text{sm}}(g)$ and $s^{\text{sm}}(kf + lg) = ks^{\text{sm}}(f) + ls^{\text{sm}}(g)$. In particular, CSM and SSM classes are motivic, that is, we have $c^{\text{sm}}(X \cup Y) = c^{\text{sm}}(X) + c^{\text{sm}}(Y) - c^{\text{sm}}(X \cap Y)$, and the same holds for s^{sm} classes.
- (ii) If $\eta: Y \rightarrow M$ is an equivariant morphism between smooth varieties then

$$\eta_*(c(TY)) = \sum_a ac^{\text{sm}}(M_a),$$

where $M_a = \{m \in M : \chi(\eta^{-1}(m)) = a\}$.

- (iii) For a closed G -invariant embedding $i: X \subset M$, with X and M both smooth, $c^{\text{sm}}(X \subset M) = i_*(c(TX))$ and $s^{\text{sm}}(X \subset M) = i_*(c(TX))/c(TM) = i_*(c(-\nu))$, where ν is the normal bundle of $X \subset M$.
- (iv) For Y and M smooth let $X \subset M$ be an invariant subvariety with an invariant Whitney stratification. Assume $\eta: Y \rightarrow M$ is (equivariant and) transversal to the strata of X . Then $s^{\text{sm}}(\eta^{-1}(X)) = \eta^*(s^{\text{sm}}(X))$.

Properties (i), (ii), and (iii) are essentially the defining properties of Ohmoto's C_*^G transformation; see [28, Section 3.2 (a)(i),(ii),(iii)]. Property (iv) follows from the equivariant Verdier–Riemann–Roch; see [28, Theorem 4.2] and [29, Proposition 3.8].

The orbit stratification of an algebraic group action with finitely many orbits is a Whitney stratification; see eg the main result of [20] and [40]. Below we will apply (iv) to such situations without mentioning the existence of Whitney stratifications.

2.4 Equivariant CSM/SSM classes in representations

Let us assume that the underlying space M is a vector space, and rename it V . Then the CSM and SSM classes of $X \subset V$ are in $H_G^*(V) = H^*(BG)$, hence they are G -characteristic classes. The main importance of this special case is the following “degeneracy locus” interpretation of SSM classes, which is a consequence of property (iv).

Let G act on V and let X be an invariant subvariety. The Borel construction applied to $X \subset V$ and $EX := EG \times_G X \subset EV := EG \times_G V$ produces fibrations over BG (with fibers X and V , respectively), and $EV \rightarrow BG$ is the universal bundle with fiber V and structure group G . Hence, for a bundle $E \rightarrow B$ with fiber V and structure group G we have the classifying maps $\rho: B \rightarrow BG$ and $\hat{\rho}: E \rightarrow EV$. We define $X(E) = \hat{\rho}^{-1}(EX)$, the “ X -points in each fiber of $E \rightarrow B$ ”. If B is smooth, a G -invariant Whitney stratification of X induces a Whitney stratification of $X(E)$.

Theorem 2.2 [29, Theorem 3.12(4)] *Let $X \subset V$ be an invariant variety of the G -representation V . Let $E \rightarrow B$ be a bundle with fiber V and structure group G , and assume B is smooth and compact. Let σ be a section which is transversal to a Whitney stratification of $X(E)$. Then the **ordinary** (that is, **nonequivariant**) SSM class of $\sigma^{-1}(X(E)) \subset B$ can be obtained as $s^{\text{sm}}(X)$ (as a G -characteristic class) evaluated at the bundle $E \rightarrow B$.*

Here are some other properties we will need:

- (v) Let $X_0 \subset X \subset V$ be invariant subvarieties and let X_0 be smooth. Assume that there is an invariant complementary subspace $W \leq V$ to T_0X_0 transversal to X . Then $c^{\text{sm}}(X)$ is divisible by $c(T_0X_0)$.
- (vi) Let $X \subset V = \mathbb{C}^n$ be an invariant (ie stable with respect to multiplication by $\lambda \in \mathbb{C}$) cone-subvariety. We have

$$c^{\text{sm}}(X) = [X] + \cdots + e(V) \quad \text{and} \quad s^{\text{sm}}(X) = [X] + \cdots,$$

that is, the smallest-degree part of both is the (equivariant) fundamental class. The class c^{sm} has finitely many nonzero components, the highest degree of which is the Euler class $e(V) = \prod_{i=1}^n w_i$ for the weights w_i of the representation. (We choose a maximal torus of G , and by “weights” we will always mean the weights of the corresponding torus action.)

(vii) Let the representation on $V = \mathbb{C}^n$ have k zero weights. That is, the zero-weight subspace $V_0 \subset V$ has dimension k . Assume W is an invariant complementary subspace to V_0 and is transversal to the invariant cone-subvariety X . Then

$$c^{\text{sm}}(X) = [X] + \cdots + \prod_{i=1}^{n-k} w_i,$$

where the w_i are the nonzero weights of V . That is, the highest-degree part of $c^{\text{sm}}(X)$ has degree $n - k$ and it is the product of the nonzero weights.

Property (v) is proved as follows (see also [39, Section 6]): from (iv) we have that $c^{\text{sm}}(X \subset V)/c(V) = c^{\text{sm}}(X \cap W \subset W)/c(W)$, and therefore $c^{\text{sm}}(X \subset V) = c(V/W)c^{\text{sm}}(X \cap W \subset W)$, but $c(V/W) = c(T_0X_0)$. The proof of (vi) is in [28, Section 4.1]. Here we prove (vii): from (iv) we know that $c^{\text{sm}}(X)/c(V) = c^{\text{sm}}(X \cap W)/c(W)$. Since $c(V) = c(W)$, applying (vi) to $X \cap W \subset W$ proves (vii).

Remark 2.3 In particular, property (vi) claims that the n^{th} component of $c^{\text{sm}}(X)$ is independent of X ; it only depends on the representation. The essence of (vii) compared to (vi) is that not only the n^{th} but also the $(n-1)^{\text{st}}, \dots, (n-k)^{\text{th}}$ components of $c^{\text{sm}}(X)$ are independent of X .

In concrete examples — eg the ones we will deal with in the paper — the existence of W (in properties (v) and (vii)) can easily be checked. In fact passing to the maximal compact subgroup (which does not affect equivariant cohomology) it can be proved in very general situations.

Example 2.4 Let the 2-torus act on \mathbb{C}^3 by $(a, b) \cdot (x, y, z) = (ax, b^2y, z)$. Let α and β be the first Chern classes of the 2-torus corresponding to a and b . Thus the weights of this representation are α , 2β , and 0 . Let $X = \{x = 0\}$, $Y = \{y = 0\}$, and $Z = \{z = 0\}$. It is instructive to verify property (vii) in the examples

$$\begin{aligned} c^{\text{sm}}(X) &= (1 + 2\beta)\alpha, & c^{\text{sm}}(Y) &= (1 + \alpha)2\beta, & c^{\text{sm}}(X \cap Y) &= 2\alpha\beta, \\ c^{\text{sm}}(X \cup Y) &= (1 + 2\beta)\alpha + (1 + \alpha)2\beta - 2\alpha\beta = \alpha + 2\beta + 2\alpha\beta. \end{aligned}$$

The claims in the first line follow from (iii) and the claim in the second follows from (i).

2.5 Interpolation characterization

Consider the linear representation V of the algebraic group G . Assume it has finitely many orbits, and assume that the representation contains the scalars, that is, orbits are

invariant under multiplication by $\lambda \in \mathbb{C}^*$. For an orbit Ω let G_Ω be the stabilizer subgroup of a point $x_\Omega \in \Omega$, and let

$$\phi_\Omega: H_G^*(V) \rightarrow H_G^*(\Omega)$$

be the restriction map. After the identifications $H_G^*(V) = H^*(BG)$ and $H_G^*(\Omega) = H_{G_\Omega}^*(x_\Omega) = H^*(BG_\Omega)$ the map $\phi_\Omega: H^*(BG) \rightarrow H^*(BG_\Omega)$ is the one induced by the inclusion $G_\Omega \subset G$.

Let T_Ω be the tangent space of Ω at x_Ω , and let $N_\Omega = V/T_\Omega$ be the “normal” space. The group G_Ω acts on T_Ω and N_Ω , hence these representations have an equivariant total Chern class (c) and an Euler class (e) in $H^*(BG_\Omega)$.

We say that the representation with finitely many orbits satisfies the *Euler condition* if $e(N_\Omega)$ is not a 0–divisor in $H^*(BG_\Omega)$ for all Ω . Let us recall two topological lemmas.

Lemma 2.5 [12, Theorem 3.7] *Let $\Theta_1, \Theta_2, \dots$ be the list of orbits which satisfy $i < j \Rightarrow \Theta_i \not\subset \bar{\Theta}_j$ in a representation satisfying the Euler condition. Suppose that $\omega \in H_G^*(V)$ is 0 when restricted to $\Theta_1 \cup \dots \cup \Theta_s$ and is also 0 when restricted to Θ_{s+1} . Then ω is 0 when restricted to $\Theta_1 \cup \dots \cup \Theta_{s+1}$. \square*

Lemma 2.6 *Let W be an invariant subspace of the G –representation V , and let $e \in H_G^*(W) = H^*(BG)$ be the equivariant Euler class of the normal bundle of $W \subset V$. If a class $\omega \in H_G^n(V) = H^n(BG)$ is supported on W (that is, it is 0 when restricted to $V - W$), then it is divisible by e .*

Proof The statement follows from the exactness of the Gysin sequence

$$H_G^{n-\text{codim}(W \subset V)}(W) \rightarrow H_G^n(W) \rightarrow H_G^n(V - W),$$

where the first map is multiplication by e , and the second map is the composition $H_G^n(W) = H_G^n(\text{pt}) = H_G^n(V) \xrightarrow{r} H_G^n(V - W)$, with r being the restriction map. \square

For a cohomology class $x = x_0 + x_1 + x_2 + \dots \in H^*(X)$, where $x_i \in H^{2i}(X)$, let $\text{deg}(x)$ be the largest i for which $x_i \neq 0$. We set $\text{deg}(0) = -\infty$.

Theorem 2.7 *Let the G –representation V contain the scalars, let it have finitely many orbits, and let it satisfy the Euler condition. Let Ω be an orbit. The following properties uniquely determine $c^{\text{sm}}(\Omega)$:*

- (I) $\phi_\Omega(c^{\text{sm}}(\Omega)) = c(T_\Omega)e(N_\Omega) \in H^*(BG_\Omega)$,
- (II) $\phi_\Theta(c^{\text{sm}}(\Omega))$ is divisible by $c(T_\Theta)$ in $H^*(BG_\Theta)$ for any orbit Θ ,
- (III) $\text{deg}(\phi_\Theta(c^{\text{sm}}(\Omega))) < \text{deg}(c(T_\Theta)e(N_\Theta))$ for any orbit Θ different from Ω .

Proof First we prove that $c^{\text{sm}}(\Omega)$ satisfies the properties.

The orbit Ω is smooth at x_Ω , hence the image of $c^{\text{sm}}(\Omega)$ at the restriction homomorphism $\phi_\Omega: H_G^*(V) \rightarrow H_G^*(\Omega) = H_{G_\Omega}^*(x_\Omega)$ is $c(T_\Omega)e(N_\Omega)$; see (iii); hence property (I) is proved.

Let

$$(4) \quad \mathbb{1}_\Omega = \sum_{\Phi \leq \Omega} d_{\Omega, \Phi} \mathbb{1}_{\bar{\Phi}},$$

where $\Phi \leq \Omega$ means that $\Phi \subset \bar{\Omega}$. Then $c^{\text{sm}}(\Omega) = \sum_{\Phi \leq \Omega} d_{\Omega, \Phi} c^{\text{sm}}(\bar{\Phi})$ and

$$\phi_\Theta(c^{\text{sm}}(\Omega)) = \sum_{\Theta \leq \Phi \leq \Omega} d_{\Omega, \Phi} \phi_\Theta(c^{\text{sm}}(\bar{\Phi})).$$

Each of the $\phi_\Theta(c^{\text{sm}}(\bar{\Phi}))$ restrictions are divisible by $c(T_\Theta)$, because of (v). This proves property (II).

Observe that the number of zero weights of G_Θ acting on the tangent space of V at M_Θ is $n - \text{deg}(c(T_\Theta)e(N_\Theta))$. Hence, for any $i \geq \text{deg}(c(T_\Theta)e(N_\Theta))$ the i^{th} component of $\phi_\Theta(c^{\text{sm}}(\bar{\Phi}))$ does not depend on Φ ; let the common value be called x_i . Then for $i \geq \text{deg}(c(T_\Theta)e(N_\Theta))$ we have that the i^{th} component of $\phi_\Theta(c^{\text{sm}}(\Omega))$ is

$$(5) \quad x_i \cdot \sum_{\Theta \leq \Phi \leq \Omega} d_{\Omega, \Phi}.$$

However, substituting M_Θ in the identity (4) we get $0 = \sum_{\Theta \leq \Phi \leq \Omega} d_{\Omega, \Phi}$. Hence expression (5) is 0 for all $i \geq \text{deg}(c(T_\Theta)e(N_\Theta))$, which proves property (III).

The proof of the uniqueness of classes satisfying (I)–(III) is an adaptation of the argument in [27, Section 3.3]. Suppose two classes satisfy the conditions above for $c^{\text{sm}}(\Omega)$, and let ω be their difference. Then for every Θ we have that $\phi_\Theta(\omega)$ is divisible by $c(T_\Theta)$ and has degree strictly less than $\text{deg}(c(T_\Theta)e(N_\Theta))$. Let $\Theta_1, \Theta_2, \dots$ be a (finite) list of orbits satisfying $i < j \Rightarrow \Theta_i \not\subset \bar{\Theta}_j$. We will prove by induction on s that ω is 0 when restricted to $\Theta_1 \cup \dots \cup \Theta_s$. For $s = 0$ the claim holds. Suppose we know this statement for $s - 1$ and want to prove it for s . Because of the induction hypotheses, ω is supported on $\Theta_s \cup \Theta_{s+1} \cup \dots$. Hence its Θ_s restriction must be divisible by $e(N_{\Theta_s})$ (Lemma 2.6). We also know that it is divisible by $c(T_{\Theta_s})$. These classes are coprime in $H^*(BG_{\Theta_s})$, therefore we have that $\phi_{\Theta_s}(\omega)$ is divisible by $c(T_{\Theta_s})e(N_{\Theta_s})$. Since its degree is strictly less than that of $c(T_{\Theta_s})e(N_{\Theta_s})$, we have that $\phi_{\Theta_s}(\omega) = 0$. Lemma 2.5 implies that ω restricted to $\Theta_1 \cup \Theta_2 \cup \dots \cup \Theta_s$ is also zero. \square

Remark 2.8 The CSM class of a variety is supported on its closure. Thus the property

$$(IV) \quad \phi_{\Theta}(c^{\text{sm}}(\Omega)) = 0 \text{ for } \Theta \not\subset \bar{\Omega}$$

holds too. It is not listed among the axioms above, because it is forced by them.

3 Matrix Schubert cells

One of our goals in this paper is to give formulas for the CSM/SSM classes of the orbits of a certain representation. These orbits will be called the matrix Schubert cells.

Let us fix nonnegative integers $k \leq n$. Consider the group $\text{GL}_k(\mathbb{C}) \times B_n^-$ acting on $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by $(A, B) \cdot M = BMA^{-1}$. Here B_n^- is the Borel subgroup of $n \times n$ lower triangular matrices. The finitely many orbits of this action are parametrized by d -element subsets $J = \{j_1 < \dots < j_d\} \subset \{1, \dots, n\}$ with $0 \leq d \leq k$. The corresponding orbit is

$$(6) \quad \Omega_J = \{M \text{ is an } n \times k \text{ matrix : rk(top } r \text{ rows of } M) = |J \cap \{1, \dots, r\}|\}.$$

A representative of the orbit Ω_J is the $n \times k$ matrix M_J whose entries are 0's, except the (j_u, u) entries are 1 ($u = 1, \dots, d$). The orbits will be called matrix Schubert cells, and their closures are usually called matrix Schubert varieties; see eg [11; 23].

For $J = \{j_1 < \dots < j_d\} \subset \{1, \dots, n\}$ we define a few subsets of the entries of $k \times n$ matrices that will be useful later. Let

$$\mathbb{A}_0 = \{(v, u) \in \{1, \dots, n\} \times \{1, \dots, k\} : u \leq d \text{ and } v = j_u\},$$

$$\mathbb{A}_1 = \{(v, u) \in \{1, \dots, n\} \times \{1, \dots, k\} : u \leq d \text{ and } v < j_u\},$$

$$\mathbb{A}_2 = \{(v, u) \in \{1, \dots, n\} \times \{1, \dots, k\} : u \leq d \text{ and } v > j_u\},$$

$$\mathbb{A}_3 = \{(v, u) \in \{1, \dots, n\} \times \{1, \dots, k\} : u > d\},$$

$$\mathbb{A}_4 = \{(v, u) \in \{1, \dots, n\} \times \{1, \dots, k\} : v = j_w \text{ and } u > w \text{ for some } w \leq d\}.$$

The set $\mathbb{T}_J = \mathbb{A}_0 \cup \mathbb{A}_2 \cup \mathbb{A}_4$ represents the directions in $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ that are in the tangent space of Ω_J at M_J , and $\mathbb{N}_J = \{1, \dots, n\} \times \{1, \dots, k\} - \mathbb{A}_0 \cup \mathbb{A}_2 \cup \mathbb{A}_4$ represents the directions normal to Ω_J at M_J . Therefore $\dim \Omega_J = |\mathbb{T}_J|$ and $\text{codim}(\Omega_J \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)) = |\mathbb{N}_J|$.

Example 3.1 For $k = 2$ and $n = 3$ there are seven orbits, corresponding to the subsets $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\}$, with representatives (\bullet or \cdot stands for 0) the

matrices

$$\begin{pmatrix} 1 & \bullet \\ \bullet & 1 \\ \bullet & \bullet \end{pmatrix}, \begin{pmatrix} 1 & \bullet \\ \bullet & \bullet \\ \bullet & 1 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet \\ 1 & \bullet \\ \bullet & 1 \end{pmatrix}, \begin{pmatrix} 1 & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}, \begin{pmatrix} \bullet & \bullet \\ 1 & \bullet \\ \bullet & \bullet \end{pmatrix}, \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \\ 1 & \bullet \end{pmatrix}, \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}.$$

In each matrix a \bullet or a 1 indicates boxes corresponding to directions tangent to Ω_J and the rest (indicated by \bullet) correspond to normal directions.

Let $\alpha_1, \dots, \alpha_k$ and β_1, \dots, β_n be the Chern roots of the group $GL_k(\mathbb{C}) \times B_n^-$, or equivalently, the homotopically equivalent reductive group $GL_k(\mathbb{C}) \times GL_1(\mathbb{C})^n$. We have

$$(7) \quad H^*(B(GL_k(\mathbb{C}) \times B_n^-)) = H^*(B(GL_k(\mathbb{C}) \times GL_1(\mathbb{C})^n)) = \mathbb{C}[\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_n]^{S_k},$$

and the weights of the representation $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ defined above are $\beta_v - \alpha_u$ for $v = 1, \dots, n$ and $u = 1, \dots, k$. The weight space of $\beta_v - \alpha_u$ is the line corresponding to the (v, u) entry of $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$.

The $(GL_k(\mathbb{C}) \times B_n^-) \simeq (GL_k(\mathbb{C}) \times GL_1(\mathbb{C})^n)$ -equivariant CSM and SSM classes of matrix Schubert cells Ω_J are hence elements of the ring (7) and its completion, respectively. To claim the result about these classes in Section 5 we first need to define some important functions in Section 4.

4 Weight functions

In this section we define some important polynomials that will be identified with CSM classes of matrix Schubert cells in Section 5.

4.1 Localization form of weight functions

Let $k \leq n$ and $I \subset \{1, \dots, k\}$, where $|I| = d \leq k$ and $I = \{i_1 < \dots < i_d\}$.

Definition 4.1 Let $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_n)$. Let $U_I(\alpha, \beta)$ be given by

$$\prod_{u=1}^d \prod_{v=i_u+1}^n (1 + \beta_v - \alpha_u) \prod_{u=d+1}^k \prod_{v=1}^n (\beta_v - \alpha_u) \prod_{u=1}^d \prod_{v=1}^{i_u-1} (\beta_v - \alpha_u) \prod_{u=1}^d \prod_{v=u+1}^k \frac{1 + \alpha_u - \alpha_v}{\alpha_u - \alpha_v}.$$

A permutation $\sigma \in S_k$ acts on a k -tuple by permuting the components. Define the “weight function”

$$W_I = W_I(\alpha; \beta) = \frac{1}{(k-d)!} \sum_{\sigma \in S_k} U_I(\sigma(\alpha); \beta).$$

Although we omitted from the notation, the function W_I depends on k and n as well; their stabilization properties will be discussed below. Despite their appearance the weight functions are polynomials with integer coefficients, in fact of degree $kn - d$.

Remark 4.2 Weight functions were used in [41] to describe q -hypergeometric solutions of the quantum Knizhnik–Zamolodchikov equations. They also appeared in joint works [18; 37; 39] with the second author, as key components in identifying cohomology rings with Bethe algebras. In these past works weight functions were only defined for $d = k$; the present $d < k$ extension is new. The reason for calling the form above the “localization form” of the weight function, together with a geometric interpretation, is explained in Appendix C.

Remark 4.3 In [39] the $d = k$ weight functions are divided by a particular factor. It is shown there that these rational functions are (in a suitable sense) representatives of CSM classes in some quotient rings that are naturally identified with cohomology rings of compact spaces. See more on this in Appendix B.

Example 4.4 For $k = 1$ and $n = 2$ we have

$$W_{\{1\}} = 1 + \beta_2 - \alpha_1, \quad W_{\{2\}} = \beta_1 - \alpha_1, \quad W_{\{\}} = (\beta_1 - \alpha_1)(\beta_2 - \alpha_1).$$

For $k = 2$ and $n = 2$ we have

$$\begin{aligned} W_{\{1,2\}} &= \frac{(1 + \beta_2 - \alpha_1)(\beta_1 - \alpha_2)(1 + \alpha_1 - \alpha_2)}{\alpha_1 - \alpha_2} + \frac{(1 + \beta_2 - \alpha_2)(\beta_1 - \alpha_1)(1 + \alpha_2 - \alpha_1)}{\alpha_2 - \alpha_1} \\ &= 1 + \beta_1 + \beta_2 + 2\beta_1\beta_2 - (\alpha_1 + \alpha_2)(\beta_1 + \beta_2) - \alpha_1 - \alpha_2 + 2\alpha_1\alpha_2. \end{aligned}$$

4.2 Residue form of weight functions

The $\beta_i = 0$ for $i = 1, \dots, n$ substitution $W_I(\alpha; 0, \dots, 0)$ of the weight function W_I will be denoted by $W_{I,\beta=0}$.

We will use residue formulas for various functions. Recall that $\text{Res}_{z=\omega} f(z) dz$ of a meromorphic form at the complex number ω is the coefficient of $(z - \omega)^{-1}$ in the Laurent expansion of f at ω , equivalently, $1/(2\pi i) \int_\gamma f(z) dz$ for a small circle γ oriented counterclockwise around ω . If f is holomorphic around ω we have $\text{Res}_{z=\omega} f(z) dz/(z - \omega) = f(\omega)$. As usual, we define $\text{Res}_{z=\infty} f(z) dz = -\text{Res}_{z=0}(f(1/z) dz/z^2)$. We will use the *residue theorem* claiming that the sum of residues of a meromorphic form on the Riemann sphere is 0. For $z = z_1, \dots, z_k$ let $\text{RES}_k(f(z) dz)$ be a shorthand notation for $\text{Res}_{z_k=\infty} \cdots \text{Res}_{z_2=\infty} \text{Res}_{z_1=\infty}(f(z) dz)$.

Lemma 4.5 For a polynomial p in k variables we have

$$(8) \quad \text{RES}_k \frac{p(-z_1, \dots, -z_k) \prod_{1 \leq i < j \leq k} (z_j - z_i) dz_1 \cdots dz_k}{\prod_{i=1}^k \prod_{j=1}^k (z_i + \omega_j)} = (-1)^k \sum_{\sigma \in S_k} \frac{p(\omega_{\sigma(1)}, \dots, \omega_{\sigma(k)})}{\prod_{1 \leq i < j \leq k} (\omega_{\sigma(j)} - \omega_{\sigma(i)})}.$$

Proof Let

$$f(z) = \frac{p(-z_1, \dots, -z_k) \prod_{1 \leq i < j \leq k} (z_j - z_i)}{\prod_{i=1}^k \prod_{j=1}^k (z_i + \omega_j)}.$$

By iterating the $\text{Res}_{z=a} f(z) dz / (z - a) = f(a)$ formula we obtain that

$$(9) \quad \text{Res}_{z_k = -\omega_{\sigma(k)}} \cdots \text{Res}_{z_1 = -\omega_{\sigma(1)}} (f(z) dz_1 \cdots dz_k) = \frac{p(\omega_{\sigma(1)}, \dots, \omega_{\sigma(k)})}{\prod_{1 \leq i < j \leq k} (\omega_{\sigma(j)} - \omega_{\sigma(i)})}.$$

for a permutation $\sigma \in S_k$. The form $f(z) dz$ has no other finite residues than the ones in (9). Hence if we add (9) for all $\sigma \in S_k$ then we obtain the sum of all finite residues of $f(z) dz$. Applying the residue theorem for z_1, z_2, \dots, z_k (at each application we pick up a (-1)) we obtain the statement of the theorem. \square

Theorem 4.6 For $d \leq k \leq n$ with $|I| = d$, let $r = k - d$ and

$$f_I = \prod_{a=1}^r z_a^{n+r-a} \prod_{a=r+1}^k z_a^{i_{k+1}-a-1} \prod_{a=r+1}^k (1 + z_a)^{n-i_{k+1}-a} \times \prod_{a=r+1}^k \prod_{b=1}^{a-1} (1 + z_b - z_a) \prod_{1 \leq b < a \leq k} (z_a - z_b).$$

We have

$$W_{I, \beta=0} = (-1)^k \text{RES}_k \left(\frac{f_I}{\prod_{u=1}^k \prod_{v=1}^k (z_u + \alpha_v)} dz_1 \cdots dz_k \right).$$

Proof We have

$$U_{I, \beta=0} = \prod_{u=1}^d (1 - \alpha_u)^{n-i_u} \prod_{u=d+1}^k (-\alpha_u)^n \prod_{u=1}^d (-\alpha_u)^{i_u-1} \prod_{u=1}^d \prod_{v=u+1}^k \frac{1 + \alpha_u - \alpha_v}{\alpha_u - \alpha_v}.$$

Temporarily denote $\alpha_u = \omega_{k+1-u}$, that is, consider the list of α_i variables backwards. After rearrangements we obtain

$$U_{I, \beta=0} = \prod_{a=1}^r (-\omega_a)^n \prod_{a=r+1}^k (-\omega_a)^{i_{k+1}-a-1} \prod_{a=r+1}^k (1 - \omega_a)^{n-i_{k+1}-a} \prod_{a=r+1}^k \prod_{b=1}^{a-1} \frac{1 + \omega_a - \omega_b}{\omega_a - \omega_b}.$$

Define

$$(10) \quad V_I = U_{I,\beta=0} \times \prod_{a=1}^r (-\omega_a)^{r-a} \prod_{a=1}^r \prod_{b=1}^{a-1} \frac{1}{\omega_a - \omega_b}.$$

We claim that

$$(11) \quad \sum_{\sigma \in S_r} V_I(\sigma(\omega_1, \dots, \omega_r), \omega_{r+1}, \dots, \omega_k) = U_{I,\beta=0}.$$

Indeed, since $U_{I,\beta=0}$ is symmetric in $\omega_1, \dots, \omega_r$, it can be pulled out of the symmetrization, and the symmetrization of the last two factors of (10) is well known to be 1.

Another interpretation of (11) is that the left-hand side of (11) equals

$$\frac{1}{r!} \sum_{\sigma \in S_r} U_{I,\beta=0}(\sigma(\omega_1, \dots, \omega_r), \omega_{r+1}, \dots, \omega_k).$$

Thus, for the weight function we obtain

$$(12) \quad W_{I,\beta=0} = \sum_{\sigma \in S_k} V_I(\sigma(\omega_1, \dots, \omega_k)).$$

Lemma 4.5 then completes the proof. □

5 CSM classes of matrix Schubert cells are weight functions

Now we calculate the CSM classes of matrix Schubert cells.

Theorem 5.1 *Consider the $(GL_k(\mathbb{C}) \times B_n^-)$ -representation $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ and the description of the orbits in Section 3. For the equivariant Chern–Schwartz–MacPherson class of the orbit Ω_I we have*

$$c^{\text{sm}}(\Omega_I) = W_I(\alpha, \beta).$$

Probably the most natural proof of Theorem 5.1 is through the classical method of resolution of singularities, using the geometry of the weight function; see Appendix C. Here we show a proof based on the interpolation characterization of CSM classes (Theorem 2.7) to illustrate this new method for future applications where manageable resolutions are not known.

Proof We will show that W_I satisfies the properties of Theorem 2.7 for the representation $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$.

Let $J = \{j_1 < \dots < j_d\}$, where $d \leq k$. By looking at the matrix M_J one finds that the maximal torus of G_{Ω_J} is of rank $n + k - d$ and the map $\phi_J: H_{\text{GL}_k(\mathbb{C}) \times B_n}^*(V) \rightarrow H^*(BG_{\Omega_J})$, composed with the inclusion $H^*(BG_{\Omega}) \subset H^*(BT_{\Omega})$ (where T_{Ω} is the maximal torus of G_{Ω}), can be described by

$$\mathbb{C}[\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_n]^{S_k} \rightarrow \mathbb{C}[\alpha_{d+1}, \dots, \alpha_k, \beta_1, \dots, \beta_n],$$

$$\alpha_u \mapsto \begin{cases} \beta_{j_u} & \text{for } u = 1, \dots, d, \\ \alpha_u & \text{for } u = d + 1, \dots, k \end{cases} \quad \text{and} \quad \beta_v \mapsto \beta_v \quad \text{for } v = 1, \dots, n.$$

Using the notation in Section 3 we have that

$$c(T_{\Omega_J}) = \phi_J \left(\prod_{(v,u) \in \mathbb{T}_J} (1 + \beta_v - \alpha_u) \right) \quad \text{and} \quad e(N_{\Omega_J}) = \phi_J \left(\prod_{(v,u) \in \mathbb{N}_J} (\beta_v - \alpha_u) \right),$$

and that $\text{deg}(c(T_{\Omega_J})e(N_{\Omega_J})) = nk - d$. It follows that the representation satisfies the Euler condition.

Let $I = \{i_1 < \dots < i_e\}$ and recall that U_I is defined to be

$$\underbrace{\prod_{u=1}^e \prod_{v=i_u+1}^n (1 + \beta_v - \alpha_u)}_{P_1} \underbrace{\prod_{u=e+1}^k \prod_{v=1}^n (\beta_v - \alpha_u)}_{P_2} \underbrace{\prod_{u=1}^e \prod_{v=1}^{i_u-1} (\beta_v - \alpha_u)}_{P_3} \underbrace{\prod_{u=1}^e \prod_{v=u+1}^k \frac{1 + \alpha_u - \alpha_v}{\alpha_u - \alpha_v}}_{P_4}.$$

We have that

$$(13) \quad \begin{aligned} \phi_J(W_I) &= \phi_J \left(\frac{1}{(k-e)!} \sum_{\sigma \in S_k} U_I(\sigma(\alpha); \beta) \right) \\ &= \frac{1}{(k-e)!} \sum_{\sigma \in S_k} U_I(\sigma(\beta_{j_1}, \dots, \beta_{j_d}, \alpha_{d+1}, \dots, \alpha_k); \beta). \end{aligned}$$

The main observation of the proof is that because of factors of P_2 and P_3 the term

$$U_{I,J,\sigma} := U_I(\sigma(\beta_{j_1}, \dots, \beta_{j_d}, \alpha_{d+1}, \dots, \alpha_k); \beta)$$

vanishes for many σ . Namely, if there exists a $u = 1, \dots, d$ such that

$$(14) \quad \sigma^{-1}(u) > e \quad \text{or} \quad (\sigma^{-1}(u) \leq e \quad \text{and} \quad i_{\sigma^{-1}(u)} > j_u)$$

then $U_{I,J,\sigma} = 0$. This condition necessarily holds if $e < d$, so for such cases (13) is 0. Therefore in the rest of the proof we will assume that $d \leq e$. Also, define S_k^*

by $\sigma \in S_k^*$ if for all $u = 1, \dots, d$ we have $\sigma^{-1}(u) \leq e$ and $i_{\sigma^{-1}(u)} \leq j_u$, and we have

$$(15) \quad \phi_J(W_I) = \frac{1}{(k-e)!} \sum_{\sigma \in S_k^*} \phi_J(U_I(\sigma(\alpha); \beta)).$$

Now we are ready to prove properties (I)–(III).

If $I = J$ (in particular $d = e$) then $\sigma \in S_k^*$ if and only if $\sigma(u) = u$ for $u = 1, \dots, e$. Hence there are $(k - e)!$ terms in (15) and each of them is

$$\begin{aligned} \phi_J \left(\prod_{(v,u) \in \mathbb{A}_2} (1 + \beta_v - \alpha_u) \prod_{(v,u) \in \mathbb{A}_3} (\beta_v - \alpha_u) \prod_{(v,u) \in \mathbb{A}_1} (\beta_v - \alpha_u) \prod_{(v,u) \in \mathbb{A}_4} \frac{(1 + \beta_v - \alpha_u)}{(\beta_v - \alpha_u)} \right) \\ = \phi_J \left(\prod_{(v,u) \in \mathbb{T}_I} (1 + \beta_v - \alpha_u) \prod_{(v,u) \in \mathbb{N}_I} (\beta_v - \alpha_u) \right) \\ = c(T_{\Omega_J})e(N_{\Omega_J}). \end{aligned}$$

This proves property (I).

To prove property (II) we need to show that $\prod_{(v,u) \in \mathbb{A}_0 \cup \mathbb{A}_2 \cup \mathbb{A}_4} \phi_J(1 + \beta_v - \alpha_u)$ divides the expression in (15). We claim that this divisibility holds for every term of (15). A term of (15) is a product of ϕ_J -images of the factors in $P_1, P_2, P_3,$ and P_4 . For $(v, u) \in \mathbb{A}_0$ we have $\phi_J(1 + \beta_v - \alpha_u) = 1$. For $(v, u) \in \mathbb{A}_2$ the factor $\phi_J(1 + \beta_v - \alpha_u)$ appears as one factor in $\phi_J(P_1)$ (because of $\sigma \in S_k^*$). If $(v, u) \in \mathbb{A}_4$ then the factor $\phi_J(1 + \beta_v - \alpha_u)$ appears either as a factor of $\phi_J(P_1)$ or $\phi_J(P_4)$ (again, because of $\sigma \in S_k^*$). The factors of $\prod_{(v,u) \in \mathbb{A}_2 \cup \mathbb{A}_4} \phi_J(1 + \beta_v - \alpha_u)$ are all different, hence we proved the divisibility property (II).

To prove property (III) recall that if $e < d$ then $\phi_J(W_I) = 0$. If $e > d$ then

$$\deg(\phi_J(W_I)) \leq \deg(W_I) = nk - e < nk - d = \deg(c(T_{\Omega_J})e(N_{\Omega_J})).$$

Let us assume that $d = e$ but $J \neq I$. Then for each term of (15) there is a $u \in \{1, \dots, d\}$ for which $j_u > i_{\sigma^{-1}(u)}$. This implies that among the factors of $\phi_J(P_1)$ one is $\phi_J(1 + \beta_{j_u} - \beta_{i_u}) = 1$. Hence

$$\deg(\phi_J(W_I)) \leq \deg(W_I) - 1 = nk - e - 1 < nk - d = \deg(c(T_{\Omega_J})e(N_{\Omega_J})). \quad \square$$

Corollary 5.2 Consider the $(GL_k(\mathbb{C}) \times B_n^-)$ -representation $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ and the description of the orbits in Section 3. For the equivariant Segre–Schwartz–MacPherson class of the orbit Ω_I we have

$$s^{\text{sm}}(\Omega_I) = \frac{W_I(\alpha, \beta)}{\prod_{u=1}^k \prod_{v=1}^n (1 + \beta_v - \alpha_u)}. \quad \square$$

The $GL_k(\mathbb{C})$ -equivariant CSM and SSM classes of Ω_I are therefore $W_{I,\beta=0}$ and $W_{I,\beta=0} / \prod_{u=1}^k (1 - \alpha_u)^n$.

6 Symmetric functions. Residue generator functions.

In Sections 7 and 8 we will give generating function descriptions of certain CSM/SSM classes. In this section we recall Schur functions, and develop the (“iterated residue”) generating function tool we will use later.

Below we will work with integer vectors $(\lambda_1, \dots, \lambda_\mu)$, and some of them will be weakly decreasing, ie satisfying $\lambda_i \geq \lambda_{i+1}$. A partition is a class of weakly decreasing integer vectors generated by the relation $(\lambda_1, \dots, \lambda_\mu) \sim (\lambda_1, \dots, \lambda_\mu, 0)$.

Let us warn the reader that certain theorems will deal with weakly decreasing integer vectors, and in those, $(3, 1)$ and $(3, 1, 0)$, for example, are *different* integer vectors.

6.1 Schur functions

Let c_i for $i = 1, 2, \dots$ be a sequence of variables, set $c_{<0} = 0$ and $c_0 = 1$, and declare $\deg(c_i) = i$. For an integer vector $\lambda = (\lambda_1, \dots, \lambda_\mu) \in \mathbb{Z}^\mu$ define

$$s_\lambda = \det(c_{\lambda_i+j-i})_{i,j=1,\dots,\mu} \in \mathbb{C}[c_1, c_2, \dots].$$

If $s_\lambda \neq 0$ then its degree is $|\lambda| = \sum \lambda_i$. We have $s_\lambda = s_{\lambda,0}$ and the straightening laws

$$(16) \quad s_{(I,a,b,J)} = -s_{(I,b-1,a+1,J)} \quad \text{and} \quad s_{(I,a,a+1,J)} = 0.$$

The collection of s_λ for partitions λ is a basis of the vector space of polynomials in c_i . For λ a partition, s_λ is called a Schur function; other s_λ will be called fake Schur functions. Later we will also deal with formal (infinite) sums of s_λ ’s, ie we formally work in the completion $\mathbb{C}[[c_1, c_2, \dots]]$. Since the straightening laws respect degree, an infinite sum of s_λ ’s make sense as long as for every n there are finitely many terms for which $|\lambda| = n$.

Certain substitutions will play a key role below. Namely, let $\alpha_1, \dots, \alpha_k$ and β_1, \dots, β_n be two finite sets of variables (all declared to have degree 1), and define

$$\rho^{k,n}: \mathbb{C}[c_1, c_2, \dots] \rightarrow \mathbb{C}[\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_n]^{S_k \times S_n}$$

by

$$(17) \quad \rho^{k,n}(1 + c_1 t + c_2 t^2 + \dots) = \frac{\prod_{i=1}^n (1 + \beta_i t)}{\prod_{i=1}^k (1 + \alpha_i t)}.$$

For example $\rho^{k,n}(s_1) = \sum_{j=1}^n \beta_j - \sum_{i=1}^k \alpha_i$, and

$$\rho^{k,0}(s_{11}) = \sum_{1 \leq i < j \leq k} \alpha_i \alpha_j \quad \text{and} \quad \rho^{0,n}(s_{11}) = \sum_{1 \leq i < j \leq n} \beta_i \beta_j + \sum_{1 \leq i \leq n} \beta_i^2.$$

The statement

$$(18) \quad \ker \rho^{k,n} = \text{span}\{s_\lambda : \lambda_{k+1} \geq n + 1\}$$

is proved (albeit not stated) in [17, Section 3.2]: indeed, the \supset part follows from the factorization formula on page 37, and the \subset part from the proposition on page 36, of [17].

Lemma 6.1 *For an integer vector $\lambda = (\lambda_1, \dots, \lambda_\mu)$ we have*

$$s_\lambda = (-1)^\mu \text{RES}_\mu \left(\prod_{i=1}^\mu z_i^{\lambda_i} \cdot \prod_{1 \leq i < j \leq \mu} \left(1 - \frac{z_i}{z_j}\right) \cdot \prod_{i=1}^\mu \sum_{u=0}^\infty \frac{c_u}{z_i^u} \cdot \prod_{i=1}^\mu \frac{dz_i}{z_i} \right),$$

$$\rho^{k,n}(s_\lambda) = (-1)^\mu \text{RES}_\mu \left(\prod_{i=1}^\mu z_i^{\lambda_i} \cdot \prod_{1 \leq i < j \leq \mu} \left(1 - \frac{z_i}{z_j}\right) \cdot \prod_{i=1}^\mu \frac{\prod_{u=1}^n (1 + \beta_u/z_i)}{\prod_{u=1}^k (1 + \alpha_u/z_i)} \cdot \prod_{i=1}^\mu \frac{dz_i}{z_i} \right).$$

Proof Using the identity $\prod_{1 \leq i < j \leq k} (1 - z_i/z_j) = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{i=1}^\mu z_i^{\sigma(i)-i}$ the right-hand side of the first line equals

$$(19) \quad (-1)^\mu \text{RES}_\mu \left(\sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{i=1}^\mu z_i^{\lambda_i + \sigma(i) - i} \prod_{i=1}^\mu \sum_{u=0}^\infty \frac{c_u}{z_i^u} \prod_{i=1}^\mu \frac{dz_i}{z_i} \right).$$

By iterated application of

$$\text{Res}_{z=\infty} \left(z^n \sum_{u=0}^\infty \frac{c_u}{z^u} \frac{dz}{z} \right) = -c_n$$

(proved by changing coordinates $z = 1/w$ and calculating the residue at $w = 0$ by checking the $(-1)^{\text{st}}$ coefficient of the Laurent expansion) we obtain further that (19) equals

$$\sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{i=1}^\mu c_{\lambda_i + \sigma(i) - i} = s_\lambda,$$

as desired. The second line follows formally from the first one by (17). □

6.2 The \mathcal{S} -operation

Let z_1, \dots, z_μ be an ordered set of variables. For a monomial $z_1^{\lambda_1} \cdots z_\mu^{\lambda_\mu}$ define

$$\mathcal{S}_{z_1, \dots, z_\mu} (z_1^{\lambda_1} \cdots z_\mu^{\lambda_\mu}) = s_{\lambda_1, \dots, \lambda_\mu}.$$

For polynomials in z_1, \dots, z_μ we extend this operation linearly. Since the *straightening rules* (16) respect the degree the operation extends formally to formal power series, resulting in infinite sums of s_λ 's, that is, formal power series in c_i 's. For example

$$(20) \quad \mathcal{S}_{z_1, z_2} \left(z_1^3 z_2^1 \cdot \sum_{i=0}^{\infty} z_2^i \right) = \sum_{i=1}^{\infty} s_{3i} = s_{31} + s_{32} + s_{33} - \sum_{i=4}^{\infty} s_{i4}.$$

Observe that the middle expression is an expansion in terms of (possibly fake) Schur functions, and the last expression is an expansion in terms of Schur functions.

We will use *certain* rational functions to encode formal power series. Namely, by convention, the rational functions of the form

$$\frac{p(z_1, \dots, z_\mu)}{\prod_{i=1}^{\mu} (1 + \kappa_i z_i)^{l_i}},$$

where p is a polynomial and $\kappa_i \in \mathbb{Z}$, will denote the formal power series obtained by replacing each $1/(1 + \kappa_i z_i)$ factor by $\sum_{j=0}^{\infty} (-\kappa_i z_i)^j$. For example, under this convention, by $\mathcal{S}_{z_1, z_2} (z_1^3 z_2 / (1 - z_2))$ we mean the same expression as (20).

Define $\mathcal{S}_z^{k,n}(f(z)) = \rho^{k,n}(\mathcal{S}_z(f(z)))$.

The following proposition — which follows directly from Lemma 6.1 — is the reason for calling the \mathcal{S} -operation the “iterated residue operation”.

Proposition 6.2 *For a polynomial or formal power series $p(z_1, \dots, z_\mu)$ we have*

$$(21) \quad (-1)^\mu \text{RES}_\mu \left(p(z) \prod_{1 \leq i < j \leq \mu} \left(1 - \frac{z_i}{z_j} \right) \prod_{i=1}^{\mu} \sum_{u=0}^{\infty} \frac{c_u}{z_i^u} \prod_{i=1}^{\mu} \frac{dz_i}{z_i} \right) = \mathcal{S}_{z_1, \dots, z_\mu} (p(z)),$$

$$(22) \quad (-1)^\mu \text{RES}_\mu \left(p(z) \prod_{1 \leq i < j \leq \mu} \left(1 - \frac{z_i}{z_j} \right) \prod_{i=1}^{\mu} \frac{\prod_{u=1}^n (1 + \beta_u / z_i)}{\prod_{u=1}^k (1 + \alpha_u / z_i)} \prod_{i=1}^{\mu} \frac{dz_i}{z_i} \right) = \mathcal{S}_{z_1, \dots, z_\mu}^{k,n} (p(z)). \quad \square$$

7 Generating functions parametrized by weakly decreasing sequences

In this section we prove generating sequence descriptions of the $\text{GL}_k(\mathbb{C})$ -equivariant CSM and SSM classes of matrix Schubert varieties, Theorem 7.6 and Corollary 7.8. These generating functions will depend on weakly decreasing integer sequences. In Section 8 these results will be improved to generating sequences depending on partitions.

It is a remarkable fact of Schubert calculus that the equivariant fundamental class of (the closure of) a matrix Schubert cells does not change when one attaches a 0 to the end of the weakly decreasing integer sequence [11; 23]. We will see below that the higher order terms of CSM and SSM classes *change* with this operation. Yet, there is one version that will depend only on a partition (see Theorem 8.5).

7.1 Conventions on integer sequences

Recall that the set

$$\mathcal{I}_{k,n} = \{I : I = \{i_1 < \dots < i_d\} \subset \{1, \dots, n\} \text{ for some } 0 \leq d \leq k\}$$

parametrizes the matrix Schubert cells of $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$. To an element $I \in \mathcal{I}_{k,n}$ we associate a weakly decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)$ of nonnegative integers by the *conversion formula*

$$\lambda_a = i_{k+1-a} - (k + 1 - a)$$

for $a = 1, \dots, k$, where by convention $i_{d+a} = n + a$ for $a = 1, \dots, k - d$.

For a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)$ of weakly decreasing integers, let

$$I_\lambda = \{i_1 < i_2 < \dots < i_k\} \subset \mathbb{Z}$$

be defined by the conversion formula (equivalent to the one above)

$$i_a = \lambda_{k+1-a} + a.$$

We say that $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)$ and the nonnegative integer n are *compatible* if the elements in I_λ larger than n form an interval (possibly empty) starting at $n + 1$; that is, if there exists a $q \geq 0$ such that $I_\lambda \cap \mathbb{Z}^{>n} = \{n + 1, n + 2, \dots, n + q\}$.

It follows that map $I \mapsto \lambda$ described above is a bijection between $\mathcal{I}_{k,n}$ and

$$\mathcal{I}'_{k,n} = \{\lambda : \lambda = (\lambda_1 \geq \dots \geq \lambda_k) \in \mathbb{N}^k \text{ with } \lambda \text{ and } n \text{ compatible}\}.$$

The inverse map $\mathcal{I}'_{k,n} \rightarrow \mathcal{I}_{k,n}$ is $\lambda \mapsto I_\lambda \cap \{1, \dots, n\}$. Observe that a given λ is compatible with any sufficiently large n .

Example 7.1 Let $k = 2$ and $n = 3$ and consider the subsets $I \subset \{1, 2, 3\}$ as in Example 3.1. The corresponding λ 's are $(0, 0)$, $(1, 0)$, $(1, 1)$, $(2, 0)$, $(2, 1)$, $(2, 2)$, $(3, 3)$, respectively.

Example 7.2 The sequence $\lambda = (3, 1)$ is compatible with n if and only if $n \geq 4$ (since $I_\lambda = \{2, 5\}$). The element corresponding to $\lambda = (3, 1)$ in $\mathcal{I}_{2,4}$ is $I = \{2\}$. For $n \geq 5$ the element corresponding to $\lambda = (3, 1)$ in $\mathcal{I}_{2,n}$ is $I = \{2, 5\}$.

7.2 Generating functions for $GL_k(\mathbb{C})$ -equivariant CSM and SSM classes

The $(GL_k(\mathbb{C}) \times B_n^-)$ -equivariant CSM/SSM classes we study are elements of

$$\mathbb{C}[\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_n]^{S_k} \quad \text{and} \quad \mathbb{C}[[\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_n]]^{S_k}.$$

By plugging in $\beta_i = 0$ for all $i = 1, \dots, n$ we obtain symmetric polynomials (power series) in $\alpha_1, \dots, \alpha_k$, hence linear combinations (formal infinite sums) of polynomials $\rho^{k,0}(s_\lambda)$. The topological counterpart of this substitution is considering equivariant cohomology only with respect to the $GL_k(\mathbb{C})$ factor of $GL_k(\mathbb{C}) \times B_n^-$.

Denote

$$c_{\beta=0}^{sm}(\Omega_I) = c^{sm}(\Omega_I)|_{\beta_v=0, v=1, \dots, n} \quad \text{and} \quad s_{\beta=0}^{sm}(\Omega_I) = s^{sm}(\Omega_I)|_{\beta_v=0, v=1, \dots, n}.$$

Our next goal is to find expressions for the Schur function expansions of these functions.

Proposition 7.3 For $\lambda \in \mathcal{I}'_{k,n}$ let $I = I_\lambda \cap \{1, \dots, n\}$ be the corresponding element in $\mathcal{I}_{k,n}$. Let

$$f_{\lambda,n} = \prod_{i=1}^k z_i^{\lambda_i+k-i} \prod_{i=1}^k (1+z_i)^{\max(0, n-k-1-\lambda_i+i)} \times \prod_{\substack{1 \leq i < j \leq k \\ \lambda_j - j \leq n - k - 1}} (1+z_i-z_j) \prod_{1 \leq i < j \leq k} (z_j-z_i).$$

We have

$$c_{\beta=0}^{sm}(\Omega_{I_\lambda \cap \{1, \dots, n\}}) = (-1)^k \text{RES}_k \left(\frac{f_{\lambda,n}}{\prod_{i=1}^k \prod_{u=1}^k (z_i + \alpha_u)} dz_1 \cdots dz_k \right).$$

Proof We have $c_{\beta=0}^{sm}(\Omega_I) = W_{I, \beta=0}$ by Theorem 5.1. For the right-hand side we have a residue description, Theorem 4.6, which is reformulated here for λ instead of I . \square

Definition 7.4 Let $k, n \in \mathbb{N}$. For $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)$, where $\lambda_k \geq 0$, define

$$\mathcal{F}_{\lambda,n}^{csm} = \prod_{i=1}^k z_i^{\lambda_i} \cdot \prod_{i=1}^k (1+z_i)^{\max(0, n-k-1-\lambda_i+i)} \cdot \prod_{\substack{1 \leq i < j \leq k \\ \lambda_j - j \leq n - k - 1}} (1+z_i-z_j),$$

$$\mathcal{F}_{\lambda,n}^{ssm} = \frac{\mathcal{F}_{\lambda,n}^{csm}}{\prod_{i=1}^k (1+z_i)^n}.$$

Observe that if n is large (precisely, $n \geq \lambda_1 + k^1$) then $\mathcal{F}_{\lambda,n}^{\text{ssm}}$ does not depend on n . The stabilized value will be called

$$(23) \quad \mathcal{F}_{\lambda,\infty}^{\text{ssm}} = \prod_{i=1}^k z_i^{\lambda_i} \cdot \prod_{i=1}^k (1 + z_i)^{-k-1-\lambda_i+i} \cdot \prod_{1 \leq i < j \leq k} (1 + z_i - z_j)$$

$$(24) \quad = \prod_{i=1}^k \left(\frac{z_i}{1 + z_i} \right)^{\lambda_i} \prod_{j=1}^k \prod_{i=1}^j \frac{1 + z_i - z_j}{1 + z_i}.$$

Example 7.5 We have

$$\begin{aligned} \mathcal{F}_{(1,0),0}^{\text{ssm}} &= z_1, & \mathcal{F}_{(1,0),1}^{\text{ssm}} &= \frac{z_1(1 + z_1 - z_2)}{(1 + z_1)(1 + z_2)}, \\ \mathcal{F}_{(1,0),2}^{\text{ssm}} &= \frac{z_1(1 + z_1 - z_2)}{(1 + z_1)(1 + z_2)^2}, & \mathcal{F}_{(1,0),\infty}^{\text{ssm}} &= \mathcal{F}_{(1,0),\geq 3}^{\text{ssm}} = \frac{z_1(1 + z_1 - z_2)}{(1 + z_1)(1 + z_2)^3}. \end{aligned}$$

The following theorem gives the generating sequences of $\text{GL}_k(\mathbb{C})$ –equivariant CSM and SSM classes of matrix Schubert cells in $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$.

Theorem 7.6 For $\lambda \in \mathcal{I}'_{k,n}$ let $I = I_\lambda \cap \{1, \dots, n\}$ be the corresponding element in $\mathcal{I}_{k,n}$. Then

$$c_{\beta=0}^{\text{sm}}(\Omega_I) = \mathcal{S}_{z_1, \dots, z_k}^{k,0}(\mathcal{F}_{\lambda,n}^{\text{csm}}) \quad \text{and} \quad s_{\beta=0}^{\text{sm}}(\Omega_I) = \mathcal{S}_{z_1, \dots, z_k}^{k,0}(\mathcal{F}_{\lambda,n}^{\text{ssm}}).$$

Proof The first statement follows from Proposition 7.3 and Proposition 6.2. The second statement follows from the first one. □

Example 7.7 Let $k = 2$ and $\lambda = (3, 1)$. Then $I_\lambda = \{2, 5\}$. Hence λ is compatible with n if and only if $n \geq 4$. The corresponding subset in $\mathcal{I}_{2,4}$ is $I = \{2\}$, and for $n \geq 5$ the corresponding subset in $\mathcal{I}_{2,n}$ is $\{2, 5\}$. Calculating Taylor series of the appropriate explicit rational functions we obtain that for $n = 4$,

$$\begin{aligned} c_{\beta=0}^{\text{sm}}(\Omega_{\{2\}}) &= \mathcal{S}_{z_1, z_2}^{2,0}(z_1^3 z_2 + (z_1^4 z_2 + z_1^3 z_2^2) + (2z_1^4 z_2^2 - z_1^3 z_2^3) + (z_1^4 z_2^3 - z_1^3 z_2^4)) \\ &= \rho^{2,0}(s_{3,1} + (s_{4,1} + s_{3,2}) + (2s_{4,2} - s_{3,3}) + s_{4,3}), \\ s_{\beta=0}^{\text{sm}}(\Omega_{\{2\}}) &= \mathcal{S}_{z_1, z_2}^{2,0}(z_1^3 z_2 - (3z_1^4 z_2 + 3z_1^3 z_2^2) + (6z_1^5 z_2 + 10z_1^4 z_2^2 + 5z_1^3 z_2^3) \\ &\quad - (10z_1^6 z_2 + 22z_1^5 z_2^2 + 17z_1^4 z_2^3 + 7z_1^3 z_2^4) + \dots) \\ &= \rho^{2,0}(s_{3,1} - (3s_{4,1} + 3s_{3,2}) \\ &\quad + (6s_{5,1} + 10s_{4,2} + 5s_{3,3})(10s_{6,1} + 22s_{5,2} + 17s_{4,3}) + \dots). \end{aligned}$$

¹Since $\lambda_1 + k = i_k$ in the language of Section 7.1 the condition is $i_k \leq n$.

For $n = 5$ we have

$$\begin{aligned}
 c_{\beta=0}^{\text{sm}}(\Omega_{\{2,5\}}) &= \mathcal{S}_{z_1, z_2}^{2,0}(z_1^3 z_2 + (z_1^4 z_2 + 2z_1^3 z_2^2) + (3z_1^4 z_2^2) \\
 &\quad + (3z_1^4 z_2^3 - 2z_1^3 z_2^4) + (z_1^4 z_2^4 - z_1^3 z_2^5)) \\
 &= \rho^{2,0}(s_{3,1} + (s_{4,1} + 2s_{3,2}) + 3s_{4,2} + 3s_{4,3} + 2s_{4,4}), \\
 s_{\beta=0}^{\text{sm}}(\Omega_{\{2,5\}}) &= \mathcal{S}_{z_1, z_2}^{2,0}(z_1^3 z_2 - (4z_1^4 z_2 + 3z_1^3 z_2^2) + (13z_1^4 z_2^2 + 5z_1^3 z_2^3 + 10z_1^5 z_2) \\
 &\quad - (20z_1^6 z_2 + 35z_1^5 z_2^2 + 22z_1^4 z_2^3 + 7z_1^3 z_2^4) + \dots) \\
 &= \rho^{2,0}(s_{3,1} - (4s_{4,1} + 3s_{3,2}) + (13s_{4,2} + 5s_{3,3} + 10s_{5,1}) \\
 &\quad - (20s_{6,1} + 35s_{5,2} + 22s_{4,3}) + \dots).
 \end{aligned}$$

If $n \geq 5$ then $\mathcal{F}_{(3,1),n}^{\text{ssm}}$ does not depend on n any longer. Hence the last formula for $s_{\beta=0}^{\text{sm}}(\Omega_{\{2,5\}})$ holds for any $n \geq 5$.

For $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ assume that n is large enough to ensure $\lambda_1 \leq n - k$. Then the set in $\mathcal{I}_{k,n}$ corresponding to λ is I_λ . Also, the elements of the matrix Schubert cell $\Omega_I \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ have full rank (ie rank k).

Corollary 7.8 *If $\lambda_1 \leq n - k$ then*

$$(25) \quad s_{\beta=0}^{\text{sm}}(\Omega_{I_\lambda}) = \mathcal{S}_{z_1, \dots, z_k}^{k,0}(\mathcal{F}_{\lambda, \infty}^{\text{ssm}}). \quad \square$$

The essence of Corollary 7.8 is that given a weakly decreasing sequence of nonnegative integers λ , there is a formula (namely the right-hand side of (25)) which expresses the $\text{GL}_k(\mathbb{C})$ -equivariant SSM class of $\Omega_{I_\lambda} \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ for all sufficiently large n . Unfortunately the expression given in Corollary 7.8 *does* depend on k , that is, it changes if we add a 0 to the end of λ . This will be improved in Section 8.

Remark 7.9 Corollary 7.8 shows the stabilization of SSM classes when $n \geq k + \lambda_1$. There is another type of stabilization of CSM classes in the $n \geq k + \lambda_1$ range. Namely, in this case

$$\mathcal{F}_{\lambda, n}^{\text{csm}} = \prod_{i=1}^k z_i^{\lambda_i} \cdot \prod_{i=1}^k (1 + z_i)^{n-k-1-\lambda_i+i} \cdot \prod_{1 \leq i < j \leq k} (1 + z_i - z_j).$$

Hence, if $\lambda = (\lambda_1, \dots, \lambda_k)$ is changed by adding 1 to each component, and n is increased by 1, then $\mathcal{F}_{\lambda, n}^{\text{csm}}$ gets multiplied by $z_1 \cdots z_k$. This means that—in the $n \geq k + \lambda_1$ range—increasing n by 1, and increasing λ by $(1^k) = (1, \dots, 1)$ changes the $\text{GL}_k(\mathbb{C})$ -equivariant CSM class of Ω_{I_λ} in a controlled way: in the Schur expansion

the partition of each Schur polynomial is increased by $(1^k) = (1, \dots, 1)$. For example for $k = 2$ and $\lambda = (3, 1)$ we get

$$\begin{aligned} c_{\beta=0}^{\text{sm}}(\Omega_{\{2,5\}} \subset \text{Hom}(2, 5)) &= \rho^{2,0}(s_{3,1} + 2s_{3,2} + s_{4,1} + 3s_{4,2} + 3s_{4,3} + 2s_{4,4}), \\ c_{\beta=0}^{\text{sm}}(\Omega_{\{3,6\}} \subset \text{Hom}(2, 6)) &= \rho^{2,0}(s_{4,2} + 2s_{4,3} + s_{5,2} + 3s_{5,3} + 3s_{5,4} + 2s_{5,5}), \\ c_{\beta=0}^{\text{sm}}(\Omega_{\{4,7\}} \subset \text{Hom}(2, 7)) &= \rho^{2,0}(s_{5,3} + 2s_{5,4} + s_{6,3} + 3s_{6,4} + 3s_{6,5} + 2s_{6,6}), \end{aligned}$$

and so on; see “lowering” and “raising” operators in [13].

8 Generating functions parametrized by partitions

Corollary 7.8 claims that $s_{\beta=0}^{\text{sm}}(\Omega_{I_\lambda})$ is obtained by applying the substitution $\rho^{k,0}$ to the generating sequence $\mathcal{S}_{z_1, \dots, z_k}(\mathcal{F}_{\lambda, \infty}^{\text{ssm}})$. However this generating sequence changes by adding a 0 to the end of λ . In this section we present a generating function independent of such change. This new generating function depends on infinitely many variables. First, in Section 8.1 we deal with the algebra of generating functions with infinitely many variables.

8.1 Increasing the number of variables in the generating function

Let $h(z_1, \dots, z_{k+1})$ be a power series in $k + 1$ variables. We have that $\mathcal{S}_{z_1, \dots, z_k, z_{k+1}}(h)$ is an infinite linear combination of Schur functions. Some of the terms correspond to partitions of length at most k — call the sum of these terms $\mathcal{S}_{z_1, \dots, z_{k+1}}^{\leq k}(h)$ — and the rest corresponds to partitions of length $k + 1$.

Lemma 8.1 *Let $f(z_1, \dots, z_k)$ and $g(z_1, \dots, z_k, z_{k+1})$ be formal power series such that $g(z_1, \dots, z_k, 0) = 1$. Then*

$$\mathcal{S}_{z_1, \dots, z_{k+1}}^{\leq k}(fg) = \mathcal{S}_{z_1, \dots, z_k}(f).$$

Proof Let $\prod_{i=1}^k z_i^{a_i}$ and $\prod_{i=1}^{k+1} z_i^{b_i}$ be monomials that occur in f and g , respectively, with nonzero coefficients. Their product $T = \prod_{i=1}^k z_i^{a_i+b_i} \cdot z_{k+1}^{b_{k+1}}$ occurs in fg with nonzero coefficient. From (16) we have that $\mathcal{S}_{z_1, \dots, z_{k+1}}(T)$ is either 0 or equal to $\pm \mathcal{S}_{z_1, \dots, z_{k+1}}(\prod_{i=1}^{k+1} z_i^{\mu_i})$, where μ is a partition, and μ_{k+1} is equal to one of

$$b_{k+1}, \quad a_k + b_k + 1, \quad a_{k-1} + b_{k-1} + 2, \quad a_{k-2} + b_{k-2} + 3, \quad \dots, \quad a_1 + b_1 + k.$$

Hence, $\mathcal{S}_{z_1, \dots, z_{k+1}}^{\leq k}(T)$ is nonzero if and only if $\mu_{k+1} = 0$. The listed integers are all necessarily positive except the first one. Hence $\mu_{k+1} = 0$ can only occur if $\mu_{k+1} = b_{k+1} = 0$. However, the $g(z_1, \dots, z_k, 0) = 1$ condition then implies that the

only monomial in g with $b_{k+1} = 0$ is the monomial 1. That is, we have that $b_i = 0$ for $i = 1, \dots, k + 1$.

We have obtained that the only way of obtaining a partition of length at most k in $\mathcal{S}_{z_1, \dots, z_{k+1}}^{\leq k}(fg)$ is by using the constant term 1 of g . This proves the lemma. \square

For a function $f(z_1, \dots, z_k)$ and $N \in \mathbb{N}$ consider

$$H_N = f(z_1, \dots, z_k) \prod_{j=1}^N \prod_{i=1}^j \frac{1 + z_i - z_j}{1 + z_i}.$$

Observe that

$$H_N = H_{N-1} \cdot \prod_{i=1}^N \frac{1 + z_i - z_N}{1 + z_i}$$

and that the factor $\prod_{i=1}^N (1 + z_i - z_N)/(1 + z_i)$ takes the value 1 if we substitute $z_N = 0$. Hence, we can apply Lemma 8.1 for $k + 1, k + 2, \dots$ and obtain that the coefficient of s_μ for any concrete partition μ stabilizes in H_N as $N \rightarrow \infty$. The sum of the stable terms will be denoted by

$$\mathcal{S}_{z_1, z_2, \dots} \left(f(z_1, \dots, z_k) \prod_{j=1}^{\infty} \prod_{i=1}^j \frac{1 + z_i - z_j}{1 + z_i} \right).$$

8.2 The \tilde{s}_λ function

Recall that a partition is an equivalence class of sequences of weakly decreasing non-negative integers with respect to the equivalence relation generated by $(\lambda_1, \dots, \lambda_k) \sim (\lambda_1, \dots, \lambda_k, 0)$. As usual, we will use a representative to denote a partition. We are ready to make a key definition of the paper.

Definition 8.2 For a partition λ denote

$$\tilde{s}_\lambda := \mathcal{S}_{z_1, z_2, \dots} \left(\prod_{i=1}^k \left(\frac{z_i}{1 + z_i} \right)^{\lambda_i} \prod_{j=1}^{\infty} \prod_{i=1}^j \frac{1 + z_i - z_j}{1 + z_i} \right).$$

Example 8.3 Some examples are given in Section 1. Another one is

$$\begin{aligned} \tilde{s}_{31} = & s_{31} - (4s_{41} + 3s_{32} + 3s_{311}) \\ & + (10s_{51} + 13s_{42} + 5s_{33} + 10s_{321} + 6s_{3111} + 13s_{411}) \\ & - (20s_{61} + 35s_{52} + 22s_{43} + 35s_{511} + 46s_{421} + 19s_{331} \\ & \quad + 10s_{322} + 28s_{4111} + 22s_{3211}) + \dots \end{aligned}$$

Observe how the partitions that occur in the subscripts grow: not only are the components larger and larger numbers but the *lengths* of the partitions are growing as well. In the usual picture of Young diagrams the shapes not only “grow to the right” but also “grow downwards”. To our best knowledge this phenomenon is new in algebraic combinatorics; it does not occur in analogous situations in the theory of equivariant fundamental classes essentially because of [14, Theorem 2.1].

The following conjecture is verified in several special cases.

Conjecture 8.4 *For every partition λ the signs in the Schur expansions of \tilde{s}_λ alternate with the degree. Namely, for a partition μ , we have that $(-1)^{|\mu|-|\lambda|}$ times the coefficient of s_μ in \tilde{s}_λ is nonnegative.*

Theorem 8.5 *Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ and $n \geq \lambda_1 + k$. Consider $\Omega_{I_\lambda} \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$. We have*

$$s_{\beta=0}^{\text{sm}}(\Omega_{I_\lambda}) = \rho^{k,0}(\tilde{s}_\lambda).$$

Proof The statement follows from Corollary 7.8 if we use the form (24) for $\mathcal{F}_{\lambda, \infty}^{\text{ssm}}$, and Lemma 8.1. □

The advantage of this theorem compared to Corollary 7.8 is that now the generator function only depends on the partition. The disadvantage is that this general generating function has infinitely many variables. The condition $n \geq \lambda_1 + k$ is equivalent to the property that the Young diagram of λ fits into a $k \times (n - k)$ rectangle, or to the property that $I_\lambda \subset \{1, \dots, n\}$, or to the property that the elements of the orbit on the left-hand side have full rank k .

8.3 SSM classes of general matrix Schubert cells in terms of \tilde{s}_λ functions

Theorem 8.5 gives the \tilde{s} -expansion of the $\text{GL}_k(\mathbb{C})$ -equivariant SSM classes of “full-rank” matrix Schubert cells, that is, those cells in $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ whose elements have rank k . While these cells are of the most interest, we will need \tilde{s} -expansions of the SSM classes of smaller rank cells too.

Let $I = \{i_1 < \dots < i_d\} \in \mathcal{I}_{k,n}$, where $|I| = d \leq k$. Recall that the corresponding $\lambda \in \mathcal{I}'_{k,n}$ has the form

$$\lambda = (\underbrace{n - d, \dots, n - d}_{k-d}, \lambda_{k-d+1}, \lambda_{k-d+2}, \dots, \lambda_k).$$

Let $\Lambda(I)$ be the set of partitions μ obtained from this λ by weakly increasing only the first $k - d$ components. That is, elements $\mu = (\mu_1, \dots, \mu_k)$ of $\Lambda(I)$ are partitions and they satisfy

- $\mu_a \geq n - d$ for $a = 1, \dots, k - d$,
- $\mu_a = \lambda_a = i_{k+1-a} - (k + 1 - a)$ for $a = k - d + 1, \dots, k$.

Theorem 8.6 *Let $I \in \mathcal{I}_{k,n}$ with $|I| = d \leq k$. For $\Omega_I \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$, we have*

$$s_{\beta=0}^{\text{sm}}(\Omega_I) = \rho^{k,0} \left(\sum_{\mu \in \Lambda(I)} \tilde{s}_\mu \right).$$

Observe that Theorem 8.5 is the $d = k$ special case of this one.

Proof For $N > n$, let $\pi: \text{Hom}(\mathbb{C}^k, \mathbb{C}^N) \rightarrow \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ be the projection defined by forgetting the bottom $N - n$ rows of an $N \times k$ matrix. The projection π is $(\text{GL}_k(\mathbb{C}) \times B_N^-)$ -equivariant, where B_N^- acts on the target through the map $B_N^- \rightarrow B_n^-$ assigning the upper-left $n \times n$ submatrix to an element of B_n^- .

Consider the cylinders $\pi^{-1}(\Omega_I) \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^N)$ for $I \in \mathcal{I}_{k,n}$. These cylinders are $(\text{GL}_k(\mathbb{C}) \times B_N^-)$ -invariant, and from the rank description of orbits (6) it follows that

$$\pi^{-1}(\Omega_I) = \bigcup_{J \in \mathcal{I}_{k,n,N}(I)} \Omega_J,$$

where

$$\mathcal{I}_{k,n,N}(I) = \{J \in \mathcal{I}_{k,N} : J \cap \{1, \dots, n\} = I\}.$$

The map π is a projection, and therefore it is transversal to the Ω_J stratification of $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ (a Whitney stratification). Hence, (iv) in Section 2.3 implies that in $(\text{GL}_k(\mathbb{C}) \times B_N^-)$ -equivariant cohomology,

$$\begin{aligned} s^{\text{sm}}(\Omega_I \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)) &= s^{\text{sm}}(\pi^{-1}(\Omega_I) \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^N)) \\ &= \sum_{J \in \mathcal{I}_{k,n,N}(I)} s^{\text{sm}}(\Omega_J \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^N)). \end{aligned}$$

Because the $(\text{GL}_k(\mathbb{C}) \times B_N^-)$ -action on $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ is through $\text{GL}_k(\mathbb{C}) \times B_n^-$, the left-hand side can be interpreted as the $(\text{GL}_k(\mathbb{C}) \times B_n^-)$ -equivariant SSM class. In particular — while the identity is in the completion of $\mathbb{C}[\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_N]^{S_k}$ — both sides only depend on the variables $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_n$.

It will be convenient to rewrite the last expression as

$$s^{\text{sm}}(\Omega_I \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)) = \sum_{\substack{J \in \mathcal{I}_{k,n,N(I)} \\ |J|=k}} s^{\text{sm}}(\Omega_J \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^N)) + \sum_{\substack{J \in \mathcal{I}_{k,n,N(I)} \\ |J|<k}} s^{\text{sm}}(\Omega_J \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^N)).$$

Now let us restrict the group action to $\text{GL}_k(\mathbb{C})$ — that is, we substitute $\beta_i = 0$ — and apply Theorem 8.5 to the terms in the first summation. We obtain

$$(26) \quad s_{\beta=0}^{\text{sm}}(\Omega_I \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)) = \sum_{\substack{\mu \in \Lambda(I) \\ \mu_1 \leq N-k}} \rho^{k,0}(\tilde{s}_\mu) + \sum_{\substack{|J|<k \\ J \cap \{1, \dots, n\} = I}} s_{\beta=0}^{\text{sm}}(\Omega_J \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^N)).$$

As $N \rightarrow \infty$, the codimensions of the Ω_J 's appearing in the second summation tend to infinity. Hence, the degree of the second summation tends to infinity (where degree means the degree of the smallest nonzero component). Therefore applying $N \rightarrow \infty$ to (26) proves the theorem. □

Example 8.7 It is instructive to compare the following two examples (compare Examples 7.2 and 7.7), both associated with the partition $\lambda = (3, 1)$ (see Section 7.1):

$$s_{\beta=0}^{\text{sm}}(\Omega_{\{2,5\}} \subset \text{Hom}(\mathbb{C}^2, \mathbb{C}^5)) = \rho^{2,0}(\tilde{s}_{(3,1)}),$$

$$s_{\beta=0}^{\text{sm}}(\Omega_{\{2\}} \subset \text{Hom}(\mathbb{C}^2, \mathbb{C}^4)) = \rho^{2,0}(\tilde{s}_{(3,1)} + \tilde{s}_{(4,1)} + \tilde{s}_{(5,1)} + \dots).$$

The fundamental class of both Ω orbits above is $\rho^{2,0}(s_{(3,1)})$. This Schur function is the smallest term of both SSM classes above. However, the full SSM classes are different. That is, while the fundamental classes of matrix Schubert varieties only depend on the associated partitions (a phenomenon observed in [11; 23]), in SSM theory this only holds for full-rank orbits.

Remark 8.8 Arguments similar to the ones used above (eg Theorem 8.6 for $k, n = \infty$ and $I = \emptyset$) show that

$$(27) \quad \sum_{\lambda} \tilde{s}_\lambda = 1.$$

In fact, using the “triangularity” property

$$(28) \quad s_\lambda^{\text{sm}} = s_\lambda + \text{higher degree terms},$$

we can see that (27) is the only linear relation among the functions \tilde{s}_λ . If we declare c_i

“of order ε^i ”, and we declare the Schur functions s_λ “positive”, then property (28) implies that the terms \tilde{s}_λ in (27) are positive. Hence the collection $\{\tilde{s}_\lambda\}_\lambda$ is a (formal power series valued) *probability distribution* on the set of partitions. In this language the GL_∞ -equivariant SSM class of an equivariant constructible function on the $(GL_\infty \times B_\infty^-)$ -representation $\text{Hom}(\mathbb{C}^\infty, \mathbb{C}^\infty)$ is the *expected value* of the constructible function. It would be interesting to see applications of this probability theory interpretation in enumerative geometry.

9 SSM classes for the A_2 quiver representation

Let $k \leq n$ be nonnegative integers, $l = n - k$, and consider the $(GL_k(\mathbb{C}) \times GL_n(\mathbb{C}))$ -representation $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ defined by $(A, B) \cdot \phi = B \circ \phi \circ A^{-1}$. This representation is also called the A_2 quiver representation. The orbits of this representation are

$$\Sigma^r = \Sigma^r_{k,n} = \{\phi \in \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) : \dim \ker \phi = r\} \quad \text{for } r = 0, \dots, k.$$

The SSM class of $\Sigma^r_{k,n}$ is a nonhomogeneous element in (the completion of)

$$H^*(B(GL_k(\mathbb{C}) \times GL_n(\mathbb{C}))) = \mathbb{C}[\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_n]^{S_k \times S_n}.$$

Theorem 9.1 *We have*

$$(29) \quad s^{\text{sm}}(\Sigma^r_{k,n}) = \rho^{k,n} \left(\sum_{\substack{\lambda_r \geq r+l \\ \lambda_{r+1} \leq r+l}} \tilde{s}_\lambda \right).$$

Another way of describing the indexing set in (29) is that it consists of partitions whose Young diagram contains the box $(r, r + l)$ but does not contain the box $(r + 1, r + l + 1)$.

Proof First we claim that there exist a formal power series \mathcal{P}^r_l in c_1, c_2, \dots only depending on r and l (not on k and n separately) such that

$$s^{\text{sm}}(\Sigma^r_{k,n}) = \rho^{k,n}(\mathcal{P}^r_l).$$

This follows from the fact that the exclusion-inclusion formula given in [31] for $s^{\text{sm}}(\Sigma^r_{k,n})$ — recalled in Theorem A.1 — only depends on r and l .²

It follows that

$$(30) \quad s^{\text{sm}}_{\beta=0}(\Sigma^r_{k,n}) = \rho^{k,0}(\mathcal{P}^r_l).$$

²Alternatively, by verifying that the inclusion $i: \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) \subset \text{Hom}(\mathbb{C}^{k+1}, \mathbb{C}^{n+1})$ is transversal to the $\Sigma^s_{k+1,n+1}$ stratification, $i^{-1}(\Sigma^r_{k+1,n+1}) = \Sigma^r_{k,n}$, and hence $i^*(s^{\text{sm}}(\Sigma^r_{k+1,n+1})) = s^{\text{sm}}(\Sigma^r_{k,n})$.

We have

$$\Sigma_{k,n}^r = \bigcup_{\substack{I \in \mathcal{I}_{k,n} \\ |I|=k-r}} \Omega_I,$$

and both the $\Sigma_{k,n}^r$ and the Ω_I sets are $\mathrm{GL}_k(\mathbb{C})$ –invariant. Thus by Theorem 8.6 we get

$$s_{\beta=0}^{\mathrm{sm}}(\Sigma_{k,n}^r) = \rho^{k,0} \left(\sum_{\substack{I \in \mathcal{I}_{k,n} \\ |I|=k-r}} \sum_{\lambda \in \Lambda(I)} \tilde{s}_\lambda \right).$$

Using the conversion formulas of Section 7.1 this is rewritten as

$$(31) \quad s_{\beta=0}^{\mathrm{sm}}(\Sigma_{k,n}^r) = \rho^{k,0} \left(\sum_{\substack{\lambda = (\lambda_1, \dots, \lambda_k) \\ \lambda_r \geq r+l, \lambda_{r+1} \leq r+l}} \tilde{s}_\lambda \right).$$

Comparing (30) to (31) and using that $\ker \rho^{k,n} = \mathrm{span}\{s_\lambda : \lambda_{k+1} \geq n+1\}$, we obtain

$$\mathcal{P}_l^r = \sum_{\substack{\lambda = (\lambda_1, \dots, \lambda_k) \\ \lambda_r \geq r+l, \lambda_{r+1} \leq r+l}} \tilde{s}_\lambda + \sum_{\lambda_{k+1} \geq 1} a_\lambda s_\lambda.$$

Since this holds for all $k \geq r$, we have

$$\mathcal{P}_l^r = \sum_{\substack{\lambda_r \geq r+l \\ \lambda_{r+1} \leq r+l}} \tilde{s}_\lambda,$$

what we wanted to prove. □

Remark 9.2 The A_2 quiver representation is the prototype of degeneracy loci theory. The fundamental class of the orbit closures for this representation satisfies $[\Sigma_{k,n}^r] = \rho^{k,n}(s_{(r+l)r})$ (called the Giambelli–Thom–Porteous formula), a positive and a very simple (one-term) expansion in terms of the “atoms” of fundamental class theory, the Schur functions. The very same positivity and simplicity are displayed in Theorem 9.1 for the SSM class, if we choose our “atoms” for the SSM theory to be the \tilde{s}_λ functions. This is one of the main messages of the present paper: the natural presentation of SSM classes is in terms of \tilde{s}_λ functions. Of course, for more complicated quivers, or for higher jet representations (singularity theory), the coefficients will be more complicated. We expect, however, that the coefficients will still be nonnegative for many geometrically relevant representations. More evidence towards this expectation will be shown in [22; 34]. Finally, this expectation, together with Conjecture 8.4, is the “two-step” positivity structure we are conjecturing for SSM classes of geometrically relevant degeneracy loci.

An exclusion-inclusion (aka sieve) type formula for $s^{\text{sm}}(\Sigma_{k,n}^r)$ was proved in [31] by Parusiński and Pragacz. For completeness, in Appendix A we reprove the Parusiński–Pragacz formula, together with an additional generating series description.

Appendix A: The Parusiński–Pragacz formula

A seminal paper on CSM/SSM classes of degeneracy loci is [31], where the authors present a sieve type formula for $s^{\text{sm}}(\Sigma_{k,n}^r)$ for the A_2 quiver representation (see Section 9). In this section we give a modern proof of their result, but essentially following the line of their arguments. The reason for giving a new proof is twofold. On the one hand we add to the results of [31] by giving a generating series description of a main ingredient. On the other hand, in Section A.1 we set up a general framework of calculating SSM classes of degeneracy loci once a fibered resolution is found; we believe this will be useful in future calculations both for quivers and singularities.

The equivariant CSM/SSM classes of $\Sigma_{k,n}^r$ are also studied in [42]. Moreover, in the recent paper [43] not only the CSM/SSM classes are calculated for the A_2 representation but also the so-called Mather classes.

For $k \leq n$ and partitions μ and ν of length at most k , let

$$D_{\mu,\nu}^{k,n} = \det\left(\left(\begin{matrix} \mu_i + k - i + \nu_j + n - j \\ \mu_i + k - i \end{matrix}\right)\right)_{i,j=1,\dots,k}$$

Theorem A.1 (essentially [31, Theorem 2.1]) *For $k \leq n$ and $l = n - k$, we have*

$$(32) \quad s^{\text{sm}}(\Sigma_{k,n}^r) = \sum_{s=r}^k (-1)^{s-r} \binom{s}{r} \Phi_{k,n}^s,$$

where

$$(33) \quad \Phi_{k,n}^s = S_{z_1,\dots}^{k,n} \left(\prod_{i=1}^s \left(\frac{z_i}{1+z_i} \right)^{s+l} \prod_{j=s+1}^{\infty} \prod_{i=1}^s \frac{1+z_i-z_j}{1+z_i} \right),$$

and also

$$(34) \quad \Phi_{k,n}^s = \rho^{k,n} \left(\sum_{l(\mu) \leq s} \sum_{l(\nu) \leq s} (-1)^{|\mu|+|\nu|} D_{\mu,\nu}^{s,s+l} s_{(s+l)^s + \mu, \nu^T} \right),$$

and

$$(35) \quad D_{\mu,\nu}^{s,s+l} \geq 0.$$

In (34) the summation runs for partitions μ and ν whose lengths (ie their numbers of nonzero parts, denoted by $l(\mu)$ and $l(\nu)$) are at most s . The symbol $(s + l)^s + \mu, \nu^T$ denotes the partition whose Young diagram is obtained from an $s \times (s + l)$ rectangle by gluing the Young diagram of μ to the right edge, and the “transpose” of the Young diagram of ν to the bottom edge. For example “ $(2 + 1)^2 + (2, 1), (3, 1)^T$ ” is the partition $(5, 4, 2, 1, 1)$.

Remark A.2 Statements (32) and (34) were proved in [31] (precisely speaking, [31, Theorem 2.1] is for the closure of Σ' , but because of additivity of SSM classes it is obviously equivalent to (32); see Theorem A.4(2)).

Statement (35) is known in relation with Segre classes of tensor products of vector bundles (see [24]). Here is a sketch of a combinatorial proof. Consider the oriented graph whose vertices are the integer points of the real plane, and whose edges are all the length-1 segments among them, oriented left/up. Consider the “source” points $P_i = (\mu_i + s - i, 0)$ for $i = 1, \dots, s$ and the “sink” points $Q_j = (0, \nu_j + (s + l) - j)$ for $j = 1, \dots, s$. Applying the Lindström–Gessel–Viennot lemma (eg [25]) to this situation interprets $D_{\mu, \nu}^{s, s+l}$ as the number of certain nonintersecting paths, that is, $D_{\mu, \nu}^{s, s+l}$ is nonnegative.

In the rest of this section — after proving some generalities about fibered resolutions — we give a full proof of Theorem A.1. Namely Theorem A.4 proves (32), Theorem A.8 proves (33), and Theorem A.9 proves (34).

A.1 Fibered resolution

Let $\Sigma \subset V$ be an invariant subvariety of the G –representation V . The G –equivariant map $\eta: \tilde{\Sigma} \rightarrow V$ is called a fibered resolution of Σ if it is a resolution of singularities of Σ , moreover, if $\tilde{\Sigma}$ is a total space of a G –vector bundle $\tilde{\Sigma} \rightarrow K$ over a smooth compact G –variety K and the resolution η factors as $\eta = \pi_V \circ i$, where $i: \tilde{\Sigma} \subset K \times V$ is a G –equivariant embedding of vector bundles and π_V is the projection to V . That is, $\eta: \tilde{\Sigma} \rightarrow V$ is a G –equivariant fibered resolution if we have a G –equivariant commutative diagram

$$(36) \quad \begin{array}{ccccc} & & \eta & & \\ & \nearrow & \text{---} & \searrow & \\ \tilde{\Sigma} & \xrightarrow{i} & K \times V & \xrightarrow{\pi_V} & V \\ & \searrow & \downarrow \pi_K & & \\ & & K & & \end{array}$$

Consider the G -equivariant quotient bundle $\nu = (K \times V \rightarrow K)/(\tilde{\Sigma} \rightarrow K)$ over K . This bundle, pulled back to $\tilde{\Sigma}$, is the normal bundle of the embedding $i: \tilde{\Sigma} \rightarrow K \times V$. The bundle ν pulled back over $K \times V$ has a natural section σ given by $\sigma(k, v) = v + i(\tilde{\Sigma}_k)$ (where $k \in K, v \in V$ and $\tilde{\Sigma}_k$ is the fiber of $\tilde{\Sigma} \rightarrow K$ over k). The section σ is transversal to the 0-section, and $\sigma^{-1}(0) = i(\tilde{\Sigma})$. Hence we have the remarkable situation that the normal bundle of $i(\tilde{\Sigma}) \subset K \times V$ extends to a bundle ν over $K \times V$.

In the cohomology calculations below we work in G -equivariant cohomology, and — as customary — we do not indicate pull-back bundles (eg ν may denote bundles over $\tilde{\Sigma}, K$, or $K \times V$, respectively). The cohomology of a total space and the base space of a vector bundle will be identified without explicit notation.

We will be concerned with two G -equivariant cohomology classes in V : the fundamental class $[\Sigma]$ of Σ in V , and the common value

$$(37) \quad \Phi_\Sigma := \frac{\eta_*(c(T\tilde{\Sigma}))}{c(V)} = \eta_*\left(\frac{c(T\tilde{\Sigma})}{c(V)}\right) = \eta_*(c(-\nu)c(TK)).$$

The first equality follows by adjunction and second one follows from the calculation

$$\frac{c(T\tilde{\Sigma})}{c(V)} = \frac{c(T\tilde{\Sigma})}{c(V)c(TK)}c(TK) = c(-\nu)c(TK).$$

Proposition A.3 *We have*

$$[\Sigma] = \int_K e(\nu) \quad \text{and} \quad \Phi_\Sigma = \int_K e(\nu)c(-\nu)c(TK).$$

Proof Since the section σ described above is transversal to the 0-section, and $\sigma^{-1}(0) = i(\tilde{\Sigma})$ we have $i_*(1) = e(\nu)$ and

$$[\Sigma] = \eta_*(1) = \pi_{V*}i_*(1) = \pi_{V*}(e(\nu)) = \int_K e(\nu),$$

proving the first statement. The second statement follows from the calculation

$$\begin{aligned} \Phi_\Sigma &= \eta_*(c(-\nu)c(TK)) = \int_K i_*(c(-\nu)c(TK)) \\ &= \int_K i_*(i^*c(-\nu))c(TK) = \int_K i_*(1)c(-\nu)c(TK) = \int_K e(\nu)c(-\nu)c(TK), \end{aligned}$$

where we used the adjunction formula, and the fact that ν extends from $\tilde{\Sigma}$ to $K \times V$. \square

Only the second statement of Proposition A.3 is relevant for the present paper — and in fact the first one follows from the second one. We included the first one (well known in the theory of fundamental classes [6; 21; 14]) for comparison purposes.

A.2 CSM/SSM classes in terms of Φ -classes

Consider the fibered resolution

$$(38) \quad \begin{array}{ccc} & \eta_r & \\ & \curvearrowright & \\ \widetilde{\Sigma}^r & \xrightarrow{i} \text{Gr}_r(\mathbb{C}^k) \times \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) \xrightarrow{\pi_2} \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) & \\ & \searrow & \downarrow \pi_1 \\ & \text{Gr}_r(\mathbb{C}^k) & \end{array}$$

of $\overline{\Sigma}^r$, where

$$\widetilde{\Sigma}^r = \{(V, \phi) \in \text{Gr}_r(\mathbb{C}^k) \times \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) : \phi|_V = 0\}$$

with the obvious embedding into $\text{Gr}_r(\mathbb{C}^k) \times \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ and projection to $\text{Gr}_r(\mathbb{C}^k)$.

Define $\Phi_{k,n}^r$ to be the class in (37) for the fibered resolution (38). The following theorem is equivalent to [31, Theorem 2.1].

Theorem A.4 *We have:*

$$(1) \quad \Phi_{k,n}^r = \sum_{s=r}^k \binom{s}{r} s^{\text{sm}}(\Sigma^s) = \sum_{s=r}^k \binom{s-1}{r-1} s^{\text{sm}}(\overline{\Sigma}^s).$$

$$(2) \quad s^{\text{sm}}(\overline{\Sigma}^r) = \sum_{s=r}^k (-1)^{s-r} \binom{s-1}{r-1} \Phi_{k,n}^s \quad \text{and} \quad s^{\text{sm}}(\Sigma^r) = \sum_{s=r}^k (-1)^{s-r} \binom{s}{r} \Phi_{k,n}^s.$$

Proof The preimage at η_r of a point in Σ^s ($r \leq s \leq k$) is $\text{Gr}_r(\mathbb{C}^s)$, whose Euler characteristic is $\binom{s}{r}$. Hence property (ii) from Section 2.3 for η_r implies

$$\eta_{r!}(c(T\widetilde{\Sigma}^r)) = \sum_{s=r}^k \binom{s}{r} c^{\text{sm}}(\Sigma^s).$$

Dividing both sides by $c(\text{Hom}(\mathbb{C}^k, \mathbb{C}^n))$ we obtain $\Phi_{k,n}^r$ on the left-hand side, and $\sum_{s=r}^k \binom{s}{r} s^{\text{sm}}(\Sigma^s)$ on the right-hand side, which proves the first equality in part (1). Using the additivity property of CSM classes, (i) from Section 2.3, we obtain

$$\begin{aligned} \sum_{s=r}^k \binom{s}{r} c^{\text{sm}}(\Sigma^s) &= \sum_{s=r}^k \binom{s}{r} (c^{\text{sm}}(\overline{\Sigma}^s) - c^{\text{sm}}(\overline{\Sigma}^{s+1})) \\ &= \sum_{s=r}^k \left(\binom{s}{r} - \binom{s-1}{r} \right) c^{\text{sm}}(\overline{\Sigma}^s) = \sum_{s=r}^k \binom{s-1}{r-1} c^{\text{sm}}(\overline{\Sigma}^s). \end{aligned}$$

Dividing by $c(\text{Hom}(\mathbb{C}^k, \mathbb{C}^n))$ proves the second equality in part (1).

Part (2) is the algebraic consequence of part (1): it follows from the fact that the inverse of the Pascal matrix $\binom{s}{r}_{s,r}$ is the matrix $((-1)^{s-r} \binom{s}{r})_{s,r}$; see eg [9]. \square

A.3 Formulas for Φ -classes

Let α_u for $u = 1, \dots, k$ and β_v for $v = 1, \dots, n$ denote the Chern roots of $\text{GL}_k(\mathbb{C})$ and $\text{GL}_n(\mathbb{C})$, respectively. Then

$$\Phi_{k,n}^s \in \mathbb{C}[\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_n]^{S_k \times S_n}.$$

Denoting the Chern roots of the tautological subbundle over $\text{Gr}_s(\mathbb{C}^k)$ by $\gamma_1, \dots, \gamma_s$, and those of the tautological quotient bundle by $\delta_1, \dots, \delta_{k-s}$, Proposition A.3 implies

$$(39) \quad \Phi_{k,n}^s(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_n) = \int_{\text{Gr}_s(\mathbb{C}^k)} \prod_{i=1}^s \prod_{v=1}^n \frac{\beta_v - \gamma_i}{1 + \beta_v - \gamma_i} \prod_{i=1}^s \prod_{j=1}^{k-s} (1 + \delta_j - \gamma_i).$$

First let us calculate a special case, $s = k$. We have

$$\begin{aligned} \Phi_{s,s+l}^s &= \prod_{u=1}^s \prod_{v=1}^{s+l} \frac{\beta_v - \alpha_u}{1 + \beta_v - \alpha_u} \\ &= \prod_{u=1}^s \prod_{v=1}^{s+l} (\beta_v - \alpha_u) \sum_{l(\mu) \leq s} \sum_{l(v) \leq s} (-1)^{|\mu|+|v|} D_{\mu,v}^{s,s+l} \rho^{s,0}(s_\mu) \rho^{0,s+l}(s_{vT}). \end{aligned}$$

Here we used the Schur function expansion of ‘‘Segre classes of a tensor product’’ from [24]. Using the ‘‘factorization formula’’ of Schur functions we obtain

$$(40) \quad \Phi_{s,s+l}^s = \rho^{s,s+l} \left(\sum_{l(\mu) \leq s} \sum_{l(v) \leq s} (-1)^{|\mu|+|v|} D_{\mu,v}^{s,s+l} s_{(s+l)^s + \mu, vT} \right).$$

Lemma A.5 (supersymmetry lemma) *We have*

$$\Phi_{k+1,n+1}^s(\alpha_1, \dots, \alpha_k, t; \beta_1, \dots, \beta_n, t) = \Phi_{k,n}^s(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_n).$$

Proof The equality follows from interpreting both sides with equivariant localization

$$\Phi_{k,n}^s = \sum_I \prod_{u \in I} \prod_{v=1}^n \frac{\beta_v - \alpha_u}{1 + \beta_v - \alpha_u} \prod_{u \in I} \prod_{w \in \bar{I}} \frac{1 + \alpha_w - \alpha_u}{\alpha_w - \alpha_u},$$

where the sum is over s -element subsets I of $\{1, \dots, k\}$, and $\bar{I} = \{1, \dots, k\} - I$. \square

Definition A.6 For $k \leq n$ nonnegative integers, $l = n - k$, and $1 \leq s \leq k$ define

$$\mathcal{F}_{k,n}^s = \prod_{i=1}^s \left(\frac{z_i}{1+z_i} \right)^{s+n-k} \prod_{j=s+1}^k \prod_{i=1}^s \frac{1+z_i-z_j}{1+z_i},$$

$$\mathcal{F}_l^s = \mathcal{F}_{\infty,\infty+l}^s = \prod_{i=1}^s \left(\frac{z_i}{1+z_i} \right)^{s+l} \prod_{j=s+1}^{\infty} \prod_{i=1}^s \frac{1+z_i-z_j}{1+z_i}.$$

Proposition A.7 For $k \leq n$ nonnegative integers, $l = n - k$, and $1 \leq s \leq k$ we have

$$\Phi_{k,n,\beta=0}^s = S_{z_1,\dots,z_k}^{k,0}(\mathcal{F}_{k,n}^s) = S_{z_1,\dots}^{k,0}(\mathcal{F}_l^s).$$

Proof Let

$$U = \prod_{u=1}^s \left(\frac{-\alpha_u}{1-\alpha_u} \right)^n \prod_{w=s+1}^k \prod_{u=1}^s \frac{1+\alpha_w-\alpha_u}{\alpha_w-\alpha_u}.$$

Substituting $\beta_i = 0$ for $i = 1, \dots, n$ in (39), the equivariant localization formula for the integral yields

$$\Phi_{k,n,\beta=0}^s = \sum_{\sigma \in S_k/S_s \times S_{k-s}} U(\sigma(\alpha_1, \dots, \alpha_k)).$$

For

$$V = U \cdot \left(\prod_{u=1}^s (-\alpha_u)^{s-u} \prod_{w=1}^s \prod_{u=1}^{w-1} \frac{1}{\alpha_w-\alpha_u} \right) \left(\prod_{u=s+1}^k (-\alpha_u)^{k-u} \prod_{w=s+1}^k \prod_{u=s+1}^{w-1} \frac{1}{\alpha_w-\alpha_u} \right),$$

we have

$$\sum_{\sigma \in S_s \times S_{k-s}} V(\sigma(\alpha_1, \dots, \alpha_k)) = U,$$

and hence

$$(41) \quad \Phi_{k,n,\beta=0}^s = \sum_{\sigma \in S_k} V(\sigma(\alpha_1, \dots, \alpha_k)).$$

Observe that

$$(42) \quad V(\sigma(\alpha_1, \dots, \alpha_k)) = \operatorname{Res}_{z_k=-\alpha_{\sigma(k)}} \cdots \operatorname{Res}_{z_2=-\alpha_{\sigma(2)}} \operatorname{Res}_{z_1=-\alpha_{\sigma(1)}} f$$

for

$$f = \prod_{i=1}^s \left(\frac{z_i}{1+z_i} \right)^n \prod_{j=s+1}^k \prod_{i=1}^s (1+z_i-z_j) \prod_{i=1}^s z_i^{s-i} \prod_{i=s+1}^k z_i^{k-i} \frac{\prod_{1 \leq j \leq i \leq k} (z_i - z_j)}{\prod_{j=1}^k \prod_{u=1}^k (z_j + \alpha_u)}$$

$$= F_{k,n}^s \cdot \prod_{1 \leq i < j \leq k} \left(1 - \frac{z_i}{z_j} \right) \prod_{i=1}^k \frac{1}{\prod_{u=1}^k (1 + \alpha_u/z_i)} \prod_{i=1}^k \frac{dz_i}{z_i}.$$

The only nonzero finite residues of the form f are the ones on the right-hand side of (42)—remember that the $1/(1 + z_i)$ factors are just abbreviations of the formal series $1 - z_1 + z_1^2 - \dots$. Hence from (41) and (42), using the residue theorem, we obtain that

$$\Phi_{k,n,\beta=0}^s = (-1)^k \text{RES}_k(f),$$

which, using (22), yields the first equality of the proposition.

The second equality follows from the first one and Lemma 8.1. □

Theorem A.8 For $k \leq n$ nonnegative integers, $l = n - k$, and $1 \leq s \leq k$, we have

$$\Phi_{k,n}^s = S_{z_1, \dots}^{k,n}(\mathcal{F}_l^s).$$

Proof Let N be a nonnegative integer and consider $\Phi_{k+N,n+N}^s$. Supersymmetry (Lemma A.5) implies that this can be written as a linear combination of Schur functions $\rho^{k+N,n+N}(s_\lambda)$. Recall from Section 8.1 that for such linear combinations f , the notation $f^{\leq m}$ is meant to be the sum of terms corresponding to partition with length at most m . By Proposition A.7,

$$\Phi_{k+N,n+N}^s(\alpha_1, \dots, \alpha_{k+N}; 0, \dots, 0) = S_{z_1, \dots}^{k+N,0}(\mathcal{F}_l^s).$$

Since $\rho^{k+N,0}(s_\lambda) = 0$ if and only if $l(\lambda) > k + N$ (see (18)) we have that

$$(\Phi_{k+N,n+N}^s)^{\leq k+N} = (S_{z_1, \dots}^{k+N,n+N}(\mathcal{F}_l^s))^{\leq k+N}.$$

Substituting $\alpha_{k+1} = \alpha_{k+2} = \dots = \alpha_{k+N} = \beta_{n+1} = \beta_{n+2} = \dots = \beta_{n+N} = 0$ and using Lemma A.5 (supersymmetry), we get

$$(\Phi_{k,n}^s)^{\leq k+N} = (S_{z_1, \dots}^{k,n}(\mathcal{F}_l^s))^{\leq k+N}.$$

Since this holds for any N , the proof is complete. □

Theorem A.9 For $k \leq n$ nonnegative integers, $l = n - k$, and $0 \leq s \leq k$, we have

$$(43) \quad \Phi_{k,n}^s = \rho^{k,n} \left(\sum_{l(\mu) \leq s} \sum_{l(v) \leq s} (-1)^{|\mu|+|v|} D_{\mu,v}^{s,s+l} s_{(s+l)^s + \mu, v^T} \right).$$

Proof By Lemma A.5 (supersymmetry) we know that

$$(44) \quad \Phi_{k,n}^s = \rho^{k,n} \left(\sum d_\lambda s_\lambda \right).$$

First we claim that if $\lambda_{s+1} > s$ then $d_\lambda = 0$.

According to Theorem A.8 we have $\Phi_{k,n}^s = \mathcal{S}_{z_1, \dots}^{k,n}(\mathcal{F}_l^s)$. The monomials occurring in the Taylor expansion of \mathcal{F}_l^s are of the form

$$z_1^{a_1} \cdots z_s^{a_s} z_{s+1}^{\varepsilon_1} z_{s+2}^{\varepsilon_2} \cdots z_{s+q}^{\varepsilon_q},$$

with all $\varepsilon_i \in \{0, 1, \dots, s\}$. Hence $\Phi_{k,n}^s$ is a sum of possibly fake Schur functions s_λ satisfying

$$(45) \quad i > s \Rightarrow \lambda_i \leq s.$$

Observe that property (45) does not change if one applies the straightening laws (16). Hence $\Phi_{k,n}^s$ is also the sum of Schur functions satisfying (45). For partitions this property is equivalent to $\lambda_{s+1} \leq s$.

We can hence improve (44), and write

$$(46) \quad \Phi_{k,n}^s = \rho^{k,n} \left(\sum_{\lambda_s \leq s} d_\lambda s_\lambda \right).$$

Let us substitute $\alpha_{s+1} = \alpha_{s+1} = \cdots = \alpha_k = \beta_{s+l+1} = \beta_{s+l+2} = \cdots = \beta_n = 0$ in (46). According to Lemma A.5 (supersymmetry) we obtain

$$(47) \quad \Phi_{s,s+l}^s = \rho^{s,s+l} \left(\sum_{\lambda_s \leq s} d_\lambda s_\lambda \right).$$

Observe that for a λ with $\lambda_s \leq s$ the Schur function $\rho^{k,n}(s_\lambda)$ is not 0. Hence each d_λ in (47) has to be the value described in (40). This proves the theorem. \square

Appendix B: Comparing CSM and SSM classes of Schubert and matrix Schubert cells

In this section we summarize the localization and residue formulas for both equivariant CSM and SSM classes of both matrix Schubert cells and Schubert cells.

Matrix Schubert cells are subsets of $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$, and their $(\text{GL}_k \times B_n^-)$ -equivariant CSM and SSM classes are elements of $\mathbb{C}[\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_n]^{S_k}$ (and its completion). Schubert cells are subsets of the Grassmannian $\text{Gr}_k(\mathbb{C}^n)$, and their B_n^- -equivariant CSM and SSM classes are elements of a quotient ring of $\mathbb{C}[\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_n]^{S_k}$.

For matrix Schubert cells let us now restrict our attention to the ones whose elements have full rank k . Then both versions of Schubert cells can be parametrized by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ with $\lambda_1 \leq n - k$ and $\lambda_k \geq 0$, or equivalently, by

subsets $\{i_1 < i_2 < \dots < i_k\}$ of $\{1, \dots, n\}$. The transition between the two parameters is $i_a = \lambda_{k+1-a} + a$.

Below we give two formulas for CSM and SSM classes of matrix and ordinary Schubert cells. The first one is the formula for the appropriate class of the given Schubert cell. The second formula is the Schur polynomial expansion of the $\beta_v = 0$ substitution. In matrix Schubert settings this means $GL_k(\mathbb{C})$ -equivariant formulas, and in the Grassmannian settings this means nonequivariant formulas. Denote $\text{Sym } f = \sum_{\sigma \in S_k} f(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k)})$.

Theorem B.1 *We have the following formulas for matrix and ordinary Schubert cells:*

(1) *CSM class of a matrix Schubert cell:*

$$\text{Sym} \prod_{u=1}^k \left(\prod_{v=i_u+1}^n (1 + \beta_v - \alpha_u) \prod_{v=1}^{i_u-1} (\beta_v - \alpha_u) \prod_{v=u+1}^k \frac{1 + \alpha_u - \alpha_v}{\alpha_u - \alpha_v} \right),$$

$$\mathcal{S}_z^{k,0} \left(\prod_{j=1}^k z_j^{\lambda_j} \prod_{j=1}^k (1 + z_j)^{n-i_{k+1-j}} \prod_{1 \leq i < j \leq k} (1 + z_i - z_j) \right);$$

(2) *CSM class of a Schubert cell:*

$$\text{Sym} \prod_{u=1}^k \left(\prod_{v=i_u+1}^n (1 + \beta_v - \alpha_u) \prod_{v=1}^{i_u-1} (\beta_v - \alpha_u) \prod_{v=u+1}^k \frac{1}{(\alpha_u - \alpha_v)(1 + \alpha_v - \alpha_u)} \right),$$

$$\mathcal{S}_z^{k,0} \left(\prod_{j=1}^k z_j^{\lambda_j} \prod_{j=1}^k (1 + z_j)^{n-i_{k+1-j}} \frac{1}{\prod_{1 \leq j < i \leq k} (1 + z_i - z_j)} \right);$$

(3) *SSM class of a matrix Schubert cell:*

$$\text{Sym} \prod_{u=1}^k \left(\frac{1}{1 + \beta_{i_u} - \alpha_u} \prod_{v=1}^{i_u-1} \frac{\beta_v - \alpha_u}{1 + \beta_v - \alpha_u} \prod_{v=u+1}^k \frac{1 + \alpha_u - \alpha_v}{\alpha_u - \alpha_v} \right),$$

$$\mathcal{S}_z^{k,0} \left(\prod_{j=1}^k \left(\frac{z_j}{1 + z_j} \right)^{\lambda_j} \prod_{j=1}^k \prod_{i=1}^j \frac{1 + z_i - z_j}{1 + z_i} \right).$$

(4) *The SSM classes of Schubert cells are represented by SSM classes of matrix Schubert cells (hence formulas of (3) are representatives of these SSM classes).*

Proof Formulas in (1) and (3) are in this paper (Theorem 5.1, Corollary 5.2, and Theorem 7.6), (4) follows from (3) via Theorem 2.2. Formulas in (2) follow from (4) — or can be deduced from results in [3; 39]. □

Let us comment on the two positivity results/conjectures known about these classes. One is the result of Huh [19] (conjectured earlier by Aluffi and Mihalcea [3]): the Schur expansion in (2) has nonnegative coefficients. The other is our Conjecture 8.4, that the Schur expansion in (3) has alternating signs.

We are not aware of any connection between the two positivity properties. One fact which makes the comparison difficult is that the three-term factors $1 + z_i - z_j$ are in the numerator and denominator, respectively, in the two cases. Another key difference is that *not all* the infinitely many coefficients of the generating sequence in (2) are positive, only the ones corresponding to partitions $\subset (n - k)^k$ — which fact does not contradict Huh’s theorem since the Schur functions corresponding to the other partitions are 0 in the quotient ring. However, our Conjecture 8.4 is about *all* the Schur coefficients of the series in (3) — even if $k \rightarrow \infty$.

Appendix C: The geometry of the weight function

In this section we give a geometric interpretation of the weight function of Definition 4.1.

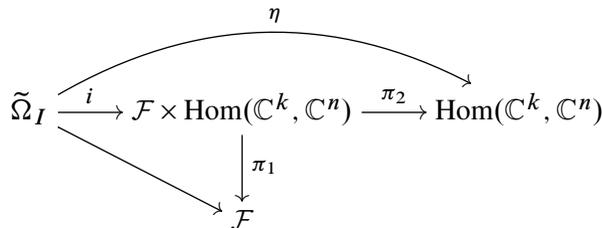
Let $k \leq n$ nonnegative integers and $I = \{i_1 < \dots < i_d\} \subset \{1, \dots, n\}$ with $|I| = d \leq k$. Consider the partial flag variety \mathcal{F} parametrizing chains of subspaces of \mathbb{C}^k :

$$V_\bullet = (V_{d+1} \subset V_d \subset V_{d-1} \subset \dots \subset V_1), \quad \text{where } \dim V_i = k + 1 - i.$$

Let $F_j = \text{span}(\epsilon_j, \epsilon_{j+1}, \dots, \epsilon_n) \subset \mathbb{C}^n$, where the ϵ_i are the standard basis vectors of \mathbb{C}^n . Define

$$\begin{aligned} \tilde{\Omega}_I &= \{(V_\bullet, \phi) \in \mathcal{F} \times \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) : \phi(V_j) \subset F_{i_j} \ (j = 1, \dots, d), \phi(V_{d+1}) = 0\}, \\ (48) \quad D_{I,j} &= \{(V_\bullet, \phi) \in \tilde{\Omega}_I : \phi(V_j) \subset F_{i_{j+1}}\} \quad \text{for } j = 1, \dots, d, \\ \tilde{\Omega}_I^o &= \tilde{\Omega}_I - \bigcup_{j=1}^d D_{I,j}. \end{aligned}$$

The $(\text{GL}_k(\mathbb{C}) \times B_n^-)$ -equivariant diagram



is a fibered resolution of $\overline{\Omega}_I \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ in the sense of Section A.1, and η restricted to $\tilde{\Omega}_I^o$ is an isomorphism to Ω_I .

Proposition C.1 We have

$$W_I = \int_{\mathcal{F}} \frac{c(\tilde{\Omega}_I)e(\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)/\tilde{\Omega}_I)c(T\mathcal{F})}{\prod_{j=1}^d (1 + [D_j])} \in H_{\text{GL}_k(\mathbb{C}) \times B_n^-}^*(\text{pt}) = \mathbb{Z}[\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_n]^{S_k}.$$

In the numerator the three factors are total Chern, Euler, and total Chern classes of bundles over \mathcal{F} . By $[D_j]$ we mean the fundamental class of the divisor D_j in $\tilde{\Omega}_J$ as an element of $H_{\text{GL}_k(\mathbb{C}) \times B_n^-}^*(\tilde{\Omega}_J) = H_{\text{GL}_k(\mathbb{C}) \times B_n^-}^*(\mathcal{F})$. The map $\int_{\mathcal{F}}$ is the push-forward to a point in equivariant cohomology.

Proof The torus fixed points of \mathcal{F} are naturally indexed by injective maps $\{1, \dots, d\} \rightarrow \{1, \dots, k\}$. For such an injective map r , let $R(r)$ be its range; then the corresponding torus fixed point $f_r \in \mathcal{F}$ is

$$(\text{span}\{\epsilon_j\}_{j \notin R(r)}, \text{span}\{\epsilon_r(d), \epsilon_j\}_{j \notin R(r)}, \text{span}\{\epsilon_r(d-1), \epsilon_r(d), \epsilon_r(d-1), \epsilon_j\}_{j \notin R(r)}, \dots).$$

We have the following restrictions to the torus fixed point f_r :

$$\begin{aligned} c(\tilde{\Omega}_I)|_{f_r} &= \prod_{u=1}^d \prod_{v=i_u}^n (1 + \beta_v - \alpha_{r(u)}), \\ e(\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)/\tilde{\Omega}_I)|_{f_r} &= \prod_{u=1}^d \prod_{v=1}^{i_u-1} (\beta_v - \alpha_{r(u)}) \cdot \prod_{u \notin R(r)} \prod_{v=1}^n (\beta_v - \alpha_u), \\ c(T\mathcal{F})|_{f_r} &= \prod_{u=1}^d \prod_{v=1}^d (1 + \alpha_{r(u)} - \alpha_{r(v)}) \cdot \prod_{u=1}^d \prod_{v \notin R(r)} (1 + \alpha_{r(u)} - \alpha_v), \\ (1 + [D_j])|_{f_r} &= 1 + \beta_{i_u} - \alpha_{r(u)}. \end{aligned}$$

Observe that the last expression appears as one of the factors in the first line. Hence, by equivariant localization, the integral displayed in the theorem equals $\sum_r \tilde{U}_r(\alpha, \beta)$, where the summation is for injective maps r , and

$$\begin{aligned} \tilde{U}_r(\alpha, \beta) &= \prod_{u=1}^d \prod_{v=i_u+1}^n (1 + \beta_v - \alpha_{r(u)}) \prod_{u=1}^d \prod_{v=1}^{i_u-1} (\beta_v - \alpha_{r(u)}) \cdot \prod_{u \notin R(r)} \prod_{v=1}^n (\beta_v - \alpha_u) \\ &\quad \times \prod_{u=1}^d \prod_{v=1}^d \frac{1 + \alpha_{r(u)} - \alpha_{r(v)}}{\alpha_{r(u)} - \alpha_{r(v)}} \cdot \prod_{u=1}^d \prod_{v \notin R(r)} \frac{1 + \alpha_{r(u)} - \alpha_v}{\alpha_{r(u)} - \alpha_v}. \end{aligned}$$

We can extend an injective map r to a permutation of $\{1, \dots, k\}$ (in $(k - d)!$ ways),

and hence we can reindex $\sum_r \tilde{U}(\alpha, \beta)$ by permutations:

$$\begin{aligned} \int_{\mathcal{F}} \frac{c(\tilde{\Omega}_I)e(\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)/\tilde{\Omega}_I)c(T\mathcal{F})}{\prod_{j=1}^d (1 + [D_j])} &= \sum_r \tilde{U}_r(\alpha, \beta) \\ &= \frac{1}{(k-d)!} \sum_{\sigma \in \mathcal{S}_k} U_I(\sigma(\alpha), \beta) = W_I(\alpha, \beta), \end{aligned}$$

for the $U_I(\alpha, \beta)$ function defined in Definition 4.1. \square

Proposition C.1 explains our terminology of calling the form presented in Definition 4.1 of the weight function a “localization form”. The geometric meaning of other sum-product-type, and residue-type formulas for various versions of weight functions in Section 4 stem from Proposition C.1.

Moreover, Proposition C.1 can also be used to give an alternative proof of Theorem 5.1, by proving the following two statements:

- Suppose in diagram (36) we have finitely many smooth normal crossing divisors $D_j \subset \tilde{\Sigma}$ such that ρ restricted to $\tilde{\Sigma} - \cup D_j$ is an isomorphism. Then

$$c^{\text{sm}}(\rho(\tilde{\Sigma} - \cup D_j)) = \frac{\int_{\mathbf{K}} c(\tilde{\Sigma})e(V/\tilde{\Sigma})c(T\mathbf{K})}{\prod(1 + [D_j])}.$$

- The $D_{I,j}$'s in (48) are smooth normal crossing divisors in $\tilde{\Omega}_I$.

The first statement is essentially Aluffi's method of calculating CSM classes via a resolution with normal crossing divisors [2]. The second statement follows from explicit coordinate calculations. Details of a proof of Theorem 5.1 along these lines are not given here, but a K-theoretic version of that argument will appear in [15].

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