

C^* -algebraic higher signatures and an invariance theorem in codimension two

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We revisit the construction of signature classes in C^* -algebra K -theory, and develop a variation that allows us to prove equality of signature classes in some situations involving homotopy equivalences of noncompact manifolds that are only defined outside a compact set. As an application, we prove a counterpart for signature classes of a codimension-two vanishing theorem for the index of the Dirac operator on spin manifolds (the latter is due to Hanke, Pape and Schick, and is a development of well-known work of Gromov and Lawson).

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1 Introduction

Let X be a connected, smooth, closed and oriented manifold of dimension d , and let π be a discrete group. Associated to each principal π -bundle \tilde{X} over X there is a *signature class*

$$\text{Sgn}_\pi(\tilde{X}/X) \in K_d(C_r^*(\pi))$$

in the topological K -theory of the reduced group C^* -algebra of π . The signature class can be defined either index-theoretically or combinatorially, using a triangulation. Its main property, which is easiest to see from the combinatorial point of view, is that if $h: Y \rightarrow X$ is a homotopy equivalence of smooth, closed, oriented manifolds that is covered by a map of principal π -bundles $\tilde{Y} \rightarrow \tilde{X}$, then

$$(1-1) \quad \text{Sgn}_\pi(\tilde{X}/X) = \text{Sgn}_\pi(\tilde{Y}/Y) \in K_d(C_r^*(\pi)).$$

This is the point of departure for the C^* -algebraic attack on Novikov's higher signature conjecture (see Rosenberg [24; 25] for surveys).

This note has two purposes. The first is to simplify some of the machinery involved in the combinatorial approach to signature classes in C^* -algebra K -theory. The main novelty

is a technique to construct signature classes that uses only finitely generated and projective modules (in part by appropriately adjusting the C^* -algebras that are involved). The second purpose is to prove a codimension-two invariance result for the signature that is analogous to a recent theorem of Hanke, Pape and Schick [7] about positive scalar curvature (which is in turn developed from well-known work of Gromov and Lawson [5, Theorem 7.5]). To prove the theorem we shall use generalizations of the signature within the context of coarse geometry. These have previously been studied closely (see for example Roe [23]), but our new approach to the signature has some definite advantages in this context (indeed it was designed with coarse geometry in mind). Here is the result. Let

$$h: N \rightarrow M$$

be a smooth, orientation-preserving homotopy equivalence between two smooth, closed, oriented manifolds of dimension $d + 2$. Let X be a smooth, closed, oriented submanifold of M of codimension two (and hence dimension d), and assume that h is transverse to X , so that the inverse image

$$Y = h^{-1}[X]$$

is a smooth, closed, oriented submanifold of N of codimension two. Suppose \tilde{X} is the universal cover of X . Then it pulls back along the map h to give a $\pi_1(X)$ -covering space \tilde{Y} of Y .

1.1 Theorem *Assume that*

- (i) $\pi_1(X) \rightarrow \pi_1(M)$ is injective;
- (ii) $\pi_2(X) \rightarrow \pi_2(M)$ is surjective; and
- (iii) the normal bundle of X in M is trivializable.

Then

$$(1-2) \quad 2(\text{Sgn}_{\pi_1(X)}(\tilde{X}/X) - \text{Sgn}_{\pi_1(X)}(\tilde{Y}/Y)) = 0 \quad \text{in } K_d(C_r^*(\pi_1(X))).$$

Let us make some remarks about this theorem. First, we do not know whether the factor of 2 in (1-2) is necessary, but in view of what is known about L -theory concerning splitting obstructions and “change of decoration”, where nontrivial 2-torsion phenomena do occur, we suspect that there are situations where the factor of 2 in (1-2) is indeed necessary.

In surgery theory (symmetric) signatures are defined in the L -theory groups of the group ring $\mathbb{Z}[\pi]$. There are comparison maps from these L -groups to our K -groups

(at least after inversion of 2; see Land and Nikolaus [16] for a thorough discussion of this issue) and the comparison maps take the respective signatures to each other (this correspondence is developed systematically by Higson and Roe [9; 10; 11]). We leave it as an open question to lift Theorem 1.1 to an equality of *L*-theoretic higher signatures.

Finally, in view of the strong Novikov conjecture, one might expect that all interesting homotopy invariants of a closed oriented manifold *M* with universal cover \tilde{M} can be derived from the signature class

$$\text{Sgn}_{\pi_1(M)}(\tilde{M}/M) \in K_*(C_r^*(\pi_1(M))).$$

Theorem 1.1 shows that $2 \text{Sgn}_{\pi_1(X)}(X) \in K_*(C_r^*(\pi_1(X)))$ is a new homotopy invariant of the ambient manifold *M*, under the conditions of the theorem. It is an interesting question to clarify the relationship between this new invariant and the signature class of \tilde{M}/M .

Here is a brief outline of the paper. In Section 2 we shall summarize the key properties of the signature class, including a new invariance property that is required for the proof of Theorem 1.1. In Section 3 we shall present the proof of Theorem 1.1. As in the work of Hanke, Pape and Schick, the proof involves coarse-geometric ideas, mostly in the form of the partitioned manifold index theorem of Roe [21]. With our new approach to the signature, the proof of the positive scalar curvature result can be adapted with very few changes. In the remaining sections we shall give our variation on the construction of the signature class, using as a starting point the approach of Higson and Roe in [9; 10].

2 Properties of the *C**-algebraic signature

In this section we shall review some features of the *C**-algebraic signature, and then introduce a new property of the signature that we shall use in the proof of Theorem 1.1. We shall not give the definition of the signature in this section, but we shall say more later, in Section 4.

2.1 Let π be a discrete group and let *X* be a smooth, closed and oriented manifold of dimension *d* (without boundary). Mishchenko, Kasparov and others have associated to every principal π -bundle \tilde{X} over *X* a *signature class* in the topological *K*-theory of the reduced *C**-algebra of the group π :

$$(2-1) \quad \text{Sgn}_{\pi}(\tilde{X}/X) \in K_d(C_r^*(\pi)).$$

2.2 The first and most important property of the signature class is that it is an *oriented homotopy invariant*: given a commuting diagram of principal π -fibrations

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{h} & X \end{array}$$

in which the bottom map is an orientation-preserving homotopy equivalence, we have

$$\text{Sgn}_\pi(\tilde{Y}/Y) = \text{Sgn}_\pi(\tilde{X}/X) \in K_d(C_r^*(\pi)).$$

2.3 Assume that X is in addition connected, and let \tilde{X} be a universal covering manifold.¹ In this case we shall abbreviate our notation and write

$$\text{Sgn}_{\pi_1(X)}(X) \in K_d(C_r^*(\pi_1(X))),$$

omitting mention of the cover. If d is congruent to zero, mod four, then we can recover the usual signature of the base manifold X from the C^* -algebraic signature $\text{Sgn}_{\pi_1(X)}(X)$ by applying the homomorphism

$$\text{Trace}_\pi: K_d(C_r^*(\pi)) \rightarrow \mathbb{R}$$

associated to the canonical trace on $C_r^*(\pi)$ as a consequence of Atiyah’s L^2 -index theorem. But the C^* -algebraic signature usually contains much more information. It is conjectured to determine the so-called higher signatures of X [18; 19; 17; 15]. This is the *strong Novikov conjecture*, and it is known in a great many cases, for example for all fundamental groups of complete, nonpositively curved manifolds [14], for all groups of finite asymptotic dimension [29], and for all linear groups [6].

2.4 Let us return to general covers. The signature class is functorial, in the sense that if $\alpha: \pi_1 \rightarrow \pi_2$ is an injective group homomorphism, and if the principal π_1 - and π_2 -bundles \tilde{X}_1 and \tilde{X}_2 are linked by a commuting diagram

$$\begin{array}{ccc} \tilde{X}_1 & \longrightarrow & \tilde{X}_2 \\ \downarrow & & \downarrow \\ X & \xlongequal{\quad} & X \end{array}$$

in which the top map is π_1 -equivariant, where π_1 acts on \tilde{X}_2 via α , then

$$\alpha_*: \text{Sgn}_{\pi_1}(\tilde{X}_1/X) \mapsto \text{Sgn}_{\pi_2}(\tilde{X}_2/X) \in K_d(C_r^*(\pi_2)).$$

¹It is not necessary to keep track of basepoints in what follows since inner automorphisms of π act trivially on $K_*(C_r^*(\pi))$.

(We recall here that the reduced group C^* -algebra is functorial for injective group homomorphisms, and therefore so is its K -theory.²)

2.5 The signature is a *bordism invariant* in the obvious sense: if W is a compact oriented smooth manifold with boundary, and if \tilde{W} is a principal π -bundle over W , then the signature class associated to the restriction of \tilde{W} to ∂W is zero; see Section 4.2 of [10].

2.6 A locally compact space Z is *continuously controlled over* \mathbb{R} if it is equipped with a proper and continuous map

$$c: Z \rightarrow \mathbb{R}.$$

If Z is a smooth and oriented manifold of dimension $d + 1$ that is continuously controlled over \mathbb{R} , and if the control map $c: Z \rightarrow \mathbb{R}$ is smooth, then the inverse image

$$X = c^{-1}[a] \subseteq Z$$

of any regular value $a \in \mathbb{R}$ is a smooth, closed, oriented manifold. Given a principal π -bundle \tilde{Z} over Z , we can restrict it to X and then form the signature class (2-1). Thanks to the bordism invariance of the signature class, the K -theory class we obtain is independent of the choice of regular value $a \in \mathbb{R}$. Indeed it depends only on the homotopy class of the proper map c .

Definition We shall call the signature class of X above the *transverse signature class* of the continuously controlled manifold Z , and denote it by

$$\text{Sgn}_{c,\pi}(\tilde{Z}/Z) := \text{Sgn}_\pi(\tilde{X}/X) \in K_d(C_r^*(\pi)).$$

2.7 To prove the main theorem of the paper we shall need to invoke an invariance property of the transverse signature class of the following sort. Suppose given a morphism

$$h: Z \rightarrow W$$

of smooth, oriented manifolds that are continuously controlled over \mathbb{R} , and suppose that h is, in suitable sense, a controlled homotopy equivalence. The conclusion is

²It is also possible to define a signature class in the K -theory of the *full* group C^* -algebra, and this version is functorial for arbitrary group homomorphisms. But since it is in certain respects more awkward to manipulate with we shall mostly confine our attention in this paper to the signature class for the reduced C^* -algebra.

that if Z and W are equipped with principal π -bundles that are compatible with the equivalence, then (up to a factor of 2 that we shall make precise in a moment) the associated transverse signature classes of Z and W are equal.

Actually we shall need to consider situations where h will be only defined on the complement of a compact set; see Paragraph 2.9. But for now we shall ignore this additional complication.

Definition Let Z and W be locally compact spaces that are continuously controlled over \mathbb{R} , with control maps c_Z and c_W . A continuous and proper map $f: Z \rightarrow W$ is *continuously controlled over \mathbb{R}* if the symmetric difference

$$c_Z^{-1}[0, \infty) \Delta f^{-1}c_W^{-1}[0, \infty)$$

is a compact subset of Z .

Thus the map $f: Z \rightarrow W$ is continuously controlled over \mathbb{R} if it is compatible with the decompositions of Z and W into positive and negative parts (according to the values of the control maps), modulo compact sets.

Definition Let Z and W be complete Riemannian manifolds, both with bounded geometry [1, Section 1].

- (a) A *boundedly controlled homotopy equivalence* from Z to W is a pair of smooth maps

$$f: Z \rightarrow W \quad \text{and} \quad g: W \rightarrow Z,$$

together with a pair of smooth homotopies

$$H_Z: Z \times [0, 1] \rightarrow Z \quad \text{and} \quad H_W: W \times [0, 1] \rightarrow W$$

between the compositions of f with g and the identity maps on Z and W , for which all the maps have bounded derivatives, meaning that, for example, $\sup_{X \in TZ, \|X\| \leq 1} \|Df(X)\| < \infty$.

- (b) If Z and W are continuously controlled over \mathbb{R} , then the above boundedly controlled homotopy equivalence is in addition *continuously controlled over \mathbb{R}* if all the above maps are continuously controlled over \mathbb{R} .
- (c) The above boundedly controlled homotopy equivalence is *compatible* with given principal π -bundles over Z and W if all the above maps lift to bundle maps.

The following result is a small extension of the partitioned manifold index theorem of Roe [21]. We shall prove it in Section 4.8.

2.8 Theorem *Let Z and W be $(d+1)$ -dimensional, complete oriented Riemannian manifolds with bounded geometry that are continuously controlled over \mathbb{R} . If \tilde{Z} and \tilde{W} are principal π -bundles over Z and W , and if there is a boundedly controlled orientation-preserving homotopy equivalence between Z and W that is continuously controlled over \mathbb{R} and compatible with \tilde{Z} and \tilde{W} , then*

$$2^\varepsilon (\text{Sgn}_{c,\pi}(\tilde{Z}/Z) - \text{Sgn}_{c,\pi}(\tilde{W}/W)) = 0$$

in $K_d(C_r^*(\pi))$, where

$$\varepsilon = \begin{cases} 0 & \text{if } d \text{ is even,} \\ 1 & \text{if } d \text{ is odd.} \end{cases}$$

2.9 As we already noted, we shall actually need a slightly stronger version of Theorem 2.8. It involves the following concept:

Definition Let Z and W be locally compact Hausdorff spaces. An *eventual map* from Z to W is a continuous and proper map

$$Z_1 \rightarrow W,$$

where Z_1 is a closed subset of Z whose complement has compact closure. Two eventual maps from Z to W are *equivalent* if they agree on the complement of some compact set in Z .

There is an obvious category of locally compact spaces and equivalence classes of eventual maps. Let us denote the morphisms as follows: $Z \rightsquigarrow W$.

Definition Two eventual morphisms

$$f_0, f_1: Z \rightsquigarrow W$$

are *homotopic* if there is an eventual morphism $g: Z \times [0, 1] \rightsquigarrow W$ for which the compositions

$$Z \rightrightarrows Z \times [0, 1] \xrightarrow{g} W$$

with the inclusions of Z as $Z \times \{0\}$ and $Z \times \{1\}$ are equal to f_0 and f_1 , respectively, in the category of locally compact spaces and eventual maps.

Homotopy is an equivalence relation on morphisms that is compatible with composition, and so we obtain a notion of *eventual homotopy equivalence*.

2.10 It is clear how to define the concept of continuously controlled eventual morphisms between locally compact spaces that are continuously controlled over \mathbb{R} , as well as eventual homotopy equivalences that are continuously controlled over \mathbb{R} . And we can speak of *boundedly controlled* eventual homotopy equivalences between bounded geometry Riemannian manifolds Z and W , and compatibility of these with principal π -bundles on Z and W .

2.11 An orientation of a manifold Z of dimension d is a class in $H_d^{\text{lf}}(Z)$, the homology of locally finite chains, that maps to a generator in each group $H_d^{\text{lf}}(Z, Z \setminus \{z\})$. We shall call the image of the orientation class under the map

$$H_d^{\text{lf}}(Z) \rightarrow \varinjlim_K H_d^{\text{lf}}(Z, K),$$

where the direct limit is over the compact subsets of Z , the associated *eventual orientation class*. The direct limit is functorial for eventual morphisms, and so we can speak of an eventual homotopy equivalence being *orientation-preserving*.

The invariance property that we shall use to prove Theorem 1.1 is as follows:

2.12 Theorem *Let Z and W be $(d+1)$ -dimensional, connected complete oriented Riemannian manifolds with bounded geometry that are continuously controlled over \mathbb{R} . If \tilde{Z} and \tilde{W} are principal π -bundles over Z and W , and if there is a boundedly controlled eventual homotopy equivalence between Z and W that is continuously controlled over \mathbb{R} , orientation-preserving, and compatible with \tilde{Z} and \tilde{W} , then*

$$2^\varepsilon (\text{Sgn}_{c,\pi}(\tilde{Z}/Z) - \text{Sgn}_{c,\pi}(\tilde{W}/W)) = 0$$

in $K_d(C_r^*(\pi))$, where

$$\varepsilon = \begin{cases} 0 & \text{if } d \text{ is even,} \\ 1 & \text{if } d \text{ is odd.} \end{cases}$$

We shall prove this result in Section 5.

2.13 Finally, we shall need a formula for the signature of a product manifold. Let X be a smooth, closed and oriented manifold of dimension p , and $\tilde{X} \rightarrow X$ a principal π -bundle over X . Similarly, let Y be a smooth, closed and oriented manifold of dimension q , and $\tilde{Y} \rightarrow Y$ a principal σ -bundle over Y . There is a natural product map

$$- \otimes -: K_p(C_r^*(\pi)) \otimes K_q(C_r^*(\sigma)) \rightarrow K_{p+q}(C_r^*(\pi \times \sigma)),$$

and we have the following product formula:

$$\text{Sgn}_{\pi \times \sigma}(\tilde{X} \times \tilde{Y} / X \times Y) = 2^{\varepsilon(X,Y)} \text{Sgn}_{\pi}(\tilde{X} / X) \otimes \text{Sgn}_{\sigma}(\tilde{Y} / Y),$$

where

$$\varepsilon(X, Y) = \begin{cases} 0 & \text{if one of } X \text{ or } Y \text{ is even-dimensional,} \\ 1 & \text{if both } X \text{ and } Y \text{ are odd-dimensional.} \end{cases}$$

This will be proved in Section 4.5.

3 Proof of the codimension-two theorem

In this section, we shall prove Theorem 1.1 using the properties of the controlled signature that were listed above.

Let $h: N \rightarrow M$ be an orientation-preserving, smooth homotopy equivalence between smooth, closed, oriented manifolds of dimension $d + 2$. Let X be a smooth, closed, oriented submanifold of M of codimension two (and hence dimension d), and assume that the smooth map $h: N \rightarrow M$ is transverse to X , so that the inverse image

$$Y = h^{-1}[X]$$

is a smooth, closed, oriented, codimension-two submanifold of N . Also assume that

- (i) $\pi_1(X) \rightarrow \pi_1(M)$ is injective;
- (ii) $\pi_2(X) \rightarrow \pi_2(M)$ is surjective; and
- (iii) the normal bundle of X in M is trivializable.

We fix a base point in X and assume that X is connected. With assumption (i) above, we shall view $\pi_1(X)$ as a subgroup of $\pi_1(M)$. Let

$$p: \hat{M} \rightarrow M$$

be the covering of M corresponding to the subgroup $\pi_1(X) \subset \pi_1(M)$. In other words, \hat{M} is the quotient space of the universal covering of M by the subgroup $\pi_1(X)$. In particular, we have the following lemma.

3.1 Lemma *The inverse image of X under the projection from \hat{M} to M is a disjoint union of copies of coverings of X . The component that contains the base point is homeomorphic to X , with homeomorphism given by the restriction of the covering projection. □*

Fix the base point copy of X in the inverse image $p^{-1}(X)$. We will use the same notation, X , for this lift of the submanifold $X \subseteq M$:

$$X \subseteq p^{-1}(X).$$

Denote by $D(X)$ a closed tubular neighborhood of X in \widehat{M} and consider the smooth manifold

$$\widehat{M} \setminus \overset{\circ}{D}(X)$$

(the circle denotes the interior) with boundary

$$\partial(\widehat{M} \setminus \overset{\circ}{D}(X)) \cong X \times S^1.$$

The following lemma is contained in the proof of [7, Theorem 4.3].

3.2 Lemma *Under the assumptions (i) and (ii) made above, the inclusion*

$$X \times S^1 \hookrightarrow \widehat{M} \setminus \overset{\circ}{D}(X)$$

induces a split injection

$$\pi_1(X \times S^1) \rightarrow \pi_1(\widehat{M} \setminus \overset{\circ}{D}(X))$$

on fundamental groups. □

Glue two copies of $\widehat{M} \setminus \overset{\circ}{D}(X)$ along $X \times S^1$ so as to form the space

$$W = \widehat{M} \setminus \overset{\circ}{D}(X) \cup_{X \times S^1} \widehat{M} \setminus \overset{\circ}{D}(X).$$

By applying the Seifert–van Kampen theorem, we obtain the following lemma, as in the proof of [7, Theorem 4.3].

3.3 Lemma *The map*

$$\pi_1(X \times S^1) \rightarrow \pi_1(W)$$

that is induced from the inclusion $X \times S^1 \hookrightarrow W$ is a split injection. □

Pull back the covering

$$p: \widehat{M} \rightarrow M$$

to N along the map h , as indicated in the following diagram:

$$\begin{array}{ccc} \widehat{N} & \xrightarrow{\widehat{h}} & \widehat{M} \\ \downarrow & & \downarrow p \\ N & \xrightarrow{h} & M \end{array}$$

We denote by

$$Y = \hat{h}^{-1}(X) \subset \hat{N}$$

the inverse image under \hat{h} of our chosen copy of X in \hat{M} .

3.4 Lemma *The normal bundle of Y in N , or equivalently in \hat{N} , is trivial.*

Proof By transversality, the normal bundle of Y in N is the pullback of the normal bundle of X in M , and hence is trivial by our assumption (iii). \square

Now let us prove Theorem 1.1, assuming Theorem 2.12 on the eventual homotopy invariance of the controlled signature (the rest of the paper will be devoted to proving Theorem 2.12).

Proof of Theorem 1.1 Form the space

$$Z = \hat{N} \setminus \mathring{D}(Y) \cup_{Y \times S^1} \hat{N} \setminus \mathring{D}(Y).$$

Here $\mathring{D}(Y)$ is the inverse image of $\mathring{D}(X)$ under \hat{h} , which is an open disk bundle of Y in \hat{N} with boundary $Y \times S^1$ thanks to Lemma 3.4.

The map $\hat{h}: \hat{N} \rightarrow \hat{M}$ induces a canonical continuous map

$$(3-1) \quad \varphi: Z \rightarrow W.$$

Although it was constructed using a process that started from a homotopy equivalence, the map φ in (3-1) is not a homotopy equivalence in general. However, we still get an eventual homotopy equivalence.

3.5 Lemma *The map φ is a boundedly controlled eventual homotopy equivalence that is continuously controlled over \mathbb{R} , with respect to the control maps*

$$c_W: W \rightarrow \mathbb{R} \quad \text{and} \quad c_Z: Z \rightarrow \mathbb{R},$$

where

$$c_W(w) = \begin{cases} d(w, X \times S^1) & \text{if } w \text{ is in the first copy } M \setminus \mathring{D}(X) \text{ in } W, \\ -d(w, X \times S^1) & \text{if } w \text{ is in the second copy } M \setminus \mathring{D}(X) \text{ in } W, \end{cases}$$

and where c_Z is constructed correspondingly. Here $d(w, X \times S^1)$ is the distance between w and $X \times S^1$ in W .

Proof The map h , its homotopy inverse $g: M \rightarrow N$, and the homotopies

$$H_1: N \times [0, 1] \rightarrow M \quad \text{and} \quad H_2: M \times [0, 1] \rightarrow N$$

between the compositions and the identity maps all lift to maps $\hat{g}, \hat{h}, \hat{H}_1, \hat{H}_2$ on \hat{M}, \hat{N} . In general, only \hat{h} will map $\hat{D}(Y) \subset \hat{N}$ to $\hat{D}(X) \subset \hat{M}$ and therefore give rise to an honest map $\phi: Z \rightarrow W$. But the other maps at least give rise to eventual maps which establish the required eventual homotopy equivalence.

We lift the Riemannian metrics from the compact manifolds N and M to \hat{N} and \hat{M} and obtain metrics on W and Z by smoothing these metrics in a compact neighborhood of $X \times S^1$ or $Y \times S^1$, respectively. Therefore, we obtain metrics of bounded geometry. All our maps are lifted from the compact manifolds M and N and therefore are boundedly controlled. By construction, they also preserve the decomposition into positive and negative parts according to the control maps c_W and c_Z (up to a compact deviation). That is, they are continuously controlled over \mathbb{R} . □

Let $r: \pi_1(W) \rightarrow \pi_1(X \times S^1)$ be the splitting of $\iota: \pi_1(X \times S^1) \rightarrow \pi_1(W)$. Let \tilde{W} be the covering space of W corresponding to the subgroup $\ker(r)$, that is, the quotient space of the universal cover of W by $\ker(r)$. Let \tilde{Z} be the covering space of Z that is the pullback of $\tilde{W} \rightarrow W$ along the map ϕ . It follows from Theorem 2.12 that

$$2^\varepsilon (\text{Sgn}_{c,\pi}(\tilde{W}/W) - \text{Sgn}_{c,\pi}(\tilde{Z}/Z)) = 0 \quad \text{in } K_d(C_r^*(\pi)),$$

where $\pi = \pi_1(X \times S^1) = \pi_1(X) \times \mathbb{Z}$ and

$$\varepsilon = \begin{cases} 0 & \text{if } d \text{ is even,} \\ 1 & \text{if } d \text{ is odd.} \end{cases}$$

Equivalently, we have

$$2^\varepsilon (\text{Sgn}_\pi(\widetilde{X \times S^1} / X \times S^1) - \text{Sgn}_\pi(\widetilde{Y \times S^1} / Y \times S^1)) = 0 \quad \text{in } K_d(C_r^*(\pi)).$$

Here $\widetilde{X \times S^1}$ is the universal cover of $X \times S^1$, which pulls back to give a covering space $\widetilde{Y \times S^1}$ over $Y \times S^1$. It is a basic computation of higher signatures that $\text{Sgn}_\mathbb{Z}(S^1)$ is a generator of $K_1(C_r^*(\mathbb{Z})) \cong \mathbb{Z}$. Moreover, we have for an arbitrary group Γ the Künneth isomorphism given by the product map

$$K_p(C_r^*(\Gamma)) \otimes K_0(C_r^*(\mathbb{Z})) \oplus K_{p-1}(C_r^*(\Gamma)) \otimes K_1(C_r^*(\mathbb{Z})) \xrightarrow{\cong} K_p(C_r^*(\Gamma \times \mathbb{Z})).$$

The theorem now follows from the product formula of signature operators (see Paragraph 2.13). □

We conclude this section with a discussion of counterexamples to some would-be strengthenings of the theorem, as well as some further comments.

Techniques from surgery theory, in particular Wall’s π - π theorem [28, Theorem 3.3], show that conditions (i) and (ii) in Theorem 1.1 are both necessary. Concerning condition (i), let us start by noting that the n -torus T^n embeds into the $(n+2)$ -sphere S^{n+2} with trivial normal bundle (indeed it maps into every $(n+2)$ -dimensional manifold with trivial normal bundle, but one embedding into the sphere will suffice). Let $f: V \rightarrow W$ be a degree-one normal map between simply connected, closed, oriented $4k$ -manifolds with distinct signatures $\text{Sgn}(V) \neq \text{Sgn}(W) \in \mathbb{Z}$. Consider the degree-one normal map of pairs

$$f \times \text{Id}: (V \times B^{n+3}, V \times S^{n+2}) \rightarrow (W \times B^{n+3}, W \times S^{n+2}).$$

According to Wall’s π - π theorem this is normally cobordant to a homotopy equivalence of pairs; let us write the homotopy equivalence on boundaries as

$$h: N \xrightarrow{\cong} W \times S^{n+2}.$$

Consider the codimension-two submanifold (with trivial normal bundle)

$$W \times T^n \subseteq W \times S^{n+2}.$$

Its signature class in the free abelian group $K(C_r^*(\mathbb{Z}^n))$ is $\text{Sgn}(W) \cdot \text{Sgn}_{\mathbb{Z}^n}(T^n)$, where $\text{Sgn}_{\mathbb{Z}^n}(T^n)$ has infinite order. However the signature class of the transverse inverse image $M = h^{-1}(W \times T^n)$ is equal to

$$\text{Sgn}_{\mathbb{Z}^n}(M) = \text{Sgn}(V) \cdot \text{Sgn}_{\mathbb{Z}^n}(T^n) \in K(C_r^*(\mathbb{Z}^n)).$$

This follows from the cobordism invariance of the signature class, using that the normal bordism between $f \times \text{id}_{S^{n+2}}$ and h restricts to a normal bordism between $f \times \text{id}_{T^n}$ and $h|_M$. Therefore (even twice) the signature classes of the two submanifolds of codimension two are distinct.

As for condition (ii), start with any degree-one normal map $f: W \rightarrow X$ such that

$$2(\text{Sgn}_{\pi_1(X)}(\tilde{W}/W) - \text{Sgn}_{\pi_1(X)}(\tilde{X}/X)) \neq 0$$

(here \tilde{X} is the universal cover of X , and \tilde{W} is the pullback to W along f). By Wall’s theorem again, the map

$$f \times \text{Id}: W \times S^2 \rightarrow X \times S^2$$

is normally cobordant to an orientation-preserving homotopy equivalence

$$h: N \xrightarrow{\cong} X \times S^2.$$

Let Y be the transverse inverse image of the codimension-two submanifold $X \times \{\text{pt}\} \subseteq X \times S^2$ under the map h . It follows from cobordism invariance of the signature class that

$$\text{Sgn}_{\pi_1(X)}(\tilde{Y}/Y) = \text{Sgn}_{\pi_1(X)}(\tilde{W}/W),$$

and therefore that

$$2(\text{Sgn}_{\pi_1(X)}(\tilde{Y}/Y) - \text{Sgn}_{\pi_1(X)}(\tilde{X}/X)) \neq 0,$$

as required.

On the other hand, it seems to be difficult to determine whether or not condition (iii) is necessary. This question is also open for the companion result about positive scalar curvature; compare in particular [5].

Finally, as we mentioned in the introduction, we do not know if the factor of 2 that appears in Theorem 1.1 is really necessary, although we suspect it is. Let $p: M \rightarrow \Sigma$ be any bundle of oriented closed manifolds over an oriented closed surface $\Sigma \neq S^2$ with fiber $X = p^{-1}(\text{pt})$. If $h: N \rightarrow M$ is a homotopy equivalence, then, whether or not the composition

$$N \xrightarrow{h} M \xrightarrow{p} \Sigma$$

is homotopic to a fiber bundle projection, and whether or not the induced map from the “fiber” $Y = (p \circ h)^{-1}(\text{pt})$ to X is cobordant to a homotopy equivalence, it follows from Theorem 1.1 that the C^* -algebraic higher signatures of Y and X coincide in $K_*(C_r^*(\pi_1(X)))$, after multiplication with 2. In the related L -theory context, examples can be constructed where the factor of 2 is really necessary, and perhaps the same is true here. Moreover it seems possible that many nontrivial examples of the theorem can be obtained from this construction.

4 Signature classes in C^* -algebra K -theory

In this section we shall review the construction of the C^* -algebraic signature class. One route towards the definition of the signature class goes via index theory and the signature operator. See for example [2, Section 7] for an introduction. Although we shall make use of the index theory approach, we shall mostly follow a different route, adapted from [9; 10], which makes it easier to handle the invariance properties of the signature class that we need.

4.1 Hilbert–Poincaré complexes

Hilbert–Poincaré complexes were introduced in [9] to adapt the standard symmetric signature constructions in *L*-theory to the context of *C**-algebra *K*-theory. We shall review the main definitions here; see [9] for more details.

4.1 Definition Let *A* be a unital *C**-algebra. A *d*-dimensional Hilbert–Poincaré complex over *A* is a complex of finitely generated (and therefore also projective) Hilbert *A*-modules and bounded, adjointable differentials

$$E_0 \xleftarrow{b} E_1 \xleftarrow{b} \dots \xleftarrow{b} E_d$$

together with bounded adjointable operators $D: E_p \rightarrow E_{d-p}$ such that

- (i) if $v \in E_p$, then $D^*v = (-1)^{(d-p)p} Dv$;
- (ii) if $v \in E_p$, then $Db^*v + (-1)^p bDv = 0$; and
- (iii) D is a homology isomorphism from the dual complex

$$E_d \xleftarrow{b^*} E_{d-1} \xleftarrow{b^*} \dots \xleftarrow{b^*} E_0$$

to the complex (E, b) .

To each *d*-dimensional Hilbert–Poincaré complex over *A* there is associated a signature in *C**-algebra *K*-theory:

$$\text{Sgn}(E, D) \in K_d(A).$$

See [9, Section 3] for the construction. If *A* is the *C**-algebra of complex numbers, and $d \equiv 0 \pmod 4$, then the signature identifies with the signature of the quadratic form induced from D on middle-dimensional homology.

The signature is a homotopy invariant [9, Section 4] and indeed a bordism invariant [9, Section 7]. These concepts will be illustrated in the following subsections.

4.2 The signature class

Let Σ be a finite-dimensional simplicial complex and denote by $\mathbb{C}[\Sigma_p]$ the vector space of finitely supported, complex-valued functions on the set of *p*-simplices in Σ . After assigning orientations to simplices we obtain in the usual way a chain complex

$$(4-1) \quad \mathbb{C}[\Sigma_0] \xleftarrow{b_1} \mathbb{C}[\Sigma_1] \xleftarrow{b_2} \dots$$

If Σ is, in addition, locally finite (meaning that each vertex is contained in only finitely many simplices) then there is also an adjoint complex

$$(4-2) \quad \mathbb{C}[\Sigma_0] \xrightarrow{b_1^*} \mathbb{C}[\Sigma_1] \xrightarrow{b_2^*} \dots,$$

in which the differentials are adjoint to those of (4-1) with respect to the standard inner product for which the delta functions on simplices are orthonormal.

If Σ is furthermore an oriented combinatorial manifold of dimension d with fundamental cycle C , then there is a duality operator

$$(4-3) \quad D: \mathbb{C}[\Sigma_p] \rightarrow \mathbb{C}[\Sigma_{d-p}]$$

defined by the usual formula $D(\xi) = \xi \cap C$ (compare [10, Section 3], where the duality operator is called \mathbb{P}). There is also an adjoint operator

$$D^*: \mathbb{C}[\Sigma_{d-p}] \rightarrow \mathbb{C}[\Sigma_p].$$

We are, as yet, in a purely algebraic, and not C^* -algebraic, context, but the operator D satisfies all the relations given in Definition 4.1, except for the first. As for the first, the operators D and $(-1)^{(d-p)p} D^*$ are chain homotopic, and if we replace D by the average

$$\frac{1}{2}(D + (-1)^{(d-p)p} D^*),$$

which we shall do from now on, then all the relations are satisfied.

In order to pass from complexes of vector spaces to complexes of Hilbert modules we shall assume the following:

4.2 Definition A simplicial complex Σ is of *bounded geometry* if there is a positive integer k such that any vertex of Σ lies in at most k different simplices of Σ .

4.3 Definition Let P be a proper metric space and let S be a set. A function

$$c: S \rightarrow P$$

is a *proper control map* if for every $r > 0$, there is a bound $N < \infty$ such that if $B \subseteq P$ is any subset of diameter less than r , then $c^{-1}[B]$ has cardinality less than N .

4.4 Definition Let P be a proper metric space, let S and T be sets, and let

$$c_S: S \rightarrow P \quad \text{and} \quad c_T: T \rightarrow P$$

be proper control maps. Suppose in addition that a discrete group π acts on P properly through isometries, and on the sets S and T , and suppose that the control maps are equivariant. A linear map

$$A: \mathbb{C}[S] \rightarrow \mathbb{C}[T]$$

is *boundedly geometrically controlled* over P if

- (i) the matrix coefficients of A with respect to the bases given by the points of S and T are uniformly bounded; and
- (ii) there is a constant $K > 0$ such that the (t, s) -matrix coefficient of A is zero whenever $d(c(t), c(s)) > K$.

We shall denote by $\mathbb{C}_{P,\pi}[T, S]$ the linear space of equivariant, boundedly geometrically controlled linear operators from $\mathbb{C}[S]$ to $\mathbb{C}[T]$.

Every boundedly geometrically controlled linear operator is adjointable, and the space $\mathbb{C}_{P,\pi}[S, S]$ is a $*$ -algebra by composition and adjoint of operators.

4.5 Definition We denote by $C_{P,\pi}^*(S, S)$ the C^* -algebra completion of $\mathbb{C}_{P,\pi}[S, S]$ in the operator norm on $\ell^2(S)$.

4.6 Remark The norm above is related to the norm in *reduced* group C^* -algebras. There is also a norm on $\mathbb{C}_{P,\pi}[S, S]$ appropriate to full group C^* -algebras, namely

$$\|A\| = \sup_{\rho} \|\rho(A): H \rightarrow H\|,$$

where the supremum is over all representations of the $*$ -algebra $\mathbb{C}_{P,\pi}[S, S]$ as bounded Hilbert space operators. See [4], where among many other things it is shown that the supremum above is finite.

Now the space $\mathbb{C}_{P,\pi}[T, S]$ in Definition 4.4 is a right module over $\mathbb{C}_{P,\pi}[S, S]$ by composition of operators.

4.7 Lemma *If there is an equivariant, injective map $T \rightarrow S$ making the diagram*

$$\begin{array}{ccc} T & \longrightarrow & S \\ c_T \downarrow & & \downarrow c_S \\ P & \xlongequal{\quad} & P \end{array}$$

commute, then $\mathbb{C}_{P,\pi}[T, S]$ is finitely generated and projective over $\mathbb{C}_{P,\pi}[S, S]$. □

Proof Define $p: \mathbb{C}[S] \rightarrow \mathbb{C}[S]$ sending $t \in T$ to t and $s \in S \setminus T$ to 0. Then $p \in \mathbb{C}_{P,\pi}[S, S]$ is an idempotent with image precisely $\mathbb{C}_{P,\pi}[T, S]$. So the image is projective and generated by the element p . \square

Assume that the hypothesis in the lemma holds (we shall always assume this in what follows). The induced module

$$C_{P,\pi}^*(T, S) = \mathbb{C}_{P,\pi}[T, S] \otimes_{\mathbb{C}_{P,\pi}[S, S]} C_{P,\pi}^*(S, S)$$

is then a finitely generated and projective module over $C_{P,\pi}^*(S, S)$. It is isomorphic to the completion of $\mathbb{C}_{P,\pi}[T, S]$ in the norm associated to the $\mathbb{C}_{P,\pi}[S, S]$ -valued, and hence $C_{P,\pi}^*(S, S)$ -valued, inner product on $\mathbb{C}_{P,\pi}[T, S]$ defined by

$$\langle A, B \rangle = A^* B.$$

So it is a finitely generated and projective Hilbert module over $C_{P,\pi}^*(S, S)$.

Let us return to the bounded geometry, combinatorial manifold Σ , which we shall now assume is equipped with an action of π . Assume that the sets Σ_p of p -simplices admit proper control maps to some P so that there is a uniform bound on the distance between the image of any simplex and the image of any of its vertices. Then the differentials in (4-1) and the duality operator (4-3) are boundedly geometrically controlled, and we obtain from them a complex

$$(4-4) \quad C_{P,\pi}^*(\Sigma_0, S) \xleftarrow{b_1} C_{P,\pi}^*(\Sigma_1, S) \xleftarrow{b_2} \dots \xleftarrow{b_d} C_{P,\pi}^*(\Sigma_d, S)$$

of finitely generated and projective Hilbert modules over $C_{P,\pi}^*(S, S)$ for any S that satisfies the hypothesis of Lemma 4.7 for all $T = \Sigma_p$. The procedure in Section 4.1 therefore gives us a signature

$$(4-5) \quad \text{Sgn}_{P,\pi}(\Sigma, S) \in K_d(C_{P,\pi}^*(S, S)).$$

If S is a free π -set with finitely many π -orbits, and if we regard the underlying set of π as a π -space by left multiplication, then the bimodule $\mathbb{C}_{P,\pi}[\pi, S]$ is a Morita equivalence

$$\mathbb{C}_{P,\pi}[S, S] \underset{\text{Morita}}{\sim} \mathbb{C}_{P,\pi}[\pi, \pi],$$

and upon completion we obtain a C^* -algebra Morita equivalence

$$(4-6) \quad C_{P,\pi}^*(S, S) \underset{\text{Morita}}{\sim} C_{P,\pi}^*(\pi, \pi).$$

But the action of π on the vector space $\mathbb{C}[\pi]$ by *right* translations gives an isomorphism of $*$ -algebras

$$\mathbb{C}[\pi] \xrightarrow{\cong} C_{P,\pi}[\pi, \pi]$$

and then by completion an isomorphism

$$(4-7) \quad C_r^*(\pi) \xrightarrow{\cong} C_{P,\pi}^*(\pi, \pi).$$

Putting (4-6) and (4-7) together we obtain a canonical isomorphism

$$(4-8) \quad K_*(C_{P,\pi}^*(S, S)) \cong K_*(C_r^*(\pi)).$$

Now let X be a smooth, closed and oriented manifold of dimension d , and let \tilde{X} be a principal π -bundle over X . Fix any triangulation of X and lift it to a triangulation Σ of \tilde{X} . In addition, fix a Riemannian metric on X and lift it to a Riemannian metric on \tilde{X} . Let P be the underlying proper metric space. If we choose S to be the disjoint union of all Σ_p , then we obtain a signature as in (4-5) above.

4.8 Definition We denote by

$$\text{Sgn}_\pi(\tilde{X}/X) \in K_d(C_r^*(\pi))$$

the signature class associated to (4-5) under the K -theory isomorphism (4-8).

4.9 Remark By using the C^* -algebra norm in Remark 4.6 we would instead obtain a signature class in $K_d(C^*(\pi))$, where $C^*(\pi)$ is the maximal group C^* -algebra.

The general invariance properties of the signature of a Hilbert–Poincaré complex imply that the signature is an oriented homotopy invariant, as in Paragraph 2.2, and a bordism invariant, as in Paragraph 2.5. Compare [10, Sections 3 and 4].

4.3 Signature classes in coarse geometry

Let P be a proper metric space. A *standard P -module* is a separable Hilbert space that is equipped with a nondegenerate representation of the C^* -algebra $C_0(P)$. It is unique up to finite-propagation unitary isomorphism. As a result, the C^* -algebra $C^*(P)$ generated by finite-propagation, locally compact operators is unique up to inner (in the multiplier C^* -algebra) isomorphism, and its topological K -theory is therefore unique up to canonical isomorphism. For all this see for example [12, Section 4].

If a discrete group π acts properly and isometrically on P , then all these definitions and constructions can be made equivariantly, and we shall denote by $C_\pi^*(P)$ the C^* -algebra generated by the π -equivariant, finite-propagation, locally compact operators.

If S is a π -set that is equipped with an equivariant proper control map to P , as in Definition 4.3, then $\ell^2(S)$ carries a π -covariant representation of $C_0(P)$ via the control map, and there is an isometric, equivariant, finite-propagation inclusion of $\ell^2(S)$ into any π -covariant standard P -module; see [12, Section 4] again. This induces an inclusion of $C_{P,\pi}^*(S, S)$ into $C_\pi^*(P)$, and a canonical homomorphism

$$(4-9) \quad K_*(C_{P,\pi}^*(S, S)) \rightarrow K_*(C_\pi^*(P)).$$

4.10 Remark If the quotient space P/π is compact, then this is an *isomorphism*, and, as we noted, in this case the left-hand side of (4-9) is canonically isomorphic to $K_*(C_r^*(\pi))$. But in general the right-hand side can be quite different from either $K_*(C_{P,\pi}^*(S, S))$ or $K_*(C_r^*(\pi))$.

4.11 Definition Let W be a complete, bounded geometry, oriented Riemannian manifold, and let \tilde{W} be a principal π -bundle over W . We shall denote by

$$\text{Sgn}_\pi(\tilde{W}/W) \in K_*(C_\pi^*(\tilde{W}))$$

the signature class obtained from a π -invariant, bounded geometry triangulation Σ of \tilde{W} by applying the map (4-9) to the class (4-5).

This is the same signature class as the one considered in [10]. Once again, the general invariance properties of the signature of a Hilbert–Poincaré complex immediately provide a homotopy invariance result:

4.12 Theorem *The signature class $\text{Sgn}_\pi(\tilde{W}/W)$ is independent of the triangulation. In fact the signature class is a boundedly controlled oriented homotopy invariant.*

4.4 The signature operator

4.13 Definition Let P be an oriented, complete Riemannian manifold. We denote by D_P the signature operator on P ; see for example [10, Section 5].

If W is an oriented, complete Riemannian manifold, and if \tilde{W} is a principal π -bundle over W , then the operator $D_{\tilde{W}}$ is a π -equivariant, first-order elliptic differential operator on \tilde{W} , and as explained in [2; 22; 23] it has an equivariant analytic index

$$\text{Ind}_\pi(D_{\tilde{W}}) \in K_d(C_\pi^*(\tilde{W})).$$

4.14 Theorem [10, Theorems 5.5 and 5.11] *If W is a complete bounded geometry Riemannian manifold of dimension d , and if \tilde{W} is a principal π -bundle over W , then*

$$\text{Sgn}_\pi(\tilde{W}/W) = \text{Ind}_\pi(D_{\tilde{W}}) \in K_d(C_\pi^*(\tilde{W})).$$

4.5 Signature of a product manifold

The signature operator point of view on the signature class makes it easy to verify the product formula in Paragraph 2.13.

If one of X or Y is even-dimensional, then the signature operator on $X \times Y$ is the product of the signature operators on X and Y ; if both X and Y are odd-dimensional, then the signature operator on $X \times Y$ is the direct sum of two copies of the product of the signature operators on X and Y . The product formula for the signature class follows from the commutative diagram

$$\begin{array}{ccc} K_p(B\pi) \otimes K_q(B\sigma) & \longrightarrow & K_{p+q}(B\pi \times B\sigma) \\ \downarrow & & \downarrow \\ K_p(C_r^*(\pi)) \otimes K_q(C_r^*(\sigma)) & \longrightarrow & K_{p+q}(C_r^*(\pi \times \sigma)) \end{array}$$

where the horizontal arrows are the products in K -homology and K -theory, and the vertical maps are index maps (or in other words assembly maps); compare [3]. The argument uses the fact that the K -homology class of the product of two elliptic operators is equal to the product of their corresponding K -homology classes.

4.6 Signature of the circle

The signature operator of S^1 is the operator $i(d/d\theta)$, which is well known to represent a generator of $K_1(B\mathbb{Z}) \cong \mathbb{Z}$, and whose index in $K_1(C_r^*(\mathbb{Z}))$ is well known to be a generator of $K_1(C_r^*(\mathbb{Z})) \cong \mathbb{Z}$.

4.7 Partitioned manifold index theorem

In this section we shall describe a minor extension of Roe’s partitioned manifold index theorem [21; 8]. Compare also [30]. We shall consider only the signature operator, but the argument applies to any Dirac-type operator.

Let A be a C^* -algebra. If I and J are closed, two-sided ideals in A with $I + J = A$,

then there is a Mayer–Vietoris sequence in K -theory:

$$\begin{array}{ccccc}
 K_0(I \cap J) & \longrightarrow & K_0(I) \oplus K_0(J) & \longrightarrow & K_0(A) \\
 \uparrow \partial & & & & \downarrow \partial \\
 K_1(A) & \longleftarrow & K_1(I) \oplus K_1(J) & \longleftarrow & K_1(I \cap J)
 \end{array}$$

See for example [12, Section 3]. The boundary maps may be described as follows. There is an isomorphism

$$I/(I \cap J) \xrightarrow{\cong} A/J,$$

and the boundary maps are the compositions

$$K_*(A) \rightarrow K_*(A/J) \cong K_*(I/(I \cap J)) \rightarrow K_{*-1}(I \cap J),$$

where the last arrow is the boundary map in the K -theory long exact sequence for the ideal $I \cap J \subseteq I$. The same map is obtained, up to sign, if I and J are switched.

Now let W be a complete Riemannian manifold that is continuously controlled over \mathbb{R} , and let \tilde{W} be a principal π -bundle over W . Let $X \subseteq W$ be the transverse inverse image of $0 \in \mathbb{R}$ and let \tilde{X} be the restriction of \tilde{W} to X .

Denote by I the ideal in $A = C_\pi^*(\tilde{W})$ generated by all finite propagation, locally compact operators that are supported in $c^{-1}[a, \infty)$ for some $a \in \mathbb{R}$, and denote by J the ideal in A generated by all finite propagation, locally compact operators that are supported on $c^{-1}(-\infty, a]$ for some $a \in \mathbb{R}$. The sum $I + J$ is equal to A .

The intersection $I \cap J$ is generated by all operators that are supported in the inverse images under the control map of compact subsets of \mathbb{R} . Any finite propagation isometry from a standard \tilde{X} module into a standard \tilde{W} -module induces an inclusion of $C_\pi^*(\tilde{X})$ into the intersection, and this inclusion induces a canonical isomorphism in K -theory; see [12, Section 5]. Following Remark 4.10, since X is compact we therefore obtain canonical isomorphisms

$$K_*(C_r^*(\pi)) \xrightarrow{\cong} K_*(C_\pi^*(\tilde{X})) \xrightarrow{\cong} K_*(I \cap J).$$

4.15 Theorem *Let W be an oriented, complete Riemannian manifold of dimension $d + 1$ that is continuously controlled over \mathbb{R} , and let \tilde{W} be a principal π -bundle over W . The Mayer–Vietoris boundary homomorphism*

$$\partial: K_{d+1}(C_\pi^*(\tilde{W})) \rightarrow K_d(C_r^*(\pi))$$

maps the index class of the signature operator $D_{\tilde{W}}$ to the index class of the signature operator $D_{\tilde{X}}$ if d is even, and to twice the index class of $D_{\tilde{X}}$ if d is odd.

Proof The argument of [21] reduces the theorem to the case of a product manifold, that is, $W = X \times \mathbb{R}$ with the control map $c: X \times \mathbb{R} \rightarrow \mathbb{R}$ given by the projection to \mathbb{R} (see also [8]). The case of the product manifold $X \times \mathbb{R}$ follows from a direct computation, as in [8]. Once again, the reason for the factor of 2, in the case when d is odd, is that signature operator on $X \times \mathbb{R}$ is *the direct sum of two copies of* the product of the signature operators of X and \mathbb{R} in this case. \square

4.16 Corollary *Assume that W has bounded geometry. The Mayer–Vietoris boundary homomorphism*

$$\partial: K_{d+1}(C_\pi^*(\tilde{W})) \rightarrow K_d(C_r^*(\pi))$$

maps $\text{Sgn}_\pi(\tilde{W}/W)$ to the transverse signature class $\text{Sgn}_{c,\pi}(\tilde{W}/W)$ if d is even, and it maps it to twice the transverse signature class of W if d is odd. \square

4.8 The transverse signature

We can now prove Theorem 2.8. Let Z and W be as in the statement. The boundedly controlled homotopy equivalence between them identifies $\text{Sgn}_\pi(\tilde{W}/W)$ with $\text{Sgn}_\pi(\tilde{Z}/Z)$. Since the equivalence is continuously controlled over \mathbb{R} there is a commuting diagram

$$\begin{array}{ccc} K_{d+1}(C_\pi^*(\tilde{W})) & \longrightarrow & K_d(C_r^*(\pi)) \\ \downarrow & & \parallel \\ K_{d+1}(C_\pi^*(\tilde{Z})) & \longrightarrow & K_d(C_r^*(\pi)) \end{array}$$

involving Mayer–Vietoris boundary homomorphisms. Thanks to Roe’s partitioned manifold index theorem, in the form of Corollary 4.16, the horizontal maps send the signature classes $\text{Sgn}_\pi(\tilde{W}/W)$ and $\text{Sgn}_\pi(\tilde{Z}/Z)$ to 2^ε times the transverse signature classes $\text{Sgn}_{c,\pi}(\tilde{W}/W)$ and $\text{Sgn}_{c,\pi}(\tilde{Z}/Z)$, respectively. So the two transverse signature classes, times 2^ε , are equal.

5 Invariance under eventual homotopy equivalences

It remains to prove Theorem 2.12, which we shall do here, and it is at this point that we shall make proper use of the complex $C_\pi^*(\Sigma_\bullet, S)$ that we introduced in Section 4.2.

5.1 Eventual signature

Let P be a proper metric space and assume that a discrete group π acts on P properly and through isometries.

5.1 Definition Let S and T be sets equipped with π -actions, and let

$$c_S: S \rightarrow P \quad \text{and} \quad c_T: T \rightarrow P$$

be π -equivariant proper control maps. A boundedly geometrically controlled linear operator $A: \mathbb{C}[S] \rightarrow \mathbb{C}[T]$ is π -compactly supported if there exists a closed subset $F \subseteq P$ whose quotient by π is compact such that $A_{t,s} = 0$ unless $c_S(s) \in F$ and $c_T(t) \in F$. Define

$$C_{P,\pi,\text{cpt}}^*(T, S) \subseteq C_{P,\pi}^*(T, S)$$

to be the operator norm-closure of the linear space of π -compactly supported, boundedly geometrically controlled operators.

When $S = T$ the space $C_{P,\pi,\text{cpt}}^*(S, S)$ is a closed two-sided ideal of the C^* -algebra $C_{P,\pi}^*(S, S)$.

5.2 Definition We define the quotient C^* -algebra

$$C_{P,\pi,\text{evtl}}^*(S, S) := C_{P,\pi}^*(S, S) / C_{P,\pi,\text{cpt}}^*(S, S)$$

and the quotient space

$$C_{P,\pi,\text{evtl}}^*(T, S) := C_{P,\pi}^*(T, S) / C_{P,\pi,\text{cpt}}^*(T, S).$$

Under the assumption $T \subseteq S$ the space $C_{P,\pi,\text{evtl}}^*(T, S)$ is a finitely generated and projective Hilbert $C_{P,\pi,\text{evtl}}^*(S, S)$ -module.

Now suppose W is a $(d+1)$ -dimensional, connected, complete, oriented Riemannian manifold of bounded geometry and that $P = \tilde{W}$ is a principal π -bundle over W . Choose a triangulation of bounded geometry of the base (compare [9, Lemma 5.8 and Proposition 5.9]), and lift to a π -invariant triangulation Σ of \tilde{W} . As before, let S be the disjoint union of all the Σ_p . We have the following Hilbert–Poincaré complex over $C_{\tilde{W},\pi,\text{evtl}}^*(S, S)$:

$$(5-1) \quad C_{\tilde{W},\pi,\text{evtl}}^*(\Sigma_0, S) \xleftarrow{b} C_{\tilde{W},\pi,\text{evtl}}^*(\Sigma_1, S) \xleftarrow{b} \dots \xleftarrow{b} C_{\tilde{W},\pi,\text{evtl}}^*(\Sigma_{d+1}, S),$$

with duality operator obtained from cap product with the fundamental cycle as before.

5.3 Definition The eventual signature class of $\tilde{W} \rightarrow W$ is defined to be

$$\text{Sgn}_{\pi,\text{evtl}}(\tilde{W} / W) := \text{Sgn}_{\pi,\text{evtl}}(\Sigma, S) \in K_{d+1}(C_{\tilde{W},\pi,\text{evtl}}^*(S, S)).$$

5.4 Theorem *The eventual signature class is invariant under oriented eventual homotopy equivalences which are boundedly controlled and compatible with the principal π -bundles.*

Proof This is a special case of [9, Theorem 4.3] once we have shown that our eventual homotopy equivalence $f: Z \rightsquigarrow W$ induces an algebraic homotopy equivalence of Hilbert–Poincaré complexes in the sense of [9, Definition 4.1]. For this, we observe that the map f and its eventual homotopy inverse $g: W \rightsquigarrow Z$, although only defined outside a compact subset, define in the usual way chain maps f_* and g_* between the $C_{\tilde{W}, \pi, \text{evtl}}^*(S, S)$ -Hilbert chain complexes of the triangulations, as compactly supported morphisms are divided out. For the construction of f_* , we need bounded control of the maps and bounded geometry of the triangulations. Similarly, the eventual homotopies can be used in the standard way to obtain chain homotopies between the identity and the composition of these induced maps, showing that f_* is a homology isomorphism. Finally, since f is orientation-preserving, it maps the (locally finite) fundamental cycle of Z , considered as a cycle relative to a suitable compact subset, to a cycle homologous to the fundamental cycle of W , again relative to a suitable compact subset. As the duality operator of the Hilbert–Poincaré chain complex $C_{\tilde{W}, \pi, \text{evtl}}^*(\Sigma_\bullet, S)$, obtained from cap product with the fundamental cycle, is determined already by such a relative fundamental cycle, the induced map f_* intertwines the two duality operators in the sense of [9, Definition 4.1]. Therefore [9, Theorem 4.3] implies the assertion. \square

5.2 Proof of Theorem 2.12

Theorem 2.12 is an almost-immediate corollary of Theorem 5.4 above. We just need the following supplementary computation:

5.5 Lemma *Assume that W is noncompact. The kernel of the homomorphism*

$$K_*(C_{\tilde{W}, \pi}^*(S, S)) \rightarrow K_*(C_{\tilde{W}, \pi, \text{evtl}}^*(S, S))$$

induced from the quotient map from $C_{\tilde{W}, \pi}^(S, S)$ to $C_{\tilde{W}, \pi, \text{evtl}}^*(S, S)$ is included in the kernel of the homomorphism*

$$K_*(C_{\tilde{W}, \pi}^*(S, S)) \rightarrow K_*(C_\pi^*(\tilde{W})).$$

Proof Since W is noncompact it contains a ray R that goes to infinity. Let $\tilde{R} \subset \tilde{W}$ be its inverse image under the projection $\tilde{W} \rightarrow W$. We have the ideal $C_\pi^*(\tilde{R} \subset \tilde{W})$ generated by operators supported in bounded neighborhoods of \tilde{R} . Because every

π -compactly supported operator is also supported in a bounded neighborhood of \tilde{R} we get a commutative diagram

$$(5-2) \quad \begin{array}{ccccc} C_{\tilde{W},\pi,\text{cpt}}^*(S, S) & \longrightarrow & C_{\tilde{W},\pi}^*(S, S) & \longrightarrow & C_{\tilde{W},\pi,\text{evtl}}^*(S, S) \\ & & \downarrow & & \\ C_{\pi}^*(\tilde{R} \subset \tilde{W}) & \longrightarrow & C_{\pi}^*(\tilde{W}) & & \end{array}$$

We have a canonical isomorphism $K_*(C_{\pi}^*(\tilde{R})) \xrightarrow{\cong} K_*(C_{\pi}^*(\tilde{R} \subset \tilde{W}))$ — see [26, Proposition 2.9] — and a standard Eilenberg swindle shows that

$$(5-3) \quad K_*(C_{\pi}^*(\tilde{R} \subset \tilde{W})) = 0.$$

See [12, Section 7, Proposition 1] or [26, Proposition 2.6]. Now we apply K -theory to the diagram (5-2). The long exact K -theory sequence of the first line of (5-2) shows that the kernel we have to consider is equal to the image of the map induced by the first arrow of this line. Naturality and commutativity imply by (5-3) that this image is mapped to zero in $K_*(C_{\pi}^*(\tilde{W}))$, as claimed. □

Proof of Theorem 2.12 If W is compact, then so is Z , and the transverse signatures of both are zero. Otherwise, from Theorem 5.4 we have that

$$\text{Sgn}_{\pi,\text{evtl}}(\tilde{W}/W) = \text{Sgn}_{\pi,\text{evtl}}(\tilde{Z}/Z) \in K_{d+1}(C_{\tilde{W},\pi,\text{evtl}}^*(S, S)),$$

and from Lemma 5.5 we conclude from this that

$$\text{Sgn}_{\pi}(\tilde{W}/W) = \text{Sgn}_{\pi}(\tilde{Z}/Z) \in K_{d+1}(C_{\pi}^*(\tilde{W})).$$

The theorem now follows from Theorem 2.8, the appropriate version of Roe’s partitioned manifold index theorem. □

5.6 Remark The proof of the positive scalar curvature result in [7] directly manipulates the spinor Dirac operator in situations where good spectral control is available only outside a compact set. This is parallel to our treatment of *eventual* homotopy equivalences. There is a large body of work that studies the homotopy invariance of signatures, and consequences, using signature operator methods rather than triangulations. For example the results of [9; 10] are analyzed from this point of view in [20], using methods in part borrowed from [13; 27]. It seems to be a challenge to prove Theorem 1.1, and in particular Theorem 2.12, using signature operator methods. But it might be useful to do so in order to study secondary invariants like ρ -invariants in the situation of Theorem 1.1.

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