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Parametrized spectra, multiplicative Thom spectra and the twisted Umkehr map

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We introduce a general theory of parametrized objects in the setting of ∞–categories. Although parametrised spaces and spectra are the most familiar examples, we establish our theory in the generality of families of objects of a presentable ∞–category parametrized over objects of an ∞–topos. We obtain a coherent functor formalism describing the relationship of the various adjoint functors associated to base-change and symmetric monoidal structures.

Our main applications are to the study of generalized Thom spectra. We obtain fiberwise constructions of twisted Umkehr maps for twisted generalized cohomology theories using a geometric fiberwise construction of Atiyah duality. In order to characterize the algebraic structures on generalized Thom spectra and twisted (co)homology, we express the generalized Thom spectrum as a categorification of the well-known adjunction between units and group rings.

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1 Introduction

In recent work with Hopkins and Rezk [3; 4], we introduced an ∞–categorical approach to parametrized spaces and spectra and showed that it provides a useful context in which to study Thom spectra and orientations. If X is a Kan complex and Sp is the ∞–category of spectra, then our model for the ∞–category of spectra parametrized by X is simply the ∞–category $\mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Sp})$ of presheaves of spectra on X, also known as the ∞–category $\mathrm{Loc}_X(\mathrm{Sp})$ of spectral-valued local systems on X. Conceptually, this approach exhibits the ∞–category Sp as the "classifying space" for bundles of spectra. In this paper, we develop this idea to give a complete theory of ∞–categories $\mathcal{C}$ parametrized over objects of an arbitrary ∞–topos, and we apply this theory to study the multiplicative properties of Thom spectra and the construction of twisted Umkehr maps.

The ∞–categorical perspective on parametrized objects is an elaboration of the modern perspective on parametrized homotopy theory (explored by Hu [32] and beautifully expounded upon by May and Sigurdsson [41]) that is based on an analogy between categories of spaces parametrized over a base space and derived categories of sheaves over a base scheme. In the context of algebraic geometry, associated to morphisms of the base scheme are collections of induced derived functors which assemble into adjoint pairs satisfying various intricate relationships. This data is organized into what is often referred to as Grothendieck's six-functor formalism, and is an essential foundation of modern work in algebraic geometry, particularly in the context of duality phenomena. We view the development of base-change functors as the basic foundational task when setting up a theory of parametrized objects.

A serious difficulty that classically arises in this context is problem of establishing the coherence of the diagrams given by this structure. Famously, the painstaking work of Conrad [17] handles some of the issues in Hartshorne's work [29] essentially by hand. A start on resolving coherence problems in the motivic context was given by Voevodsky using his formalism of cross-functors [48]. Voevodsky explains that coherence can be handled either via fibered categories (eg the Grothendieck construction) or using a good theory of 2–functors. Following these outlines, a great deal of hard work in the motivic context has developed this coherence theory; see Ayoub [9] and Cisinski and Déglise [16]. In the case of parametrized spectra, efforts in this direction can be found in May and Sigurdsson [41, Sections 13 and 17].

One basic point of departure for this paper is the observation that solutions to the kind of coherence problems which arise in these sorts of situations are precisely the sorts of issues that the ∞–categorical formalism is well placed to resolve. Specifically, ∞–functors are a natural generalization of 2–functors, and at the heart of Lurie's treatment of quasicategories is a generalization of the Grothendieck construction. Lurie's approach depends on a correspondence from functors from a ∞–category $\mathcal{C}$ into the ∞–category Cat_∞ of ∞–categories to ∞–categories fibered over $\mathcal{C}$. From this perspective, the right way to describe these functor formalisms is in terms of sheaves valued in the ∞–category of symmetric monoidal presentable ∞–categories and symmetric monoidal functors which admit *both* left and right adjoints. Whereas the algebrogeometric context is very difficult to study, the construction of the functor formalism and coherence is very straightforward in the topological setting. More generally, when studying parametrized objects over arbitrary ∞–topoi one can take advantage of the fact that ∞–topoi are accessible left exact localizations of ∞–categories that are freely generated under colimits.

1.1 Objects parametrized over ∞–topoi

Since our intended applications are primarily topological and differential-geometric in nature, we take as our motivating example the case of objects parametrized over the ∞–category of spaces; we will focus on this case in the introduction. However, in the body of the paper we state our results in terms of an arbitrary ∞–topos, equipped with the cartesian symmetric monoidal structure. Relevant examples of ∞–topoi (other than spaces) include spaces with the action of a topological group G, presheaves on the orbit ∞–category $\mathrm{Pre}(\mathrm{Orb}_G)$, or sheaves of spaces on a Grothendieck site (such as the site associated to a topological space).

Let $\mathcal{S}$ denote the ∞–category of spaces. Since $\mathcal{S}$ is freely generated under colimits by its final object (the point), for any ∞–category $\mathcal{M}$ with small limits the ∞–category of limit-preserving functors $\mathcal{S}^{\mathrm{op}} \to \mathcal{M}$ is equivalent (via evaluation at the point) to $\mathcal{M}$ itself. If $\mathcal{C}$ is an object of $\mathcal{M}$, then we will write

$$\mathcal{C}_{/(-)}\colon \mathcal{S}^{\mathrm{op}} \to \mathcal{M}$$

for the corresponding functor under this equivalence. Now, to say that $\mathcal{C}_{/(-)}$ preserves limits in $\mathcal{S}^{\mathrm{op}}$ is to say that it satisfies descent in $\mathcal{S}$, and so we call such a functor a *sheaf* on $\mathcal{S}$ with values in $\mathcal{M}$. The main case of interest is when $\mathcal{M}$ is the ∞–category

$\widehat{\mathrm{Cat}}_\infty$ of (not necessarily small) ∞–categories itself, in which case any ∞–category $\mathcal{C}$ uniquely determines, and is determined by, a sheaf

$$\mathcal{C}_{/(-)}\colon \mathcal{S}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$$

of ∞–categories on $\mathcal{S}$.

If $f\colon S \to T$ is a map of spaces, we write f^* for the induced functor $\mathcal{C}_{/T} \to \mathcal{C}_{/S}$. We now restrict attention to *presentable* ∞–categories $\mathcal{C}$, essentially without loss of generality: if $\mathcal{C}$ is any ∞–category, then $\mathcal{C}$ embeds fully faithfully into $\mathrm{Pre}(\mathcal{C})$, which is presentable. (Although note that making this precise involves set-theoretic technicalities.) Then for formal reasons f^* has a left adjoint $f_!$ and a right adjoint f_*. In certain cases, there is even a further right adjoint $f^!$ of f_*, for instance when f is proper in the sense that its homotopy fibers are compact. Notably, this hypothesis holds in the case of a smooth family of compact manifolds $S \to T$, a central example in the study of twisted Umkehr maps.

Additionally, many examples of interest are equipped with multiplicative structure on $\mathcal{C}$. If $\mathcal{C}^\otimes$ is a presentable symmetric monoidal ∞–category then $\mathcal{C}^\otimes_{/S}$ has a tensor bifunctor which commutes with colimits in each variable. For each object $X \in \mathcal{C}_{/S}$, the "left multiplication by X" functor

$$X \otimes_S (-)\colon \mathcal{C}_{/S} \to \mathcal{C}_{/S}$$

admits a right adjoint

$$F_S(X,-)\colon \mathcal{C}_{/S} \to \mathcal{C}_{/S},$$

and we assemble all of this structure in the following omnibus theorem:

Theorem 1.1 *A presentable symmetric monoidal ∞–category $\mathcal{C}^\otimes$ uniquely determines, and is determined by, a sheaf of presentable symmetric monoidal ∞–categories*

$$\mathcal{C}^\otimes_{/(-)}\colon \mathcal{S}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

together with left adjoints $f_!$ to the restrictions f^* for arbitrary maps of spaces $f\colon S \to T$, and right adjoints $f^!$ to the pushforwards f_* for proper maps of spaces $f\colon S \to T$, satisfying certain coherences and relations detailed in Section 3. Moreover, for any space S, $\mathcal{C}^\otimes_{/S}$ is equivalent to the symmetric monoidal ∞–category $\mathrm{Fun}(S^{\mathrm{op}},\mathcal{C})^\otimes$ of $\mathcal{C}^\otimes$–valued presheaves on S.

There are versions of this theorem that hold with spaces replaced by an arbitrary ∞–topos and $\mathcal{C}^\otimes$ a presentable $\mathcal{O}^\otimes$–monoidal ∞–category for an ∞–operad $\mathcal{O}^\otimes$ equipped with a fixed map $\mathbb{E}_1^\otimes \to \mathcal{O}^\otimes$. (See Theorems 5.10 and 6.4 for the precise statements.)

In Appendix B, we show that when restricted to the case of parametrized spectra, Theorem 1.1 generalizes the homotopical structure underlying the theory of parametrized spectra in May and Sigurdsson [41]. Moreover, there are distinct advantages to the ∞–categorical context when dealing with multiplicative structures; these were not handled in full generality in [41] due to the ferocious point-set technical difficulties.

1.2 The twisted Umkehr map and multiplicative Thom spectra

One of our primary motivations for this treatment of parametrized homotopy theory is to characterize the multiplicative properties of the Thom spectrum functor. We explain our foundational results in this direction below, but we now turn to describe the most interesting application, the construction of twisted Umkehr maps.

We begin by recalling the construction of the Thom spectrum functor in our framework. Let R be an $\mathbb{E}_n$–ring spectrum, and let Mod_R be the ∞–category of right R–modules. Within Mod_R is the full subgroupoid spanned by the invertible R–modules, Pic_R. Given a space X and a map $f\colon X \to \mathrm{Pic}_R$, in Ando, Blumberg, Gepner, Hopkins and Rezk [3; 4; 2] we defined the Thom spectrum of f to be the colimit Mf of the composite map

$$X \xrightarrow{f} \mathrm{Pic}_R \to \mathrm{Mod}_R .$$

Regarding such a map as classifying a twisted form of the trivial R–line bundle over X, we can consider the associated R–module Thom spectrum Mf to be the f–twisted and R–stable homotopy type of X, and define twisted homology and cohomology accordingly.

Definition 1.2 Let R be an $\mathbb{E}_n$–ring spectrum with $n > 1$ and let $\alpha\colon X \to \mathrm{Pic}_R$ be a map. The *α–twisted R–homology and R–cohomology groups* of X are given by

$$R_\alpha(X) = \pi_0 \operatorname{map}_R(R, M\alpha) \cong \pi_0 M\alpha,$$
$$R^\alpha(X) = \pi_0 \operatorname{map}_R(M(-\alpha), R).$$

Here $-\alpha$ denotes the inverse of α in the grouplike $\mathbb{E}_{n-1}$–space Pic_R (ie the involution given by taking an invertible R–module M to its R–linear dual $D_R M$) and $\operatorname{map}_R(-,-)$ denotes the mapping space in the ∞–category Mod_R of right R–modules.

Note that this differs in two ways from the convention used in [4; 2]. First, Pic_R need not decompose as $\mathbb{Z} \times B\mathrm{GL}_1(R)$, so it is potentially problematic to specify maps $f\colon X \to \mathrm{Pic}_R$ in terms of maps $\alpha\colon X \to B\mathrm{GL}_1(R)$ and $n\colon X \to \mathbb{Z}$; moreover, even if $\pi_0 \operatorname{Pic}_R \cong \mathbb{Z}$, the induced map $\operatorname{Pic}_R \to \mathbb{Z}$ does not necessarily admit a splitting as grouplike $\mathbb{E}_{n-1}$–spaces. Second, in keeping with the usual convention in ordinary homology and cohomology that $R_n(*) \cong R^{-n}(*)$, as well the convention that twisted cohomology should be the space of sections of the associated bundle of spectra, it is necessary to dualize the twist before taking the Thom spectrum.

Nevertheless, given a invertible bundle of R–modules $f\colon X^{\mathrm{op}} \to \operatorname{Pic}_R$, which we view via the inclusion $\operatorname{Pic}_R \to \operatorname{Mod}_R$ as an object of the stable ∞–category $\operatorname{Fun}(X^{\mathrm{op}}, \operatorname{Mod}_R)$ of bundles of R–modules over X, we write

$$R^{n+f}(X) = R^{\Sigma^n f}(X)$$

for the twisted cohomology of X with respect to the suspended twist

$$\Sigma^n f \in \operatorname{Fun}(X^{\mathrm{op}}, \operatorname{Mod}_R).$$

Note that since X is an ∞–groupoid, there is an equivalence $X \simeq X^{\mathrm{op}}$, so that we may (and sometime will) regard twists as covariant functors $X \to \operatorname{Pic}_R$.

In fact, in some situations it is useful to have a generalization of the notion of Thom spectrum in which "noninvertible" twists are also allowed. By a noninvertible twist we mean an arbitrary bundle of module spectra. Most notably, such noninvertible twists arise naturally from the geometry in Stolz and Teichner's program [46] for relating twisted field theories to twisted cohomology theories; a main motivating example is Witten's work [51] on Chern–Simons theory. The bulk of our work extends to this more general setting; we discuss this is in detail in Section 3.2.

We now want to construct Umkehr maps for twisted cohomology theories. We begin by recalling how this works in the untwisted case. For convenience, we switch to using exponential notation for Thom spectra; eg given a twist $\alpha\colon X \to \operatorname{Pic}_R$, the Thom spectrum will be written X^{α}. Now let X be a compact manifold with tangent bundle T. The Pontryagin–Thom construction gives a stable map

$$\mathrm{PT}(X)\colon \mathbb{S} \to X^{-T} \simeq DX$$

dual to the map $X \to *$. If $f\colon X \to B$ is a fiber bundle of d–dimensional compact manifolds with tangent bundle along the fiber T_f, then this construction generalizes to give a stable map

$$\mathrm{PT}(f)\colon B_+ \to X^{-T_f}.$$

For a ring spectrum R, there is a map in cohomology

$$R^*(X^{-T_f}) \to R^*(B_+)$$

and composing with a Thom isomorphism $R^{*+d}X \cong R^*(X^{-T_f})$ induced from an orientation, we obtain an *Umkehr* map

$$R^{*+d}(X) \to R^*(B).$$

Recently it has become important in a number of contexts to consider twisted generalizations of these constructions (see for example Carey and Wang [15], Freed and Witten [24] and Wang [49]). We explain how to provide twisted Umkehr maps for any sufficiently multiplicative generalized cohomology theory. The basic strategy is as follows. Composing, a twist $\alpha\colon B \to \mathrm{Pic}_R$ gives rise to a twist

$$X \xrightarrow{\alpha f} \mathrm{Pic}_R .$$

If R is an E_n–ring spectrum, the category of twists is an E_{n-1}–monoidal category and we show that the Thom spectrum functor applied to the twist lands in E_{n-1}–ring spectra. In particular, we can make sense of the generalized R–module Thom spectrum $X^{-T_f+\alpha f}$. Provided we can construct a twisted Pontryagin–Thom transfer map

$$\mathrm{PT}(f,\alpha)\colon B^{\alpha} \to X^{-T_f+\alpha f},$$

a Thom isomorphism $R^{*+d}(X) \cong R^*(X^{-T_f+\alpha f})$ then induces the twisted Umkehr map

$$R^{*+d}(X) \to R^*(B^{\alpha}) \cong R^{*-\alpha}(B).$$

The key idea of the construction is to show that the Pontryagin–Thom map can be realized as the pushforward of a fiberwise map

$$\mathrm{PT}(f_{/B})\colon \mathbb{S}_B \to D(f_{/B}),$$

along the map $p\colon B \to *$ to obtain the map $\mathrm{PT}(f)$. (Here $\mathbb{S}_B$ denotes the sphere spectrum over B and D denotes the fiberwise functional dual.) In order to use this description, we also need to be able to provide a geometric interpretation of $D(f_{/B})$ in various cases, notably for smooth families of compact manifolds. This is surprisingly difficult from a purely homotopical viewpoint, as it involves grappling with the functoriality of the Atiyah duality map in order to construct a parametrized version. Our approach involves ideas related to Hu's study [32] of the dualizing complex in the setting of parametrized stable homotopy theory.

Given such a fiberwise Pontryagin–Thom map, we can twist by α via fiberwise smashing to get the map

$$\mathrm{PT}(f_{/B}) \wedge_B \alpha\colon \mathbb{S}_B \wedge_B \alpha \to D(f_{/B}) \wedge_B \alpha$$

of R–module spectra over B. Applying the Thom spectrum functor, ie the pushforward $p_!$ associated to $p\colon B \to *$, now yields the map $\mathrm{PT}(f,\alpha)$ as well as in many cases a geometric description of the target.

Remark 1.3 The basic idea that in geometric circumstances the Pontryagin–Thom map arises from a fiberwise construction goes all the way back to the origins of the classical Umkehr map, as can be seen in the "families" index theorems of Atiyah and Singer [8]. This idea is also explicit in Becker and Gottlieb's classic paper [11], for example. May and Sigurdsson have a beautiful exposition of a geometric fiberwise construction in the setting of a "bundle theory" for parametrized spectra in [41].

1.3 Categorification of the Picard group and multiplicative Thom spectra

We now explain applications of our work on parametrized objects to foundational results on the multiplicative structure of the Thom spectrum functor. The monoidal structure we have studied so far on $\mathcal{C}^{\otimes}_{/S}$ in Theorem 1.1 is pointwise on S, ie induced from the diagonal map $S \to S \times S$. For the applications to Thom spectra, we will develop a multiplicative theory of objects parametrized over *monoidal* spaces, where the monoidal product of parametrized objects involves the monoidal product on the base.

Our approach involves a categorification of the notion of Picard group, classically defined as the group of invertible objects in a symmetric monoidal category. To explain the strategy for categorification, recall that in algebra the units functor GL_1 arises from the free/forgetful adjunction

$$\mathbb{Z}[-]\colon (\text{monoids}) \to \mathrm{Alg}(\mathrm{Mod}_{\mathbb{Z}}).$$

The restriction of this to $\mathbb{Z}[-]\colon (\text{groups}) \to \mathrm{Alg}(\mathrm{Mod}_{\mathbb{Z}})$ is then the left adjoint of the units functor GL_1. To produce our version of the Picard group, we will categorify this adjunction, as follows. (Note that, as will become evident, the natural categorification of the units is the Picard space, not the classical space of units of a ring spectrum.)

Fix a suitable ∞–operad $\mathcal{O}$ (eg we require that there exist a map $\eta\colon \mathbb{E}_1 \to \mathcal{O}$). If S is an $\mathcal{O}$–algebra, consider the *covariant* functor

$$\mathrm{Pre}\colon \mathcal{S} \to \mathrm{Pr}^{\mathrm{L}}$$

whose value at S is $\mathrm{Pre}(S) = \mathrm{Fun}(S^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{S}_{/S}$ and which takes $f\colon S \to T$ to the left adjoint $f_!$. This functor extends to a symmetric monoidal functor

$$\mathrm{Pre}\colon \mathcal{S}^{\otimes} \to (\mathrm{Pr}^{\mathrm{L}})^{\otimes},$$

and so induces a functor

$$\mathrm{Pre}\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{S}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L}}),$$

where here $\mathrm{Alg}_{\mathcal{O}}(-)$ denotes the ∞–category of $\mathcal{O}$–algebras. Since $\mathcal{S}$ is the unit of the symmetric monoidal structure on Pr^{L}, it is appropriate to think of Pr^{L} as the ∞–category $\mathrm{Mod}_{\mathcal{S}}$ of $\mathcal{S}$–modules, and of Pre as the free $\mathcal{S}$–module functor, analogous to the free $\mathbb{Z}$–module functor.

Let $\mathrm{Alg}^{\mathrm{gp}}_{\mathcal{O}}(\mathcal{S})$ be the full subcategory of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{S})$ on the grouplike algebra objects. We construct an analogous right adjoint Pic for the composite functor

$$\mathrm{Pre}\colon \mathrm{Alg}^{\mathrm{gp}}_{\mathcal{O}}(\mathcal{S}) \hookrightarrow \mathrm{Alg}_{\mathcal{O}}(\mathcal{S}) \xrightarrow{\mathrm{Pre}} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L}}).$$

Definition 1.4 Let $\mathcal{O}^{\otimes}$ be an ∞–operad equipped with a map from $\mathbb{E}_1^{\otimes}$ and let $\mathcal{R}^{\otimes}$ be an $\mathcal{O}$–monoidal ∞–category. Define $\mathrm{Pic}(\mathcal{R})$ to be the maximal grouplike ∞–groupoid in the $\mathcal{O}$–monoidal ∞–category of invertible objects of $\mathcal{R}^{\otimes}$.

The categorified Picard group describes the right adjoint to Pre.

Theorem 1.5 *The Picard ∞–groupoid defines a functor*

$$\mathrm{Pic}\colon \mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Alg}^{\mathrm{gp}}_{\mathcal{O}}(\mathcal{S})$$

that is right adjoint to the free $\mathcal{O}$–monoidal $\mathcal{S}$–module functor Pre*.*

Lurie [37, 6.3.5.17] has proved a conjecture of Mandell which implies that for $n > 1$, $\mathbb{E}_n$–algebras admit $\mathbb{E}_{n-1}$–monoidal module categories. Applying Theorem 1.5 in the context of $\mathcal{R}^{\otimes} = \mathrm{Mod}_R^{\otimes}$ for an $\mathbb{E}_n$–ring spectrum R with $n > 1$ — in which case $\mathrm{Pic}(\mathrm{Mod}_R^{\otimes}) = \mathrm{Pic}_R$ — now leads to a multiplicative characterization of the Thom spectrum functor in terms of the categorification of Pic. Recall that the generalized Thom spectrum functor $\mathcal{S}_{/\mathrm{Pic}_R} \to \mathrm{Mod}_R$ is defined on objects by assigning the pushforward $p_! f$ to $f\colon X \to \mathrm{Pic}_R$, where $p\colon X \to *$ is the terminal map.

Theorem 1.6 *The functor of $\mathbb{E}_{n-1}$–monoidal presentable ∞–categories*

$$\mathcal{S}_{/\mathrm{Pic}_R} \to \mathrm{Mod}_R,$$

arising from the counit of the adjunction of Theorem 1.5, coincides with the generalized Thom spectrum functor (after forgetting the multiplicative structure).

An immediate corollary is the following generalization of Lewis's theorem about multiplicative structures on Thom spectra:

Theorem 1.7 *Let R be an $\mathbb{E}_n$–ring spectrum, with $n > 1$. Then Pic_R is an $\mathbb{E}_{n-1}$–space, and if $f\colon X \to \mathrm{Pic}_R$ is $\mathbb{E}_m$–monoidal for some $m < n$, then the Thom spectrum Mf has the structure of an $\mathbb{E}_m$–ring spectrum.*

We also derive a characterization of the multiplicative properties of the Thom isomorphism. Lewis showed that an $\mathbb{E}_n$–orientation gives rise to an $\mathbb{E}_n$ Thom isomorphism [35, IX.7.4]. We generalize Lewis's result as follows:

Corollary 1.8 *Let R be an $\mathbb{E}_n$–ring spectrum with $n > 1$ and let $f\colon X \to \mathrm{GL}_1 R$ be an $\mathbb{E}_m$–monoidal map for some $m < n$. Suppose that Mf admits an $\mathbb{E}_m$–orientation over a spectrum R, ie an $\mathbb{E}_m$–algebra map $Mf \to R$. Then the composite of the Thom diagonal and the orientation*

$$Mf \to \Sigma^\infty_+ X \wedge Mf \to \Sigma^\infty_+ X \wedge R$$

is an equivalence of $\mathbb{E}_m$–ring spectra.

The categorification of the Picard group can itself be categorified to produce a description of the Brauer group; we give a sketch of this theory as well as its applications in "twisted parametrized homotopy theory" [18] in Appendix A.

1.4 Parametrized homotopy theory and the tangent bundle

Finally, we note that, from another point of view, underlying the notion of parametrized homotopy theory is Lurie's notion of the tangent bundle $p\colon T_{\mathcal{C}} \to \mathcal{C}$ of a presentable ∞–category $\mathcal{C}$ [37]. The fiber of $p\colon T_{\mathcal{C}} \to \mathcal{C}$ over the object S of $\mathcal{C}$ is the stabilization of the slice $\mathcal{C}_{/S}$ over S. In the topological context, ie when $\mathcal{C}$ is the ∞–category of spaces, $\mathcal{C}_{/S}$ is the ∞–category of spaces parametrized over S and $\mathrm{Stab}(\mathcal{C}_{/S})$ is the ∞–category of spectra parametrized over S. A map $f\colon S \to T$ induces restriction maps

$$f^*\colon \mathrm{Stab}(\mathcal{C}_{/T}) \to \mathrm{Stab}(\mathcal{C}_{/S})$$

which admit both left and right adjoints, namely left and right Kan extension, or induction and coinduction, written $f_!$ and f_*, respectively.

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2 Background on ∞–categories

In this section we give a very brief overview of our use of the framework of $(\infty, 1)$–categories. There are now many well-studied models for ∞–categories, including Rezk's Segal spaces [44], the Segal categories [30; 47] of Simpson and Tamsamani, the "quasicategories" (weak Kan complexes) of Boardman and Vogt and the homotopy theory of simplicial categories as studied by Dwyer and Kan [21] and Bergner [12]. Many models are known to be equivalent (see [13] for a nice discussion of the situation). Very little of the work of this paper depends on model-specific details; given certain basic structural properties, one could carry out most of our arguments in any model. We have chosen to use quasicategories as a model for ∞–categories, as developed by Joyal [34] and Lurie [36; 37], in large part because of Lurie's extensive foundational work on multiplicative structures.

2.1 ∞–Operads and symmetric monoidal ∞–categories

We now quickly review the theory of ∞–operads as we will apply it in the body of the paper, following [37, Section 2]. Let Γ denote the category with objects the pointed sets $\{*, 1, 2, \dots, n\}$ for each natural number $n \in \mathbb{N}$ and morphisms the pointed maps of sets. An ∞–operad is then specified by an ∞–category $\mathcal{O}^{\otimes}$ and a functor

$$p\colon \mathcal{O}^{\otimes} \to \mathrm{N}(\Gamma)$$

satisfying certain conditions [37, 2.1.1.10]. Here N denotes the homotopy coherent nerve functor, which in this case coincides with the usual nerve.

The definition of an ∞–operad is the generalization of the notion of a multicategory (colored operad). In fact, there is a general correspondence result which associates to a simplicial multicategory an operadic nerve which is an ∞–operad provided that each morphism simplicial set of the multicategory is a Kan complex [37, 2.1.1.27]. To obtain the generalization of a classical operad we restrict to ∞–operads equipped with an essentially surjective functor $\Delta^0 \to p^{-1}(\{*, 1\})$. To make sense of this, note that $p^{-1}(\{*, 1\})$ should be thought of as the "underlying" ∞–category $\mathcal{O}$ associated to $\mathcal{O}^{\otimes}$, which should contain only a single (equivalence class of an) object for the ∞–version of an ordinary operad.

The identity map $\mathrm{N}(\Gamma) \to \mathrm{N}(\Gamma)$ is an ∞–operad; this is the analogue of the $\mathbb{E}_\infty$–operad. More generally, we can define a topological category $\widetilde{\mathbb{E}}[k]$ [37, 5.1.0.2] such that there is a natural functor $\mathrm{N}(\widetilde{\mathbb{E}}[k]) \to \mathrm{N}(\Gamma)$ which is an ∞–operad. We refer to the resulting ∞–operads as the $\mathbb{E}_k$–operads, as there is an equivalence between $\mathbb{E}_k$ and the nerve of the classical little k–cubes operad. These operads are equipped with natural maps $\mathbb{E}_m \to \mathbb{E}_n$ for $m \leq n$.

A *symmetric monoidal* ∞–category is then an ∞–category $\mathcal{C}^{\otimes}$ equipped with a cocartesian fibration of ∞–operads [37, 2.1.2.18]

$$p\colon \mathcal{C}^{\otimes} \to \mathrm{N}(\Gamma).$$

The "underlying" ∞–category is obtained as the fiber $\mathcal{C} = p^{-1}(\{*, 1\})$. In abuse of terminology, we will say that an ∞–category $\mathcal{C}$ is a symmetric monoidal ∞–category if it is equivalent to $p^{-1}(\{*, 1\})$ for some symmetric monoidal ∞–category $\mathcal{C}^{\otimes}$. More generally, if $\mathcal{O}^{\otimes}$ is an ∞–operad and $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a cocartesian fibration of ∞–operads such that the composite

$$\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to \mathrm{N}(\Gamma)$$

exhibits $\mathcal{C}^{\otimes}$ as an ∞–operad [37, 2.1.2.13], then $\mathcal{C}$ is an $\mathcal{O}$–monoidal ∞–category.

Given a symmetric monoidal model category $\mathcal{C}$, we can associate to it a symmetric monoidal ∞–category $\mathrm{N}(\mathcal{C}^c)[W^{-1}]^{\otimes}$ with underlying ∞–category $\mathrm{N}(\mathcal{C}^c)[W^{-1}]$ [37, 4.1.3.6]. (See Appendix B for further discussion of the passage from model categories to ∞–categories.)

For symmetric monoidal ∞–categories $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$, we have two associated categories of functors between them:

(1) the ∞–category of ∞–operad maps $\mathrm{Alg}_{\mathcal{C}}(\mathcal{D})$, which is the analogue of lax symmetric monoidal functors [37, 2.1.2.7], and

(2) the ∞–category $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ of symmetric monoidal functors, which is the analogue of strong symmetric monoidal functors [37, 2.1.3.7].

We now turn to the description of operadic algebras in an $\mathcal{O}$–monoidal ∞–category. For a fibration $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ of ∞–operads and a map of ∞–operads $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$, we define an $\mathcal{O}'$–algebra object of $\mathcal{C}$ over $\mathcal{O}$ to be a map of ∞–operads $A\colon \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ over $\mathcal{O}$ such that $q \circ A$ is α [37, 2.1.3.1]. The ∞–category of $\mathcal{O}'$–algebra objects in $\mathcal{C}$ over $\mathcal{O}$, denoted by $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$, is the full subcategory of the functor category $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$ spanned by the maps of ∞–operads. Several special cases are worth noting, as they arise most frequently in practice:

(1) When $\mathcal{O}$ is the commutative ∞–operad, $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$; that is, this case covers the situation of $\mathcal{O}'$–algebras in a symmetric monoidal ∞–category.

(2) When α is the identity map we write $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ for $\mathrm{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C})$. When in addition $\mathcal{O}$ is the commutative ∞–operad, we will write $\mathrm{CAlg}(\mathcal{C})$ to denote the category $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ — these are the commutative algebra objects in $\mathcal{C}$.

A particularly interesting class of symmetric monoidal structures on ∞–categories comes from *cartesian monoidal structures*. Any ∞–category with finite products admits a unique cartesian symmetric monoidal structure; the monoidal product is given by the categorical product [37, Section 2.4.1]. When $\mathcal{C}$ is a cartesian symmetric monoidal ∞–category and $\mathcal{O}$ is an ∞–operad, we often write $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ in place of $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ [37, Section 2.4.2].

Finally, we will primarily be interested in algebras over ∞–operads which are *unital* and *coherent*. A unital ∞–operad [37, 2.3.1.1] has an essentially unique "nullary" operation. For instance, the $\mathbb{E}_n$–operads are unital for any n. As one might expect, algebras over unital operads are equipped with well-behaved unit maps. Coherent ∞–operads $\mathcal{O}^{\otimes}$ [37, Section 3.3.1] satisfy technical conditions that ensure that the categories of modules over $\mathcal{O}$–algebras admit suitably monoidal products. Once again, the $\mathbb{E}_n$–operads are coherent.

2.2 The ∞–category of presentable ∞–categories

In this section, we quickly review the ∞–category of presentable ∞–categories and introduce some notation. Let Cat_{∞} denote the ∞–category of ∞–categories. An ∞–category $\mathcal{C}$ is presentable if there exists a regular cardinal κ and a small ∞–category $\mathcal{D}$

with κ–small colimits such that $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{D})$; ie $\mathcal{C}$ is the free completion of $\mathcal{D}$ under κ–filtered colimits [36, 5.5.1.1]. The theory of presentable ∞–categories is an analogue of the theory of combinatorial model categories.

We will be working with functors to the ∞–category of presentable ∞–categories. Recall from [37, Section 6.3.1] that the ∞–category Pr^{L} of presentable ∞–categories and left-adjoint functors is complete and cocomplete, with limits created in Cat_∞. The ∞–category Pr^{L} is closed, as the ∞–category of functors $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ is itself a presentable ∞–category [36, 5.5.3.8]. Furthermore, Pr^{L} is tensored over spaces, with a convenient description of the tensor: given a presentable ∞–category $\mathcal{C}$ and a space S, the presentable ∞–category $S \otimes \mathcal{C}$ is naturally equivalent to $\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{C})$.

In addition, Pr^{L} can be given the structure of a symmetric monoidal ∞–category, with product that we will denote by $\otimes$ and with unit $\mathcal{S}$, the ∞–category of spaces. The fact that $\mathcal{S}$ is the unit implies that it is canonically a commutative algebra object and that the forgetful functor

$$\mathrm{Mod}_{\mathcal{S}}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Pr}^{\mathrm{L}}$$

is an equivalence, where here $\mathrm{Mod}_{\mathcal{S}}(\mathrm{Pr}^{\mathrm{L}}) = \mathrm{Mod}_{\mathcal{S}}^{\mathrm{Comm}}(\mathrm{Pr}^{\mathrm{L}})$ is the ∞–category of modules over $\mathcal{S}$ in the symmetric monoidal structure on Pr^{L}.

The ∞–category Pr^{L} has as a subcategory the ∞–category $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ of stable presentable ∞–categories and colimit-preserving functors. Recall that the ∞–category $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$ of stable presentable ∞–categories also admits a symmetric monoidal structure with product we will denote by $\otimes$ and unit the ∞–category Sp of spectra [37, Section 6.3.1]. There is a map of commutative algebra objects

$$\mathcal{S} \to \mathcal{S}_* \to \mathrm{Sp}$$

in Pr^{L}, where $\mathcal{S}_*$ is the ∞–category of pointed spaces. Both of these maps induce localizations in the sense that the change of scalar endofunctors $(-)\otimes_{\mathcal{S}}\mathcal{S}_*$ and $(-)\otimes_{\mathcal{S}}\mathrm{Sp}$ of $\mathrm{Mod}_{\mathcal{S}}$ are idempotent: clearly $\mathcal{S}_* \otimes_{\mathcal{S}} \mathcal{S}_* \simeq \mathcal{S}_*$, and the same is true for Sp since $\mathrm{Sp} \simeq \mathcal{S}_*[\Sigma^{-1}]$.

Given an arbitrary ∞–operad $\mathcal{O}^{\otimes}$, a (not necessarily small) $\mathcal{O}$–monoidal ∞–category is an object of $\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathrm{Cat}}_\infty)$, and an $\mathcal{O}$–monoidal presentable ∞–category is an object of $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L}})$. In particular, associated to an $\mathbb{E}_n$ algebra R in a symmetric monoidal ∞–category $\mathcal{C}$, there is an ∞–category Mod_R of right R–modules. A central theorem in the subject (verifying a conjecture of Mandell) is that an $\mathbb{E}_n$–algebra R induces an $\mathbb{E}_{n-1}$–monoidal structure on Mod_R with unit R such that the tensor product commutes with colimits in each variable; moreover, the functor $R \mapsto \mathrm{Mod}_R$ from $\mathbb{E}_n$–algebras to $\mathbb{E}_{n-1}$–monoidal ∞–categories is fully faithful [37, 6.3.5.17]. Furthermore, in an $\mathbb{E}_n$–monoidal ∞–category $\mathcal{C}$, for $m \le n$ the map of ∞–operads $\mathbb{E}_m \to \mathbb{E}_n$ implies that we have an $\mathbb{E}_{n-m}$–monoidal ∞–category $\mathrm{Alg}_{\mathbb{E}_m/\mathbb{E}_n}(\mathcal{C})$ of $\mathbb{E}_m$ algebra objects in $\mathcal{C}$.

3 Parametrized spaces and spectra

In this section, we review explicit ∞–categorical models of parametrized spaces, spectra and R–modules for an $\mathbb{E}_n$–ring spectrum R from [3; 2; 4]. Our main goal is to provide the necessary background for Section 4, in which we relate this formalism to geometric constructions in order to study the twisted Umkehr maps. In Section 5, we develop the general theory of arbitrary ∞–categories parametrized over arbitrary ∞–topoi, which specializes to the definitions and theorems given in this section.

We begin with the definition of parametrized spaces.

Definition 3.1 Let T be a Kan complex, which we view as an object of $\mathcal{S}$ (the ∞–category of ∞–groupoids). Define the ∞–category of spaces over S to be the ∞–category

$$\mathcal{S}_{/T} = \mathrm{Fun}(T^{\mathrm{op}}, \mathcal{S})$$

of presheaves of spaces on T, and the ∞–category of pointed spaces over T (or ex-spaces) as the ∞–category

$$(\mathcal{S}_{/T})_* = \mathrm{Fun}(T^{\mathrm{op}}, \mathcal{S})_* \simeq \mathrm{Fun}(T^{\mathrm{op}}, \mathcal{S}_*) = (\mathcal{S}_*)_{/T}$$

of presheaves of pointed spaces on S.

Remark 3.2 The ∞–category of ∞–categories admits an autoequivalence $(-)^{\mathrm{op}}$ which sends $\mathcal{C}$ to $\mathcal{C}^{\mathrm{op}}$, which when restricted to the full subcategory of ∞–groupoids comes equipped with a natural equivalence $(-)^{\mathrm{op}} \to \mathrm{id}$. In particular, any ∞–groupoid T comes equipped with a canonical involution $T^{\mathrm{op}} \simeq T$, so that $\mathrm{Fun}(T^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}(T, \mathcal{S})$. For this reason, we will sometime ignore the $(-)^{\mathrm{op}}$ and write $\mathcal{S}_{/T} \simeq \mathrm{Fun}(T, \mathcal{S})$, and more generally adopt a similar convention for presheaves on T valued in an arbitrary ∞–category $\mathcal{C}$, such as pointed spaces or spectra.

Definition 3.3 Let T be a Kan complex. The ∞–category of spectra over T is defined to be the ∞–category

$$\mathrm{Sp}_{/T} = \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Sp})$$

of presheaves of spectra on T.

In both of these cases, associated to a map $S \to T$ we have functors $f^*\colon \mathcal{S}_{/T} \to \mathcal{S}_{/S}$ and $f^*\colon \mathrm{Sp}_{/T} \to \mathrm{Sp}_{/S}$ such that there are adjoint pairs $(f_!, f^*)$ and (f^*, f_*). Moreover, there is a closed symmetric monoidal structure providing fiberwise smash products $\wedge_B$ and mapping spaces and spectra $F_B(-,-)$. As explained in Proposition 3.4 (see also Proposition 6.8), these functors satisfy the following formulas:

Proposition 3.4 *Let $\mathcal{C}$ denote either $\mathcal{S}$ or Sp and let $f\colon S \to T$ be a map of spaces, X an object of $\mathcal{C}_{/S}$ and Y and Z objects of $\mathcal{C}_{/T}$. Then there are natural equivalences*

(1) $f^*(Y \otimes_T Z) \simeq f^*Y \otimes_S f^*Z$,

(2) $F_T(Y, f_*X) \simeq f_* F_S(f^*Y, X)$,

(3) $f^* F_T(Y, Z) \simeq F_S(f^*Y, f^*Z)$,

(4) $f_!(f^*Y \otimes_S X) \simeq Y \otimes_T f_!X$,

(5) $F_T(f_!X, Y) \simeq f_* F_S(X, f^*Y)$.

The ∞–category $\mathrm{Sp}_{/T}$ of spectra parametrized over T can be understood as the stabilization of the ∞–category $\mathcal{S}_{/T}$ of spaces parametrized over S. Indeed, we have the following proposition:

Proposition 3.5 *Let T be a Kan complex. Then we have a natural equivalence*

$$\mathrm{Stab}(\mathcal{S}_{/T}) \simeq \mathrm{Sp}_{/T}$$

of symmetric monoidal stable presentable ∞–categories.

Proof First observe that Pr^{L} is tensored over the ∞–category of spaces via the formula $T \otimes \mathcal{C} \simeq \mathrm{Fun}(T^{\mathrm{op}}, \mathcal{C})$, and this tensor structure is compatible with the tensor product on Pr^{L} by Proposition 5.5 below (where we also recall the notion of the tensor product on Pr^{L}). Moreover, according to [37, 4.8.1.23], the stabilization endofunctor $\mathrm{Stab}\colon \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}$ is given by the formula $\mathrm{Stab}(\mathcal{C}) := \mathrm{Sp}(\mathcal{C}) \simeq \mathcal{C} \otimes \mathrm{Sp}$. As shown in [26, Section 3], this is a smashing localization of Pr^{L}, hence symmetric monoidal, with essential image precisely the full subcategory of Pr^{L} consisting of the stable presentable ∞–categories. We therefore obtain a chain of equivalences

$$\mathrm{Stab}(\mathcal{S}_{/T}) \simeq (T \otimes \mathcal{S}) \otimes \mathrm{Sp} \simeq T \otimes (\mathcal{S} \otimes \mathrm{Sp}) \simeq \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Sp}) \simeq \mathrm{Sp}_{/T}$$

of symmetric monoidal stable presentable ∞–categories. □

Remark 3.6 It is evident from the argument for the previous proposition that it extends to any presentable ∞–category $\mathcal{C}$ and its stabilization, and the resulting equivalence is compatible with $\mathcal{O}$–monoidal structures.

Finally, we describe parametrized module spectra.

Definition 3.7 Let R be an $\mathbb{E}_1$–ring spectrum, let T be a Kan complex and let Mod_R denote the stable presentable ∞–category of right R–modules. The ∞–category of parametrized R–module spectra (ie bundles of R–modules) over T is the stable presentable ∞–category $(\mathrm{Mod}_R)_{/T}$.

In practice, however, R is often more than an $\mathbb{E}_1$–ring spectrum; rather, it may be an $\mathbb{E}_n$–ring spectrum for some $1 \le n \le \infty$. In this case, the category of parametrized module spectra inherits a multiplicative structure where the product combines the product on R–modules with the diagonal map on the base space. Roughly speaking, the product is produced by taking the "external" product to get an R–module over $T \times T$ and pulling back along the diagonal map $T \to T \times T$.

Proposition 3.8 *Let R be an $\mathbb{E}_n$–ring spectrum, with $n > 0$, and let T be a Kan complex. Then the ∞–category $(\mathrm{Mod}_R)_{/T}$ of parametrized R–module spectra over T is the underlying ∞–category of an $\mathbb{E}_{n-1}$–monoidal stable presentable ∞–category $(\mathrm{Mod}_R)^{\otimes}_{/T}$.*

Proof Let Mod_R denote the ∞–category of right R–modules, which is an $\mathbb{E}_{n-1}$–algebra object of Pr^{L} and in particular an $\mathbb{E}_{n-1}$–monoidal ∞–category. Then

$$\mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Mod}_R)^{\otimes} = \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Mod}_R^{\otimes}) \times_{\mathrm{Fun}(T^{\mathrm{op}}, \mathrm{N}(\Gamma))} \mathrm{N}(\Gamma)$$

is an $\mathbb{E}_{n-1}$–monoidal ∞–category such that the underlying ∞–category

$$(\mathrm{Mod}_R)_{/T} = \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Mod}_R)$$

is stable and presentable. By construction, the operations in $\mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Mod}_R)^{\otimes}$ are computed pointwise, so that the tensor product commutes with colimits in each variable. Hence, $(\mathrm{Mod}_R)^{\otimes}_{/S} = \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Mod}_R)^{\otimes}$ is an $\mathbb{E}_{n-1}$–monoidal stable presentable ∞–category. □

3.1 Twists and twisted cohomology

We begin by recalling the notion of a twist (see also [2]). Given an $\mathbb{E}_{n+1}$–ring spectrum R, we can form the $\mathbb{E}_{n+1}$–space $\mathrm{Pic}(R)$ as the maximal ∞–groupoid in the category of R–modules spanned by the invertible objects (see Definition 7.5).

Definition 3.9 Let R be an E_{n+1}–ring spectrum and let B be a space (regarded as a Kan complex). A twist of R over B is a map

$$\alpha\colon B \to \mathrm{Pic}_R \to \mathrm{Mod}_R$$

classifying a bundle of invertible R–modules over B. Notice that the ∞–category $\mathcal{S}_{/\,\mathrm{Pic}_R}$ of twists is $\mathbb{E}_n$–monoidal.

We can interpret this using our definition of Thom spectra and twisted cohomology. Let $p\colon B \to *$ denote the terminal map. As explained in [3; 2], $M\alpha = p_!(R_B \wedge_{R_B} \alpha)$ is the generalized Thom spectrum of the map α; by definition, its R–cohomology is the α–twisted R–cohomology of B,

$$R^{k-\alpha}(B) = \pi_0 \operatorname{map}_R(M\alpha, \Sigma^k R).$$

Remark 3.10 When R is the K–theory spectrum (real or complex), the question arises of comparing our version of twisted K–theory to the Atiyah–Segal construction of twisted K–theory in terms of Fredholm operators. In [2, Section 5], we interpret the Atiyah–Segal construction as associating to a twist $f\colon X \to B\mathrm{GL}_1 A$ the spectrum

$$\Gamma_X(f) \simeq \mathrm{Sp}_{/X}(\mathbb{S}_X, f),$$

ie the spectrum of sections of f. Further, we explain how this spectrum is equivalent to the Thom spectrum functor applied to the twist $-f$ (the image under the involution $-1\colon B\mathrm{GL}_1 A \to B\mathrm{GL}_1 A$).

One might worry however that the geometric aspects of the Atiyah–Segal construction associate to a twist $X \to K(\mathbb{Z}/2, 2)$ a composite other than that induced by the inclusion $K(\mathbb{Z}/2, 2) \to B\mathrm{GL}_1 KO$, and so there could be a potential discrepancy. But in [7], Antieau, Gomez and the third author prove that up to homotopy any map $j\colon K(\mathbb{Z}/2, 2) \to B\mathrm{GL}_1 KO$ is either trivial or the canonical inclusion (and similarly for KU).

Remark 3.11 There are many other interesting examples of twisted cohomology theories.

(1) The spectrum tmf of topological modular forms comes equipped with a canonical map $K(\mathbb{Z}, 4) \to \mathrm{Pic}_{\mathrm{tmf}}$; see [2] for the details of construction of this map. Note that in [2] we used the connected component $B\mathrm{GL}_1(\mathrm{tmf})$ of the identity in the grouplike $\mathbb{E}_\infty$–space $\mathrm{Pic}_{\mathrm{tmf}}$, instead of the whole Picard space, which is of course sufficient since $K(\mathbb{Z}, 4)$ is connected.

(2) Another (conjecturally) telescopic height 2–theory is the algebraic K–theory $K(ku)$ of the connective topological K–theory spectrum ku. It is also equipped with a map $K(\mathbb{Z},4) \to B\mathrm{GL}_1(K(ku)) \simeq \mathrm{Pic}_{K(ku)}$ constructed as follows: delooping, it suffices to produce an $\mathbb{A}_\infty$–map $K(\mathbb{Z},3) \to \mathrm{GL}_1(K(ku))$. Using the composition $B\mathrm{GL}_1(ku) \to B\mathrm{GL}(ku) \to \mathbb{Z} \times B\mathrm{GL}(ku)^+ \simeq \Omega^\infty K(ku)$, which is a map of $\mathbb{E}_\infty$–spaces for the multiplicative structure on $\Omega^\infty K(ku)$, we obtain this map by delooping the $\mathbb{E}_\infty$–map $K(\mathbb{Z},2) \to \mathrm{GL}_1(ku)$.

(3) The family of $\mathbb{E}_\infty$–ring spectra (defined for each prime p and positive integer n) studied by C Westerland [50] admit twists by $K(\mathbb{Z}_p,n)$. These spectra R_n are defined as the homotopy fixed points $E_n^{hS\mathbb{G}_n^\pm}$ of the Lubin–Tate spectra E_n, and admit Snaith-style presentations of the form $R_n \simeq L_{K(n)}\Sigma^\infty_+ K(\mathbb{Z}_p, n+1)[\rho^{-1}]$. They therefore come equipped with canonical $\mathbb{E}_\infty$–maps $K(\mathbb{Z}_p, n+2) \to \mathrm{Pic}_{R_n}$.

The monoidal structure on the category of twists gives rise to a product in twisted cohomology:

Proposition 3.12 *Let R be an $\mathbb{E}_{n+1}$–ring spectrum. For any space X, the $\mathbb{E}_n$–monoidal structure on $\mathcal{S}_{/\mathrm{Pic}_R}$ gives rise to a product map*

$$R^\alpha(X) \otimes R^\beta(X) \to R^{\alpha+\beta}(X).$$

3.2 Noninvertible twists

In many situations it is useful to have a generalization of the notion of Thom spectrum in which "noninvertible" twists are also allowed. By a noninvertible twist we mean a not necessarily invertible twist, ie an arbitrary bundle of module spectra (or even an object of an arbitrary presentable symmetric monoidal stable ∞–category) over a space. In classical geometry, this is the analogue of a vector bundle, which is tensor-invertible if and only if the vector bundle is a line bundle. For the remainder of this section, by *twist* we will always mean a *not necessarily invertible twist*, unless otherwise specified.

For example, in Stolz and Teichner's program [46] for relating field theories to cohomology theories, twisted field theories (as initially defined in [46] and elaborated upon in [33]) naturally arise and are conjecturally connected to certain twisted cohomology theories. Such twisted field theories are also considered in a slightly different setting, where they are referred to as *relative field theories*, by Freed and Teleman [23].

Although the twists that arise have natural geometric descriptions, they are often not invertible; for instance, many arise as the pushforward of invertible twists (see

Proposition 3.16). The main motivating example is Witten's work [51] on Chern–Simons theory, which is a twist for Wess–Zumino–Witten theory. In other words, Wess–Zumino–Witten theory is twisted by Chern–Simons theory in the sense of [46, Definition 5.2].

Almost all of the foundational work of this paper applies in this more general setting. In this section, we record some definitions and basic results relevant to the broader context of noninvertible twists.

Definition 3.13 Let R be an $\mathbb{E}_{n+1}$–ring spectrum. A (not necessarily invertible) twist of R–theory over a space X is a functor $\tau\colon X^{\mathrm{op}} \to \mathrm{Mod}_R$.

Of course, τ is said to be *invertible* if it factors through the inclusion $\mathrm{Pic}_R \to \mathrm{Mod}_R$ of the invertible R–module spectra. Note that since X is a space, regarded as an ∞–groupoid, or equivalently as an ∞–category in which all arrows are invertible, the mapping spaces

$$\mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Mod}_R)^{\sim} \simeq \mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Mod}_R^{\sim})$$

are equivalent. This means that invertibility is a property of the twist τ, namely factoring through the full subgroupoid $\mathrm{Pic}_R \subseteq \mathrm{Mod}_R^{\sim}$, as opposed to extra structure.

Definition 3.14 The *generalized Thom spectrum* of the twist $\tau\colon X^{\mathrm{op}} \to \mathrm{Mod}_R$ is the colimit of τ; equivalently, the left Kan extension $p_!\tau$ of τ along the projection $p\colon X \to *$ from X to the point.

Given an $\mathbb{E}_1$–ring spectrum R and a twist $\tau\colon X^{\mathrm{op}} \to \mathrm{Mod}_R$ of R over X, we can define the τ–twisted homology and cohomology of X, as R–module spectra, via the formulae

$$R_\tau(X) := p_!\tau \quad \text{and} \quad R^\tau(X) := p_*\tau.$$

We record the following easy facts:

Proposition 3.15 *Let R be an $\mathbb{E}_\infty$–ring spectrum and let $\tau\colon X^{\mathrm{op}} \to \mathrm{Mod}_R$ be a twist of R. Suppose that τ factors through the inclusion $\mathrm{Mod}_R^{\omega} \subseteq \mathrm{Mod}_R$ of the full subcategory of dualizable R–module spectra. Then*

$$R^{D\tau}(X) \simeq \mathrm{Mod}_R(R_\tau(X), R)$$

as R–module spectra.

Proof Using the fact that $p^*\colon \mathrm{Mod}_R \to \mathrm{Mod}_{R_X} \simeq \mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Mod}_R)$ is a symmetric monoidal functor which admits both left and right adjoints, we obtain equivalences

$$\mathrm{Mod}_R(p_!\tau, R) \simeq \mathrm{Mod}_{R_X}(\tau, R_X) \simeq \mathrm{Mod}_{R_X}(R_X, D\tau) \simeq p_* D\tau \simeq R^{D\tau}(X).$$

Note that the second equivalence follows from the fact that τ is dualizable, as an R–module is dualizable if and only if it is compact. □

The following trivial observation illustrates how noninvertible twists arise quite naturally from partial pushforwards of possibly invertible twists. Hence, actual R–module Thom spectra, ie the ones associated to invertible twists, can typically be written as the "generalized" Thom spectrum associated to a noninvertible twist.

Proposition 3.16 *Let X and Y be spaces and write $p\colon X \to *$ and $q\colon Y \to *$ for the projections to the point. Consider the (not necessarily commutative) diagram*

$$\begin{array}{ccc} Y^{\mathrm{op}} & \xrightarrow{\ \tau\ } & \mathrm{Mod}_R \\ {\scriptstyle f}\big\downarrow & \overset{f_!\tau}{\nearrow} & \big\downarrow \\ X^{\mathrm{op}} & \xrightarrow{\ p\ } & * \end{array}$$

obtained from a bundle of R–modules $\tau\colon Y^{\mathrm{op}} \to \mathrm{Mod}_R$ on Y by forming the left Kan extension $f_!\tau\colon X^{\mathrm{op}} \to \mathrm{Mod}_R$ of τ along f. Then $q_!\tau \simeq p_! f_! \tau$.

Proof Since $q \simeq p \circ f$, this is immediate from the universal property of the left Kan extension. More precisely,

$$\mathrm{map}(p_! f_! \tau, M) \simeq \mathrm{map}(\tau, f^* p^* M) \simeq \mathrm{map}(\tau, q^* M) \simeq \mathrm{map}(q_!\tau, M)$$

for any R–module M. □

4 Twisted cohomology theories and the twisted Umkehr map

Let $f\colon X \to B$ be a bundle of smooth manifolds and let T_f be the bundle of tangent vectors along the fiber, say of dimension d. The Pontryagin–Thom construction gives rise to a stable map from the suspension spectrum of B to the Thom spectrum of $-T_f$,

$$\mathrm{PT}(f)\colon \Sigma^\infty_+ B \to X^{-T_f}, \tag{4.1}$$

which we'll call the "Pontryagin–Thom transfer" associated to f. In the presence of a ring spectrum theory R and a Thom isomorphism

$$R^*(\Sigma^\infty_+ X) \to R^{*-d}(X^{-T_f})$$

one then has an Umkehr homomorphism

$$R^*(\Sigma^\infty_+ X) \to R^{*-d}(\Sigma^\infty_+ B).$$

A twisted version of this Umkehr map plays a role in a number of contexts, including the K–theoretic analysis of anomalies in string theory [49; 24] and the Umkehr map in Grojnowski's equivariant elliptic cohomology (which plays a key role in the proof of Witten's rigidity theorems and the derivation of the Kac–Weyl character formula [45; 1; 25]). These examples lead to the following generalization:

Suppose that R is an E_∞–ring spectrum, and let $\alpha\colon B \to \mathrm{Pic}_R$ classify an invertible R_B–module. One can ask for a "twisted Umkehr map"

$$R^{*+T_f-\alpha}(\Sigma^\infty_+ X) \to R^{*-\alpha}(\Sigma^\infty_+ B).$$

For example if $-T_f + \alpha$ is null-homotopic, then a choice of null-homotopy should determine a map

$$R^*(\Sigma^\infty_+ X) \to R^{*-\alpha}(\Sigma^\infty_+ B),$$

and if α itself is null, one expects to recover the Umkehr map.

The purpose of this section is to construct such a twisted Umkehr map. In what follows, we write D_B to denote the fiberwise functional dual (ie $D_B X$ denotes the internal mapping object $F_B(\Sigma^\infty_B X_+, \mathbb{S}_B)$ in the category $\mathrm{Sp}_{/B}$). The key points are:

(1) The Pontryagin–Thom transfer map (4.1) arises from a map of spectra

$$\mathbb{S}_B \to D_B X$$

over B by pushforward along the map $p\colon B \to *$, and the twisted Umkehr map arises by smashing with the bundle classified by α

$$\alpha \to \alpha \wedge_B D_B X$$

and then pushing forward along p,

$$p_!\alpha \to p_!(\alpha \wedge_B D_B X).$$

(2) Identifying $p_! D_B(X) \simeq X^{-T_f}$ requires a parametrized version of Atiyah duality

$$D_B X \simeq \mathbb{S}_B^{-T_f},$$

where here $\mathbb{S}_B^{-T_f}$ is the inverse of the parametrized spectrum obtained from the bundle of tangents along the fiber, from which one concludes that

$$p_! D_B X \simeq X^{-T_f}.$$

4.1 The Becker–Gottlieb transfer

We begin by recalling from [41, Section 15.3] the construction of the Becker–Gottlieb transfer in the setting of parametrized homotopy theory. The transfer map arises from the categorical trace associated to the diagonal map of a stably dualizable space X. Specifically, we have the composite

$$\mathbb{S} \xrightarrow{\eta} X \wedge DX \to DX \wedge X \xrightarrow{\mathrm{id} \wedge \Delta} DX \wedge X \wedge X \xrightarrow{\epsilon \wedge \mathrm{id}} \mathbb{S} \wedge X \simeq X,$$

where η and ϵ are the coevaluation and evaluation maps exhibiting the duality. These transfer maps satisfy a series of compatibility relations — see [41, 15.2.4]; the required conditions on the triangulation of the homotopy category hold here, either by comparison or as can be shown directly.

The key observation about duality in the parametrized setting is that we can characterize dualizability fiberwise because the smash product is computed pointwise.

Lemma 4.2 *Let B be a space and $X \in \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{Sp})$ a parametrized spectrum. Then X is dualizable if and only if for each $b \in B$, the value X_b of X at b is a dualizable spectrum.*

In particular, given a map $f: X \to B$ of spaces with stably dualizable (homotopy) fibers, such as a proper fibration, by adjoining a disjoint basepoint we get a diagonal map on $\Sigma_B^\infty X_+$ and so (as above) a transfer map

$$\mathbb{S}_B \to \Sigma_B^\infty X_+.$$

Pushing forward along the map $B \to *$ now yields the classical transfer map

$$\Sigma_+^\infty B \to \Sigma_+^\infty X.$$

Note that we can easily recover Dwyer's generalization of the transfer [20]. Specifically, let R be an $\mathbb{E}_\infty$–ring spectrum and suppose that $f: E \to B$ has homotopy fiber F

such that $R \wedge \Sigma^\infty_+ F$ is a dualizable object in the category of R–modules. Then the construction of the transfer in this setting gives rise to an R–module transfer map

$$R \wedge \Sigma^\infty_+ B \to R \wedge \Sigma^\infty_+ E.$$

Proposition 4.3 *Let R be an $\mathbb{E}_\infty$–ring spectrum and let $f\colon E \to B$ be a map of spaces with homotopy fiber F such that $R \wedge \Sigma^\infty_+ F$ is a dualizable object in* Mod_R. *Then the diagonal map on E gives rise to a map of R–modules over B*

$$R_B \to R \wedge \Sigma^\infty_B E_+$$

such that the pushforward along $B \to *$ is the R–module transfer

$$R \wedge \Sigma^\infty_+ B \to R \wedge \Sigma^\infty_+ E.$$

4.2 Duality and the Pontryagin–Thom map

The construction of the transfer map given in the previous section is formal and does not immediately reveal the geometric content of the transfer. To this end, we now turn to recall the geometric constructions of the dual and the Umkehr map.

As above, recall that $\mathcal{S}$ denotes the ∞–category of spaces and Sp denotes the ∞–category of spectra. We let $\mathbb{S}$ denote the sphere spectrum and, for a space X, we write DX for the Spanier–Whitehead dual

$$DX = F(\Sigma^\infty_+ X, \mathbb{S})$$

of the spectrum $\Sigma^\infty_+ X$; this specifies a contravariant functor from spaces to spectra

$$D\colon \mathcal{S}^{\mathrm{op}} \to \mathrm{Sp},$$

which we regard as a presheaf of spectra on $\mathcal{S}$. Since applying D to the unique map of spaces $p\colon X \to *$ gives a map of spectra $\mathbb{S} \to DX$, in fact D specifies a functor

$$\phi\colon \mathcal{S}^{\mathrm{op}} \to \mathrm{Sp}_{\mathbb{S}/},$$

where $\mathrm{Sp}_{\mathbb{S}/}$ denotes the category of spectra under $\mathbb{S}$. If X is a compact manifold, then the Pontryagin–Thom construction and Atiyah duality give a wonderful formula for the map $\mathbb{S} \to DX$. Take an embedding $X \to \mathbb{R}^N$ with normal bundle ν_X, and form the Pontryagin–Thom construction (collapse to a point the complement of a tubular neighborhood of X in $\mathbb{R}^N$) to get a map

$$S^N \to X^{\nu_X}.$$

Desuspending N times yields the desired stable map, the Pontryagin–Thom map

$$\mathrm{PT}\colon \mathbb{S} \to X^{-T},$$

where T denotes the tangent bundle of X; the following proposition records the classical comparison of X^{-T} to the dual of X;

Proposition 4.4 *There is an equivalence $X^{-T} \to DX$ such that the diagram*

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\mathrm{PT}} & X^{-T} \\ & {\scriptstyle \phi(X)} \searrow & \downarrow \\ & & D(X) \end{array}$$

commutes up to homotopy.

Suppose that R is an E_∞–ring spectrum and we have a Thom isomorphism

$$R^*(X_+) \cong R^{*-d}(X^{-T}).$$

The Umkehr map associated to the map $X \to *$ and the Thom isomorphism for R is given by the composition

$$\begin{array}{ccccc} R^*(X_+) & \longrightarrow & R^{*-d}(X^{-T}) & & \\ & & \cong \downarrow & \overset{\mathrm{PT}^*}{\searrow} & \\ & & R^{*-d}(DX) & \xrightarrow[\phi(X)]{} & R^{*-d}(S^0) \end{array}$$

Now suppose that $f\colon X \to B$ is a smooth and proper family of manifolds over B; in other words, for each $b \in B$, the fiber X_b is a smooth and compact manifold which varies continuously over B in the sense that X is classified by a map $B^{\mathrm{op}} \to \mathcal{M}$, where $\mathcal{M}$ denotes the ∞–category of smooth compact manifolds with morphisms the diffeomorphisms. Here $\mathcal{M}$ can be described as the coherent nerve of the ordinary category of smooth compact manifolds and diffeomorphisms.

Remark 4.5 $\mathcal{M}$ is an ∞–groupoid which, when regarded as a space, decomposes as the sum

$$\mathcal{M} \simeq \coprod_{[M]} B\mathrm{Diff}(M),$$

indexed over diffeomorphism classes of smooth manifolds, where $\mathrm{Diff}(M)$ denotes the (topological) group of diffeomorphisms of a representative M for the class $[M]$.

If $B = BG$ is connected, then this amounts to an A_∞–map $G \to \mathrm{Diff}(M)$ for some smooth and compact manifold M, and $X \simeq M/G$ is the homotopy quotient of M by its G–action.

The composition

$$B^{\mathrm{op}} \to \mathcal{M} \to \mathcal{S} \xrightarrow{\phi} \mathrm{Sp}_{\mathbb{S}/}$$

gives an object of $(\mathrm{Sp}_{\mathbb{S}/})_{/B} \simeq (\mathrm{Sp}_{/B})_{\mathbb{S}_B/}$, that is, a B–parametrized spectrum

$$\phi_{X/B}\colon \mathbb{S}_B \to D_B(X)$$

under $\mathbb{S}_B$ whose value at $b \in B$ is

$$\phi(X_b)\colon \mathbb{S} \to D(X_b).$$

This is the fiberwise dual of the map $X_b \to *$.

Pushing forward along $p\colon B \to *$, we obtain a map

$$\Sigma^\infty_+ B \simeq p_! p^* \mathbb{S} \simeq p_! \mathbb{S}_B \to p_! D_B(X).$$

To identify $p_! D_B(X)$ with X^{-T_f}, and so obtain the Pontryagin–Thom transfer map (4.1) as advertised, we need a parametrized form of Atiyah duality.

4.3 Parametrized manifolds and fiberwise Atiyah–Milnor–Spanier duality

Unfortunately, the usual construction of the Atiyah duality equivalence $X^{-TX} \to DX$ does not have attractive functoriality and naturality properties; this makes it difficult to implement a naive construction of fiberwise Atiyah duality. In this section, we give a new approach to work of [32; 41], which shows how to calculate the fiberwise dual of a bundle of manifolds $X \to B$ in terms of a suitable parametrized version of the Pontryagin–Thom construction.

Again let $f\colon X \to B$ be a continuous family of smooth compact manifolds, classified by a map $B \to \mathcal{M}$, the ∞–category of smooth compact manifolds. Let

$$T_f = T_{X/B}$$

denote the bundle of tangents along the fiber of $f\colon X \to B$. A classical construction of this object proceeds by passage to the associated principal bundle of f; see [11, Section 4] or [41, Section 3.3] for more detailed discussion. Alternatively and more relevantly for our work, one can consider the pullback $X \times_B X$; then T_f is specified as the normal bundle of the image of the diagonal $X \to X \times_B X$. We write $\mathbb{S}_X^{T_{X/B}}$ or $\mathbb{S}_X^{T_f}$ for the suspension spectrum of the associated sphere bundle; it is a bundle of spectra over X.

We will show that $f_! \Sigma_X^V \mathbb{S}_X^{-T_f}$ is naturally equipped with an equivalence to $D_B X$, where here V is a euclidean space in which the fiber F embeds. Our approach relies on the observation (which we learned from [32]) that the suspension spectrum of the cofiber

$$C_f = \mathrm{hocofib}(X \times_B X - \Delta \to X \times_B X) \simeq (X \times_B X / X \times_B X - \Delta)$$

gives a model for $\mathbb{S}_X^{T_X}$, where Δ denotes the image of the diagonal $X \to X \times_B X$ and we regard $X \times_B X$ as a space over X via the projection onto the first coordinate; this makes the diagonal into a map over X.

We begin by considering the case in which $B = *$. Then the classical observation (see eg [42, Section 10] or [14, Section 12]) that for a compact manifold M, the normal bundle of the diagonal embedding $M \to M \times M$ is homeomorphic to the tangent bundle of M implies the following lemma:

Lemma 4.6 *The (homotopy) cofiber $C_f = \Sigma_X^\infty (X \times X / X \times X - \Delta)$ is equivalent to the tangent sphere bundle $\mathbb{S}_X^T$.*

Since the case when $B = *$ describes the fiber over an arbitrary base B, we obtain the following description:

Corollary 4.7 *The (homotopy) cofiber $C_f = \Sigma_X^\infty (X \times_B X / X \times_B X - \Delta)$ is equivalent to the tangent sphere bundle $\mathbb{S}_X^{T_{X/B}}$.*

We now turn to analyzing the dual of C_f. Again, we begin by studying the case in which $B = *$, so that X is just a smooth compact manifold. We choose a smooth embedding of X in a euclidean space V with normal bundle ν.

Lemma 4.8 *For $f \colon X \to *$, the parametrized spectrum C_f is dualizable and in fact invertible, with inverse given by $\Sigma_X^{-V} \mathbb{S}^\nu$.*

Proof This follows from the evident equivalence

$$\mathbb{S}_X^{T_{X/B}} \wedge \mathbb{S}_X^\nu \to S^V,$$

induced from the fact that $\tau \oplus \nu$ is the trivial bundle. □

Since dualizability is detected fiberwise, this has as an immediate corollary invertibility for the case of general $f\colon X \to B$ with manifold fibers.

Corollary 4.9 *The parametrized spectrum C_f is dualizable and in fact invertible.*

Remark 4.10 The central observation of [32] is that the inverse of C_f gives a model for a dualizing complex in the sense of Grothendieck, adapted to the Wirthmüller setting.

We now exhibit a natural map

$$\theta\colon f_! D_X C_f \to D_B X$$

and then show that it is an equivalence; this provides a geometric model of the functional dual $D_B X$.

Construction 4.11 The natural evaluation map

$$C_f \wedge_X D_X C_f \to \mathbb{S}_X,$$

composed with the map to the cofiber

$$\Sigma^\infty_X (X \times_B X) \to C_f,$$

produces a natural composite

$$\Sigma^\infty_X (X \times_B X) \wedge_X D_X C_f \to \mathbb{S}_X.$$

We have canonical natural equivalences $\Sigma^\infty_X (X \times_B X) \simeq f^* \Sigma^\infty_B X_+$ and $f^* \mathbb{S}_B \simeq \mathbb{S}_X$, and so we may rewrite the map as

$$f^* \Sigma^\infty_B X_+ \wedge_X D_X C_f \to f^* \mathbb{S}_B.$$

Passing to adjoints, we have

$$D_X C_f \to F_X(f^* \Sigma^\infty_B X_+, f^* \mathbb{S}_B).$$

Next, using the natural equivalence $F_X(f^* \Sigma^\infty_B X_+, f^* \mathbb{S}_B) \simeq f^* F_B(\Sigma^\infty_B X_+, \mathbb{S}_B)$, we obtain the map

$$D_X C_f \to f^* D_B X,$$

and, finally, applying the $(f_!, f^*)$ adjunction yields the desired map θ.

Proposition 4.12 *The natural map*

$$\theta\colon f_! D_X C_f = f_! D_X(X \times_B X / X \times_B X - \Delta) \to D_B X$$

is an equivalence.

Proof Since equivalences of parametrized spectra are detected fiberwise, it suffices to restrict to the fiber over each point $b \in B$. Equivalently, we can assume that $B = *$. In this case, the map reduces to

$$f_! D_X(X \times X / X \times X - \Delta) \to DX.$$

By the discussion in [41, Section 19.6], $D_X(X \times X / X \times X - \Delta) \simeq \Sigma_X^{-V} \mathbb{S}_X^{\nu}$, and since $f_! \Sigma_X^{-V} \mathbb{S}_X^{\nu} \simeq \Sigma^{-V} X^{\nu} \simeq X^{-T_X}$, Atiyah duality abstractly implies the equivalence we want.

We need to check that the map in question is homotopic to the standard map inducing the Atiyah duality equivalence. First, recall that the evaluation map

$$\epsilon\colon X^{\nu} \wedge X_+ \to S^V$$

which induces the Atiyah duality equivalence $X^{-T_X} \to DX$ is induced from the Pontryagin–Thom construction applied to the composite of the diagonal $X \to X \times X$ and the zero section $X \times X \to \nu \times X$ (note that the normal bundle of the composition of these two embeddings is trivial). Next, the composite

$$X \times X \to X \times X / X \times X - \Delta$$

is a model for the Pontryagin–Thom collapse map associated to the embedding of the diagonal. Therefore, under the equivalences $D_X(X \times X / X \times X - \Delta) \simeq \Sigma_X^{-V} \mathbb{S}_X^{\nu}$ (which is not natural) and $\Sigma_X^{\infty}(X \times X) \simeq f^* \Sigma_+ X$ (which is natural), the map

$$f_!\big(\Sigma_X^{\infty}(X \times X) \wedge_X D_X(X \times X / X \times X - \Delta)\big) \to \mathbb{S}$$

is homotopic to the map

$$f_!(f^* \Sigma_+ X \wedge_X \Sigma_X^{-V} \mathbb{S}_X^{\nu}) \to \mathbb{S},$$

which, when expressed as

$$\Sigma_+ X \wedge \Sigma_X^{-V} X^{\nu} \to \mathbb{S},$$

is the usual Atiyah duality map. □

Corollary 4.13 *A choice of equivalence $C_f \simeq \mathbb{S}_X^{T_f}$ induces an equivalence of spectra over B,*

$$f_! \mathbb{S}_X^{-T_f} \simeq D_B X.$$

Pushing this equivalence forward along the map $p\colon B \to *$ yields an equivalence of spectra,

$$p_! D_B X \simeq X^{-T_f}.$$

Proof The parametrized spectrum $\mathbb{S}_X^{-T_f}$ is classified by a map

$$X^{\mathrm{op}} \to \mathrm{Pic}_{\mathbb{S}},$$

and the Thom spectrum X^{-T_f} is by definition the left Kan extension along $pf\colon X \to *$; that is, we have

$$X^{-T_f} = (pf)_! \mathbb{S}^{-T_f} \cong p_! f_! \mathbb{S}^{-T_f}. \qquad \square$$

Definition 4.14 The *parametrized Pontryagin–Thom transfer* is the composition

$$\mathrm{PT}_{/B}(f)\colon \mathbb{S}_B \xrightarrow{\phi_{X/B}} D_B X \simeq f_! \mathbb{S}_X^{-T_f}.$$

Pushing forward the parametrized Pontryagin–Thom transfer map yields the Umkehr map.

Definition 4.15 The *Pontryagin–Thom transfer map* associated to $f\colon X \to B$ is the map

$$\mathrm{PT}(f)\colon \Sigma_+^\infty B \to X^{-T_f}$$

obtained by applying the pushforward $p_!$ to the parametrized Pontryagin–Thom transfer

$$\mathrm{PT}_{/B}(f)\colon \mathbb{S}_B \to f_! \mathbb{S}_X^{-T_f}.$$

If R is a ring spectrum, then a Thom isomorphism $R^*(\Sigma_+^\infty X) \simeq R^{*-d}(X^{-T_f})$ determines an *Umkehr* map

$$R^*(\Sigma_+^\infty X) \simeq R^{*-d}(X^{-T_f}) \to R^{*-d}(\Sigma_+^\infty B).$$

Remark 4.16 One also expects to have Umkehr maps arising from embeddings of manifolds. We construct these as follows: Let $j\colon W \to M$ be an embedding of manifolds with normal bundle ν and let $p\colon M \to *$ be the map to a point. To obtain a similar view of the Umkehr map j, we need to realize the geometric Pontryagin–Thom map

$$\Sigma^\infty_+ M \to \Sigma^\infty W^\nu$$

as $p_! t$, where t is a map of spectra over M. Now, if S^ν is the parametrized space associated to ν, then

$$W^\nu \simeq p_! j_! S^\nu,$$

and this suggests that the map we seek is a suspension of a map of the form

$$\alpha\colon S^0_M \to j_! S^\nu.$$

The required map is constructed by May and Sigurdsson [41, 18.6.3 and 18.6.5].

4.4 Twisting the Umkehr map

The Pontryagin–Thom transfer map $\mathrm{PT}(f)$ of Definition 4.15 arises from a map of spectra parametrized by B, and so we can twist it. Specifically, given a twist $\alpha\colon B \to \mathrm{Pic}_R \to \mathrm{Mod}_R$ and a generalized Umkehr map

$$R_B \to X,$$

we can twist the map by fiberwise smash to obtain a map

$$R_B \wedge_{R_B} \alpha \to X \wedge_{R_B} \alpha.$$

We now specialize to the geometric example considered in Section 4.3. We continue to let $f\colon X \to B$ denote a family of compact manifolds over a space B, and to write p for the map $B \to *$. Given a twist α, we can form the map of R–modules over B

$$\mathrm{PT}_{/B}(f) \wedge_{R_B} \mathrm{id}\colon R_B \wedge_{R_B} \alpha \to f_! \mathbb{S}_X^{-T_f} \wedge_{R_B} \alpha.$$

Applying the pushforward $p_!\colon (\mathrm{Mod}_R)_{/B} \to \mathrm{Mod}_R$ gives rise to the twisted Pontryagin–Thom transfer. Notice that the abstract formulation is obviously natural; the naturality of the geometric version must be expressed in terms of the natural models discussed above.

Definition 4.17 The twisted Pontryagin–Thom transfer map is defined as

$$\mathrm{PT}(f,\alpha) = p_!\big(\mathrm{PT}_{/B}(f) \wedge_{R_B} \mathrm{id}\colon R_B \wedge_{R_B} \alpha \to f_! \mathbb{S}_X^{-T_f} \wedge_{R_B} \alpha\big).$$

As for $f_!\mathbb{S}_X^{-T_f} \wedge_{R_B} \alpha$, the projection formula yields

$$f_!\mathbb{S}_X^{-T_f} \wedge_{R_B} \alpha \simeq f_!(R_X \wedge_{\mathbb{S}_X} \mathbb{S}_X^{-T_f}) \wedge_{R_B} \alpha \simeq f_!((R \wedge_{\mathbb{S}_X} \mathbb{S}_X^{-T_f}) \wedge_{R_X} f^*\alpha),$$

and so

$$p_!(f_!\mathbb{S}_X^{-T_f} \wedge_{R_B} \alpha) \simeq p_! f_!((R_X \wedge_{\mathbb{S}_X} \mathbb{S}_X^{-T_f}) \wedge_{R_X} f^*\alpha) = X^{-T_f+\alpha f}$$

is the R–module Thom spectrum whose R–module cohomology is the cohomology of X, twisted by the sum of

$$X \xrightarrow{-T_f} \mathrm{Pic}_{\mathbb{S}} \to \mathrm{Pic}_R \to \mathrm{Mod}_R$$

and

$$X \xrightarrow{f} B \xrightarrow{\alpha} \mathrm{Pic}_R \to \mathrm{Mod}_R\,.$$

Summarizing, we have the following proposition:

Proposition 4.18 *Let $f\colon X \to B$ be a family of compact manifolds and let $\alpha\colon B \to \mathrm{Pic}_R \to \mathrm{Mod}_R$ be a parametrized invertible R–module over B. Then we have an equivalence of R–modules*

$$X^{-T_f+\alpha f} \simeq p_!(f_!\mathbb{S}_X^{-T_f} \wedge_{R_B} \alpha),$$

and so the twisted Pontryagin–Thom transfer $\mathrm{PT}(X,\alpha)$ *may be viewed as a map of R–modules*

$$\mathrm{PT}(X,\alpha)\colon B^\alpha \to X^{-T_f+\alpha f}.$$

Passing to R–cohomology gives a twisted Umkehr map

$$R^{*+Tf-\alpha}(X) \to R^{*-\alpha}(B).$$

We close with an example, motivated by [49; 24]. Suppose that α makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{-Tf} & \mathrm{Pic}_{\mathbb{S}} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ B & \xrightarrow{\alpha} & \mathrm{Pic}_R \end{array}$$

homotopy commute. A choice of homotopy between the two compositions determines an equivalence of R–modules

$$X^{-Tf+\alpha f} \simeq \Sigma^\infty_+ X \wedge R,$$

so the twisted Pontryagin–Thom transfer map takes the form

$$\mathrm{PT}(X,\alpha)\colon B^{\alpha} \to \Sigma^{\infty}_{+} X \wedge R,$$

and passing to R–cohomology yields a twisted Umkehr map

$$R^{*}(X) \to R^{*-\alpha}(B).$$

The following instance of this construction was described in our paper [2], and was inspired by the work of Freed and Witten [24] and Carey and Wang [15; 49].

Let $j\colon D \to X$ be an embedded submanifold, let ν be the normal bundle of j, and suppose that D carries a complex vector bundle ξ. If ν carries a Spin^{c}–structure, then we can form the K–theory pushforward

$$j_{!}\colon K(D) \to K(X).$$

In that situation, Minasian and Moore [43] and Witten [52] discovered that it is sensible to think of the K–theory class

$$j_{!}(\xi) \in K(X)$$

as the "charge" of the D–brane D with Chan–Paton bundle ξ.

Let $b\colon K(\mathbb{Z}/2,2) \to K(\mathbb{Z},3)$ be the indicated Bockstein; then $B\mathrm{Spin}$ is the fiber in the sequence

$$\begin{array}{ccccc} B\mathrm{Spin} \longrightarrow & B\mathrm{SO} & \xrightarrow{\quad bw_2 \quad} & & K(\mathbb{Z},3) \\ & & {\scriptstyle w_2}\searrow & & \nearrow{\scriptstyle b} \\ & & & K(\mathbb{Z}/2,2) & \end{array}$$

Suppose that ν does not carry a Spin^{c}–structure, but we have a map $H\colon X \to K(\mathbb{Z},3)$ making the diagram

$$\begin{array}{ccc} D & \xrightarrow{\ \nu\ } & B\mathrm{SO} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle bw_2} \\ X & \xrightarrow[H]{} & K(\mathbb{Z},3) \end{array}$$

commute up to homotopy. A homotopy $bw_2 \simeq Hj$ determines an isomorphism

$$K^{*}(D) \cong K^{*+H}(D^{\nu})$$

(since $\nu = -Tj$). Using the construction of May and Sigurdsson described in Remark 4.16 together with the discussion of twisted Umkehr maps above, we have a twisted Umkehr map

(4.19) $$j_!\colon K^*(D) \to K^{*+H}(X).$$

The class $j_!(\xi) \in K^{*+H}(X)$ is evidently an analogue of the charge in this situation. The discovery of the condition that there exists a class H on X such that $H|_D = W_3(\nu)$ is due to Freed and Witten [24].

5 Parametrized ∞–category theory

We now switch gears and develop the necessary foundations to support the applications described in the previous sections. Specifically, we make rigorous what we mean by a family of objects of an ∞–category $\mathcal{C}$ parametrized over an object of a "geometric" ∞–category $\mathcal{X}$. It turns out that the properties of the category of spaces that are required to provide a good theory of parametrized objects are precisely encoded in the notion of an ∞–topos, so we work in this generality. Therefore, our results will apply not only to spaces but to the ∞–topoi of sheaves on manifolds, schemes, spaces, etc.

5.1 ∞–Topoi

The general theory of parametrized objects works not only over the ∞–category of spaces, but equally well over an arbitrary ∞–topos. For the purposes of this paper, we will not need a detailed construction of the ∞–category of ∞–topoi as in [36]; rather, it will suffice to note that an ∞–topos $\mathcal{X}$ is a presentable ∞–category which arises as an accessible left-exact localization of a presheaf ∞–category, and the ∞–category of morphisms between two ∞–topoi $\mathcal{X}$ and $\mathcal{Y}$ (ie the *geometric morphisms*) is the full subcategory of $\mathrm{Fun}^{\mathrm{R}}(\mathcal{X}, \mathcal{Y})$ consisting of those right-adjoint functors $f_*\colon \mathcal{X} \to \mathcal{Y}$ whose left adjoint $f^*\colon \mathcal{Y} \to \mathcal{X}$ preserves finite limits (ie is *left exact*). Finally, the terminal ∞–topos is the ∞–category $\mathcal{S}$ of spaces, and a point of *stalk* of an ∞–topos $\mathcal{X}$ is a geometric morphism $f_*\colon \mathcal{S} \to \mathcal{X}$, or equivalently a finite limit- and arbitrary colimit-preserving functor $f^*\colon \mathcal{X} \to \mathcal{S}$.

The key feature of an ∞–topos $\mathcal{X}$ is that it satisfies a very strong form of descent. A succinct way of expressing this fact is as follows: adopting the notation of [36], we write $\mathcal{O}_X = \mathrm{Fun}(\Delta^1, \mathcal{X})$ for the ∞–category of arrows of $\mathcal{X}$ and $p\colon \mathcal{O}_{\mathcal{X}} \to \mathcal{X}$ for the cartesian fibration which assigns to an object $f\colon S \to T$ in $\mathcal{O}_{\mathcal{X}}$ its target T in $\mathcal{X}$. Clearly the fiber of this cartesian fibration over the object T is precisely $\mathcal{X}_{/T}$, the slice over T, which is itself an ∞–topos. Straightening this cartesian fibration, we obtain a functor

$$\mathcal{X}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$$

which is a *sheaf* in the sense that it preserves limits in $\mathcal{X}^{\mathrm{op}}$; in other words, if $T \simeq \operatorname{colim} T_\alpha$ is a colimit diagram in $\mathcal{X}$, then the induced map

$$\mathcal{X}_{/T} \to \lim_{\alpha} \mathcal{X}_{/T_\alpha}$$

is an equivalence in $\widehat{\mathrm{Cat}}_\infty$. In fact, this descent condition characterizes ∞–topoi amongst locally cartesian closed presentable ∞–categories (see eg [27] for details about locally cartesian closed ∞–categories). Indeed, presentability and locally cartesian closure implies that the slice functor

$$\mathcal{X}_{/(-)}\colon \mathcal{X}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty,$$

obtained by straightening the target fibration $\mathrm{Fun}(\Delta^1, \mathcal{X}) \to \mathcal{X}$, factors through the full subcategory $\mathrm{Pr}^{\mathrm{L}} \subset \widehat{\mathrm{Cat}}_\infty$, or equivalently that colimits in $\mathcal{X}$ are universal. The fact that such an $\mathcal{X}$ is an ∞–topos follows from [36, Theorem 6.1.3.9 and Proposition 6.1.3.19].

If $\mathcal{X}$ is an ∞–topos, or more generally any presentable ∞–category, then the Yoneda embedding $\mathcal{X} \to \mathrm{Fun}^{\lim}(\mathcal{X}^{\mathrm{op}}, \mathcal{S})$ is an equivalence of ∞–categories. Here we use the fact that the functor represented by an object X of $\mathcal{X}$ preserves limits in $\mathcal{X}^{\mathrm{op}}$. When $\mathcal{X}$ is an ∞–topos, a limit-preserving functor $\mathcal{X}^{\mathrm{op}} \to \mathcal{S}$ is called a *sheaf* of spaces on $\mathcal{X}$, so that $\mathcal{X}$ is naturally equivalent to the ∞–category of sheaves of spaces on $\mathcal{X}$. Of course, we can also consider sheaves valued in a general ∞–category $\mathcal{C}$.

5.2 Parametrized objects

Let $\mathcal{C}$ be a presentable ∞–category, which we will view as the ∞–category in which our sheaves on $\mathcal{X}$ take values. Following [36, 6.3.5.16], we define the ∞–category of $\mathcal{C}$–valued sheaves on $\mathcal{X}$ as

$$\mathrm{Shv}_{\mathcal{C}}(\mathcal{X}) = \mathrm{Fun}^{\lim}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}),$$

the ∞–category of limit-preserving functors from $\mathcal{X}^{\mathrm{op}}$ to $\mathcal{C}$. In light of the discussion above, the target fibration $p\colon \mathcal{O}_{\mathcal{X}} \to \mathcal{X}$ can be viewed (via the straightening functor) as a $\widehat{\mathrm{Cat}}_\infty$–valued sheaf on $\mathcal{X}$. In the special case in which $\mathcal{C} = \mathcal{S}$ is the ∞–category of spaces, we simply write $\mathrm{Shv}(\mathcal{X})$ in place of $\mathrm{Shv}_{\mathcal{S}}(\mathcal{X})$, and the Yoneda embedding induces an equivalence

$$\mathcal{X} \simeq \mathrm{Fun}^{\lim}(\mathcal{X}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Shv}(\mathcal{X})$$

from $\mathcal{X}$ to sheaves of spaces on $\mathcal{X}$.

We are now in a position to define objects of an ∞–category $\mathcal{C}$ parametrized over objects of an ∞–topos $\mathcal{X}$.

Definition 5.1 Let $\mathcal{X}$ be an ∞–topos and let $\mathcal{C}$ be a presentable ∞–category. Then a family of objects of $\mathcal{C}$ parametrized by an object S of $\mathcal{X}$ is a $\mathcal{C}$–valued sheaf $\mathcal{F}\colon (\mathcal{X}_{/S})^{\mathrm{op}} \to \mathcal{C}$ on $\mathcal{X}_{/S}$. The ∞–category of families of objects of $\mathcal{C}$ parametrized by an object S of $\mathcal{X}$ is the ∞–category $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X}_{/S})$ of $\mathcal{C}$–valued sheaves on $\mathcal{X}_{/S}$.

We will typically abbreviate the terminology and refer to these parametrized families simply as an object of $\mathcal{C}$ parametrized by, or over, an object S of the ∞–topos $\mathcal{X}$; similarly, we abbreviate notation and often write $\mathcal{C}_{/S} = \mathrm{Shv}_{\mathcal{C}}(\mathcal{X}_{/S})$ for the ∞–category of objects of $\mathcal{C}$ parametrized over S.

Our first order of business is to verify that when $\mathcal{X}$ is an ∞–topos of presheaves on spaces, Definition 5.1 recovers the notions of parametrized spaces and spectra from Section 3. As a first check, it must be the case that if $\mathcal{X} = \mathcal{S}$ is the ∞–topos of spaces, then the ∞–category $\mathcal{C}_{/S}$ of objects of $\mathcal{C}$ parametrized over a contractible space S is equivalent to $\mathcal{C}$ itself. Since the projection $\mathcal{S}_{/S} \to \mathcal{S}$ determines an equivalence of ∞–topoi, this fact follows from the following proposition:

Proposition 5.2 *Let $\mathcal{C}$ be an ∞–category with all small limits. The evaluation at the point induces an equivalence*

$$\mathrm{Shv}_{\mathcal{C}}(\mathcal{S}) \to \mathcal{C}$$

between $\mathcal{C}$ and the ∞–category of $\mathcal{C}$–valued sheaves on $\mathcal{S}$.

Proof The ∞–category $\mathcal{S}^{\mathrm{op}}$ is freely generated under limits by the point and the ∞–category $\mathcal{C}$ admits all small limits. □

Now we verify that if one takes $\mathcal{X} = \mathrm{Pre}(S)$, the ∞–topos of presheaves of spaces on an ∞–groupoid S, we recover the notion of parametrized space or spectrum we worked with previously.

Lemma 5.3 *Let $\mathcal{C}$ be a (possibly large) ∞–category and let T be an ∞–groupoid. The ∞–category $\mathrm{Fun}(T^{\mathrm{op}}, \mathcal{C})$ of $\mathcal{C}$–valued presheaves on T is naturally equivalent to the ∞–category $\mathrm{Shv}_{\mathcal{C}}(\mathcal{S}_{/T})$ of $\mathcal{C}$–values sheaves on $\mathcal{S}_{/T}$.*

Proof The two notions are canonically equivalent via the equivalences (natural in T and $\mathcal{C}$)

$$\mathrm{Fun}(T^{\mathrm{op}},\mathcal{C}) \simeq \mathrm{Fun}^{\lim}(\mathrm{Pre}(T)^{\mathrm{op}},\mathcal{C}) \simeq \mathrm{Fun}^{\lim}(\mathcal{S}_{/T}^{\mathrm{op}},\mathcal{C}) \simeq \mathrm{Shv}_{\mathcal{C}}(\mathcal{S}_{/T}),$$

since, as in the proof of Proposition 5.2, $\mathrm{Pre}(T)$ is freely generated under colimits by the image of the Yoneda embedding $T \to \mathrm{Pre}(T)$. □

Therefore, the notation $\mathcal{C}_{/T}$ is unambiguous. Note that since T is an ∞–groupoid, $T \simeq T^{\mathrm{op}}$, so that we may also regard a local system on T as a covariant functor $T \to \mathcal{C}$.

Remark 5.4 If T is connected, then $T \simeq BG$, where $G \simeq \Omega T$ is the loop space of T (at a chosen basepoint). In this case, a functor $BG^{\mathrm{op}} \to \mathcal{C}$ is an object of $\mathcal{C}$ equipped with a right action of the ∞–group (grouplike monoidal ∞–groupoid) G. Note that if $\mathcal{C} = \mathcal{S}$ is the ∞–category of spaces, then our notion of a space parametrized over BG is a functor $BG^{\mathrm{op}} \to \mathcal{S}$, or a (naive) G–space. Of course, since BG is a space, we also have the slice ∞–category $\mathcal{S}_{/BG}$ of spaces over BG. These ∞–categories are canonically equivalent by the straightening construction of [36, 2.2.1.2].

As a special case of [37, Proposition 6.3.1.16], we have the following description of sheaves in terms of the symmetric monoidal product of presentable ∞–categories. Recall that the tensor product $\mathcal{A}\otimes\mathcal{B}$ of presentable ∞–categories $\mathcal{A}$ and $\mathcal{B}$ satisfies the universal property that, for any other presentable ∞–category $\mathcal{C}$, $\mathrm{Fun}^{\mathrm{L}}(\mathcal{A}\otimes\mathcal{B},\mathcal{C}) \subset \mathrm{Fun}(\mathcal{A}\times\mathcal{B},\mathcal{C})$ is the full subcategory spanned by those functors $f\colon \mathcal{A}\times\mathcal{B} \to \mathcal{C}$ which preserve colimits separately in each variable. Moreover, there is a canonical equivalence $\mathcal{A}\otimes\mathcal{B} \simeq \mathrm{Fun}^{\lim}(\mathcal{B}^{\mathrm{op}},\mathcal{A})$. If $\mathcal{C}$ is presentable and $\mathcal{D}$ admits all small limits, the inclusion of the full subcategory $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}},\mathcal{D}) \subseteq \mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}},\mathcal{D})$ of right adjoint functors into limit-preserving functors is also essentially surjective, hence an equivalence of ∞–categories. This follows from the equivalences

$$\mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}},\mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathcal{C},\mathcal{D}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{D}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}},\mathcal{D}),$$

where the middle equivalence uses [36, 5.5.2.10], the fact that any colimit-preserving functor from a presentable ∞–category to an ∞–category with all small colimits automatically admits a right adjoint. There are canonical equivalences of presentable ∞–categories

$$\mathcal{C}\otimes\mathcal{D} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}},\mathcal{D}) \simeq \mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}},\mathcal{D}),$$

where the first is the content of [37, 4.8.1.17] and the second follows from the preceding discussion.

Proposition 5.5 *Let $\mathcal{X}$ be an ∞–topos and let $\mathcal{C}$ be a presentable ∞–category. Then there is a canonical equivalence of presentable ∞–categories*

$$\mathcal{C} \otimes \mathcal{X} \simeq \mathrm{Shv}_{\mathcal{C}}(\mathcal{X}).$$

5.3 Objects parametrized by presheaves

The ∞–topos $\mathrm{Pre}(\mathcal{T})$ of presheaves of spaces on a small ∞–category $\mathcal{T}$ has the effect of formally adding all small colimits to $\mathcal{T}$. Therefore, if X is an object of $\mathrm{Pre}(\mathcal{T})$, then writing $\mathcal{T}_{/X}$ for the pullback

$$\begin{array}{ccc} \mathcal{T}_{/X} & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow \\ \mathrm{Pre}(\mathcal{T})_{/X} & \longrightarrow & \mathrm{Pre}(\mathcal{T}) \end{array}$$

the resulting fully faithful functor $\mathcal{T}_{/X} \to \mathrm{Pre}(\mathcal{T})_{/X}$ induces a colimit-preserving functor

$$\mathrm{Pre}(\mathcal{T}_{/X}) \to \mathrm{Pre}(\mathcal{T})_{/X}.$$

By construction, this functor is fully faithful, since it is the pullback of the fully faithful Yoneda embedding $\mathcal{T} \to \mathrm{Pre}(\mathcal{T})$ along the projection $\mathrm{Pre}(\mathcal{T})_{/X} \to \mathrm{Pre}(\mathcal{T})$. It is also essentially surjective: given a map $X \to Y$ in $\mathrm{Pre}(\mathcal{T})$, writing $X \simeq \mathrm{colim}\, U_\alpha$ as a colimit of representable presheaves U_α, we obtain a diagram in $\mathcal{T}_{/Y}$ whose colimit in $\mathrm{Pre}(\mathcal{T}_{/Y})$ is sent to its colimit X in $\mathrm{Pre}(\mathcal{T})_{/Y}$. Hence, it is an equivalence of ∞–categories.

The equivalence $\mathrm{Pre}(\mathcal{T})_{/X} \simeq \mathrm{Pre}(\mathcal{T}_{/X})$ can be formulated more conceptually. Recall that an object U of an ∞–category $\mathcal{C}$ is said to be *completely compact* if the associated corepresentable functor $\mathrm{map}_{\mathcal{C}}(U, -)\colon \mathcal{C} \to \mathrm{Sp}$ commutes with small colimits [36, 5.1.6.5].

Proposition 5.6 *Let $\mathcal{X}$ be an ∞–topos and let $\mathcal{T} \subset \mathcal{X}$ denote the full subcategory of completely compact objects of $\mathcal{X}$. Then $\mathcal{X}$ is a presheaf ∞–topos if and only if the colimit-preserving functor $\mathrm{Pre}(\mathcal{T}) \to X$ induced by the inclusion $\mathcal{T} \subset \mathcal{X}$ is an equivalence. In particular, any presheaf ∞–topos is freely generated under colimits by its full subcategory of completely compact objects.*

Proof Clearly $\mathcal{X}$ is a presheaf ∞–topos if $\mathrm{Pre}(\mathcal{T}) \to \mathcal{X}$ is an equivalence. Conversely, if $\mathcal{X}$ is a presheaf ∞–topos, then $\mathcal{X} \simeq \mathrm{Pre}(\mathcal{T}')$ for some small ∞–category $\mathcal{T}' \subseteq \mathcal{X}$. Since the objects of $\mathcal{T}'$ are completely compact, we see that $\mathcal{T}' \subset \mathcal{T}$, so $\mathrm{Pre}(\mathcal{T}') \subset \mathrm{Pre}(\mathcal{T})$

is fully faithful. It therefore suffices to show that $f\colon \mathrm{Pre}(\mathcal{T}) \to \mathcal{X}$ is fully faithful. To this end, choose $X \in \mathrm{Pre}(\mathcal{T})$, and write $X \simeq \operatorname{colim} U_\alpha$ for $U_\alpha \in \mathcal{T}$. Then, for any $U \in \mathcal{T}$,

$$\begin{aligned} \mathrm{map}_{\mathrm{Pre}(\mathcal{T})}(U, X) &\simeq \mathrm{map}_{\mathrm{Pre}(\mathcal{T})}(U, \operatorname{colim}_\alpha U_\alpha) \simeq \operatorname{colim}_\alpha \mathrm{map}_{\mathcal{T}}(U, U_\alpha) \\ &\simeq \operatorname{colim}_{\mathcal{X}} \mathrm{map}(U, U_\alpha) \simeq \operatorname{colim}_{\mathcal{X}} \mathrm{map}(U, f(X)) \end{aligned}$$

since f preserves colimits and U is completely compact. □

If $\mathcal{X} = \mathcal{S}_{/T}$ is the slice ∞–topos of spaces over T, then an object $S \to T$ of $\mathcal{X}$ is completely compact if and only if S is contractible [36, 5.1.6.9]. The following corollary is an immediate consequence:

Corollary 5.7 *Let $\mathcal{X} \simeq \mathrm{Pre}(\mathcal{T})$ be a presheaf ∞–topos. Then, for any (possibly large) ∞–category $\mathcal{C}$, there is a canonical equivalence*

$$\mathrm{Shv}_{\mathcal{C}}(\mathcal{X}) \simeq \mathrm{Pre}_{\mathcal{C}}(\mathcal{T}).$$

In particular, if $\mathcal{X} = \mathcal{S}_{/T}$, then $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X}) \simeq \mathrm{Pre}_{\mathcal{C}}(T)$.

5.4 The base-change functors: f^* and its adjoints $f_!$ and f_*

In practice, it is useful to require more structure on $\mathcal{C}$ than that of an arbitrary ∞–category. The first and most useful assumption is that $\mathcal{C}$ is presentable, which is to say that $\mathcal{C}$ has all small colimits and $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$ for some infinite regular cardinal κ (here $\mathcal{C}^\kappa$ denotes the full subcategory of κ–compact objects in $\mathcal{C}$). Since maps between presentable ∞–categories are typically taken to be colimit-preserving, they admit right adjoints by definition. We want our theory of parametrized objects to reflect this; that is, when $\mathcal{C}$ is presentable, each of the base-change functors $f^*\colon \mathcal{C}_{/T} \to \mathcal{C}_{/S}$ should admit a right adjoint $f_*\colon \mathcal{C}_{/S} \to \mathcal{C}_{/T}$.

Proposition 5.8 *Let $\mathcal{X}$ be an ∞–topos and let $\mathcal{C}$ be a presentable ∞–category. Then there exists a unique sheaf of presentable ∞–categories and left-adjoint functors*

$$\mathcal{C}_{/(-)}\colon \mathcal{X}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$$

on $\mathcal{X}$ whose value at the object $S \in \mathcal{X}$ is equivalent to the presentable ∞–category

$$\mathcal{C}_{/S} \simeq \mathcal{C} \otimes \mathcal{X}_{/S} \simeq \mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$$

of $\mathcal{C}$–valued sheaves on $\mathcal{X}_{/S}$.

Proof Since the restriction functors $f^*\colon \mathcal{X}_{/S} \to \mathcal{X}_{/T}$ are left-adjoint functors of presentable ∞–categories, it is clear that, tensoring with $\mathcal{C}$, we obtain a unique functor $\mathcal{X}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$. Thus, it only remains to see that this functor preserves limits. Equivalently, since $\mathcal{C} \otimes (-)\colon \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}$ commutes with colimits, we can check instead that the left adjoints $f_!$ of the restrictions f^* induce colimit decompositions

$$\operatorname{colim}_\alpha \mathcal{X}_{/U_\alpha} \to \mathcal{X}_{/T}$$

in Pr^{L} for any colimit diagram $\operatorname{colim}_\alpha U_\alpha \simeq T$ in $\mathcal{X}$. Since the forgetful functor $\mathrm{Pr}^{\mathrm{L}} \to \widehat{\mathrm{Cat}}_\infty$ preserves limits, this follows immediately from descent, ie that the restrictions induce an equivalence $\mathcal{X}_{/T} \simeq \lim_\alpha \mathcal{X}_{/U_\alpha}$ in $\widehat{\mathrm{Cat}}_\infty$. □

The restriction functors $f^*\colon \mathcal{X}_{/T} \to \mathcal{X}_{/S}$ admit both left and right adjoints $f_!$ and f^*, respectively. Let $\mathrm{Pr}^{\mathrm{L,R}}$ denote the subcategory of Pr^{L} whose objects are again presentable ∞–categories, but whose morphisms consist of those functors $\mathcal{C} \to \mathcal{D}$ which are simultaneously both left and right adjoints (equivalently, by the adjoint functor theorem, those functors $\mathcal{C} \to \mathcal{D}$ which preserve all limits and colimits).

Lemma 5.9 *Each of the arrows in the cartesian square*

$$\begin{array}{ccc} \mathrm{Pr}^{\mathrm{L,R}} & \longrightarrow & \mathrm{Pr}^{\mathrm{L}} \\ \downarrow & & \downarrow \\ \mathrm{Pr}^{\mathrm{R}} & \longrightarrow & \widehat{\mathrm{Cat}}_\infty \end{array}$$

preserve and detect small limits.

Proof Both arrows to $\widehat{\mathrm{Cat}}_\infty$ preserve limits by [36, Theorems 5.5.3.13 and 5.5.3.18], and their proofs reveal that they also detect limits. □

Theorem 5.10 *Let $\mathcal{X}$ be an ∞–topos and $\mathcal{C}$ a presentable ∞–category. Then the sheaf $\mathcal{C}_{/(-)}\colon \mathcal{X}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$ of presentable ∞–categories on $\mathcal{X}$ factors through the subcategory $\mathrm{Pr}^{\mathrm{L,R}} \subset \mathrm{Pr}^{\mathrm{L}}$. In particular, there exists a unique sheaf of presentable ∞–categories and left- and right-adjoint functors*

$$\mathcal{C}_{/(-)}\colon \mathcal{X}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L,R}}$$

on $\mathcal{X}$ whose value at the object $S \in \mathcal{X}$ is equivalent to the ∞–category $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X}_{/S})$ of $\mathcal{C}$–valued sheaves on $\mathcal{X}_{/S}$.

Proof By Proposition 5.2, to specify a limit-preserving functor $F\colon \mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, it is enough to specify the image of the initial object $*$, which we take to be $\mathcal{C}$. For a given space S, we have that $\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{C}) \simeq \lim_S \mathcal{C}$, which shows that the value of F on S is equivalent to $\mathcal{C}_{/S}$. The result now follows from the factorization obtained in Lemma 5.9 above. □

Thus far, we have constructed, for each ∞–topos $\mathcal{X}$ and presentable ∞–category $\mathcal{C}$, a "three-functor formalism" for the theory of objects of $\mathcal{C}$ parametrized over objects of $\mathcal{X}$. There are a number of "Beck–Chevalley-type" relations which occur when given a pullback square in $\mathcal{X}$; see [41, Propositions 2.2.11 and 11.4.8] for a treatment in the context of (pointed) spaces and spectra parametrized over spaces.

Proposition 5.11 (Beck–Chevalley conditions) *Suppose given a cartesian square*

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ U & \xrightarrow[i]{} & V \end{array}$$

in an ∞–topos $\mathcal{X}$. Then there are canonical natural equivalences $g_! f^* \simeq i^* h_!$ and $i^* h_* \simeq g_* f^*$ of functors $\mathcal{X}_{/T} \to \mathcal{X}_{/U}$ that are interchanged by adjunction.

Proof Using the commutativity of the square and (co)unit transformations, it is easy to construct natural transformations $g_! f^* \to i^* h_!$ and $i^* h_* \to g_* f^*$. Moreover, by adjunction and symmetry, the second transformation is an equivalence if and only if the first is an equivalence. Thus, it only remains to show that the transformation $g_! f^* \to i^* h_!$ induces an equivalences upon evaluation at an object X of $\mathcal{X}_{/T}$. But the projection $\mathcal{X}_{/T} \to \mathcal{X}$ is conservative, meaning it is enough to check this in $\mathcal{X}$ itself, where it follows from the equivalences

$$X \times_T S \simeq X \times_T T \times_V U \simeq X \times_V U. \qquad \square$$

Corollary 5.12 *Let $\mathcal{X}$ be an ∞–topos and let $\mathcal{C}$ be an $\mathcal{X}$–module in Pr^{L}. Suppose given a cartesian square*

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ U & \xrightarrow[i]{} & V \end{array}$$

in $\mathcal{X}$, as above. Then there are canonical natural equivalences $g_! f^* \simeq i^* h_!$ and $i^* h_* \simeq g_* f^*$ of functors $\mathcal{C}_{/T} \to \mathcal{C}_{/U}$ that are interchanged by adjunction.

5.5 The proper pushforward and its right adjoint

We now suppose that $\mathcal{C}$ is a stable presentable ∞–category. In this case, it turns out that for proper geometric morphisms $p\colon \mathcal{X} \to \mathcal{Y}$ of ∞–topoi, the induced functor $p_*\colon \mathrm{Shv}_{\mathcal{C}}(\mathcal{X}) \to \mathrm{Shv}_{\mathcal{C}}(\mathcal{Y})$ preserves *all* colimits, so that p_* itself admits a right adjoint $p^!\colon \mathrm{Shv}_{\mathcal{C}}(\mathcal{Y}) \to \mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$. To show this, we make use of the relative symmetric monoidal product on presentable ∞–categories; there is a product $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}$ if $\mathcal{R}$ is a commutative algebra object of Pr^{L} and $\mathcal{A}$ and $\mathcal{B}$ are (right) $\mathcal{R}$–modules, which enjoys a similar universal property to the absolute notion but with colimit-preserving replaced by $\mathcal{R}$–linearity.

Proposition 5.13 *Let $\mathcal{C}$ be a compactly generated stable ∞–category and $p\colon \mathcal{X} \to \mathcal{Y}$ a proper geometric morphism of ∞–topoi. Then the induced functor $p_*\colon \mathrm{Shv}_{\mathcal{C}}(\mathcal{X}) \to \mathrm{Shv}_{\mathcal{C}}(\mathcal{Y})$ admits a right adjoint $p^!\colon \mathrm{Shv}_{\mathcal{C}}(\mathcal{Y}) \to \mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$.*

Proof By [36, Remark 7.3.1.5], $p_*\colon \mathcal{X} \to \mathcal{Y}$ preserves filtered colimits, so that $p_*\colon \mathrm{Sp} \otimes \mathcal{X} \to \mathrm{Sp} \otimes \mathcal{Y}$ is a map of Sp–modules in Pr^{L} and in particular preserves all colimits. Now, since $\mathcal{C}$ is stable and presentable, $\mathcal{C}$ is also an Sp–module in Pr^{L}, so that

$$p_*\colon \mathcal{C} \otimes \mathcal{X} \simeq \mathcal{C} \otimes_{\mathrm{Sp}} \mathrm{Sp} \otimes \mathcal{X} \to \mathcal{C} \otimes_{\mathrm{Sp}} \mathrm{Sp} \otimes \mathcal{Y} \simeq \mathcal{C} \otimes \mathcal{Y}$$

is again a map of Sp–modules in Pr^{L} and in particular preserves all colimits. It follows that $p_*\colon \mathrm{Shv}_{\mathcal{C}}(\mathcal{X}) \simeq \mathcal{C} \otimes \mathcal{X} \to \mathcal{C} \otimes \mathcal{Y} \simeq \mathrm{Shv}_{\mathcal{C}}(\mathcal{Y})$ admits a right adjoint $p^!\colon \mathrm{Shv}_{\mathcal{C}}(\mathcal{Y}) \to \mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$. □

Definition 5.14 Let $\mathcal{X}$ be an ∞–topos. Then a map $p\colon S \to T$ in $\mathcal{X}$ is proper if $p_*\colon \mathcal{X}_{/S} \to \mathcal{X}_{/T}$ is a proper morphism of ∞–topoi.

For example, in the ∞–topos of spaces, a map is proper if the fibers are compact. Given a proper map $p\colon S \to T$ in $\mathcal{X}$, it follows that the pushforward p_* admits a right adjoint $p^!$. This gives a series of adjunctions $p_!$, p^*, p_*, $p^!$, each of which is right adjoint to the functor on its left.

5.6 The tangent bundle of an ∞–topos

In this subsection, we provide an interpretation of parametrized spaces and spectra over an arbitrary ∞–topos in terms of the notion of tangent bundles of ∞–categories.

Let $\mathcal{X}$ be an ∞–topos and let

$$\mathcal{O}_{\mathcal{X}} \simeq \mathrm{Fun}(\Delta^1, \mathcal{X})$$

denote the source of the presentable fibration $p\colon \mathcal{O}_{\mathcal{X}} \to \mathcal{X}$ over $\mathcal{X}$ associated to "target" map $d_0\colon \Delta^1 \to \Delta^0$. Note that p is a presentable cartesian fibration since $\mathcal{X}$ admits pullbacks and each of the fibers $\mathcal{X}_{/X}$ over $X \in \mathcal{X}$ is presentable (even an ∞–topos), and that the fiber of the projection $p\colon \mathcal{O}_{\mathcal{X}} \to \mathcal{X}$ is precisely the ∞–category $\mathcal{S}_{/X}$ of spaces over X.

The objects are $\mathcal{S}_{/X}$ not literally spaces fibered over X, for the simple reason that X itself need not be a space, but we may nevertheless reasonably regard them as spaces over X for the following reason. If $\mathcal{X} = \mathcal{S}$, then a space over $X \in \mathcal{X}$ is precisely an X–indexed family (a functor $X^{\mathrm{op}} \to \mathcal{X}$) of objects of $\mathcal{X}$, so that this literally still holds if X is a "space" in $\mathcal{X}$. Furthermore, even if X is an arbitrary object of $\mathcal{X}$, and therefore not necessarily a space, or even in the image of the canonical colimit-preserving functor $\mathcal{S} \to \mathcal{X}$ given by sending the point to the terminal object of $\mathcal{X}$, we may still identify $\mathcal{X}_{/X}$ with the ∞–category of sheaves of spaces on $\mathcal{X}_{/X}$.

Lemma 5.15 *Let $\mathcal{X}$ be an ∞–topos and let $t\colon \mathcal{X} \to \mathcal{S}$ denote the unique geometric morphism from $\mathcal{X}$ to the ∞–category $\mathcal{S}$ of spaces.*

(1) *If $X \simeq t^*S$ for some $S \in \mathcal{S}$, then $\mathcal{X}_{/X} \simeq \mathrm{Fun}(S^{\mathrm{op}}, \mathcal{X})$.*
(2) *For any object $X \in \mathcal{X}$, $\mathcal{X}_{/X} \simeq \mathrm{Fun}^{\lim}(\mathcal{X}_{/X}^{\mathrm{op}}, \mathcal{S})$ is the ∞–topos of bundles of spaces over $\mathcal{X}_{/X}$.*

The target fibration $p\colon \mathcal{O}_{\mathcal{X}} \to \mathcal{X}$ is an unstable version of the tangent bundle $q\colon \mathrm{T}_{\mathcal{X}} \to \mathcal{X}$ of $\mathcal{X}$; see [37, Definition 7.3.1.9] for details. Since p is a presentable fibration, we may regard it as the unstraightening of the (limit-preserving) functor $\mathcal{S}_{/\mathcal{X}}\colon \mathcal{X}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}}$. Stabilizing, we arrive at a functor $\mathrm{Sp}_{\mathcal{X}}\colon \mathcal{X}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}}_{\mathrm{St}}$, which unstraightens to the tangent bundle $q\colon \mathrm{T}_{\mathcal{X}} \to \mathcal{X}$. We record this in the following proposition.

Proposition 5.16 *Let $\mathcal{X}$ be an ∞–topos. Then the fiber of the projection*

$$p\colon \mathcal{O}_{\mathcal{X}} \to \mathcal{X}$$

over $X \in \mathcal{X}$ is the ∞–category $\mathcal{S}_{/X}$ of spaces over X, and the fiber of the projection

$$q\colon \mathrm{T}_{\mathcal{X}} \to \mathcal{X}$$

over $X \in \mathcal{X}$ is the ∞–category $\mathrm{Sp}_{/X}$ of spectra over X.

6 Algebraic structures on parametrized objects

Certain results concerning Thom spectra only make sense when the spectra themselves admit $\mathbb{E}_\infty$–algebra structures. The algebraic structures in question are encoded by the action of an ∞–operad $\mathcal{O}^\otimes$. Hence, to make this precise, it would be useful to know when a given operadic structure on the ∞–category $\mathcal{C}$ determines an operadic structure on ∞–category $\mathcal{C}_{/(-)}$ of objects of $\mathcal{C}$ parametrized over objects of $\mathcal{X}$.

6.1 The closed monoidal structure

We fix an ∞–operad $\mathcal{O}^\otimes$, an ∞–topos $\mathcal{X}$ and an $\mathcal{O}$–monoidal presentable ∞–category $\mathcal{C}^\otimes$. The goal now is to construct, for each object S of $\mathcal{X}$, an $\mathcal{O}$–monoidal ∞–category $\mathcal{C}^\otimes_{/S}$ with underlying ∞–category $\mathcal{C}_{/S}$ such that the restriction functor $f^*\colon \mathcal{C}_{/T} \to \mathcal{C}_{/S}$ induced by a map $f\colon S \to T$ in $\mathcal{X}$ is $\mathcal{O}$–monoidal.

Let $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L,R}})$ denote the pullback

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L,R}}) & \longrightarrow & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L}}) \\ \downarrow & & \downarrow \\ \mathrm{Pr}^{\mathrm{L,R}} & \longrightarrow & \mathrm{Pr}^{\mathrm{L}} \end{array}$$

that is, the subcategory of $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L}})$ consisting of those $\mathcal{O}$–monoidal functors which are also right adjoints. To analyze the behavior of limits in $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L,R}})$ we require the following technical lemma:

Lemma 6.1 *The subcategory $\widehat{\mathrm{Cat}}{}^{\mathrm{R}}_\infty \subset \widehat{\mathrm{Cat}}_\infty$ spanned by the complete ∞–categories and the limit-preserving functors is stable under pullbacks.*

Proof Suppose given a pullback diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{C} & \xrightarrow{i} & \mathcal{D} \end{array}$$

in $\widehat{\mathrm{Cat}}_\infty$ such that $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are complete ∞–categories and h and i are limit-preserving functors. We first show $\mathcal{A}$ is complete, which amounts to showing that the constant diagram functor $\mathcal{A} \to \mathrm{Fun}(K, \mathcal{A})$ admits a right adjoint $\lim\colon \mathrm{Fun}(K, \mathcal{A}) \to \mathcal{A}$. Since the corresponding result holds for $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ by assumption, we obtain a map

$$\mathrm{Fun}(K, \mathcal{A}) \simeq \mathrm{Fun}(K, \mathcal{B}) \times_{\mathrm{Fun}(K, \mathcal{D})} \mathrm{Fun}(K, \mathcal{C}) \to \mathcal{B} \times_{\mathcal{D}} \mathcal{C} \simeq \mathcal{A},$$

which is easily seen to be right adjoint to $\mathcal{A} \to \mathrm{Fun}(K, \mathcal{A})$ since mapping spaces in a limit of ∞–categories are computed as the limit of the mapping spaces. To see that the projections $f\colon \mathcal{A} \to \mathcal{B}$ and $g\colon \mathcal{A} \to \mathcal{C}$ preserve limits, we note that, by construction, $\lim\colon \mathrm{Fun}(K, \mathcal{A}) \to \mathcal{A}$ is the pullback of the diagram of maps $\lim\colon \mathrm{Fun}(K, \mathcal{B}) \to \mathcal{B}$ and $\lim\colon \mathrm{Fun}(K, \mathcal{C}) \to \mathcal{C}$ over $\lim\colon \mathrm{Fun}(K, \mathcal{D}) \to \mathcal{D}$, so this is clear.

Finally, given a commutative diagram

$$\begin{array}{ccc} \mathcal{A}' & \xrightarrow{f} & \mathcal{B} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{C} & \xrightarrow{i} & \mathcal{D} \end{array}$$

in $\widehat{\mathrm{Cat}}_\infty^{\mathrm{R}}$, we must show that the ∞–groupoid of limit-preserving functors from $\mathcal{A}'$ to $\mathcal{A}$ over $\mathcal{B} \to \mathcal{D} \leftarrow \mathcal{C}$ is contractible. This follows from the fact that these functors form a full ∞–subgroupoid of the ∞–groupoid of all functors from $\mathcal{A}'$ to $\mathcal{A}$ over $\mathcal{B} \to \mathcal{D} \leftarrow \mathcal{C}$ coupled with the observations that this latter ∞–groupoid is contractible and that the unique such functor preserves limits. □

Remark 6.2 A similar argument shows that the subcategory $\widehat{\mathrm{Cat}}_\infty^{\mathrm{R}} \subset \widehat{\mathrm{Cat}}_\infty$ is stable under all small limits.

Since $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L,R}})$ is the pullback of a diagram of complete ∞–categories and limit-preserving functors, Lemma 6.1 has the following corollary:

Corollary 6.3 *The ∞–category $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L,R}})$ admits all small limits and the inclusion of the subcategory $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L,R}}) \subset \mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L}})$ preserves them.*

Our main foundational theorem is the following result, which follows from Corollary 6.3 and the proof of Theorem 5.10.

Theorem 6.4 *There exists a unique sheaf of presentable $\mathcal{O}$–monoidal ∞–categories $\mathcal{C}^{\otimes}_{/(-)}$ on $\mathcal{X}$ whose value at the object S of $\mathcal{X}$ is the $\mathcal{O}$–monoidal ∞–category $\mathcal{C}^{\otimes}_{/S}$ of $\mathcal{C}^{\otimes}$–valued sheaves on S.*

Remark 6.5 In the symmetric monoidal context, the right adjoint f_* is lax $\mathcal{O}$–monoidal and the left adjoint $f_!$ is oplax $\mathcal{O}$–monoidal.

We now further suppose that $\mathcal{O}^{\otimes}$ comes equipped with a fixed map $\mathbb{E}_1^{\otimes} \to \mathcal{O}^{\otimes}$. This implies (by restriction along this map) that any $\mathcal{O}$–monoidal ∞–category $\mathcal{C}^{\otimes}$ is equipped with a distinguished monoidal structure $\otimes$. The following lemma is immediate from the adjoint functor theorem:

Lemma 6.6 *Let $\mathcal{C}^{\otimes}$ be a monoidal presentable ∞–category. Then $\mathcal{C}^{\otimes}$ is closed in the sense that, for each object X of $\mathcal{C}$, the left and right multiplication functors $X \otimes (-)\colon \mathcal{C} \to \mathcal{C}$ and $(-) \otimes X\colon \mathcal{C} \to \mathcal{C}$ admit right adjoints.*

Writing $\otimes$ for the monoidal product obtained by restriction along the map $\mathbb{E}_1^{\otimes} \to \mathcal{O}^{\otimes}$, for each object $X \in \mathcal{C}$, we write

$$F(X,-)\colon \mathcal{C} \to \mathcal{C}$$

for the right adjoint of the right multiplication functor $(-) \otimes X\colon \mathcal{C} \to \mathcal{C}$. As S varies over all spaces, the base-change functors and closed tensor structures collectively give rise to a (not necessarily symmetric) sort of "Wirthmüller context" [22].

In the context of parametrized spaces, f^* is a symmetric monoidal functor. The situation of a symmetric monoidal functor with left and right adjoints gives rise to a series of compatibility formulas (eg the projection formula). Following [22], we now abstract this relationship into what we will refer to as a Wirthmüller context.

We continue to fix an ∞–operad $\mathcal{O}^{\otimes}$ over $\mathbb{E}_1$ and an $\mathcal{O}$–monoidal ∞–category $\mathcal{C}^{\otimes}$. To say more, we now suppose that the map $\mathbb{E}_1^{\otimes} \to \mathcal{O}^{\otimes}$ factors through $\mathbb{E}_{\infty}^{\otimes}$, which is to say that $\mathcal{C}^{\otimes}$ is a *symmetric* monoidal presentable ∞–category [36, Proposition 4.1.1.20]. Specializing the definition of $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L,R}})$ to the case of the terminal ∞–operad, we obtain the ∞–category $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,R}})$ of symmetric monoidal presentable ∞–categories and symmetric monoidal functors which are simultaneously left and right adjoints.

Definition 6.7 A Wirthmüller context is a $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,R}})$–valued sheaf on $\mathcal{X}$, that is, a limit-preserving functor

$$\mathcal{X}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,R}}).$$

Some of the useful consequences of the existence of a Wirthmüller context are summarized in the following standard proposition:

Proposition 6.8 *Let $\mathcal{X}$ be an ∞–topos, $\mathcal{C}^{\otimes}$ a symmetric monoidal presentable ∞–category, $f\colon S \to T$ a morphism in $\mathcal{X}$, X an object of $\mathcal{C}_{/S}$, and Y and Z objects of $\mathcal{C}_{/T}$. Then there are natural equivalences*

(1) $f^*(Y \otimes_T Z) \simeq f^*Y \otimes_S f^*Z$,

(2) $F_T(Y, f_*X) \simeq f_* F_S(f^*Y, X)$,

(3) $f^* F_T(Y, Z) \simeq F_S(f^*Y, f^*Z)$,

(4) $f_!(f^*Y \otimes_S X) \simeq Y \otimes_T f_!X$,

(5) $F_T(f_!X, Y) \simeq f_* F_S(X, f^*Y)$.

Proof As explained in [22], we can deduce all of these equivalences from (1) and the projection formula (4). First, the equation (1) follows immediately from the fact that $f^*\colon \mathcal{C}^{\otimes}_{/T} \to \mathcal{C}^{\otimes}_{/S}$ is a strong symmetric monoidal functor. For (4), recall that we have a natural composite

$$(f^*Y \otimes_S X) \to (f^*Y \otimes_S f^*Y f_!X) \to f^*(Y \otimes_T f_!X),$$

where the first map is the unit of the adjunction and the second map comes from the fact that f^* is strong monoidal. The projection formula arises from the adjoint of this composite. It is immediate that this map is an equivalence whenever $\mathcal{C} = \mathrm{Pre}(\mathcal{T})$ is a presheaf ∞–topos since (replacing $\mathcal{X}$ with $\mathcal{C} \otimes \mathcal{X} \simeq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{X})$) the projection formula holds inside any ∞–topos: $Y \times_T S \times_S X \simeq Y \times_T X$. To conclude (4) in general, we use the fact that any symmetric monoidal presentable ∞–category $\mathcal{C}$ is a symmetric monoidal accessible localization of some $\mathrm{Pre}(\mathcal{T})$; since the localization functor is a strong symmetric monoidal left adjoint [37, 2.2.1.9], we can deduce (4) from the result for presheaves.

We now explain how to obtain the remaining equivalences. Let $f^*\colon \mathcal{B}^{\otimes} \to \mathcal{A}^{\otimes}$ be a morphism of commutative algebra objects in $\mathrm{Pr}^{\mathrm{L,R}}$. By the relative adjoint functor theorem [37, 8.3.2.7], $f_*\colon \mathcal{A}^{\otimes} \to \mathcal{B}^{\otimes}$ is lax symmetric monoidal. Using [10, Theorem 1.4] to study the opposite category, we can analogously deduce that $f_!\colon \mathcal{A}^{\otimes} \to \mathcal{B}^{\otimes}$ is oplax symmetric monoidal. Now suppose we are given an object X of $\mathcal{A}$ and objects Y and Z of $\mathcal{B}$. Then (2) follows from (1) because

$$\begin{aligned} \mathrm{map}(Z, F(Y, f_*X)) &\simeq \mathrm{map}(Z \otimes Y, f_*X) \simeq \mathrm{map}(f^*Z \otimes f^*Y, X) \\ &\simeq \mathrm{map}(f^*Z, F(f^*Y, X)) \simeq \mathrm{map}(Z, f_*F(f^*Y, X)), \end{aligned}$$

and (3) follows from (2) and (4) since the unit applied to Z gives a map

$$F(Y, Z) \to F(Y, f_*f^*Z) \simeq f_*F(f^*Y, f^*Z)$$

whose adjoint is the map $f^*F(Y, Z) \to F(f^*Y, f^*Z)$, which is an equivalence because

$$\begin{aligned} \mathrm{map}(X, f^*F(Y, Z)) &\simeq \mathrm{map}(f_!X, F(Y, Z)) \simeq \mathrm{map}(f_!X \otimes Y, Z) \\ &\simeq \mathrm{map}(f_!(X \otimes f^*Y), Z) \simeq \mathrm{map}(X \otimes f^*Y, f^*Z) \\ &\simeq \mathrm{map}(X, F(f^*Y, f^*Z)). \end{aligned}$$

Finally, (5) follows because

$$\begin{aligned}\operatorname{map}(Z, F(f_! X, Y)) &\simeq \operatorname{map}(Z \otimes f_! X, Y) \simeq \operatorname{map}(f_!(f^* Z \otimes X), Y)\\ &\simeq \operatorname{map}(f^* Z \otimes X, f^* Y) \simeq \operatorname{map}(f^* Z, F(X, f^* Y))\\ &\simeq \operatorname{map}(Z, f_* F(X, f^* Y)).\end{aligned}$$

(See also [41, 2.2.2 and 11.4.1] for a verification in the particular case of parametrized spaces and spectra.) □

We can also consider this situation when $\mathcal{C}^{\otimes}$ is an $\mathbb{E}_n$–monoidal presentable ∞–category. In this case, provided that $n > 2$, the theory is the same because the equivalences of Proposition 6.8 arise as isomorphisms in the homotopy category, and for $n > 2$ an $\mathbb{E}_n$–monoidal presentable ∞–category has a closed symmetric monoidal homotopy category. (We also use the fact that $\mathbb{E}_n$–monoidal functors induce symmetric monoidal functors on the homotopy category in this case.) We suspect that analogous formulas hold for the braided monoidal case $n = 2$ and even the monoidal case $n = 1$.

Finally, when $\mathcal{X} = \mathcal{S}$, we have the following basic existence result as a corollary of Theorem 6.4:

Corollary 6.9 *A Wirthmüller context over $\mathcal{S}$ determines, and is determined by, a symmetric monoidal presentable ∞–category $\mathcal{C}^{\otimes}$.*

Remark 6.10 This generalizes in a obvious way to presheaf ∞–topoi, and in a straightforward but less obvious way to ∞–topoi. We leave the details to the interested reader.

6.2 Parametrized objects over spaces with multiplicative structure

In this section, we study the multiplicative structures that arise on ∞–categories of parametrized objects over $\mathbb{E}_n$–spaces. In the previous sections, the multiplicative structure on $\mathcal{C}_{/S}$ was obtained pointwise, or, equivalently, from the evident external product via pullback along the diagonal $\Delta\colon S \to S \times S$ of the base space. Here, the multiplicative structures arise from an actual product on S itself.

Recall that the ∞–categorical Day convolution product [37, Section 6.3] is a consequence of the existence of a symmetric monoidal functor from spaces to presentable ∞–categories. The relevant functor on objects agrees with the sheaf $\mathcal{S}_{/(-)}\colon \mathcal{S} \to \mathrm{Pr}^{\mathrm{L}}$ on objects, but takes maps of Kan complexes $f\colon S \to T$ to the left adjoint $f_!\colon \mathcal{S}_{/S} \to \mathcal{S}_{/T}$

of f^*. Since we will be interested in ∞–categories of modules over an $\mathcal{O}$–monoidal presentable ∞–category $\mathcal{R}$, for the remainder of this section we will replace Pr^{L} with the equivalent ∞–category $\mathrm{Mod}_{\mathcal{S}}$ of $\mathcal{S}$–module objects in Pr^{L}.

Proposition 6.11 *There is a unique colimit-preserving functor*

$$\mathrm{Pre}\colon \mathcal{S} \to \mathrm{Mod}_{\mathcal{S}}$$

whose value at the space T is the ∞–topos $\mathcal{S}_{/T}$ of presheaves of spaces on T.

Proof The ∞–category of spaces $\mathcal{S}$ is freely generated under colimits by the one-point space [36, 5.1.5.8]. Since any space T is equivalent to the T–indexed colimit of the constant diagram on the point, it follows that

$$\mathrm{Pre}(T) \simeq T \otimes \mathcal{S}. \qquad \square$$

We have the following proposition as a consequence of the properties of the ∞–categorical Day convolution [37, 6.3.1.2]:

Proposition 6.12 *The functor* $\mathrm{Pre}\colon \mathcal{S} \to \mathrm{Mod}_{\mathcal{S}}$ *extends to a symmetric monoidal functor*

$$\mathrm{Pre}\colon \mathcal{S}^{\otimes} \to \mathrm{Mod}_{\mathcal{S}}^{\otimes}.$$

It follows from Proposition 6.12 that the functor Pre preserves multiplicative structures:

Corollary 6.13 *Let $\mathcal{O}$ be an ∞–operad and let X be an $\mathcal{O}$–algebra object of $\mathcal{S}$. Then* $\mathrm{Pre}(X)$ *is an $\mathcal{O}$–algebra object of* $\mathrm{Mod}_{\mathcal{S}}$.

We now assume that $\mathcal{O}$ is a unital and coherent ∞–operad. Since $\mathrm{Mod}_{\mathcal{S}}$ is a symmetric monoidal ∞–category, we can consider $\mathcal{O}$–algebra objects in $\mathrm{Mod}_{\mathcal{S}}$. We require $\mathcal{O}$ to be unital since we will need to consider the unit map $\eta\colon \mathcal{S} \to \mathcal{R}$ of an $\mathcal{O}$–algebra object $\mathcal{R}$ of $\mathrm{Mod}_{\mathcal{S}}$.

Recall that if $\mathcal{R}$ is an object of $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_{\mathcal{S}})$ then $\mathrm{Mod}_{\mathcal{R}}^{\mathcal{O}}$ is the ∞–category of $\mathcal{R}$–module objects in $\mathrm{Mod}_{\mathcal{S}}$ [37, Section 3.3.3].

Proposition 6.14 *Let $\mathcal{O}$ be a coherent and unital ∞–operad and let $\mathcal{R}$ be an $\mathcal{O}$–algebra object of* $\mathrm{Mod}_{\mathcal{S}}$. *Then there exists a unique colimit-preserving functor*

$$\mathrm{Pre}_{\mathcal{R}}\colon \mathcal{S} \to \mathrm{Mod}_{\mathcal{R}}^{\mathcal{O}}$$

whose value on the point is the "free rank-one" $\mathcal{R}$–module $\mathcal{R}$.

Proof The functor

$$\eta^*\colon (-) \otimes_{\mathcal{S}} \mathcal{R}\colon \mathrm{Mod}_{\mathcal{S}}^{\mathcal{O}} \to \mathrm{Mod}_{\mathcal{R}}^{\mathcal{O}}$$

preserves colimits, as it is left adjoint to the restriction

$$\eta_*\colon \mathrm{Mod}_{\mathcal{R}}^{\mathcal{O}} \to \mathrm{Mod}_{\mathcal{S}}^{\mathcal{O}}$$

along $\eta\colon \mathcal{S} \to \mathcal{R}$. Thus, the composite

$$\mathrm{Pre}_{\mathcal{R}} \simeq \eta^* \circ \mathrm{Pre}\colon \mathcal{S} \to \mathrm{Mod}_{\mathcal{S}}^{\mathcal{O}} \to \mathrm{Mod}_{\mathcal{R}}^{\mathcal{O}}$$

preserves colimits, and sends the one-point space $*$ to $\mathcal{R} \simeq \mathcal{S} \otimes_{\mathcal{S}} \mathcal{R}$. □

The functor $\mathrm{Pre}_{\mathcal{R}}$ also preserves multiplicative structures:

Corollary 6.15 *Let $\mathcal{O}$ be a coherent ∞–operad and let X be an $\mathcal{O}$–algebra object of $\mathcal{S}$. Then $\mathrm{Pre}_{\mathcal{R}}(X)$ is an $\mathcal{O}$–algebra object of $\mathrm{Mod}_{\mathcal{R}}^{\mathcal{O}}$.*

The main example of this phenomenon which will be of interest to us is the case of an $\mathcal{O}$–algebra object $\mathcal{R}$ of $\mathrm{Mod}_{\mathrm{Sp}}$, where $\mathcal{O}$ is a coherent ∞–operad under $\mathbb{E}_1$. Then $\mathcal{R}$ is in particular an associative algebra object of $\mathrm{Mod}_{\mathrm{Sp}}$, and so it has a Picard ∞–groupoid $\mathrm{Pic}(\mathcal{R})$, the full subgroupoid of $\mathcal{R}$ spanned by the invertible objects.

7 Picard spaces

In this section we define and study the Picard ∞–groupoid of an $\mathcal{O}$–monoidal stable presentable ∞–category $\mathcal{R}$ (for suitable ∞–operads $\mathcal{O}^{\otimes}$) and the categories of parametrized objects over Picard ∞–groupoids. Roughly speaking, we define the Picard ∞–groupoid of $\mathcal{R}$ as the space of invertible objects in $\mathcal{R}$; the work of the section is to keep track of the multiplicative structure inherited from $\mathcal{R}$. The main theorem of this section describes Pic as participating in an adjunction that (when specialized to modules over an $\mathbb{E}_n$–ring spectrum) gives rise to the Thom spectrum functor as the counit. In the special case where R is an $\mathbb{E}_n$–ring spectrum, $\mathrm{Mod}_R^{\otimes}$ inherits the structure of a presentable $\mathbb{E}_{n-1}$–monoidal ∞–category, and

$$\mathrm{Pic}_R = \mathrm{Pic}(\mathrm{Mod}_R^{\otimes})$$

is an $\mathbb{E}_{n-1}$–space which comes equipped with an $\mathbb{E}_{n-1}$–map $i\colon B\mathrm{GL}_1(R) \to \mathrm{Pic}_R \subset \mathrm{Mod}_R$ which is the delooping of the $\mathbb{E}_n$–map $\mathrm{GL}_1(R) \simeq \mathrm{Aut}_{\mathrm{Mod}_R}(R)$. In particular, i is the inclusion of the connected component of the identity $B\mathrm{GL}_1(R)$ in Pic_R.

Note that contrary to the standard convention, our Picard ∞–groupoids will be grouplike $\mathcal{O}$–spaces, not necessarily grouplike $\mathbb{E}_\infty$–spaces. As a consequence, we begin by recalling some details concerning grouplike $\mathbb{E}_1$–spaces. In any ∞–topos (in particular, such as the ∞–category of spaces), there is a notion of a grouplike $\mathbb{E}_1$–space [37, 5.1.3.2]. Specifically, we have the following characterization [37, 5.1.3.5]:

Definition 7.1 An $\mathbb{E}_1$–space X is said to be *grouplike* if the monoid $\pi_0 X$ is a group. Given a map $\eta\colon \mathbb{E}_1 \to \mathcal{O}$ of coherent ∞–operads, we say that an $\mathcal{O}$–monoidal space X is *grouplike* if $\eta^* X$ is a grouplike $\mathbb{E}_1$–space.

Given any $\mathcal{O}$–monoidal space X, we can restrict to the maximal grouplike subspace of X.

Lemma 7.2 *For an $\mathcal{O}$–monoidal space X, there is a maximal grouplike subspace $\mathrm{GL}_1 X$. That is, the inclusion*

$$\mathrm{Mon}^{\mathrm{gp}}_{\mathcal{O}}(\mathcal{S}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{S})$$

of grouplike $\mathcal{O}$–monoidal spaces into $\mathcal{O}$–monoidal spaces has a right adjoint GL_1 given by passage to the maximal grouplike $\mathcal{O}$–monoidal space.

Proof The inclusion functor preserves colimits [37, 5.1.3.5] and therefore the adjoint functor theorem implies that there exists a right adjoint GL_1. We can explicitly identify this as follows: Given an $\mathcal{O}$–monoidal space X, $\pi_0(X)$ is a monoid. The maximal grouplike space $\mathrm{GL}_1 X$ is the full subgroupoid obtained by passage to the invertible elements of $\pi_0(X)$ (ie the maximal group contained in $\pi_0(X)$). Since any product of invertible objects in $\pi_0(X)$ is invertible, the criterion of [37, 2.2.1.1] implies that this space is itself $\mathcal{O}$–monoidal. Because $\mathrm{GL}_1 X$ is a full subgroupoid of X, it is clear that any map from a grouplike $\mathcal{O}$–monoidal space uniquely factors through it. $\square$

More generally, given any $\mathcal{O}$–monoidal ∞–category $\mathcal{R}$, we can pass to the full subcategory of invertible objects in $\mathcal{R}$, which we will denote by $\mathcal{R}^\times$. Explicitly, this can be built as the pullback

$$\begin{array}{ccc} (\mathcal{R}^{\otimes})^{\times} & \longrightarrow & \mathcal{R}^{\otimes} \\ \downarrow & & \downarrow \\ \mathrm{Ho}(\mathcal{R}^{\otimes})^{\times} & \longrightarrow & \mathrm{Ho}(\mathcal{R}^{\otimes}) \end{array} \tag{7.3}$$

where $\mathrm{Ho}(\mathcal{R}^{\otimes})^{\times}$ denotes the full monoidal subcategory of the (ordinary) monoidal category $\mathrm{Ho}(\mathcal{R}^{\otimes})$ spanned by the invertible objects. This is the $\mathcal{O}$–monoidal ∞–category of invertible objects. The same argument as in the proof of Lemma 7.2 proves the following lemma:

Lemma 7.4 *For an $\mathcal{O}$–monoidal ∞–category $\mathcal{R}$, the full ∞–subcategory $\mathcal{R}^{\times}$ of invertible objects is an $\mathcal{O}$–monoidal ∞–category.*

However, we want the Picard object to be a space. Recall that the inclusion of ∞–groupoids into ∞–categories preserves products and has a right adjoint; explicitly, if $\mathcal{C}$ is an ∞–category, then $\mathcal{C}^{\simeq}$ is the subcategory of $\mathcal{C}$ consisting of the invertible morphisms.

We can now define the Picard ∞–groupoid of an $\mathbb{E}_1$ object in Pr^{L}.

Definition 7.5 Let $\mathcal{R}$ be an Sp–algebra in Pr^{L}. Then $\mathrm{Pic}(\mathcal{R})$ is the maximal grouplike ∞–groupoid $(\mathcal{R}^{\times})^{\simeq}$ inside the monoidal ∞–category $\mathcal{R}^{\times}$. When $\mathcal{R} = \mathrm{Mod}_R$ for an $\mathbb{E}_n$–ring spectrum R, with $n > 1$, we typically write Pic_R in place of $\mathrm{Pic}(\mathcal{R})$.

When applied to the category of modules over a commutative ring spectrum R, Definition 7.5 recovers the usual construction of the Picard group. In fact, we can perform this construction in either order. First, given $\mathcal{R}$, pass to the full ∞–subcategory $\mathcal{R}^{\times}$ of invertible objects in $\mathcal{R}$, and then take the maximal ∞–groupoid in $\mathcal{R}^{\times}$. Equivalently, given $\mathcal{R}$, pass to the maximal ∞–groupoid $\mathcal{R}^{\simeq}$ contained in $\mathcal{R}$, then pass to the largest grouplike object inside $\mathcal{R}^{\simeq}$.

Furthermore, if $\mathcal{R}$ is a closed symmetric monoidal stable ∞–category, we can characterize $\mathrm{Pic}(\mathcal{R})$ as a subspace of the subcategory of dualizable objects in $\mathcal{R}$. (See for example [40, Section 2] for an excellent discussion of this perspective on the level of homotopy categories.) In this case, the inverse of $X \in \mathrm{Pic}(\mathcal{R})$ is the functional dual $F_{\mathcal{R}}(X, 1)$. The point is that the equivalences witnessing the invertibility of X are duality data; this follows from [40, 2.9] since ∞–categorical duality can be detected on the homotopy category. It is not difficult to extend the description of (7.3) and the inverse to the situation when $\mathcal{R}$ has weaker monoidal structures, but to state the results requires a discussion of duality in these settings which we do not wish to pursue herein.

In order to obtain a multiplicative structure, we would like to describe $\mathrm{Pic}(\mathcal{R})$ more explicitly as part of an adjunction. To make this precise, we first need the following result, which allows us to control the size of the Picard group.

Lemma 7.6 *Let $\mathcal{A}$ be a monoidal presentable ∞–category. Then there exists a regular cardinal κ such that the inclusion*

$$\mathrm{Pic}(\mathcal{A}^{\kappa}) \subseteq \mathrm{Pic}(\mathcal{A})$$

is an equivalence of ∞–groupoids. In particular, $\mathrm{Pic}(\mathcal{A})$ *is essentially small.*

Proof By [37, Lemma 6.3.7.12], there exists a regular cardinal κ such that $\mathcal{A}$ is κ–presentable, the unit $1_{\mathcal{A}}$ of $\mathcal{A}$ is κ–compact and the full subcategory $\mathcal{A}^{\kappa} \subset \mathcal{A}$ consisting of the κ–compact objects is a monoidal subcategory. Let $A \in \mathrm{Pic}(\mathcal{A})$ be an invertible object of $\mathcal{A}$. Since $\mathcal{A} \simeq \mathrm{Ind}_{\kappa}(\mathcal{A}^{\kappa})$, we have $A = \mathrm{colim}_I\, A_i$ for some κ–filtered diagram of κ–compact objects of $\mathcal{A}$. Since A has an inverse B, $1 \simeq A \otimes B \simeq \mathrm{colim}_I(A_i \otimes B)$, and since 1 is κ–compact, the equivalence $1 \to \mathrm{colim}_I(A_i \otimes B)$ factors through a κ–small stage $J \subset I$. But then $1 \simeq \mathrm{colim}_J(A_j \otimes B)$ implies that

$$\mathrm{colim}_J\, A_j \simeq B^{-1} \simeq \mathrm{colim}_I\, A_i,$$

so that A is a κ–small colimit of κ–compact objects and hence itself is κ–compact. □

This now permits us to give the following characterization of Pic:

Theorem 7.7 *Let $\mathcal{O}$ be a coherent ∞–operad equipped with a map $\mathbb{E}_1 \to \mathcal{O}$. Then*

$$\mathrm{Pic}\colon \mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Alg}_{\mathcal{O}}^{\mathrm{gp}}(\mathcal{S})$$

is right adjoint to the free presentable ∞–category functor

$$\mathrm{Pre}\colon \mathrm{Alg}_{\mathcal{O}}^{\mathrm{gp}}(\mathcal{S}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L}}).$$

Proof First observe that the free presentable ∞–category functor $\mathrm{Pre}\colon \mathcal{S} \to \mathrm{Pr}^{\mathrm{L}}$ is symmetric monoidal, and therefore preserves all operadic structures. Moreover, the Yoneda embedding $G \to \mathrm{Pre}(G)$ extends to a map of $\mathbb{E}_1$–algebras, and therefore factors through the maximal subgroupoid $\mathrm{Pic}(\mathrm{Pre}(G)) \subset \mathrm{Pre}(G)$ spanned by the invertible objects and morphisms. Since the Yoneda embedding is natural, we obtain a unit transformation $\eta_G\colon G \to \mathrm{Pic}(\mathrm{Pre}(G))$, natural in grouplike $\mathcal{O}$–monoidal spaces (provided $\mathcal{O}$ is equipped with a map $\mathbb{E}_1 \to \mathcal{O}$, which is given by hypothesis).

To check that the adjunction exists as stated, let G be a grouplike $\mathcal{O}$–monoidal space and $\mathcal{R}$ a presentable $\mathcal{O}$–monoidal ∞–category. Restriction along the Yoneda embedding $G \to \mathrm{Pre}(G)$ induces equivalences

$$\mathrm{map}_{\mathrm{Alg}_{\mathcal{O}}(\mathrm{Pr}^{\mathrm{L}})}(\mathrm{Pre}(G), \mathcal{R}) \to \mathrm{map}_{\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathrm{Cat}}_{\infty})}(G, U(\mathcal{R})) \to \mathrm{map}_{\mathrm{Alg}_{\mathcal{O}}^{\mathrm{gp}}(\mathcal{S})}(G, \mathrm{Pic}(\mathcal{R})),$$

where $U\colon \mathrm{Pr}^{\mathrm{L}} \to \widehat{\mathrm{Cat}}_\infty$ denotes the underlying ∞–category functor, which is lax symmetric monoidal and therefore also preserves operadic structures. In particular, the composite is an equivalence, so the adjunction follows as claimed. □

The unit of the adjunction is the Yoneda embedding $G \to \mathcal{R}[G]$. The counit of the adjunction is the map

$$\mathcal{S}_{/\operatorname{Pic}(\mathcal{R})} \to \mathcal{R}$$

adjoint to the identity map $\operatorname{Pic}(\mathcal{R}) \to \operatorname{Pic}(\mathcal{R})$. As a functor between presentable ∞–categories, this map preserves colimits and is uniquely determined by the image of $\operatorname{Pic}(\mathcal{R})$ in $\mathcal{R}$.

Remark 7.8 When $\mathcal{O}$ is a model for the $\mathbb{E}_n$–operad, we have the following specialization: the functor

$$\operatorname{Pic}\colon \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_n}(\mathcal{S})$$

is corepresented by the $\mathbb{E}_n$–monoidal ∞–category $\mathcal{S}[\Omega^n \Sigma^n_+ *]$. This is because $\Omega^n \Sigma^n_+ *$ is the free grouplike $\mathbb{E}_n$–space on a single generator $*$.

Passing to the stable setting, the argument for Theorem 7.7 yields the following:

Theorem 7.9 *Let $\mathbb{E}_1 \to \mathcal{O}$ be a map of coherent ∞–operads and let $\mathcal{R}$ be a stable presentable $\mathcal{O}$–monoidal ∞–category. Then the canonical map*

$$\mathrm{Sp}_{/\operatorname{Pic}(\mathcal{R})} \to \mathcal{R}$$

is a map of stable presentable $\mathcal{O}$–monoidal ∞–categories.

The work of this section amounts to a categorification of the classical theory of the space of units of a ring spectrum. The multiplicative structure on $\operatorname{Pic}(\mathcal{R})$ is such that the canonical map

$$\mathrm{Sp}_{/\operatorname{Pic}(\mathcal{R})} \to \mathcal{R},$$

adjoint to the inclusion $\operatorname{Pic}(\mathcal{R}) \to \mathcal{R}$ of the invertible objects, is an $\mathcal{O}$–algebra map. Conceptually, this is a categorification of the adjunction which defines GL_1. Just as the underlying infinite loop space functor $\Omega^\infty\colon \mathrm{Sp} \to \mathcal{S}$ is right adjoint to the symmetric monoidal suspension spectrum functor $\Sigma^\infty_+\colon \mathcal{S} \to \mathrm{Sp}$, the forgetful functor

$$\mathrm{map}_{\mathrm{Sp}}\colon (\mathrm{Sp}, -)\colon \mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}} \to \mathcal{S}$$

is right adjoint to the symmetric monoidal functor

$$\mathrm{Pre}_{\mathrm{Sp}}\colon \mathcal{S} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}.$$

8 Multiplicative properties of the Thom spectrum functor

In this section, we apply the work of the previous section in the context of the generalized Thom spectrum functor. Theorem 7.9 has the following immediate consequence, which proves Theorem 1.6; this is a generalization of Lewis's theorem about multiplicative structures on Thom spectra.

Corollary 8.1 *Let $\mathbb{E}_1 \to \mathcal{O}$ be a map of ∞–operads and let $\mathcal{R}$ be a stable presentable $\mathcal{O}$–monoidal ∞–category. The composite functor*

$$\mathcal{S}_{/\operatorname{Pic}(\mathcal{R})} \to \mathrm{Sp}_{/\operatorname{Pic}(\mathcal{R})} \to \mathcal{R}$$

is a map of presentable $\mathcal{O}$–monoidal ∞–categories. Moreover, if $\mathcal{R} = \mathrm{Mod}_R$ for an $\mathbb{E}_n$–algebra object R of Sp, with $n > 1$, then this composite is equivalent to the generalized Thom spectrum functor, ie the colimit.

As an application, we use this to prove R–module generalizations of Lewis's results about the multiplicative properties of the Thom isomorphism theorem. Lewis proved [35, Section IX.7.4] that given an $\mathbb{E}_n$ classifying map $f\colon X \to B\mathrm{GL}_1\mathbb{S}$ such that Mf admits an $\mathbb{E}_n$–orientation over R (ie an $\mathbb{E}_n$–map $Mf \to R$), then the map inducing the Thom isomorphism is an $\mathbb{E}_n$–map. We now provide a concise proof of the analogous results for generalized Thom spectra over $B\mathrm{GL}_1 R$.

Assume that R is an $\mathbb{E}_{n+1}$–ring spectrum and f is an object of the ∞–category $\mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{S}_{/B\mathrm{GL}_1 R})$, ie an $\mathbb{E}_n$–map of spaces

$$f\colon X \to B\mathrm{GL}_1 R.$$

One of the main theorems of our previous work on Thom spectra and units [3; 4; 5] shows that an orientation of the Thom spectrum Mf is specified by a map $P \to \mathrm{GL}_1 R$ in $\mathrm{Mod}_{\mathrm{GL}_1 R}$, where here P is the pullback of the universal principal $\mathrm{GL}_1 R$–bundle along f and $\mathrm{Mod}_{\mathrm{GL}_1 R}$ is the ∞–category of $\mathrm{GL}_1 R$–modules in spaces. This suggests the following generalization of an orientation to the setting of $\mathbb{E}_n$–maps:

Definition 8.2 Assume that R is an $\mathbb{E}_{n+1}$–ring spectrum for $n > 0$. Let P be an object in $\mathrm{Alg}_{/\mathbb{E}_n}(\mathrm{Mod}_{\mathrm{GL}_1 R})$. Then the space of $\mathbb{E}_n$–orientations of P is the space of $\mathbb{E}_n$–algebra maps $P \to \mathrm{GL}_1 R$ in $\mathrm{Alg}_{/\mathbb{E}_n}(\mathrm{Mod}_{\mathrm{GL}_1 R})$.

It is convenient to view the Thom spectrum functor in this light; the following proposition is an immediate consequence of the straightening/unstraightening equivalence [36, Theorem 2.2.1.2].

Proposition 8.3 *There is an equivalence of $\mathbb{E}_n$–monoidal ∞–categories*

$$\mathcal{S}_{/B\mathrm{GL}_1 R} \simeq \mathrm{Mod}_{\mathrm{GL}_1 R},$$

and hence an equivalence of ∞–categories

$$\mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{S}_{/B\mathrm{GL}_1 R}) \simeq \mathrm{Alg}_{/\mathbb{E}_n}(\mathrm{Mod}_{\mathrm{GL}_1 R}).$$

As a consequence, the Thom spectrum functor can be written as the composite

$$\mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{S}_{/B\mathrm{GL}_1 R}) \to \mathrm{Alg}_{/\mathbb{E}_n}(\mathrm{Sp}_{/B\mathrm{GL}_1 R}) \to \mathrm{Alg}_{/\mathbb{E}_n}(\mathrm{Sp}),$$

where the first map is the stabilization.

Now, given any object P in $\mathrm{Alg}_{/\mathbb{E}_n}(\mathrm{Mod}_{\mathrm{GL}_1 R})$, we have a version of the Thom diagonal, given by the $\mathbb{E}_n$–map

$$\Delta\colon P \xrightarrow{\mathrm{id}\times *} P \times (\mathrm{GL}_1 R \times X),$$

where here $\mathrm{GL}_1 R \times X$ is the free $\mathrm{GL}_1 R$–module on the space X. We use the fact that both X and $\mathrm{GL}_1 R$ are based spaces.

Applying the Thom spectrum functor now yields a map

$$Mf \to Mf \wedge (R \wedge X_+),$$

of $\mathbb{E}_n$–ring spectra.

On the other hand, given an orientation $P \to \mathrm{GL}_1 R$, applying the Thom spectrum functor produces a map

$$Mf \to R$$

of $\mathbb{E}_n$–ring spectra. Putting these together, we get the composite

$$Mf \to Mf \wedge (R \wedge \Sigma^\infty_+ X) \to R \wedge (R \wedge \Sigma^\infty_+ X) \to R \wedge \Sigma^\infty_+ X,$$

which is a map of $\mathbb{E}_n$–ring spectra realizing the Thom isomorphism:

Theorem 8.4 *An $\mathbb{E}_n$–orientation $P \to \mathrm{GL}_1 R$ in $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}_{\mathrm{GL}_1 R})$ gives rise to a map of $\mathbb{E}_n$–ring spectra*

$$Mf \to R \wedge \Sigma^\infty_+ X$$

which is an equivalence and realizes the Thom isomorphism.

Appendix A The Brauer group and twisted parametrized spectra

In this section, we indicate how to categorify the definition of the Picard group; this produces a delooping which should be regarded as the Brauer group of an $\mathbb{E}_\infty$–ring spectrum. The connections to the classical definitions of the Brauer group have been studied by the third author with various collaborators [6; 28]. We do not go into detail about any of the applications of this here, other than to briefly observe that this definition allows us to situate the work of Douglas on "twisted parametrized spectra" [18] in our context.

Definition A.1 Let R be a $\mathbb{E}_\infty$–ring spectrum. The Brauer group of R is

$$\mathrm{Br}_R = \mathrm{Pic}(\mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{cg}}),$$

the Picard ∞–groupoid of the symmetric monoidal ∞–category Mod_R^{ω} of compactly generated Mod_R–modules in Pr^{L}, the ∞–category of presentable ∞–categories.

It is straightforward to check that the Brauer group of R provides a delooping of the Picard group.

Lemma A.2 *Let R be an $\mathbb{E}_\infty$–ring spectrum. There is a natural equivalence*

$$\mathrm{Pic}_R \simeq \Omega \mathrm{Br}_R.$$

The Brauer group now provides the proper context to define twisted parametrized spectra:

Definition A.3 (haunts and specters) For a commutative $\mathbb{S}$–algebra R, the ∞–category of R–haunts over a space X is given by the ∞–category (actually, ∞–groupoid) $(\mathrm{Br}_R)_{/X} = (\mathrm{Pic}_{\mathrm{Mod}_R})_{/X}$ of Mod_R–torsors over X. For a given haunt $\mathcal{H}$ on a space X, the ∞–category of specters is the limit of the composite

$$X \to \mathrm{Br}_R \to \mathrm{Mod}_{\mathrm{Mod}_R}.$$

It is now possible to reprove the theorems of Douglas using Definition A.3 and the work of this paper, although we do not carry out this project here.

Appendix B Comparison to the May–Sigurdsson model

In this section, we show that our theories of parametrized spaces and spectra are compatible with those of May and Sigurdsson [41]. As a consequence, one can produce a parametrized spectrum in our context from point-set data (eg sequences of parametrized spaces linked by fiberwise suspension), and a functor from a point-set functor which is homotopical (eg a Quillen functor). Conversely, homotopical conclusions about May and Sigurdsson's setup follow from our results.

To make the comparison, one produces from a model category $\mathcal{C}$ an associated ∞–category. When $\mathcal{C}$ is a simplicial model category, one way to do this is to restrict to the full subcategory of cofibrant–fibrant objects $\mathcal{C}^{\mathrm{cf}}$; then the simplicial nerve [36, 1.1.5.5] $\mathrm{N}(\mathcal{C}^{\mathrm{cf}})$ is an ∞–category. Although any combinatorial model category is Quillen equivalent to a simplicial model category [19], this replacement process can be inconvenient. Furthermore, very few functors preserve cofibrant–fibrant objects; this is a particular problem when studying (symmetric) monoidal model categories.

More recently, Section 1.3.3 of [37] provides an analogue of the Dwyer–Kan simplicial localization. Starting with a (not necessarily simplicial) model category $\mathcal{C}$, one passes to an ∞–category via the ordinary nerve applied to the full subcategory of cofibrant objects and subsequently inverts the weak equivalences:

$$\mathrm{N}(\mathcal{C}^{\mathrm{c}})[W^{-1}].$$

Given a simplicial model category $\mathcal{C}$, there is an equivalence of ∞–categories

$$\mathrm{N}(\mathcal{C}^{\mathrm{cf}}) \simeq \mathrm{N}(\mathcal{C}^{\mathrm{c}})[W^{-1}],$$

which implies that we can apply either process as needed [37, 1.3.3.7].

Recall that $\mathcal{S}_{/S}$ can be described via the straightening and unstraightening correspondence as the ∞–category associated to the model category $\mathrm{Set}_{\Delta/S}$ (with the projective model structure), and $\mathcal{S}_{*/S}$ can analogously be described as the ∞–category of pointed objects in $\mathrm{Set}_{\Delta/S}$.

This provides a comparison to the ∞–categories associated to the May–Sigurdsson categories of parametrized spaces $\mathrm{Top}_{/B}$ and $(\mathrm{Top}_{/B})_*$ over a space B.

Proposition B.1 *Let B be a space. There are equivalences of symmetric monoidal ∞–categories*

$$\mathcal{S}^{\otimes}_{/\Pi_\infty B} \simeq \mathrm{N}(\mathrm{Set}_{\Delta/\Pi_\infty B})[W^{-1}]^{\otimes} \simeq \mathrm{N}(\mathrm{Top}_{/B})[W^{-1}]^{\otimes}$$

and

$$(\mathcal{S}_{/\Pi_\infty B})_*^{\otimes} \simeq \mathrm{N}((\mathrm{Set}_{\Delta/\Pi_\infty B})_*)[W^{-1}]^{\otimes} \simeq \mathrm{N}((\mathrm{Top}_{/B})_*)[W^{-1}]^{\otimes}.$$

Proof For a space B, the projective model structure on $\mathrm{Top}_{/B}$ (in which fibrations and weak equivalences are detected by the forgetful functor to Top with the standard model structure) is Quillen equivalent to the corresponding simplicial model category structure on simplicial sets over $\Pi_\infty B$, which in turn is Quillen equivalent to the simplicial model category of simplicial presheaves on the simplicial category $\mathfrak{C}[\Pi_\infty B]$ (with the projective model structure) [36, 2.2.1.2]. (Here $\mathfrak{C}$ denotes the left adjoint to the simplicial nerve; it associates a simplicial category to a simplicial set [36, 1.1.5].)

Next, we have a comparison

$$\mathrm{St}\colon \mathrm{N}\,\mathrm{Set}_{\Delta/\Pi_\infty B} \to \mathrm{Fun}(\Pi_\infty B^{\mathrm{op}}, \mathrm{N}\,\mathrm{Set}_\Delta);$$

the map, called the *straightening* functor, rigidifies a fibration over $\Pi_\infty B$ into a presheaf of ∞–groupoids on $\Pi_\infty B$ whose value at the point b is equivalent to the fiber over b [36, 3.2.1].

Finally, the symmetric monoidal structure on $\mathcal{S}_{/S}$ is cartesian and therefore unique [37, 2.4.1.9]. Thus, we can promote this equivalence to an equivalence of symmetric monoidal ∞–categories. The result for pointed objects follows. □

To complete the comparison to the model of May and Sigurdsson, we need to study the base-change functors. Almost all of the subtlety and difficulty of the foundational portion of their work arises from the complexities of topological spaces (which they must contend with in order to handle the equivariant setting) and the fact that it is impossible to have a model structure in which the pairs $(f_!, f^*)$ and (f^*, f_*) are simultaneously Quillen adjunctions.

Although the point-set category $(\mathrm{Top}_{/B})_*$ of ex-spaces has a model structure induced by the standard model structure on Top (which they refer to as the q–model structure), one of the key insights of May and Sigurdsson is that for the purposes of stable parametrized homotopy theory it is essential to work with the (Quillen equivalent) qf–model structure [41, 6.2.6].

The situation is easier in the simplicial setting: For a map $f\colon A \to B$, we can obtain point-set models of the functors f^*, f_* and $f_!$ by considering model categories of simplicial presheaves. We must still confront the fact that

$$f^*\colon \mathrm{Fun}(\mathfrak{C}[\Pi_\infty B^{\mathrm{op}}], \mathrm{Set}_\Delta) \to \mathrm{Fun}(\mathfrak{C}[\Pi_\infty A^{\mathrm{op}}], \mathrm{Set}_\Delta)$$

is a *right* Quillen functor for the *projective* model structure, with left adjoint $f_!$, and a *left* Quillen functor for the *injective* model structure, with right adjoint f_*, on the above categories of simplicial presheaves. Nonetheless, this suffices to produce the desired adjoint pairs on the level of ∞–categories.

Theorem B.2 *The Wirthmüller context we construct in Corollary 6.9 on $(\mathcal{S}_{/S})_*$ is compatible with that of May and Sigurdsson.*

Proof To see this, observe that it suffices to check this for f^*; compatibility then follows formally for the adjoints f_* and $f_!$. Thus, we need to check that the right derived functor of $f^*\colon (\mathrm{Top}_{/B})_* \to (\mathrm{Top}_{/A})_*$ in the qf–model structure is compatible with the right derived functor of

$$f^*\colon \mathrm{Fun}(\mathfrak{C}[\Pi_\infty B^{\mathrm{op}}], \mathrm{Set}_\Delta) \to \mathrm{Fun}(\mathfrak{C}[\Pi_\infty A^{\mathrm{op}}], \mathrm{Set}_\Delta)$$

in the projective model structure. By the work of [41, 9.3], it suffices to check the compatibility for f^* in the q–model structure. Since both versions of f^* that arise here are Quillen right adjoints, this amounts to the verification that the diagram

$$\begin{array}{ccc} \mathrm{Fun}(\mathfrak{C}[\Pi_\infty B^{\mathrm{op}}], \mathrm{Set}_\Delta) & \xrightarrow{f^*} & \mathrm{Fun}(\mathfrak{C}[\Pi_\infty A^{\mathrm{op}}], \mathrm{Set}_\Delta) \\ \big\downarrow{\scriptstyle \mathrm{Un}} & & \big\downarrow{\scriptstyle \mathrm{Un}} \\ \mathrm{Set}_{\Delta/B} & \xrightarrow{f^*} & \mathrm{Set}_{\Delta/A} \end{array}$$

commutes when applied to fibrant objects, where here Un denotes the unstraightening functor (which is the right adjoint of the Quillen equivalence). Finally, this follows from [36, 2.2.1.1].

The promotion of this comparison to the symmetric monoidal structure is a consequence of the fact that f^* preserves products and the fact that the cartesian symmetric monoidal structure is unique. □

Therefore, in order to compare our model of parametrized spectra over B to the May–Sigurdsson model, we will work with the corresponding formal stabilization of model categories. Specifically, given a left proper cellular model category $\mathcal{C}$ and an endofunctor of $\mathcal{C}$, Hovey constructs a cellular model category $\mathrm{Sp}^{\mathbb{N}}\mathcal{C}$ of spectra [31]. When the $\mathcal{C}$ is additionally a simplicial symmetric monoidal model category, the endofunctor given by the tensor with S^1 yields a simplicial symmetric monoidal model category of symmetric spectra $\mathrm{Sp}^{\Sigma}\mathcal{C}$ (in addition to the simplicial model category

$\mathrm{Sp}^{\mathbb{N}}\mathcal{C}$ of spectra). These models of the stabilization are functorial in left Quillen functors which are suitably compatible with the respective endofunctors (see [31, 5.2]).

Proposition B.3 *Let $\mathcal{C}$ be a left proper cellular simplicial model category and write $\mathrm{Sp}^{\mathbb{N}}\mathcal{C}$ for the cellular simplicial model category of spectra generated by the tensor with S^1. Then there is an equivalence of ∞–categories*

$$\mathrm{N}((\mathrm{Sp}^{\mathbb{N}}\mathcal{C})^{\mathrm{c}})[W^{-1}] \simeq \mathrm{Stab}(\mathrm{N}(\mathcal{C}^{\mathrm{c}})[W^{-1}]).$$

When $\mathcal{C}$ is a simplicial symmetric monoidal model category, this equivalence extends to an equivalence

$$\mathrm{N}((\mathrm{Sp}^{\Sigma}\mathcal{C})^{\mathrm{c}})[W^{-1}]^{\otimes} \simeq \mathrm{Stab}(\mathrm{N}(\mathcal{C}^{\mathrm{c}})[W^{-1}]^{\otimes})$$

of symmetric monoidal ∞–categories.

Proof The functors $\mathrm{Ev}_n\colon \mathrm{Sp}^{\mathbb{N}}\mathcal{C} \to \mathcal{C}$ which associate to a spectrum its n^{th} space A_n induce a functor

$$f\colon \mathrm{N}((\mathrm{Sp}^{\mathbb{N}}\mathcal{C})^{\mathrm{c}})[W^{-1}] \to \lim\{\cdots \xrightarrow{\Omega} \mathrm{N}(\mathcal{C}^{\mathrm{c}}_*)[W^{-1}] \xrightarrow{\Omega} \mathrm{N}(\mathcal{C}^{\mathrm{c}}_*)[W^{-1}]\} \\ \simeq \mathrm{Stab}(\mathrm{N}(\mathcal{C}^{\mathrm{c}})[W^{-1}]),$$

which is evidently essentially surjective. To see that it is fully faithful, it suffices to check that for cofibrant–fibrant spectrum objects A and B in $\mathrm{Sp}^{\mathbb{N}}\mathcal{C}$, there is an equivalence of mapping spaces

$$\mathrm{map}(A,B) \simeq \mathrm{holim}\{\cdots \xrightarrow{\Omega} \mathrm{map}(A_1,B_1) \xrightarrow{\Omega} \mathrm{map}(A_0,B_0)\},$$

where $\Omega\colon \mathrm{map}(A_{n+1},B_{n+1}) \to \mathrm{map}(A_n,B_n)$ acts as

$$A_{n+1} \to B_{n+1} \mapsto A_n \simeq \Omega A_{n+1} \to \Omega B_{n+1} \simeq B_n.$$

Since any cofibrant A is a retract of a cellular object, inductively we can reduce to the case in which $A = F_m X$, ie A is the shifted suspension spectrum on a cofibrant object X of $\mathcal{C}_*$. Then $\mathrm{map}(A,B) \simeq \mathrm{map}(X,B_m)$ by adjunction. The latter is in turn equivalent to $\mathrm{map}(\Sigma^{n-m}X, B_n)$, where we interpret $\Sigma^{n-m}X = *$ for $m > n$, in which case the homotopy limit is equivalent to that of the homotopically constant (above degree n) tower whose n^{th} term is $\mathrm{map}(\Sigma^{n-m}X, B_n)$.

In the symmetric monoidal setting, the fact that $\mathrm{Stab}\,\mathcal{C}$ is the initial stable symmetric monoidal ∞–category which accepts a symmetric monoidal ∞–functor from $\mathcal{C}$

coupled with the equivalence between prespectra and symmetric spectra implies the desired comparison. □

May and Sigurdsson construct a symmetric monoidal stable model structure on the category $\mathscr{S}_B$ of orthogonal spectra in $(\mathrm{Top}_{/B})_*$ [41, 12.3.10]. This model structure is based on the qf–model structure on ex-spaces, leveraging the diagrammatic viewpoint of [39; 38]. Similarly, they construct a stable model structure on the category $\mathscr{P}_B$ of prespectra in $(\mathrm{Top}_{/B})_*$. The forgetful functor $\mathscr{S}_B \to \mathscr{P}_B$ induces a Quillen equivalence [41, 12.3.10]. The following comparison is now essentially an immediate consequence of Proposition B.3, using the standard comparison between orthogonal spectra and symmetric spectra [39].

Theorem B.4 *Let B be a topological space. There is an equivalence of symmetric monoidal ∞–categories between the ∞–category associated to the model category of orthogonal spectra and the ∞–category of parametrized spectra,*

$$\mathrm{N}(\mathscr{S}_B)[W^{-1}]^{\otimes} \simeq \mathrm{Fun}(\Pi_\infty B^{\mathrm{op}}, \mathrm{Sp})^{\otimes}.$$

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