Geometry & Topology 22 (2018) 3979–4011

Quasi-projectivity of even Artin groups

RUBÉN BLASCO-GARCÍA
JOSÉ IGNACIO COGOLLUDO-AGUSTÍN

Even Artin groups generalize right-angled Artin groups by allowing the labels in the defining graph to be even. We give a complete characterization of quasi-projective even Artin groups in terms of their defining graphs. Also, we show that quasi-projective even Artin groups are realizable by $K(\pi,1)$ quasi-projective spaces.

14F45, 20F36; 14H30, 32S50, 57M05

Introduction

As suggested in Catanese [6], a group is said to be quasi-projective (resp. quasi-Kähler) if it is the fundamental group of a smooth, connected, quasi-projective (resp. quasi-Kähler) space, that is, the complement of a hypersurface in a projective (resp. Kähler) variety. The question of classification of quasi-projective groups, which today is referred to as Serre's question, has been frequently alluded to since Zariski [25] and van Kampen [18] proposed it for complements of curves in the projective plane. The search for properties of such groups goes back to Enriques [13] and Zariski [26, Chapter VIII]. This has developed in the search for obstructions for a group to be quasi-projective (resp. quasi-Kähler) starting with Morgan [22], Kapovich and Millson [19], Arapura [1; 2], Libgober [20], Dimca [10], Dimca, Papadima and Suciu [12], and Artal, Matei and the second author [3; 4].

In this paper we concentrate on the possible characterization of quasi-projective Artin groups, as stated in [12, page 451]. Any proof of such results requires the use of obstructions to disregard the negative cases as well as the constructive part of finding realizations for the positive cases.

A first approach to this problem is given in [12, Theorem 11.7], where quasi-projective right-angled Artin groups are characterized by complete multipartite graphs corresponding to direct products of free groups. In the more general case of even Artin groups — that is, Artin groups associated to even-labeled graphs — the label plays an important role and not all multipartite graphs produce quasi-projective Artin groups.

Published: 6 December 2018 DOI: 10.2140/gt.2018.22.3979

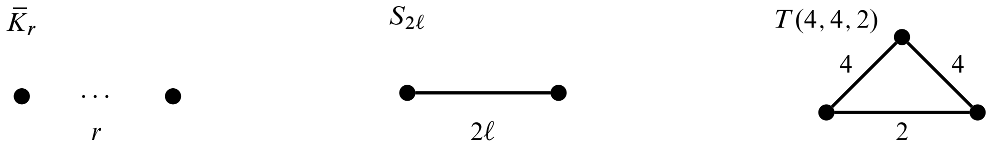

Figure 1: QP–irreducible graphs of type $\bar{K}_r$, $S_{2\ell}$, and $T(4,4,2)$

In order to describe such graphs we define the concept of QP–*irreducible graph*. In this context, *graph* means for us simple graph. Let us denote by $\mathcal{G}_{\mathrm{QP}}$ the family of labeled graphs whose associated Artin groups are quasi-projective. Given two labeled graphs $\Gamma_1 = (V_1, E_1, m_1)$, $\Gamma_2 = (V_2, E_2, m_2)$ we define their 2–join $\Gamma_1 *_2 \Gamma_2 = (V, E, m)$ as the labeled graph given by the join of Γ_1 and Γ_2 whose connecting edges have all label 2; that is,

$$m(e) = \begin{cases} m_i(e) & \text{if } e \in E_i, \\ 2 & \text{if } e \in E \setminus (E_1 \cup E_2). \end{cases}$$

We say $\Gamma \in \mathcal{G}_{\mathrm{QP}}$ is a QP–irreducible graph if Γ is not a 2–join of two graphs in $\mathcal{G}_{\mathrm{QP}}$. Denote by $\bar{K}_r$ a disjoint graph with r vertices and no edges. Also denote by S_m the graph given by two vertices joined by an edge with label m. Finally, denote by $T(4,4,2)$ the triangle as shown in Figure 1. It will be shown that these are the only QP–irreducible even graphs. In other words, the main result of this paper is the following.

Theorem 1 *Let $\Gamma = (V, E, 2\ell)$ be an even-labeled graph and $\mathbb{A}_\Gamma$ its associated even Artin group. Then the following are equivalent:*

(1) *$\mathbb{A}_\Gamma$ is quasi-projective, that is, $\Gamma \in \mathcal{G}_{\mathrm{QP}}$.*

(2) *Γ is the 2–join of finitely many copies of $\bar{K}_r$, $S_{2\ell}$ and T.*

Moreover, if $\Gamma \in \mathcal{G}_{\mathrm{QP}}$, then $\mathbb{A}_\Gamma = \pi_1(X)$, where $X = \mathbb{P}^2 \setminus \mathcal{C}$ is a curve complement.

The $K(\pi,1)$ conjecture for an Artin group $\mathbb{A}_\Gamma$ claims that a certain space which appears as a quotient of the Coxeter arrangement by the action of the Coxeter group associated to Γ is an Eilenberg–Mac Lane space whose fundamental group is $\mathbb{A}_\Gamma$, or a $K(\mathbb{A}_\Gamma, 1)$ space; see for instance [24] for a detailed explanation of this conjecture. In the context of quasi-projective groups, we can also ask ourselves whether or not a quasi-projective Artin group is realizable by an Eilenberg–Mac Lane space.

Conjecture (quasi-projective $K(\pi,1)$ conjecture) *Any quasi-projective Artin group $\mathbb{A}_\Gamma$ can be realized as $\mathbb{A}_\Gamma = \pi_1(X)$ for a smooth, connected, quasi-projective Eilenberg–Mac Lane space X.*

The other main result of this paper is a positive answer to the quasi-projective $K(\pi,1)$ conjecture for even Artin groups.

Theorem 2 *Quasi-projective even Artin groups satisfy the quasi-projective $K(\pi,1)$ conjecture.*

This paper is organized as follows: In Section 1 the general definitions of (even) Artin groups and quasi-projective groups will be given as well as the notion of characteristic varieties as an invariant of a group. Section 2 will be devoted to studying kernels of cyclic subgroups of Artin groups, called cocyclic subgroups. Section 3 focuses on the problem of finding QP–irreducible graphs. The main theorems will be proved in Section 4.

Acknowledgements Blasco-García is partially supported by the Departamento de Industria e Innovación del Gobierno de Aragón and Fondo Social Europeo PhD grant and the Spanish Government MTM2015-67781-P(MINECO/FEDER). Both authors are partially supported by the Spanish Government MTM2016-76868-C2-2-P and Grupo de Referencia Álgebra y Geometría from the Gobierno de Aragón.

1 Settings and definitions

1.1 Artin groups

Artin groups are an interesting family of groups from both an algebraic and a topological point of view.

We recall the definition of the Artin group associated to a labeled graph $\Gamma=(V,E,\ell)$, where (V,E) is a graph and $m\colon E\to\mathbb{Z}_{\geq 2}$ is the label map assigning an integer $m_e=m(e)\in\mathbb{Z}_{\geq 2}$ to each edge $e\in E$ of Γ.

Definition 1.1 (Artin groups) Let $\Gamma=(V,E,m)$ be a labeled graph. The *Artin group* $\mathbb{A}_\Gamma$ associated to Γ has the presentation

$$\mathbb{A}_\Gamma=\big\langle v\in V \mid \langle u,v\rangle^{m_e}=\langle v,u\rangle^{m_e},\ e=\{u,v\}\in E\big\rangle, \tag{1}$$

where $\langle uv\rangle^{m_e}$ is the alternating product of length m_e beginning with u, that is,

$$\langle u,v\rangle^{m_e}=\underbrace{(uv\cdots)}_{m_e}.$$

Note that $\langle u,v\rangle^{2\ell}=(uv)^{\ell}$.

Right-angled Artin groups are defined as Artin groups in which all the edges of the graph have label 2.

Remark 1.2 As a word of caution, nonadjacent vertices have no associated relation. This notation differs in other contexts where nonadjacencies are replaced by ∞–labeled edges, edges with label 2 are removed, and labels 3 are erased.

One special subfamily of Artin groups which we will use in this paper is the family of even Artin groups.

Definition 1.3 (even Artin groups) We say that an Artin group associated to the graph $\Gamma = (V, E, m)$ is *even* if its labels m_e, for $e \in E$, are all even numbers. This will be oftentimes be denoted as $\Gamma = (V, E, 2\ell)$.

1.2 Quasi-projective groups

The main focus of this paper is the study of those groups that can appear as fundamental groups in an algebraic geometry context, in particular as fundamental groups of smooth connected quasi-projective varieties. Recall that a *quasi-projective* variety is the complement of a hypersurface in a projective variety defined simply as the zero locus of a finite number of homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_n]$.

Definition 1.4 A group G is *quasi-projective* if $G = \pi_1(X)$ for a smooth connected quasi-projective variety X.

Example 1.5 Since the fundamental group of the complement of a smooth plane curve of degree d in $\mathbb{P}^2$ is the cyclic group $\mathbb{Z}_d$ and the complement of two lines in $\mathbb{P}^2$ has the homotopy type of $\mathbb{C}^*$, all cyclic groups are quasi-projective. Moreover, since the complement of $r+1$ irreducible smooth curves $C_0, \dots, C_r$ of degrees $d_i = \deg C_i$ intersecting transversally has fundamental group

$$\pi_1(\mathbb{P}^2 \setminus C_0 \cup \cdots \cup C_r) = \mathbb{Z}^r \oplus \mathbb{Z}_d,$$

where $d = \gcd(d_0, \dots, d_r)$, one immediately obtains that all abelian groups are quasi-projective.

This example points out that the quasi-projective variety whose fundamental group realizes a quasi-projective group is clearly not unique in any geometrical sense, since the torsion part d can be obtained in many different ways.

Example 1.6 At the other end of abelianization properties, the free group of rank r is also quasi-projective since it can be realized as the fundamental group of the complement of $r+1$ points in the complex projective line $\mathbb{P}^1$.

The following important properties of quasi-projective groups are well known.

Proposition 1.7 (1) *If G is a quasi-projective group and $K \subset G$ is a finite-index subgroup of G, then K is also a quasi-projective group.*

(2) *If G_1 and G_2 are quasi-projective groups, then $G_1 \times G_2$ is also a quasi-projective group.*

1.3 Serre's question for Artin groups

The question about deciding whether a certain group is quasi-projective is known as *Serre's question*. This question is solved for right-angled Artin groups, but almost nothing is known for more general Artin groups.

Theorem 1.8 [12, Theorem 11.7] *The right-angled Artin group $\mathbb{A}_\Gamma$ is quasi-projective if and only if $\mathbb{A}_\Gamma$ is a product of finitely generated free groups, ie $\mathbb{A}_\Gamma = \mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_r}$.*

The direct implication is proved by exploiting the obstructions of resonance varieties of quasi-projective groups. The converse is achieved by realizing such groups as fundamental groups of quasi-projective varieties, built as products of complements of points in $\mathbb{C}$.

In fact, this result can be interpreted in terms of the graphs via the 2–join construction as follows.

Definition 1.9 Consider two labeled graphs Γ_1 and Γ_2. The 2*–join* of Γ_1 and Γ_2, denoted by $\Gamma = \Gamma_1 *_2 \Gamma_2$, is the labeled graph Γ defined as the join of the graphs and whose label map is defined as

$$m(e) = \begin{cases} m_i(e) & \text{if } e \in E(\Gamma_i), \\ 2 & \text{otherwise.} \end{cases}$$

The Artin group of a 2–join is the product of the Artin groups, that is,

$$\mathbb{A}_{\Gamma_1 *_2 \Gamma_2} = \mathbb{A}_{\Gamma_1} \times \mathbb{A}_{\Gamma_2}. \tag{2}$$

From Example 1.5 note that a free abelian group of rank r is an Artin group corresponding to a complete right-angled Artin group of r vertices, or 2–joins of r points. From Example 1.6 note that a free group of rank r is also an Artin group corresponding to a totally disconnected graph of r vertices. Using (2), Theorem 1.8 can be rewritten as follows.

Theorem 1.10 *Let Γ be a right-angled graph. Then $\mathbb{A}_\Gamma$ is quasi-projective if and only if Γ is the 2–join of finitely many totally disconnected graphs.*

For triangle Artin groups and general-type Artin groups, partial results on their quasi-projectivity are given in [3]. Among those we describe the following results for Artin groups of type $\mathbb{A}_{S_{2\ell}}$ and $\mathbb{A}_T$ as in Figure 1.

Theorem 1.11 [3, Chapter 5] *The Artin groups $\mathbb{A}_{S_{2\ell}} = \langle a, b \mid (ab)^\ell = (ba)^\ell \rangle$ and $\mathbb{A}_T = \langle a, b, c \mid abab = baba,\ acac = caca,\ bc = cb \rangle$ are quasi-projective.*

Our objective in this paper is to give a similar characterization to Theorem 1.10 for even Artin groups.

1.4 Characteristic varieties

Characteristic varieties are a sequence of invariants of a group. They were introduced by Hillman [16] for links and systematically studied by Cohen and Suciu [9] for hyperplane arrangement complements, Libgober [21] for plane curve complements and, from a different point of view, by Arapura [1] for Kähler manifolds using jumping loci of cohomology of local systems. It should also mentioned that the connection between Alexander modules and cohomology of local systems was first proved by E Hironaka [17].

For expository reasons we will mainly follow [21] and we will only provide specific references for the more specialized results. Let X be a finite CW–complex and $G = \pi_1(X)$ its fundamental group. For the sake of simplicity, we assume that the abelianization $H_1(G) = G/G'$ of G is torsion-free, say $H_1(G) = \mathbb{Z}^r$. Consider the universal abelian cover $\widetilde{X} \xrightarrow{\phi} X$, where $\mathrm{Deck}(\phi) = \mathbb{Z}^r$ is generated by $t_1, \dots, t_r \in \mathrm{Deck}(\phi)$. Since $\mathrm{Deck}(\phi)$ acts on $H_1(\widetilde{X})$, the group $H_1(\widetilde{X})$ inherits a module structure over the ring $\Lambda = \mathbb{Z}[\mathrm{Deck}(\phi)] = \mathbb{Z}[\mathbb{Z}^r]$. This module $M_X = H_1(\widetilde{X})$ is called the

Alexander module of X. As any Λ–module, M_X has a sequence of invariants given by the Fitting ideals or analogously by the sequence of annihilators of its exterior powers

$$I_k = \mathrm{Ann}_\Lambda\big(\textstyle\bigwedge^k M_X\big) \subset \Lambda,$$

where $\mathrm{Ann}_R(A) = \{r \in R \mid ra = 0 \text{ for all } a \in A\} \subset R$ is by definition the annihilator ideal of an R–module A. After tensoring Λ by $\mathbb{C}$, a new ring $\Lambda^{\mathbb{C}}$ is obtained over which one can take an algebrogeometric point of view and consider the zero locus of $I_r \otimes \Lambda^{\mathbb{C}}$ inside the torus $\mathrm{Spec}\,\Lambda^{\mathbb{C}} = (\mathbb{C}^*)^r$.

Definition 1.12 We define the sequence of *characteristic varieties* of X as

$$V_1(X) := Z(I_1) \supset \cdots \supset V_k(X) := Z(I_k) \supset \cdots,$$

where $Z(I_k) \subset (\mathbb{C}^*)^r$ is the zero locus of I_k.

There is an alternative way to define characteristic varieties using Fitting ideals.

Definition 1.13 Let $\varphi\colon A_2 \to A_1$ be a map of free modules over a ring R. We define the ideal $\widetilde{F}_k(\varphi) \subset R$ as the image of the canonical map

$$\textstyle\bigwedge^k A_2 \otimes \bigwedge^k A_1^* \to R$$

induced by φ.

Definition 1.14 Let M be a finitely presented module over R and consider a free resolution

$$\varphi\colon A_2 \to A_1 \to M \to 0$$

of M such that A_1 (resp. A_2) is a finitely generated R–module of rank r (resp. s). For every integer $k \geq 0$ we define the k^{th} Fitting ideal of M to be

$$F_k(M) := \widetilde{F}_{r-k}(\varphi).$$

Proposition 1.15 *Under the above conditions, the sequence $V_k(X)$ of characteristic varieties coincides with the zero locus of the Fitting ideals of its Alexander module $F_k(M_X)$.*

Proof This is an immediate consequence of [5, Corollary 1.3]. □

Characteristic varieties of quasi-projective spaces have the following property.

Proposition 1.16 [1; 11] *The irreducible components of the characteristic varieties associated to a quasi-projective group G are algebraic translated tori by torsion points. That is, they are intersections of polynomials of the form*

$$P(t_1,\dots,t_r)=\prod_i (t_1^{n_1}\cdots t_r^{n_r}-\nu_i),$$

where ν_i is a root of unity.

Moreover, the intersection of two such irreducible components is a finite union of torsion points.

From the computational point of view, a third way to calculate the sequence of characteristic varieties from a finite presentation of a group

$$G=\pi_1(X)=\langle a_1,\dots,a_n : R_1=\dots=R_m=1\rangle \tag{3}$$

is provided via Fox calculus; see [14].

Formally, one associates a matrix

$$A=\left(\frac{\partial R_i}{\partial a_j}\right)_{1\le i\le m,\,1\le j\le n},$$

to the presentation (3), where the derivative of a word in the letters $a_1,\dots,a_n$ is obtained by extending the following defining properties by linearity:

$$\frac{\partial uv}{\partial a_j}=\frac{\partial u}{\partial a_j}+\phi(u)\frac{\partial v}{\partial a_j},\qquad \frac{\partial 1}{\partial a_j}=0 \qquad\text{and}\qquad \frac{\partial a_i}{\partial a_j}=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{otherwise.}\end{cases}$$

The matrix A is called the *Alexander matrix* associated with (3) and it turns out to be the matrix of the free resolution of a module which is not the Alexander module, but the *Alexander invariant* $\widetilde{M}_X=H_1(\widetilde{X},\phi^{-1}(p))$, which is the relative homology of the universal abelian cover of X relative to the preimage of a point as a Λ–module exactly as was done for the Alexander module M_X. As in knot theory, both invariants are related; see for instance [7, Chapter 1].

Proposition 1.17 *The sequence of characteristic varieties of X can be calculated via Fox calculus as*

$$V_k(X)\setminus\mathbb{1}=Z(F_{k+1}(M_X))\setminus\mathbb{1}=Z(F_{k+1}(\widetilde{M}_X))\setminus\mathbb{1}.$$

The computational advantage of $F_{k+1}(\widetilde{M}_X)$ is that it can be computed from the Alexander matrix A of a free resolution of $\widetilde{M}_X$ as follows:

$$F_{k+1}(\widetilde{M}_X) = \begin{cases} \Lambda & \text{if } k > n, \\ 0 & \text{if } k \leq \max\{0, n-m\}, \\ (\text{minors of order } n-k \text{ of } A) & \text{otherwise.} \end{cases}$$

2 Preliminaries

Characteristic varieties of even Artin groups are too similar to those of quasi-projective groups and hence they cannot be used to tell them apart. However, some of their finite-index subgroups can be detected as not quasi-projective. This is why we present a study of a certain type of subgroup of even Artin groups that will be key in the discussion on quasi-projectivity.

2.1 Cocyclic subgroups of even Artin groups

Let us consider the even Artin group associated to $\Gamma = (V, E, 2\ell)$.

The Artin group associated to Γ has a presentation $\mathbb{A}_\Gamma = \langle v \in V \mid A_{\ell_e}(e) \text{ for } e \in E \rangle$, where $A_{\ell_e}(e)$ denotes the relation $(uv)^{\ell_e} = (vu)^{\ell_e}$ with $e = \{u, v\}$. Let us fix a vertex, say $u \in V$, and an integer $k > 1$; our purpose is to give a presentation of the index k subgroup $\mathbb{A}_{\Gamma,u,k}$ of $\mathbb{A}_\Gamma$ defined as the kernel of the morphism

$$\alpha_{u,k} \colon \mathbb{A}_\Gamma \to \mathbb{Z}_k, \qquad v \mapsto \begin{cases} 1 & \text{if } v = u, \\ 0 & \text{otherwise.} \end{cases}$$

As suggested by an anonymous referee, one can think of these as finite-index normal subgroups of a group that appear as the kernel of a surjection onto a finite cyclic group, and refer to them as *cocyclic subgroups*.

Note that for any $v \in V$, the conjugation of v by u^i is in $\mathbb{A}_{\Gamma,u,k}$, $v_i := u^i v u^{-i}$. Also, $\bar{u} := u^k$ will be in the kernel of $\alpha_{u,k}$. In order to write a presentation for $\mathbb{A}_{\Gamma,u,k}$ we need some notation. Let us denote by $\langle x, y \rangle_{i,\varepsilon}^l$ a formal word in the letters $\{x_0, \dots, x_{k-1}, y\}$ as follows:

$$\langle x, y \rangle_{i,\varepsilon}^l = (x_i \cdots x_{k-1} y x_0 \cdots x_{i-1})^c x_i \cdots x_{i+r-1} x_{i+r}^{\varepsilon},$$

where $l = ck + r$, $i \in \mathbb{Z}_k$ and $\varepsilon = 0, 1$. Note that $\langle x, y \rangle_{i,\varepsilon}^l$ can be thought of as a cyclic product of the elements $x_0, \dots, x_{k-1}$ and y starting at x_i and with length $c(k+1) + r + \varepsilon$.

Also, let us consider the set

$$V_{2,u} = \{v \in V \mid e = \{u, v\} \in V,\ m_e = 2\ell_e = 2\}$$

of vertices in V adjacent to u with label 2. The remaining vertices will be denoted by $W = V \setminus (\{u\} \cup V_{2,u})$.

One obtains the following presentation for $\mathbb{A}_{\Gamma,u,k}$.

Theorem 2.1 *The cocyclic subgroup $\mathbb{A}_{\Gamma,u,k}$ is generated by*

$$\{\overline{u}\} \cup V_{2,u} \cup \bigcup_{w \in W} \{w_0, \dots, w_{k-1}\},$$

and the following is a complete set of relations:

(R1) $A_1(v, \overline{u})$ *for* $v \in V_{2,u}$.

(R2) $A_{\ell_e}(v, v')$ *for* $v, v' \in V_{2,u}$, $e = \{v, v'\} \in E$.

(R3) $A_{\ell_e}(v, w_i)$ *for* $v \in V_{2,u}$, $w \in W$, $i \in \mathbb{Z}_k$, $e = \{v, w\} \in E$.

(R4) $A_{\ell_e}(w_i, w'_i)$ *for* $w, w' \in W$, $i \in \mathbb{Z}_k$, $e = \{w, w'\} \in E$.

(RB) $B^i_{\ell_e,k}(w, \overline{u})$ *for* $w \in W \cap \mathrm{lk}(u)$, $i \in \mathbb{Z}_k$, $e = \{u, w\} \in E$.

Here $\ell_e = c_e k + r_e$ and $B^i_{\ell_e}(w, \overline{u})$ is the relation

$$\langle w, \overline{u}\rangle^{\ell_e}_{i,\varepsilon} = \langle w, \overline{u}\rangle^{\ell_e}_{i+1,\varepsilon} \quad \textit{with } \varepsilon = \begin{cases} 0 & \text{if } 0 \le i < k - r_e, \\ 1 & \text{otherwise.} \end{cases}$$

Proof The proof is a direct application of the Reidemeister–Schreier theorem (see [15, Theorem 2.1]) to obtain a presentation of $\mathbb{A}_{\Gamma,u,k}$ as the kernel of $\alpha_{u,k}$

$$\mathbb{A}_{\Gamma,u,k} \xhookrightarrow{j} \mathbb{A}_\Gamma \xrightarrow{\alpha_{u,k}} \mathbb{Z}_k.$$

Consider the Reidemeister's section $s\colon \mathbb{Z}_k \to \mathbb{A}_\Gamma$ of the map $\alpha_{u,k}$ given as $s(i) := u^i$. Then $\mathbb{A}_{\Gamma,u,k}$ admits a presentation generated by the letters

$$\{\overline{u}\} \cup \bigcup_{v \in V} \{v_0, \dots, v_{k-1}\},$$

where $j(\overline{u}) = u^k$ and $j(v_i) = u^i v u^{-i}$, whose relations are

(1) $A_{\ell_e}(v_i, w_i)$ for $v, w \in V$ and $i \in \mathbb{Z}_k$ if $e = \{v, w\} \in E$;

(2) $B^i_{\ell_e,k}(w, \overline{u})$ for $i \in \mathbb{Z}_k$ if $v \in \mathrm{lk}(u)$.

However, note that if $v \in V_{2,u}$, then $u^i v u^{-i} = v_i = v_j = u^j v u^{-j}$, implying a reduction in the set of generators, which now becomes

$$\{\overline{u}\} \cup V_{2,u} \cup \bigcup_{w \in W} \{w_0, \dots, w_{k-1}\}.$$

Finally, note that the only relations affected by this elimination of generators are those of type $A_{\ell_e}(v_i, w_i)$ for $v \in V_{2,u}$, which now become $A_{\ell_e}(v, w_i)$, and those of type $B^i_{1,k}(v, \overline{u})$ for $v \in V_{2,u}$, which are now reduced to $A_1(v, \overline{u})$, as stated. □

Remark 2.2 Our purpose will be to study the characteristic varieties of the cocyclic subgroups. As presented in Section 1.4 these are subvarieties of $\operatorname{Spec} \mathbb{C}[G/G']$, for $G = \mathbb{A}_{\Gamma,u,k}$. First we will describe the abelianization of $\mathbb{A}_{\Gamma,u,k}$. Since G is finitely presented consider $\mathbb{F} \to G$ the map from the free group $\mathbb{F}$ in the generators of G. The kernel K of this homomorphism is a free subgroup generated by the set of relations in G. Consider $G \xrightarrow{\Phi_G} G/G'$, $g \to t_g$, the abelianization map (with a multiplicative structure). According to Theorem 2.1 the abelianization $G/G' = \Phi_{\mathbb{F}}(\mathbb{F})/\Phi_{\mathbb{F}}(K)$ is generated by

$$\{t_{\overline{u}}\} \cup \{t_v\}_{v \in V_{2,u}} \cup \bigcup_{w \in W} \{t_{w,0}, \dots, t_{w,k-1}\},$$

where for convenience $t_{w,i}$ is used to denote t_{w_i}. Note that (R1)–(R4) considered as words in the free group $\mathbb{F}$ belong in fact to $\mathbb{F}'$ and hence their image by the abelianitation map $\Phi_{\mathbb{F}}$ is trivial. On the other hand, the words $B^i_{\ell_e,k}(w, \overline{u})$, $w \in W \cap \operatorname{lk}(u)$, produce the following relations in homology:

$$t_{w,i} = t_{w,i+d_e} = \cdots = t_{w,i+nd_e} \quad \text{if } e = \{u, w\},\ d_e = \gcd(\ell_e, k), \tag{4}$$

Definition 2.3 The presentation described in Theorem 2.1 will be referred to as the *standard presentation* of $\mathbb{A}_{\Gamma,u,k}$.

2.2 Fox calculus on the cocyclic subgroups $\mathbb{A}_{\Gamma,u,k}$

2.2.1 Fox derivatives of a standard presentation We want to describe the Fox derivatives of the relations of a standard presentation of the subgroup $\mathbb{A}_{\Gamma,u,k}$.

The first set of relations (R1)–(R4) in Theorem 2.1 are classical Artin relations. In order to describe their Fox derivatives we introduce the polynomial

$$p_l(t) = \frac{t^l - 1}{t - 1}$$

and as above, we denote by t_g the homology class of an element g. In the following results we present the Fox derivatives of certain relations of type $W_1 = W_2$; by this we mean the derivative of the abstract word $W_1 W_2^{-1}$.

Lemma 2.4 *Under the above conditions,*

$$\frac{\partial A_{\ell_e}(a,b)}{\partial g} = \begin{cases} -(t_b-1)\,p_{\ell_e}(t_a t_b) & \text{if } g=a,\\ (t_a-1)\,p_{\ell_e}(t_a t_b) & \text{if } g=b,\\ 0 & \text{otherwise.} \end{cases}$$

In order to describe the derivatives of relations of type (RB), let us use the conventions

$$\overline{t}_{w,i,j} = \begin{cases} t_{w,i}\cdots t_{w,j-1} & \text{if } 0\le i<j\le k,\\ 1 & \text{if } i=j,\\ \overline{t}_w/\overline{t}_{w,j,i} & \text{if } 0\le j<i\le k, \end{cases}$$

where $t_{w,i} = t_{w_i}$, with $w_i = u^i w u^{-i}$, $t_0 = t_{\overline{u}}$ and $\overline{t}_w = \overline{t}_{w,0,k}$.

Lemma 2.5 *Under the above conditions,*

$$\frac{\partial B^i_{\ell_e,k}(w,\overline{u})}{\partial g} =$$
$$\begin{cases} \overline{t}_{w,i,k}(1-t^{-1}_{w,i+r_e+\varepsilon})\,p_{c_e+\varepsilon}(t_0\overline{t}_w) & \text{if } g=\overline{u},\\ \overline{t}_{w,i,j}(1-t^{-1}_{w,i+r_e+\varepsilon})\,p_{c_e}(t_0\overline{t}_w) & \text{if } g=w_j,\ j<i,\\ \left(1-\dfrac{t_0\overline{t}_w}{t_{w,i+r_e+\varepsilon}}\right)p_{c_e}(t_0\overline{t}_w)+(t_0\overline{t}_w)^{c_e} & \text{if } g=w_i,\\ \overline{t}_{w,i,j}(1-t^{-1}_{w,i+r_e+\varepsilon})\,p_{c_e+1}(t_0\overline{t}_w) & \text{if } g=w_j,\ i<j<i+r_e+\varepsilon,\\ \overline{t}_{w,i,j}(1-t^{-1}_{w,i+r_e+\varepsilon})\,p_{c_e}(t_0\overline{t}_w)-\dfrac{\overline{t}_{w,i,i+r_e+\varepsilon}}{t_{w,i+r_e+\varepsilon}}(t_0\overline{t}_w)^{c_e} & \text{if } g=w_j,\ i<j=i+r_e+\varepsilon,\\ \overline{t}_{w,i,j}(1-t^{-1}_{w,i+r_e+\varepsilon})\,p_{c_e}(t_0\overline{t}_w) & \text{if } g=w_j,\ j<i+r_e+\varepsilon. \end{cases}$$

Proof The proof is straightforward. We will work out a sample case. Assume $g = w_i$. Then

$$\frac{\partial B^i_{\ell_e,k}(w,\overline{u})}{\partial w_i} = \frac{\langle w,\overline{u}\rangle^{\ell_e}_{i,\varepsilon}(\langle w,\overline{u}\rangle^{\ell_e}_{i+1,\varepsilon})^{-1}}{\partial w_i}.$$

First let us calculate $\langle w,\overline{u}\rangle_{i,\varepsilon}^{\ell_e}/\partial w_i$. It is straightforward that

$$\frac{\langle w,\overline{u}\rangle_{i,\varepsilon}^{\ell_e}}{\partial w_i}=p_{c_e}(t_0\overline{t}_w)+(t_0\overline{t}_w)^{c_e},\qquad \frac{\langle w,\overline{u}\rangle_{i+1,\varepsilon}^{\ell_e}}{\partial w_i}=\overline{t}_{w,i+1,i}\,p_{c_e}(t_0\overline{t}_w).$$

Now, using the multiplication rule and

$$0=\frac{\partial uu^{-1}}{\partial v}=\frac{\partial u}{\partial v}+t_u\frac{\partial u^{-1}}{\partial v},$$

one obtains

$$\frac{(\langle w,\overline{u}\rangle_{i+1,\varepsilon}^{\ell_e})^{-1}}{\partial w_i}=-\frac{\overline{t}_{w,i+1,i}\,p_{c_e}(t_0\overline{t}_w)}{(t_0\overline{t}_w)^{c_e}\overline{t}_{w,i,i+r_e+\varepsilon}}.$$

Therefore,

$$\begin{aligned}\frac{\partial B^i_{\ell_e,k}(w,\overline{u})}{\partial w_i}&=p_{c_e}(t_0\overline{t}_w)+(t_0\overline{t}_w)^{c_e}+(t_0\overline{t}_w)^{c_e}t_{w,i,i+r_e+\varepsilon}\frac{(\langle w,\overline{u}\rangle_{i+1,\varepsilon}^{\ell_e})^{-1}}{\partial w_i}\\&=p_{c_e}(t_0\overline{t}_w)+(t_0\overline{t}_w)^{c_e}-(t_0\overline{t}_w)^{c_e}t_{w,i,i+r_e+\varepsilon}\frac{(\overline{t}_{w,i+1,i})\,p_{c_e}(t_0\overline{t}_w)}{(t_0\overline{t}_w)^{c_e}t_{w,i,i+r_e+\varepsilon}}\\&=p_{c_e}(t_0\overline{t}_w)+(t_0\overline{t}_w)^{c_e}-\frac{(t_0\overline{t}_w)^{c_e}}{t_{w,i,i+r_e+\varepsilon}}p_{c_e}\\&=\left(1-\frac{t_0\overline{t}_w}{t_{w,i+r_e+\varepsilon}}\right)p_{c_e}(t_0\overline{t}_w)+(t_0\overline{t}_w)^{c_e}.\end{aligned}$$

□

2.2.2 Alexander matrices for cocyclic subgroups of even Artin groups Given an even labeled graph $\Gamma=(V,E,2\ell)$, let us fix $u\in V$ and an integer $k>1$. We will denote by M_Γ (resp. $M_{\Gamma,u,k}$) the Alexander matrix associated to the Artin presentation of $\mathbb{A}_\Gamma$ (resp. the standard presentation of $\mathbb{A}_{\Gamma,u,k}$ given in Section 2.1). The purpose of this section is to describe some relevant properties of both M_Γ and $M_{\Gamma,u,k}$.

Among these properties, the most relevant for our purposes refers to their rank. Note that since these matrices have coefficients in a ring of Laurent polynomials $R=\mathbb{C}[\mathbb{Z}^m]$, a matrix $A\in\mathrm{Mat}(R)$ has rank at least r if and only if there is a value $p=(t_1,\dots,t_m)$ in $\mathbb{C}^m$ such that $A\otimes R/\mathfrak{m}_p\in\mathrm{Mat}(\mathbb{C})$ has an $r\times r$ nonzero minor, where $\mathfrak{m}_p$ denotes the maximal ideal at p. This operation will be called *evaluating* and will be used oftentimes to simplify notation.

Lemma 2.6 *The rank of the Alexander matrix M_Γ defined above is exactly $|V|-1$.*

Proof Consider M_T, the row submatrix of M_Γ given by the $|V|-1$ relations determined by the edges of a maximal tree T in Γ. Since M_T clearly has rank $|V|-1$, the matrix M_Γ has rank at least $|V|-1$.

To see the equality, consider $\overline{\Gamma}=(V,\overline{E},2\overline{\ell})$, the completion of the graph Γ obtained from Γ by adding an edge of label 2 for every pair of disconnected vertices. The matrix $M_{\overline{\Gamma}}$ associated to this graph contains M_Γ as a submatrix. Choosing any vertex $v\in V$, we will show that the $|V|-1$ rows associated to the relations involving v generate the remaining rows.

Consider $e=\{w,w'\}\in\overline{E}$. Using Lemma 2.4, the row f_e associated to the classical Artin relation $A_{\overline{\ell}_e}(w,w')$ has the form

$$p_{\overline{\ell}_e}(t_w t_{w'})(0\ \cdots\ 0\ \ 1-t_{w'}\ \ 0\ \cdots\ 0\ \ t_w-1\ \ 0\ \cdots\ 0), \tag{5}$$

where the nonzero elements are at the columns corresponding to the vertices w and w' respectively.

Note that since $\overline{\Gamma}$ is a complete labeled even graph, the three vertices $v,w,w'\in\overline{V}=V$ form a triangle, that is, $e=\{w,w'\}$, $e_1=\{v,w\}$, $e_2=\{v,w'\}$. Moreover, the rows f_e, f_{e_1} and f_{e_2} satisfy the linear combination

$$\frac{t_v-1}{p_{\overline{\ell}_e}(t_w t_{w'})}f_e+\frac{t_{w'}-1}{p_{\overline{\ell}_{e_1}}(t_v t_w)}f_{e_1}+\frac{t_w-1}{p_{\overline{\ell}_{e_2}}(t_v t_{w'})}f_{e_2}=0.$$

Thus, $M_{\overline{\Gamma}}$ has rank less than or equal to $|V|-1$. Since M_Γ is a submatrix of $M_{\overline{\Gamma}}$, the result follows. □

Notation 2.7 Recall from Theorem 2.1 that the generators of a standard presentation of $\mathbb{A}_{\Gamma,u,k}$ can be distinguished in three type-groups $\{\overline{u}\}\cup V_{2,u}\cup W_{k,u}$, where

$$V_{2,u}=\{v\in V\mid e=\{u,v\}\in V,\ \ell_e=1\},$$
$$W_{k,u}=\{w_{i,j}\mid w_i\in W=V\setminus(\{\overline{u}\}\cup V_{2,u}),\ j\in\mathbb{Z}_k\}.$$

In the sequel, the elements in $V_{2,u}$ will be denoted by $v_1,\dots,v_m$, where m is the number of vertices adjacent to u with label 2. Analogously, the elements of $W_{k,u}$ will be denoted by $w_{i,j}$, where $w_i\in W$ for $1\le i\le n=|V|-m-1$, and $j\in\mathbb{Z}_k$.

From the results of the two previous sections, we immediately obtain the following description of the Alexander matrix $M_{\Gamma,u,k}$.

Lemma 2.8 *The Alexander matrix $M_{\Gamma,u,k}$ of $\mathbb{A}_{\Gamma,u,k}$ associated to its standard presentation has the form*

$$
\begin{array}{c}
\begin{array}{cccccccc} w_{*,0} & w_{*,1} & \cdots & w_{*,k-1} & \qquad v_1 & \cdots & v_m \qquad & \bar{u} \end{array}\\
\left(\begin{array}{cccc|c|c}
0 & 0 & \cdots & 0 & A_k & 0\\
A'_0 & 0 & \cdots & 0 & A_0 & 0\\
0 & A'_1 & \cdots & 0 & A_1 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & A'_{k-1} & A_{k-1} & 0\\
\hline
0 & 0 & \cdots & A'_k & \begin{matrix} t_{\bar u}-1 & & \\ & \ddots & \\ & & t_{\bar u}-1\end{matrix} & \begin{matrix}1-t_{v_m}\\ \vdots \\ 1-t_{v_m}\end{matrix}\\
\hline
\multicolumn{6}{c}{M_B}
\end{array}\right),
\end{array}
$$

where:

(1) $w_{*,j}$ *denotes the set of columns associated with all the generators of type* $w_{i,j} \in W_{k,u}$ *for a fixed* $j \in \mathbb{Z}_k$*, with* $w_i \in W$*, as in Notation 2.7.*

(2) A_k *is the Artin matrix corresponding to relations of type (R2) in Theorem 2.1 with respect to the generators* $\{v_1, \dots, v_m\}$*.*

(3) *The submatrices* A'_j *and* A_j *are such that the matrix* $(A'_j \mid A'_j)$ *is the Alexander matrix of the relations of type (R3) in Theorem 2.1, ie their rows are of the form*

$$f_{a,b} \equiv p_{c_{ab}} t_a t_b \big(0 \ \cdots \ 0 \ 1-t_b \ 0 \ \cdots \ 0 \ t_a-1 \ 0 \ \cdots \ 0\big)$$

for $a = v_l \in V_{2,u}$ *and* $b = w_{i,j} \in W_{k,u}$*.*

(4) *The submatrix* M_B *is the Alexander matrix associated to the relations of type (RB) in Theorem 2.1. Note that this is a block matrix whose blocks are the submatrices* $M_{B(w,u)}$ *associated to the relations of type* $B^i_{\ell_e,k}(w,\bar{u})$ *for* $i \in \mathbb{Z}_k$ *and edges* $\{u, w\} \in E$*.*

Lemma 2.9 *The submatrix* $M_{B(w,u)}$ *has maximal rank.*

Proof As was mentioned above, we are assuming $\{u, w\} \in E$. Let us distinguish two cases, depending on whether or not ℓ_e is a multiple of k.

(1) Assume $\ell_e \equiv 0 \bmod k$. Using Lemma 2.5 and evaluating

$$t_{w,0}=t_{w,1}=\cdots=t_{w,k-2}=1$$

in $M_{B(w,u)}$, we obtain the following upper-triangular matrix, which has maximal rank:

$$M=\begin{array}{c}
\begin{array}{cccccc} w_0 & w_1 & \cdots & w_{k-2} & w_{k-1} & \overline{u} \end{array}\\
\left(\begin{array}{ccccc|c}
t_{\overline{u}}-1 & t_{\overline{u}}-1 & \cdots & t_{\overline{u}}-1 & t_{\overline{u}}-1 & 1-t_{w,k-1}\\
1-t_{w,k-1}t_{\overline{u}} & 0 & \cdots & 0 & 0 & 0\\
0 & 1-t_{w,k-1}t_{\overline{u}} & \cdots & 0 & 0 & 0\\
0 & 0 & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 1-t_{w,k-1}t_{\overline{u}} & 0 & 0
\end{array}\right)
\end{array}.$$

(2) Assume $\ell_e \not\equiv 0 \bmod k$. Write $\ell_e=c_ek+r_e$, with $0<r_e<k$. Analogously to the previous case, using Lemma 2.5 and evaluating now at

$$t_{w,0}=t_{w,1}=\cdots=t_{w,k-2}=t_{w,k-1}=1,$$

we obtain the matrix

$$M=\begin{array}{c}
\begin{array}{cccccccccc} w_0 & \cdots & w_{r-1} & w_r & \cdots & w_{k-r-1} & w_{k-r} & \cdots & w_{k-1} & \overline{u} \end{array}\\
\left(\begin{array}{ccccccccc|c}
1 & & & -t_{\overline{u}}^{c} & & & & & & 0\\
 & \ddots & & & \ddots & & & & & \vdots\\
 & & \ddots & & & \ddots & & & & \vdots\\
 & & & \ddots & & & \ddots & & & \vdots\\
 & & & & \ddots & & & \ddots & & \vdots\\
 & & & & & 1 & & & -t_{\overline{u}}^{c} & \\
\hline
-t_{\overline{u}}^{c+1} & & & & & & 1 & & & \vdots\\
 & \ddots & & & & & & \ddots & & \\
 & & -t_{\overline{u}}^{c+1} & & & & & & 1 & 0
\end{array}\right)
\end{array}.$$

Formally, $t_{\overline{u}}=0$ produces a matrix of maximal rank and hence the result follows using small enough values of $t_{\overline{u}}$. □

Remark 2.10 In the previous lemma, the submatrix of $M_{B(w,\overline{u})}$ resulting from deleting the column $\overline{u}$ has maximal rank. Therefore, in order to study the rank of $M_{\Gamma,u,k}$, and after row operations, one can assume that $M_{B(w,\overline{u})}$ is equivalent to

$$\begin{array}{cccccc|c} & w_0 & w_1 & \cdots & w_{k-2} & w_{k-1} & \overline{u} \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} * & * & \cdots & * & * & * \\ 0 & * & \cdots & * & * & * \\ 0 & \vdots & \ddots & * & \vdots & \vdots \\ 0 & 0 & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & * & * \end{array}\right).$$

Recall that the corank of a matrix M is defined as

$$\operatorname{corank}(M) = \#\operatorname{columns}(M) - \operatorname{rank}(M).$$

Then one has the following result on the corank of $M_{\Gamma,u,k}$.

Lemma 2.11 *Under the conditions above,* $\operatorname{corank}(M_{\Gamma,u,k}) \leq 1$.

Proof Let us consider $\Gamma_u = \Gamma \setminus \{u\}$. We will first assume that Γ_u is connected. Following the notation above, $V_{2,u} = \{v_1, \dots, v_m\}$ denotes the set of vertices adjacent to u with label 2 and $W = \{w_1, \dots, w_n\}$ denotes the set of remaining vertices of Γ_u. We will consider the matrix M obtained by eliminating the column corresponding to u from the Alexander matrix $M_{\Gamma,u,k}$ (which has $(nk+m+1)-1 = nk+m$ columns). We will prove the result by showing that M has maximal rank.

(1) If $n = 0$, the matrix M becomes

$$M = \begin{array}{c} \begin{array}{ccc} v_1 & \cdots & v_m \end{array} \\ \begin{pmatrix} 1-t_{\overline{u}} & & \\ & \ddots & \\ & & 1-t_{\overline{u}} \\ * & * & * \\ \vdots & \vdots & \vdots \\ * & * & * \end{pmatrix} \end{array},$$

which has maximal rank.

(2) If $n \neq 0$, consider a spanning tree T on Γ_u.

(a) Assume $m \neq 0$. In this case we will describe certain submatrices of M_Γ which will appear as blocks in $M_{\Gamma,u,k}$ of the appropriate rank.

In order to do this, note that T will contain at least n edges $e_1, \dots, e_n$ with the property that each e_i involves at least one vertex in W and $W \subset V(\{e_1, \dots, e_n\})$. Let us denote by $S \subset T$ the forest containing the edges $e_1, \dots, e_n$. Note that S defines a submatrix M_0 of M_{Γ_u}. We will show that columns and rows can be ordered in such a way that M_0 is upper-triangular, every diagonal element is nonzero, and the columns associated to the vertices W come first.

This can be easily seen by induction. In the case that Γ_u has only two vertices, say v and w (this is by hypothesis the minimum number of vertices), and only one edge, the matrix M_0 is a row matrix of type (5) whose columns can be reordered as wanted. Now suppose the result is true for $\lambda - 1$ vertices and consider the case when Γ_u has exactly λ vertices. Choose a vertex w' in $V(S)$ of degree 1. Note that, by definition, S must contain at least one such vertex in W, so one can assume $w' \in W$. Then $S \setminus \{w'\}$ verifies the result. The matrix M_0 results from the latter after adding one column (associated to w') and one row f (associated to the edge containing w'). Note that placing w' as the first column and f as the first row concludes the proof.

Also note that the submatrix M_n of M_0 resulting from keeping only the columns associated to the vertices in W appears as is in k blocks in $M_{\Gamma,u,k}$ corresponding to the copies of the vertices in W and the relations associated to the edges of S. This produces a square submatrix M_k of $M_{\Gamma,u,k}$ of size kn and nonzero determinant. Finally, let us add to M_k the columns associated to all vertices in $V_{2,u}$ placed at the end. Since every $v_i \in V_{2,u}$ is adjacent to u with label 2 the relations associated to these edges result in rows producing an upper-triangular square submatrix M of size $kn + m$ whose determinant is nonzero as below:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} & W & & \phantom{1-t_{\bar u}} & V & \phantom{1-t_{\bar u}} \end{array} \\
M = & \left(\begin{array}{cccccc}
M_n & & & & & \\
0 & \ddots & & & & \\
\vdots & 0 & M_n & & & \\
\hline
\vdots & \vdots & \ddots & 1 - t_{\bar{u}} & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
0 & 0 & 0 & 0 & 0 & 1 - t_{\bar{u}}
\end{array}\right).
\end{array}
$$

This ends this case.

(b) If $m = 0$, then the spanning tree T consists of n vertices and $n - 1$ edges. Let us consider a vertex w_n in Γ adjacent to u (there must be at least one since Γ is connected). Consider the Alexander submatrix $M_{B(w_n,\overline{u})}$ associated with relations of type (RB), which by Remark 2.10 is equivalent to

$$\begin{array}{c} \begin{array}{ccccc} w_{n,0} & w_{n,1} & \cdots & w_{n,k-2} & w_{n,k-1} \end{array} \\ \begin{pmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & \ddots & \ddots & * & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & * \end{pmatrix} \end{array}.$$

Let $f_1,\dots,f_k$ denote the k rows of this matrix.

On the other hand, let M_T be the $(n-1)\times n$ submatrix M_Γ associated to T. Let us order the relations in such a way that M_T is upper-triangular with nonzero diagonal elements and its last column corresponds to w_n —in other words, the vertex associated to the first column must have degree 1.

For each group of copies of the vertices $w_{j,p}$, there is a copy of the tree T with an analogous matrix $M_{T,p}$. Now, one can write the Alexander matrix M_Γ in the following way: the first rows correspond to the matrix M_T, then the row f_1 completed with zeroes where necessary, then the rows corresponding to the matrix $M_{T,1}$, then the row f_2. The final rows correspond to the matrix $M_{T,k-1}$ and the row f_k. This matrix is clearly upper-triangular and it has maximal rank $kn=kn+m$.

Summarizing, if Γ_u is connected, then $\operatorname{rank}(M_{\Gamma,u,k})\geq nk+m$, and it follows that $\operatorname{corank}(M_{\Gamma,u,k})\leq 1$.

Assume now that Γ_u is not connected. Denote by $\Gamma_1,\dots,\Gamma_s$ its connected components.

Then, the Alexander matrix $M_{\Gamma,u,k}$ after removing the column $\overline{u}$ is of the form

$$\begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_s \end{array} \\ \begin{pmatrix} M_{C_1} & 0 & \cdots & 0 \\ 0 & M_{C_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M_{C_s} \end{pmatrix} \end{array},$$

where M_{C_i} corresponds to a connected graph. The result follows from the connected case since the matrix is block-diagonal. □

Lemma 2.12 *Assume $e=\{w,u\}\in E$ is such that $\ell_e\equiv 0 \bmod k$. Then the matrix $M_{B(w,\overline{u})}$ has rank 1 over $\Lambda/\mathfrak{p}$, where $\mathfrak{p}$ is the ideal generated by $1-t_{\overline{u}}t_{w,0}\cdots t_{w,k-1}$.*

Proof By Lemma 2.5 we know that $M_{B(w,\overline{u})}$ is a multiple by $p_{c_e}(t_{\overline{u}}\overline{t}_w)$ of the matrix

$$M=\begin{pmatrix} 1-\overline{t}_{w,0,k} & t_{\overline{u}}-1 & \cdots & \overline{t}_{w,0,k-1}(t_{\overline{u}}-1)\\ \overline{t}_{w,1,k}(t_{w,0}-1) & 1-\overline{t}_{w,1,0} & \cdots & \overline{t}_{w,1,k-1}(t_{w,0}-1)\\ \vdots & \vdots & \ddots & \vdots\\ \overline{t}_{w,k-1,k}(t_{w,k-2}-1) & \overline{t}_{w,k-1,0}(t_{w,k-2}-1) & \cdots & t_{w,k-2}-1 \end{pmatrix}.$$

Note that M can be written, mod $\mathfrak{p}$, as

$$\begin{pmatrix} t_{\overline{u}}^{-1}(t_0-1) & t_{\overline{u}}-1 & \cdots & \overline{t}_{w,0,k-1}(t_{\overline{u}}-1)\\ t_{\overline{u}}^{-1}t_{w,0}^{-1}(t_{w,0}-1) & t_{w,0}^{-1}(t_{w,0}-1) & \cdots & \overline{t}_{w,1,k-1}(t_{w,0}-1)\\ \vdots & \vdots & \ddots & \vdots\\ t_{\overline{u}}^{-1}t_{w,0}^{-1}\cdots t_{w,k-2}^{-1}(t_{w,k-2}-1) & t_{w,0}^{-1}\cdots t_{w,k-2}^{-1}(t_{w,k-2}-1) & \cdots & t_{w,k-2}-1 \end{pmatrix}.$$

If f_j denotes the j^{th} row of M, note that

$$(t_{\overline{u}}-1)t_{w,0}\cdots t_{w,j-2}f_j=(t_{w,j-2}-1)f_1$$

for any $j=2,\dots,k-1$, and thus the result follows. □

3 QP–irreducible graphs

As mentioned in the introduction, a graph is called *quasi-projective* — or a QP–graph — if its associated Artin group is in $\mathcal{G}_{\mathrm{QP}}$. The purpose of this section is to describe the simplest QP–graphs, referred to as QP–*irreducible* graphs, for even Artin groups.

Definition 3.1 We call Γ a QP–*irreducible graph* if $\mathbb{A}_\Gamma$ is quasi-projective and it cannot be obtained as a 2–join of two quasi-projective graphs.

By Proposition 1.7(2), the 2–join of QP–graphs must be a QP–graph. However, in general, properties of Artin groups are not easily read from subgraphs. This result allows one to read an obstruction to quasi-projectivity from certain subgraphs of graphs.

Definition 3.2 We say that Γ_1 is a v–*subgraph* of Γ if Γ_1 is obtained from Γ by deleting some vertices. We will denote it by $\Gamma_1\subset_v\Gamma$. In this situation Γ is called a v–*supergraph* of Γ_1.

Lemma 3.3 *Let $\mathbb{A}_{\Gamma_1}$ be the Artin group of $\Gamma_1 = (V_1, E_1, m_1)$. Assume that for certain $k \in \mathbb{Z}_{\geq 2}$ and $u \in \Gamma_1$, the subgroup $\widehat{G}_k := \mathbb{A}_{\Gamma_1,u,k} \subset \mathbb{A}_{\Gamma_1}$ has the property that there exist two ideals $\widehat{I}_1, \widehat{I}_2 \subset \widehat{\Lambda}_k := \mathbb{C}[H_1(\widehat{G}_k)]$ such that*

(C1) $Z(\widehat{I}_i) \subset V_{r_i}(\widehat{G}_k)$, $r_i \geq 1$ *for* $i = 1, 2$,

(C2) $\dim(Z(\widehat{I}_1 + \widehat{I}_2)) \geq 1$, *and*

(C3) (a) *either* $Z(\widehat{I}_1 + \widehat{I}_2) \subset V_r(\widehat{G}_k)$ *for* $r > \max\{r_1, r_2\}$,

(b) *or* $\widehat{I}_1, \widehat{I}_2$ *are prime ideals of* $\widehat{\Lambda}_k$.

Then $\mathbb{A}_{\Gamma_1}$ is not quasi-projective.

Moreover, if $\Gamma = (V, E, m)$ is any v–supergraph of Γ_1 such that m_e is even for any $e = \{v, w\} \in E$ with $v \in V_1$ and $w \in V \setminus V_1$, then $\mathbb{A}_\Gamma$ is not quasi-projective.

Proof Let us prove the first part by contradiction. Assuming that $\mathbb{A}_{\Gamma_1}$ is quasi-projective would imply that the cocyclic group $\mathbb{A}_{\Gamma_1,u,k}$ is also quasi-projective by Proposition 1.7(1). The strategy of this proof is to reach a contradiction on the quasi-projectivity of $\mathbb{A}_{\Gamma_1,u,k}$ by finding two irreducible components of its characteristic variety intersecting in a positive dimensional component and thus contradicting Proposition 1.16. Let us assume that $r_1 \geq r_2$. Note that the set of zeroes $Z(\widehat{I}_i)$ may be nonirreducible, but, using condition (a) in the statement, there exists an irreducible component, say H_1 (resp. H_2), in $Z(\widehat{I}_1)$ (resp. $Z(\widehat{I}_2)$) which is not contained in $Z(\widehat{I}_2)$ (resp. $Z(\widehat{I}_1)$). By condition (C2) their intersection $H_1 \cap H_2$ has dimension greater than or equal to 1.

To prove the *moreover* part, we will show that $\mathbb{A}_\Gamma$ also satisfies the hypotheses of the first part, that is, that there exist two ideals $I_1, I_2 \subset \Lambda_k := \mathbb{C}[H_1(G_k)]$ satisfying conditions (C1)–(C3) for the subgroup $G_k := \mathbb{A}_{\Gamma,u,k} \subset \mathbb{A}_\Gamma$. Note that the condition on the parity of the labels joining vertices from V_1 and $V \setminus V_1$ ensures the existence of a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & G_k \arrow[r] & G_\Gamma \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \widehat{G}_k \arrow[r] & G_{\Gamma_1} \arrow[r] & \mathbb{Z} \arrow[r] & 1
\end{tikzcd}$$

which allows for the existence of a morphism $H_1(G_k) \to H_1(\widehat{G}_k)$ extending to a morphism $\Lambda_k \to \widehat{\Lambda}_k$. Moreover, $\widehat{\Lambda}_k = \Lambda_k / I$ for a certain ideal. In order to describe it let us decompose V as a disjoint union $V = V_1 \cup \widetilde{V}_{2,u} \cup W$, where

$$\widetilde{V}_{2,u} = \big\{ v \in V \setminus V_1 \mid e = \{u, v\} \in E,\ m_e = 2 \big\}.$$

Then

$$I = \mathrm{Ideal}\big(\{t_v - 1 \mid v \in V \setminus \widetilde{V}_{2,u}\} \cup \{t_{w,j} - 1 \mid w \in W,\ j \in \mathbb{Z}_k\}\big).$$

Since the tensor product is right exact, the matrix $\widehat{M}_{\Gamma,u,k} = M_{\Gamma,u,k} \otimes \Lambda/I$ determines the Alexander $\widehat{\Lambda}_k$–module of $\widehat{G}_k$. We claim that

$$\widehat{M}_{\Gamma,u,k} = \left(\begin{array}{c|c} M_{\Gamma_1,u,k} & 0 \\ \hline 0 & A' \end{array}\right). \tag{6}$$

In order to check this, first note that the submatrix of $\widehat{M}_{\Gamma,u,k}$ whose rows are associated to the edges of Γ_1 has the form

$$\left(\begin{array}{c|c} M_{\Gamma_1,u,k} & 0 \end{array}\right).$$

The claim will follow if we prove that the remaining rows, associated to the edges in $E \setminus E_1$, have the property that any entry in a column in V_1 is in $\widehat{I}$. The latter is a consequence of (5) and Lemma 2.5.

Finally, note that if condition (a) (resp. (b)) is satisfied for $\widehat{I}_i$, then also condition (a) (resp. (b)) is satisfied for $I_i = I + \widehat{I}_i$ using (6) (resp. using that $Z(I_i) = Z(\widehat{I}_i) \times \{1\}$ is irreducible). Therefore the ideals $I_1, I_2 \subset \Lambda_k$ also satisfy the conditions of the statement for $\mathbb{A}_\Gamma$, and the result follows. $\square$

Remark 3.4 By Theorem 1.10, the only QP–irreducible right-angled graphs are sets of r disconnected vertices, $\bar{K}_r$. On the other hand, we have established by Theorem 1.11 that both the segment $S_{2\ell}$ with label 2ℓ (for $\ell > 1$) and the triangle $T(4,4,2)$ are also QP–irreducible graphs.

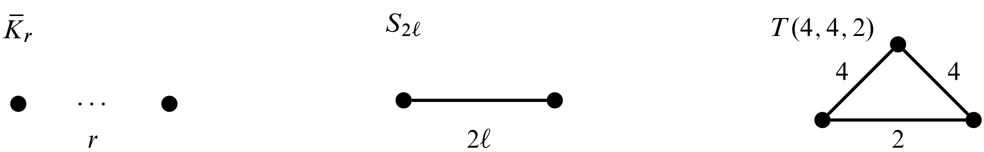

Figure 2: QP–irreducible graphs of type $\bar{K}_r$, $S_{2\ell}$ and $T(4,4,2)$

The purpose of this section is to show that the only QP–irreducible graphs are $\bar{K}_r$, $S_{2\ell}$ (with $\ell > 1$), and $T(4,4,2)$.

First we can assume that our graph has at least three vertices, otherwise it is QP–irreducible if and only if it is disconnected $\bar{K}_r$ (with $r = 1, 2$) or a segment $S_{2\ell}$ (with $\ell > 1$). The second reduction is given in [3] for *strictly even* graphs, that is, even graphs that are not right-angled. We recall it here.

Theorem 3.5 [3, Theorem 5.26] *If Γ is a strictly even, noncomplete graph with at least three vertices, then A_Γ is not quasi-projective.*

This result is shown by proving that the characteristic varieties of the Artin groups of noncomplete strictly even graphs contain two irreducible components having a positive-dimensional intersection, which contradicts Proposition 1.16.

Note that QP–irreducible even graphs — other than a point — are necessarily strictly even. Hence, the purpose of the rest of the section is to study complete QP–irreducible graphs of three or more vertices.

3.1 Complete QP–irreducible graphs with 3 vertices

Theorem 3.6 *Assume that Γ is an even v–supergraph of $T(2r, 2k, 2\ell)$ with $r \geq 3$ and $k \geq 2$; see Figure 3. Then $\mathbb{A}_\Gamma$ is not quasi-projective.*

Proof Without loss of generality, one can assume $r \geq k \geq \ell$. Four separate cases will be considered. The first case will be shown in detail. The remaining cases follow analogously.

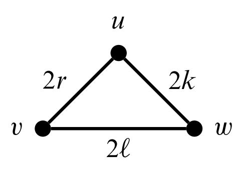

Figure 3

(1) In the case $r \geq 4$, $k \geq 2$, consider the index r subgroup $\mathbb{A}_{T,u,r} \subset \mathbb{A}_T$, where $T = T(2r, 2k, 2\ell)$. According to Lemma 2.8 $M_{T,u,r}$ has two B–Artin submatrices $M_{B(v,\bar{u})}$ and $M_{B(w,\bar{u})}$ of r rows each and an Artin submatrix of r rows (corresponding to the r relations $A_\ell(v_i, w_i)$ for $i \in \mathbb{Z}_r$). Hence $M_{T,u,r}$ is a $3r \times (2r+1)$ matrix whose corank is ≤ 1 by Lemma 2.11. Define $p = 1 - t_{\bar{u}} \overline{t_v}$ and consider the ideals

$$I_1 = (p, t_{v,0} - 1, t_{v,1} - 1, t_{w,0} - 1, t_{w,1} - 1),$$
$$I_2 = (p, t_{v,0} - 1, t_{v,2} - 1, t_{w,0} - 1, t_{w,2} - 1).$$

Note that $\operatorname{rank}(M_{B(v,\bar{u})}|_{I_i}) = 1$ by Lemma 2.12. In addition, note that the first two rows of $M_{T,u,r}|_{I_1}$ are zero and so are the first and third rows of $M_{T,u,r}|_{I_2}$. Summarizing, $M_{T,u,r}|_{I_i}$ contains three submatrices $M_{i,A}$, $M_{i,B(v,\bar{u})}$ and $M_{i,B(w,\bar{u})}$, where $\operatorname{rank}(M_{i,A}) \leq r-2$, $\operatorname{rank}(M_{i,B(v,\overline{u})}) = 1$ and $\operatorname{rank}(M_{i,B(w,\overline{u})}) \leq r$. Therefore $\operatorname{rank}(M_{T,u,r}|_{I_i}) \leq 2r-1$. Since $M_{T,u,r}$ has $2r+1$ columns, one has

$$\operatorname{corank} M_{T,u,r}|_{I_i} \geq 2 > \operatorname{corank} M_{T,u,r}.$$

Also,

$$\operatorname{corank} M_{T,u,r}|_{I_1+I_2} < \max\{\operatorname{corank} M_{T,u,r}|_{I_1}, \operatorname{corank} M_{T,u,r}|_{I_2}\},$$

since $I_1 + I_2$ makes one extra row vanish, which is originally independent from the others. Finally, $Z(I_1 + I_2)$ has dimension ≥ 1 since the variable $t_{\overline{u}}$ is free ($r \geq 4$, so P gives a relation between $t_{v,3}$ and $t_{\overline{u}}$ but does not fix any of them). Therefore, by Lemma 3.3, $\mathbb{A}_\Gamma$ is not quasi-projective.

According to Remark 2.2, if $\gcd(k,r)=1$ (resp. $\gcd(k,r)=2$), then $t_{w,0}=t_{w,1}=t_{w,2}$ (resp. $t_{w,0} = t_{w,2}$). However, the variables $t_{v,i}$ are all different due to the choice of r, the label of the edge $\{u,v\}$, as the index of the finite subgroup. Hence, the ideals I_1 and I_2 satisfy the properties of Lemma 3.3 anyway.

(2) The case $r=3$, $k=3$ can be treated by considering $\mathbb{A}_{T,u,3} \subset \mathbb{A}_T$ and the ideals

$$\begin{aligned} I_1 &= (t_{\overline{u}}-1, t_{v,1}-1, t_{v,2}-1, t_{w,0}-1, t_{w,1}-1), \\ I_2 &= (t_{\overline{u}}-1, t_{v,1}-1, t_{v,2}-1, t_{w,0}-1, t_{w,2}-1). \end{aligned}$$

(3) The case $r=3$, $k=\ell=2$ follows after considering the subgroup $\mathbb{A}_{T,u,2} \subset \mathbb{A}_T$ of index 2 and the ideals

$$\begin{aligned} I_1 &= (1-t_{\overline{u}}\overline{t}_w, 1-t_{\overline{u}}\overline{t}_v, p_0), \\ I_2 &= (1-t_{\overline{u}}\overline{t}_w, 1-t_{\overline{u}}\overline{t}_v, p_1), \end{aligned}$$

with $p_i = 1 + t_{w,i}t_{v,i} + t_{w,i}^2 t_{v,i}^2$ for $i=0,1$.

(4) Finally, if $r=3$, $k=2$ and $\ell=1$, the result follows after considering the subgroup $\mathbb{A}_{T,v,3} \subset \mathbb{A}_T$ of index 3 and the ideals

$$\begin{aligned} I_1 &= (p_0, p_1, 1-t_{\overline{v}}\overline{t}_u), \\ I_2 &= (p_0, p_2, 1-t_{\overline{v}}\overline{t}_u), \end{aligned}$$

where $p_i = 1 + t_w t_{u,i}$ for $i=0,1,2$. □

Theorem 3.7 *Assume Γ is an even v–supergraph of $T(4,4,4)$. Then $\mathbb{A}_\Gamma$ is not quasi-projective.*

Proof Consider $T = T(4,4,4)$ with vertices $V = \{u, v, w\}$ and the index 2 cocyclic subgroup $\mathbb{A}_{T,u,2} \subset \mathbb{A}_T$. By Lemma 2.8, the Alexander matrix of the associated group is

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \quad v_0 \quad & \quad w_0 \quad & \quad v_1 \quad & \quad w_1 \quad & \quad \overline{u} \quad \end{array} \\
M_{T,u,2} = & \begin{pmatrix}
p_0(1-t_{w,0}) & p_0(t_{v,0}-1) & 0 & 0 & 0 \\
0 & 0 & p_1(1-t_{w,1}) & p_1(t_{v,1}-1) & 0 \\
t_{\overline{u}}-1 & 0 & t_{v,0}(t_{\overline{u}}-1) & 0 & 1-t_{v,0}t_{v,1} \\
1-t_{v,1}t_{\overline{u}} & 0 & t_{v,0}-1 & 0 & t_{v,1}(t_{v,0}-1) \\
0 & t_{\overline{u}}-1 & 0 & t_{w,0}(t_{\overline{u}}-1) & 1-t_{w,0}t_{w,1} \\
0 & 1-t_{w,1}t_{\overline{u}} & 0 & t_{w,0}-1 & t_{w,1}(t_{w,0}-1)
\end{pmatrix},
\end{array}
$$

with $p_i = 1 + t_{v,i}t_{w,i}$ for $i = 0, 1$. By Lemma 2.11, $M_{T,u,2}$ has corank ≤ 1. Consider the ideals

$$I_1 = (1 - t_{\overline{u}}\overline{t}_v, 1 - t_{\overline{u}}\overline{t}_w, p_0),$$
$$I_2 = (1 - t_{\overline{u}}\overline{t}_v, 1 - t_{\overline{u}}\overline{t}_w, p_1).$$

By Lemma 2.12, it is clearly seen that $M_{T,u,2}|_{I_i}$ has corank 2. Therefore

$$2 = \max\{\operatorname{corank} M_{T,u,2}|_{I_1}, \operatorname{corank} M_{T,u,2}|_{I_2}\} > 1 \geq \operatorname{corank} M_{T,u,2}.$$

It is also easy to see that $M_{T,u,2}|_{I_1+I_2}$ has corank 3, which implies

$$\operatorname{corank} M_{T,u,2}|_{I_1+I_2} = 3 > \max\{\operatorname{corank} M_{T,u,2}|_{I_1}, \operatorname{corank} M_{T,u,2}|_{I_2}\} = 2.$$

Moreover, $Z(I_1+I_2)$ has dimension ≥ 1 since I_1+I_2 is generated by four polynomials in five variables. Therefore, by Lemma 3.3, $\mathbb{A}_\Gamma$ is not quasi-projective. $\square$

The previous results combined prove the following.

Corollary 3.8 *The only strictly even complete* QP*–graphs with three vertices are $T(2\ell, 2, 2)$ with $\ell \geq 2$ and $T(4,4,2)$. Moreover, the latter is the only* QP*–irreducible even graph with three vertices.*

3.2 QP–irreducible even graphs with 4 vertices

As an immediate consequence of Theorems 3.6, 3.7 and 3.5, the only candidates for QP–irreducible even graphs with 4 vertices must be complete v–supergraphs of either $T(2\ell, 2, 2)$ or $T(4,4,2)$. Figure 4 shows all such possible graphs.

This list can easily be obtained using the following observation.

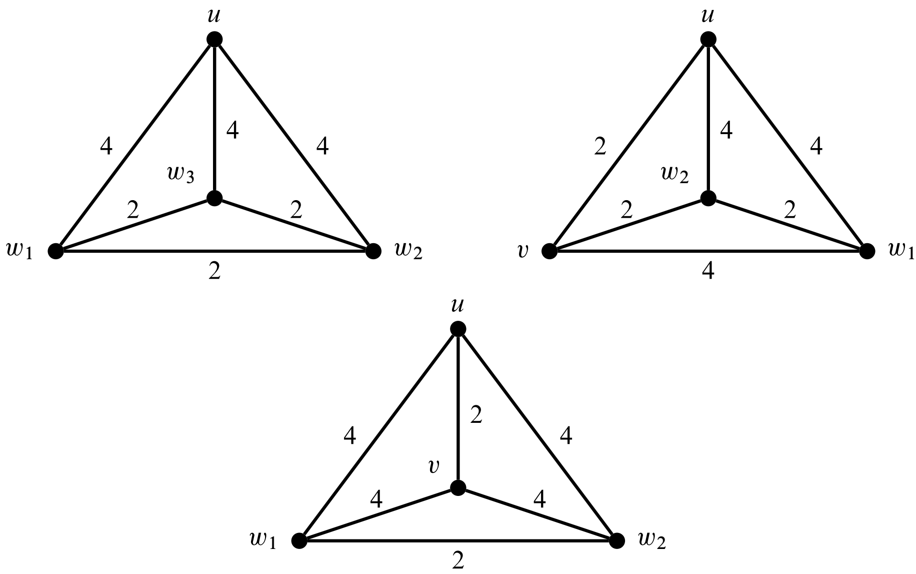

Figure 4

Lemma 3.9 *Any* QP*–irreducible even graph with at least* 3 *vertices has labels no larger than* 4*.*

Proof By Theorem 3.5 one can assume the graph Γ is complete. Assume $m_e \geq 6$ for some edge $e \in E$ of Γ. By Theorem 3.6 all edges adjacent to e must have a label 2. Since Γ is complete, $\Gamma = \{e\} *_2 \Gamma'$, where Γ' is the resulting v–subgraph after deleting the vertices of e. □

Note that the 4–graph in Figure 4, top-left, is the only candidate containing $T(2,2,2)$, Figure 4, top-right, is the only candidate containing $T(4,4,2)$ but no $T(2,2,2)$, and Figure 4, bottom, is the only candidate containing $T(4,4,2)$ but no $T(2\ell,2,2)$.

We are going to see that these three candidates cannot be QP–irreducible graphs.

Theorem 3.10 *There are no* QP*–irreducible even graphs of four vertices.*

Moreover, an even graph containing any of the graphs in Figure 4 as a v*–subgraph is not quasi-projective.*

Proof As discussed above, one only needs to rule out the list of graphs shown in Figure 4. We will do this separately and using similar arguments. For this reason we will show only the case of Figure 4, top-left, in detail.

Figure 4, top-left Consider the index 2 subgroup $\mathbb{A}_{\Gamma,u,2} \subset \mathbb{A}_\Gamma$. Its Alexander matrix is

$$M_{\Gamma,u,2} =$$

$$\begin{array}{c}
\begin{array}{ccccccc} w_{1,0} & w_{2,0} & w_{3,0} & w_{1,1} & w_{2,1} & w_{3,1} & \bar{u} \end{array} \\
\begin{pmatrix}
1-t_{2,0} & t_{1,0}-1 & 0 & 0 & 0 & 0 & 0 \\
1-t_{3,0} & 0 & t_{1,0}-1 & 0 & 0 & 0 & 0 \\
0 & 1-t_{3,0} & t_{2,0}-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1-t_{2,1} & t_{1,1}-1 & 0 & 0 \\
0 & 0 & 0 & 1-t_{3,1} & 0 & t_{1,1}-1 & 0 \\
0 & 0 & 0 & 0 & 1-t_{3,1} & t_{2,1}-1 & 0 \\
t_{\bar{u}}-1 & 0 & 0 & t_{1,0}(t_{\bar{u}}-1) & 0 & 0 & 1-t_{1,0}t_{1,1} \\
1-t_{1,1}t_{\bar{u}} & 0 & 0 & 1-t_{1,0} & 0 & 0 & t_{1,1}(t_{1,0}-1) \\
0 & t_{\bar{u}}-1 & 0 & 0 & t_{2,0}(t_{\bar{u}}-1) & 0 & 1-t_{2,0}t_{2,1} \\
0 & 1-t_{2,1}t_{\bar{u}} & 0 & 0 & t_{2,0}-1 & 0 & t_{2,1}(t_{2,0}-1) \\
0 & 0 & t_{\bar{u}}-1 & 0 & 0 & t_{3,0}(t_{\bar{u}}-1) & 1-t_{3,0}t_{3,1} \\
0 & 0 & 1-t_{3,1}t_{\bar{u}} & 0 & 0 & t_{3,0}-1 & t_{3,1}(t_{3,0}-1)
\end{pmatrix}
\end{array},$$

where $t_{i,j}$ denotes $t_{w_{i,j}}$. We now consider the prime ideals

$$I_1 = (t_{\bar{u}}-1, t_{1,1}-1, t_{2,0}-1, t_{2,1}-1, t_{3,0}-1),$$
$$I_2 = (t_{\bar{u}}-1, t_{1,0}-1, t_{1,1}-1, t_{2,1}-1, t_{3,0}-1).$$

Note that $\operatorname{corank}(M_{\Gamma,u,2}|_{I_1}) = \operatorname{corank}(M_{\Gamma,u,2}|_{I_2}) = 2$, $\operatorname{corank}(M_{\Gamma,u,2}|_{I_1+I_2}) = 4$ and

$$Z(I_1+I_2)=\{(t_{\bar{u}}, t_{1,0}, t_{1,1}, t_{2,0}, t_{2,1}, t_{3,0}, t_{3,1}) = (1,1,1,1,1,1,\lambda) \mid \lambda \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^7.$$

The result follows from Lemma 3.3 and the fact that $\dim Z(I_1+I_2) = 1$.

Figure 4, top-right Consider $\mathbb{A}_{\Gamma,u,2} \subset \mathbb{A}_\Gamma$ and the prime ideals

$$I_1 = (t_v-1, t_{\bar{u}}-1, t_{1,1}-1, t_{2,0}-1),$$
$$I_2 = (t_{\bar{u}}-1, t_{1,1}-1, t_{2,0}-1, 1+t_{1,0}t_v).$$

Figure 4, bottom Consider the subgroup $\mathbb{A}_{\Gamma,u,2} \subset \mathbb{A}_\Gamma$ and the prime ideals

$$I_1 = (t_{\bar{u}}-1, t_{1,0}-1, t_{2,1}-1, 1+t_{1,1}t_v),$$
$$I_2 = (t_{\bar{u}}-1, t_{1,0}-1, t_{2,1}-1, 1+t_{2,0}t_v). \qquad \square$$

3.3 QP–irreducible even graphs with more than 4 vertices

As a consequence of the results obtained in the previous sections, no quasi-projective even Artin group can contain any of the following:

(1) A vertex with two edges with labels $2r$, $r \geq 3$ and $2k$, $k \geq 2$; see Theorems 3.6 and 3.5.

(2) A triangle $T(4,4,4)$; see Theorem 3.7.

(3) A three-edge tree of labels $4,4,4$; see Theorems 3.5, 3.7 and 3.10.

Theorem 3.11 *There are no* QP*–irreducible even graphs with more than* 3 *vertices.*

Proof The result follows for graphs with four vertices by the previous section.

For any QP–irreducible even graph Γ with more than four vertices, note the following:

- Γ must be complete by Theorem 3.5.
- If Γ contains an edge e with label $m_e = 2r$, $r \geq 3$, then by (1) above, $\Gamma = \{e\} *_2 \widehat{\Gamma}$ and hence Γ is not QP–irreducible.
- If Γ contains an edge e with label $m_e = 4$, then either $\Gamma = \{e\} *_2 \widehat{\Gamma}$ (see (2) above) or $\Gamma = T(4,4,2) *_2 \widehat{\Gamma}$ (see (3)). □

4 Proofs of the main theorems

4.1 Proof of Theorem 1

As a consequence of Theorems 1.10 and 1.11, graphs of type $\overline{K}_r$, $S_{2\ell}$ and $T(4,4,2)$ as in Figure 2 are QP–irreducible. Moreover, by Corollary 3.8 and Theorem 3.11 these are the only ones. Using (2) and Proposition 1.7(2) any 2–join QP–irreducible graphs is quasi-projective. This completes the first part of the proof.

For the *moreover* part it is enough to check that the product of two fundamental groups of curve complements is also the fundamental group of a curve complement. This is a consequence of the following result.

Theorem 4.1 [23] *Let C_1 and C_2 be plane algebraic curves in $\mathbb{C}^2$. Assume that the intersection $C_1 \cap C_2$ consists of $d_1 d_2$ distinct points, where the d_i $(i = 1,2)$ are the respective degrees of C_1 and C_2. Then the fundamental group $\pi_1(\mathbb{C}^2 \setminus (C_1 \cup C_2))$ is isomorphic to the product of $\pi_1(\mathbb{C}^2 \setminus C_1)$ and $\pi_1(\mathbb{C}^2 \setminus C_2)$.*

4.2 Proof of Theorem 2

Since the product of two $K(\pi,1)$ spaces is also a $K(\pi,1)$ space, it is enough to prove the result for the QP–irreducible even graphs $\bar{K}_r$, $S_{2\ell}$ and $T(4,4,2)$. The graph $\bar{K}_r$ is associated to the free group $\mathbb{F}_r$ of rank r, which can be realized as the fundamental group of the complement to r points in $\mathbb{C}$, which is an Eilenberg–Mac Lane space.

The group $\mathbb{A}_{S_{2\ell}}$ associated to the segment graph $S_{2\ell}$ is the fundamental group of the complement X to the affine curve $\{y-x^\ell\}\cup\{y=0\}$. In projective coordinates X can be seen as the complement to the projective curve $\mathcal{C}=\{yz(yz^{\ell-1}-x^\ell)=0\}\subset\mathbb{P}^2$, that is, $X=\mathbb{P}^2\setminus\mathcal{C}$. Consider the projection $\pi\colon \mathbb{P}^2\setminus\{[1:0:0]\}\to\mathbb{P}^1$ defined by $[x:y:z]\mapsto[y:z]$. Note that $\pi|_X\colon X\to\mathbb{P}^1\setminus\{[0:1],[1:0]\}$ is a well-defined, locally trivial fibration and moreover, the fiber at each point $[y:z]$ is homeomorphic to $\mathbb{C}\setminus\{\ell \text{ points}\}$. Thus X is also an Eilenberg–Mac Lane space.

Finally, the triangle Artin group $\mathbb{A}_T$ associated to the triangle $T=T(4,4,2)$ is the fundamental group of the complement X to the affine curve

$$\{y-x^2\}\cup\{2x-y-1=0\}\cup\{2x+y+1=0\}$$

as in [3, Example 5.11]. Using the identification $\mathbb{C}^2=\mathbb{P}^2\setminus\{z=0\}$ we can think of X as the complement of a smooth conic and three tangent lines in the complex projective plane. After an appropriate change of coordinates, X can be given as $\mathbb{P}^2\setminus\mathcal{C}$, where $\mathcal{C}=\{F(x,y,x)=xyz(x^2+y^2+z^2-2(xy+xz+yz))=0\}\subset\mathbb{P}^2$.

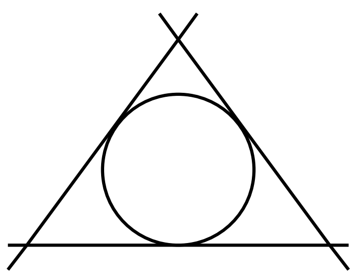

Figure 5: Projective curve $\mathcal{C}=\{F=0\}$

We will consider a 4-fold cover X_4 of X. Since the higher homotopy groups of X and X_4 are isomorphic, it is enough to show that X_4 is an Eilenberg–Mac Lane space. Consider the Kummer morphism $\kappa\colon\mathbb{P}^2\to\mathbb{P}^2$ defined by $\kappa([x:y:z])=[x^2:y^2:z^2]$. Note that κ is a 4:1 ramified cover and its ramification locus is $R=\{xyz=0\}$. Now $R\subset\mathcal{C}$ so κ defines an unramified cover on $X_4=\kappa^{-1}(X)$. Moreover, the preimage of $\mathcal{C}$ by κ is a product of 7 lines, three of which are the axis $xyz=0$ and four of which are the preimage of the conic $(x^2+y^2+z^2-2(xy+xz+yz))=0$. In particular,

$$\begin{aligned}\mathcal{C}_2=\kappa^{-1}(\mathcal{C})&=\{F(x^2,y^2,z^2)=0\}\\&=\{xyz(x+y+z)(x+y-z)(x-y+z)(x-y-z)=0\}.\end{aligned}$$

Geometrically this corresponds to a Ceva arrangement — formed by the six lines of a generic pencil of conics — with an extra line passing through two out of the three double points. In our equations, the pencil of conics can be defined as $F_{[\alpha:\beta]}=\alpha((x+z)^2-y^2)-\beta((x-z)^2-y^2)$. Note that for $\alpha=\beta=1$ one obtains $F_{[1:1]}=4xz$. The rational map $\pi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ defined by the pencil, where $\pi^{-1}[\alpha:\beta]=\{F_{[\alpha:\beta]}=0\}$, that is, $\pi([x:y:z])=[(x-z)^2-y^2:(x+z)^2-y^2]$, is not defined at the basepoints of the pencil. Since the curve $\mathcal{C}_2$ contains these basepoints, one obtains that $\pi|_{X_4}$ is well defined, where $X_4=\kappa^{-1}(X)=\mathbb{P}^2\setminus\mathcal{C}_2$.

After our discussion above, recall that the special fibers of π are the six lines

$$\{xz(x+y+z)(x+y-z)(x-y+z)(x-y-z)=0\}.$$

Finally note that the line $y=0$ is a multisection since $\pi|_{y=0}$ is defined by

$$\pi([x:0:z])=[(x-z)^2:(x+z)^2],$$

which is 2:1 and ramifies only at $[0:1]$ and $[1:0]$, therefore the map

$$\begin{aligned}&\pi|_{X_4}\colon X_4\to\mathbb{P}^1\setminus\{[0:1],[1:0],[1:1]\},\\&[x:y:z]\mapsto[(x-z)^2-y^2:(x+z)^2-y^2],\end{aligned}$$

is a well-defined locally trivial fibration whose generic fiber is the smooth conic of the pencil with six points removed (the four basepoints and the two points of intersection with the multisection $\{y=0\}$). Therefore X_4 is an Eilenberg–Mac Lane space.

5 An example

To end this paper we take a closer look into the triangle Artin group $\mathbb{A}_T$, $T=T(4,4,2)$ using geometrical methods coming from its quasi-projectivity property.

First we will show that $\mathbb{A}_T$ is not an extension of free groups. To do so we first study the surjections of $\mathbb{A}_T$ onto a free group $\mathbb{F}_r$ of rank r. Any surjection of groups $G_1\twoheadrightarrow G_2$ induces an injection of characteristic varieties $V_i(G_2)\hookrightarrow V_i(G_1)$ via the change of base $*\otimes_{\mathbb{C}[G_2/G_2']}\mathbb{C}[G_1/G_1']$ that turns an ideal in $\mathbb{C}[G_2/G_2']$ into an ideal in $\mathbb{C}[G_1/G_1']$; see [21]. An Alexander matrix of $\mathbb{A}_T$ can be obtained immediately from Lemma 2.4 and (5) as

$$M_{\mathbb{A}_T} = \begin{pmatrix} -(t_0t_1+1)(t_1-1) & (t_0t_1+1)(t_0-1) & 0 \\ -(t_0t_2+1)(t_2-1) & 0 & (t_0t_2+1)(t_0-1) \\ 0 & -t_2+1 & t_1-1 \end{pmatrix},$$

and thus its characteristic variety $V_1(\mathbb{A}_T) = \mathbb{T}_1 \cup \mathbb{T}_2 \cup \mathbb{T}_3$ is the zero set of the Fitting ideal generated by the 2×2–minors of $M_{\mathbb{A}_T}$, where

$$\begin{aligned} \mathbb{T}_1 &= \{(-t^{-1}, t, 1) \mid t \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3, \\ \mathbb{T}_2 &= \{(-t^{-1}, 1, t) \mid t \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3, \\ \mathbb{T}_3 &= \{(-t^{-1}, t, t) \mid t \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3 \end{aligned}$$

are three one-dimensional complex tori in $(\mathbb{C}^*)^3$. Since the characteristic variety of the free group $\mathbb{F}_r$ has dimension r, this implies that the only possible surjection $\mathbb{A}_T \twoheadrightarrow \mathbb{F}_r$ is restricted to $r = 1$.

Note that any short exact sequence

$$1 \to \mathbb{F}_s \to \mathbb{A}_T \to \mathbb{Z} \to 0$$

splits and the action of $\mathbb{Z}$ on $\mathbb{A}_T$ is trivial in homology. Therefore $\mathbb{A}_T = \mathbb{F}_s \rtimes \mathbb{Z}$ is called an *IA–product of free groups* and by [8, Corollary 3.4] the Poincaré polynomial $P_{\mathbb{A}_T}(t)$ of $\mathbb{A}_T$ should factor as a product of linear terms in $\mathbb{Z}[t]$. However, since the complement $X = \mathbb{P}^2 \setminus \mathcal{C}$ of the conic and three tangent lines shown in Figure 5 is a $K(\mathbb{A}_T, 1)$–space, it is enough to calculate $P_X(t)$. One can easily check that $h_0(X) = 1$ and $h_1(X) = 3$. Moreover, using the additivity of the Euler characteristic,

$$\begin{aligned} \chi(X) &= \chi(\mathbb{P}^2) - \sum \chi(\mathcal{C}_i) + \#\operatorname{Sing}(\mathcal{C}) = 3 - 4\chi(\mathbb{P}^1) + 3 = 1 \\ &= h_0(X) - h_1(X) + h_2(X) = -2 + h_2(X), \end{aligned}$$

where the $\mathcal{C}_i$ are the irreducible components of $\mathcal{C}$ and $\chi(\mathcal{C}_i) = \chi(\mathbb{P}^1) = 2$ since they are all rational curves. Therefore $h_2(X) = 3$ and thus

$$P_{\mathbb{A}_T}(t) = P_X(t) = 3t^2 + 3t + 1,$$

which is not a product of linear factors in $\mathbb{Z}[t]$.

However, as shown in the proof of Theorem 2 (see Section 4.2) its 4–fold cover X_4 is the complement of a line arrangement of fibered type, whose fundamental group $\pi_1(X_4)$ is a finite-index normal subgroup of $\mathbb{A}_T$, which is an IA–free product of free groups $\mathbb{F}_3 \rtimes \mathbb{F}_3$.

References

[1] **D Arapura**, *Geometry of cohomology support loci for local systems, I*, J. Algebraic Geom. 6 (1997) 563–597 MR

[2] **D Arapura**, **P Bressler**, **M Ramachandran**, *On the fundamental group of a compact Kähler manifold*, Duke Math. J. 68 (1992) 477–488 MR

[3] **E Artal Bartolo**, **J I Cogolludo-Agustín**, **D Matei**, *Quasi-projectivity, Artin–Tits groups, and pencil maps*, from "Topology of algebraic varieties and singularities" (J I Cogolludo-Agustín, E Hironaka, editors), Contemp. Math. 538, Amer. Math. Soc., Providence, RI (2011) 113–136 MR

[4] **E Artal Bartolo**, **J I Cogolludo-Agustín**, **D Matei**, *Characteristic varieties of quasi-projective manifolds and orbifolds*, Geom. Topol. 17 (2013) 273–309 MR

[5] **D A Buchsbaum**, **D Eisenbud**, *What annihilates a module?*, J. Algebra 47 (1977) 231–243 MR

[6] **F Catanese**, *Fibred Kähler and quasi-projective groups*, Adv. Geom. 3 (2003) S13–S27 MR

[7] **J I Cogolludo-Agustín**, *Topological invariants of the complement to arrangements of rational plane curves*, Mem. Amer. Math. Soc. 756, Amer. Math. Soc., Providence, RI (2002) MR

[8] **D C Cohen**, **A I Suciu**, *Homology of iterated semidirect products of free groups*, J. Pure Appl. Algebra 126 (1998) 87–120 MR

[9] **D C Cohen**, **A I Suciu**, *Characteristic varieties of arrangements*, Math. Proc. Cambridge Philos. Soc. 127 (1999) 33–53 MR

[10] **A Dimca**, *On the irreducible components of characteristic varieties*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 15 (2007) 67–73 MR

[11] **A Dimca**, **Ş Papadima**, **A I Suciu**, *Alexander polynomials: essential variables and multiplicities*, Int. Math. Res. Not. 2008 (2008) art. id. rnm119 MR

[12] **A Dimca**, **Ş Papadima**, **A I Suciu**, *Topology and geometry of cohomology jump loci*, Duke Math. J. 148 (2009) 405–457 MR

[13] **F Enriques**, *Sulla costruzione delle funzioni algebriche di due variabili possedenti una data curva di diramazione*, Ann. Mat. Pura Appl. 1 (1924) 185–198 MR

[14] **R H Fox**, *Free differential calculus, I: Derivation in the free group ring*, Ann. of Math. 57 (1953) 547–560 MR

[15] **R H Fox**, *Free differential calculus, III: Subgroups*, Ann. of Math. 64 (1956) 407–419 MR

[16] **J A Hillman**, *Alexander ideals of links*, Lecture Notes in Math. 895, Springer (1981) MR

[17] **E Hironaka**, *Alexander stratifications of character varieties*, Ann. Inst. Fourier (Grenoble) 47 (1997) 555–583 MR

[18] **E R van Kampen**, *On the fundamental group of an algebraic curve*, Amer. J. Math. 55 (1933) 255–260 MR

[19] **M Kapovich**, **J J Millson**, *On representation varieties of Artin groups, projective arrangements and the fundamental groups of smooth complex algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. 88 (1998) 5–95 MR

[20] **A S Libgober**, *Groups which cannot be realized as fundamental groups of the complements to hypersurfaces in* $\mathbb{C}^N$, from "Algebraic geometry and its applications" (C L Bajaj, editor), Springer (1994) 203–207 MR

[21] **A Libgober**, *Characteristic varieties of algebraic curves*, from "Applications of algebraic geometry to coding theory, physics and computation" (C Ciliberto, F Hirzebruch, R Miranda, M Teicher, editors), NATO Sci. Ser. II Math. Phys. Chem. 36, Kluwer, Dordrecht (2001) 215–254 MR

[22] **J W Morgan**, *The algebraic topology of smooth algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. 48 (1978) 137–204 MR

[23] **M Oka**, **K Sakamoto**, *Product theorem of the fundamental group of a reducible curve*, J. Math. Soc. Japan 30 (1978) 599–602 MR

[24] **L Paris**, *Lectures on Artin groups and the* $K(\pi,1)$ *conjecture*, from "Groups of exceptional type, Coxeter groups and related geometries" (N S N Sastry, editor), Springer Proc. Math. Stat. 82, Springer (2014) 239–257 MR

[25] **O Zariski**, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. 51 (1929) 305–328 MR

[26] **O Zariski**, *Algebraic surfaces*, 2nd edition, Ergeb. Math. Grenzgeb. 61, Springer (1971) MR

Departamento de Matemáticas, IUMA, Universidad de Zaragoza
Zaragoza, Spain

rubenb@unizar.es, jicogo@unizar.es

Proposed: Lothar Göttsche — Received: 21 February 2017
Seconded: Dan Abramovich, Walter Neumann — Revised: 2 January 2018