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Volumes of $\mathrm{SL}_n(\mathbb{C})$–representations of hyperbolic 3–manifolds

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Let M be a compact oriented three-manifold whose interior is hyperbolic of finite volume. We prove a variation formula for the volume on the variety of representations of $\pi_1(M)$ in $\mathrm{SL}_n(\mathbb{C})$. Our proof follows the strategy of Reznikov's rigidity when M is closed; in particular, we use Fuks's approach to variations by means of Lie algebra cohomology. When $n=2$, we get Hodgson's formula for variation of volume on the space of hyperbolic Dehn fillings. Our formula also recovers the variation of volume on the space of decorated triangulations obtained by Bergeron, Falbel and Guilloux and Dimofte, Gabella and Goncharov.

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1 Introduction

Let M be a compact oriented three-manifold whose interior admits a complete hyperbolic metric of finite volume. There is a well-defined notion of volume of a representation of its fundamental group $\pi_1(M)$ in $\mathrm{SL}_n(\mathbb{C})$ — see Definition 22 for instance — and here we view the volume as a function defined on the variety of representations $\mathrm{Hom}(\pi_1(M), \mathrm{SL}_n(\mathbb{C}))$. Bucher, Burger and Iozzi [7] have shown that the volume is maximal precisely at the composition of the lifts of the holonomy with the irreducible representation $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$. If M is furthermore closed, then this volume function is constant on connected components of $\mathrm{Hom}(\pi_1(M), \mathrm{SL}_n(\mathbb{C}))$ (see Reznikov [24]) but in the noncompact case the volume can vary locally. When $n=2$ this variety of representations (up to conjugation) contains the space of hyperbolic structures on the manifold, and the volume has been intensively studied in this case, starting with the seminal work of Neumann and Zagier [23]; in particular, a variation formula was obtained in Hodgson's thesis [17, Chapter 5], by means of Schläfli's variation formula for polyhedra in hyperbolic space. The variation of the volume was also discussed by Bergeron, Falbel and Guilloux [1] when $n=3$, and by Dimofte, Gabella and Goncharov [10] for general n, through the study of decorated ideal triangulations of manifolds.

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The purpose of this paper is to produce an infinitesimal formula for the variation of the volume in $\operatorname{Hom}(\pi_1(M), \mathrm{SL}_n(\mathbb{C}))$ for arbitrary n and for differentiable deformations of any representation, independently of the existence of decorated triangulations. The variety of representations has deformations that are nontrivial up to conjugation; more precisely, the component of $\operatorname{Hom}(\pi_1(M), \mathrm{SL}_n(\mathbb{C}))/\mathrm{SL}_n(\mathbb{C})$ that contains the representation of maximal volume has dimension $(n-1)k$ — see Menal-Ferrer and Porti [20] — where k is the number of components of ∂M. Our results are proved in $\operatorname{Hom}(\pi_1(M), \mathrm{SL}_n(\mathbb{C}))$, but they apply with no change to $\operatorname{Hom}(\pi_1(M), \mathrm{PSL}_n(\mathbb{C}))$.

The boundary ∂M of M consists of $k \geq 1$ tori, $T_1^2, \dots, T_k^2$. Fix the orientation of ∂M corresponding to the outer normal, as in Stokes theorem, and choose ordered generators l_i and m_i of $\pi_1(T_i^2)$, so that if we view them as oriented curves, they generate the induced orientation. For instance, for the exterior of an oriented knot in S^3, we can take l_1 as a longitude and m_1 as a meridian, with l_1 following the orientation of the knot and m_1 as describing the positive sense of rotation. For a complex number $z \in \mathbb{C}$, denote by $\Re(z)$ and $\Im(z)$ its real and imaginary parts, respectively. Assume now ρ_t is a differentiable path of representations in $\operatorname{Hom}(\pi_1(M), \mathrm{SL}_n(\mathbb{C}))$ parametrized by $t \in I \subset \mathbb{R}$. As a consequence of the Lie–Kolchin theorem, there exist 1–parameter families of matrices $A_i(t) \in \mathrm{SL}_n(\mathbb{C})$ and of upper-triangular matrices $a_i(t), b_i(t) \in \mathfrak{sl}_n(\mathbb{C})$ such that

$$(1) \quad \rho_t(l_i) = A_i(t) \exp(a_i(t)) A_i(t)^{-1} \quad \text{and} \quad \rho_t(m_i) = A_i(t) \exp(b_i(t)) A_i(t)^{-1}.$$

Our main result states:

Theorem 1 *Assume that $A_i(t)$, $a_i(t)$ and $b_i(t)$ as in (1) are differentiable. Then the volume is differentiable and*

$$\frac{d}{dt}\operatorname{vol}(M, \rho_t) = \sum_{i=1}^{k} \operatorname{tr}(\Re(b_i)\Im(\dot{a}_i) - \Re(a_i)\Im(\dot{b}_i)).$$

For $n = 2$ this formula is precisely Hodgson's formula in the Dehn filling space, and for $n = 3$ it is equivalent to the variation on the space of decorated triangulations obtained by Bergeron, Falbel and Guilloux [1; 15] for $n = 3$, and by Dimofte, Gabella and Goncharov for general n [10]. See Section 6.3 below.

The hypothesis on differentiability of $A_i(t)$, $a_i(t)$, and $b_i(t)$ in Theorem 1 is necessary, as the volume form is not differentiable on $\operatorname{Hom}(\pi_1(M), \mathrm{SL}_n(\mathbb{C}))$; see Lemma 40 below. Notice that the volume formulas of [23; 1; 10] are defined in spaces of decorated triangulations; these are not open subsets of $\mathrm{Hom}(\pi_1(M), \mathrm{SL}_n(\mathbb{C}))/\mathrm{SL}_n(\mathbb{C})$ but rather branched coverings of it. A decoration yields a choice of Borel subgroup containing the representation of the peripheral subgroup, thus a differential path of decorated triangulations implies differentiability of the terms in (1). In the appropriate context, the choice of Borel subgroups amounts to work in the so-called augmented variety of representations; see Dubois and Garoufalidis [11].

Our argument is a generalization of Reznikov's proof of the rigidity of the volume for closed manifolds [24]. At the heart of Reznikov's argument is the fact that the volume of a representation ρ can be seen as a characteristic class of the horizontal foliation on the total space of the flat principal bundle on M induced by ρ. This characteristic class comes from a cohomology class of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, ie it is induced by a class in $\mathrm{H}^3(\mathfrak{g})$. The study of the variation of this characteristic class then relies on results by Fuks [12]; he shows in particular that the variation of volume itself can be interpreted as a characteristic class of a foliation and this class stems from a cohomology class in $\mathrm{H}^2(\mathfrak{g}; \mathfrak{g}^\vee)$, where $\mathfrak{g}^\vee$ is the dual Lie algebra, viewed as a $\mathfrak{g}$–module. But since $\mathfrak{g}$ is semisimple, this cohomology group is trivial, as follows from a classical result of Cartier [9] — see Corollary 42 — hence the volume for M compact is locally constant. We aim to follow the same outline in the nonclosed case, which technically amounts to extending the homological tools used by Reznikov to a relative setting. Next we explain the plan of this work.

Firstly, in Section 2 we develop the homological tools needed for our construction: we give a definition of cohomology groups of an object relative to a family of subobjects. As it is difficult to find a single place in the literature where all the relative versions of the maps we need are explained, we start by defining in a unified way the relative cohomology constructions we will use; this is inspired by the work of Bieri and Eckmann [2] on relative cohomology of groups, but with a stronger emphasis on the pair object–subobject. The relative cohomology groups are devised in such a way that, by definition, if G is an object and $\{A\}$ is a family of subobjects, then the cohomology of G relative to $\{A\}$ fits into a long exact sequence

$$\cdots \to \mathrm{H}^n(G; \{A\}) \to \mathrm{H}^n(G) \to \prod \mathrm{H}^n(A) \to \mathrm{H}^{n+1}(G; \{A\}) \to \cdots .$$

We also discuss the relations between our definitions and previously existing notions of relative cohomology groups.

Secondly, in Section 3 we use the relative cohomological tools of the previous section to give relative versions of the constructions of Fuks [12] on characteristic classes of foliations and variations of those. This gives the conceptual framework in which we can state and prove our formula. Up to this point we work in a general context so as to pave the way for future applications.

In the compact case the volume of a representation $\rho\colon \pi_1(M) \to \mathrm{SL}_n(\mathbb{C})$ is defined as a pullback of a universal hyperbolic volume class in the continuous cohomology group $\mathrm{H}^3_c(\mathrm{SL}_n(\mathbb{C}))$; since the peripheral subgroups of a noncompact finite-volume hyperbolic manifold are all abelian, the cohomology group where we want to look for a universal relative volume class is $\mathrm{H}^3_c(\mathrm{SL}_n(\mathbb{C}); \{B\})$, the continuous cohomology groups of $\mathrm{SL}_n(\mathbb{C})$ relative to the family $\{B\}$ consisting of its Borel subgroups. This program for constructing the volume is carried out and explained in Section 4, where we also show that the definition through relative cohomology corresponds to the common definitions in the literature, for instance the one given in [7] via the use of the transfer map in continuous-bounded cohomology. The key point for our construction is the crucial fact that continuous-bounded cohomology of an amenable group is trivial, hence we have a canonical isomorphism $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_n(\mathbb{C}); \{B\}) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_n(\mathbb{C}))$ that allows to interpret the classical universal hyperbolic volume cohomology class as a relative cohomology class.

The study of the variation of the volume requires us then to find explicit cocycle representatives for the relative volume cohomology class. This is the object of Section 5. The main ingredient in this part of our work is the fact, underlying the van Est isomorphism connecting the continuous cohomology of a real connected Lie group G with maximal compact subgroup K and the cohomology of its Lie algebra, that $\Omega^*_{\mathrm{dR}}(G/K)^G$, the equivariant de Rham complex of the symmetric space G/K, computes the continuous cohomology of G; our cocycle will then appear as a bounded differential 3–form on G/K with a specific choice of trivialization on each Borel subgroup. Here we also show how to express the volume and its variation as a characteristic class on the total space of the flat bundle induced by a representation.

Finally, in Section 6 we collect our efforts and prove our variation formula and give some consequences.

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2 Relative cohomology

Our approach to define relative cohomology relies on the following three crucial points:

(1) The existence of functorial cochain complexes that compute the cohomology groups we want to relativize.

(2) The fact that given a family of objects $(A_i)_{i\in I}$ and coefficients V_i and functorial cochain complexes $C^*(A_i;V_i)$, the product chain complex $\prod_{i\in I} C^*(A_i;V_i)$ has as n–cohomology the product of cohomologies $\prod_{i\in I} \mathrm{H}^n(A_i;V_i)$.

(3) The fact that the cone of a cochain map between chain complexes is functorial in the homotopy category of complexes of $\mathbb{R}$–vector spaces.

2.1 The cone construction

Consider two cochain complexes of $\mathbb{R}$–vector spaces, ie differentials increase degree by one, and a chain map $f\colon K^* \to L^*$. By definition $\mathrm{Cone}(f)^*$, the cone of f, is the cochain complex given by

$$\mathrm{Cone}(f)^n = L^{n-1} \oplus K^n \quad \text{and} \quad d_{\mathrm{Cone}(f)} = \begin{pmatrix} -d_L & f \\ 0 & d_K \end{pmatrix},$$

where $d_{\mathrm{Cone}(f)}$ acts on column vectors.

One checks that, as expected in any reasonable definition of a *relative* cocycle, an element $\binom{l}{k} \in L^{n-1} \oplus K^n$ is an n–cocycle if and only if k is a cocycle in K^n and $d_L(l) = f^n(k)$. For such a pair we will call k the *absolute* part and l the *relative* part.

This construction is functorial in the following sense. If we have a commutative square of maps of chain complexes

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ A & \xrightarrow[g]{} & B \end{array}$$

then we have an induced chain map $\mathrm{Cone}(r,s)\colon \mathrm{Cone}(f)^* \to \mathrm{Cone}(g)^*$, given by

$$\mathrm{Cone}(r,s) = \begin{pmatrix} s & 0 \\ 0 & r \end{pmatrix}.$$

The main use of $\mathrm{Cone}(f)^*$ is that its homology interpolates between that of L and that of K; indeed, by construction there is a short exact sequence of complexes

$$0 \to L[-1] \to \mathrm{Cone}(f) \to K \to 0,$$

where $L[-1]$ is the shifted complex $L[-1]^n = L^{n-1}, d_{L[-1]} = -d_L$. This sequence splits in each degree and by standard techniques gives rise to a long exact sequence in cohomology

$$\cdots \to \mathrm{H}^{n-1}(L) \to \mathrm{H}^n(\mathrm{Cone}(f)) \to \mathrm{H}^n(K) \xrightarrow{\delta} \mathrm{H}^n(L) \to \cdots .$$

One checks directly by unwinding the definitions that the connecting homomorphism δ: $\mathrm{H}^*(K) \to \mathrm{H}^*(L)$ coincides with $\mathrm{H}^*(f)$. As expected, if we are given a morphism (r, s) between maps of cochain complexes, then we will have an induced commuting ladder in cohomology

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathrm{H}^{n-1}(L) & \longrightarrow & \mathrm{H}^n(\mathrm{Cone}(f)) & \longrightarrow & \mathrm{H}^n(K) & \xrightarrow{\delta} & \mathrm{H}^n(L) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \mathrm{H}^{n-1}(s)} & & \downarrow{\scriptstyle \mathrm{H}^n(\mathrm{Cone}(r,s))} & & \downarrow{\scriptstyle \mathrm{H}^n(r)} & & \downarrow{\scriptstyle \mathrm{H}^n(s)} & \\
\cdots \longrightarrow & \mathrm{H}^{n-1}(B) & \longrightarrow & \mathrm{H}^n(\mathrm{Cone}(g)) & \longrightarrow & \mathrm{H}^n(A) & \xrightarrow{\delta} & \mathrm{H}^n(B) & \longrightarrow \cdots
\end{array}$$

2.2 Relative cohomology

Definition 2 Let H^* be our cohomology theory, possibly with coefficients (eg discrete group cohomology). If the cohomology theory admits coefficients, we assume that the functorial cochain complexes computing the cohomology with coefficients are functorial in both variables.

Let G be an object (Lie algebra, Lie group, manifold etc) and $(A_i)_{i\in I}$ a family of subobjects, possibly with repetitions. If the theory admits coefficients, we consider also a coefficient V for the object G, coefficients W_i for each object A_i and maps between coefficients compatible with the inclusions $A_i \hookrightarrow G$, so that we have an induced map $C^*(G; V) \to C^*(A_i; W_i)$ for each $i \in I$.

Then we define the relative cohomology of G with coefficients in V with respect to $\{A_i\}$ and $\{W_i\}$ and we denote by $\mathrm{H}^*(G, \{A_i\}; V, \{W_i\})$ the cohomology of the cone of the canonical map $C^*(G, V) \to \prod_{i\in I} C^*(A_i, W_i)$.

As usual, if both coefficients V and the W_i are the ground field $\mathbb{R}$, then we simply write $\mathrm{H}^*(G, \{A_i\})$ for the relative cohomology group. Concretely, a relative n–cocycle in $C^n(G, \{A_i\}; V, \{W_i\})$ is a pair $(c, \{a_i\}_{i\in I})$, where c is an ordinary n–cocycle for G with coefficients in V which is a coboundary (ie trivial) on each subobject A_i when the coefficients are restricted to W_i, together with a specific trivialization a_i on each subobject A_i.

The following properties of the relative cohomology groups are immediate from the functoriality of the cochain complexes $C^*(G;V)$ and $C^*(A_i;W_i)$ and that of the cone construction:

Proposition 3 (1) *The relative cohomology groups* $\mathrm{H}^*(G,\{A_i\};V,\{W_i\})$ *are functorial in both pairs* $(G,(A_i)_{i\in I})$ *and* $(V,\{W_i\})$.

(2) *The relative cohomology groups fit into a long exact sequence*

$$\cdots \to \prod_{i\in I} \mathrm{H}^{n-1}(A_i;W_i) \xrightarrow{\delta} \mathrm{H}^n(G,\{A_i\};V,\{W_i\}) \to \mathrm{H}^n(G;V) \to \prod_{i\in I} \mathrm{H}^n(A_i,W_i) \to \cdots .$$

(3) *If* $J \subset I$ *is a subset of the indexing family for the subobjects* A_i, *then we have an induced natural transformation in relative cohomology*

$$\mathrm{H}^*(G,\{A_i\}_{i\in I};V,\{W_i\}_{i\in I}) \to \mathrm{H}^*(G,\{A_i\}_{i\in J};V,\{W_i\}_{i\in J}).$$

2.3 Examples

The different objects and cohomologies we have in mind are:

(1) Continuous or smooth cohomology of a Lie group Here G is a Lie group, for our purposes $\mathrm{SL}_n(\mathbb{C})$, and each A_i is a closed subgroup, for us a Borel subgroup of G. We take for $C^*(G;\mathbb{R})$ the continuous or smooth *normalized* bar resolution $C_c^*(G;\mathbb{R})$ or $C_\infty^*(G;\mathbb{R})$ [4, Chapter IX]. In this case, by a classical result of Hochschild and Mostow, the canonical inclusion map $C_\infty^*(G;\mathbb{R}) \to C_c^*(G;\mathbb{R})$ is a quasi-isomorphism.

Another functorial way to compute the cohomology underlies the van Est theorem (see Section 5.2). Given a semisimple Lie group with associated symmetric space G/K, the subcomplex of the de Rham complex of G–invariant differential forms computes the continuous cohomology of G, that is, $\mathrm{H}^*(\Omega_{\mathrm{dR}}(G/K)^K) \simeq \mathrm{H}_c^*(G;\mathbb{R})$. This resolution is functorial in the category of pairs semisimple Lie group–maximal compact subgroup.

(2) Continuous-bounded cohomology Since the only case we are interested in is for Lie groups with trivial coefficients, we may use the cochain complex of continuous-bounded functions $C_{\mathrm{cb}}^*(G;\mathbb{R})$.

(3) Cohomology of discrete groups Here G is a discrete group, A_i is a family of subgroups and $C^*(G;\mathbb{R})$ stands for the usual bar resolution. This can of course be viewed as a particular case of continuous cohomology.

(4) De Rham cohomology of manifolds In this case G is a smooth manifold, A_i is a family of smooth submanifolds, typically the connected components of the boundary, and $C^*(G;\mathbb{R}) = \Omega^*_{\mathrm{dR}}(G)$ is the de Rham complex of smooth differential forms on M.

(5) Lie group cohomology [25, Chapter 7] Here G and A are respectively a real Lie algebra and a family of Lie subalgebras. For $C^*(G;\mathbb{R})$ we use the so-called *standard resolution* of Chevalley and Eilenberg. It is only in this case that we will need to consider nontrivial coefficients.

For some of these theories one can find in the literature other relative cohomology theories, and the one presented here coincides with these except for one important case: Lie algebra cohomology. Let us review briefly this.

Relative cohomology for discrete groups This has been defined by Bieri and Eckmann [2]. Their construction defines the relative cohomology $\mathrm{H}^*(G,\{A_i\})$ as the absolute cohomology of the group G with coefficients in a specific nontrivial G–module. As they explain in [2, page 282], their construction is isomorphic to ours, up to a sign in the long exact sequence of the pair $(G;\{A_i\}_{i\in I})$. Fortunately for us this gives in our case a reformulation of their geometric interpretation of relative group cohomology without sign problems:

Theorem 4 [2, Theorem 1.3] *Let the pair (X,Y) be an Eilenberg–Mac Lane pair $K(G,\{A_i\};1)$. Then the relative cohomology sequences of X modulo Y and of G modulo $\{A_i\}_{i\in I}$ are isomorphic. More precisely, one has a commuting ladder with vertical isomorphisms*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathrm{H}^n(G,\{A_i\}) & \longrightarrow & \mathrm{H}^n_{\mathrm{dR}}(G) & \longrightarrow & \Pi_i\, \mathrm{H}^n(A_i) & \longrightarrow & \mathrm{H}^{n+1}(G,\{A_i\}) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \mathrm{H}^n(X,Y) & \longrightarrow & \mathrm{H}^n(X) & \longrightarrow & \mathrm{H}^n(Y) & \longrightarrow & \mathrm{H}^{n+1}(X,Y) & \longrightarrow \cdots \end{array}$$

where the cohomology in the bottom is the usual long exact sequence in singular cohomology.

We will be particularly interested in the case where $X = M$ is a manifold whose interior is hyperbolic of finite volume and $Y = \partial M$ is its boundary, in which case Y is a finite disjoint union of tori, ie copies of $K(\mathbb{Z}^2,1)$.

De Rham cohomology Given a manifold M and smooth submanifold A, a usual way to define relative cohomology groups $\mathrm{H}^*_{\mathrm{dR}}(M,A)$ is to consider the kernel $\Omega^*_{\mathrm{dR}}(M,A)$

of the canonical map $\Omega^*_{\mathrm{dR}}(M) \to \Omega^*_{\mathrm{dR}}(A)$ between de Rham complexes induced by the inclusion. This gives rise to a levelwise split short exact sequence of complexes

$$0 \to \Omega^*_{\mathrm{dR}}(M, A) \to \Omega^*_{\mathrm{dR}}(M) \to \Omega^*_{\mathrm{dR}}(A) \to 0,$$

where the surjectivity uses the tubular neighborhood to extend any differential form on A to a form on M. As these are chain complexes of $\mathbb{R}$–vector spaces, the usual argument based on the snake lemma gives rise to a long exact sequence

$$\cdots \to \mathrm{H}^n_{\mathrm{dR}}(M, A) \to \mathrm{H}^n_{\mathrm{dR}}(M) \to \mathrm{H}^n_{\mathrm{dR}}(A) \to \mathrm{H}^{n+1}_{\mathrm{dR}}(M, A) \to \cdots .$$

The relative de Rham cohomology can also be defined using the cone construction; cf [5]. There is a canonical map

$$\Omega^*_{\mathrm{dR}}(M, A) \to \mathrm{Cone}(\Omega^*_{\mathrm{dR}}(M) \to \Omega^*_{\mathrm{dR}}(A)),$$

which maps a differential form ω of degree n that is zero on A to

$$(0, \omega) \in \Omega^{n-1}_{\mathrm{dR}}(A) \oplus \Omega^n_{\mathrm{dR}}(M) = \Omega_{\mathrm{dR}}(M, \{A\}).$$

It is immediate to check that this is a map of chain complexes, compatible with the restriction map and the connecting homomorphisms, and hence gives a commutative ladder

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathrm{H}^n_{\mathrm{dR}}(M, A) & \longrightarrow & \mathrm{H}^n_{\mathrm{dR}}(M) & \longrightarrow & \mathrm{H}^n_{\mathrm{dR}}(A) & \longrightarrow & \mathrm{H}^{n+1}_{\mathrm{dR}}(M, A) & \longrightarrow \cdots \\
& \downarrow & & \| & & \| & & \downarrow & \\
\cdots \longrightarrow & \mathrm{H}^n_{\mathrm{dR}}(M, \{A\}) & \longrightarrow & \mathrm{H}^n_{\mathrm{dR}}(M) & \longrightarrow & \mathrm{H}^n_{\mathrm{dR}}(A) & \longrightarrow & \mathrm{H}^{n+1}_{\mathrm{dR}}(M, \{A\}) & \longrightarrow \cdots
\end{array}$$

where in the bottom we denote by $\mathrm{H}^*_{\mathrm{dR}}(M, \{A\})$ "our" relative cohomology groups. Applying the five lemma we conclude that this canonical map is a quasi-isomorphism.

Lie algebra cohomology This is an important case where our relative groups do not coincide with the usual ones. Given a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$, Chevalley and Eilenberg [25] define the relative Lie algebra cohomology via the (now known as) relative Chevalley–Eilenberg complex

$$\mathrm{H}^*(\mathfrak{g}, \mathfrak{h}) = \mathrm{H}^*\big(\mathrm{Hom}_{\mathfrak{h}\text{–mod}}\big(\textstyle\bigwedge^* \mathfrak{g}/\mathfrak{h}, \mathbb{R}\big)\big).$$

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras of a Lie group G and a closed subgroup H, the relationship between the cohomologies $\mathrm{H}^*(\mathfrak{g})$, $\mathrm{H}^*(\mathfrak{h})$ and $\mathrm{H}^*(\mathfrak{g}, \mathfrak{h})$ parallels the relationship between the cohomologies of the spaces in the fibration sequence

$$H \to G \to G/H.$$

In particular, there is a Hochschild–Serre spectral sequence relating these cohomologies, in contrast with the long exact sequence in our case.

To distinguish our definition and to avoid an unnecessary clash with standard notation, even in the case we have a family of subobjects consisting of a single element, we will denote our relative version as $\mathrm{H}^*(\mathfrak{g},\{\mathfrak{h}\})$ and keep the usual notation $\mathrm{H}^*(\mathfrak{g},\mathfrak{h})$ for the cohomology of the complex $\mathrm{Hom}_{\mathfrak{h}\text{–mod}}\big(\bigwedge^*\mathfrak{g}/\mathfrak{h},\mathbb{R}\big)$.

2.4 The case of continuous-bounded cohomology

Continuous-bounded cohomology produces cohomology groups that are naturally Banach spaces, and this is an important feature of the theory. As we will have to consider uncountable families of subgroups, there is no hope that we could give some metric to our relative cohomology groups $\mathrm{H}^*_{\mathrm{cb}}(G,\{A_i\})$ in any way compatible with the metrics on the absolute cohomology groups, for the space $\prod_{i\in I}\mathrm{H}^*_{\mathrm{cb}}(A_i)$ will not usually be metrizable. However, we are only interested in these relative cohomology groups as tools interpolating between the cohomology of a group and the cohomologies of subgroups in a given family and we will not enter the subtler point of the metric.

Notation 5 As a general rule we will write cohomology with coefficients separated by semicolons, eg $\mathrm{H}^3(\mathrm{SL}_n(\mathbb{C});\mathbb{C}^n)$, unless we are dealing with the ground field $\mathbb{R}$ as coefficients, in which case we will usually omit them, and write $\mathrm{H}^3_c(\mathrm{SL}_n(\mathbb{C}))$ instead of $\mathrm{H}^3_c(\mathrm{SL}_n(\mathbb{C});\mathbb{R})$. For cochain complexes we will however keep the reference to the coefficients in all cases.

3 Relative characteristic and variation maps

In this section we explain how one can "relativize" Fuks's construction [12, Chapter 3, Paragraph 1] of a characteristic class of a foliation, and more generally of a manifold with $\mathfrak{g}$–structure, and the way he handles their variation.

3.1 Relative characteristic classes

Given a smooth principal G–bundle E and a flat connection $\nabla\in\Omega^1_{\mathrm{dR}}(E,\mathfrak{g})$ on E with values in a Lie algebra $\mathfrak{g}$, the absolute characteristic class map is given on the cochain level by

$$\mathrm{Char}_\nabla\colon C^*(\mathfrak{g};\mathbb{R})\to\Omega^*_{\mathrm{dR}}(E),\quad \alpha\mapsto(X_1,\dots,X_n)\rightsquigarrow\alpha(\nabla X_1,\dots,\nabla X_n).$$

This construction is contravariantly functorial in both variables $\mathfrak{g}$ and E; flatness of ∇ implies this is in fact a chain map, ie it commutes with the differentials.

Fix a family of Lie subalgebras $\{\mathfrak{b}\}$ of $\mathfrak{g}$ and a family of smooth closed submanifolds $\{A\} \subset E$, for instance the family of connected components of the boundary of E. Assume that the flat connection ∇ on A restricts to a flat connection with values in $\mathfrak{b}^A$, an element in the chosen family of Lie subalgebras. Then, by functoriality of the map Char_∇, we have for each $A \subset E$ a commutative diagram

$$\begin{array}{ccc} C^*(\mathfrak{g};\mathbb{R}) & \xrightarrow{\mathrm{Char}_\nabla} & \Omega^*_{\mathrm{dR}}(E) \\ \downarrow & & \downarrow \\ C^*(\mathfrak{b}_A;\mathbb{R}) & \xrightarrow{\mathrm{Char}_{\nabla|_A}} & \Omega^*_{\mathrm{dR}}(A) \end{array}$$

By functoriality of the cone construction we get a *relative* characteristic class cochain map

$$C^*(\mathfrak{g},\{\mathfrak{b}\}) \xrightarrow{\mathrm{Char}_{\nabla,\{\nabla|_A\}}} \Omega^*_{\mathrm{dR}}(E,\{A\}).$$

3.2 Variation of characteristic classes

Let us again briefly recall Fuks's framework in the absolute case [12, Chapter 3, pages 241–246]. We consider a 1–parameter family of flat connections ∇_t on a manifold E with values in a *fixed* Lie algebra $\mathfrak{g}$. Given a Lie algebra cohomology class $[\omega] \in \mathrm{H}^*(\mathfrak{g};\mathbb{R})$, we want to understand the variation of the cohomology class $\mathrm{Char}_{\nabla_t}(\omega) \in \mathrm{H}^*_{\mathrm{dR}}(E)$ as t varies.

Assume that the connection ∇_t depends differentiably on t, then its derivative at $t=0$, denoted by $\dot{\nabla}_0$, is again a connection with values in $\mathfrak{g}$. The characteristic class $\mathrm{Char}_{\nabla_t}(\alpha)$ then also depends differentiably on the parameter t and, assuming α is of degree n, its derivative at $t=0$ is directly computed to be the de Rham cohomology class of the form obtained by the Leibniz derivative rule

$$\mathrm{Var}_{\nabla_t}(\alpha)\colon (X_1,\dots,X_n) \mapsto \sum_{i=1}^{n} \alpha(\nabla_0 X_1,\dots,\nabla_0 X_{i-1},\dot{\nabla}_0 X_i,\nabla_0 X_{i+1},\dots \nabla_0 X_n).$$

From this we get a cochain map

$$\mathrm{Var}_{\nabla_t}\colon C^*(\mathfrak{g};\mathbb{R}) \to \Omega^*_{\mathrm{dR}}(E), \quad \alpha \mapsto \mathrm{Var}_{\nabla_t}(\alpha).$$

The family of connections ∇_t can also be seen as a single connection on E but with values in the algebra of currents

$$\widetilde{\mathfrak{g}} = \mathcal{C}^1(\mathbb{R}, \mathfrak{g}).$$

The associated characteristic class map

$$\mathrm{Char}_{\nabla_t} \colon \mathrm{H}^*(\widetilde{\mathfrak{g}}) \to \mathrm{H}^*_{\mathrm{dR}}(E)$$

factors the variation map in a very nice way. Consider the following two cochain maps:

(1) The map

$$\begin{aligned} \mathrm{var} \colon C^n(\mathfrak{g}; \mathbb{R}) &\to C^{n-1}(\mathfrak{g}; \mathfrak{g}^\vee), \\ \alpha &\mapsto (g_1, \dots, g_{n-1}) \rightsquigarrow \big(h \mapsto \alpha(g_1, \dots, g_{n-1}, h)\big), \end{aligned}$$

where $\mathfrak{g}^\vee$ denotes the dual vector space $\mathrm{Hom}(\mathfrak{g}, \mathbb{R})$; this is canonically a left $\mathfrak{g}$–module by setting $(g\phi)(h) = -\phi([g, h])$.

(2) The Fuks map [12, Chapter 3 page 244] is a cochain map, in fact a split monomorphism

$$\mathrm{F} \colon C^{n-1}(\mathfrak{g}; \mathfrak{g}^\vee) \to C^n(\widetilde{\mathfrak{g}}),$$

that sends a cochain $\alpha \in C^{n-1}(\mathfrak{g}; \mathfrak{g}^\vee)$ to the cochain

$$(\phi_1, \dots, \phi_n) \mapsto \sum_{i=1}^{n} (-1)^{n-i} [\alpha(\phi_1(0), \dots, \phi_{i-1}(0), \widehat{\phi_i(0)}, \phi_{i+1}(0), \dots, \phi_n(0))](\dot{\phi}_i(0)).$$

By direct computation one shows that the following diagram of cochain maps commutes:

$$\begin{array}{ccccc} C^n(\mathfrak{g}; \mathbb{R}) & \xrightarrow{\mathrm{var}} & C^{n-1}(\mathfrak{g}; \mathfrak{g}^\vee) & \xrightarrow{\mathrm{F}} & C^n(\widetilde{\mathfrak{g}}; \mathbb{R}) \\ & & & & \downarrow{\scriptstyle \mathrm{Char}_{\nabla_t}} \\ & & \xrightarrow{\quad \mathrm{Var}_{\nabla_t} \quad} & & \Omega^n_{\mathrm{dR}}(E) \end{array}$$

Let us now relativize the construction above. We have fixed a relative cocycle $(\omega, \{\beta\}) \in C^n(\mathfrak{g}, \{\mathfrak{b}\})$, a 1–parameter family of connections ∇_t on a manifold E and a family of closed submanifolds $\{A\}$ in E. Assume that for each value of t the restriction of ∇_t to A takes values in the Lie subalgebra $\mathfrak{b}_t^A \in \{\mathfrak{b}\}$. Then, for each value of the parameter t, we have as data a relative de Rham cocycle with absolute part

$$(X_1, \dots X_n) \mapsto \omega(\nabla_t X_1, \dots, \nabla_t X_n),$$

and relative part given on each submanifold A by

$$(Y_1, \dots, Y_{n-1}) \mapsto \beta_t^A(\nabla_t Y_1, \dots, \nabla_t Y_{n-1}).$$

The instant variation of this class is given by computing the usual limit. For the absolute part ω we get the same result as in the nonrelative case,

$$\mathrm{Var}_{\nabla_t}(\omega).$$

For the relative part we have to compute the limit as $t \to 0$ of

$$\Delta(\beta, t) = \frac{1}{t}(\beta_t^A(\nabla_t Y_1, \dots, \nabla_t Y_{n-1}) - \beta_0^A(\nabla_0 Y_1, \dots, \nabla_0 Y_{n-1})). \tag{2}$$

Here we are stuck, as the usual tricks that lead to a Leibniz-type derivation formula in this case do not work: the problem lies in the fact that the class β_t^A also depends on the time t. To overcome this difficulty we will impose the following coherence condition on the connection with respect to the family of Lie subalgebras $\mathfrak{b}_t^A$:

Definition 6 Assume $\mathfrak{g}$ is the Lie algebra of a connected Lie group G. Consider on a manifold E with a family of submanifolds A a one-parameter family of connections ∇_t. Assume that for each A, the restriction $\nabla_0|_A$ lies in the Lie subalgebra $\mathfrak{b}^A$. We say that the connection varies *coherently* along the submanifolds A with respect to the family $\{\mathfrak{b}\}$ if and only if the following holds:

There is a subgroup $H \subset G$ such that for each subspace A there exists a differentiable 1–parameter family of elements h_t of H, with $h_0 = \mathrm{Id}$, such that for each value of the parameter t the connection $\widetilde{\nabla}_t^A = \mathrm{Ad}_{h_t} \nabla_t|_A$ takes values in the Lie subalgebra at the origin $\mathfrak{b}^A$.

This condition will force us to restrict our treatment of the variation of a relative characteristic class in two ways:

1. Firstly, we will only consider classes whose global part is an H–invariant cocycle, where H is the group defined above.
2. Secondly, given a connection that varies coherently along the submanifolds A with respect to the family $\{\mathfrak{b}\}$, we will ask for the relative part of the cocycle to satisfy the coherence condition

$$\forall (Y_1, \dots, Y_{n-1}) \quad \beta_t^A(\nabla_t Y_1, \dots, \nabla_t Y_{n-1}) = \beta_0^A(\widetilde{\nabla}_t^A Y_1, \dots, \widetilde{\nabla}_t^A Y_{n-1}).$$

Definition 7 We say that the characteristic class *varies coherently* with the connection if the previous two conditions are satisfied.

Under this assumption we can pursue the computation in (2) above:

$$\begin{aligned}\lim_{t\to 0}\Delta(\beta,t) &= \lim_{t\to 0}\frac{1}{t}(\beta_t^A(\nabla_t Y_1,\dots,\nabla_t Y_{n-1})-\beta_0^A(\nabla_0 Y_1,\dots,\nabla_0 Y_{n-1}))\\ &= \lim_{t\to 0}\frac{1}{t}(\beta_0^A(\widetilde{\nabla}_t^A Y_1,\dots,\widetilde{\nabla}_t^A Y_{n-1})-\beta_0^A(\widetilde{\nabla}_0^A Y_1,\dots,\widetilde{\nabla}_0^A Y_{n-1}))\\ &= \sum_{j=1}^{n-1}\beta_0^A(\widetilde{\nabla}_0^A Y_1,\dots,\widetilde{\nabla}_0^A Y_{j-1},\dot{\widetilde{\nabla}}_0^A Y_j,\widetilde{\nabla}_0^A Y_{j+1},\dots\widetilde{\nabla}_t^A Y_{n-1})\\ &= \mathrm{Var}_{\widetilde{\nabla}_t}(\beta_0^A).\end{aligned}$$

Observe that, since ω is H–invariant, for any vector fields $(X_1,\dots,X_n)$ on E we have

$$\omega(\nabla_t X_1,\dots,\nabla_t X_n)=\omega(\widetilde{\nabla}_t X_1,\dots,\widetilde{\nabla}_t X_n),$$

and in particular, as differential forms,

$$d\ \mathrm{Var}_{\widetilde{\nabla}_t}(\beta_t^A)=j_A^*(\mathrm{Var}_{\nabla_t}(\omega))=j_A^*(\mathrm{Var}_{\widetilde{\nabla}_t}(\omega)),$$

where $j_A\colon A\hookrightarrow E$ is the inclusion. Hence, the data

$$(\mathrm{Var}_{\nabla_t}(\omega),\{\mathrm{Var}_{\widetilde{\nabla}_t}(\beta^A)\})=\mathrm{Var}_{\nabla_t,\{\widetilde{\nabla}\}}(\omega,\{\beta\})$$

is indeed a relative differential form on $(E,\{A\})$.

We will now relativize the maps var and F involved in Fuks factorization of the map Var_{∇_t}.

Lemma 8 *Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra and $H\subset G$ a subgroup. Then the cochain complexes $C^*(\mathfrak{g};\mathbb{R})$, $C^*(\mathfrak{g};\mathfrak{g}^\vee)$ and $C^*(\widetilde{\mathfrak{g}};\mathbb{R})$ are cochain complexes of H–modules, where the action of H is induced by its adjoint action on $\mathfrak{g}$. Moreover, the maps* var *and* F *are compatible with the action of H.*

Proof This is an immediate consequence of the fact that the above chain complexes are functorial in the variable $\mathfrak{g}$ and the adjoint action is by automorphisms of Lie algebras. □

Notation 9 Denote by $C_H^*(\mathfrak{g};\mathbb{R})$, $C_H^*(\mathfrak{g};\mathfrak{g}^\vee)$ and $C_H^*(\widetilde{\mathfrak{g}})$ the subspace of fixed points under the action of H of the vector spaces $C^*(\mathfrak{g})$, $C^*(\mathfrak{g};\mathfrak{g}^\vee)$ and $C^*(\widetilde{\mathfrak{g}};\mathbb{R})$, respectively.

Definition 10 Denote by

$$C_H^*(\mathfrak{g};\{\mathfrak{b}\}) = \mathrm{Cone}\Big(C_H^*(\mathfrak{g};\mathbb{R}) \to \prod C^*(\mathfrak{b};\mathbb{R})\Big),$$
$$C_H^*(\mathfrak{g},\{\mathfrak{b}\};\mathfrak{g}^\vee,\{\mathfrak{b}^\vee\}) = \mathrm{Cone}\Big(C_H^*(\mathfrak{g};\mathfrak{g}^\vee) \to \prod C^*(\mathfrak{b};\mathfrak{b}^\vee)\Big)$$

the cones taken along the maps induced by the inclusions $\mathfrak{b} \to \mathfrak{g}$.

Notice that in the above definition we do not ask a priori for the chains on the Lie algebras $\mathfrak{b}$ to be invariant in any way.

Proposition 11 *Via the cone construction the chain maps* var *and* F *induce relative chain maps*

$$\mathrm{var}\colon C_H^*(\mathfrak{g},\{\mathfrak{b}\}) \to C_H^{*-1}(\mathfrak{g},\{\mathfrak{b}\};\mathfrak{g}^\vee,\{\mathfrak{b}^\vee\})$$

and

$$\mathrm{F}\colon C_H^{*-1}(\mathfrak{g},\{\mathfrak{b}\};\mathfrak{g}^\vee,\{\mathfrak{b}^\vee\}) \to C_H^*(\tilde{\mathfrak{g}},\{\tilde{\mathfrak{b}}\}).$$

Proof This follows from the functoriality of the cone construction and the commutativity of the two squares

$$\begin{array}{ccc} C_H^*(\mathfrak{g};\mathbb{R}) & \longrightarrow & C_H^{*-1}(\mathfrak{g};\mathfrak{g}^\vee) \\ \downarrow & & \downarrow \\ C^*(\mathfrak{b};\mathbb{R}) & \longrightarrow & C^{*-1}(\mathfrak{b};\mathfrak{b}^\vee) \end{array} \quad \text{and} \quad \begin{array}{ccc} C_H^{*-1}(\mathfrak{g};\mathfrak{g}^\vee) & \longrightarrow & C_H^*(\tilde{\mathfrak{g}};\mathbb{R}) \\ \downarrow & & \downarrow \\ C^{*-1}(\mathfrak{b};\mathfrak{b}^\vee) & \longrightarrow & C^*(\tilde{\mathfrak{b}};\mathbb{R}) \end{array}$$

for any Lie subalgebra $\mathfrak{b} \subset \mathfrak{g}$. □

Proposition 12 *With the notation of Definition 6, the maps* $\mathrm{Char}_{\nabla_t}\colon C_H^*(\tilde{\mathfrak{g}};\mathbb{R}) \to \Omega^*_{\mathrm{dR}}(E)$ *and* $\mathrm{Char}_{\tilde{\nabla}_t^A}\colon C_H^*(\widetilde{\mathfrak{b}^A};\mathbb{R}) \to \Omega^*_{\mathrm{dR}}(A)$ *induce a map in relative cohomology*

$$\mathrm{Char}_{\nabla_t,\{\tilde{\nabla}_t^A\}}\colon C_H^*(\tilde{\mathfrak{g}},\{\tilde{\mathfrak{b}}^A\}) \to \Omega^*_{\mathrm{dR}}(E,\{A\})$$

which is compatible with the restrictions and inflation maps, where

$$C_H^*(\tilde{\mathfrak{g}},\{\tilde{\mathfrak{b}}^A\}) = \mathrm{Cone}\Big(C_H^*(\tilde{\mathfrak{g}};\mathbb{R}) \hookrightarrow C^*(\tilde{\mathfrak{g}};\mathbb{R}) \xrightarrow{\text{rest.}} \prod_A C^*(\tilde{\mathfrak{b}}^A;\mathbb{R})\Big).$$

Proof By functoriality of the cone construction, it is enough to show that for each A the following diagram, where the vertical maps are the restriction maps, commutes:

$$\begin{array}{ccc} C_H^*(\tilde{\mathfrak{g}};\mathbb{R}) & \xrightarrow{\mathrm{Char}_{\nabla_t}} & \Omega^*_{\mathrm{dR}}(E) \\ \downarrow & & \downarrow \\ C^*(\tilde{\mathfrak{b}}^A;\mathbb{R}) & \xrightarrow[\mathrm{Char}_{\tilde{\nabla}_t^A}]{} & \Omega^*_{\mathrm{dR}}(A) \end{array}$$

which is achieved by a trivial diagram chasing. □

Summing up the results in this section we have shown that:

Theorem 13 *Let $(\omega,\{\beta\})\in C_H^*(\mathfrak{g},\{\mathfrak{b}\})$ vary coherently along a connection ∇_t on E. The variation chain map* Var: $C^*(\mathfrak{g};\mathbb{R})\to\Omega^*_{\mathrm{dR}}(E)$ *induces via the cone construction a relative variation chain map*

$$\mathrm{Var}_{\nabla_t,\{\tilde{\nabla}_t\}}\colon C_H^*(\mathfrak{g},\{\mathfrak{b}\})\to\Omega^*_{\mathrm{dR}}(E,\{A\})$$

whose induced map in cohomology computes the derivative at $t=0$ of the cohomology classes $\mathrm{Char}_{\nabla_t}(\omega,\{\beta\})\in \mathrm{H}^*_{\mathrm{dR}}(E,\{A\})$.

We also have a Fuks-type factorization of the variation map:

Theorem 14 *The relative variation map factors as*

$$\begin{array}{ccccc} C_H^*(\mathfrak{g},\{\mathfrak{b}\}) & \xrightarrow{\mathrm{var}} & C_H^{*-1}(\mathfrak{g},\{\mathfrak{b}\};\mathfrak{g}^\vee,\{\mathfrak{b}^\vee\}) & \xrightarrow{\mathrm{F}} & C_H^*(\tilde{\mathfrak{g}},\{\tilde{\mathfrak{b}}\}) \\ & \searrow^{\mathrm{Var}} & & & \downarrow{\scriptstyle \mathrm{Char}_{\nabla_t,\{\tilde{\nabla}_t\}}} \\ & & & & \Omega^*_{\mathrm{dR}}(E,\{A\}) \end{array}$$

and this factorization is compatible with the restriction and connecting homomorphisms.

4 The volume of a representation

4.1 Setup and notation

Let us start with some definitions and notation involving the structure of the groups $\mathrm{SL}_n(\mathbb{C})$. We will regard these groups as real Lie groups. Recall then that for each $n\geq 2$, the group $\mathrm{SU}(n)\subset\mathrm{SL}_n(\mathbb{C})$ is a maximal compact subgroup. Let $D_n\subset\mathrm{SL}_n(\mathbb{C})$

denote the subgroup of diagonal matrices; then $D_n \cap \mathrm{SU}(n) = T$ is a maximal real torus isomorphic to $(\mathbb{S}^1)^{n-1}$. By definition a Borel subgroup of $\mathrm{SL}_n(\mathbb{C})$ is a maximal solvable subgroup; the Borel subgroups are also the stabilizers of complete flags in $\mathbb{C}^n$. The Gram–Schmidt process then shows that the subgroup $\mathrm{SU}(n)$ acts transitively on complete flags, and hence that all Borel subgroups are pairwise conjugated in $\mathrm{SL}_n(\mathbb{C})$ by elements in $\mathrm{SU}(n)$.

We fix as our model Borel subgroup $B \subset \mathrm{SL}_n(\mathbb{C})$ the subgroup of upper-triangular matrices. In particular, the transitive action by conjugation of $\mathrm{SU}(n)$ on the set of all Borel subgroups provides each of these with a specified choice of a maximal compact subgroup. Denote by $U_n \subset B$ the subgroup of unipotent matrices; this is a normal subgroup and gives B the structure of a semidirect product $B = U_n \rtimes D_n$.

Again by the Gram–Schmidt process, the inclusion $B \hookrightarrow \mathrm{SL}_n(\mathbb{C})$ induces a homeomorphism of homogeneous manifolds $B/T \simeq \mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$. For $n = 2$, this symmetric space is hyperbolic space. For normalization purposes, let us recall that

$$\begin{pmatrix} e^{(l+i\theta)/2} & 0 \\ 0 & e^{-(l+i\theta)/2} \end{pmatrix} = \exp \begin{pmatrix} \frac{1}{2}(l+i\theta) & 0 \\ 0 & -\frac{1}{2}(l+i\theta) \end{pmatrix} \tag{3}$$

acts on $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2) \simeq \mathcal{H}^3$ as the composition of a loxodromic isometry of translation length l composed with a rotation of angle θ along the same axis; see [19, Section 12.1].

Let π denote the fundamental group of M, the compact three-manifold with nonempty boundary, whose interior is hyperbolic of finite volume. The $k \geq 1$ boundary components are tori, $\partial M = T_1^2 \sqcup T_2^2 \cdots \sqcup T_k^2$. For each boundary component of M fix a path from the basepoint of M to the boundary; this gives us a definite choice of a peripheral system $P_1, \dots, P_k$ in π, where $P_i \simeq \pi_1(T_i^2)$.

Let's now fix a representation $\rho\colon \pi \to \mathrm{SL}_n(\mathbb{C})$ for some $n \geq 2$. Since each peripheral subgroup P_i is abelian and the Borel subgroups of $\mathrm{SL}_n(\mathbb{C})$ are maximal solvable subgroups, the image of the restriction of ρ to P_i lies in a Borel subgroup. Fix for each peripheral subgroup P_i such a Borel subgroup B_i.

4.2 Some known results in bounded and continuous cohomology

The continuous cohomology of the groups $\mathrm{SL}_n(\mathbb{C})$ has a rather simple structure:

Proposition 15 [3] *Let* $n \geq 1$ *be an integer; then* $\mathrm{H}_c^*(\mathrm{SL}_n(\mathbb{C}))$ *is an exterior algebra,*

$$\mathrm{H}_c^*(\mathrm{SL}_n(\mathbb{C})) = \bigwedge \langle x_{n,j} \mid 1 \leq j \leq n-1 \rangle,$$

over so-called Borel classes $x_{n,j}$ of degree $2j+1$. These classes are stable: if $j_n\colon \mathrm{SL}_n(\mathbb{C}) \to \mathrm{SL}_{n+1}(\mathbb{C})$ denotes the inclusion in the upper-left corner, then for $j \le n$, $j_n^*(x_{n+1,j}) = x_{n,j}$.

Remark 16 For $\mathrm{SL}_2(\mathbb{C})$ the Borel class x_1 is also known as the hyperbolic volume class and we denote it by $\mathrm{vol}_{\mathcal{H}^3}$. It is completely determined by stability and the requirement that on $\mathrm{SL}_2(\mathbb{C})$ it is represented by the cocycle

$$(A, B, C, D) \mapsto \int_{(A*,B*,C*,D*)} d\,\mathrm{vol}_{\mathcal{H}^3},$$

where $(A*, B*, C*, D*)$ denotes the hyperbolic oriented tetrahedron with geodesic faces spanned by the four images of the basepoint $* \in \mathcal{H}^3$ by A, B, C and D, respectively and $d\,\mathrm{vol}_{\mathcal{H}^3}$ is the hyperbolic volume form. Notice that this cocycle is bounded by the maximal volume of an ideal tetrahedron. See for instance [14, Section 3] for a thorough discussion of volumes of hyperbolic manifolds and continuous cocycles.

Compared to the relatively simple structure of continuous cohomology, the continuous-bounded cohomology of $\mathrm{SL}_n(\mathbb{C})$ is considerably more complicated and largely unknown; see Monod [21]. Nevertheless, fitting our purposes well we have the following:

Proposition 17 [22] *The canonical comparison map $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_n(\mathbb{C})) \to \mathrm{H}^3_c(\mathrm{SL}_n(\mathbb{C}))$ is surjective.*

For continuous-bounded we have also the following crucial feature, which applies in particular to the Borel and unipotent subgroups of $\mathrm{SL}_n(\mathbb{C})$:

Proposition 18 *Let G denote an amenable Lie group, eg abelian or solvable; then $\mathrm{H}^*_{\mathrm{cb}}(G) = 0$ for $* > 0$.*

We are now ready to define the volume of our representation $\rho\colon \pi \to \mathrm{SL}_n(\mathbb{C})$. The long exact sequence in continuous cohomology for the pair $(\mathrm{SL}_n(\mathbb{C}), \{B_i\})$, where $\{B_i\}$ stands for the family of Borel subgroups we have fixed, together with Proposition 18 gives immediately:

Proposition 19 *For $* \ge 2$, the map induced by forgetting the relative part induces an isomorphism*

$$\mathrm{H}^*_{\mathrm{cb}}(\mathrm{SL}_n(\mathbb{C}), \{B_i\}) \xrightarrow{\simeq} \mathrm{H}^*_{\mathrm{cb}}(\mathrm{SL}_n(\mathbb{C})).$$

Remark 20 Under the hypothesis of Proposition 19 above, and since the groups B_i are the Borel subgroups of $\mathrm{SL}_n(\mathbb{C})$, by Corollary 3 in [26] the long exact sequence in continuous cohomology of Proposition 3 splits into short exact sequences

$$0 \to \prod_i \mathrm{H}_c^{*-1}(B_i) \to \mathrm{H}_c^*(\mathrm{SL}_n(\mathbb{C}), \{B_i\}) \to \mathrm{H}_c^*(\mathrm{SL}_n(\mathbb{C})) \to 0.$$

Moreover, since all Borel subgroups are conjugated, all the groups $\mathrm{H}_c^*(B_i)$ are isomorphic to one another. However, since $\mathrm{H}_c^*(B_i) \neq 0$, for instance for $* = 1$, we do not have in general an isomorphism as for continuous *bounded* cohomology.

Comparing continuous cohomology and bounded continuous cohomology for the pair $(\mathrm{SL}_n(\mathbb{C}), \{B_i\})$ gives us a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_{\mathrm{cb}}^3(\mathrm{SL}_n(\mathbb{C}), \{B_i\}) & \xrightarrow{\ \sim\ } & \mathrm{H}_{\mathrm{cb}}^3(\mathrm{SL}_n(\mathbb{C})) \\ \downarrow & & \downarrow \\ \mathrm{H}_c^3(\mathrm{SL}_n(\mathbb{C}), \{B_i\}) & \longrightarrow & \mathrm{H}_c^3(\mathrm{SL}_n(\mathbb{C})) \end{array}$$

This shows that the continuous-bounded cohomology class $\mathrm{vol}_{\mathcal{H}}$ has a canonical representative as a continuous bounded relative class $\mathrm{vol}_{\mathcal{H},\partial} \in \mathrm{H}_c^3(\mathrm{SL}_n(\mathbb{C}), \{B_i\})$.

The representation ρ induces a map of pairs $\rho\colon (\pi, \{P_i\}) \to (\mathrm{SL}_n(\mathbb{C}), \{B_i\})$ by construction, hence by functoriality we have an induced map in continuous cohomology

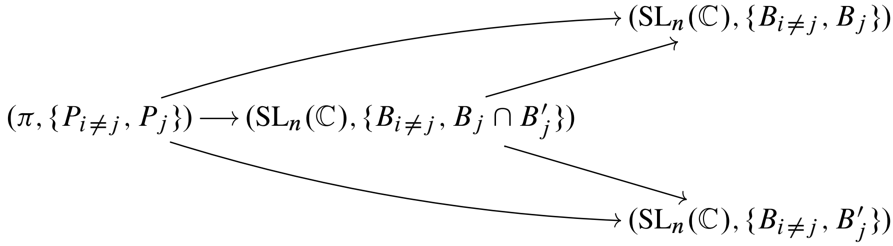
$$\mathrm{H}_c^3(\mathrm{SL}_n(\mathbb{C}), \{B_i\}) \xrightarrow{\rho^*} \mathrm{H}_c^3(\pi, \{P_i\}).$$

But, for discrete groups, continuous cohomology and ordinary group cohomology coincide, so we have a well-defined class, up to a possible ambiguity given by the choice of the Borel subgroups B_i,

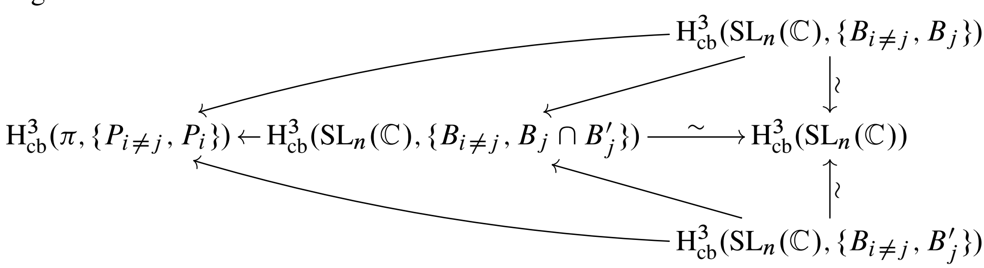
$$\rho^*(\mathrm{vol}_{\mathcal{H},\partial}) \in \mathrm{H}^3(\pi, \{P_i\}).$$

Proposition 21 *The class* $\rho^*(\mathrm{vol}_{\mathcal{H},\partial}) \in \mathrm{H}_c^3(\pi, \{P_i\})$ *is independent of the possible choice of a different family of Borel subgroups* $\{B_i\}$.

Proof Let us assume for clarity that we have two possible choices B_j and B'_j for the Borel subgroup that contains $\rho(P_j)$, and that we make a unique choice for the rest of the peripheral subgroups. We denote the two families of subgroups by $\{B_{i\neq j}, B_j\}$ and $\{B_{i\neq j}, B'_j\}$. Because Borel subgroups are closed in $\mathrm{SL}_n(\mathbb{C})$, their intersection, as $B_i \cap B'_i$, is also amenable. The restriction of ρ to the peripheral subgroup P_j factors in both cases through this intersection, so we have a commutative diagram of group homomorphisms



Together with the forgetful isomorphisms to the absolute cohomology of $\mathrm{SL}_n(\mathbb{C})$, and given the fact that the subgroups involved are all amenable, we have a commutative diagram



and this finishes the proof. □

Now M is a $K(\pi,1)$ and each boundary component is a $K(P_i,1)$ for the corresponding peripheral subgroup; in particular, $\mathrm{H}^3(\pi,\{P_i\}) \simeq \mathrm{H}^3(M;\partial M) \simeq \mathbb{R}$ by Theorem 4, due to Bieri and Eckmann, and this leads to our compact definition of the volume of a representation (for a more precise statement see Definition 38):

Definition 22 Let $\rho\colon \pi \to \mathrm{SL}_n(\mathbb{C})$ be a representation of the fundamental group of a finite-volume hyperbolic 3–manifold. Then, evaluating on our fixed fundamental class $[M,\partial M] \in \mathrm{H}_3(M,\partial M)$, we set

$$\mathrm{Vol}(\rho) = \langle \rho^*(\mathrm{vol}_{\mathcal{H},\partial}), [M,\partial M]\rangle.$$

In [6], Bucher, Burger and Iozzi prove that the volume of a representation $\pi \to \mathrm{SL}_n(\mathbb{C})$ is maximal at the composition of the irreducible representation $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$ with a lift of the holonomy. Their definition, as ours, relies on continuous-bounded cohomology and are clearly equivalent: their transfer argument is replaced here by an isomorphism through a relative cohomology group. The passage through continuous cohomology seems for the moment rather useless; it will however be crucial in our next section, the study of the variation of the volume.

5 Variation of the volume class

We follow Reznikov's idea [24] to prove rigidity of the volume in the compact case. We will first show that the volume class can be viewed as a characteristic class on the total space of the flat bundle defined by the representation, then find explicit relative cocycles representing $\mathrm{vol}_{\mathcal{H},\partial}$ and, finally, apply the machinery of Section 3.

Let us start with some more notation. In the previous section we defined a series of Lie subgroups of $\mathrm{SL}_n(\mathbb{C})$; we now pass to their Lie algebras, all viewed as real Lie algebras:

Lie group	Lie algebra	Description as subgroup
$\mathrm{SL}_n(\mathbb{C})$	$\mathfrak{sl}_n$	
$\mathrm{SU}(n)$	$\mathfrak{su}_n$	fixed maximal compact subgroup
B	$\mathfrak{b}_n$	fixed Borel subgroup of upper triangular matrices
D_n	$\mathfrak{h}_n + i\,\mathfrak{h}_n$	subgroup of diagonal matrices in B
$T = D_n \cap \mathrm{SU}(n)$	$\mathfrak{h}_n$	maximal torus in $\mathrm{SU}(n)$ (and in B and $\mathrm{SL}_n(\mathbb{C})$)
U_n	$\mathfrak{ut}_n$	subgroup of unipotent elements in B.

For explicit formulas, we will need a concrete basis for the real Lie algebra $\mathfrak{su}_n$. Recall that $\mathfrak{su}_n = \{X \in M_n(\mathbb{C}) \mid X + {}^t\overline{X} = 0 \text{ and } \mathrm{tr}(X) = 0\}$.

There is a standard $\mathbb{R}$–basis of $\mathfrak{su}_2$, orthogonal with respect to the Killing form,

$$h = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}, \quad e = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}.$$

From this we can construct an analogous basis for $\mathfrak{su}_n$; we only give here the nonzero entries of the matrices.

(1) For an integer $1 \le s \le n-1$ let h_s denote the matrix with a coefficient $\frac{i}{2}$ in diagonal position s and a coefficient $-\frac{i}{2}$ in diagonal position n. It will be convenient to write $h_{st} = h_s - h_t$.

(2) For any pair of integers $1 \le s < t \le n$ let e_{st} have coefficient in row s and column t equal to $\frac{1}{2}$ and coefficient in row t and column s equal to $-\frac{1}{2}$.

(3) For any pair of integers $1 \le s < t \le n$ let f_{st} denote the matrix which has coefficient in row s and column t equal to $\frac{i}{2}$ and coefficient in row t and column s equal to $\frac{i}{2}$.

Notice that the matrices h_s generate the Lie subalgebra $\mathfrak{h}$, the Lie algebra of the real torus T. The dual basis elements will be denoted by $h_s^\vee$, $e_{st}^\vee$ and $h_{st}^\vee$. With these conventions, for $n=2$, $h=h_1$, $e=e_{12}$ and $f=f_{12}$.

Analogously, for $\mathfrak{b}$, the Lie algebra of upper-triangular matrices with zero trace, we have a basis made of the matrices h_s and ih_s for $1\le s\le n-1$, and for $1\le k<l\le n$, the matrices ur_{kl} (upper-real) which are equal to 1 in row k and column l and $\mathrm{ui}_{kl}=i\,\mathrm{ur}_{kl}$ (upper-imaginary matrices). We have $\mathrm{ur}_{kl}=e_{kl}-if_{kl}$ and $\mathrm{ui}_{kl}=ie_{kl}+f_{kl}$.

The following result provides us with the right cochain complex in which to find our cocycle representatives; beware that the relative cohomology of Lie algebras in the statement is not the one we defined in Section 2, but the classical one as defined for instance in Weibel [25, Chapter 7].

Proposition 23 (van Est isomorphism for trivial coefficients [16]) *Let G be a connected real Lie group. Denote by $\mathfrak{g}$ its Lie algebra and $\mathfrak{k}$ the Lie algebra of a maximal compact subgroup $K\subset G$. Then, for all m, there is a canonical isomorphism $\mathrm{H}_c^m(G;\mathbb{R})\simeq \mathrm{H}^m(\mathfrak{g},\mathfrak{k};\mathbb{R})$. More precisely, the de Rham cochain complex of left-invariant differential forms*

$$0\to\Omega^0_{\mathrm{dR}}(G/K)^G\to\cdots\to\Omega^n_{\mathrm{dR}}(G/K)^G\to\cdots$$

computes both cohomologies.

Functoriality of the cone construction allows us to extend van Est isomorphism to relative cohomology as follows. Fix a connected Lie group G and a family of connected closed subgroups $\{B_i\}$. Pick for each index i a maximal compact subgroup $K_i\subset B_i$ and fix a maximal compact subgroup $K\subset G$. Then, by maximality, for each index i there is an element $g_i\in G$ such that $K_i\subset g_iKg_i^{-1}$. Then the composite

$$j_{g_i}\colon B_i/K_i\to G/g_iKg_i^{-1}\xrightarrow{c_{g_i}} G/K,$$

where the second map is induced by conjugation by g_i, induces a cochain map

$$\Omega^*_{\mathrm{dR}}(G/K)^G\to\Omega^*_{\mathrm{dR}}(B_i/K_i)^{B_i}$$

which in — lets say — continuous cohomology is the map induced by the inclusion $B_i\to G$. Indeed, it is clear for the first map using van Est isomorphism with the maximal subgroups $g_iKg_i^{-1}$ in G and K_i in B_i, and as for the second map, by construction it induces in cohomology the map that is induced by conjugation by g_i and this is well known to be the identity. Let us denote the first composite by $j_{g_i}\colon B_i/K_i\to G/K$.

Denote by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{b}_i$ and $\mathfrak{k}_i$ the Lie algebras of G, K, B_i and K_i, respectively. Then an immediate application of the five lemma and van Est isomorphism gives us:

Corollary 24 (relative van Est isomorphism) *With the above notation and conventions, the cone on the map*

$$\Omega^G_{\mathrm{dR}}(G/K) \xrightarrow{\Pi j^*_{g_i}} \Pi\Omega(B_i/K_i)^{B_i}$$

computes both the relative continuous cohomology groups $\mathrm{H}^*_c(G,\{B_i\};\mathbb{R})$ *and the unaesthetic relative Lie cohomology groups* $\mathrm{H}^*(\mathfrak{g},\mathfrak{k},\{\mathfrak{b}_i,\mathfrak{k}_i\};\mathbb{R})$. *In particular, both these relative cohomology groups are canonically isomorphic.*

Recall that the volume class comes from a bounded cohomology class, so its de Rham representative will be rather special and can be explicitly detected thanks to the following result of Burger and Iozzi [8, Proposition 3.1]:

Proposition 25 [8] *Let G be a connected semisimple Lie group with finite center, let K be a maximal compact subgroup, let G/K be the associated symmetric space and let $L \subset G$ be any closed subgroup. Then there exists a map*

$$\delta^*_{\infty,L}\colon \mathrm{H}^*_{\mathrm{cb}}(L;\mathbb{R}) \to \mathrm{H}^*(\Omega_{\mathrm{dR},\infty}(G/K)^L)$$

such that the diagram

$$\begin{array}{ccccc} \mathrm{H}^*_{\mathrm{cb}}(L;\mathbb{R}) & \xrightarrow{c^*_L} & \mathrm{H}^*_c(L;\mathbb{R}) & \xleftarrow{\sim} & \mathrm{H}^*(\Omega_{\mathrm{dR}}(G/K)^L) \\ & {\scriptstyle \delta_{\infty,L}}\searrow & & \nearrow{\scriptstyle i_{\infty,L}} & \\ & & \mathrm{H}^*(\Omega_{\mathrm{dR},\infty}(G/K)^L) & & \end{array}$$

commutes, where $\Omega_{\mathrm{dR},\infty}(G/K)$ is the de Rham complex of bounded differential forms with bounded differential and $i_{\infty,L}$ is the map induced in cohomology by the inclusion of complexes $\Omega_{\mathrm{dR},\infty}(G/K) \hookrightarrow \Omega_{\mathrm{dR}}(G/K)$.

5.1 A relative cocycle representing $\mathrm{vol}_{\mathcal{H},\partial}$

We will apply the relative van Est isomorphism in the particular case where $G = \mathrm{SL}_n(\mathbb{C})$, $K = \mathrm{SU}(n)$ and $\{B\}$ is the family of all Borel subgroups in cohomological degree 3. Here the situation is simpler, as, for any Borel subgroup, $B \cap \mathrm{SU}(n)$ is a maximal torus and in our case this is also a maximal compact subgroup of B, so the "conjugation" part of the statement can be avoided and we simply use as maximal compact subgroup of B the intersection $B \cap \mathrm{SU}(n)$.

In particular, to represent the class $\mathrm{vol}_{\mathcal{H},\partial}$, we look for a relative cocycle whose absolute part lies in $\Omega^3_{\mathrm{dR}}(\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n))^{\mathrm{SL}_n(\mathbb{C})}$ and whose relative part lies in the groups $\Omega^2_{\mathrm{dR}}(B/(B \cap \mathrm{SU}(n)))^B$.

We take now advantage of the fact that all pairs (B,T) where B is a Borel subgroup and T a maximal torus in B are conjugated in $\mathrm{SL}_n(\mathbb{C})$, so in fact we only need to determine the relative part for our standard Borel B of upper-triangular matrices; if β is a relative part for this particular subgroup and B' is another Borel, there exists an element $g \in \mathrm{SL}_n(\mathbb{C})$ that conjugates (B,T) and $(B', \mathrm{SU}(n) \cap B')$; then conjugation by g induces a homeomorphism $c_g\colon B/T \to B'/(B' \cap \mathrm{SU}(n))$, hence the relative part for B' is given by $c^*_{g^{-1}}(\beta)$.

5.1.1 The absolute part Let $K^{\mathbb{R}}_{\mathfrak{sl}_n}$ be the real Killing form of the real Lie algebra $\mathfrak{sl}_n$. With respect to this form we have an orthogonal decomposition $\mathfrak{sl}_n = \mathfrak{su}_n \oplus i\,\mathfrak{su}_n$. We let

$$\mathrm{pr}_{\mathfrak{su}_n}\colon \mathfrak{sl}_n \to \mathfrak{su}_n, \quad A \mapsto \tfrac{1}{2}(A - {}^t\overline{A}),$$
$$\mathrm{pr}_{i\,\mathfrak{su}_n}\colon \mathfrak{sl}_n \to i\,\mathfrak{su}_n, \quad A \mapsto \tfrac{1}{2}(A + {}^t\overline{A}),$$

be the canonical projections.

The behavior of these projections with respect to the Lie bracket is given by

$$\mathrm{pr}_{\mathfrak{su}}([a,b]) = [\mathrm{pr}_{\mathfrak{su}}a, \mathrm{pr}_{\mathfrak{su}}b] + [\mathrm{pr}_{i\,\mathfrak{su}}a, \mathrm{pr}_{i\,\mathfrak{su}}b], \tag{4}$$
$$\mathrm{pr}_{i\,\mathfrak{su}}([a,b]) = [\mathrm{pr}_{\mathfrak{su}}a, \mathrm{pr}_{i\,\mathfrak{su}}b] + [\mathrm{pr}_{i\,\mathfrak{su}}a, \mathrm{pr}_{\mathfrak{su}}b]. \tag{5}$$

The tangent space at the class of Id in the symmetric space $\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$ is canonically identified with $i\,\mathfrak{su}_n$, and the induced action of $\mathrm{SU}(n)$ on this tangent space is easily checked to be the adjoint action. Let us now consider the rescaling of the *complex* Killing form on $\mathfrak{sl}_n$, $A, B \rightsquigarrow \mathrm{tr}(AB)$. This gives rise to a complex-valued alternating 3–form, sometimes known as the (here rescaled) Cartan–Killing form, $\mathrm{CK}^{\mathbb{C}}_{\mathfrak{sl}_n}\colon (A,B,C) \mapsto \mathrm{tr}(A[B,C])$. It is folklore knowledge that "the hyperbolic volume is the imaginary part of this Cartan–Killing form" (see Yoshida [27] for a precise statement when $n=2$ or Reznikov [24]); let us turn this into a precise statement. We fix our attention in the following part of the de Rham complex:

$$\Omega^2_{\mathrm{dR}}(\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n))^{\mathrm{SL}_n(\mathbb{C})} \to \Omega^3_{\mathrm{dR}}(\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n))^{\mathrm{SL}_n(\mathbb{C})} \to \Omega^4_{\mathrm{dR}}(\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n))^{\mathrm{SL}_n(\mathbb{C})}.$$

Lemma 26 *The vector space* $\Omega^2_{\mathrm{dR}}(\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n))^{\mathrm{SL}_n(\mathbb{C})}$ *is trivial.*

Proof By transitivity of the action, an alternating 2–form on the homogeneous space $\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$ is completely determined by what happens at the class of the identity, ie by a unique element in $\big(\bigwedge^2(i\mathfrak{su}_n)^\vee\big)^{\mathrm{SU}(n)}$. As $\mathrm{SU}(n)$–modules, $i\mathfrak{su}^\vee$ and $\mathfrak{su}^\vee$ are isomorphic, and via the real Killing form on $\mathrm{SU}(n)$, a symmetric nondegenerate form, the Lie algebra $\mathfrak{su}$ and its dual are also isomorphic $\mathrm{SU}(n)$–modules. So to prove the statement it is enough to show that $\big(\bigwedge^2\mathfrak{su}_n\big)^{\mathrm{SU}(n)}=0$. Let $\phi\colon \mathfrak{su}(n)\wedge\mathfrak{su}(n)\to\mathbb{R}$ be a skew-symmetric invariant form. Invariance by the adjoint action of $\mathrm{SU}(n)$ is equivalent to

$$\phi([X,Y],Z)+\phi(Y,[X,Z])=0 \quad \text{for all } X,Y,Z\in\mathfrak{su}(n).$$

Combined with skew-symmetry of both ϕ and the Lie bracket, this equality yields

$$\phi([X,Y],Z)=\phi([X,Z],Y)=-\phi([Z,X],Y) \quad \text{for all } X,Y,Z\in\mathfrak{su}(n).$$

Namely, $\phi([X,Y],Z)$ changes the sign when the entries $X,Y,Z\in\mathfrak{su}(n)$ are cyclically permuted, therefore it vanishes. Then $\phi=0$ because $\mathfrak{su}(n)$ is simple. □

For a manifold X, denote by $Z^n_{\mathrm{dR}}(X)\subset\Omega^n_{\mathrm{dR}}(X)$ the subspace of closed forms.

Corollary 27 *The canonical quotient map*

$$Z^3_{\mathrm{dR}}(\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n))^{\mathrm{SL}_n(\mathbb{C})}\to \mathrm{H}^3_c(\mathrm{SL}_n(\mathbb{C}))\simeq\mathbb{R}$$

is an isomorphism.

Since $\mathrm{H}^3_c(\mathrm{SL}_n(\mathbb{C}))\simeq\mathbb{R}$ by Borel's computations, there is a unique closed form on $\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$ that represents the class $\mathrm{vol}_{\mathcal{H}}$. There is an obvious candidate for such a form, it is given on the tangent space at Id by

$$\textstyle\bigwedge^3 i\mathfrak{su}_n\to\mathbb{R},\quad (A,B,C)\mapsto 2i\,\mathrm{tr}(A[B,C])=-2\Im\,\mathrm{tr}(A[B,C]).$$

Then $\varpi_n\colon\mathfrak{sl}_n\to\mathbb{R}$ is the composition of the projection $\mathrm{pr}_{i\mathfrak{su}}\colon\mathfrak{sl}_n\to i\mathfrak{su}_n$ with this form, that is,

$$\varpi\colon \textstyle\bigwedge^3\mathfrak{sl}_n\to\mathbb{R},\quad (A,B,C)\mapsto 2i\,\mathrm{tr}(\mathrm{pr}_{i\mathfrak{su}}(A)[\mathrm{pr}_{i\mathfrak{su}}(B),\mathrm{pr}_{i\mathfrak{su}}(C)]).$$

That this form is alternating and invariant under the adjoint action of $\mathrm{SU}(n)$ is an immediate consequence of the fact that the Cartan–Killing form $(A,B,C)\mapsto\mathrm{tr}(A[B,C])=\mathrm{tr}(ABC-ACB)$ is alternating and $\mathrm{SU}(n)$–invariant, and that the adjoint action of

$\mathrm{SU}(n)$ respects the decomposition $\mathfrak{sl}_n = \mathfrak{su}_n \oplus i\,\mathfrak{su}_n$. Observe that by construction this form is compatible with the inclusions $\mathfrak{sl}_n \to \mathfrak{sl}_{n+1}$: if we denote the form defined by $\mathfrak{sl}_n$ by ϖ_n then $\varpi_{n+1}|_{\mathfrak{sl}_n} = \varpi_n$, in line with the stability result of Borel in degree 3. We only have to check that this is a cocycle when viewed as a classical relative cocycle in the Lie algebra cohomology of $\mathfrak{sl}_n/\mathfrak{su}_n = i\,\mathfrak{su}_n$ (ie gives rise to a closed form), that it is not trivial and fixes the normalization constant; this will done by comparing it with the hyperbolic volume form for $n = 2$.

Lemma 28 *The alternating 3–form $\varpi \in \mathrm{Hom}\big(\bigwedge^3 \mathfrak{sl}_n, \mathbb{R}\big)$ is a cocycle.*

Proof By definition of the differential in the Cartan–Chevalley complex — see Weibel [25, Chapter 7] — and since $[i\,\mathfrak{su}_n, i\,\mathfrak{su}_n] \subset \mathfrak{su}_n$, the differential in this cochain complex is in fact trivial, so any element in $\mathrm{Hom}\big(\bigwedge^3 i\,\mathfrak{su}_n; \mathbb{R}\big)$ is a cocycle. □

Lemma 29 *Via the canonical isomorphism $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2) \simeq \mathcal{H}^3$, the form ϖ_2 is mapped to the hyperbolic volume form $d\,\mathrm{vol}_{\mathcal{H}^3}$.*

Proof We use the half-space model $\mathcal{H}^3 = \{z + tj \mid z \in \mathbb{C},\ t \in \mathbb{R},\ t > 0\}$, so that the action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C}) \cong \mathbb{C} \cup \{\infty\}$ extends conformally by isometries. In particular, $\mathrm{SU}(2)$ is the stabilizer of the point j, and we use the natural map from $\mathfrak{sl}_2$ to the tangent space $T_j\mathcal{H}^3$ that maps $a \in \mathfrak{sl}_2$ to the vector $\frac{d}{dt}\exp(ta)j\,|_{t=0}$. From this construction, $\mathfrak{su}_2$ is mapped to zero and $i\,\mathfrak{su}_2$ is naturally identified to the tangent space to $\mathcal{H}^3$ at j. Thus, the form induced by the volume form is the result of composing a form on $i\,\mathfrak{su}_2$ with the projection $\mathfrak{sl}_2(\mathbb{C}) \to i\,\mathfrak{su}_2$. By $\mathrm{SU}(2)$–invariance, it suffices to check that its evaluation at an orthonormal basis is 1. The ordered basis

$$\left\{ \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \right\} \tag{6}$$

of $i\,\mathfrak{su}_2$ is mapped to $\{1, i, j\}$ via the isomorphism $i\,\mathfrak{su}_2 \cong T_j\mathcal{H}^3$, which is a positively oriented orthonormal basis, and ϖ evaluated at the basis (6) is 1. □

Remark 30 The cocycle has the precise form

$$\varpi = -\sum_{j<k} (ih_{jk})^\vee \wedge (ie_{jk})^\vee \wedge (if_{jk})^\vee.$$

Fixing a pair of indexes $1 \le j < k \le n$ fixes a Lie subalgebra in $\mathfrak{su}_n$ isomorphic to $\mathfrak{su}_2$. The restriction of ϖ to each of these $\frac{1}{2}n(n-1)$ copies of $\mathfrak{su}_2$ is exactly the corresponding hyperbolic volume form.

Remark 31 The imaginary part of the Cartan–Killing form, $(x,y,z)\mapsto \Im\,\mathrm{tr}([x,y]z)$ for all $x,y,z\in\mathfrak{sl}_n$, is cohomologous to $-2\varpi_n$, but it does not come from a bounded cocycle in $\mathrm{SL}_n\mathbb{C}$ (see [27, Lemma 3.1] for $n=2$).

5.1.2 The relative part We now turn to the relative part of our cocycle. For this we have to understand the restriction of the form $\varpi\in\Omega^3_{\mathrm{dR}}(\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n))$ along the canonical map $B/T_n\to\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$ induced by the inclusion of an arbitrary Borel subgroup B. As all Borel subgroups are conjugated in $\mathrm{SL}_n(\mathbb{C})$ by an element of $\mathrm{SU}(n)$, provided by the Gram–Schmidt process, and the form ϖ is $\mathrm{SU}(n)$–invariant, it is enough to treat the case of our fixed Borel B of upper-triangular matrices. As we will see, because we require our trivializations to come from a bounded class, there will be only one choice, and this uniqueness will then provide the coherence condition we need for computing the variation.

Lemma 32 *The vector space* $\Omega^1_{\mathrm{dR}}(B/T)^B$ *is generated by the closed* 1*–forms* $ih^\vee_s$*. In particular, the differential* $\Omega^1_{\mathrm{dR}}(B/T)^B\to\Omega^2_{\mathrm{dR}}(B/T)^B$ *is trivial and* $\mathrm{H}^1_c(B;\mathbb{R})=\mathbb{R}^{n-1}$*.*

Proof As before, by transitivity an element in $\Omega^1_{\mathrm{dR}}(B/T)^B$ is determined by its restriction to the tangent space to the identity, $\mathfrak{b}_n/\mathfrak{h}_n$, ie by a form on this tangent space invariant under the induced action by the torus T. The Borel Lie algebra $\mathfrak{b}_n$, the Lie algebra of the torus $\mathfrak{h}_n$ and the Lie algebra of strictly upper-triangular matrices $\mathfrak{ut}_n$ fit into a commutative diagram with exact row of T–modules,

$$\begin{array}{ccccccccc} & & & & \mathfrak{ut}_n & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & \mathfrak{h}_n & \longrightarrow & \mathfrak{b}_n & \longrightarrow & \mathfrak{b}_n/\mathfrak{h}_n & \longrightarrow & 0 \end{array}$$

We view a T–invariant form on $\mathfrak{b}_n/\mathfrak{h}_n$ as a T–invariant form $\psi\colon\mathfrak{b}_n\to\mathbb{R}$ which is trivial on $\mathfrak{h}_n$. The action of T is readily checked to be induced by the conjugation action of T on B, hence invariance is equivalent to

$$\forall t\in\mathfrak{h}_n\quad\forall b\in\mathfrak{b}_n\qquad\psi([t,b])=0.$$

But $[\mathfrak{h}_n,\mathfrak{b}_n]=\mathfrak{ut}_n$, hence ψ is in fact a form on $\mathfrak{b}_n/\mathfrak{ut}_n$. It is finally straightforward to check that indeed the $n-1$–forms $h^\vee_s$ are both closed and linearly independent. □

Lemma 33 *The space* $\Omega^2_{\mathrm{dR}}(B/T)^B$ *has a basis given by*

(1) *the* $\frac{1}{2}n(n-1)$ *forms* $\mathrm{ur}^\vee_{kl}\wedge\mathrm{ui}^\vee_{kl}$ *for all* $1\le k<l\le n$*;*

(2) *the* $\frac{1}{2}(n-1)(n-2)$ *closed forms* $ih^\vee_s\wedge ih^\vee_r$ *for all* $1\le s<r\le n-1$*.*

Proof Such a form, say ϕ, is exactly a T–invariant and alternating 2–form on $\mathfrak{b}_n/\mathfrak{h}_n$. As a T–module, $\mathfrak{b}_n/\mathfrak{h}_n = i\,\mathfrak{h}_n \oplus \mathfrak{ut}_n$, hence $\bigwedge^2 \mathfrak{b}_n/\mathfrak{h}_n = \bigwedge^2 i\,\mathfrak{h}_n \oplus i\,\mathfrak{h}_n \wedge \mathfrak{ut}_n \oplus \mathfrak{ut}_n \wedge \mathfrak{ut}_n$. Moreover, we have that $[\mathfrak{h}_n, i\,\mathfrak{h}_n] = 0$ and $[\mathfrak{h}_n, \mathfrak{ut}_n] = \mathfrak{ut}_n$. By derivation of the invariance condition,

$$\forall a \in \mathfrak{h}_n \ \ \forall X, Y \in \mathfrak{ut}_n \quad \phi([a, X], Y) + \phi(X, [a, Y]) = 0.$$

From this equation one gets immediately that all forms in $i\,\mathfrak{h}_n \wedge i\,\mathfrak{h}_n$ are invariant, and, by further close inspection, that ϕ on $i\,\mathfrak{h}_n \wedge \mathfrak{ut}_n$ is 0.

A direct and straightforward computation shows that on $\mathfrak{ut}_n \wedge \mathfrak{ut}_n$ the forms appearing in point (1) are the unique invariant 2–forms on this space.

Linear independence is immediate by checking on suitable elements of $\mathfrak{b}_n/\mathfrak{h}_n$. □

As a corollary, the trivialization we are looking for is a linear combination of the forms in Lemma 33. Let us find first a suitable candidate. Given matrices $x, y \in \mathfrak{b}$, write them as $x = x_d + x_u$ and $y = y_d + y_u$ with $x_d, y_d \in \mathfrak{h}_n + i\,\mathfrak{h}_n$ diagonal and $x_u, y_u \in \mathfrak{ut}_n$ unipotent (strictly upper-triangular). Define

$$\beta\colon \mathfrak{b}_n \times \mathfrak{b}_n \to \mathbb{R}, \quad (x, y) \mapsto \tfrac{1}{4}\Im\, \mathrm{tr}(x_u {}^t\overline{y}_u - {}^t\overline{x}_u y_u) = \tfrac{1}{4} i\, \mathrm{tr}({}^t\overline{x}_u y_u - x_u {}^t\overline{y}_u). \tag{7}$$

For $(a_{kl}), (b_{kl}) \in \mathfrak{b}_n$ (ie $a_{kl} = b_{kl} = 0$ for $k > l$), (7) is equivalent to

$$\beta((a_{kl}), (b_{kl})) = \frac{i}{4} \sum_{k<l} (\overline{a}_{kl} b_{kl} - a_{kl} \overline{b}_{kl}) = \frac{1}{2} \sum_{k<l} \Im(a_{kl} \overline{b}_{kl}),$$

so

$$\beta = \frac{1}{2} \sum_{k<l} \mathrm{ur}_{kl} \wedge \mathrm{ui}_{kl}.$$

In particular, in this formula coefficients in the diagonal do not occur. A straightforward computation yields:

Lemma 34 *The coboundary of β is the restriction of ϖ to the Borel subalgebra:*

$$\delta(\beta) = \varpi|_{\mathfrak{b}_n}.$$

Proposition 35 *The form β above is the unique* bounded *2–form $\beta \in \Omega^2_{\mathrm{dR}}(B/T)^B$ such that $d\beta = \varpi|_B$. It is characterized by the fact that it is the unique trivialization that is 0 on the intersection $B \cap B^-$, where B^- is the opposite Borel subgroup of lower-triangular matrices.*

Proof Since Lemma 33 gives a basis for $\Omega^2_{\mathrm{dR}}(B/T)^B$, any other invariant trivialization of ϖ restricted to $\mathfrak{b}_n$ differs from β by a term of the form

$$\sum_{s,r} \gamma_{sr} i h_s^\vee \wedge i h_r^\vee.$$

To show that the coefficients γ_{sr} are all 0, observe that fixing a pair of indexes s and r, the exponentials of the elements ih_s and ih_r give us a flat $\mathbb{R}^2 \subset B/T$. On this flat the volume form is trivial by direct inspection, and so are the forms $\mathrm{ur}_{kl}^\vee \wedge \mathrm{ui}_{kl}^\vee$ and $ih_p^\vee \wedge ih_q^\vee$ if $\{p,q\} \neq \{s,r\}$. So our invariant form on this flat restricts to the multiple $\gamma_{sr} i h_s^\vee \wedge i h_r^\vee$ of the euclidean volume form; this is bounded if and only if $\gamma_{sr} = 0$.

So the unique candidate for a bounded trivialization is β, and since we know that there has to be one bounded trivialization, this is it. □

As a form in $\Omega^2_{\mathrm{dR}}(B/T)^B$, β corresponds to the construction of Weinhard in [26, Corollary 2.4], by means of a Poincaré lemma with respect to an ideal point.

Summarizing, the class $\mathrm{vol}_{\mathcal{H}^3,\partial} \in \mathrm{H}^3_c(\mathrm{SL}_n(\mathbb{C}); \{B_i\})$ is represented in the relative de Rham complex $\Omega^*_{\mathrm{dR}}(\mathrm{SL}_n(\mathbb{C})/B)^{\mathrm{SL}_n(\mathbb{C})} \oplus \bigoplus_i \Omega^{*-1}_{\mathrm{dR}}(B_i/B_i \cap \mathrm{SU}(n))^{B_i}$ by a relative cocycle, where:

(1) The absolute part is given by the invariant 3–form

$$\varpi\colon \textstyle\bigwedge^3 \mathfrak{sl}_n \to \mathbb{R}, \quad (A,B,C) \mapsto -2i\ \mathrm{tr}(\mathrm{pr}_{i\mathfrak{su}} A[\mathrm{pr}_{i\mathfrak{su}} B, \mathrm{pr}_{i\mathfrak{su}} C]).$$

(2) The relative part is given on the copy $\Omega^2_{\mathrm{dR}}(B_i/B_i \cap \mathrm{SU}(n))^{B_i}$ determined by the Borel subgroup B_i, by choosing an arbitrary element $h_i \in \mathrm{SU}(n)$ such that $h_i^{-1} B h_i = B_i$, then extending by invariance the 2–form on $T_{\mathrm{Id}}(B_i/B_i \cap \mathrm{SU}(n))$ defined by $\beta_i = \mathrm{Ad}^*_H(\beta)$, where

$$\beta\colon \mathfrak{b}_n \times \mathfrak{b}_n \to \mathbb{R}, \quad (x,y) \mapsto \tfrac{1}{4} i\ \mathrm{tr}({}^t\overline{x}_u y_u - x_u {}^t\overline{y}_u).$$

Here $x_u, y_u \in \mathfrak{ut}_n$ are the respective unipotent parts of x and y.

By construction the data $(\varpi, \{\beta_i\})$ forms a relative 2–cocycle on $\mathfrak{sl}_n$ relative to the family of Borel Lie subalgebras $\{\mathfrak{b}_i\}$.

5.1.3 Volume and the Veronese embedding As an application let us show a formula relating the volume of a finite-volume hyperbolic 3–manifold and the volume of its defining representation composed with the unique irreducible rank n representation of $\mathrm{SL}_2(\mathbb{C})$ induced by the Veronese embedding. This formula is proved in [7, Proposition 21], with different techniques (see also [13, Theorem 1.15]).

Let $\sigma_n\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$ denote the n–dimensional irreducible representation. Namely, σ_n is the $(n-1)^{\mathrm{st}}$ symmetric product, induced by the Veronese embedding $\mathbb{CP}^1 \to \mathbb{CP}^{n-1}$.

Proposition 36 [7] *For $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$, $\mathrm{vol}(\sigma_n \circ \rho) = \binom{n+1}{3}\mathrm{vol}(\rho)$.*

Recall that given *any* family of Borel subgroups $\{B\}$, the map that forgets the relative part induces a natural isomorphism in continuous cohomology

$$\mathrm{H}^3_c(\mathrm{SL}_n(\mathbb{C}), \{B\}) \to \mathrm{H}^3_c(\mathrm{SL}_n(\mathbb{C})).$$

Therefore, to prove Proposition 36, by the van Est isomorphism we only need to understand the effect of the induced map $\sigma_n\colon \mathfrak{sl}_2 \to \mathfrak{sl}_n$ on the absolute part ϖ of the volume cocycle. Denote by ϖ_n this absolute part, seen as a cocycle on $\mathfrak{sl}_n$, to emphasize its dependence on the index n.

Lemma 37 *Let $\sigma_n\colon \mathfrak{sl}_2 \to \mathfrak{sl}_n$ denote the representation of Lie algebras induced by the irreducible representation that comes from the Veronese embedding. Then*

$$\sigma_n^*(\varpi_n) = \binom{n+1}{3}\varpi_2.$$

Proof The result is a consequence of the fact that $\sigma_n(i\,\mathfrak{su}_2) \subset i\,\mathfrak{su}(\mathfrak{n})$ and the equalities, for $a, b \in \mathfrak{sl}_2(\mathbb{C})$,

$$[\sigma_n(a), \sigma_n(b)] = \sigma_n([a,b]), \quad \mathrm{tr}(\sigma_n(a)\sigma_n(b)) = \binom{n+1}{3}\,\mathrm{tr}(ab).$$

The first equality is just a property of Lie algebra representations. For the second one, compute the image of a basis of $\mathfrak{sl}_2(\mathbb{C})$:

$$\sigma_n\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix} n-1 & & & 0\\ & n-3 & & \\ & & \ddots & \\ 0 & & & 1-n\end{pmatrix},$$

$$\sigma_n\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix} 0 & n-1 & & & 0\\ & 0 & n-2 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1\\ 0 & & & & 0\end{pmatrix} \quad \text{and} \quad \sigma_n\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = \begin{pmatrix} 0 & & & & 0\\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & n-1 & 0\end{pmatrix}.$$

By bilinearity, we just need to check the formula on the basis, which is straightforward from the sums

$$(n-1)^2+(n-3)^2+\cdots+(1-n)^2 = 2\binom{n+1}{3},$$
$$(n-1)1+(n-2)2+\cdots+1(n-1) = \binom{n+1}{3}. \qquad \square$$

5.2 The volume as a characteristic class

In this section we recall briefly how a differentiable deformation of a representation translates into a differentiable deformation of a connection on the associated flat principal bundle. We will also recall how integration on M of pullbacks of invariant cocycles on $\mathrm{SL}_n(\mathbb{C})$ by using a developing map gives the interpretation of the volume form as a characteristic class.

Recall that $\pi=\pi_1(M)$ is the fundamental group of a compact manifold M whose interior caries a hyperbolic metric of finite volume. In particular, the boundary ∂M, if not empty, is a disjoint union of finitely many tori $T_1 \sqcup \cdots \sqcup T_k$. Since π is a discrete group, associated to our fixed representation $\rho\colon \pi \to \mathrm{SL}_n(\mathbb{C})$ there is a flat principal fibration

$$\mathrm{SL}_n(\mathbb{C}) \hookrightarrow E_\rho \twoheadrightarrow M.$$

The total space E_ρ is constructed as

$$E_\rho = \widetilde{M} \times \mathrm{SL}_n(\mathbb{C})/\pi,$$

where $\widetilde{M}$ is the universal covering space of M, with $\gamma\cdot(x,g)=(\gamma x,\rho(\gamma)g)$ for $\gamma\in\pi$, $x\in\widetilde{M}$ and $g\in\mathrm{SL}_n(\mathbb{C})$. The natural flat connection

$$\nabla\colon TE_\rho \to \mathfrak{sl}_n$$

is induced by the composition of the projection to the second factor of $T(\widetilde{M}\times\mathrm{SL}_n(\mathbb{C})) \cong T\widetilde{M}\times T\mathrm{SL}_n(\mathbb{C})$ and the identification $T_g\mathrm{SL}_n(\mathbb{C}) \cong \mathfrak{sl}_n$ via l_{g*}, where l_g denotes left multiplication by $g\in\mathrm{SL}_n(\mathbb{C})$.

Notice that E_ρ is a noncompact manifold with boundary ∂E_ρ that fibers over ∂M. Recall that in Section 4.1 we have fixed a path from our basepoint in M to a basepoint on each boundary component. Fix a basepoint on each covering space of each boundary component ∂M_i; this induces commutative diagrams by sending the basepoint to the chosen path to ∂M_i,

$$\begin{array}{ccc} \widetilde{\partial M_i} & \hookrightarrow & \widetilde{M} \\ \downarrow & & \downarrow \\ \partial M_i & \hookrightarrow & M \end{array}$$

Since the restriction of ρ to each parabolic subgroup $P_i \simeq \pi_1(\partial M_i)$ takes values in a Borel subgroup B_i, over the component ∂M_i, this restricted fibration

$$\mathrm{SL}_n(\mathbb{C}) \hookrightarrow \partial E_\rho \twoheadrightarrow \partial M_i$$

is obtained by extending the fiber from the flat fibration

$$B_i \hookrightarrow \partial\widetilde{M_i} \times_\rho B_i \twoheadrightarrow \partial M_i$$

along the inclusion $B_i \hookrightarrow \mathrm{SL}_n(\mathbb{C})$. In particular, the flat connection ∇ restricted to a component ∂M_i takes values in the Lie algebra $\mathfrak{b}_i$ of the chosen Borel B_i.

As M is aspherical, $\dim M \leq 3$ and $\mathrm{SL}_n(\mathbb{C})$ is 2–connected, by Whitehead's theorem there exists a ρ–equivariant map that sends the basepoint in $\widetilde{M}$ to Id,

$$D\colon \widetilde{M} \to \mathrm{SL}_n(\mathbb{C}).$$

By precomposing this map with our fixed inclusions of the universal covering spaces of the boundary components, we get for each of those a compatible developing map

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{D} & \mathrm{SL}_n(\mathbb{C}) \\ \uparrow & & \uparrow \\ \widetilde{\partial M_i} & \xrightarrow[D_i]{} & B_i \end{array}$$

On the one hand the developing map induces a trivialization Θ_ρ of the flat bundle or, equivalently, a section s_ρ to the fibration map:

$$\begin{array}{ccc} (x,g) & \longmapsto & (x, D(x)g) \\ \widetilde{M} \times \mathrm{SL}_n(\mathbb{C}) & \longrightarrow & \widetilde{M} \times \mathrm{SL}_n(\mathbb{C}) \\ \downarrow & & \downarrow \\ M \times \mathrm{SL}_n(\mathbb{C}) & \xrightarrow{\Theta_\rho} & E_\rho \end{array} \qquad \begin{array}{ccc} x & \longmapsto & (x, D(x)) \\ \widetilde{M} & \longrightarrow & \widetilde{M} \times \mathrm{SL}_n(\mathbb{C}) \\ \downarrow & & \downarrow \\ M & \xrightarrow{s_\rho} & E_\rho \end{array}$$

Both maps are of course related:

$$s_\rho = \Theta_\rho \circ s,$$

where $s\colon M \to M \times \mathrm{SL}_n(\mathbb{C})$ is the constant section of the trivial bundle, given by fixing $\mathrm{Id} \in \mathrm{SL}_n(\mathbb{C})$ as second coordinate. The composition of the section with the flat connection

$$\nabla \circ (s_\rho)_*\colon TM \to \mathfrak{sl}_n$$

is used to evaluate characteristic classes of $\mathfrak{sl}_n$.

The trivialization Θ_ρ is used to pull back the connection on E_ρ to the trivial bundle:

$$\nabla_\rho \stackrel{\text{def}}{=} \nabla \circ (\Theta_\rho)_*\colon T(M \times \mathrm{SL}_n\mathbb{C}) \to \mathfrak{sl}_n.$$

In this way, when we deform ρ, we deform ∇_ρ on the trivial bundle, because

$$\nabla_\rho \circ s_* = \nabla \circ (s_\rho)_*.$$

On the other hand, the developing map models the map induced in continuous cohomology by the representation ρ in the following way. Recall from [4, Proposition 5.4 and Corollary 5.6] that if N is a smooth manifold on which G acts properly smoothly then the complex $\Omega^*_{\mathrm{dR}}(N)^G$ computes the continuous cohomology of G. Moreover, the map in continuous cohomology induced by a continuous homomorphism $\rho\colon G \to H$ can be computed by considering a ρ–equivariant map $R\colon N \to M$, where N is a G–manifold as above and M an H–manifold. By definition this is exactly what the developing map D is with respect to the continuous map $\rho\colon \pi \to \mathrm{SL}_n(\mathbb{C})$. Indeed, by the above-cited result, we have the known fact that the canonical inclusion $\Omega^*_{\mathrm{dR}}(\widetilde{M})^{\pi_1(M)} \to \Omega^*_{\mathrm{dR}}(M)$ is a quasi-isomorphism.

The same discussion holds true for each boundary component since each of these is a $K(\mathbb{Z}^2, 1)$, and the compatibility of the developing maps on M and on its boundary components imply that they induce via the cone construction the map

$$\rho*\colon \mathrm{H}^*_c(\mathrm{SL}_n(\mathbb{C}), \{B\}) \to \mathrm{H}^3(M, \partial M).$$

Let us be slightly more precise and let us revisit our previous definition, Definition 22, of the volume. At the level of de Rahm cochains, the volume class $\mathrm{vol}_{\mathcal{H},\partial}$ is represented by the relative cocycle (ϖ, β) constructed in Section 5.1. Since evaluation on the fundamental class translates in de Rahm cohomology into integrating, by Stokes' formula and the above discussion:

Definition 38 Let $\rho\colon \pi \to \mathrm{SL}_n(\mathbb{C})$ be a representation of the fundamental a 3–manifold M whose interior is an hyperbolic manifold of finite volume. Denote the boundary components of M by $T_1 \sqcup \cdots \sqcup T_k$. Fix a system of peripheral subgroups P_i

in π and for each such group fix a Borel subgroup $B_i \subset \mathrm{SL}_n(\mathbb{C})$ such that $\rho(P_i) \subset B_i$. Denote by D the developing map associated to ρ and by D_r its restriction to the universal cover of the boundary component T_r. Then

$$\mathrm{Vol}(\rho) = \int_M D^*(\varpi) - \sum_{r=1}^{k} \int_{T_r} D_r^*(\beta_r), \tag{8}$$

where the differential forms $D^*(\varpi)$ and $D_r^*(\beta_r)$ descend from the universal covers to differential forms on the manifolds by equivariance.

Now, since $\mathrm{SL}_n(\mathbb{C})$ is 2–connected, the Leray–Serre spectral sequence in relative cohomology gives us a short exact sequence

$$0 \to \mathrm{H}^3(M, \partial M) \to \mathrm{H}^3(E_\rho, \partial E_\rho) \to \mathrm{H}^3(\mathrm{SL}_n\mathbb{C}) \to 0.$$

In particular, the volume class $\rho^*(\mathrm{vol}_{\mathcal{H},\partial})$ defined in Section 4 can be seen as a class in $\mathrm{H}^3(E_\rho, \partial E_\rho)$. The key observation of Reznikov in [24] is that in this larger group the volume class can be interpreted as a characteristic class associated to the foliation of E_ρ induced by the flat connection.

Proposition 39 *Denote by $j^*\colon \mathrm{H}^3(M, \partial M) \to \mathrm{H}^3(E_\rho, \partial E_\rho)$ the morphism induced by the projection $E_\rho \to M$ in de Rham cohomology. Then*

$$j^*(\rho^*(\mathrm{vol}_{\mathcal{H},\partial})) = \mathrm{Char}_{\nabla_\rho, \nabla|_{\partial M_i}}(\varpi, \{\beta_i\}).$$

Proof First observe that

$$\mathrm{Char}_{\nabla_\rho, \nabla|_{\partial M_i}}(\varpi, \{\beta_i\}) \in \ker\big(\mathrm{H}^3(E_\rho, \partial E_\rho) \to \mathrm{H}^3(\mathrm{SL}_n(\mathbb{C}))\big) = \mathrm{Im}\, j^*.$$

Indeed, by construction, the restriction of this characteristic class to the fiber $\mathrm{SL}_n(\mathbb{C})$ is given by the form ϖ. But the inclusion $\mathrm{SU}(n) \to \mathrm{SL}_n(C)$ is a weak equivalence, hence induces an isomorphism in cohomology, and since ω only depends on the projection on $i\mathfrak{su}_n$, the restriction of ϖ to $\mathrm{SU}(n)$ is the trivial form. So to check the equality we only need to show that after composing with the map induced by the section s_ρ both sides of the equation agree. Recall that by construction $(\omega, \{\beta_i\})$ is a relative cocycle that represents the hyperbolic form in $\mathrm{H}^3_c(\mathrm{SL}_n(\mathbb{C}), \{B\})$. Hence, by the discussion on the map D the class $\rho^*(\mathrm{vol}_{\mathcal{H},\partial})$ is represented by the cocycle $D^*((\omega, \{\beta_i\}))$.

To finish the proof it is enough to show that we have a commutative diagram

$$\begin{array}{ccccc}
C^3(\mathfrak{sl}_n,\{\mathfrak{b}\}) & \xrightarrow{\mathrm{Char}_{\nabla_\rho,\nabla|_{\partial M_i}}} & \Omega^3_{\mathrm{dR}}(E_\rho,\partial E_\rho) & \xrightarrow{\Theta_\rho^*} & \Omega^3_{\mathrm{dR}}(M\times\mathrm{SL}_n(\mathbb{C}),\{\partial M_i\times B_i\}) \\
\downarrow & & & \searrow^{s_\rho^*} & \downarrow{\scriptstyle s^*} \\
\Omega^3_{\mathrm{dR}}(\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n),\{B_i/T_i\}) & & \longrightarrow & & \Omega^3_{\mathrm{dR}}(M,\partial M)
\end{array}$$

where the bottom row is induced by D and the quasi-isomorphisms $\Omega^*_{\mathrm{dR}}(\widetilde{M})^\pi \to \Omega^*_{\mathrm{dR}}(M)$ and $\Omega^*_{\mathrm{dR}}(\widetilde{\partial M_i})^{\pi_1(\partial M_i)} \to \Omega^*_{\mathrm{dR}}(\partial M_i)$.

As this is a diagram on the chain level in relative cohomology, it is enough to check that the corresponding absolute maps yield commutative diagrams and are compatible, ie, for the absolute part,

$$\begin{array}{ccccc}
C^3(\mathfrak{sl}_n) & \xrightarrow{\mathrm{Char}_{\nabla_\rho}} & \Omega^3_{\mathrm{dR}}(E_\rho) & \xrightarrow{\Theta_\rho^*} & \Omega^3_{\mathrm{dR}}(M\times\mathrm{SL}_n(\mathbb{C})) \\
\downarrow & & & \searrow^{s_\rho^*} & \downarrow{\scriptstyle s^*} \\
\Omega^3_{\mathrm{dR}}(\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)) & \xrightarrow{D^*} & \Omega^3_{\mathrm{dR}}(\widetilde{M})^\pi & \longrightarrow & \Omega^3_{\mathrm{dR}}(M)
\end{array}$$

and, analogously for the relative part,

$$\begin{array}{ccccc}
C^2(\mathfrak{b}_i) & \xrightarrow{\mathrm{Char}_{\nabla|_{\partial M_i}}} & \Omega^2_{\mathrm{dR}}(\partial E_\rho) & \xrightarrow{\Theta_\rho^*} & \Omega^2_{\mathrm{dR}}(\partial M_i\times B) \\
\downarrow & & & \searrow^{s_\rho^*} & \downarrow{\scriptstyle s^*} \\
\Omega^2_{\mathrm{dR}}(B_i/T_i) & \xrightarrow{D^*} & \Omega^2_{\mathrm{dR}}(\widetilde{\partial M})^{\pi_1(\partial M)} & \longrightarrow & \Omega^2_{\mathrm{dR}}(\partial M)
\end{array}$$

Both the proof of commutativity of the diagrams and the compatibility are now elementary diagram chases. □

6 The variation formula

We are now ready to collect our efforts; but first a word of caution on the smoothness of the variety of representations. The algebraic variety $\mathrm{Hom}(\pi_1(M),\mathrm{SL}_n(\mathbb{C}))$ is not differentiable in general; in fact, for M compact the singularities that appear can be as wild as possible; for a discussion of the singularities, see for instance [18]. Nevertheless, by Whitney's theorem the algebraic variety $\mathrm{Hom}(\pi_1(M),\mathrm{SL}_n(\mathbb{C}))$ is generically smooth (ie the nonsmooth locus is of Lebesgue measure zero). Even restricted to the smooth locus, the volume function itself is not everywhere differentiable as is transparent from previous work of Neumann and Zagier. More precisely, let us check:

Lemma 40 [23] *For $n=2$ and a manifold with a single boundary component, the volume function is not differentiable at the defining representation.*

Recall that the defining representation is the one corresponding to the complete hyperbolic structure on the interior of M. In [23], Neumann and Zagier use a parameter $u \in \mathbb{C}$ in a neighborhood of the origin to parametrize a neighborhood of the complete structure in the moduli space of hyperbolic ideal triangulations. As noticed in their work [23], u and $-u$ correspond to the same hyperbolic metric on the interior of M. In fact, $\mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ is locally parametrized by

$$\mathrm{trace}(\rho_u(l)) = \pm 2\cosh\bigl(\tfrac{1}{2}u\bigr) = \pm\bigl(2+\tfrac{1}{4}u^2+O(u^4)\bigr),$$

where ρ_u denotes the holonomy of the structure with parameter u. In particular, ρ_0 is the defining representation. Then Neumann and Zagier define an analytic function $v(u)$ such that $\mathrm{trace}(\rho_u(m)) = \pm 2\cosh\bigl(\tfrac{1}{2}v\bigr)$ and prove that $v = \tau u + O(u^3)$, where $\tau \in \mathbb{C}$ is the so-called cusp length with $\Im(\tau) > 0$, and

$$\mathrm{vol}(\rho_u) = \mathrm{vol}(\rho_0) + \tfrac{1}{4}\Im(u\bar{v}) + O(|u|^4) = \mathrm{vol}(\rho_0) + \tfrac{1}{4}\Im(\tau)|u|^2 + O(|u|^4).$$

Thus, by choosing a local parameter $z = 2\cosh\bigl(\tfrac{1}{2}u\bigr) - 2 = \tfrac{1}{4}u^2 + O(u^4)$ in a neighborhood of the origin, the volume function has an expansion of the form

$$\mathrm{vol}(\rho_u) - \mathrm{vol}(\rho_0) = -\Im(\tau)|z| + O(|z|^2).$$

Hence, the volume is not a differentiable function on $\mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$, as $z \mapsto |z|$ is not differentiable at $z=0$. The volume is also not differentiable on the variety of representations, because the projection $\mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C})) \to \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ is a fibration in a neighborhood of ρ_0.

This being said, let us go back now to our variation formula. The following two subsections conclude the proof of the main theorem.

6.1 The variation comes from the boundary

Recall that the group $\mathrm{SU}(n)$ acts transitively by conjugation on the set of Borel subgroups of $\mathrm{SL}_n(\mathbb{C})$. Then, by uniqueness of the trivialization β, proved in Proposition 35, the trivializations of the volume form on two different Borel subgroups B_1 and B_2, say β_1 and β_2, are compatible in the sense that if $H \in \mathrm{SU}(n)$ is chosen such that $HB_2H^{-1} = B_1$, then $\beta_2(b,b') = \beta_1(\mathrm{Ad}_H\, b, \mathrm{Ad}_H\, b') = \mathrm{Ad}_H^*(\beta_1)$ for any $b, b' \in \mathfrak{b}_2$.

Let $\rho_t\colon \pi_1(M) \to \mathrm{SL}_n(\mathbb{C})$ be a differentiable family of representations. As we discussed before, we may think of the associated flat bundles E_{ρ_t} as being the flat bundle E_{ρ_0} but with a varying family of connections ∇_t. The uniqueness property discussed in the previous paragraph is precisely the coherence requirement of Definition 6 with respect to the subgroup $H = \mathrm{SU}(n)$. Consistent with our conventions at the end of Section 3.2, we will decorate with a subscript as in $\mathrm{H}^*_{\mathrm{SU}(n)}$ the cohomology of the complexes $C^*_{\mathrm{SU}(n)}(\mathfrak{g};\mathbb{R})$ etc defined in Notation 9 and Definition 10 of Section 3.2.

We can now apply the results of Section 5.2 to compute the variation of the volume. By the construction of the factorization of the variation map

$$\begin{array}{ccccc}
\mathrm{H}^*_{\mathrm{SU}(n)}(\mathfrak{su}_n,\{\mathfrak{b}\}) & \xrightarrow{\ \mathrm{var}\ } & \mathrm{H}^{*-1}_{\mathrm{SU}(n)}(\mathfrak{su}_n,\{\mathfrak{b}\};\mathfrak{su}_n^\vee,\{\mathfrak{b}^\vee\}) & \xrightarrow{\ \mathrm{F}\ } & \mathrm{H}^*_{\mathrm{SU}(n)}(\widetilde{\mathfrak{su}}_n,\{\widetilde{\mathfrak{su}}_n\}) \\
 & & & & \Big\downarrow \mathrm{Char}_{\nabla_t,\{\tilde{\nabla}_t\}} \\
 & & \xrightarrow{\qquad\qquad \mathrm{Var} \qquad\qquad} & & \mathrm{H}^*_{\mathrm{dR}}(M,\partial M)
\end{array}$$

we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}^2_{\mathrm{SU}(n)}(\mathfrak{sl}_n;\mathbb{R}) & \longrightarrow & \prod \mathrm{H}^2(\mathfrak{b};\mathbb{R}) & \longrightarrow & \mathrm{H}^3_{\mathrm{SU}(n)}(\mathfrak{sl}_n,\{\mathfrak{b}\};\mathbb{R}) & \longrightarrow & \mathrm{H}^3_{\mathrm{SU}(n)}(\mathfrak{sl}_n;\mathbb{R}) \\
\Big\downarrow \mathrm{var} & & \Big\downarrow \mathrm{var} & & \Big\downarrow \mathrm{var} & & \Big\downarrow \mathrm{var} \\
\mathrm{H}^1_{\mathrm{SU}(n)}(\mathfrak{sl}_n;\mathfrak{sl}_n^\vee) & \longrightarrow & \prod \mathrm{H}^1(\mathfrak{b};\mathfrak{b}^\vee) & \longrightarrow & \mathrm{H}^2_{\mathrm{SU}(n)}(\mathfrak{sl}_n,\{\mathfrak{b}\};\mathfrak{sl}_n^\vee,\{\mathfrak{b}^\vee\}) & \longrightarrow & \mathrm{H}^2_{\mathrm{SU}(n)}(\mathfrak{sl}_n;\mathfrak{sl}_n^\vee) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\mathrm{H}^2(M;\mathbb{R}) & \longrightarrow & \prod \mathrm{H}^2(\partial M) & \longrightarrow & \mathrm{H}^3(M,\partial M) & \longrightarrow & \mathrm{H}^3(M) \simeq 0
\end{array}$$

Let us recall the following lemma by Cartier [9, Lemme 1]:

Lemma 41 *Let V be a vector space on a field k and A be a family of endomorphisms of V. Assume that V is completely reducible. Denote by $V^\sharp$ the subspace of those vectors annihilated by all the $X \in A$ and by V^0 the subspace generated by the vectors Xv for $X \in A$ and $v \in V$.*

(1) $V = V^\sharp \oplus V^0$.

(2) *If V is equipped with a differential d that commutes to the $X \in A$ and such that Xv is a boundary if v is a cycle, then the homology with respect to this boundary gives $H(V) \simeq H(V^\sharp)$.*

Corollary 42 *For $* \geq 1$ and any $n \geq 2$,*

$$\mathrm{H}^*(\mathfrak{sl}_n;\mathfrak{sl}_n^\vee) \simeq 0 \simeq \mathrm{H}^*_{\mathrm{SU}(n)}(\mathfrak{sl}_n;\mathfrak{sl}_n^\vee).$$

Proof That $\mathrm{H}^*(\mathfrak{sl}_n;\mathfrak{sl}_n^\vee)\simeq 0$ is the direct application that Cartier makes of his lemma, given that $\mathfrak{su}_n$ is semisimple.

For the second isomorphism, we apply Lemma 41 to the (acyclic!) complex $V = C^*(\mathfrak{sl}_n;\mathfrak{sl}_n^\vee)$ viewed as a (graded) vector space acted upon by $\mathrm{SU}(n)$. Since $\mathrm{SU}(n)$ is compact, V is indeed completely reducible. Moreover, by functoriality of the complex, its differential commutes with the action of the elements in $\mathrm{SU}(n)-\mathrm{Id}$. If v is a cycle and $X\in\mathrm{SU}(n)-\mathrm{Id}$, then Xv is a cycle, and by acyclicity of this complex, it is a boundary. Observe that being annihilated by $A-\mathrm{Id}$ is the same as being fixed by A, hence Lemma 41 tells us that the embedding $C^*_{\mathrm{SU}(n)}(\mathfrak{sl}_n;\mathfrak{sl}_n^\vee)\hookrightarrow C^*(\mathfrak{sl}_n;\mathfrak{sl}_n^\vee)$ is a quasi-isomorphism. □

As a consequence our diagram above boils down to

$$\begin{array}{ccccccc}
 & & \prod \mathrm{H}^2(\mathfrak{b};\mathbb{R}) & \longrightarrow & \mathrm{H}^3_{\mathrm{SU}(n)}(\mathfrak{sl}_n,\{\mathfrak{b}\};\mathbb{R}) & \longrightarrow & \mathrm{H}^3_{\mathrm{SU}(n)}(\mathfrak{sl}_n;\mathbb{R}) \\
 & & \downarrow{\scriptstyle \mathrm{var}} & & \downarrow{\scriptstyle \mathrm{var}} & & \downarrow{\scriptstyle \mathrm{var}} \\
0 & \longrightarrow & \prod \mathrm{H}^1(\mathfrak{b};\mathfrak{b}^\vee) & \longrightarrow & \mathrm{H}^2_{\mathrm{SU}(n)}(\mathfrak{sl}_n,\{\mathfrak{b}\};\mathfrak{sl}_n^\vee,\{\mathfrak{b}^\vee\}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^2(M;\mathbb{R}) & \longrightarrow & \prod \mathrm{H}^2(\partial M) & \longrightarrow & \mathrm{H}^3(M,\partial M) & \longrightarrow & 0
\end{array}$$

In particular, the variation of the volume class is the image of a cohomology class in $\prod \mathrm{H}^2(\partial M)$. To see which one, we have to find an inverse to the isomorphism

$$\prod \mathrm{H}^1(\mathfrak{b};\mathfrak{b}^\vee)\to \mathrm{H}^2_{\mathrm{SU}(n)}(\mathfrak{sl}_n,\{\mathfrak{b}\};\mathfrak{sl}_n^\vee,\{\mathfrak{b}^\vee\}).$$

Unraveling the definitions, it is given by the following construction: The map

$$C^2_{\mathrm{SU}(n)}(\mathfrak{sl}_n,\{\mathfrak{b}\};\mathfrak{sl}_n^\vee,\{\mathfrak{b}^\vee\})\to C^2_{\mathrm{SU}(n)}(\mathfrak{sl}_n,;\mathfrak{sl}_n^\vee).$$

simply forgets the relative part, and acyclicity on the right-hand side means that the absolute part $\mathrm{var}(\varpi)$ of the relative cocycle $\mathrm{var}(\varpi,\{\beta\})$ is a coboundary, say $\mathrm{var}(\alpha)=d\gamma$. Then the preimage of $\mathrm{var}(\varpi,\{\beta\})$ in $\prod \mathrm{H}^1(\mathfrak{b};\mathfrak{b}^\vee)$ is given by the class of the family $\mathrm{var}(\beta)-i^*\gamma$, where i^* is the map induced by the inclusion $\mathfrak{b}\to\mathfrak{su}$.

Lemma 43 *The image of ϖ, the absolute part of the volume cocycle, under the map* $\mathrm{var}\colon C^3_{\mathrm{SU}(n)}(\mathfrak{sl}_n;\mathbb{R})\to C^2(\mathfrak{sl}_n;\mathfrak{sl}_n^\vee)$ *is the coboundary of the cochain*

$$\gamma\colon \mathfrak{sl}_n\to\mathfrak{sl}_n^\vee,\quad g\mapsto h\rightsquigarrow i\,\mathrm{tr}(\mathrm{pr}_{i\,\mathfrak{su}_n}(g)\mathrm{pr}_{\mathfrak{su}_n}(h)),$$

where $\mathrm{pr}_{\mathfrak{su}_n}\colon\mathfrak{sl}_n\to\mathfrak{su}_n$ *and* $\mathrm{pr}_{i\,\mathfrak{su}_n}\colon\mathfrak{sl}_n\to i\,\mathfrak{su}_n$ *are the canonical projections associated to the orthogonal decomposition* $\mathfrak{sl}_n=\mathfrak{su}_n\oplus i\,\mathfrak{su}_n$.

Proof For $x_1, x_2 \in \mathfrak{sl}_n$,

$$d(\gamma)(x_1,x_2) = x_1\gamma(x_2) - x_2\gamma(x_1) - \gamma([x_1,x_2]).$$

Recall that for $\theta \in \mathfrak{g}^\vee$ and $x, y \in \mathfrak{g}$, we have $(x\theta)(y) = -\theta([x,y])$. Hence, for $x_1, x_2, x_3 \in \mathfrak{sl}_n$,

$$\begin{aligned} d(\gamma)(x_1,x_2)(x_3) &= -\gamma(x_2)([x_1,x_3]) + \gamma(x_1)([x_2,x_3]) - \gamma([x_1,x_2])(x_3) \\ &= i\ \mathrm{tr}\big(-\mathrm{pr}_{i\,\mathfrak{su}_n}(x_2)\mathrm{pr}_{\mathfrak{su}_n}([x_1,x_3]) + \mathrm{pr}_{i\,\mathfrak{su}_n}(x_1)\mathrm{pr}_{\mathfrak{su}_n}([x_2,x_3]) \\ &\qquad\qquad - \mathrm{pr}_{i\,\mathfrak{su}_n}([x_1,x_2])\mathrm{pr}_{\mathfrak{su}_n}(x_3)\big). \end{aligned}$$

Since $[\mathfrak{su}_n, \mathfrak{su}_n] \subset \mathfrak{su}_n$, $[i\,\mathfrak{su}_n, i\,\mathfrak{su}_n] \subset \mathfrak{su}_n$ and $[i\,\mathfrak{su}_n, \mathfrak{su}_n] \subset i\,\mathfrak{su}_n$,

$$\begin{aligned} &\delta(\gamma)(x_1,x_2)(x_3) \\ &\quad = i\ \mathrm{tr}\Big(-\mathrm{pr}_{i\,\mathfrak{su}_n}(x_2)\big([\mathrm{pr}_{\mathfrak{su}_n}(x_1), \mathrm{pr}_{\mathfrak{su}_n}(x_3)] + [\mathrm{pr}_{i\,\mathfrak{su}_n}(x_1,)\mathrm{pr}_{i\,\mathfrak{su}_n}(x_3)]\big) \\ &\qquad\qquad + \mathrm{pr}_{i\,\mathfrak{su}_n}(x_1)\big([\mathrm{pr}_{\mathfrak{su}_n}(x_2), \mathrm{pr}_{\mathfrak{su}_n}(x_3)] + [\mathrm{pr}_{i\,\mathfrak{su}_n}(x_2), \mathrm{pr}_{i\,\mathfrak{su}_n}(x_3)]\big) \\ &\qquad\qquad\qquad - \big([\mathrm{pr}_{\mathfrak{su}_n}(x_1), \mathrm{pr}_{i\,\mathfrak{su}_n}(x_2)] - [\mathrm{pr}_{i\,\mathfrak{su}_n}(x_1), \mathrm{pr}_{\mathfrak{su}_n}(x_2)]\big)\mathrm{pr}_{\mathfrak{su}_n}(x_3)\Big) \\ &\quad = 2i\ \mathrm{tr}\big(\mathrm{pr}_{i\,\mathfrak{su}_n}(x_1)[\mathrm{pr}_{i\,\mathfrak{su}_n}(x_2), \mathrm{pr}_{i\,\mathfrak{su}_n}(x_3)]\big). \end{aligned}$$

Here we have used that $(A,B,C) \mapsto \mathrm{tr}(A[B,C])$ is alternating. □

Each Borel Lie algebra $\mathfrak{b}_n$ fits into a split exact sequence of Lie algebras

$$0 \to \mathfrak{ut}_n \to \mathfrak{b}_n \to \mathfrak{t}_n \to 0.$$

We have a splitting $\mathfrak{t}_n = \mathfrak{h}_n \oplus i\,\mathfrak{h}_n$. Denote by $\mathrm{pr}_{\mathfrak{h}_n}$ (resp. $\mathrm{pr}_{i\,\mathfrak{h}_n}$) the projection onto $\mathfrak{h}_n$ (resp. $i\,\mathfrak{h}_n$).

Proposition 44 *The variation of the volume of a representation is given by the sum over the components of* ∂M *of the integral over each component of the image of the cohomology class of the* 1*–cocycle in* $C^1(\mathfrak{b}_n; \mathfrak{b}_n^\vee)$

$$\zeta\colon \mathfrak{b}_n \to \mathfrak{b}_n^\vee, \quad x \mapsto y \rightsquigarrow i\ \mathrm{tr}(\mathrm{pr}_{i\,\mathfrak{h}_n}(x)\mathrm{pr}_{\mathfrak{h}_n}(y)),$$

under the map

$$\mathrm{H}^1(\mathfrak{b}_n; \mathfrak{b}_n) \to \mathrm{H}^2(\partial M).$$

Proof As $\mathrm{var}(\varpi)$ is the coboundary of γ, the cocycle $(\mathrm{var}(\varpi), \{\mathrm{var}(\beta_r)\})$ is cohomologous to $\big(0, \{\mathrm{var}(\beta_r) - i^*(\gamma)\}\big)$. Therefore, as the integral on the boundary ∂M

appears subtracted in Definition 38, the variation of volume is

$$-\sum_{r=1}^{k}\int_{T_r}(s_\rho^*\circ \mathrm{Char}_{\nabla_t}\circ \mathrm{F})(\mathrm{var}(\beta_r)-i^*(\gamma)).$$

Hence, we need to prove that $\zeta = i^*(\gamma)-\mathrm{var}(\beta)$. Given $x,y\in\mathfrak{b}_n$, write

$$x = x_d + x_u \quad \text{and} \quad y = y_d + y_u$$

with $x_u, y_u \in \mathfrak{u}_n$ and $x_d, y_d \in \mathfrak{h}+i\mathfrak{h}$ diagonal, their Chevalley–Jordan decomposition. Notice that $\mathrm{pr}_{i\mathfrak{su}_n}(x_d)=\mathrm{pr}_{i\mathfrak{h}}(x)$ and $\mathrm{pr}_{\mathfrak{su}_n}(y_d)=\mathrm{pr}_{\mathfrak{h}}(y)$ are diagonal, hence their product with elements of $\mathfrak{ut}_n$ and ${}^t\mathfrak{ut}_n$ have trace zero. Therefore,

$$\gamma(x)(y) = i\,\mathrm{tr}(\mathrm{pr}_{i\mathfrak{h}_n}(x)\mathrm{pr}_{\mathfrak{h}_n}(y)) + \gamma(x_u)(y_u) = \zeta(x)(y)+\gamma(x_u)(y_u).$$

As $\mathrm{pr}_{i\mathfrak{su}}(x_u)=\frac{1}{2}(x_u+{}^t\overline{x}_u)$, $\mathrm{pr}_{\mathfrak{su}}(y_u)=\frac{1}{2}(y_u-{}^t\overline{y}_u)$, and the trace vanishes on $\mathfrak{u}_n$,

$$\gamma(x_u)(y_u)=\tfrac{1}{4}i\,\mathrm{tr}((x_u+{}^t\overline{x}_u)(y_u-{}^t\overline{y}_u))=\tfrac{1}{4}i\,\mathrm{tr}({}^t\overline{x}_u y_u - x_u{}^t\overline{y}_u)=\beta(x,y).$$

Hence, $i^*(\gamma)=\zeta+\mathrm{var}(\beta)$, as claimed. □

Observe that this form we have to integrate does only depend on the projection on $\mathfrak{b}_n/\mathfrak{ut}_n$. Recall that corresponding to the above split exact sequence of $\mathfrak{b}_n$ we have a split short exact sequence of Lie groups

$$1\to U_n\to B_n\to T_n\to 1,$$

where U_n stands for the unipotent matrices, and the sequence is split by the semisimple matrices in B_n. Then the fact that the cochain ζ only depends on the projection onto $\mathfrak{t}_n$ means precisely that the variation of the volume depends on the restriction of the representation $\rho\colon P_i\to B_i$ only through its projection on B_i/U_n, a representation with values in an abelian group.

As an immediate corollary we have that if for each peripheral subgroup the restriction of the representation ρ takes values in unipotent subgroups of $\mathrm{SL}_n(\mathbb{C})$ and the deformation of ρ is also boundary unipotent, then the volume does not vary:

Corollary 45 *The volume function restricted to the subspace of boundary unipotent representations is locally constant.*

We now turn to a more explicit formula for the variation of the volume as encoded on each torus.

6.2 Deforming representations on the torus

Let $\{\alpha, \beta\}$ be a generating set of the fundamental group of the 2–torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. They act on the universal covering $\alpha, \beta\colon \mathbb{R}^2 \to \mathbb{R}^2$ as the integer lattice of translations: $\alpha(x, y) = (x+1, y)$ and $\beta(x, y) = (x, y+1)$.

By the Lie–Kolchin theorem, the image $\rho(\pi_1(T^2))$ is contained in a Borel subgroup B_n and up to conjugation we assume that its variation is contained in a fixed subgroup. The class we want to evaluate vanishes in $\mathfrak{u}_n$, so we do not need to understand the whole perturbation of ρ in B_n but just its projection to $\pi\colon B_n \to B_n/U_n = \Delta_n \cong (\mathbb{C}^*)^{n-1}$. Write

$$\pi(\rho(\alpha)) = \exp(a), \pi(\rho(\beta)) = \exp(b) \in \Delta_n,$$

where $a, b \in \mathfrak{sl}_n(\mathbb{C})$ are diagonal matrices. Notice that there is an indeterminacy of the logarithm — the nontrivial entries (diagonal) of a and b are only well defined up addition of a term in $2\pi i\mathbb{Z}$ — but this does not affect the final result.

Since Δ_n is abelian, for such a representation we have a ρ–equivariant map

$$D\colon \mathbb{R}^2 \to B_n/U_n, \quad (x, y) \mapsto \exp(xa + yb).$$

Then

$$(\nabla \circ (s_\rho)_*)\Big(\frac{\partial}{\partial x}\Big) = a \quad \text{and} \quad (\nabla \circ (s_\rho)_*)\Big(\frac{\partial}{\partial y}\Big) = b.$$

We vary the representation by varying a and b, so

$$(\dot\nabla \circ (s_\rho)_*)\Big(\frac{\partial}{\partial x}\Big) = \dot a \quad \text{and} \quad (\dot\nabla \circ (s_\rho)_*)\Big(\frac{\partial}{\partial y}\Big) = \dot b.$$

Lemma 46 *For* $c \in C^1(\mathfrak{g}, \mathfrak{g}^\vee)$ *and a variation as above,*

$$\int_{\partial M} (s_\rho)^*(\mathrm{Char}_{\nabla_t}(\mathrm{F}(c))) = c(a)(\dot b) - c(b)(\dot a).$$

Proof For Z_1 and Z_2 vector fields on $E_\rho|_{\partial M}$,

$$\mathrm{Char}_{\nabla_t}(\mathrm{F}(c))(Z_1, Z_2) = c(\nabla(Z_1))(\dot\nabla(Z_2)) - c(\nabla(Z_2))(\dot\nabla(Z_1)).$$

Setting $Z_1 = (s_\rho)_*\big(\frac{\partial}{\partial x}\big)$ and $Z_2 = (s_\rho)_*\big(\frac{\partial}{\partial y}\big)$, we have $\nabla(Z_1) = a$, $\dot\nabla(Z_1) = \dot a$, $\nabla(Z_2) = b$ and $\dot\nabla(Z_2) = \dot b$, hence

$$(s_\rho)^*(\mathrm{Char}_{\nabla_t}(\mathrm{F}(c))) = (c(a)(\dot b) - c(b)(\dot a))\,dx \wedge dy.$$

As $\int_{\partial M} dx \wedge dy = 1$, the lemma follows. □

Corollary 47 *If $a, b, \dot a, \dot b \in \mathfrak{b}_n$, then the evaluation of the cocycle ζ that is as in Proposition 44 is given by*

$$\operatorname{tr}(\Re(b)\Im(\dot a) - \Re(a)\Im(\dot b)),$$

where $\Re$ and $\Im$ denote the usual real and imaginary parts of the coefficients.

Proof By Lemma 46 and Proposition 44, the evaluation of ζ is

$$i(\operatorname{tr}(\operatorname{pr}_{i\mathfrak{h}_n}(a)\operatorname{pr}_{\mathfrak{h}_n}(\dot b)) - \operatorname{pr}_{i\mathfrak{h}_n}(b)\operatorname{pr}_{\mathfrak{h}_n}(\dot a)).$$

Let $\operatorname{pr}_{\mathfrak{h}+i\mathfrak{h}}\colon \mathfrak{b}_n \to \mathfrak{h} + i\mathfrak{h}$ denote the projection to the diagonal part, then, as $\mathfrak{h} \subset \mathfrak{su}(n)$ is the subalgebra of diagonal matrices with zero real part,

$$\operatorname{pr}_{\mathfrak{h}} = i\Im \circ \operatorname{pr}_{\mathfrak{h}+i\mathfrak{h}}, \quad \operatorname{pr}_{i\mathfrak{h}} = \Re \circ \operatorname{pr}_{\mathfrak{h}+i\mathfrak{h}}.$$

Thus,

$$\begin{aligned} i\operatorname{tr}(\operatorname{pr}_{i\mathfrak{h}_n}(a)\operatorname{pr}_{\mathfrak{h}_n}(\dot b) - \operatorname{pr}_{i\mathfrak{h}_n}(b)\operatorname{pr}_{\mathfrak{h}_n}(\dot a)) &= i\operatorname{tr}(\Re(a) i\Im(\dot b) - \Re(b) i\Im(\dot a)) \\ &= -\operatorname{tr}(\Re(a)\Im(\dot b) - \Re(b)\Im(\dot a)). \qquad \square \end{aligned}$$

This concludes the proof of the main theorem.

6.3 Comparison with other variation formulas

When $n = 2$, we can write

$$a = \begin{pmatrix} \frac{1}{2}(l_1 + i\theta_1) & 0 \\ 0 & -\frac{1}{2}(l_1 + i\theta_1) \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} \frac{1}{2}(l_2 + i\theta_2) & 0 \\ 0 & -\frac{1}{2}(l_2 + i\theta_2) \end{pmatrix}.$$

Hence, $\exp(a)$ is an hyperbolic isometry with translation length l_1 and rotation angle θ_1, and so is $\exp(b)$ with parameters l_2 and θ_2. Then, by Corollary 47, the contribution to the variation of volume of the corresponding torus component is

$$\operatorname{tr}(\Re(b)\Im(\dot a) - \Re(a)\Im(\dot b)) = \tfrac{1}{2}(l_2\dot\theta_1 - l_1\dot\theta_2),$$

which is precisely Hodgson's formula in [17], as he derived from Schläfli's formula for the variation of the volume for polyhedra in hyperbolic space.

Still in the case $n = 2$, Neumann and Zagier [23] study the space of hyperbolic structures on a manifold by studying triangulations by ideal hyperbolic simplices. To each hyperbolic ideal triangulation there is a natural assignment of a holonomy representation in $\mathrm{PSL}_2(\mathbb{C})$, and its volume is just the addition of the volumes of the tetrahedra involved.

For an arbitrary value of n, variational formulas for the volume have been obtained in remarkable works by several authors using spaces of decorated ideal triangulations and the Bloch group; see for instance [13]. Here we shall briefly describe the approach of [1; 10] and relate their formulas to ours.

For $n = 3$, Bergeron, Falbel and Guilloux [1] consider ideal hyperbolic tetrahedra with an additional decoration by flags in $\mathbb{P}^2(\mathbb{C})$ (see also [13]). Under some compatibility conditions one gets back the manifold equipped with a decorated hyperbolic structure, to which one can associate a holonomy in $\mathrm{PSL}_3(\mathbb{C})$, as well as a flag to each peripheral subgroup (equivalently this yields a Borel subgroup for the holonomy of each peripheral subgroup). Pushing this data to the Bloch group gives a volume for the holonomy.

Firstly the volume in [1] is $\frac{1}{4}$ of ours; they chose a normalization of the volume such that composing with the irreducible representation $\sigma_3\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_3(\mathbb{C})$ does not change the volume (in our case, by Proposition 36 it is multiplied by 4). Secondly, they have a different choice of coordinates in $\mathrm{PSL}_3(\mathbb{C})$: the holonomy of the peripheral elements m and l is given respectively by

$$\begin{pmatrix} 1/A^* & * & * \\ 0 & 1 & * \\ 0 & 0 & A \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1/B^* & * & * \\ 0 & 1 & * \\ 0 & 0 & B \end{pmatrix}; \tag{9}$$

see [1, Section 5.5.2]. Then Proposition 11.1.1 of [1] states that each end contributes to the variation of volume by a term

$$\tfrac{1}{12}\Im(d\log \wedge_{\mathbb{Z}} \log)(2A \wedge_{\mathbb{Z}} B + 2A^* \wedge_{\mathbb{Z}} B^* + A^* \wedge_{\mathbb{Z}} B + A \wedge_{\mathbb{Z}} B^*), \tag{10}$$

where $\wedge_{\mathbb{Z}}$ stands for the wedge product as $\mathbb{Z}$–modules of the space of analytic functions on the space of decorated structures, and

$$\Im(d\log \wedge_{\mathbb{Z}} \log)(f \wedge_{\mathbb{Z}} g) = \Im(\log|g| \cdot d(\log f) - \log|f| \cdot d(\log g)) \tag{11}$$

for any pair of analytic functions f and g. Then, after a change of coordinates in $\mathrm{PSL}_3(\mathbb{C})$, it is straightforward to check that (10) is $\frac{1}{4}$ of Corollary 47 for $\mathrm{SL}_3(\mathbb{C})$.

When $n \geq 3$, Dimofte, Gabella and Goncharov [10] also consider the space of framed flat connections. This yields decorated ideal triangulations by means of flags in $\mathbb{P}^{n-1}(\mathbb{C})$ and they generalize (10). In their work, the holonomy of the peripheral elements l

and m (resp. a and b in our setting) is given by

$$\begin{pmatrix} 1 & 0 & 0 & & 0 \\ * & l_1 & 0 & & 0 \\ * & * & l_1 l_2 & & 0 \\ & & & \ddots & \\ * & * & * & & l_1 \cdots l_{n-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & & 0 \\ * & m_1 & 0 & & 0 \\ * & * & m_1 m_2 & & 0 \\ & & & \ddots & \\ * & * & * & & m_1 \cdots m_{n-1} \end{pmatrix};$$

see [10, (3.42)]. If one denotes by κ the Cartan matrix of size $n-1$ given by

$$\kappa_{ij} = \begin{cases} 2 & \text{for } i = j, \\ -1 & \text{for } i = j \pm 1, \\ 0 & \text{otherwise,} \end{cases}$$

then the contribution of each peripheral group to the variation of volume is [10, (4.52) and (4.53)]

$$\log d \arg \sum_{i,j=1}^{n} (\kappa^{-1})_{ij} l_i \wedge m_j. \tag{12}$$

Here [10, 4.60],

$$\log d \arg(f \wedge g) = \log|f|\, d \arg g - \log|g|\, d \arg f$$

is the exact the analog of (11).

Again, an easy computation shows that (12) is the same formula as Corollary 47.

To conclude, our work gets back exactly the same formula as in [1; 10] but with the advantage that we do not have to bother about the existence of decorated ideal triangulations (the existence of nondegenerate ideal triangulations for the complete structure still remains conjectural).

References

[1] **N Bergeron**, **E Falbel**, **A Guilloux**, *Tetrahedra of flags, volume and homology of* SL(3), Geom. Topol. 18 (2014) 1911–1971 MR

[2] **R Bieri**, **B Eckmann**, *Relative homology and Poincaré duality for group pairs*, J. Pure Appl. Algebra 13 (1978) 277–319 MR

[3] **A Borel**, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. 57 (1953) 115–207 MR

[4] **A Borel**, **N R Wallach**, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Annals of Mathematics Studies 94, Princeton Univ. Press (1980) MR

[5] **R Bott**, **L W Tu**, *Differential forms in algebraic topology*, Graduate Texts in Mathematics 82, Springer (1982) MR

[6] **M Bucher**, **M Burger**, **A Iozzi**, *A dual interpretation of the Gromov–Thurston proof of Mostow rigidity and volume rigidity for representations of hyperbolic lattices*, from "Trends in harmonic analysis" (M A Picardello, editor), Springer INdAM Ser. 3, Springer (2013) 47–76 MR

[7] **M Bucher**, **M Burger**, **A Iozzi**, *The bounded Borel class and complex representations of* 3*–manifold groups*, preprint (2014) arXiv

[8] **M Burger**, **A Iozzi**, *Bounded differential forms, generalized Milnor–Wood inequality and an application to deformation rigidity*, Geom. Dedicata 125 (2007) 1–23 MR

[9] **P Cartier**, *Théorie des algèbres semi-simples*, from "Séminaire Sophus Lie 1954–55", volume 1, École Normale Supérieure, Paris (1955) Exposé 7

[10] **T Dimofte**, **M Gabella**, **A B Goncharov**, *K–decompositions and* 3*d gauge theories*, J. High Energy Phys. (2016) 151 pp MR

[11] **J Dubois**, **S Garoufalidis**, *Rationality of the* $\mathrm{SL}(2,\mathbb{C})$*–Reidemeister torsion in dimension* 3, Topology Proc. 47 (2016) 115–134 MR

[12] **D B Fuks**, *Cohomology of infinite-dimensional Lie algebras*, Consultants Bureau, New York (1986) MR

[13] **S Garoufalidis**, **D P Thurston**, **C K Zickert**, *The complex volume of* $\mathrm{SL}(n,\mathbb{C})$*–representations of* 3*–manifolds*, Duke Math. J. 164 (2015) 2099–2160 MR

[14] **A Goncharov**, *Volumes of hyperbolic manifolds and mixed Tate motives*, J. Amer. Math. Soc. 12 (1999) 569–618 MR

[15] **A Guilloux**, *Représentations et structures géométriques*, habilitation, Université Pierre et Marie Curie (2015) Available at `https://webusers.imj-prg.fr/~antonin.guilloux/these/hdr.pdf`

[16] **G Hochschild**, **G D Mostow**, *Cohomology of Lie groups*, Illinois J. Math. 6 (1962) 367–401 MR

[17] **C Hodgson**, *Degeneration and regeneration of geometric structures on* 3*–manifolds*, PhD thesis, Princeton University (1986)

[18] **M Kapovich**, **J J Millson**, *On representation varieties of* 3*–manifold groups*, Geom. Topol. 21 (2017) 1931–1968 MR

[19] **C Maclachlan**, **A W Reid**, *The arithmetic of hyperbolic* 3*–manifolds*, Graduate Texts in Mathematics 219, Springer (2003) MR

[20] **P Menal-Ferrer**, **J Porti**, *Twisted cohomology for hyperbolic three manifolds*, Osaka J. Math. 49 (2012) 741–769 MR

[21] **N Monod**, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Mathematics 1758, Springer (2001) MR

[22] **N Monod**, *Stabilization for* SL_n *in bounded cohomology*, from "Discrete geometric analysis" (M Kotani, T Shirai, T Sunada, editors), Contemp. Math. 347, Amer. Math. Soc., Providence, RI (2004) 191–202 MR

[23] **W D Neumann**, **D Zagier**, *Volumes of hyperbolic three-manifolds*, Topology 24 (1985) 307–332 MR

[24] **A Reznikov**, *Rationality of secondary classes*, J. Differential Geom. 43 (1996) 674–692 MR

[25] **C A Weibel**, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics 38, Cambridge Univ. Press (1994) MR

[26] **A Wienhard**, *Remarks on and around bounded differential forms*, Pure Appl. Math. Q. 8 (2012) 479–496 MR

[27] **T Yoshida**, *The* η*–invariant of hyperbolic* 3*–manifolds*, Invent. Math. 81 (1985) 473–514 MR

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