The normal closure of big Dehn twists and plate spinning with rotating families

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We study the normal closure of a big power of one or several Dehn twists in a mapping class group. We prove that it has a presentation whose relators consist only of commutators between twists of disjoint support, thus answering a question of Ivanov. Our method is to use the theory of projection complexes of Bestvina, Bromberg and Fujiwara, together with the theory of rotating families, simultaneously on several spaces.

Introduction

Consider a closed orientable surface Σ of negative Euler characteristic. The mapping class group of Σ, denoted by $\mathrm{MCG}(\Sigma)$, is the quotient of the group of orientation-preserving homeomorphisms by the path-connected component of the identity. A classical theorem of Dehn and Nielsen indicates a natural isomorphism between this group and a subgroup of index 2 of the outer automorphism group of $\pi_1(\Sigma)$.

As the Riemann uniformization theorem makes $\pi_1(\Sigma)$ act as a lattice on the hyperbolic plane, one can argue that $\mathrm{MCG}(\Sigma)$ is (in a sense) some hyperbolic analogue of $\mathrm{SL}_2(\mathbb{Z})$ which is of index 2 in the automorphism group of $\mathbb{Z}^2$, a lattice in the euclidean plane.

However, contrarily to $\mathrm{SL}_2(\mathbb{Z})$, some nontrivial elements of $\mathrm{MCG}(\Sigma)$ have large centralizer. For instance, consider a simple closed curve α on Σ and a tubular neighborhood of it $\alpha^{(t)} \simeq [-\epsilon, \epsilon] \times \alpha \hookrightarrow \Sigma$, and define a (simple) Dehn twist τ

as the identity in $\Sigma \setminus \alpha^{(t)}$, and as a full twist on $\alpha^{(t)}$, namely, identifying α with S^1, the map $[(\eta, e^{i\theta}) \mapsto (\eta, e^{i(\theta + (\eta + \epsilon)\pi/\epsilon)})]$. A Dehn twist will obviously commute with any mapping class whose support is disjoint from this tube, and therefore with a lot of other Dehn twists. By a theorem of Dehn, $\mathrm{MCG}(\Sigma)$ is generated by Dehn twists around simple closed curves, thus by an intricate set of generators linked by commutation relations, but also braid relations and lantern relations. These differences can lead to modification of the expected analogy with the euclidean case in order to include $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ (generated by elementary matrices).

Thurston, and Nielsen, (see the discussion in Handel and Thurston [9]) classified mapping classes into three cases, those of finite order, those that are reducible in the sense that they have infinite order and that some nontrivial power preserves the homotopy class of a simple closed curve, and finally the pseudo-Anosov. The pseudo-Anosov mapping classes happen to be the hyperbolic isometries of an action of $\mathrm{MCG}(\Sigma)$ on an important graph, the curve graph of Σ, which is Gromov hyperbolic; see Masur and Minsky [17]. They are, in many ways, the witnesses that some phenomena of rank one happen in $\mathrm{MCG}(\Sigma)$ that are similar to the structure of $\mathrm{SL}_2(\mathbb{Z})$, and its action on the modular tree. On the other hand, Dehn twists are as reducible as it is possible to be. They are, or should be, the witnesses of some phenomena of higher rank, similar to the structure of $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$.

Here is an illustration of the difference of behaviors. If one considers a finite collection of pseudo-Anosov elements, one can show that, after taking suitable powers, the group they generate is free; see Ivanov [11] and McCarthy [18]. This is a ping-pong argument, for instance on the boundary of Teichmüller space, or on the curve graph. If one considers a finite collection of Dehn twists around simple closed curves, then Koberda [13] proved the beautiful ping-pong result that the group generated by some powers of these Dehn twists is a right-angled Artin group: a group whose presentation over the given generating set is a collection of commutators, the obvious ones (two Dehn twists commute if their curves are disjoint).

The case of normal subgroups is our interest. If $n \geq 3$, by the Margulis normal subgroup theorem, all normal subgroups of $\mathrm{SL}_n(\mathbb{Z})$ are finite or of finite index. In $\mathrm{SL}_2(\mathbb{Z})$ it is not the case: this group is virtually free, and has uncountably many nonisomorphic quotients.

It is a natural question to ask whether (and how) these phenomena are seen in $\mathrm{MCG}(\Sigma)$. What can be the normal closure of a power of a pseudo-Anosov, the normal closure

of a power of a Dehn twist, and the group generated by all k^{th} powers of all simple Dehn twists? Farb [7, Section 2.4] and Ivanov [12, Section 3] each asked this question in the case of a pseudo-Anosov, attributing it to Long, McCarthy and Penner. Ivanov [12, Section 12] also asked what he calls the deep relation question, that is, whether all relations among certain powers of Dehn twists must derive from obvious commutation relations.

In Dahmani, Guirardel and Osin [6, Section 5], we answered the first question: there is an integer $N = N(\Sigma)$ such that for any pseudo-Anosov mapping class γ, the normal closure $\langle\!\langle \gamma^N \rangle\!\rangle_{\text{MCG}(\Sigma)}$ is free, and consists only of pseudo-Anosov elements and the identity. This is in line with what happens in $\text{SL}_2(\mathbb{Z})$, for each infinite order element.

We are interested in the question of the closure of a power of a Dehn twist, and in the group generated by certain powers of all (simple) Dehn twists, as in Ivanov's deep relation problem. A naive expectation along the lines of the analogy with $\text{SL}_n(\mathbb{Z})$, and the Margulis normal subgroup theorem, could be to expect each such normal subgroup to be a finite index subgroup. Whereas it is the case for squares of Dehn twists [10], it is not the case for large powers (see Humphries [10], Funar [8], Coulon [5, Theorem 6.17]; see also Stylianakis [19] and Masbaum [16] for the case of powers of half-Dehn twists on punctured spheres). Another expectation could be, in light of the finite-type situation, and ping pong arguments, to expect infinitely generated right-angled Artin groups. Again, this is not the case in general (see Clay, Brendle, Leininger and Margalit [4; 3]; Brendle and Margalit proved restrictions on the automorphism group of certain of these normal subgroups that forbid them to be right-angled Artin groups). However, we indeed prove that there is no need of relations other than the obvious ones.

Theorem 1 *For every orientable closed surface Σ, there is an integer N_0 such that for any multiple N of N_0:*

- *For any Dehn twist τ, the normal closure of τ^N in the mapping class group of Σ has a partially commutative presentation, built on an infinite set of generators that are conjugates of τ^N, so that the relators are commutations between pairs of conjugates of τ^N that have disjoint underlying curves.*

- *The group generated by all N^{th} powers of all simple Dehn twists has a partially commutative presentation, built on an infinite set of generators that are N^{th} powers of Dehn twists, and whose relators are commutations between pairs of conjugates of the generators that have disjoint underlying curves.*

The difference with an infinitely generated right-angled Artin group is that some elements in the commutator relators are not in the generating set, but merely conjugates of elements in the generating set. We recover that the normal closure is far from being of finite index in $\mathrm{MCG}(\Sigma)$, for instance because it has abelianization of infinite rank (the relators being in the derived subgroup of the free group over the set of generators).

In our point of view, this result and its departure from the complexity of normal subgroups of $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ (granted by the Margulis normal subgroup theorem) reinforce Coulon [5] and Funar [8] in witnessing a dent in the analogy between $\mathrm{MCG}(\Sigma)$ and $\mathrm{SL}_n(\mathbb{Z})$. It also answers Ivanov's question on deep relations.

Let us discuss the proof of this theorem.

In Dahmani, Guirardel and Osin [6] the structure of the normal closure of a big pseudo-Anosov was studied with the help of rotating families. Consider G a group acting by isometries on a space X. A rotating family in G on X is a collection of subgroups (the rotation groups) that is closed under conjugacy and such that each of them fixes a certain point in X (thus inducing some kind of rotation around this point). Take ρ in one of these subgroups, fixing c. One may measure an analogue of the angle of rotation of ρ by taking x at distance 1 from c, and measuring the infimal length between x and ρx of paths outside the ball of radius 1 around c. If X is Gromov-hyperbolic (for a small hyperbolicity constant), if the fixed points of the different rotation groups are sufficiently far from each other, and if the angles of rotations are sufficiently big, the group generated by all the rotation groups is a free product of a selection of them. In [6] we applied this theory to the action of $\mathrm{MCG}(\Sigma)$ on a cone-off of the curve graph of Σ. The rotation groups were the conjugates of the big pseudo-Anosov considered.

The rotating family argument can be explained as follows. One analyzes the structure of groups generated by more and more rotation groups, to discover that they arrange as a sequence of free products. Starting from a quasiconvex set W (that will change over time) that is at first a small ball around a single fixed point of a single rotation group, one sets G_W to be the group generated by the rotation groups whose centers are in W, and one makes W grow until it (almost) touches another center of rotation, for some other group. Call S a G_W–transversal of the newly approached centers of rotation. Then one *unfolds* W into W' by taking its images by the group $G_{W'}$ (thus generated by the new rotations, and the rotation already with center in W). Because of hyperbolicity, and of largeness of angles of rotations involved, the resulting space is still quasiconvex, with almost the same constant — with a little repair, it has the same

quasiconvexity constant indeed. Actually W' has the structure of a tree whose vertices are the images of W by the group $G_{W'}$, and the images of points in S by $G_{W'}$, thus giving by Bass–Serre duality the structure of free product of G_W and the rotation groups around points of S (edge stabilizers are trivial since no element can fix two different centers of rotation). Then, one takes the new W as W' and starts over. In the direct limit, the group generated by all rotations has been described as a free product of a selection of rotation groups.

Bestvina, Bromberg and Fujiwara [2], using a system of subsurface projections, discovered that there is a normal finite index subgroup G_0 of $\mathrm{MCG}(\Sigma)$ that acts on some spaces quasi-isometric to trees, and on which Dehn twists behave like large rotation subgroups. It has been observed by several people that this implies that the normal closure of a certain power of a Dehn twist in G_0 is free, using the argument of [6]. However, it is far from obvious how to promote this structural feature to the normal closure in $\mathrm{MCG}(\Sigma)$.

In this paper, we use several quasitrees as above, one for each left coset of G_0 in $\mathrm{MCG}(\Sigma)$. The group G_0 acts on each of them, but its action is twisted by the automorphism of G_0 that is the conjugation by elements g_i, for $i = 1, \ldots, m$, realizing a transversal of G_0 in $\mathrm{MCG}(\Sigma)$. If τ^N is a Dehn twist in G_0, the normal closure of τ^N in $\mathrm{MCG}(\Sigma)$ equals the normal closure of the collection $\{g_i \tau^N g_i^{-1} : i = 1, \ldots, m\}$ in G_0. Each $g_i \tau^N g_i^{-1}$ is a legitimate rotation on the quasitree associated to g_i.

The argument of [6] is then performed simultaneously on each of the m quasitrees. Instead of one convex subset that grows, and gets unfolded in a hyperbolic space, we have m convex sets $\mathcal{W}_1, \ldots, \mathcal{W}_m$ in the m quasitrees. Each of them is invariant by the group generated by the rotations around rotation points in all of them. One looks for a rotation point R that is nearby one of these sets, and in a certain sense, nearby all of them (although they do not live in the same quasitrees, this still makes sense in the framework of projection systems). Then, one unfolds our convex sets in all coordinates $i = 1, \ldots, m$. A funny phenomenon happens. The unfolding in the coordinate of R provides a nice tree, as the argument of [6], and the convexity of the result is quantitatively very good. This tree gives the structure of the new group by Bass–Serre duality, and reveals that only commutation relations are involved. There is no reason that the unfolding in all other coordinates produces something resembling a tree, and could in principle destroy the convexity of $\mathcal{W}_j$. However, using the properties of the projection system, we show that the result is still somehow convex (less convex than before though). The game is then to unfold in the different quasitrees at regular

intervals of time in the process, and to control the degradation of the convexity so that the repair can wait until a new unfolding occurs. It is a game of plate spinning.

The quasitrees that we will use come from projection complexes defined in [2]. We wrote the argument in this axiomatic language, to avoid dealing with useless hyperbolicity constants. In the end, even if the spaces are indeed quasitrees, this fact does not appear in the argument. The axioms of projection systems are extensively used though, and they contain the information that the geometric space is a quasitree. We will thus prove a statement similar to Theorem 1, namely Theorem 2.2, that gives the structure of groups generated by composite rotating families. There is actually more information coming from this composite rotating family structure, for instance the Greendlinger property (see Definition 2.4), which describes how an element in the group can be shortened in some coordinate of the composite projection system.

1 Composite projection systems

1.1 Projection systems

Let us recall a part of the axiomatic construction of [2].

Definition 1.1 [2] A projection system is a set $\mathbb{Y}$, with a constant $\theta > 0$, and for each $Y \in \mathbb{Y}$, a function $d_Y^\pi \colon \mathbb{Y} \setminus \{Y\} \times \mathbb{Y} \setminus \{Y\} \to \mathbb{R}_+$ satisfying the following axioms:

- **Symmetry** $d_Y^\pi(X, Z) = d_Y^\pi(Z, X)$ for all X, Y, Z.
- **Triangle inequality** $d_Y^\pi(X, Z) + d_Y^\pi(Z, W) \geq d_Y^\pi(X, W)$ for all X, Y, Z, W.
- **Behrstock inequality** $\min\{d_Y^\pi(X, Z), d_Z^\pi(X, Y)\} \leq \theta$ for all X, Y, Z.
- **Properness** $\{Y : d_Y^\pi(X, Z) > \theta\}$ is finite for all X, Z.

In this work one also assumes

- **Separation** $d_Y^\pi(Z, Z) \leq \theta$ for all Z, Y.

Observe that if the axioms are true for some θ, they hold for all larger θ.

From this rudimentary axiomatic set, Bestvina, Bromberg and Fujiwara manage to extract meaningful geometry, by modifying the functions d_Y^π into some functions d_Y that satisfy many more properties, usually encapsulated in the statement that the projection complex of $\mathbb{Y}$, for a suitable parameter K, is a quasitree.

One should think of d_Y (or d_Y^π) as an angular measure between X and Z seen from Y. The axioms fit in this viewpoint: the Behrstock inequality says that if the angle at Y between X and Z is large, then from the point of view of Z, the items Y and X look aligned.

Let us review very quickly the procedure of [2] to produce the functions d_Y. Given θ for which the axioms hold, Bestvina, Bromberg and Fujiwara define $\mathcal{H}(X, Z)$ to be the set of pairs (X', Z') such that both d_X^π and d_Z^π between them is strictly larger than 2θ, and one also include the pairs (X, Z') if $d_Z^\pi(X, Z') > 2\theta$, symmetrically the pairs (X', Z) if $d_X^\pi(X', Z) > 2\theta$, and finally the pair (X, Z) itself.

Then $d_Y(X, Z)$ is defined to be the infimum of d_Y^π over $\mathcal{H}(X, Z)$.

For each K, the set $\{Y : d_Y(X, Z) \geq K\}$ is denoted by $\mathbb{Y}_K(X, Z)$.

Theorem 3.3 in [2] states that there exists Θ and $\kappa \geq \theta$, depending only on θ, such that for all X, Y, Z, W:

- **Symmetry** $d_Y(X, Z) = d_Y(Z, X)$.
- **Coarse equality** $d_Y^\pi - \kappa \leq d_Y \leq d_Y^\pi$.
- **Coarse triangle inequality** $d_Y(X, Z) + d_Y(Z, W) \geq d_Y(X, W) - \kappa$.
- **Behrstock inequality** $\min\{d_Y(X, Z), d_X(Y, Z)\} \leq \kappa$.
- **Properness** $\{V : d_V(X, Z) > \Theta\}$ is finite.
- **Monotonicity**   If $d_Y(X, Z) \geq \Theta$ then both $d_W(X, Y)$ and $d_W(Z, Y)$ are at most $d_W(X, Z)$.
- **Order** $\mathbb{Y}_\Theta(X, Z) \cup \{X, Z\}$ is totally ordered by an order $\dot{<}$ such that X is least, Z is greatest, and if $Y_0 \dot{<} Y_1 \dot{<} Y_2$, then

$$d_{Y_1}(X, Z) - \kappa \leq d_{Y_1}(Y_0, Y_2) \leq d_{Y_1}(X, Z),$$

  and

$$d_{Y_0}(Y_1, Y_2) \leq \kappa \quad \text{and} \quad d_{Y_2}(Y_1, Y_0) \leq \kappa.$$

Then choosing K larger than Θ, the projection complex $\mathcal{P}_K(\mathbb{Y})$ is defined as follows: it is a graph whose vertices are the elements of $\mathbb{Y}$ and where X and Z span an edge if and only if $\mathbb{Y}_K(X, Z) = \varnothing$. Then [2, Theorem 3.16] states that for sufficiently large K, $\mathcal{P}_K(\mathbb{Y})$ is connected and quasi-isometric to a tree for its path metric.

1.2   Composite projection systems

In this work, we are concerned with a composite situation.

1.2.1 Definitions and projection complexes Let $\mathbb{Y}_*$ be the disjoint union of finitely many countable sets $\mathbb{Y}_1, \ldots, \mathbb{Y}_m$. Their indices $i = 1, \ldots, m$ are called the coordinates. Given $Y \in \mathbb{Y}_*$, denote by $i(Y)$ its coordinate: $Y \in \mathbb{Y}_{i(Y)}$.

Definition 1.2 A composite projection system on a countable set $\mathbb{Y}_* = \bigsqcup_{i=1}^m \mathbb{Y}_i$ is the data of (i) a constant $\theta > 0$, (ii) a family of subsets (the active set for Y) $\mathrm{Act}(Y) \subset \mathbb{Y}_*$, for $Y \in \mathbb{Y}_*$, such that $\mathbb{Y}_{i(Y)} \subset \mathrm{Act}(Y)$, and (iii) a family of functions

$$d_Y^\pi \colon (\mathrm{Act}(Y) \setminus \{Y\} \times \mathrm{Act}(Y) \setminus \{Y\}) \to \mathbb{R}_+$$

satisfying symmetry, the triangle inequality, the Behrstock inequality for θ whenever both quantities are defined, properness for θ when restricted to each $\mathbb{Y}_i$, the separation property for θ, and also three other properties related to the map Act:

- **Symmetry in action** $X \in \mathrm{Act}(Y)$ if and only if $Y \in \mathrm{Act}(X)$.

- **Closeness in inaction**   If $X \notin \mathrm{Act}(Z)$, then for all $Y \in \mathrm{Act}(X) \cap \mathrm{Act}(Z)$, $d_Y^\pi(X, Z) \leq \theta$.

- **Finite filling**   For all $\mathcal{Z} \subset \mathbb{Y}_*$, there is a finite collection of elements X_j in $\mathcal{Z}$ such that $\bigcup_j \mathrm{Act}(X_j)$ covers $\bigcup_{X \in \mathcal{Z}} \mathrm{Act}(X)$.

The closeness in inaction can be understood as a complement to the Behrstock inequality: "if $d_Y^\pi(X, Z) > \theta$, then $d_X^\pi(Y, Z)$ is *defined* and is less than θ".

Applying [2] (as recalled in the previous subsection) we get, for each coordinate $i \leq m$, and for a suitable choice of θ, a modified function $d_Y \colon \mathbb{Y}_i \times \mathbb{Y}_i \to \mathbb{R}_+$. This function is unfortunately not defined on $\mathrm{Act}(Y) \setminus \mathbb{Y}_i$, but d_Y^π is defined on it, and thus we choose to define $d_Y^\lhd(X, Z)$ to be d_Y if both X and Z are in $\mathbb{Y}_i$, and d_Y^π otherwise.

We then define $\mathbb{Y}_M^j(X, Z) = \{Y \in \mathbb{Y}_j \cap \mathrm{Act}(X) \cap \mathrm{Act}(Z) : d_Y^\lhd(X, Z) \geq M\}$. The elements X, Y and Z need not be in the same coordinate.

In the following we first choose θ such that the construction of [2] applies for all coordinates $\mathbb{Y}_i$, and this provides the constants Θ and κ (suitable for all coordinates).

Then we choose $c_* > 1000(\Theta + \kappa)$, and $\Theta_P = c_* + 21m\kappa$. One can choose $K > \Theta_P$ sufficiently large to get quasitrees in all coordinates, but this is not important for us.

Finally, choose $\Theta_{\mathrm{Rot}} > 2c_* + 2\Theta_P + 20(\kappa + \Theta)$ for later purpose.

To keep track of the constants, it is worth keeping in mind that

$$\Theta_{\mathrm{Rot}} \gg 2\Theta_P \gg 2c_* \gg 20(\Theta + \kappa) \gg \theta.$$

1.2.2 Group in the picture

An *automorphism* of a composite projection system is a map $\psi\colon \mathbb{Y}_* \to \mathbb{Y}_*$

- that induces a bijection on each $\mathbb{Y}_i$,
- that sends $\mathrm{Act}(Y)$ to $\mathrm{Act}(\psi(Y))$,
- such that $d_Y^{\sphericalangle}(X, Z) = d_{\psi(Y)}^{\sphericalangle}(\psi(X), \psi(Z))$ for all Y and all $X, Z \in \mathrm{Act}(Y)$.

A *rotation* around $X \in \mathbb{Y}_*$ in a composite projection system $\mathbb{Y}_*$ is an automorphism ψ such that $\psi(X) = X$, and such that for all $Y \in \mathbb{Y}_* \setminus \mathrm{Act}(X)$, and for all $W, Z \in \mathrm{Act}(Y)$, $\psi(Y) = Y$, and $d_Y^{\sphericalangle}(W, Z) = d_Y^{\sphericalangle}(\psi(W), \psi(Z))$.

We assume that a group G acts on the composite projection system by automorphisms.

Let us denote by G_X the stabilizer of $X \in \mathbb{Y}_*$.

We say that a subgroup $\Gamma_X < G_X$ has *proper isotropy* if for all $N > 0$ there is a finite subset $F(N)$ of Γ_X such that if $\gamma \in \Gamma_X \setminus F(N)$, and if $Y \in \mathrm{Act}(X)$, then $d_X^{\pi}(Y, \gamma Y) > N$.

1.2.3 Betweenness and orbit estimates

Lemma 1.3 (betweenness is transitive) If $d_Y^{\sphericalangle}(X, Z) > 2\kappa$ and $d_Z^{\sphericalangle}(Y, T) > 2\kappa$, then Z is in $\mathrm{Act}(X)$ and $d_Z^{\sphericalangle}(X, T) \geq d_Z^{\sphericalangle}(Y, T) - 2\kappa$.

If $d_Y^{\sphericalangle}(X, Z) > 10\kappa$ and $d_Z^{\sphericalangle}(X, T) > 10\kappa$, then $d_Y^{\sphericalangle}(X, T) \geq d_Y^{\sphericalangle}(X, Z) - 2\kappa$.

Proof By the Behrstock inequality, one has $d_Z^{\sphericalangle}(X, Y) \leq \kappa$ in both cases. For the first implication, by the triangle inequality, $d_Z^{\sphericalangle}(X, T) \geq d_Z^{\sphericalangle}(Y, T) - d_Z^{\sphericalangle}(X, Y) - \kappa$.

For the second implication, $d_Z^{\sphericalangle}(Y, T)$ is within 2κ of $d_Z^{\sphericalangle}(X, T)$. The Behrstock inequality gives that $d_Y^{\sphericalangle}(Z, T) \leq \kappa$ and therefore $d_Y^{\sphericalangle}(X, T) \geq d_Y^{\sphericalangle}(X, Z) - 2\kappa$. □

Lemma 1.4 (orbit estimates or transfer in a coordinate) *Assume that Γ_X has proper isotropy. For the finite subset $F = F(10\kappa)$ of Γ_X, for all $Y \in \mathrm{Act}(X)$, all X' in either $\mathrm{Act}(Y)$ or $\mathrm{Act}(X)$ and all $\gamma \in \Gamma_X \setminus F$, either $d_Y^{\sphericalangle}(X', X) \leq \kappa$ or $d_Y^{\sphericalangle}(\gamma X', X) \leq \kappa$.*

Proof Let us first treat the case of $X' \in \mathrm{Act}(Y)$. If $d_Y^{\sphericalangle}(X', X) \leq \kappa$ we are done. Assume that $d_Y^{\sphericalangle}(X', X) > \kappa$. By closeness in inaction, $X' \in \mathrm{Act}(X)$, and by the Behrstock inequality (and because $\kappa \geq \theta$), one has $d_X^{\sphericalangle}(X', Y) \leq \kappa$. By proper isotropy (and the coarse triangle inequality), $d_X^{\sphericalangle}(\gamma X', Y) > 5\kappa$. Thus, by the Behrstock inequality again, $d_Y^{\sphericalangle}(\gamma X', X) \leq \kappa$.

Now assume that X' is not in $\mathrm{Act}(Y)$ but is in $\mathrm{Act}(X)$. Since $Y \in \mathrm{Act}(X)$ we can measure $d_X^{\vartriangleleft}(X', Y)$ and (since Γ_X preserves $\mathrm{Act}(X)$) also $d_X^{\vartriangleleft}(\gamma X', Y)$. By proper isotropy, $d_X^{\vartriangleleft}(X', \gamma X') \geq 10\kappa$ and therefore at least one of the quantities $d_X^{\vartriangleleft}(X', Y)$ and $d_X^{\vartriangleleft}(\gamma X', Y)$ is larger than 4κ. Assume for instance that $d_X^{\vartriangleleft}(X', Y) \geq 4\kappa$. Then by the Behrstock inequality, $d_Y^{\vartriangleleft}(X', X) \leq \kappa$.   □

To simplify notation, we will say that a property is true for almost all elements of a group if the property holds for all elements outside a certain finite subset of the group. Using Lemma 1.4 four times, together with the triangle inequality, one gets:

Lemma 1.5 (orbit estimates for proper isotropy) *Let X_1, X_2, X_1' and X_2' be such that $X_1, X_2 \in \mathrm{Act}(Y)$. Assume that X_i' is in either $\mathrm{Act}(Y)$ or $\mathrm{Act}(X_i)$.*

If the groups Γ_{X_1} and Γ_{X_2} have proper isotropy, then for almost all elements $\gamma_1 \in \Gamma_{X_1}$ and $\gamma_2 \in \Gamma_{X_2}$, one has

$$d_Y^{\vartriangleleft}(\gamma_1(X_1'), \gamma_2(X_2')) - 4\kappa \leq d_Y^{\vartriangleleft}(X_1, X_2) \leq d_Y^{\vartriangleleft}(\gamma_1(X_1'), \gamma_2(X_2')) + 4\kappa.$$

Recall that we chose $K > 2\Theta + \kappa$.

Proposition 1.6 (ellipticity) *Given $X \in \mathbb{Y}_*$, and any $j \leq m$, the group G_X has an orbit in $\mathcal{P}_K(\mathbb{Y}_j)$ of diameter at most 1.*

Proof If $j = i(X)$, and more generally, if G_X fixes an element $Y \in \mathbb{Y}_j$, the result is obvious. Assume then that $\mathbb{Y}_j \subset \mathrm{Act}(X)$.

The group G_X preserves the set $\{Z \in \mathbb{Y}_j : \mathbb{Y}_{K_0}^j(X, Z) = \varnothing\}$ for any K_0 hence for $K_0 = (K - \kappa)/2 \geq \Theta$. Consider Z_a and Z_b in this set. We claim that $\mathbb{Y}_K^j(Z_a, Z_b)$ is empty. Assume $Y \in \mathbb{Y}_K^j(Z_a, Z_b)$. Since $Y \in \mathrm{Act}(X)$ we can consider $d_Y^{\vartriangleleft}(Z_a, X)$ and $d_Y^{\vartriangleleft}(Z_b, X)$. By the triangle inequality, $d_Y^{\vartriangleleft}(Z_a, X) + d_Y^{\vartriangleleft}(Z_b, X) \geq d_Y^{\vartriangleleft}(Z_a, Z_b) - \kappa \geq K - \kappa$. Thus, one of them needs to be larger than $(K - \kappa)/2$, hence Y is either in $\mathbb{Y}_{K_0}(X, Z_a)$ or in $\mathbb{Y}_{K_0}(X, Z_b)$, and this is a contradiction to our assumption.   □

Proposition 1.7 (induced orders) *Consider $X, Z \in \mathbb{Y}_*$, with $Z \in \mathrm{Act}(X)$. Assume that Γ_X and Γ_Z are infinite subgroups of G_X and G_Z with proper isotropy.*

For all $i \leq m$, for all $M \geq \Theta + 12\kappa$, the set $\mathbb{Y}_M^i(X, Z)$ is finite, and carries a partial order $\doteq$ that is given by the order of $\mathbb{Y}_{M-4\kappa}^i(\gamma_X(X^i), \gamma_Z Z^i)$, for arbitrary X^i and Z^i, in $\mathbb{Y}_i$, and almost all $\gamma_X \in \Gamma_X$ and $\gamma_Z \in \Gamma_Z$.

Proof Let us first check that the set is finite. We may assume that there are $X^i \in \mathrm{Act}(X) \cap \mathbb{Y}_i$ and $Z^i \in \mathrm{Act}(Z) \cap \mathbb{Y}_i$, otherwise $\mathbb{Y}_M^i(X, Z)$ is empty. By Lemma 1.4,

there exist $\gamma_X \in \Gamma_X$ and $\gamma_Z \in \Gamma_Z$ such that each $Y \in \mathbb{Y}^i_M(X,Z)$ is in one of the four sets $\mathbb{Y}^i_{M-3\kappa}(\eta_X X^i, \eta_Z Z^i)$, for $\eta_X \in \{1, \gamma_X\}$ and $\eta_Z \in \{1, \gamma_Z\}$. The union of these four sets is finite by the properness axiom.

We now need to check that the order on $\mathbb{Y}^i_{M-4\kappa}(\gamma_X(X^i), \gamma_Z Z^i)$ includes all $\mathbb{Y}^i_M(X,Z)$ and does not depend on the choice of the points X^i and Z^i. By Lemma 1.5, for arbitrary choice of points, and for any $Y \in \mathbb{Y}^i_M(X,Z)$, there is a finite set of Γ_X and of Γ_Z such that for all elements γ_X and γ_Z outside these finite sets, $Y \in \mathbb{Y}^i_{M-4\kappa}(\gamma_X X^i, \gamma_Z Z^i)$ (the finite sets depend on the choice of X^i and Z^i though). Since $\mathbb{Y}^i_M(X,Z)$ is finite, we may find a finite set of Γ_X and Γ_Y suitable for all of them. Thus, for almost all γ_X and γ_Z, all $\mathbb{Y}^i_{M-4\kappa}(\gamma_X(X^i), \gamma_Z(Z^i))$ is ordered, and the order, once the points X^i and Z^i are chosen, does not depend on γ_X and γ_Z.

Assume that for two different choices of points X^i and Z^i, namely (X^i_a, Z^i_a) and (X^i_b, Z^i_b), the orders are different, and take Y_1 and Y_2 such that $Y_1 \dot{<}_a Y_2$ for the first order, and $Y_2 \dot{<}_b Y_1$ for the other.

$Y_1 \dot{<}_a Y_2$ means that $d_{Y_1}(Y_2, \gamma_Z(Z^i_a)) \le \kappa$. By the orbit estimate, $d^{\vartriangleleft}_{Y_1}(Y_2, Z) \le 5\kappa$ for suitable γ_Z.

$Y_2 \dot{<}_b Y_1$ means that $d_{Y_1}(Y_2, \gamma_X(X^i_b)) \le \kappa$. By the orbit estimate, $d^{\vartriangleleft}_{Y_1}(Y_2, X) \le 5\kappa$. Finally, by the coarse triangle inequality, $d^{\vartriangleleft}_{Y_1}(Z, X) \le 11\kappa$, contradicting the assumption that Y_1 is in $\mathbb{Y}^i_M(X,Z)$. $\qquad\square$

1.3 Convexity

Definition 1.8 (convexity) Let $L > 10\kappa$. We say that a subset $\mathcal{W} \subset \mathbb{Y}_*$ is L–convex if for all i, all $X, Z \in \mathcal{W} \cap \mathbb{Y}_i$ and all j, the set $\mathbb{Y}^j_L(X,Z)$ is a subset of $\mathcal{W}$.

Let now $\mathcal{L} = (L(1), \dots, L(m))$ be an m–tuple of positive numbers. A subset $\mathcal{W}$ of $\mathbb{Y}_*$ is called $\mathcal{L}$–convex if for all $X, Z \in \mathcal{W}$ with the same coordinate $i(X) = i(Z)$ and for all j, the set $\mathbb{Y}^j_{L(j)}(X,Z)$ is a subset of $\mathcal{W}$.

Note that being L–convex, for $L > 0$, is equivalent to being $(L, \dots, L)$–convex.

Definition 1.9 Let $\mathcal{W} \subset \mathbb{Y}_*$ nonempty, and $R \in \mathbb{Y}_* \setminus \mathcal{W}$ for which $\mathrm{Act}(R) \cap \mathcal{W}$ is nonempty. Let $L \ge 10\kappa$. Define $\mathbb{Y}_L(\mathcal{W}, R)$ as the set of $Y \in \mathbb{Y}_*$ satisfying

- $Y \in \mathrm{Act}(R)$,
- $Y \notin \mathcal{W}$,
- $\mathcal{W} \cap \mathrm{Act}(R) \cap \mathrm{Act}(Y) \ne \varnothing$,
- $Y \in \mathbb{Y}^{i(Y)}_L(X, R)$ for all $X \in \mathcal{W} \cap \mathrm{Act}(R) \cap \mathrm{Act}(Y)$.

Proposition 1.10 *Assume for each $X \in \mathcal{W}$ that $\mathcal{W}$ is invariant by an infinite group Γ_X of rotations around X, with proper isotropy. If $L \geq \Theta + 12\kappa$, then for all R for which it is defined, the set $\mathbb{Y}_L(\mathcal{W}, R)$ is finite.*

Proof From the definition, $\mathbb{Y}_L(\mathcal{W}, R) \subset \bigcup_i \bigcap_{X \in \mathrm{Act}(R) \cap \mathcal{W}} \left(\mathbb{Y}_L^i(X, R) \cup (\mathbb{Y}_i \setminus \mathrm{Act}(X)) \right)$. By the finite filling assumption on the projection system, there is a finite collection of elements $X_j \in \mathcal{W} \cap \mathrm{Act}(R)$ such that $\bigcup_j \mathrm{Act}(X_j)$ covers $\bigcup_{\mathcal{W} \cap \mathrm{Act}(R)} \mathrm{Act}(X)$.

In particular, $\mathbb{Y}_L(\mathcal{W}, R)$ is inside a finite union of sets of the form $\mathbb{Y}_L^i(X_j, R)$, which are finite by Proposition 1.7. □

Proposition 1.11 *Assume for each $X \in \mathcal{W}$ that $\mathcal{W}$ is invariant by an infinite group Γ_X of rotations around X, with proper isotropy. Let $L \geq \Theta + 12\kappa$.*

If $\mathcal{W}$ is $(L - 6\kappa)$–convex and $S \in \mathbb{Y}_L(\mathcal{W}, R)$, then $\mathbb{Y}_L(\mathcal{W}, S) \subset \mathbb{Y}_{L-2\kappa}(\mathcal{W}, R)$.

Moreover, if $\mathcal{W}'$ contains $\mathcal{W}$, then $\mathbb{Y}_L(\mathcal{W}', R) \subset \mathbb{Y}_L(\mathcal{W}, R)$.

Proof Let $Y \in \mathbb{Y}_L(\mathcal{W}, S)$ in coordinate i. There exists $X \in \mathcal{W} \cap \mathrm{Act}(Y) \cap \mathrm{Act}(S)$ such that $d_Y^{\triangleleft}(X, S) \geq L$.

Assume that $\widetilde{X} \in \mathrm{Act}(R) \cap \mathrm{Act}(Y) \cap \mathcal{W}$. If it is not in $\mathrm{Act}(S)$, then $d_Y^{\triangleleft}(\widetilde{X}, S) < \kappa$ and $d_Y^{\triangleleft}(\widetilde{X}, X) > L - 2\kappa$. Transferring $\widetilde{X}$ in the coordinate of X (by invariance under $\Gamma_{\widetilde{X}}$), one has $d_Y^{\triangleleft}(\widetilde{X}_{i(X)}, X) > L - 6\kappa$. By convexity, $Y \in \mathcal{W}$ though we assumed otherwise. Therefore, $\widetilde{X} \in \mathrm{Act}(S)$. Therefore, by definition of $\mathbb{Y}_L(\mathcal{W}, S)$, one has $d_Y^{\triangleleft}(\widetilde{X}, S) \geq L$, but also $d_S^{\triangleleft}(\widetilde{X}, R) \geq L$. It follows by transitivity of betweenness (Lemma 1.3) that $d_Y^{\triangleleft}(\widetilde{X}, R) \geq L - 2\kappa$.

The second assertion is a direct consequence of the definition. □

Proposition 1.12 *Assume for each $X \in \mathcal{W}$ that $\mathcal{W}$ is invariant by an infinite group Γ_X of rotations around X, with proper isotropy. If $\mathrm{Act}(R) \cap \mathcal{W}$ is not empty, then for all $L \geq (2m + 12)\kappa + \Theta$, there exists $Z \in \mathbb{Y}_L(\mathcal{W}, R)$ such that $\mathbb{Y}_{L-2m\kappa}(\mathcal{W}, Z) = \varnothing$.*

Proof Let us say that R has k L–links to $\mathcal{W}$ if $\{i : \mathbb{Y}_L(R, \mathcal{W}) \cap \mathbb{Y}_i \neq \varnothing\}$ has k elements.

For any such index i, take a minimal item Z_i in $\mathbb{Y}_L(R, \mathcal{W}) \cap \mathbb{Y}_i$ for the order of Proposition 1.7. Then, by Proposition 1.11, $\mathbb{Y}_{L-2\kappa}(\mathcal{W}, Z_i)$ is included in $\mathbb{Y}_L(R, \mathcal{W})$; thus Z_i has at most $(k - 1)$ $(L - 2\kappa)$–links to $\mathcal{W}$.

Iterating this choice at most m times, we find an element Z that has no $(L - 2m\kappa)$–links to $\mathcal{W}$. Therefore $\mathbb{Y}_{L-2m\kappa}(\mathcal{W}, Z) = \varnothing$. □

Proposition 1.13 *Let $L \geq \Theta + 12\kappa$. Consider $\mathcal{W}$, and assume it is L–convex and that for all $X \in \mathcal{W}$, there is a $\Gamma_X < G_X$, infinite, that leaves $\mathcal{W}$ invariant and that has proper isotropy.*

If $\mathbb{Y}_{L'}(\mathcal{W}, R)$ is well defined and empty, then $\mathcal{W} \cup \{R\}$ is $(L+L'+5\kappa)$–convex.

Proof If $\mathcal{W} \cap \mathbb{Y}_{i(R)}$ is empty, there is nothing to prove. We assume it is nonempty. Consider $Y \in \mathbb{Y}_{L+L'+5\kappa}(R, X)$ for some $X \in \mathcal{W} \cap \mathbb{Y}_{i(R)}$, and assume that $Y \notin \mathcal{W}$. Notice that $Y \in \mathrm{Act}(R)$ though, and $X \in \mathrm{Act}(R)$ since they have same coordinate. Hence, $X \in \mathcal{W} \cap \mathrm{Act}(R) \cap \mathrm{Act}(Y)$.

Let X' be any other element of $\mathcal{W} \cap \mathrm{Act}(R) \cap \mathrm{Act}(Y)$. Transfer X' in the coordinate $i = i(R)$, inside $\mathcal{W}$, by $\Gamma_{X'}$. There exists $X_i' \in \mathbb{Y}_i \cap \mathcal{W}$ such that $d_Y^{\preceq}(X', X_i') \leq \kappa$. But, $\mathcal{W}$ being L–convex, one has $d_Y^{\preceq}(X', X_i') \leq L$. It follows by the triangle inequality that $d_Y^{\preceq}(R, X') \geq L' + 2\kappa$. Since this is true for all X' as above, it follows that $Y \in \mathbb{Y}_{L'+2\kappa}(\mathcal{W}, R)$, contradicting our assumption. □

2 Composite rotating families and windmills

We proceed to adapt the rotating families study of [6] to the context of composite projection systems.

2.1 Definition of composite rotating family and of composite windmill

Definition 2.1 A composite rotating family on a composite projection system, endowed with an action of a group G by isomorphisms, is a family of subgroups Γ_Y, for $Y \in \mathbb{Y}_*$, such that:

- For all $X \in \mathbb{Y}_*$, the subgroup $\Gamma_X < G_X = \mathrm{Stab}_G(X)$ is an infinite group of rotations around X, with proper isotropy.
- For all $g \in G$, and all $X \in \mathbb{Y}_*$, one has $\Gamma_{gX} = g\Gamma_X g^{-1}$.
- If $X \notin \mathrm{Act}(Z)$ then Γ_X and Γ_Z commute.
- For all i, for all $X, Y, Z \in \mathbb{Y}_i$, if $d_Y(X, Z) \leq \Theta_P$ then for all $g \in \Gamma_Y \setminus \{1\}$, $d_Y(X, gZ) \geq \Theta_{\mathrm{Rot}}$.

We will show the following:

Theorem 2.2 *Consider $\mathbb{Y}_*$ a composite projection system. If $\{\Gamma_Y : Y \in \mathbb{Y}_*\}$ is a composite rotating family for sufficiently large Θ_{Rot}, then the group Γ_{Rot} generated by $\bigcup_{Y \in \mathbb{Y}_*} \Gamma_Y$ has a partially commutative presentation.*

More precisely, two presentations of Γ_{Rot} are

$$\Gamma_{\mathrm{Rot}} \simeq \left\langle \bigcup_{Y \in \mathbb{Y}_*} \Gamma_Y \;\middle|\; [\Gamma_Y, \Gamma_{Y'}] = 1 \text{ for all } Y \in \mathbb{Y}_* \text{ and } Y' \notin \mathrm{Act}(Y), \right.$$
$$\left. \Gamma_{gY} = g \Gamma_Y g^{-1} \text{ for all } Y \in \mathbb{Y}_* \text{ and } g \in \Gamma_{\mathrm{Rot}} \right\rangle$$

and, for a certain $\mathcal{S} \subset \mathbb{Y}_*$,

$$\Gamma_{\mathrm{Rot}} \simeq \left\langle \bigcup_{Y \in \mathcal{S}} \Gamma_Y \;\middle|\; [s, ws'w^{-1}] = 1 \text{ for all } Y, Y' \in \mathcal{S}, \; w/Y \notin \mathrm{Act}(wY'), \; s \in Y, \; s' \in Y' \right\rangle.$$

In these presentations, we consider implicit the relations of the groups Γ_Y that appear in the generating sets. Moreover the expression $\Gamma_{gY} = g\Gamma_Y g^{-1}$ refers to the following precise collection of formal relations: for all γ in Γ_Y, for all $g \in \Gamma_{\mathrm{Rot}}$, given the element $\gamma' \in \Gamma_{gY}$ equal to $g\gamma g^{-1}$ (which exists by definition of composite rotating family), we add the relation $(\gamma')^{-1} g\gamma g^{-1} = 1$. It is somewhat tautological, but necessary, in a presentation over this generating set. The point of the second presentation is to avoid these tautological relations by reducing the generating set to a certain set of representatives of conjugacy classes of groups Γ_Y.

Unfortunately, it is not so easy to describe a priori the subset $\mathcal{S}$. It is constructed recursively in a number of steps, by taking at each step orbit representatives of a certain subset of $\mathbb{Y}_*$ under the action of the group generated by the Γ_Y that have been collected so far in the process. In principle, it probably can be enumerated explicitly, but at the cost of a certain complexification of the exposition.

The following result is, in our point of view, an incarnation of the Greendlinger lemma, from the small cancellation theories. If one considers a relation γ of the quotient group, one can find in it a large part of a defining relation γ_s. Compare to [6, Section 5.1.3].

Let us consider Γ_{Rot} as in the previous theorem, and $\gamma \in \Gamma_{\mathrm{Rot}}$. A principal coordinate for γ is a coordinate $i \leq m$ for which $d_R(X, \gamma X) > \Theta_{\mathrm{Rot}} - 2\Theta_P - \kappa$ for all $X \in \mathbb{Y}_i$ (the constants are somewhat ad hoc, chosen for the counting arguments to flow properly). In this case, a shortening pair (R, γ_s) for γ in a principal coordinate i, at $X \in \mathbb{Y}_i$, is a pair consisting of a element R of $\mathbb{Y}_i$, and of an element $\gamma_s \in \Gamma_R$ such that $d_R(X, \gamma_s \gamma X) \leq 2\Theta_P + 3\kappa$.

Theorem 2.3 *Consider $\mathbb{Y}_*$ a composite projection system. If $\{\Gamma_Y : Y \in \mathbb{Y}_*\}$ is a composite rotating family for sufficiently large Θ_{Rot}, let Γ_{Rot} be the group generated by $\bigcup_{Y \in \mathbb{Y}_*} \Gamma_Y$.*

Then for all $\gamma \in \Gamma_{\mathrm{Rot}} \setminus \{1\}$, there is a principal coordinate $i(\gamma) \le m$ for γ and a shortening pair (R, γ_s) for γ in that coordinate.

A major tool for analyzing rotating families is the concept of windmills. We are going to use composite windmills.

Let us fix $\mathcal{L}$ the m–tuple

$$\mathcal{L} = (c_* + 20(m-1)\kappa, \ c_* + 20(m-2)\kappa, \ \ldots, \ c_* + 20\kappa, \ c_*).$$

Let σ be the cyclic shift $\sigma(i) = (i-1)$ on $\mathbb{Z}/m\mathbb{Z}$, and define $\mathcal{L}_j = \sigma^{j-1}(\mathcal{L})$ to be obtained by shifting the coordinates of the m–tuple.

Thus $\mathcal{L}_i$ reaches its maximum $c_* + 20(m-1)\kappa$ at the coordinate i and minimum c_* at $i-1$. Note that the maximum of $\mathcal{L}$ is less than $\Theta_P - \kappa$.

Definition 2.4   A composite windmill is a collection $(\mathcal{W}_1, \ldots, \mathcal{W}_m, G_W, j_0)$ in which:

- G_W is the subgroup of G generated by a set of subgroups $\{\Gamma_Y : Y \in \bigcup_{i \in I_*} \mathcal{W}_i\}$ for I_* either $\{1, \ldots, m\}$ or $\{1, \ldots, m\} \setminus \{j_0\}$.

- $\mathcal{W}_i$ is a subset of $\mathbb{Y}_i$ for all i, invariant under G_W.

- j_0 satisfies $1 \le j_0 \le m$ and is called the principal coordinate.

- $\bigcup_i \mathcal{W}_i$ is $\mathcal{L}_{j_0}$–convex.

- The group G_W has a partially commutative presentation, that is, a presentation of the form

$$G \simeq \langle \mathcal{S} \mid \mathcal{R} \rangle,$$

   where $\mathcal{S}$ is the union over a subset $\mathcal{W}_*$ of $\mathcal{W}$ of generating sets for Γ_X, $X \in \mathcal{W}_*$, and $\mathcal{R}$ consists of words over the alphabet $\mathcal{S} \cup \mathcal{S}^{-1}$ of the form $[s, ws'w^{-1}]$ for w a word over $\mathcal{S} \cup \mathcal{S}^{-1}$. Moreover, if $X, X' \in \mathcal{W}_*$, $s \in \Gamma_X$ and $s' \in \Gamma_{X'}$, the word $[s, ws'w^{-1}]$ is in $\mathcal{R}$ if and only if $wX' \notin \mathrm{Act}(X)$.

- (Greendlinger property) For each $\gamma \in G_W$ there is an $i(\gamma) \le m$ such that for all $X \in \mathcal{W}_{i(\gamma)}$, either $\gamma \in \Gamma_X$ or there is an $R \in \mathcal{W}_{i(\gamma)}$ such that $d_R(X, \gamma X) > \Theta_{\mathrm{Rot}} - 2\Theta_P - \kappa$. Moreover, there is a $\gamma_s \in \Gamma_R$ such that $d_R(X, \gamma_s \gamma X) \le 2\Theta_P + 3\kappa$ (the pair (R, γ_s) is called a shortening pair for γ at X).

We say that the composite windmill has full group if G_W is the subgroup of G generated by $\{\Gamma_Y : Y \in \bigcup_{i=1}^{m} \mathcal{W}_i\}$.

If we do not mention it, our windmills will be full. Only in specific circumstances do we need nonfull windmills. Indeed, we will use the case of a nonfull group only at most once by coordinate, when initiating the process in each coordinate.

Proposition 2.5 *In a composite windmill $\mathcal{W}$, for all i such that $\mathcal{W}_i$ is nonempty, $\mathcal{W}_i$ is connected in $\mathcal{P}_K(\mathbb{Y}_i)$.*

Proof Consider two points X and X' in $\mathcal{W}_i$. By [2, Theorem 3.7] (more precisely the first claim in its proof), there exists a path $X_1, \ldots, X_n = X'$ between them such that $X_j \in \mathbb{Y}_K^i(X, X')$ for each j. Since $K > \max(\mathcal{L})$, it follows that each X_j is in $\mathcal{W}_i$. □

We say that a windmill $\mathcal{W}'$ (with its representative set $\mathcal{W}'_*$ used for the presentation of the definition) is *constructed over* $\mathcal{W}$ if $\mathcal{W} \subset \mathcal{W}'$ and if the set of representatives $\mathcal{W}'_*$ contains the set of representatives $\mathcal{W}_*$. Note that this is transitive: if $\mathcal{W}''$ is constructed over $\mathcal{W}'$, and $\mathcal{W}'$ is constructed over $\mathcal{W}$, then $\mathcal{W}''$ is constructed over $\mathcal{W}$.

2.2 Osculations of two kinds

We define osculators of two types:

- An osculator of type *gap* of a composite windmill $(\mathcal{W}_1, \ldots, \mathcal{W}_m, G_W, j_0)$ is an element R of $\mathbb{Y}_{j_0} \setminus \mathcal{W}_{j_0}$ such that there exists $i \leq m$ and $X_i, Z_i \in \mathcal{W}_i$ that are in $\mathrm{Act}(R)$ such that $d_R^\lhd(X_i, Z_i) > c_*/2 - 20\kappa$.

- An osculator of type *neighbor* of a composite windmill $(\mathcal{W}_1, \ldots, \mathcal{W}_m, G_W, j_0)$ is an element R of $\mathbb{Y}_{j_0} \setminus \mathcal{W}_{j_0}$ such that $\mathbb{Y}_{c_*/2}(\mathcal{W}, R) = \varnothing$.

Lemma 2.6 *Consider a composite windmill $\mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_m, G_W, j_0)$, assume that $\mathcal{W}_{j_0} \neq \varnothing$, and let $R \in \mathbb{Y}_{j_0}$ be an osculator of type gap.*

Let $Y \in \mathbb{Y}_i$ be in $\mathrm{Act}(R)$. Then there exists $X \in \mathcal{W}_{j_0}$ such that $d_Y^\lhd(X, R) \leq \kappa$.

Proof If R is an osculator of type gap, there are $X', Z' \in \mathcal{W}_i$, for some i, such that $d_R^\lhd(X', Z') > c_*/2 - 20\kappa$.

Let $X_0 \in \mathcal{W}_{j_0}$, and consider its orbit under the groups $\Gamma_{X'}$ and $\Gamma_{Z'}$, which preserve $\mathcal{W}_{j_0}$. We may use Lemma 1.5 to find $X'^{(j_0)}$ and $Z'^{(j_0)}$ in these orbits, hence in $\mathcal{W}_{j_0}$, such that $d_R^\lhd(X'^{(j_0)}, Z'^{(j_0)}) > c_*/2 - 24\kappa$.

By the coarse triangle inequality, for at least one point among $X'^{(j_0)}$ and $Z'^{(j_0)}$, say $X'^{(j_0)}$, we have $d_R^\lhd(Y, X'^{(j_0)}) > c_*/4 - 13\kappa$. The Behrstock inequality gives $d_Y^\lhd(R, X'^{(j_0)}) \leq \kappa$. □

Lemma 2.7 *Let $\mathcal{W}$ be a composite windmill, and let R_1 and R_3 be two osculators of $\mathcal{W}$. Assume $\mathcal{W}_{j_0} \neq \varnothing$, and let $X_2 \in \mathcal{W}_{j_0}$.*

If R_3 is of type neighbor and $\mathcal{W}$ is $((c_*/2)-20\kappa)$–convex, then $d_{R_1}(X_2, R_3) \leq c_*$.

If R_3 is of type gap, then $d_{R_1}(X_2, R_3) \leq \Theta_P$.

Proof For R_3 an osculator of neighbor type, the result follows from Proposition 1.13.

If R_3 is an osculator of type gap, the proof is slightly more involved. There is an i, and there are $X, Z \in \mathcal{W}_i$ such that $d_{R_3}^{\triangleleft}(X, Z) > c_*/2 - 20\kappa$.

Since $\mathcal{W}_{j_0}$ is nonempty, and invariant for Γ_X and Γ_Z, we can apply Lemma 1.5 and find $X^{(j_0)}, Z^{(j_0)} \in \mathcal{W}_{j_0}$ such that $d_{R_3}(X^{(j_0)}, Z^{(j_0)}) \geq d_{R_3}^{\triangleleft}(X, Z) - 4\kappa$, which is at least $c_*/2 - 24\kappa$. By the coarse triangle inequality, at least one of the quantities $d_{R_3}(R_1, X^{(j_0)})$ and $d_{R_3}(R_2, Z^{(j_0)})$ is greater than $c_*/4 - 13\kappa$. Say it is $d_{R_3}(R_1, X^{(j_0)})$. The Behrstock inequality then gives that $d_{R_1}(R_3, X^{(j_0)}) \leq \kappa$, and again the coarse triangle inequality gives $d_{R_1}(X^{(j_0)}, X_2) \geq d_{R_1}(X_2, R_3) - \kappa$. Since the first is bounded by the maximal convexity constant of $\mathcal{W}$, the result follows. $\qquad\square$

2.3 The unfolding in the different coordinates

Given a composite windmill $\mathcal{W}$, we will define its unfolding.

Observe first the following, which justifies the subsequent definition of admissible set of osculators.

Lemma 2.8 *If $\mathcal{W}$ is a composite windmill, it has some gap osculator if and only if it is not $((c_*/2)-20\kappa)$–convex.*

Assume that for all $R \in \mathbb{Y}_*$, $\mathrm{Act}(R) \cap \mathcal{W} \neq \varnothing$. If $\mathcal{W}$ is $((c_*/2)-20\kappa)$–convex, and yet does not contain $\mathbb{Y}_*$, then there exists a neighbor osculator.

Proof The first assertion is direct from the definitions. To prove the second assertion, take $X \notin \mathcal{W}$. By Proposition 1.12 there is a Z in $\mathbb{Y}_{(c_*/2)+2m\kappa}(\mathcal{W}, X) \cup \{X\}$ such that $\mathbb{Y}_{c_*/2}(\mathcal{W}, Z) = \varnothing$. It is therefore a neighbor osculator of $\mathcal{W}$. $\qquad\square$

We define now admissible sets of osculators of a composite windmill $\mathcal{W}$ that does not cover the entire set $\mathbb{Y}_*$.

If $\mathcal{W}$ is not $((c_*/2)-20\kappa)$–convex, then the (only) admissible set of osculators for $\mathcal{W}$ is the set $\mathcal{R}_{\mathrm{gap}}$ of osculators of type gap in $\mathbb{Y}_{j_0}$. Note that it can be the empty set if the gap osculators are not in the coordinate j_0.

If $\mathcal{W}$ is $((c_*/2)-20\kappa)$–convex (but does not cover the entire set $\mathbb{Y}_*$), then an admissible set of osculators for $\mathcal{W}$ is a set $\mathcal{R} = \{G_W R\}$ for a choice of an osculator R (necessarily of type neighbor).

We define the unfolding of $\mathcal{W}$ as follows.

Definition 2.9 (unfolding) Let $\mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_m, G_W, j_0)$ be a composite windmill that does not contain the entire set $\mathbb{Y}_*$, and let $\mathcal{R}$ be an admissible set of osculators. For each i, define $\mathcal{W}_i'$ to be the union of all the images of $\mathcal{W}_i$ by elements of the group $G_{W'}$ generated by $G_W \cup \{\bigcup_{\mathcal{R}} \Gamma_R\}$. Then $(\mathcal{W}_1', \ldots, \mathcal{W}_m', G_{W'}, j_0 + 1)$, where $j_0 + 1$ is taken modulo m, is the unfolding of $\mathcal{W}$.

If $\mathcal{W}$ contains $\mathbb{Y}_*$, its unfolding is $\mathcal{W}' = \mathcal{W}$.

Here is an obvious lemma:

Lemma 2.10 (trivial unfolding) *Let $\mathcal{R}$ be a choice of an admissible set of osculators of $\mathcal{W}$. If $\mathcal{R}$ is empty, then the unfolding $\mathcal{W}' = (\mathcal{W}_1, \ldots, \mathcal{W}_m, G_W, j_0 + 1)$ is a composite windmill.*

We thus concentrate on the case where $\mathcal{R}$ is nonempty.

In the case where $\mathcal{W}_{j_0}$ is empty, we include here a convexity result for an intermediate step in the construction: adding an admissible set of osculators $\mathcal{R}$, which produces a nonfull composite windmill.

Lemma 2.11 *Assume that $\mathcal{W}$ is a full composite windmill of principal coordinate j_0, with $\mathcal{W}_{j_0} = \varnothing$.*

Let $\mathcal{W}_{j_0}^s$ be a set $\mathcal{R}$ of admissible osculators as defined above, assumed nonempty.

For all other coordinates, let $\mathcal{W}_i^s = \mathcal{W}_i$.

Then $\mathcal{W}^s = (\mathcal{W}_1^s, \mathcal{W}_2^s, \ldots, \mathcal{W}_m^s, G_W, j_0)$ is a nonfull composite windmill of principal coordinate j_0. If moreover $\mathcal{R}$ is the orbit of a neighbor osculator, and if $\mathcal{W}$ is $((c_*/2)-20\kappa)$–convex, then $\mathcal{W}^s$ is B–convex, for $B = c_*/2 + 10\kappa \leq \inf \mathcal{L}$.

Proof If $\mathcal{R} = \varnothing$, there is nothing to prove. Consider the case of the orbit of a neighbor osculator. It suffices to check that $\mathcal{W}_{j_0}^s (= G_W R)$ is convex in the sense that for all $\gamma \in G_W$ and all i the set $\mathbb{Y}_B^i(R, \gamma R)$ is in $\mathcal{W}_i$.

By the Greendlinger property, given γ, there exists a j and $Y_j \in \mathcal{W}_j$ such that $d_{Y_j}(R, \gamma R) > \Theta_{\mathrm{Rot}} - 2\Theta_P - \kappa$, or $R = \gamma R$ (if R is not active for all the shortening pairs of γ).

Of course we consider only the first case of the alternative.

Assume that some $Y \in \mathbb{Y}_i$ is in $\mathbb{Y}_B^{(i)}(R, \gamma R)$.

If $Y \notin \mathrm{Act}(Y_i)$, then one can use a shortening pair at Y_i to reduce the length of γ in its principal coordinate, and this shortening pair gives γ' such that $d_Y^{\preceq}(R, \gamma R) = d_Y^{\preceq}(R, \gamma' R)$. Thus, $Y \in \mathbb{Y}_B^{(i)}(R, \gamma' R)$ as well, and by performing this reduction sufficiently many times, we may assume that $Y \in \mathrm{Act}(Y_i)$.

By Lemma 1.4, either R or γR approximates by κ the projection of Y_j on Y.

Say that $d_Y^{\preceq}(\gamma R, Y_j) \leq \kappa$. By osculation, if $Y \notin \mathcal{W}_i$, one has $d_Y^{\preceq}(Y_j, R) \leq c_*/2$. Therefore, one has $d_Y^{\preceq}(\gamma R, R) \leq d_Y^{\preceq}(\gamma R, Y_j) + d_Y^{\preceq}(R, Y_j) + \kappa \leq c_*/2 + 2\kappa$, which is less than B.

If $d_Y^{\preceq}(R, Y_j) \leq \kappa$, one has $d_Y^{\preceq}(R, \gamma R)$ is within 2κ of $d_Y^{\preceq}(Y_j, \gamma R)$, which equals $d_{\gamma^{-1}Y}^{\preceq}(\gamma^{-1}Y_j, R)$. Of course, $Y \notin \mathcal{W}_i$ if and only if $\gamma^{-1}Y \notin \mathcal{W}_i$; hence, if this is the case, then $d_Y^{\preceq}(Y_j, \gamma R) \leq c_*/2$ and $d_Y^{\preceq}(\gamma R, R) \leq c_*/2 + 2\kappa \leq B$, by osculation of R.

In the case where $\mathcal{R}$ is the set of gap osculators, the proof is similar. Indeed, if R_1 is a gap between X_1 and Z_1, and R_2 is a gap between X_2 and Z_2, and if Y is between R_1 and R_2, so that $d_Y^{\preceq}(R_1, R_2) \geq c_* + 20(m-1)\kappa \; (= \mathcal{L}_{j_0}(j_0))$, then Y is also between X_1 (or Y_1) and X_2 (or Y_2), so that, say, $d_Y^{\preceq}(X_1, X_2) \geq c_* + 20(m-1)\kappa - 3\kappa$. One can transfer X_2 in the coordinate of X_1, by Lemma 1.4, in $\mathcal{W}$ (in the Γ_{X_2}–orbit of X_1). The convexity of $\mathcal{W}$ then shows that $Y \in \mathcal{W}$. $\qquad\square$

The aim of the remainder of Section 2.3 is to prove the following:

Proposition 2.12 *If $\mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_m, G_{W'}, j_0)$ is a (full) composite windmill and $\mathcal{R}$ an admissible set of osculators, then the unfolding $\mathcal{W}' = (\mathcal{W}'_1, \ldots, \mathcal{W}'_m, G_{W'}, j_0 + 1)$ is a (full) composite windmill, and $\mathcal{W}'_*$ can be chosen to contain $\mathcal{W}_*$ (in other words, $\mathcal{W}'$ is constructed over $\mathcal{W}$).*

2.3.1 Unfolding a tree

Proposition 2.13 (principal coordinate tree) *Consider a full composite windmill $\mathcal{W}$, of principal coordinate j_0.*

Let $\mathcal{R} \neq \varnothing$ be an admissible set of osculators as defined in the previous section. If $\mathcal{W}_{j_0} = \varnothing$, let $\mathcal{W}_{j_0}^s = \mathcal{R}$, and otherwise let $\mathcal{W}_{j_0}^s = \mathcal{W}_{j_0}$.

There exists a $G_{W'}$–tree T, bipartite, with black and white vertices, with an equivariant injective map $\psi\colon T \to \mathcal{P}(\mathbb{Y}_{j_0})$ (the set of subsets of $\mathbb{Y}_{j_0}$) that sends black vertices to images of osculators by $G_{W'}$, and white vertices to images of $\mathcal{W}^s_{j_0}$ by $G_{W'}$, and that sends the neighbors (in T) of the preimage of $\mathcal{W}^s_{j_0}$ to $\mathcal{R}$.

Moreover, for any pair of distinct white vertices w_1 and w_2, any black vertex v in the interval between them (in T), and any $X_1 \in \psi(w_1)$ and $X_2 \in \psi(w_2)$, one has $d_{\psi(v)}(X_1, X_2) \geq \Theta_{\mathrm{Rot}} - 2\Theta_P - \kappa$.

Finally, if w_1 and w_2 are white vertices for which the path from a black vertex v starts by the same edge, then for any $X_1 \in \psi(w_1)$ and $X_2 \in \psi(w_2)$, one has $d_{\psi(v)}(X_1, X_2) \leq 2\Theta_P + 3\kappa$.

Proof Take a transversal $\mathcal{R}^t$ of $\mathcal{R}$ under the action of G_W. For each $R \in \mathcal{R}^t$, let $(G_W)_R$ the subgroup of G_W generated by $\bigcup_{X \in \mathcal{W} \setminus \mathrm{Act}(R)} \Gamma_X$.

Set T to be the Bass–Serre tree of the (abstract) graph of groups whose vertex groups are G_W and the groups $\Gamma_R \times (G_W)_R$, for $R \in \mathcal{R}^t$, and the edges are the pairs (G_W, R), for $R \in \mathcal{R}^t$, and the edge groups are the groups $(G_W)_R$.

Let $\widetilde{G_{W'}}$ be the fundamental group of this graph of groups. The group $G_{W'}$ is a quotient of this group, since it is generated by G_W and the stabilizers of elements R of $\mathcal{R}^t$, which, by assumption (Definition 2.1), are direct sums of their rotation group with the groups $(G_W)_R$.

The tree T is endowed with a $\widetilde{G_{W'}}$–action, bipartite, and with an equivariant (with respect to $\widetilde{G_{W'}} \twoheadrightarrow G_{W'}$) map $\psi\colon T \to \mathcal{P}(\mathbb{Y}_j)$ that sends black vertices to images of elements of $\mathcal{R}$ by $G_{W'}$, and white vertices to images of $\mathcal{W}^s_{j_0}$ by $G_{W'}$.

We need to show that it is injective, and at the same time, we will show the estimate of the end of the statement.

Consider a path p of T, starting and ending at white vertex. Up to cyclic permutation, and up to the group action, we may assume that the path p starts at the vertex fixed by G_W, and its second vertex is fixed by some $R_1 \in \mathcal{R}^t$, and that its length is even.

Let us denote by $p_0, p_1, \ldots, p_N$ the consecutive vertices of p, and let X_{2i} be a choice of a element of $\psi(p_{2i})$, and $R_{2i+1} = \psi(p_{2i+1})$.

The monotonicity property in the coordinate j_0 says that if $d_Y(X, Z) \geq \Theta$ then $d_W(X, Z) \geq d_W(X, Y)$.

We will use this in an induction to establish that for all k odd, all i in $1 \leq i \leq \frac{1}{2}(N - k)$ and all j in $1 \leq j \leq \frac{1}{2}(k - 1)$, one has

$$d_{R_k}(R_{k-2j}, R_{k+2i}) \geq \Theta_{\text{Rot}} - 2\Theta_P - \kappa,$$

$$d_{R_k}(X_{k-2j+1}, X_{k+2i-1}) \geq \Theta_{\text{Rot}} - 2\Theta_P - \kappa \quad \text{for all } X_s \in \psi(p_s).$$

The case $i, j = 1$ happens as follows. Fix a k.

We first show how a black vertex separates two adjacent white vertices. Note that there is a $X'_{k+1} \in \psi(p_{k+1})$ that equals $g X_{k-1}$ for some $g \in \Gamma_{R_k} \setminus \{0\}$. By convexity of $\mathcal{W}^s_{j_0}$ (which is guaranteed by assumption, or by Lemma 2.11 if $\mathcal{W}_{j_0}$ is empty), $d_{R_k}(X_{k+1}, X'_{k+1}) \leq \Theta_P$. Also, $d_{R_k}(X_{k-1}, X'_{k+1}) \geq \Theta_{\text{Rot}}$, by assumption on the rotating groups. Thus, $d_{R_k}(X_{k-1}, X_{k+1}) \geq \Theta_{\text{Rot}} - \Theta_P - \kappa$, the second inequality.

By Lemma 2.7, $d_{R_k}(X_{k+1}, R_{k+2}) \leq \Theta_P$ and $d_{R_k}(X_{k-1}, R_{k-2}) \leq \Theta_P$. By the triangle inequality, we get $d_{R_k}(R_{k-2}, R_{k+2}) \geq \Theta_{\text{Rot}} - 2\Theta_P - \kappa$. We have both inequalities.

Assume that the inequalities are proven for all (i, j) such that $i + j \leq i_0$ (and for all k), and let us choose k and (i, j) with $i + j \leq i_0$, and prove the inequality for $(i + 1, j)$.

Set $Y = R_{k+2i}$, and $W = R_k$. In the following we set either $Z = R_{2i+k+2}$ or X_{2i+k+1}, and either $X = R_{k-2j}$ or $X = X_{k-2j+1}$.

By the inductive assumption for $k' = k + 2i$, $i' = 1$ and $j' = i$, one has $d_Y(W, Z) \geq \Theta_{\text{Rot}} - 2\Theta_P - \kappa$.

Also for k, i and j the induction gives $d_W(Y, X) \geq \Theta_{\text{Rot}} - 2\Theta_P - \kappa$. The Behrstock inequality then provides $d_Y(W, X) \leq \kappa$ and therefore $d_Y(X, Z) \geq \Theta_{\text{Rot}} - 2\Theta_P - 3\kappa$. This is still far above Θ. One thus may apply the monotonicity property and obtain $d_W(X, Z) \geq d_W(X, Y)$. In other words,

$$d_{R_k}(R_{k-2j}, R_{k+2i+2}) \geq \Theta_{\text{Rot}} - 2\Theta_P - \kappa.$$

The inequality is also proven for $(i, j + 1)$ in the same manner, symmetrically. This finishes the induction.

We have obtained $d_{R_1}(X_0, R_{N-1}) \geq \Theta_{\text{Rot}} - \Theta_P$ for $i = N/2 - 1$ and $k = 1$, and it follows that $d_{R_1}(X_0, X_N) \geq \Theta_{\text{Rot}} - 2 \times \Theta_P - \kappa$, which is the estimate of the statement.

If we assume that p is mapped to a loop, then $\mathcal{W}_{j_0}$ contains both X_0 and X_N, and not R_1 (it is an osculator), and the convexity of $\mathcal{W}_{j_0}$ imposes $\Theta_{\text{Rot}} - 2 \times \Theta_P - \kappa \leq \Theta_P$, meaning $\Theta_{\text{Rot}} \leq \Theta_P + \kappa$. which contradicts our choice of Θ_{Rot}.

It also follows from this analysis that if w_1 and w_2 are white vertices of T and v is a black vertex between them, then $d_{\psi(v)}(X_1, X_2) \geq \Theta_{\text{Rot}} - 2\Theta_P - \kappa$ (in our induction above). A final use of the Behrstock inequality provides that whenever a path from v to a white vertex w_1 has more than three edges, if v' is the first black vertex after v on this path, and if $X_1 \in \psi(w_1)$, then $d_{\psi(v)}(X_1, \psi(v')) \leq \kappa$. It follows from that and Lemma 2.7 that if w_2 is another white vertex w_1 whose path from v starts at the same edge, $d_{\psi(v)}(X_1, \psi(v')) \leq 2\Theta_P + 3\kappa$.   □

The former proposition allows to define, for each element γ of $G_{W'}$, its principal coordinate, and its principal tree. Indeed, if $\gamma \in G_{W'}$ is not conjugated to G_W, the proposition shows that it is either loxodromic or the stabilizer of a black vertex on the tree T. Then we define its principal coordinate as j_0 and its principal tree as T. If it is in G_W, or conjugate in it, its principal coordinate and its principal tree are defined inductively, according to the process of unfoldings of composite windmills.

2.3.2 Preservation of convexity

Proposition 2.14 (convexity of $\mathcal{W}'$) *Let $\mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_m, G_W, j_0)$ be a composite windmill (possibly nonfull).*

Assume that $\mathcal{R}$ is an admissible set of osculators, and $\mathcal{W}'$ the unfolding defined in Definition 2.9

If $\mathcal{R}$ consists of the orbit of a neighbor, then $\mathcal{W}'$ is c_*–convex.

If $\mathcal{R}$ consists of gap osculators, then $\mathcal{W}'$ is $\mathcal{L}_{j_0+1}$–convex.

The case of $\mathcal{R} = \varnothing$ is trivial, so we assume it is not empty.

Proof If $\mathcal{R}$ consists of the orbit of a neighbor, let $A_j = c_*$ for all j. If $\mathcal{R}$ consists of gaps, let $A_j = \mathcal{L}_{j_0}(j) + 20\kappa$ (which is less than $\mathcal{L}_{j_0}(j+1)$).

Let $X, Z \in \mathcal{W}_i'$, and consider $Y \in \mathbb{Y}_{A(j)}^j(X, Z)$.

Our main claim is that Y is a $G_{W'}$–translate of one of the following types of elements:

- Y' for which there exist $X_f, Z_f \in \mathcal{W}_{j_0}$ such that $d_{Y'}^{\triangleleft}(X_f, Z_f) \geq A(j) - 10\kappa$;
- Y' for which there exists $X_f \in \mathcal{W}_{j_0}$ and R an osculator of $\mathcal{W}$ in $\mathcal{W}_{j_0}'$ such that $d_{Y'}^{\triangleleft}(X_f, R) \geq A(j) - 10\kappa$;
- Y' for which there exist R_1, R_2 osculators of $\mathcal{W}$ in $\mathcal{W}_{j_0}'$ such that $d_{Y'}^{\triangleleft}(R_1, R_2) \geq A(j) - 10\kappa$.

We will prove this claim then finish the proof of the proposition.

Transfer of X and Z to $\mathbb{Y}_{j_0}$ In $\mathcal{W}'$, the groups Γ_X and Γ_Z preserve $\mathcal{W}'_{j_0}$, which is not empty (it contains $\mathcal{R}$). So, by Lemma 1.5, there are $X^{(j_0)}$ and $Z^{(j_0)}$ in $\mathcal{W}'_{j_0}$ such that $d_Y^{\lhd}(X^{(j_0)}, Z^{(j_0)}) \geq A(j) - 4\kappa$.

The interval in T Taking ψ^{-1} of $X^{(j_0)}$ and of $Z^{(j_0)}$ produces two vertices in the principal coordinate tree T of Proposition 2.13. More precisely, each of $X^{(j_0)}$ and $Z^{(j_0)}$ is in the image of a black vertex of T, or in the image of a white vertex of T. This thus give two vertices of T, which we (slightly abusively) denote by $\psi^{-1}(X^{(j_0)})$ and $\psi^{-1}(Z^{(j_0)})$.

If these vertices are adjacent, we have achieved the second point of the claim. If these vertices are the same, we have achieved the first point of the claim. If these vertices are different, both black with only one white vertex in the interval, we have achieved the third point of the claim.

Thus, we may assume that there is at least one black vertex of T in the open interval $(\psi^{-1}(X^{(j_0)}), \psi^{-1}(Z^{(j_0)}))$. Let $R_1, \dots, R_N$ the images by ψ of these black vertices, in order starting from the side of $\psi^{-1}(X^{(j_0)})$.

By Proposition 2.13, we have, for all i, that $d_{R_i}(X^{(j_0)}, Z^{(j_0)}) > \Theta_{\mathrm{Rot}} - 2\Theta_P - \kappa$, which is $> 50\kappa$.

Reduction to the case where $R_i \in \mathrm{Act}(Y)$ If Y is equal to one of the R_i then we fall in the first possibility of the main claim. Thus, let us assume that Y is different from all the R_i.

We may assume that Y is in $\mathrm{Act}(R_i)$ for all i. Indeed if it was not, one could use an element of Γ_{R_i} to reduce the length of the path p, without changing the value of the projection distance $d_Y^{\lhd}(X^{(j_0)}, Z^{(j_0)})$ since Γ_{R_i} leaves d_Y^{π} invariant.

Transfer of Y in $\mathbb{Y}_{j_0}$ We may apply Lemma 1.4 again, and find an element $Y^{(j_0)}$ in $\mathbb{Y}_{j_0}$ (far in an orbit of Γ_Y) such that, for all i, one has $d_{R_i}^{\lhd}(Y, Y^{(j_0)}) \leq 4\kappa$.

Position of $Y^{(j_0)}$ in the order Fix $0 < i \leq N$. Since $d_{R_i}(X^{(j_0)}, Z^{(j_0)}) > 50\kappa$, either $d_{R_i}(X^{(j_0)}, Y^{(j_0)})$ or $d_{R_i}(Y^{(j_0)}, Z^{(j_0)})$ is larger than 24κ.

All R_i are in $\mathbb{Y}_{50\kappa}(X^{(j_0)}, Y^{(j_0)})$, and therefore they satisfy the order property in this set, which coincides with the ordering of their indices. By this order property and the Behrstock inequality, if for some i one has $d_{R_i}(Y^{(j_0)}, X^{(j_0)}) > 5\kappa$, then for all $i' < i$,

one still has $d_{R_i}(Y^{(j_0)}, X^{(j_0)}) > 5\kappa$. Similarly if $d_{R_i}(Y^{(j_0)}, Z^{(j_0)}) > 5\kappa$ then for all greater i'' the same holds.

Thus we have three cases: either $d_{R_1}(Y^{(j_0)}, X^{(j_0)}) \leq 5\kappa$ or $d_{R_N}(Y^{(j_0)}, Z^{(j_0)}) \leq 5\kappa$, or there exists $i \geq 1$, the largest such that $d_{R_i}(Y^{(j_0)}, X^{(j_0)}) > 5\kappa$ and $i < N$.

By symmetry, and translation by an element of $G_{W'}$ the first and second cases have the same resolution. Let us treat the first one. By the triangle inequality, $d_{R_1}(Z^{(j_0)}, Y^{(j_0)}) > \Theta_{\text{Rot}} - 10\kappa - 2\Theta_P$, which is still greater than 20κ.

Returning to Y: $d_{R_1}^{\preceq}(Z^{(j_0)}, Y) > 16\kappa$. By the Behrstock inequality, $d_Y^{\preceq}(Z^{(j_0)}, R_1) < \kappa$, and finally by the triangle inequality, $d_Y^{\preceq}(X^{(j_0)}, R_1) \geq A(j) - 2\kappa$. We are in the second point of the claim if $X^{(j_0)}$ is in a white vertex, and in the third point if it is a black vertex.

We therefore turn to the case in which there exists $i \geq 1$, the largest such that $d_{R_i}(Y^{(j_0)}, X^{(j_0)}) > 5\kappa$ and $i < N$.

One has

$$d_{R_{i+1}}(Y^{(j_0)}, Z^{(j_0)}) > \Theta_{\text{Rot}} - 2\Theta_P - 10\kappa,$$
$$d_{R_{i+1}}^{\preceq}(Y, Z^{(j_0)}) > \Theta_{\text{Rot}} - 2\Theta_P - 14\kappa,$$
$$d_Y^{\preceq}(R_{i+1}, Z^{(j_0)}) \leq \kappa$$

and

$$d_{R_i}(Y^{(j_0)}, X^{(j_0)}) \geq 5\kappa,$$
$$d_{R_i}^{\preceq}(Y, X^{(j_0)}) \geq \kappa,$$
$$d_Y^{\preceq}(R_i, X^{(j_0)}) \leq \kappa.$$

So, $d_Y^{\preceq}(R_i, R_{i+1}) \geq A(j) - 4\kappa$. We have the third point of the claim, and the claim is established.

We need to finish the proof of the proposition. There are several cases to treat. The easiest is when the first case of the claim occurs.

In that case, if $j = j_0$, then Y' is actually a gap osculator, hence in $\mathcal{W}_{j_0}'$. If $j \neq j_0$, by convexity of $\mathcal{W}$, it is in $\mathcal{W}_j$.

Assume now that the second case occurs.

If R is of type neighbor, it simply contradicts Proposition 1.13.

If R is an osculator of type gap between X_0 and X_1, and $j = j_0$, one easily gets that R is an osculator of type gap between X_f and either X_0 or X_1 (any one for which $d_R(Y', X_\epsilon)$ is larger than κ; by the triangle inequality, there must be at least one).

If $j \neq j_0$, we may use the same argument. $Y' \in \mathrm{Act}(R)$, and therefore $d_R^{\preceq}(Y', X_\epsilon)$ is larger than κ for ϵ equal to either 0 or 1. Then $d_{Y'}^{\preceq}(R, X_\epsilon) < \kappa$, and by the triangle inequality, $d_{Y'}^{\preceq}(X_f, X_\epsilon) \geq A(j) - 12\kappa (> \mathcal{L}_{j_0}(j))$. It follows by convexity of $\mathcal{W}$ that $Y' \in \mathcal{W}_j$.

Finally, assume that the third case occurs.

Assume that R_2 is an osculator of type gap, between X_0 and X_1. Then, again with the same reasoning, $Y' \in \mathrm{Act}(R_2)$ and there is an ϵ for which it is in $\mathrm{Act}(X_\epsilon)$ and $d_{Y'}^{\preceq}(R_2, X_\epsilon)$ is less than κ. Thus $d_{Y'}^{\preceq}(R_1, X_\epsilon) \geq A(j) - 12\kappa$, and we are back to the second case of the claim, with a slightly lower constant. The proof goes nevertheless through, and the desired conclusion holds.

Finally, assume that R_2 is of type neighbor. Then both R_1 and R_2 are of type neighbor, and $R_2 = \gamma R_1$ for some $\gamma \in \Gamma_W$. Let us rename $R_1 = R$, call $i = i(Y')$, and let j be the principal coordinate of γ (for the Greendlinger property). Let $Z \in \mathcal{W}_j$ be the vertex of a shortening pair for γ for which $Z \in \mathrm{Act}(Y') \cap \mathrm{Act}(R)$ (there exists one, otherwise one can reduce the length of γ in its principal tree by a shortening pair at Z). Thus, $d_Z^{\preceq}(R, \gamma R) > \Theta_{\mathrm{Rot}} - 2\Theta_P - 2\kappa$.

Suppose $d_{Y'}^{\preceq}(R, \gamma R) > c_* - 10\kappa$. There are then two possible cases: either $d_{Y'}^{\preceq}(R, Z) > c_*/2 - 6\kappa$ or $d_{Y'}^{\preceq}(\gamma R, Z) > c_*/2 - 6\kappa$ (or both).

In the first case, $d_Z^{\preceq}(R, Y') \leq \kappa$. Thus $d_Z^{\preceq}(Y', \gamma R) > \kappa$, and so $d_{Y'}^{\preceq}(\gamma R, Z) < \kappa$.

Recall that $Z \in \mathrm{Act}(R) \cap \mathrm{Act}(Y')$. Thus $d_{Y'}^{\preceq}(Z, R) > c_* - 2\kappa$, and $Y' \in \mathbb{Y}_{c_* - 2\kappa}(Z, R)$. Now let Z' any other element of $\mathcal{W}$ in $\mathrm{Act}(R) \cap \mathrm{Act}(Y')$. By $((c_*/2) - 20\kappa)$–convexity of $\mathcal{W}$, one has $d_{Y'}^{\preceq}(Z, Z') \leq c_*/2 - 20\kappa$ and therefore $Y' \in \mathbb{Y}_{c_* - 2\kappa - (c_*/2) + 21\kappa}(Z', R)$. In other words, $Y' \in \mathbb{Y}_{(c_*/2) + 19\kappa}(\mathcal{W}, R)$ and this contradicts the fact that R is a neighbor.

The second case is similar, after composing with the automorphism γ^{-1}. $\qquad\square$

2.3.3 The unfolding is a windmill

Proposition 2.15 *If $\mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_m, G_W, j_0)$ is a composite windmill, and if $\mathcal{W}' = (\mathcal{W}_1', \ldots, \mathcal{W}_m', G_{W'}, j_0 + 1)$ is an unfolding over an admissible set of osculators, then $\mathcal{W}'$ is a composite windmill.*

Moreover, the set $\mathcal{W}_*'$ of the fifth point of Definition 2.4 can be assumed to contain the set $\mathcal{W}_*$ (in other words, $\mathcal{W}'$ is constructed over $\mathcal{W}$).

Proof The first three points follow by construction. The fourth point (convexity) is the result of Proposition 2.14. The sixth point is a consequence of Proposition 2.13. The same proposition introduces an action of $G_{W'}$ on a tree T which is Bass–Serre dual to a presentation of $G_{W'}$ as the fundamental group of a graph of group, with one vertex v_0 carrying the group G_W and the other vertices $v_{[R]}$, for $[R] \in \mathcal{R}/G_W$, adjacent to a single edge whose other end is v_0, carrying the group $\Gamma_R \times (G_W)_R$, if R is a representative of the orbit $[R]$. $\qquad\square$

2.4 Towers of windmills, and accessibility

2.4.1 Starting point We start the process by selecting $\mathcal{W}(0)$ to be a maximal collection of mutually inactive elements in $\mathbb{Y}_*$. Thus, whenever $\mathcal{W}(0)_j \neq \varnothing$, it is reduced to a single point.

We choose $j_0 = 1$. It is clear that $\mathcal{W}(0)$ defines a composite windmill, where for all i, $\mathcal{W}(0)_i$ is either empty or a singleton, and where G_W is the direct product of the groups G_X, for $X \in \mathcal{W}(0)$ (there are at most m direct factors).

$\mathcal{W}(0)$ is κ–convex, and $\mathrm{Act}(R) \cap \mathcal{W}(0) \neq \varnothing$ for all R, by maximality of $\mathcal{W}(0)$. Recall that by choice, $c_* > 25\kappa + 2\Theta$, hence by Proposition 1.12, there exists a neighbor osculator in $\mathbb{Y}_{(c_*/2)+2m\kappa}(\mathcal{W}(0), R)$.

2.4.2 The process Recall that we assumed $\mathbb{Y}_*$ to be countable.

We will work with indices in the *set* of countable ordinals: we will define $\mathcal{W}(k)$ for k any countable ordinal (not necessarily a number). We adopt the notation

$$\mathcal{W}(k) = (\mathcal{W}(k)_1, \ldots, \mathcal{W}(k)_m, G_{\mathcal{W}(k)}, j_k).$$

We will write $\mathcal{W}(k) \subset \mathcal{W}(k')$ to mean that $\mathcal{W}(k)_i \subset \mathcal{W}(k')_i$ for all $i \leq m$. This is not an order relation; however, note that for full windmills, if $\mathcal{W}(k) \subset \mathcal{W}(k') \subset \mathcal{W}(k)$, and if $\mathcal{W}(k)$ is fixed, there are only m possibilities for $\mathcal{W}(k')$ (corresponding to the values of $j_{k'}$). We will also write $\mathcal{W}(k) \subsetneqq \mathcal{W}(k')$ if $\mathcal{W}(k) \subset \mathcal{W}(k')$ and one of the inclusions $\mathcal{W}(k)_i \subset \mathcal{W}(k')_i$ is strict.

We have chosen $\mathcal{W}(0)$. In order to define $\mathcal{W}(k)$ for k any countable ordinal, we treat separately the case of k a successor of some ordinal, and the case of k a limit ordinal.

For any countable ordinal k, we define $\mathcal{W}(k+1)$ to be the unfolding of $\mathcal{W}(k)$ (as in Definition 2.9) over an admissible set of osculators. Recall that if there is no gap

osculator at all, one may need to choose a certain neighbor osculator to define a choice of admissible set of osculators. We could, but do not, impose the choice.

Note that by maximality of $\mathcal{W}(0)$, Lemma 2.8 can be applied to show that such a choice is always possible for all $\mathcal{W}(k)$.

Lemma 2.16 *If $\mathcal{W}(k)$ is a composite windmill, then $\mathcal{W}(k+1)$ is still a composite windmill, constructed over $\mathcal{W}(k)$.*

Proof This follows from Proposition 2.12 if the set of osculators is nonempty, and from Lemma 2.10 otherwise. $\square$

We now define $\mathcal{W}(\alpha)$ assuming that α is a limit ordinal and that all $\mathcal{W}(k)$, for $k < \alpha$, have been defined and satisfy $\mathcal{W}(k) \subset \mathcal{W}(k')$ for all $k < k'$.

We consider $\mathcal{W}(\alpha)_i = \bigcup_{k<\alpha} W(k)_i$ for each $i \le m$, and $G_{W(\alpha)} = \bigcup_{k<\alpha} G_{W(k)}$, and we set $j_\alpha = 1$.

Lemma 2.17 *Suppose that α is a limit countable ordinal such that for all $k < \alpha$, $\mathcal{W}(k)$ is a composite windmill and for all $k < \alpha$, the windmill $\mathcal{W}(k+1)$ is constructed over $\mathcal{W}(k)$. Then $\mathcal{W}(\alpha)$ is a composite windmill, constructed over $\mathcal{W}(k)$, for all $k < \alpha$.*

Proof One easily check that all the points, except possibly the fifth (on the partially commutative presentation), of the definition of composite windmill (Definition 2.4) are satisfied after taking a direct union. Assume that the fifth point is not satisfied. Consider then α_0, the smallest ordinal such that this point fails. The ordinal α_0 is a limit ordinal (otherwise Lemma 2.16 says that $\mathcal{W}(\alpha_0)$ is a composite windmill constructed over an earlier $\mathcal{W}(k)$). Fix $k_0 < \alpha_0$. For all $k < k_0$, the windmill $\mathcal{W}(k)$ is contained in $\mathcal{W}(k_0)$.

Note that by definition, for each $i \le m$,

$$\mathcal{W}(\alpha_0)_i = \bigcup_{k_0 < k < \alpha_0} W(k)_i \quad \text{and} \quad G_{W(\alpha_0)} = \bigcup_{k_0 < k < \alpha_0} G_{W(k)}.$$

Since for all $k' > k$ less than α_0, the windmill $\mathcal{W}(k')$ is constructed over $\mathcal{W}(k)$, we obtain a presentation of $G_{W(\alpha_0)}$ by increasing union of the generating sets of $G_{W(k)}$ (each of which contains that of $G_{W(k_0)}$), and by increasing union of the relators of $G_{W(k)}$. The fifth point of Definition 2.4 is then satisfied by $\mathcal{W}(\alpha_0)$, and it is a composite windmill constructed over $\mathcal{W}(k_0)$. Since this is true for all $k_0 < \alpha_0$, we obtain a contradiction with the definition of α_0. $\square$

2.4.3 Accessibility

Lemma 2.18 *Let $\mathcal{I}$ be the set of countable ordinals k such that for all $k' < k$, $\mathcal{W}(k') \subsetneq \mathcal{W}(k)$. Then $\mathcal{I}$ is countable. Moreover, for each k_1 and k_2 in $\mathcal{I}$, consecutive in $\mathcal{I}$, there are at most m ordinals between k_1 and k_2.*

Proof For each $k \in \mathcal{I}$, unless it is its maximal element, one can associate its successor $s(k)$ in $\mathcal{I}$, and therefore an element X_k in $\mathbb{Y}_*$ in $\mathcal{W}(s(k))$ but not in $\mathcal{W}(k)$. The assignment of X_k is obviously injective on $\mathcal{I}$, and $\mathbb{Y}_*$ is countable, thus $\mathcal{I}$ is countable.

For the second assertion, assume that there are $m + 1$ consecutive countable ordinals $k_1, \dots, k_{m+1}$ outside $\mathcal{I}$, all less than some $k_t \in \mathcal{I}$. Then by the pigeonhole argument, for two of them, k and k', one has $\mathcal{W}(k) = \mathcal{W}(k')$. Thus, by the rules of construction of $\mathcal{W}(k+1)$, one has that $\mathcal{W}(k) \subset \mathcal{W}(k+r) \subset \mathcal{W}(k)$ for all $r \in \mathbb{N}$, or equivalently, for all r, that $\mathcal{W}(k + r + 1) \subset \mathcal{W}(k + r) \subset \mathcal{W}(k + r + 1)$. Since we take direct limits for limit ordinals, this holds also for all countable ordinals r. However, k_t is a countable ordinal, and therefore $\mathcal{W}(k_t + 1) \subset \mathcal{W}(k_t) \subset \mathcal{W}(k_t + 1)$, contradicting that $k_t \in \mathcal{I}$. $\square$

Lemma 2.19 *There is a countable ordinal k_{top} such that $\mathbb{Y}_* \subset \mathcal{W}(k_{\text{top}})$.*

Proof By Lemma 2.18, the supremum of $\mathcal{I}$ is still a countable ordinal. Call this ordinal k_{top}. Then $\mathcal{W}(k_{\text{top}})$ is well defined. Assume that $\mathbb{Y}_* \not\subset \mathcal{W}(k_{\text{top}})$. Then it follows from Lemma 2.8 that $\mathcal{W}(k_{\text{top}})$ is not $((c_*/2)-20\kappa)$–convex. Therefore, there is a gap osculator in one of the coordinates, and this coordinate is reached while $r \leq m$. This is a contradiction to the definition of k_{top}. Thus, $\mathbb{Y}_* \subset \mathcal{W}(k_{\text{top}})$. $\square$

2.5 End of the proof of Theorems 2.2 and 2.3

Consider $\mathcal{W}(k_{\text{top}})$ from Lemma 2.19. Assume it is not a composite windmill. Then there is a smallest ordinal k_1 such that $\mathcal{W}(k_1)$ is not a composite windmill. If k_1 is not a limit ordinal, it is of the form $k_0 + 1$ for k_0 such that $\mathcal{W}(k_0)$ is a composite windmill. Lemma 2.16 leads to a contradiction. If k_1 is a limit ordinal, then Lemma 2.17 leads to a contradiction. Thus $\mathcal{W}(k_{\text{top}})$ is a composite windmill.

Since $\mathcal{W}(k_{\text{top}})$ contains all elements of $\mathbb{Y}_*$, Theorems 2.2 and 2.3 follow from the definition of composite windmill.

3   Conclusion, application to Dehn twists, and Theorem 1

Let Σ be an orientable closed surface of genus greater than 2. Consider its mapping class group $\mathrm{MCG}(\Sigma)$.

Bestvina, Bromberg and Fujiwara produced a finite coloring of the set of simple closed curves of Σ such that two curves of same color intersect, and a finite-index normal subgroup G_0 of $\mathrm{MCG}(\Sigma)$ that preserves the coloring. G_0 is called the color-preserving group. After refinement of the colors, we actually may assume that the colors are in correspondence with the cosets of G_0. We denote the colors by $\{1, \dots, m\}$.

Let c and c' be simple closed curves. If they intersect, the projection of c' on c is the family of elements in the arc complex of the annulus around c (that is, the cover of Σ associated to c) that come from lifts of c'. They are all disjoint. If c'' is another simple closed curve intersecting c, then $d_c^\pi(c', c'')$ is the diameter in the curve graph of the union of the projections of c' and c'' on the annulus around c.

The functions d_c^π define a composite projection system on the set of all (homotopy classes of) simple closed curves. Indeed, let $\mathrm{Act}(c)$ be the set of curves intersecting c. Clearly d_c^π is symmetric, and satisfies the separation property. The symmetry in action and the closeness in inaction properties are also direct consequences of definitions. The finite filling property is a consequence of the fact that all sequences of subsurfaces up to isotopy, increasing under inclusion, are eventually stationary. Finally, d_c^π satisfies the triangle inequality since it is a diameter of projections, and the Behrstock inequality; see [1]; see also [14; 15]. Properness is ensured by [2, Lemma 5.3]

We can now define two composite projection systems with composite rotating families. The first one, defined on $\mathbb{Y}_*$, is the set $\mathfrak{S}$ of *all* homotopy classes of simple closed curves of Σ.

Let us define $\mathbb{Y}_i$ to be the subset of this set of simple closed curves of color i in the Bestvina–Bromberg–Fujiwara coloring, and $\mathbb{Y}_*$ their union. It is, as we just said, a composite projection system on which G_0 acts by automorphisms.

Performing the construction of [2] and the choices as after Definition 1.2, we have constants Θ, κ, c_*, Θ_P and Θ_{Rot}.

We select N_1 such that all N_1–powers of Dehn twists in $\mathrm{MCG}(\Sigma)$ are in G_0. This is possible since there are only finitely many $\mathrm{MCG}(\Sigma)$–orbits of simple closed curves in Σ, and G_0 has finite index. Then we select N_2 a multiple of N_1 such that for all simple

closed curves c, the Dehn twist $\tau_c^{N_2}$ around c satisfies $d_c(c', \tau_c^{N_2} c') > \Theta_{\text{Rot}} + 2\Theta_P$ if c' is a curve of the same color as c (hence intersecting c). Since d_c is comparable with d_c^{π}, by definition of the latter, there exists such an exponent N_2. Then it follows that, for all $k \in \mathbb{N}$, the collection $\{\Gamma_c = \langle \tau_c^{kN_2} \rangle : c \in \mathfrak{S}\}$, is a composite rotating family.

The second composite projection system is a subsystem, invariant for G_0, provided by the $\text{MCG}(\Sigma)$–orbit of a simple closed curve $c_0 \in \mathfrak{S}$. Namely, the composite rotating family is the collection $\{\Gamma_c : c \in (\text{MCG}(\Sigma)c_0) \subset \mathfrak{S}\}$.

It is straightforward that both families are composite rotating families.

One can then apply Theorem 2.2. In the first case, one obtains that the group generated by the $(kN_2)^{\text{th}}$ powers of all Dehn twists has a partially commutative presentation, which is the second point of Theorem 1. In the case of the second composite rotating family, one obtains that the group generated by all $(kN_2)^{\text{th}}$ powers of all Dehn twists that are $\text{MCG}(\Sigma)$–conjugated to τ_{c_0} has a partially commutative presentation. This latter group is the normal closure of $\tau_{c_0}^{kN_2}$ in $\text{MCG}(\Sigma)$. We therefore obtained Theorem 1.

Acknowledgements

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