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Indicability, residual finiteness, and simple subquotients of groups acting on trees

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We establish three independent results on groups acting on trees. The first implies that a compactly generated locally compact group which acts continuously on a locally finite tree with nilpotent local action and no global fixed point is virtually indicable; that is to say, it has a finite-index subgroup which surjects onto $\mathbb{Z}$. The second ensures that irreducible cocompact lattices in a product of nondiscrete locally compact groups such that one of the factors acts vertex-transitively on a tree with a nilpotent local action cannot be residually finite. This is derived from a general result, of independent interest, on irreducible lattices in product groups. The third implies that every nondiscrete Burger–Mozes universal group of automorphisms of a tree with an arbitrary prescribed local action admits a compactly generated closed subgroup with a nondiscrete simple quotient. As applications, we answer a question of D Wise by proving the nonresidual finiteness of a certain lattice in a product of two regular trees, and we obtain a negative answer to a question of C Reid, concerning the structure theory of locally compact groups.

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1 Introduction

Given a group G acting by automorphisms on a graph X, the *local action* of G at a vertex $v \in VX$ is the permutation group induced by the action of the vertex

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stabilizer $G_{(v)}$ on the set of edges $E(v)$ emanating from v. Various results from the literature show how restrictions on the local action impact the global properties of the group G. This phenomenon is strikingly illustrated by the work of M Burger and S Mozes [4; 5] on lattices in products of trees.

Our first main result provides an illustration of this paradigm in the case that X is a tree. A (topological) group is called *virtually indicable* if it has a finite-index (finite-index open) subgroup admitting a (continuous) surjective homomorphism onto the infinite cyclic group. A tree is called *leafless* if it has no vertex of valency 1.

Theorem 1.1 (see Theorem 3.1) *Let G be a topological group with a continuous, cocompact action by automorphisms on an infinite locally finite leafless tree T. Suppose that the local action $F(v)$ of G at every vertex v is such that the subgroup of $F(v)$ generated by its point stabilizers is intransitive on $E(v)$. Then G is virtually indicable.*

For every finite transitive permutation group F of degree d that is generated by its point stabilizers, there exists an infinite simple group acting transitively on the (undirected) edges of the d–regular tree whose local action at every vertex is isomorphic to F; see [4, Proposition 3.2.1] or the discussion preceding Theorem 1.8 below. The condition on the local action in Theorem 1.1 can therefore not be weakened. By considering an $\mathrm{Aut}(T_3)$–equivariant embedding of the trivalent tree T_3 in the 4–regular tree, one sees that the hypothesis of minimality of the G–action on T is also necessary in the theorem, even if G is compactly generated.

A special class of permutation groups satisfying the local condition of the theorem is that of nilpotent groups; see Lemma 3.3. In that particular case, the minimality assumption on the G–action can be replaced by the assumption that G be compactly generated, yielding the following result.

Corollary 1.2 *Let H be a locally compact group with a continuous action by automorphisms on a locally finite tree T. If the local action of H at every vertex is nilpotent, then every compactly generated closed subgroup $G \leq H$ either fixes a vertex or edge or is virtually indicable.*

In particular, if H is a closed subgroup of $\mathrm{Aut}(T)$ whose local action at every vertex is nilpotent, then every compactly generated noncompact closed subgroup of H is virtually indicable. As pointed out to us by Y Cornulier, that conclusion cannot be strengthened by the claim that H has a finite-index subgroup all of whose compactly generated noncompact closed subgroups are indicable. Indeed, a locally compact group all of whose compactly generated noncompact closed subgroups are indicable, has an open locally elliptic radical, hence an open amenable radical. On the other hand, a closed subgroup of $\mathrm{Aut}(T)$ with nilpotent local action at every vertex can be simple and nonamenable; see Section 2.

A natural framework in which these results are relevant is that of lattices in products of trees. More generally, if Γ is a finitely generated lattice in a product of the form $\mathrm{Aut}(T) \times H$, where T is a locally finite tree and H is a locally compact group, then Γ is virtually indicable as soon as the local action of Γ at every vertex of T is nilpotent. Our second main result shows that in the latter situation the lattice Γ cannot be residually finite, unless it is reducible.

Theorem 1.3 (see Corollary 6.4) *Let T be a locally finite leafless tree and H be a compactly generated totally disconnected locally compact group with a trivial amenable radical. Let $\Gamma \leq \mathrm{Aut}(T) \times H$ be a cocompact lattice whose projection to H has a nondiscrete image. If the Γ–action on T is vertex-transitive and the local action of Γ at a vertex of T is nilpotent, then the projection of Γ to H is noninjective, and Γ is not residually finite.*

The condition of vertex-transitivity of Γ on T can be removed if one strengthens slightly the hypothesis on the local action; see Corollary 6.4.

Theorem 1.3 applies in particular to D Wise's iconic example of an irreducible lattice in $\mathrm{Aut}(T_4) \times \mathrm{Aut}(T_6)$, where T_d denotes the d–regular tree, which we call the *Wise lattice*; see Wise [27; 26, Example 4.1]. In [27, Main Theorem 7.5], Wise proves that his lattice has an inseparable finitely generated subgroup, which he uses to prove that the double of Γ over that subgroup, which is an irreducible lattice in $\mathrm{Aut}(T_8) \times \mathrm{Aut}(T_6)$, is not residually finite. We show that the Wise lattice itself already fails to be residually finite, thereby resolving a problem posed by Wise [27, Problem 10.19].

Corollary 1.4 *The Wise lattice $\Gamma \leq \mathrm{Aut}(T_4) \times \mathrm{Aut}(T_6)$ is not residually finite.*

Remark 1.5 Corollary 1.4 is independently proved in [3] by Bondarenko and Kivva, via considering square complexes associated to automata.

Theorem 1.3 is deduced from a general statement on irreducible lattices in products of locally compact groups, a special case of which is the following (see also Theorem 5.14 for another related result of independent interest).

Theorem 1.6 (see Theorem 5.13) *Let $G = G_1 \times \cdots \times G_n$ be a product of nondiscrete compactly generated totally disconnected locally compact groups that have a trivial amenable radical and no infinite discrete quotient. Let $\Gamma \leq G$ be a cocompact lattice whose projection to G_i is dense for all i. If Γ is residually finite, then every compact open subgroup of G has a compact normalizer, and G has a trivial quasicenter.*

For our last main result, we study the universal groups of Burger and Mozes [4, Section 3.2]. These groups depend on a choice of a finite permutation group F and are denoted by $U(F)$. For F with degree d, the group $U(F)$ is a closed vertex-transitive subgroup of $\mathrm{Aut}(T_d)$. Whenever F does not act freely, the subgroup generated by pointwise edge-stabilizers, denoted by $U(F)^+$, is an abstractly simple nondiscrete closed subgroup of $\mathrm{Aut}(T_d)$. When F is transitive and generated by its point stabilizers, the group $U(F)^+$ is a compactly generated nondiscrete simple group acting edge-transitively on T; see Proposition 2.11. For such F, the group $U(F)^+$ therefore belongs to the following interesting class of groups.

Definition 1.7 Let $\mathscr{S}$ denote the class of nondiscrete totally disconnected locally compact groups that are topologically simple and compactly generated.

When F is not transitive or not generated by its point stabilizers, the group $U(F)^+$ is not compactly generated; eg see Corollary 2.13. We show that $U(F)$ nonetheless admits a group in $\mathscr{S}$ as a subquotient.

Theorem 1.8 (see Theorem 4.16) *Let F be a finite permutation group which does not act freely. Then $U(F)$ has a compactly generated closed subgroup H admitting a discrete normal subgroup D such that H/D is a nondiscrete compactly generated simple group.*

Combining Theorems 1.1 and 1.8, we obtain a negative answer to a question asked by Colin Reid [22, Question 2]; see Section 4.5.

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2 Preliminaries

2.1 Graphs and Bass–Serre theory

Following J P Serre [23], a *graph* is a tuple $\Gamma = (V\Gamma, E\Gamma, o, r)$ consisting of a vertex set $V\Gamma$, a directed edge set $E\Gamma$, a map $o\colon E \to V$ assigning to each edge an *initial vertex*, and a bijection $r\colon E \to E$, denoted by $e \mapsto \bar{e}$ and called *edge reversal*, such that $r^2 = \mathrm{id}$ and $e \neq \bar{e}$. Edge reversal and the initial vertex map together give a terminal vertex map $t(e) := o(\bar{e})$. A graph *automorphism* is a permutation of $V\Gamma$ and $E\Gamma$ which respects the maps o and r. For a graph Y and a vertex $w \in VY$, we define $E_Y(w)$ to be the edges with origin w. When clear from context, we suppress the subscript Y. We say a graph Y is *d–regular* if $|E_Y(w)| = d$ for all $w \in VY$.

For graphs X and Y, a *graph homomorphism* $\phi\colon X \to Y$ is given by two functions $\phi_V\colon VX \to VY$ and $\phi_E\colon EX \to EY$ such that $\phi_V \circ o = o \circ \phi_E$ and $\phi_E(\bar{e}) = \overline{\phi_E(e)}$. We say a graph X is a *covering graph* of a graph Y if there is a graph homomorphism $\phi\colon X \to Y$ such that ϕ_V is surjective and $(\phi_E)\restriction_{E_X(v)}\colon E_X(v) \to E_Y(\phi_V(v))$ is a bijection for all $v \in VX$. We call the homomorphism ϕ a *covering map*. The automorphism group $\mathrm{Aut}(X)$ of the graph X is endowed with the topology of pointwise convergence for its natural action on the set $VX \sqcup EX$, viewed as a discrete set. In particular, if X is connected and locally finite, then $\mathrm{Aut}(X)$ is a second countable totally disconnected locally compact group.

Theorem 2.1 [23, Section 5.4] *Let X be a connected graph and $H \leq \mathrm{Aut}(X)$ be a closed subgroup acting without edge inversion. The following hold:*

(1) *There is a covering map $\phi\colon T \to X$, where T is the a tree. If X is additionally n–regular, then T is n–regular.*

(2) *There is a closed subgroup $G \leq \mathrm{Aut}(T)$ and a continuous surjective homomorphism $\Phi\colon G \to H$ with the following properties:*

(a) $\mathrm{Ker}(\Phi)$ *is discrete. In particular, if H is unimodular, then so is G.*
(b) *The following diagrams commute for all $g \in G$:*

$$\begin{array}{ccc} VT & \xrightarrow{g} & VT \\ \big\downarrow \phi_V & & \big\downarrow \phi_V \\ VX & \xrightarrow{\Phi(g)} & VX \end{array} \qquad \begin{array}{ccc} ET & \xrightarrow{g} & ET \\ \big\downarrow \phi_E & & \big\downarrow \phi_E \\ EX & \xrightarrow{\Phi(g)} & EX \end{array}$$

We call the map Φ given by Theorem 2.1 the *covering homomorphism*. The group G is called the *lift* of H to T.

Throughout this paper, given a group G acting on a set X, the pointwise fixator of $Y \subset X$ is denoted by $G_{(Y)}$. When X is a graph, we set

$$G^+ := \langle G_{(e)} \mid e \in ET \rangle.$$

We record the following fact from Bass–Serre theory.

Proposition 2.2 *Let T be a tree and H be a locally compact group acting continuously on T with compact edge stabilizers. The quotient graph T/H^+ is a tree.*

Proof The group H^+ acts without inversion on T, since it is generated by edge stabilizers. Bass–Serre theory ensures that H^+ is isomorphic to the fundamental group of a graph of groups whose underlying graph is $X := T/H^+$; see [23, Theorem 13]. Additionally, the fundamental group of a graph of groups maps onto the fundamental group of the underlying graph; see [23, Section 5.1].

Suppose toward a contradiction that X admits a cycle. The fundamental group of X thus maps onto $\mathbb{Z}$, hence $\mathbb{Z}$ is a quotient of H^+. Any homomorphism of a locally compact group to $\mathbb{Z}$ is continuous via [1, Corollary 3]. Any homomorphism $H^+ \to \mathbb{Z}$ therefore has a trivial image, since H^+ is generated by compact subgroups and $\mathbb{Z}$ has no nontrivial finite subgroups. This is absurd. We conclude that X is indeed a tree. □

2.2 Normal subgroups of groups acting on trees

The following basic fact seems to be due to J Tits. For a proof of the first two claims, the reader may consult [16, Lemma 4.2]. The claim on nonamenability follows easily from the existence of discrete free subgroups afforded by a standard ping-pong argument; a detailed proof may be found in [17, Theorem 1].

Proposition 2.3 (Tits) *Let T be a tree with more than two ends. If $G \leq \mathrm{Aut}(T)$ acts minimally without a fixed end, then every nontrivial normal subgroup N of G is such that it acts minimally without a fixed end, has a trivial centralizer in $\mathrm{Aut}(T)$, and is not amenable.*

There is an important normal subgroup of $G \leq \mathrm{Aut}(T_n)$, where T_n is the n–regular tree. Define $\sim$ on VT_n by $v \sim w$ if and only if $d(v,w)$ is even. This is a G–equivariant equivalence relation on VT_n which partitions VT_n into two parts. The subgroup G^*, of index 2 in G, is the collection of g that do not interchange the parts. The subgroup G^* acts on T_n without edge inversion.

2.3 Burger–Mozes groups

Let Y be a graph and $d > 0$ be an integer. A *coloring of degree d* of Y is a map $c\colon EY \to [d]$ such that for every $v \in VY$, the restriction

$$c_v := c\restriction_{E_Y(v)}\colon E_Y(v) \to [d]$$

is either a bijection or is constant. A vertex v is called *c–regular* or *c–singular* accordingly. The coloring c is called *regular* if all vertices are c–regular. In that case Y is d–regular.

Let $g \in \mathrm{Aut}(Y)$, c be a coloring of degree d, and $v \in VY$ be a vertex such that v and gv are both c–regular. We may then define the *local action* of g at v as the permutation of $[d]$ given by

$$\sigma_c(g,v) := c_{g(v)} \circ g \circ c_v^{-1}.$$

When clear from context or unimportant, we suppress the subscript c. The local action enjoys two important properties, the proofs of which are easy exercises:

$$\sigma(gh,v) = \sigma(g,hv)\sigma(h,v), \tag{1}$$

$$\sigma(g^{-1},v) = \sigma(g,g^{-1}v)^{-1} \tag{2}$$

for all automorphisms g and h preserving the set of c–regular vertices.

Definition 2.4 Let $d > 2$ and let T be the d–regular tree. For a permutation group $F \leq \mathrm{Sym}(d)$ and $c\colon ET \to [d]$ a regular coloring, the *Burger–Mozes group* is

$$U_c(F) := \{g \in \mathrm{Aut}(T) \mid \sigma_c(g,v) \in F \text{ for all } v \in VT\}.$$

We write $U_c((F,[d]))$ when we wish to emphasize the permutation representation of F.

The group $U_c(F)$ depends on the coloring c, and this dependence is somewhat mysterious. For instance, it is easy to construct regular colorings such that $U_c(F)$ is finite. There is a class of colorings, however, for which we have good control over the resulting group.

Definition 2.5 A coloring c of a graph Y is *legal* if $c(e) = c(\overline{e})$ for each edge $e \in EY$.

Proposition 2.6 *Let $F, F' \leq \mathrm{Sym}(d)$ and let c, c' be regular legal colorings of the d–regular tree T. If F and F' are isomorphic as permutation groups, then $U_c(F)$ is conjugate to $U_{c'}(F')$ by some $g \in \mathrm{Aut}(T)$. In particular, the isomorphism type of $U_c(F)$ is independent of the choice of regular legal coloring.*

Proof In the case $F = F'$, the required assertion is proved by Burger and Mozes in [4, Section 3.2]. Assume now that F and F' are distinct and isomorphic. There thus exists $h \in \mathrm{Sym}(d)$ with $hFh^{-1} = F'$. Let d be the coloring defined by $d(e) := hc(e)$. Clearly, d is again a regular legal coloring. For $g \in U_c(F)$ and $v \in VT$, we see that

$$\sigma_d(g, v) = d_{g(v)} \circ g \circ d_v^{-1} = hc_{g(v)} \circ g \circ c_v^{-1} h^{-1} = h\sigma_c(g, v)h^{-1}.$$

We conclude that $\sigma_d(g, v) \in F'$ for all $g \in U_c(F)$ and $v \in VT$. Hence, $U_c(F) \leq U_d(F')$. The converse inclusion is similar, and thus $U_c(F) = U_d(F')$. We conclude from our initial observation that $U_c(F)$ is indeed conjugate to $U_{c'}(F')$. □

For c a regular legal coloring, one easily verifies that $U_c(\{1\})$ acts vertex transitively on T. Therefore, $U_c(F)$ acts vertex transitively on T for any regular legal coloring c and permutation group F.

Definition 2.7 For $d \geq 3$, $F \leq \mathrm{Sym}(d)$, and c some (equivalently, any) regular legal coloring, we call the group $U_c(F)$ the *Burger–Mozes universal group* with local action prescribed by F and denote it by $U(F)$.

The term "universal" is justified by [4, Proposition 3.2.2], where it is shown that if $F \leq \mathrm{Sym}(d)$ is transitive, then every vertex-transitive subgroup $H \leq \mathrm{Aut}(T_d)$ whose local action at some vertex is isomorphic to F is contained in $U_c(F)$ for some regular legal coloring c. Adapting the argument there, we obtain a slightly more general fact, which covers the case where F is intransitive.

Proposition 2.8 (compare [4, Proposition 3.2.2]) *Let T be a tree and $H \leq \mathrm{Aut}(T)$. Suppose that we have a bijection $k\colon E(v) \to [d]$ for some $v \in VT$ and $d > 2$ and set*

$$F := \{\sigma_k(g, v) \mid g \in H_{(v)}\} \leq \mathrm{Sym}(d).$$

The following assertions hold:

(i) *There exists a coloring $\widetilde{k}$ of T satisfying the following properties:*
  (a) $\widetilde{k} \restriction_{E(v)} = k$.
  (b) *The set of $\widetilde{k}$–regular vertices coincides with the H–orbit of v.*
  (c) $\sigma_{\widetilde{k}}(g, w) \in F$ *for all* $g \in H$ *and* w *in the* H*–orbit of* v.

(ii) *If H is vertex-transitive, then $H \leq U_{\widetilde{k}}(F)$ and $\widetilde{k}$ is a regular coloring.*

(iii) *In addition, if either F is transitive or every edge of T is inverted by some element of H, then $\widetilde{k}$ can be chosen to be legal.*

Proof (i) Let $H.v$ denote the orbit of v under the action of H. For each $w \in H.v$, fix $h_w \in H$ such that $h_w(w) = v$. For $w = v$, let us take $h_v = 1$. We now define $\widetilde{k}\colon ET \to [d]$ by

$$\widetilde{k}(e) := \begin{cases} 0 & \text{if } o(e) \notin H.v, \\ k(h_{o(e)}(e)) & \text{if } o(e) \in H.v. \end{cases}$$

That $\widetilde{k}$ is a coloring satisfying properties (a) and (b) is obvious from the definition of $\widetilde{k}$.

To establish property (c), take h_w to be as in the definition of $\widetilde{k}$. We now compute

$$\sigma_{\widetilde{k}}(h_w, w) = \widetilde{k}_v \circ h_w \circ \widetilde{k}_w^{-1} = k \circ h_w \circ (k \circ h_w)^{-1} = 1.$$

We deduce that $\sigma_{\widetilde{k}}(h_w, w) = 1$ and that $\sigma_{\widetilde{k}}(h_w^{-1}, v) = 1$. For an arbitrary $g \in H$ and $w \in H.v$, the element $h_{g(w)} g h_w^{-1}$ fixes v, so by definition of F,

$$\sigma_{\widetilde{k}}(h_{g(w)} g h_w^{-1}, v) \in F.$$

On the other hand,

$$\begin{aligned} \sigma_{\widetilde{k}}(h_{g(w)} g h_w^{-1}, v) &= \sigma_{\widetilde{k}}(h_{g(w)}, g h_w^{-1}(v)) \sigma_{\widetilde{k}}(g h_w^{-1}, v) \\ &= \sigma_{\widetilde{k}}(h_{g(w)}, g(w)) \sigma_{\widetilde{k}}(g, h_w^{-1}(v)) \sigma_{\widetilde{k}}(h_w^{-1}, v) \\ &= 1 \cdot \sigma_{\widetilde{k}}(g, w) \cdot 1 \\ &= \sigma_{\widetilde{k}}(g, w). \end{aligned}$$

We deduce that $\sigma_{\widetilde{k}}(g, w) \in F$ for all $g \in H$ and $w \in H.v$.

(ii) This is immediate from (i).

(iii) Let $\widetilde{k}$ be the coloring given by (i). For all $i, j \in [d]$, if there is some element of F that carries i to j, then fix $g_{ij} \in F$ such that $g_{ij}(i) = j$. We assume that $g_{ii} = 1$. Observe that we have such a g_{ij} whenever $i = \widetilde{k}(e)$ and $j = \widetilde{k}(\overline{e})$ for some edge e, by our hypotheses.

Fix $w \in VT$ and for each vertex $v \in VT \setminus \{w\}$, let e_v be the edge with origin v on the geodesic from v to w. We now define a legal coloring $c\colon ET \to [d]$ on $E(v)$ by induction on $d(v,w)$ such that for each $e \in ET$ there is a $g \in F$ such that $c(e) = g\widetilde{k}(e)$. For the base case, we set $c(f) := \widetilde{k}(f)$ for all $f \in E(w)$. Suppose that we have defined c on $E(v)$ for all $v \in B_n(w)$. Take v such that $d(v,w) = n+1$. Since $t(e_v) \in B_n(w)$, the coloring c is defined on $\overline{e_v}$. Say that $c(\overline{e_v}) = g\widetilde{k}(\overline{e_v}) = g(j)$ and $\widetilde{k}(e_v) = i$. We set $c(f) := gg_{ij}\widetilde{k}(f)$ for $f \in E(w)$. It follows that c is a legal coloring.

It is clear that the coloring c satisfies (a) and (b) of (i). Let us argue for (c). Taking $g \in H$ and $w \in H.v$,

$$\sigma_c(g,w) = c_{g(w)} \circ g \circ c_w = z\widetilde{k}_{g(w)} \circ g \circ \widetilde{k}_w^{-1} y = z\sigma_{\widetilde{k}}(g,w)y$$

for some $z, y \in F$. Since $\sigma_{\widetilde{k}}(g,w) \in F$ by claim (i), we infer that $\sigma_c(g,w) \in F$, verifying (c). □

We finish this subsection with some supplementary results that will be useful in recognizing when a subgroup $H \le \mathrm{Aut}(T)$ is conjugate to a subgroup of $U(F)$.

Lemma 2.9 *Let T be a tree and $H \le \mathrm{Aut}(T)$ be a closed subgroup. Let $v \in VT$ be such that the action of $H_{(v)}$ on $E(v)$ has a unique fixed point e. If H is unimodular, then for every $h \in H$ with $hv = t(e)$, we have $he = \overline{e}$.*

Proof We prove the contrapositive. Suppose that $he \neq \overline{e}$. The subgroup $H_{(e)}$ fixes hv, and hence it also fixes he. We deduce that $H_e \le H_{he} = hH_eh^{-1}$. On the other hand the unique fixed point of H_{hv} on $E(hv)$ is $he \neq \overline{e}$, so $H_e \neq H_{he}$. The compact open subgroup H_{he} is thus conjugate to a proper subgroup of itself, preventing H from being unimodular. □

Corollary 2.10 *Let $d \ge 2$ and let $F \le \mathrm{Sym}(d+1)$ be a permutation group fixing 0 and acting transitively on $\{1,\dots,d\}$. Let $H \le \mathrm{Aut}(T_{d+1})$ be a vertex-transitive, unimodular, closed subgroup whose local action is isomorphic to F. Then there is a regular legal coloring c of T_{d+1} such that H is a subgroup of $U_c(F)$.*

Proof Let $v \in VT$ and $e \in E(v)$. Since H is vertex-transitive, there exists $h \in H$ with $hv = t(e)$. If e is the unique fixed point of $H_{(v)}$ on $E(v)$, then $h(e) = \bar{e}$ by Lemma 2.9. Otherwise, let $e' \neq e$ be the unique fixed point of $H_{(v)}$ on $E(v)$. Thus $he' \neq he$ is the unique fixed point of $H_{(hv)}$ on $E(hv)$. By Lemma 2.9, we have $he' \neq \bar{e}$, so he and $\bar{e}$ lie in the same $H_{(hv)}$–orbit on $E(hv)$. There is thus $g \in H_{hv}$ with $ghe = \bar{e}$.

In either case, we have shown that the edge $e \in E(v)$ can be inverted. Since e was arbitrary, we conclude from Proposition 2.8(iii) that $H \le U_c(F)$ for some regular legal coloring c. □

2.4 The group G^+

Given a tree T and a subgroup $G \le \mathrm{Aut}(T)$, recall that $G^+ = \langle G_{(e)} \mid e \in ET \rangle$. The subgroup G^+ is normal, and if G is closed in $\mathrm{Aut}(T)$, then G^+ is open (hence closed) in G. In particular, G is discrete if and only if G^+ is discrete. The group G^+ plays an important role in the setting of the groups $U(F)$.

Proposition 2.11 [4, Proposition 3.2.1] *Let $F \le \mathrm{Sym}(d)$ with $d > 2$.*

(i) *$U(F)$ is discrete if and only if F acts freely if and only if $U(F)^+$ is trivial.*

(ii) *If F does not act freely, then $U(F)^+$ is abstractly simple.*

(iii) *$[U(F) : U(F)^+]$ is finite if and only if $[U(F) : U(F)^+] = 2$ if and only if F is transitive and generated by its point stabilizers.*

Remark 2.12 We direct the reader to [11] for detailed proofs of Proposition 2.11.

It is important to notice that $U(F)^+$ need not be compactly generated. Indeed, we have the following (see Definition 1.7 for the definition of $\mathscr{S}$).

Corollary 2.13 *$U(F)^+ \in \mathscr{S}$ if and only if F is transitive and generated by its point stabilizers.*

Proof The "if" part follows from Proposition 2.11. For the converse, observe that $U(F)^+$ is a nontrivial normal subgroup of the vertex-transitive group $U(F)$, so $U(F)^+$ does not preserve any nonempty proper subtree by Proposition 2.3. Since $U(F)^+$ is compactly generated, it follows from [6, Lemma 2.4] that $U(F)^+$ acts cocompactly on T. Therefore, $U(F)/U(F)^+$ is compact, hence finite since $U(F)^+$ is open. The group F is thus transitive and generated by its point stabilizers by Proposition 2.11. □

3 Virtual indicability

Theorem 3.1 *Let G be a topological group with a continuous action by automorphisms on an infinite locally finite leafless tree T. We assume that G has finitely many orbits of vertices, and that for every vertex $v \in VT$, the local action $F(v) \leq \mathrm{Sym}(E(v))$ of G at v is such that the subgroup of $F(v)$ generated by its point stabilizers is intransitive on $E(v)$. Then G is virtually indicable.*

Proof Let $\varphi\colon G \to \mathrm{Aut}(T)$ be the induced homomorphism and set $H := \overline{\varphi(G)}$. The group H^+ is open in H, so the restriction of the quotient map $H \to H/H^+$ to the dense subgroup $\varphi(G)$ is surjective. It thus suffices to show that H/H^+ is virtually indicable.

If H does not contain any hyperbolic element, then by [24, Proposition 3.4], either H fixes a vertex or inverts an edge, or H fixes an end and preserves each horoball centered at that end. In either case, we get a contradiction with the hypotheses that T is infinite leafless and that the G–action has finitely many orbits of vertices. Thus H contains hyperbolic elements.

Assume next that H fixes an end $\xi \in \partial T$. Since H contains a hyperbolic element, it permutes the horoballs centered at ξ nontrivially. The Busemann homomorphism[1] associated to ξ yields a continuous, surjective homomorphism $H \to \mathbb{Z}$ vanishing on H^+. The group H/H^+ is thus virtually indicable, as desired.

We assume henceforth that H does not fix an end. The quotient graph $X := T/H^+$ is a tree by Proposition 2.2, and the natural action of the discrete quotient group H/H^+ on X is proper and cocompact. We shall argue that X is an infinite tree by showing that each vertex of X has degree at least 2. It then follows that H/H^+ is virtually an infinite free group and is thus virtually indicable, as required.

Fix a vertex $w_0 \in VT$, let W be the H–orbit of w_0 in VT, and say that m is the degree of w_0 in T. Our assumption on the local action of G ensures that $m \geq 2$. Fix $c_0\colon E(w_0) \to [m]$ a bijection and set $F := \{\sigma_{c_0}(g, w_0) \mid g \in H_{(w_0)}\}$. The number of orbits of F on $[m]$ is a lower bound for the degree of the image of w_0 in the quotient graph X. Therefore, if F is not transitive on $[m]$, then the image of w_0 in the quotient graph X has degree at least 2. Let us assume that F is transitive on $[m]$. Proposition 2.8 provides a coloring $c\colon ET \to [m]$ extending c_0 such that for each

[1] Fixing a representative ray $x_0, x_1, \ldots$ of the end ξ, the Busemann homomorphism is the map $H \to \mathbb{Z}$ defined by $g \mapsto \lim_{i\to\infty}(d(g(x_i), x_0) - i - 1)$.

$w \in W$, the restriction $c \restriction_{E(w)}\colon E(w) \to [m]$ is bijective, for each $y \in VT \setminus W$, the restriction $c \restriction_{E(y)}\colon E(y) \to [m]$ is constant, and $\sigma_c(g, w) \in F$ for all $g \in H$ and $w \in W$. As F acts transitively, Proposition 2.8 ensures further that k can be chosen in such a way that $k(e) = k(\overline{e})$ for all $e \in ET$.

Let F^+ be the subgroup of F generated by the point stabilizers. Recall that by hypothesis, F^+ is intransitive and let $B_1, \dots, B_p$ list the F^+–orbits on $[m]$. The F^+–orbits form an F–equivariant equivalence relation on $[m]$, and as is customary, we call the B_i *blocks*. Let $\pi\colon F \to \mathrm{Sym}(\{B_1, \dots, B_p\})$ be the F–action on the blocks. It is easy to see that $\mathrm{Ker}(\pi) = F^+$ and that the F/F^+–action on the blocks is free.

Fix $g \in H$ and suppose that $w, w' \in W$ are such that every vertex different from w and w' on the geodesic $[w, w']$ is not in W. We infer that $c(e) = c(f)$ for all edges e and f on $[w, w']$, since $c(e) = c(\overline{e})$ for all edges e. The pair gw and gw' also enjoys the same condition on $[gw, gw']$, so $c(e) = c(f)$ for all edges e and f on $[gw, gw']$. If $\sigma_c(g, w) \in \mathrm{Ker}(\pi)$, then the elements $c(e)$ and $c(ge)$ belong to the same block for all $e \in E(w)$. Therefore, $c(e)$ and $c(ge)$ belong to the same block for all edges e on $[w, w']$. In particular, the edge $f \in E(w')$ on the geodesic from w' to w is such that $c(f)$ and $c(gf)$ belong to the same block. Thus, $\sigma_c(g, w')$ fixes a block, and hence $\sigma_c(g, w')$ belongs to $\mathrm{Ker}(\pi)$, since the F/F^+–action on the blocks is free. The obvious induction argument on $d(w, w')$ now shows for all $g \in H$ and $w, w' \in W$, $\sigma_c(g, w) \in \mathrm{Ker}(\pi)$ if and only if $\sigma_c(g, w') \in \mathrm{Ker}(\pi)$.

Take $g \in H$ which fixes pointwise an edge e in T. The local action $\sigma_c(g, o(e))$ must fix a block, so $\sigma_c(g, o(e)) \in \mathrm{Ker}(\pi)$, since the action of F/F^+ on the blocks is free. By the previous paragraph, $\sigma_c(g, w) \in \mathrm{Ker}(\pi)$ for all $w \in W$. As H^+ is generated by edge fixators, it follows that $\sigma_c(g, w) \in \mathrm{Ker}(\pi)$ for every $g \in H^+$ and $w \in W$. The degree of the image of w_0 in X is thus at least the number of blocks, which is at least 2.

We conclude all vertices of the tree X have degree at least 2, completing the proof. □

Remark 3.2 The hypothesis that T be locally finite in Theorem 3.1 is essential. Indeed, consider the action of the free product $G = S_1 * S_2$ of two infinite simple groups S_1 and S_2 on its Bass–Serre tree T, which is locally infinite since S_1 and S_2 are infinite. The G–action on T enjoys all the properties required by Theorem 3.1, but G is not virtually indicable: indeed G is perfect and does not have any proper subgroup of finite index.

A class of finite permutation groups satisfying the condition in the theorem is provided by the nilpotent groups.

Lemma 3.3 *Let $F \le \mathrm{Sym}(\Omega)$ be a permutation group on a finite set Ω of size at least 2. If F is nilpotent, then the subgroup of F generated by the point stabilizers is intransitive.*

Proof As the result is clear for F intransitive, we may assume that F is transitive. There exists an F–invariant equivalence relation $\sim$ on Ω such that the F–action on $\Omega/\sim$ is primitive and nontrivial. The only nilpotent primitive permutation groups are cyclic of prime order. Letting F^+ be the subgroup of F generated by the point stabilizers, the F–action on the $\sim$–equivalence classes is through a quotient that acts freely. We conclude that the F^+–action on the $\sim$–equivalence classes is trivial. In particular, every F^+–orbit is entirely contained in some $\sim$–equivalence class, so F^+ is intransitive. □

Proof of Corollary 1.2 We assume that G fixes neither a vertex nor an edge. Since G is compactly generated, it follows that G contains a hyperbolic element. If G fixes an end ξ of T, the Busemann homomorphism at ξ yields an infinite cyclic quotient of G, so G is virtually indicable. We thus assume that G also does not fix any end. There exists a minimal nonempty G–invariant subtree X in T. Since the G–action is fixed-point-free, the tree X is infinite, and X is locally finite since T is locally finite. The local action of the image of G in $\mathrm{Aut}(X)$ at every vertex v of X is a quotient of the local action of $G \le \mathrm{Aut}(T)$ at v. The group G thus acts on X with a nilpotent local action at every vertex. By Lemma 3.3, all hypotheses of Theorem 3.1 are satisfied, and the conclusion follows. □

4 Simple subquotients

In this section, we prove Theorem 1.8. That is, we show every nondiscrete Burger–Mozes group admits a subquotient belonging to the class $\mathscr{S}$.

4.1 Reduction to cyclic groups of order p acting on $p+1$ points

The first step in the proof of Theorem 1.8 consists in reducing the problem from all nonfree permutation groups F to the rather odd class of permutation groups $(C_p, [p+1])$ of cyclic groups of prime order p acting on $p+1$ points. Notice that the cyclic group C_p has only one faithful permutation representation on $[p+1]$ up to conjugacy. Thus, by Proposition 2.6, the group $U_c((C_p,[p+1]))$ is uniquely defined and does not depend on the choice of the regular legal coloring c. We shall denote it by $U(C_p)$.

The required reduction step is realized by the following observation.

Lemma 4.1 *Let $F \leq \mathrm{Sym}(d)$ be a permutation group and c be a regular legal coloring of the d–regular tree T. If F does not act freely, then $U_c(F)$ contains a closed subgroup isomorphic to $U(C_p)$ for some prime p.*

Proof Let $i \in [d]$ be such that $F_{(i)}$ is nontrivial. Replacing the coloring by $\sigma \circ c$ for a suitable permutation σ of $[d]$, we may assume that $i = 0$. Let p be a prime dividing the order of $F_{(0)}$ and let $C_p \leq F_{(0)}$ be a nontrivial cyclic subgroup of order p. We may find a set $\Omega \subseteq [d]$ of size $p+1$ such that $0 \in \Omega$ and C_p cyclically permutes $\Omega \setminus \{a\}$. By changing the coloring again if needed, we may assume that $\Omega = [p+1]$.

Fix a vertex $v \in VT$ and let X be the subtree spanned by the edges colored by $[p+1]$. The tree X is a copy of T_{p+1}, and the restriction of the ambient coloring to X gives an action of $U(C_p)$ on X. Taking $g \in U(C_p)$, we may extend the action of g on X to the whole tree T by declaring the local action to be trivial on $VT \setminus VX$. We thus deduce that $U(C_p) \leq U(F)$. □

The following subsidiary fact will be useful to identify $U(C_p)$ in a context where certain a priori illegal colorings are allowed.

Lemma 4.2 *Let $F \leq \mathrm{Sym}(p+1)$ be a permutation group which fixes 0 and transitively permutes $\{1, \dots, p\}$ and let T be the $(p+1)$–regular tree. If c is a regular coloring such that $c(e) = 0$ implies $c(\overline{e}) = 0$, then $U_c(F) = U_d(F)$, where d is a regular legal coloring. In particular, if p is a prime, we have $U_c(C_p) = U(C_p)$.*

Proof Fix a vertex $v \in VT$ and for each vertex $w \neq v$, define e_w to be the edge with origin w on the geodesic from w to v. For each $i, j \in \{1, \dots, p\}$, fix $g_{ij} \in F$ such that $g_{ij}(i) = j$; for $i = j$, we take the element g_{ij} to be trivial.

We define the coloring d by defining the value of d on each $e \in E(w)$ by induction on $d(v, w)$. For the base case, $w = v$, we put $d(e) = c(e)$ for each $e \in E(v)$. Suppose we have defined d on $E(w)$ for all $w \in B_n(v)$. Take w such that $d(w, v) = n+1$. If $c(\overline{e_w}) = 0$, then we set $d(e) = c(e)$ for all $e \in E(w)$. If $c(\overline{e_w}) = j \neq 0$, then we set $d(e) := g_{kj} c(e)$ for $e \in E(w)$, where $k := c(e_w)$. The function d is clearly a legal coloring.

Taking $g \in U_c(F)$, let us compute the local action of g according to d:

$$\sigma_d(g, v) = d_{g(v)} \circ g \circ d_v = z c_{g(v)} \circ g \circ c_v^{-1} y = z \sigma_c(g, v) y$$

for some $z, y \in F$. Since $g \in U_c(F)$, we infer that $\sigma_c(g, v) \in F$, hence $\sigma_d(g, v) \in F$. We conclude that $U_c(F) \leq U_d(F)$. The converse inclusion is similar. □

4.2 Frobenius groups

In view of Corollary 2.13, the group $U(C_p)^+$ is not a member of $\mathscr{S}$. In fact, by Theorem 3.1, we know that it is virtually indicable. In order to show that $U(C_p)$ has a simple subquotient in $\mathscr{S}$, we shall show that $U(C_p)$ has a simple subquotient of the form $U(F)^+$, where F is a Frobenius group of a specific kind that will be associated to p. Let us first recall the definition of Frobenius groups.

Definition 4.3 A *Frobenius* group is a transitive permutation group $(F,[n])$ such that the action is not free, but the stabilizer of every ordered pair of distinct points is trivial. A point stabilizer $F_{(i)}$ is called a *Frobenius complement*.

We shall need the existence of certain Frobenius groups. To experts the next theorem is likely obvious, but we sketch a proof. We thank Reid for pointing out to us this family of examples.

Theorem 4.4 *For each prime $p>2$, there is a finite Frobenius group F such that C_p is the Frobenius complement and $[F:C_p]$ is a power of 2.*

Proof Since p is coprime to 2, we may find a nontrivial irreducible representation $\phi\colon C_p \to \mathrm{GL}_n(\mathbb{F}_2)$, where $\mathbb{F}_2$ is the field with two elements. The representation ϕ induces an action $C_p \curvearrowright \mathbb{F}_2^n$ which is fixed-point-free. Consider the semidirect product $F := \mathbb{F}_2^n \rtimes_\phi C_p$.

We now argue that the action of F on the set of left cosets F/C_p shows that F is a Frobenius group. Certainly this action is transitive and has nontrivial point stabilizers. Suppose that $C_p \cap hC_ph^{-1}$ is nontrivial. Since C_p has no proper nontrivial subgroups, we deduce that $h \in N_F(C_p)$. The element h has the form (a,x), where $a \in \mathbb{F}_2^n$ and $x \in C_p$, and since $x \in N_F(C_p)$, the element a is in $N_F(C_p)$. Considering the conjugate $(a,1)(1,x)(a^{-1},1)$, it follows that $\phi(x)$ fixes a, and since $\phi(x)$ generates C_p, C_p in fact fixes a. We conclude that $a=1$ as C_p acts fixed-point-freely on $\mathbb{F}_2^n$, so $h \in C_p$. Two point stabilizers are thus trivial.

The second claim of the proposition is immediate. □

The next proposition is also likely well known. We again sketch a proof for completeness.

Proposition 4.5 *A finite Frobenius group is generated by its point stabilizers.*

Proof Let $F \le \mathrm{Sym}(n)$ be a Frobenius group and $L \le F$ be the subgroup generated by the point stabilizers. If i and j are in distinct L–orbits, then the stabilizer $L_{(i)} = F_{(i)}$

acts freely on the L–orbit of j, whose size is thus a multiple of $p = |F_{(0)}|$. On the other hand, $L_{(j)}$ has exactly one fixed point in the L–orbit of j, so the size of that orbit is congruent to 1 modulo p. This contradiction shows that L acts transitively. Thus $F = LF_{(0)} = L$, as required. □

Corollary 4.6 *For any nontrivial finite Frobenius group F, the group $U(F)^+$ is an abstractly simple open subgroup of index 2 in $U(F)$. In particular, $U(F)^+$ belongs to $\mathscr{S}$.*

Proof This is immediate from Proposition 4.5 and Corollary 2.13. □

4.3 Colorings and blow-ups

We now develop machinery to build new graphs from trees. This technique will, in particular, allow us to change the local action in stages to arrive at a subquotient of $U(C_p)$ which has the form $U(F)^+$ for F a Frobenius group.

Definition 4.7 Let T_n be the n–regular tree with $n \geq 3$ and c be a regular coloring of T_n by $[n]$. The *blow-up* of T_n relative to c is the graph $B_c(T_n)$ defined by $VB_c(T_n) := VT_n \times [n]$ and $((v,i),(w,j)) \in EB_c(T_n)$ if and only if either $v = w$ and $i \neq j$, or $(v,w) \in ET_n$, $c((v,w)) = i$, and $c((w,v)) = j$. When unimportant or clear from context, we suppress the subscript c.

The blow-up operation replaces the vertices of T_n by complete graphs on n vertices; see Figure 1. There is also a canonical action of $\mathrm{Aut}(T_n)$ on $B_c(T_n)$ by graph automorphisms: $g((v,i)) := (g(v), \sigma_c(g,v)(i))$, where $\sigma_c(g,v)$ is the local action relative to c. The blow-up of an n–regular tree is an n–regular graph. We note a useful condition to ensure groups acting on blow-ups do not invert edges.

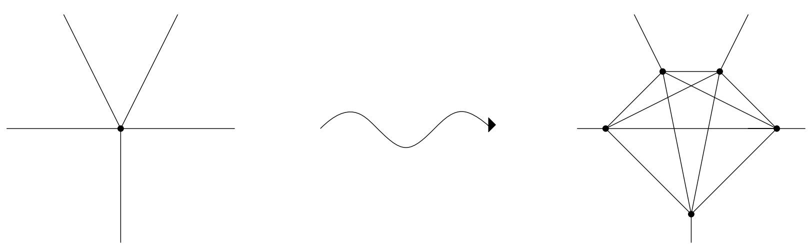

Figure 1: The blow-up of T_5

Lemma 4.8 *Let T_n with $n \geq 3$, let c be a regular coloring, and suppose that $H \leq \mathrm{Aut}(T_n)$ acts on T_n without edge inversion. If $\langle \sigma_c(h,v) \rangle$ acting on $[n]$ has no size 2 orbits for all $v \in VT_n$ and $h \in H_{(v)}$, then H acts on $B_c(T_n)$ without edge inversion.*

Proof Fix $h \in H$. Edges in $B_c(T_n)$ have either the form $((v,i),(w,j))$, where $(v,w) \in ET_n$, $c((v,w)) = i$, and $c((w,v)) = j$, or the form $((v,i),(v,j))$ for some $i \neq j$.

For edges of the form $((v,i),(w,j))$,

$$h\big(((v,i),(w,j))\big) = \big((h(v), \sigma_c(h,v)(i)), (h(w), \sigma_c(h,w)(j))\big).$$

Since h does not invert edges of T_n, we see that h cannot invert such an edge.

For edges of the form $((v,i),(v,j))$ for some $i \neq j$,

$$h\big(((v,i),(v,j))\big) = \big((h(v), \sigma_c(h,v)(i)), h(v), \sigma_c(h,v)(j)\big).$$

If h inverts such an edge, then $h(v) = v$, $\sigma_c(h,v)(i) = j$, and $\sigma_c(h,v)(j) = i$, which implies that $\langle \sigma_c(h,v) \rangle$ has an orbit of size 2. We deduce that h also does not invert these edges. The lemma is now verified. □

Definition 4.9 Let T_n be the n–regular tree, c be a regular coloring of T_n, and $\mathcal{P} := \{O_0, \dots, O_{d-1}\}$ be an ordered partition of $[n]$. The *partition blow-up* of T_n with respect to c and $\mathcal{P}$ is the graph $B_{c,\mathcal{P}}(T_n)$ defined as follows: $VB_{c,\mathcal{P}}(T_n) := VT_n \times \mathcal{P}$ and $((v, O_i), (w, O_j)) \in EB_c(T_n)$ if and only if either $v = w$ and $i - j = \pm 1 \bmod d$, or $(v,w) \in ET_n$, $c((v,w)) \in O_i$, and $c((w,v)) \in O_j$.

An example of a partition blow-up is depicted in Figure 2. We stress that in the case of an ordered partition into singletons, the partition blow-up does not coincide with the blow-up defined previously. The partition blow-up of T_4 with respect to $\mathcal{P} := \{\{1\},\{2\},\{3\},\{4\}\}$, for instance, is a 3–regular graph.

Given a partition $\mathcal{Q}$ of $[n]$, the *Young subgroup* associated to $\mathcal{Q}$ is the group of permutations of $[n]$ which setwise fix the parts of the partition. Letting $P \leq \mathrm{Sym}([n])$ be the Young subgroup associated to $\mathcal{P}$, the Burger–Mozes group $U_c(P)$ acts on $B_{c,\mathcal{P}}(T_n)$ by $g((v, O_i)) = (g(v), O_i)$. Taking $U_c^+(P)$, it follows that $U_c^+(P)$ acts on $B_{c,\mathcal{P}}(T_n)$ without edge inversion.

We need to consider covering trees of the blow-ups. Lifting colorings to covering trees thereby becomes important.

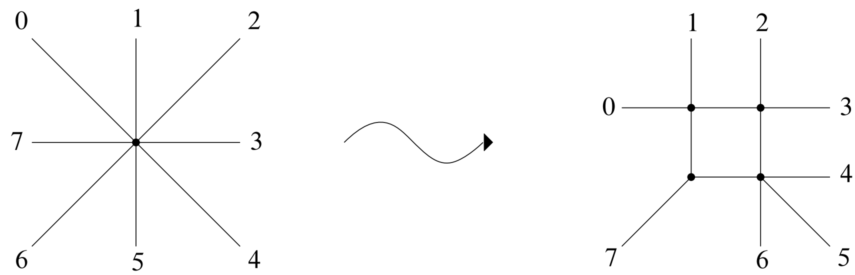

Figure 2: The partition blow-up of T_8 relative to the ordered partition $\{\{0,1\},\{2,3\},\{4,5,6\},\{7\}\}$

Definition 4.10 Let X be an n–regular graph with a coloring c and $\phi\colon T_n \to X$ the covering map, where T_n is the n–regular tree. We call $\widetilde{c} := c \circ \phi$ the *lifted coloring* on T_n induced by c.

Lemma 4.11 *Let X be an n–regular graph with a regular coloring c, let $\phi\colon T_n \to X$ be the covering map, and let $\Phi\colon H \to \mathrm{Aut}(X)$ be the covering homomorphism afforded by Theorem 2.1. Then the lifted coloring $\widetilde{c}$ is a coloring of T_n, and*

$$\sigma_{\widetilde{c}}(g,v) = \sigma_c(\Phi(g), \phi(v))$$

for all $g \in H$ and $v \in VT_n$.

Proof That $\widetilde{c}$ is a coloring is immediate since ϕ_E is a bijection when restricted to $E_{T_n}(v)$.

For the second claim, the following diagram commutes for all $g \in H$ and $v \in VT_n$, via Theorem 2.1:

$$\begin{array}{ccc} E_{T_n}(v) & \xrightarrow{\;g\;} & E_{T_n}(g(v)) \\ \big\downarrow \phi_E & & \big\downarrow \phi_E \\ E_X(\phi(v)) & \xrightarrow{\Phi(g)} & E_X(\phi(g(v))) \end{array}$$

Defining $\phi_v := (\phi_E)\restriction_{E(v)}\colon E_{T_n}(v) \to E_X(\phi(v))$, we see that

$$\begin{aligned} \sigma_{\widetilde{c}}(g,v) &= c_{\phi(g(v))} \circ \phi_{g(v)} \circ g \circ \phi_v^{-1} \circ c_{\phi(v)}^{-1} \\ &= c_{\phi(g(v))} \circ \Phi(g) \circ c_{\phi(v)}^{-1} \\ &= c_{\Phi(g)(\phi(v))} \circ \Phi(g) \circ c_{\phi(v)}^{-1} \\ &= \sigma_c(\Phi(g), \phi(v)). \end{aligned}$$

□

4.4 Simple subquotients

We are now prepared to prove the main theorem of this section, Theorem 4.16, from which Theorem 1.8 will easily follow. The proof of Theorem 4.16 makes heavy use of blow-ups, and the argument is inspired by Lemma 9.2 from J Huang's paper [13].

We begin by proving the $p=2$ case of the main technical theorem; this case requires separate analysis to deal with edge inversions.

Proposition 4.12 *For $C_2 \leq \mathrm{Sym}(3)$, the group $U(C_2)$ contains a compactly generated closed subgroup H admitting a discrete normal subgroup D such that H/D is isomorphic to $\mathrm{Aut}(T_3)^+$.*

Proof Let $G := \mathrm{Aut}(T_3)^+$. Consider the G–action on the blow-up X of the trivalent tree T_3. This action inverts edges, so we take $\widehat{X}$ the barycentric subdivision of X. By Theorem 2.1, the group G lifts to a unimodular closed subgroup J of the automorphism group of the covering tree T of $\widehat{X}$.

The tree T has vertices of degree 2 and degree 3; denote the set of degree 3 vertices by V_3. By construction, the local action of J at v for $v \in V_3$ is the cyclic group C_2 acting on three points. Define a new tree T' by $VT' := V_3$ and $(v,w) \in ET'$ if and only if either $(v,w) \in ET$ or the geodesic from v to w uses only degree 2 vertices, other than v and w. The graph T' is the 3–regular tree, $J \leq \mathrm{Aut}(T')$ and acts vertex transitively, and the local action of J at any $v \in T'$ is the cyclic group C_2.

The kernel of the covering homomorphism $\Phi\colon J \to G$ is discrete. In view of the unimodularity of J, it follows from Corollary 2.10 that J is contained in $U_c(C_2) = U(C_2)$ for some regular legal coloring c of T_3. The group J is thus a compactly generated closed subgroup of $U(C_2)$ that admits $\mathrm{Aut}(T_3)^+$ as a quotient modulo a discrete normal subgroup. □

For the $p>2$ case, we proceed by proving three lemmas. For $G \leq \mathrm{Aut}(T)$, recall from Section 2.2 that G^* is the subgroup of G that preserves the natural bipartition of VT.

Lemma 4.13 *For $p>2$ a prime, suppose $(F,[n])$ is transitive and such that $F_{(0)} = C_p$ and suppose c is a regular legal coloring of T_n. Letting $(C_p,[n]) := (F_{(0)},[n])$, there is a regular coloring d of T_n such that*

1. *for all $e \in ET_n$, $d(e)=0$ if and only if $d(\overline{e})=0$, and*
2. *there is a closed subgroup $K \leq U_d((C_p,[n]))^*$ admitting a continuous homomorphism $\Phi\colon K \to U_c((F,[n]))^*$ with discrete kernel and finite-index open image.*

Proof Set $G := U_c((F,[n]))^*$, let X be the blow-up $B_c(T_n)$, and recall that G acts on X by graph automorphisms. Since the vertex stabilizers of F equal C_p for $p > 2$, the conditions of Lemma 4.8 are satisfied. We conclude that G acts on X without edge inversion.

For each $i \in [n]$, fix $g_i \in F$ such that $g_i(i) = 0$ and say that $g_0 = 1$; we may find these elements since $(F,[n])$ is transitive. Recalling that each $v \in VX$ has the form (u,i) for some $u \in VT_n$ and $i \in [n]$, we define a coloring a on X by

$$a(e) := g_{\pi_2(o(e))}\pi_2(t(e)),$$

where $\pi_2\colon VT_n \times [n] \to [n]$ is the projection onto the second coordinate. That c is a regular legal coloring ensures that a is a regular coloring. The coloring a is also such that $a(e) = 0$ if and only if $a(\bar{e}) = 0$, again since c is a legal coloring.

Setting $a_{(v,i)} := a\restriction_{E_X((v,i))}$, the map $a_{(v,i)}^{-1}\colon [n] \to E_X((v,i))$ is such that

$$a_{(v,i)}^{-1}(j) = \big((v,i),(u,g_i^{-1}(j))\big),$$

where $u = v$ if and only if $j \neq 0$. From this observation and a simple, but tedious computation, it follows that

$$\sigma_a(g,(v,i)) = g_{\sigma_c(g,v)(i)} \cdot \sigma_c(g,v) \cdot g_i^{-1}.$$

The elements g_j are in F for all $j \in [n]$, so $\sigma_a(g,(v,i)) \in F$. Additionally, $\sigma_a(g,(v,i))(0) = 0$. We deduce that $\sigma_a(g,(v,i)) \in F_{(0)}$ for all $(v,i) \in VX$. Recalling that $F_{(0)} = C_p$, the local action of G on X is $(C_p,[n]) = (F_{(0)},[n])$.

The covering tree of X is T_n. Let d be the lift of the coloring a to T_n. It follows that $d(e) = 0$ if and only if $d(\bar{e}) = 0$ and that d is a regular coloring. Applying Theorem 2.1, we obtain $H \leq \mathrm{Aut}(T_n)$ the covering group of G and $\Phi\colon H \to G$ the covering homomorphism. The subgroup H is closed, and the map Φ has a discrete kernel. Via Lemma 4.11, $\sigma_d(g,v)$ is an element of $(C_p,[n])$ for all $g \in H$ and $v \in VT_n$, so in fact $H \leq U_d((C_p,[n]))$. Taking $K := H \cap U_d((C_p,[n]))^*$ now satisfies the lemma. □

We now make use of the partition blow-up.

Lemma 4.14 *For $p > 2$ a prime and $n > 3$ even, suppose that $(C_p,[n])$ is a permutation group with more than two orbits such that 0 is the only fixed point and suppose that c is a regular coloring of T_n such that $c(e) = 0$ if and only if $c(\bar{e}) = 0$.*

Letting $(C_p, [p+2])$ *be the permutation group given by the cycle* $(2, \dots, p+1)$, *there is a regular coloring* d *of* T_{p+2} *such that*

(1) *for all* $e \in ET_{p+2}$ *and* $i \in \{0, 1\}$, $d(e) = i$ *if and only if* $d(\overline{e}) = i$, *and*

(2) *there is a closed subgroup* $H \leq U_d((C_p, [p+2]))^*$ *admitting a continuous homomorphism* $\Phi\colon K \to U_c((C_p, [n]))^*$ *with discrete kernel and finite-index open image.*

Proof Set $G := U_c((C_p, [n]))^*$. Let $O_0, \dots, O_{l-1}$ list the orbits of C_p on $[n]$ such that $O_0 = \{0\}$ and observe that l is an even number with $l > 2$, by our hypotheses. Take $B_{c,\mathcal{P}}(T_n)$ the partition blow-up of T_n with respect to $\mathcal{P} := \{O_0, \dots, O_{l-1}\}$ and the coloring c. The group G acts on $B_{c,\mathcal{P}}(T_n)$, because C_p setwise fixes the parts of the partition. This action is also without edge inversion, because G acts on T_n without edge inversion.

The part O_0 consists of only one element, so we modify $B_{c,\mathcal{P}}(T_n)$ as to ensure that for every vertex v, there are $p+2$ edges with origin v. To this end, we proceed as follows. Delete each edge e of the form $e = ((v, O_0), (w, O_0))$ and add new, distinct edges $e_2, \dots, e_{p+1}$ to $EB_{c,\mathcal{P}}(T_n)$ such that $o(e_i) = (v, O_0)$ and $t(e_i) = (w, O_0)$. Since $c((v, w)) = 0$ implies $c((w, v)) = 0$, we may define $\overline{e_i} := (\overline{e})_i$. We call the resulting graph Y, and this graph is $(p+2)$–regular. The group G acts on Y via extending the action on $B_{c,\mathcal{P}}(T_n)$ by declaring that $g(e_i) := (g(e))_i$.

Fixing x a generator for C_p, the element x has a cycle decomposition $s_1 \cdots s_{l-1}$, where the s_i are pairwise disjoint p–cycles and s_i is the p–cycle that permutes O_i. Fix $h_i \in \mathrm{Sym}([n])$ such that $h_i s_i h_i^{-1}$ is the p–cycle $(2, \dots, p+1)$. In particular, $h_i(O_i) = \{2, \dots, p+1\}$. We now define a regular coloring $a\colon EY \to [p+2]$ of Y. Fix a vertex $(v, O_i) \in VY$ and let $f \in E_Y((v, O_i))$.

(1) If $i = 0$, then

$$a(f) := \begin{cases} k & \text{if } f = e_k \text{ for some } k \in \{2, \dots, p+1\}, \\ 1 & \text{if } f = ((v, O_0), (v, O_{l-1})), \\ 0 & \text{if } f = ((v, O_0), (v, O_1)). \end{cases}$$

(2) If $i = l-1$, then

$$a(f) := \begin{cases} h_{l-1}c((v, w)) & \text{if } f = ((v, O_{l-1}), (w, O_j)) \text{ with } v \neq w, \\ 0 & \text{if } f = ((v, O_{l-1}), (v, O_{l-2})), \\ 1 & \text{if } f = ((v, O_{l-1}), (v, O_0)). \end{cases}$$

(3) If $i \neq l-1$ and is odd, then

$$a(f) := \begin{cases} h_i c((v,w)) & \text{if } f = ((v,O_i),(w,O_j)) \text{ with } v \neq w, \\ 0 & \text{if } f = ((v,O_i),(v,O_{i-1})), \\ 1 & \text{if } f = ((v,O_i),(v,O_{i+1})). \end{cases}$$

(4) If $i \neq 0$ and is even, then

$$a(f) := \begin{cases} h_i c((v,w)) & \text{if } f = ((v,O_i),(w,O_j)) \text{ with } v \neq w, \\ 1 & \text{if } f = ((v,O_i),(v,O_{i-1})), \\ 0 & \text{if } f = ((v,O_i),(v,O_{i+1})). \end{cases}$$

The map a is a regular coloring, and furthermore, $a(e) = z$ implies that $a(\overline{e}) = z$ for $z \in \{0,1\}$. The latter claim follows since $l-1$ is odd.

Let us compute the local action $\sigma_a(g,(v,O_i))$ for $g \in G$. If $i = 0$, then it is immediate that $\sigma_a(g,(v,O_0)) = 1$. As the remaining cases are similar, we compute the local action $\sigma_a(g,(v,O_i))$ for $i \neq 0$ and even. We see that

$$a^{-1}_{(v,O_i)}(k) = \begin{cases} ((v,O_i),(v,O_{i-1})) & \text{if } k = 1, \\ ((v,O_i),(v,O_{i+1})) & \text{if } k = 0, \\ ((v,O_i),(w,O_j)) & \text{if } c(v,w) = h_i^{-1}(k) \text{ and } k \neq 0,1. \end{cases}$$

It follows immediately that $\sigma_a(g,(v,O_i))(k) = k$ for $k \in \{0,1\}$. For $k \notin \{0,1\}$, we see that

$$\sigma_a(g,(v,O_i))(k) = a\big(((g(v),O_i),(g(w),O_j))\big)$$

such that $c(v,w) = h_i^{-1}(k)$. The value $c((g(v),g(w))$ equals $\sigma_c(g,v)(c(v,w))$, so

$$a\big(((g(v),O_i),(g(w),O_j))\big) = h_i \sigma_c(g,v) h_i^{-1}(k).$$

By our choice of h_i, we conclude that $h_i \sigma_c(g,v) h_i^{-1}$ acts as some power of the p–cycle $(2,\dots,p+1)$ on $[p+2]$. For all $g \in G$ and $(v,O_i) \in VY$, it is thus the case that $\sigma_c(g,(v,O_i)) \in (C_p,[p+2])$.

The covering tree of Y is T_{p+2}. Let d be the lift of the coloring a to T_{p+2}. It follows that $d(e) = i$ if and only if $d(\overline{e}) = i$ for $i \in \{0,1\}$ and that d is regular. Applying Theorem 2.1, we obtain the covering group $H \leq \mathrm{Aut}(T_{p+2})$ of G and the covering homomorphism $\Phi\colon H \to G$. The subgroup H is closed, and the map Φ has a discrete kernel. Via Lemma 4.11, $\sigma_d(g,v)$ is an element of $(C_p,[p+2])$ for all $g \in H$ and $v \in VT_{p+2}$, so in fact $H \leq U_d((C_p,[p+2]))$. Taking $K := H \cap U_d((C_p,[p+2]))^*$ now satisfies the lemma. □

Lemma 4.15 *For $p > 2$ a prime, suppose $(C_p, [p+2])$ is a permutation group given by the cycle $(2, \dots, p+1)$ and suppose c is a regular coloring of T_{p+2} such that $c(e) = i$ if and only if $c(\overline{e}) = i$ for $i \in \{0, 1\}$. There is a closed subgroup $H \leq U(C_p)$ admitting a continuous epimorphism $\Phi\colon H \to U_c((C_p, [p+2]))^*$ with discrete kernel.*

Proof Set $G := U_c((C_p, [p+2]))^*$. Let $W := \{0, 1\}$ and $U := \{2, \dots, p+1\}$ and let $B_{c,\mathcal{P}}(T_{p+2})$ be the partition blow-up of T_{p+2} with respect to $\mathcal{P} := \{W, U\}$ and the coloring c. The group G acts on $B_{c,\mathcal{P}}(T_{p+2})$, because C_p setwise fixes the parts of the partition. This action is also without edge inversion, because G acts on T_{p+2} without edge inversion.

The part W consists of only two elements, so we modify $B_{c,\mathcal{P}}(T_{p+2})$ as to ensure that for every vertex $(v, L) \in VB_{c,\mathcal{P}}(T_{p+2})$, there are $p+1$ edges with origin (v, L). To this end, we proceed as follows. For each edge e of the form $e = ((v, W), (w, W))$ with $c((v, w)) = 0$, we delete the edge e and add new edges $e_1, \dots, e_{p-1}$ to $EB_{c,\mathcal{P}}(T_{p+2})$ such that $o(e_i) = (v, W)$ and $t(e_i) = (w, W)$. For the vertex (v, W), there is also an edge $((v, W), (u, W))$ where $c((v, u)) = 1$. We rename this edge e_p. Since $c((v, w)) = c((w, v))$ whenever $c((v, w)) \in \{0, 1\}$, we may define $\overline{e_i} := (\overline{e})_i$. We call the resulting graph Z, and G acts on Z via extending the action on $B_{c,\mathcal{P}}(T_{p+2})$ by declaring that $g(e_i) := (g(e))_i$. This action clearly also does not invert edges.

Fix $h \in \mathrm{Sym}([p+2])$ such that $h(2, \dots, p+1)h^{-1} = (1, \dots, p)$. For $f \in E_Z((v, W))$, we define

$$a(f) := \begin{cases} i & \text{if } f = e_i \text{ for some } i \in \{1, \dots, p\}, \\ 0 & \text{if } f = ((v, W), (v, U)). \end{cases}$$

For $f \in E_Z((v, U))$, we define

$$a(f) := \begin{cases} 0 & \text{if } f = ((v, U), (v, W)), \\ hc((v, w)) & \text{if } f = ((v, U), (w, U)). \end{cases}$$

The map a is a regular coloring of Z by $[p+1]$ and $a(e) = 0$ implies $a(\overline{e}) = 0$.

Let us now compute the local action $\sigma_a(g, (v, L))$ for $g \in G$. It is immediate that if $L = W$, then $\sigma_a(g, (v, W)) = 1$. For $L = U$, we note that

$$a^{-1}_{(v,U)}(k) = \begin{cases} ((v, U), (v, W)) & \text{if } k = 0, \\ ((v, U), (w, U)) & \text{if } k = hc(v, w) \text{ and } k \neq 0. \end{cases}$$

It follows immediately that $\sigma_a(g, (v, U))(0) = 0$. For $k \neq 0$,

$$\sigma_a(g, (v, U))(k) = a\big(((g(v), U), (g(w), U))\big)$$

such that $c(v,w) = h^{-1}k$. The value $c((g(v), g(w))$ equals $\sigma_c(g,v)(c(v,w))$, so

$$a\big(((g(v),U),(g(w),U))\big) = h\sigma_c(g,v)h^{-1}(k).$$

By our choice of h, we conclude that $h\sigma_c(g,v)h^{-1}$ acts as some power of the p–cycle $(1,\dots,p)$ on $[p+1]$. We conclude that for all $g \in G$ and $(v,L) \in VZ$, $\sigma_a(g,(v,L)) \in (C_p,[p+1])$, where $(C_p,[p+1])$ is given by the cycle $(1,\dots,p)$.

The covering tree of Z is T_{p+1}. Let d be the lift of the coloring a to T_{p+1}. It follows that $d(e) = 0$ if and only if $d(\overline{e}) = 0$. Applying Theorem 2.1, we obtain the covering group $H \le \mathrm{Aut}(T_{p+1})$ of G and the covering homomorphism $\Phi\colon H \to G$. The subgroup H is closed, and the map Φ has a discrete kernel. Via Lemma 4.11, $\sigma_d(g,v)$ is an element of $(C_p,[p+1])$ for all $g \in H$ and $v \in VT_{p+1}$, so in fact $H \le U_d((C_p,[p+1]))$. Lemma 4.2 ensures that $U_d((C_p,[p+1])) = U(C_p)$, so the lemma is verified. □

We are now prepared to prove the main technical theorem of this section. The hypotheses of case (2) of the next theorem are satisfied by the Frobenius groups found in Theorem 4.4.

Theorem 4.16 *Suppose that p is a prime and $C_p \le \mathrm{Sym}(p+1)$.*

(1) *If $p = 2$, then $U(C_p)$ contains a compactly generated closed subgroup H admitting a discrete normal subgroup D such that H/D is isomorphic to $\mathrm{Aut}(T_3)^+$.*

(2) *If $p > 2$ and $F \le \mathrm{Sym}(n)$ is a Frobenius group such that the Frobenius complement is C_p and has index a power of 2, then $U(C_p)$ contains a compactly generated closed subgroup H admitting a discrete normal subgroup D such that H/D is isomorphic to $U(F)^+$.*

Proof The case of $p = 2$ is already established in Proposition 4.12. Let us then suppose that $p > 2$ and $F \le \mathrm{Sym}(n)$ is a Frobenius group such that the Frobenius complement is C_p and has index a power of 2.

Fix c_1 a legal coloring of T_n, let $G_1 := U_{c_1}(F)^+$, and set $(C_p,[n]) := (F_{(0)},[n])$. By Lemma 4.13, there is a coloring c_2 of T_n such that $U_{c_2}((C_p,[n]))^*$ has a closed subgroup $G_2 \le U_{c_2}((C_p,[n]))^*$ admitting a continuous homomorphism $\Phi_1\colon G_2 \to G_1$ with discrete kernel and finite-index open image. As G_1 is compactly generated, we may take G_2 to be compactly generated, and as G_1 is simple, Φ_1 is indeed an epimorphism. Additionally, $c_2(e) = 0$ if and only if $c_2(\overline{e}) = 0$.

If $(C_p, [n])$ has two orbits on $[n]$, then $n = p + 1$. Lemma 4.2 supplies a legal coloring d such that $U_{c_2}((C_p, [p+1])) = U_d((C_p, [p+1])) = U(C_p)$. Therefore, $U(C_p)$ contains a compactly generated closed subgroup G_2 admitting a discrete normal subgroup D such that G_2/D is isomorphic to $G_1 = U(F)$, as required.

Let us henceforth assume that $(C_p, [n])$ has more than two orbits on $[n]$. Since $(F, [n])$ is a Frobenius group, $(C_p, [n])$ has exactly one fixed point, namely 0. We are thus in a position to apply Lemma 4.14. Letting $(C_p, [p+2])$ be the permutation group given by the cycle $(2, \dots, p+1)$, there is a coloring c_3 of T_{p+2} such that $U_{c_3}((C_p, [p+2]))^*$ has a closed subgroup $H \leq U_{c_3}((C_p, [p+2]))^*$ admitting a continuous homomorphism $\Phi\colon H \to U_{c_2}((C_p, [n]))^*$ with discrete kernel and finite-index open image. Additionally, $c_3(e) = i$ if and only if $c_3(\overline{e}) = i$ for $i \in \{0, 1\}$. We may find a closed compactly generated $G_3 \leq H$ such that $\Phi_2 := \Phi\restriction_{G_3}\colon G_3 \to G_2$ has a discrete kernel and a finite-index open image.

By Lemma 4.15, $U(C_p)$ has a closed subgroup H admitting a continuous epimorphism $\Phi\colon H \to U_{c_3}((C_p, [p+2]))^*$ with discrete kernel. We may find a closed compactly generated $G_4 \leq H$ such that $\Phi_3 := \Phi\restriction_{G_4}\colon G_4 \to G_3$ is surjective with discrete kernel. The map $\Psi\colon G_4 \to G_1$ by $\Psi := \Phi_1 \circ \Phi_2 \circ \Phi_3\colon G_4 \to G_1$ is continuous and has a discrete kernel and finite-index open image. As G_1 is simple, Ψ is indeed onto. We conclude that $U(C_p)$ admits a compactly generated closed subgroup G_4 admitting a discrete normal subgroup D such that G_4/D is isomorphic to $U(F)^+$. The theorem is now verified. □

Corollary 4.17 *Let $F \leq \mathrm{Sym}(d)$ be a permutation group that does not act freely. Then $U(F)$ contains a compactly generated closed subgroup H admitting a discrete normal subgroup D such that H/D is topologically simple and nondiscrete. In particular, $U(F)$ admits a subquotient in $\mathscr{S}$.*

Proof This is immediate from Lemma 4.1 and Theorem 4.16. □

4.5 Elementary groups and relative Tits cores

The class of elementary groups, denoted by $\mathscr{E}$, is the smallest class of totally disconnected, locally compact, second countable (t.d.l.c.s.c.) groups that contains the second countable profinite groups and the countable discrete groups and that is closed under the elementary operations; see [25]. (These operations are taking closed subgroups, Hausdorff quotients, group extensions, and countable directed unions of open subgroups.) The class of elementary groups is disjoint from the class $\mathscr{S}$ comprising the nondiscrete compactly generated topologically simple t.d.l.c.s.c. groups.

We say that a topological group H admits a group G as a *subquotient* if there is some closed subgroup $K \leq H$ such that K admits G as a continuous quotient. Admitting a subquotient a group in $\mathscr{S}$ is sufficient to be nonelementary; see [25]. The following consequence of Corollary 4.17 implies that the nondiscrete Burger–Mozes universal groups $U(F)$ all admit groups in $\mathscr{S}$ as subquotients and are thus nonelementary.

Corollary 4.18 *For F a finite permutation group, the following are equivalent:*

(1) *F acts freely.*

(2) *$U(F)$ is discrete.*

(3) *$U(F)$ is elementary.*

(4) *$U(F)$ does not admit a group in $\mathscr{S}$ as a subquotient.*

Proof That (1) implies (2) follows from Proposition 2.11(i), and (2) implies (3) is immediate. The contrapositive of (3) implies (4) is given by [25, Proposition 6.5]. Finally, Corollary 4.17 gives the contrapositive of (4) implies (1). □

All known examples of nonelementary groups are thus because they admit some group in $\mathscr{S}$ as a subquotient. One naturally asks if admitting a subquotient in $\mathscr{S}$ is a necessary condition to be nonelementary.

Question 4.19 For G a t.d.l.c.s.c. group, if G is nonelementary, then is there a compactly generated closed $H \leq G$ such that H has a continuous quotient in $\mathscr{S}$?

One possible approach to Question 4.19 is via a stronger formulation due to Reid. A positive answer to Reid's question, Question 4.20 below, would imply the positive answer to Question 4.19. For a locally compact group G, recall that $g \in G$ is *periodic* if $\langle g \rangle$ is relatively compact. For any element $g \in G$, we define the *contraction group* of g as

$$\mathrm{con}(g) := \bigl\{x \in G \mid \lim_{n\to+\infty} g^n x g^{-n} = 1\bigr\}.$$

The *relative Tits core* of g in G, denoted by $G_g^\dagger$, is defined by

$$G_g^\dagger := \overline{\langle \mathrm{con}(g) \cup \mathrm{con}(g^{-1}) \rangle}.$$

Question 4.20 (Reid [22, Question 2]) Let G be a t.d.l.c.s.c. group. If G is nonelementary, then is there $g \in G$ nonperiodic and $n \geq 1$ such that g^n is an element of the closure of the relative Tits core of g?

The Burger–Mozes groups $U(F)$ with F nilpotent provide examples demonstrating that the answer to Question 4.20 is negative.

Corollary 4.21 *For F nilpotent, every nonperiodic element $g \in U(F)$ is such that $g^n \notin G_g^\dagger$ for all $n \neq 0$.*

Proof In any locally compact group G, the group $\overline{\langle G_g^\dagger, g\rangle}$ is compactly generated for any $g \in G$. Let now $G = U(F)$ and let $g \in U(F)$ be a nonperiodic element. The group $H := \overline{\langle G_g^\dagger, g\rangle}$ is thus a compactly generated closed subgroup of $U(F)$, and g is hyperbolic. Appealing to Corollary 1.2, H admits an infinite discrete quotient H/O. Since $G_g^\dagger$ is topologically generated by contraction groups, it follows that $G_g^\dagger \leq O$. We deduce that $H/\overline{G_g^\dagger}$ is infinite, and thus, $g^n \notin \overline{G_g^\dagger}$ for any nonzero power of g. □

If in addition F does not act freely, then $U(F)$ is nonelementary by Corollary 4.18, so we obtain the following consequence of Theorem 3.1, yielding a negative answer to Question 4.20.

Corollary 4.22 *For any nilpotent permutation group (F, Ω) such that F does not act freely on Ω, the Burger–Mozes universal group $U(F)$ is nonelementary, and every nonperiodic element $g \in U(F)$ is such that $g^n \notin G_g^\dagger$ for all nonzero n.*

5 Lattices

5.1 Intersecting lattices with subgroups

The following basic facts are well known.

Proposition 5.1 *Let G be a locally compact group, $O \leq G$ be an open subgroup and $H \leq G$ be a closed subgroup.*

(i) *If H is cocompact in G, then $H \cap O$ is cocompact in O.*

(ii) *If H is of finite covolume in G, then $H \cap O$ is of finite covolume in O.*

(iii) *If H is a lattice in G, then $H \cap O$ is a lattice in O.*

Proof All the assertions follow by observing that the openness of O implies that the image of the canonical projection

$$O/H \cap O \to G/H$$

is both open and closed. $\square$

5.2 The amenable radical and centralizers of lattices

The following fundamental fact is essentially due to H Furstenberg; see for example [10, Lemma 16.7]. For completeness, we give a proof via the main result of [2].

Theorem 5.2 *Let G be a second countable locally compact group. If G has a trivial amenable radical, then every closed subgroup of finite covolume has a trivial amenable radical.*

Proof Let H be a closed subgroup of finite covolume in G and $R \leq H$ be the amenable radical of H. The orbit map under the conjugation action of G induces a continuous surjective map $G/H \to \{gRg^{-1} \mid g \in G\} \subset \mathbf{Sub}(G)$. Pushing forward the finite invariant measure on G/H, we get an amenable invariant random subgroup (IRS) of G. The main result of [2] ensures that this IRS is contained in the amenable radical of G, which is trivial by hypothesis. We conclude that R is trivial. $\square$

The following algebraic consequences will be useful.

Corollary 5.3 *Let G be a σ–compact locally compact group and $H \leq G$ be a closed subgroup of finite covolume. Then the centralizer $\mathrm{C}_G(H)$ is contained in the amenable radical of G. In particular, if G has trivial amenable radical, then H has trivial centralizer.*

Proof It suffices to prove the statement in the special case where the amenable radical of G is trivial: indeed, if $\mathrm{C}_G(H)$ is not contained in the amenable radical $R(G)$, then the closure of the image of H in $G/R(G)$ is a closed subgroup of finite covolume with a nontrivial centralizer in $G/R(G)$. We assume henceforth that $R(G) = \{1\}$.

By [12, Theorem 8.7], the group G has a compact normal subgroup K such that G/K is second countable. Since G has a trivial amenable radical, the group K is trivial, hence G is second countable. Let $a \in \mathrm{C}_G(H)$ and set $A = \overline{\langle a \rangle}$ and $B = \overline{AH}$. The subgroup B is closed, is of finite covolume in G, and contains A in its center. Theorem 5.2 implies that A is trivial. We conclude that $\mathrm{C}_G(H)$ is trivial. $\square$

Corollary 5.4 *Let G be a locally compact group with a discrete amenable radical. If $\Gamma \leq G$ is a finitely generated lattice, then $\mathrm{N}_G(\Gamma)/\Gamma$ is finite. In particular, $\mathrm{N}_G(\Gamma)$ is a finitely generated lattice.*

Proof Since G contains a finitely generated lattice, it is compactly generated by [7, Lemma 2.12]. In particular, G is σ–compact. By [12, Theorem 8.7], every identity neighborhood of G contains a compact normal subgroup K such that G/K is second countable. Applying this fact to an identity neighborhood that intersects trivially the amenable radical $R(G)$, which is discrete by hypothesis, we deduce that G is second countable. By Corollary 5.3, we have $\mathrm{C}_G(\Gamma) \leq R(G)$, so that $\mathrm{C}_G(\Gamma)$ is discrete, hence countable since G is second countable. The kernel of natural continuous map $\mathrm{N}_G(\Gamma) \to \mathrm{Aut}(\Gamma)$ coincides with $\mathrm{C}_G(\Gamma)$. As Γ is finitely generated, its automorphism group is countable, so $\mathrm{N}_G(\Gamma)$ is a countable locally compact group. Hence, $\mathrm{N}_G(\Gamma)$ is discrete by the Baire category theorem. Since Γ is of finite covolume in G, it is of finite covolume in $\mathrm{N}_G(\Gamma)$ (see [19, Lemma I.1.6]), hence it is of finite index in $\mathrm{N}_G(\Gamma)$. The required assertions now follow. □

5.3 The discrete residual and the quasicenter

The *discrete residual* of a topological group G, denoted by $\mathrm{Res}(G)$, is the intersection of all open normal subgroups of G. A group whose discrete residual is trivial is called *residually discrete*. Residually discrete groups are exactly the groups such that each nontrivial element is nontrivial in some discrete quotient.

The *quasicenter* of a locally compact group G, denoted by $\mathrm{QZ}(G)$, is the set of elements whose centralizer is open. It is a characteristic (but not necessarily closed) subgroup of G containing all discrete normal subgroups.

Lemma 5.5 *For G_1 and G_2 locally compact groups, we have*

$$\mathrm{Res}(G_1 \times G_2) = \mathrm{Res}(G_1) \times \mathrm{Res}(G_2) \quad \textit{and} \quad \mathrm{QZ}(G_1 \times G_2) = \mathrm{QZ}(G_1) \times \mathrm{QZ}(G_2).$$

Proof If N_1 and N_2 are open normal subgroups of G_1 and G_2, respectively, then the product $N_1 \times N_2$ is open and normal in $G_1 \times G_2$. This implies that $\mathrm{Res}(G_1 \times G_2) \leq \mathrm{Res}(G_1) \times \mathrm{Res}(G_2)$. Conversely, let $N \leq G_1 \times G_2$ be an open normal subgroup. There is a compact identity neighborhood K_i in G_i such that $K_1 \times K_2$ is contained in N. Let N_i be the smallest normal subgroup of G_i containing K_i. Since the subgroup N_i contains an identity neighborhood of G_i, it is open. Since $K_i \times \{1\} \subset N$, we have $N_i \times \{1\} \subset N$, and thus $N_1 \times N_2 \leq N$. We conclude that $\mathrm{Res}(G_1 \times G_2) \geq \mathrm{Res}(G_1) \times \mathrm{Res}(G_2)$.

For the quasicenter, notice that the inclusion $\mathrm{QZ}(G_1 \times G_2) \geq \mathrm{QZ}(G_1) \times \mathrm{QZ}(G_2)$ is clear. The reverse inclusion follows from the fact that the canonical projections of $G_1 \times G_2$ to G_1 and G_2 are open maps. □

We shall need two further facts.

Proposition 5.6 [8, Corollary 4.1] *A compactly generated totally disconnected locally compact group is residually discrete if and only if it has a basis of identity neighborhoods consisting of compact open normal subgroups.*

Corollary 5.7 *Let G be a compactly generated locally compact group with a trivial amenable radical. If every open normal subgroup of G is of finite index, then the discrete residual $\mathrm{Res}(G)$ has a trivial centralizer.*

Proof By Proposition 5.6, the compactly generated residually discrete (hence totally disconnected) quotient group $G/\mathrm{Res}(G)$ is compact-by-discrete. Since G does not have any infinite discrete quotient, the same holds for $G/\mathrm{Res}(G)$, and we infer that $G/\mathrm{Res}(G)$ is compact. In particular $G/\mathrm{Res}(G)$ carries a G–invariant measure of finite volume. In other words $\mathrm{Res}(G)$ is a closed subgroup of finite covolume in G. Its centralizer is thus trivial by Corollary 5.3. □

5.4 Residually finite irreducible lattices in products

A fundamental discovery of Burger and Mozes [5] is that an irreducible lattice in the product of two trees which is residually finite must have injective projection to both factors under natural, mild conditions. That fact is generalized to lattices in products of groups acting on CAT(0) spaces in [9, Section 2.B]. In this section, we present a purely algebraic version of that fact, which is based on similar arguments.

Throughout this section, we let $G_1, \dots, G_n$ be nontrivial locally compact groups and denote by

$$\pi_i \colon G_1 \times \cdots \times G_n \to G_i$$

the canonical projection to the i^{th} factor.

A group is called *Noetherian* if it satisfies the ascending chain condition on subgroups. A group is Noetherian if and only if all its subgroups are finitely generated. Finite groups are obvious examples of Noetherian groups.

Lemma 5.8 *Let $\Gamma \le G_1 \times G_2$ be a lattice such that $\pi_1(\Gamma)$ is dense in G_1. Assume that the discrete residual* $\mathrm{Res}(G_1)$ *has a trivial centralizer in G_1. If Γ is residually Noetherian* (eg residually finite), *then the restriction $\pi_2\restriction_\Gamma\colon \Gamma \to G_2$ is injective.*

Proof The kernel of $\pi_2\restriction_\Gamma$ is a discrete subgroup of $G_1 \times G_2$ of the form $N_1 \times \{1\}$ for some subgroup $N_1 \le G_1$. In particular, N_1 is discrete in G_1. Since $N_1 \times \{1\}$ is normal in Γ, we see that N_1 is normalized by $\pi_1(\Gamma)$. The normalizer of N_1 is closed, so N_1 is in fact normal in $G_1 = \overline{\pi_1(\Gamma)}$.

Let now M be a normal subgroup of Γ such that Γ/M is Noetherian. Observe that $M \cap (N_1 \times \{1\})$ is a discrete subgroup of $G_1 \times G_2$ of the form $M_1 \times \{1\}$ with $M_1 \le G_1$ discrete. Moreover, since M and $N_1 \times \{1\}$ are normal in Γ, it follows that M_1 is normal in $G_1 = \overline{\pi_1(\Gamma)}$. The quotient

$$N_1/M_1 \cong N_1 \times \{1\}/M \cap (N_1 \times \{1\})$$

is isomorphic to a subgroup of Γ/M and is thus finitely generated since Γ/M is Noetherian. Hence, its automorphism group is finite or countable.

The conjugation action of G_1 on N_1/M_1 yields a continuous homomorphism $G_1 \to \mathrm{Aut}(N_1/M_1)$. The kernel of that homomorphism is a closed subgroup of G_1 of finite or countable index, so the kernel is open by the Baire category theorem. This implies that $[\mathrm{Res}(G_1), N_1] \le M_1$. We thus have

$$[\mathrm{Res}(G_1), N_1] \times \{1\} \le M_1 \times \{1\} \le M.$$

Since this is valid for all normal subgroups M of Γ such that Γ/M is Noetherian, that Γ is residually Noetherian implies that $[\mathrm{Res}(G_1), N_1] = \{1\}$. We conclude that N_1 is contained in the centralizer of $\mathrm{Res}(G_1)$, which is trivial by hypothesis. □

5.5 Normalizers of compact open subgroups

Lemma 5.9 *Let $\Gamma \le G_1 \times G_2$ be a lattice. Assume that G_2 is totally disconnected and that at least one of the following conditions hold:*

1. *Γ is cocompact, G_2 is compactly generated, and every cocompact lattice in G_2 has a discrete centralizer in G_2.*
2. *G_2 has Kazhdan's property* (T), *and every lattice in G_2 has a discrete centralizer.*

If $\overline{\pi_1(\Gamma)}$ has a compact open normal subgroup (eg $\pi_1(\Gamma)$ is discrete in G_1), *then $\pi_2(\Gamma)$ is discrete in G_2.*

Proof The group $O_1 := \overline{\pi_1(\Gamma)}$ has a compact open normal subgroup K_1. Additionally, Γ is a lattice in $O_1 \times G_2$ which is cocompact if Γ is cocompact in $G_1 \times G_2$. The natural projection $\varphi\colon O_1 \times G_2 \to O_1/K_1 \times G_2$ is proper since K_1 is compact. Thus, $\Lambda := \varphi(\Gamma)$ is a lattice in $O_1/K_1 \times G_2$. Since O_1/K_1 is discrete, it follows from Proposition 5.1 that $\Lambda_2 := \Lambda \cap (\{1\} \times G_2)$ is a lattice in $\{1\} \times G_2$, and Λ_2 is cocompact in $\{1\} \times G_2$ if Γ is cocompact.

Let U_2 be a compact open subgroup of G_2 such that $\{1\} \times U_2$ intersects Λ trivially. The subgroup $\Lambda_1 := \Lambda \cap (O_1/K_1 \times U_2)$ is a lattice in $O_1/K_1 \times U_2$ by Proposition 5.1. Since Λ_2 is a normal subgroup of Λ, we see that $\Lambda_1\Lambda_2$ is a discrete subgroup. Moreover, $\Lambda_1\Lambda_2$ is a lattice in $\Lambda_1(\{1\} \times G_2)$, and $\Lambda_1(\{1\} \times G_2)$ is of finite covolume in $(O_1/K_1) \times G_2$. We conclude that $\Lambda_1\Lambda_2$ is a lattice in $(O_1/K_1) \times G_2$, in view of [19, Lemma I.1.6], hence $\Lambda_1\Lambda_2$ is of finite index in Λ.

Let $V_2 \le U_2$ be the closure of the projection of Λ_1 to G_2 and Λ'_2 be the projection of Λ_2 to G_2. The group V_2 is compact and normalizes Λ'_2, which is discrete. If hypothesis (1) holds, then Λ'_2 is a cocompact lattice in the compactly generated group G_2 and is thus finitely generated with a discrete centralizer in G_2. If hypothesis (2) holds, then Λ'_2 is also finitely generated with a discrete centralizer in G_2.

The compact group V_2 acts by conjugation on the finitely generated group Λ'_2, so V_2 admits an open subgroup of finite index which centralizes Λ'_2. This open subgroup must be discrete since $\mathrm{C}_{G_2}(\Lambda'_2)$ is discrete. Hence the compact group V_2 is discrete, hence finite. We conclude that Λ_1 has finite image in G_2. Therefore, $\Lambda_1\Lambda_2$ has discrete image in G_2, so Λ has discrete image in G_2 as $[\Lambda : \Lambda_1\Lambda_2]$ is finite. The projection $\pi_2\colon O_1 \times G_2 \to G_2$ factorizes through $\varphi\colon O_1 \times G_2 \to O_1/K_1 \times G_2$. Hence, Γ and Λ have the same image in G_2. That is to say, $\pi_2(\Gamma)$ is discrete. □

Lemma 5.10 *Let $\Gamma \le G_1 \times G_2$ be a lattice. Assume that G_2 is totally disconnected and that at least one of the following conditions hold:*

1. *Γ is cocompact, G_2 is compactly generated, and every cocompact lattice in G_2 has a discrete centralizer in G_2.*
2. *G_2 has Kazhdan's property* (T), *and every lattice in G_2 has a discrete centralizer.*

If $\pi_2\restriction_\Gamma\colon \Gamma \to G_2$ is injective, then every compact open subgroup of G_1 has a compact normalizer in G_1.

Proof Let $K_1 \le G_1$ be a compact open subgroup and set $O_1 := \mathrm{N}_{G_1}(K_1)$. The intersection $\Gamma_{O_1} := \Gamma \cap (O_1 \times G_2)$ is a lattice in $O_1 \times G_2$ by Proposition 5.1, and it is additionally cocompact if Γ is cocompact. Notice that $K_1 \cap \overline{\pi_1(\Gamma_{O_1})}$ is a compact open normal subgroup of $\overline{\pi_1(\Gamma_{O_1})}$. In view of Lemma 5.9, the projection of Γ_{O_1} to G_2 is discrete. It follows that we may find an open subgroup $L \leq G_2$ such that $\Gamma_{O_1} \cap (O_1 \times L) = \Gamma_{O_1} \cap (O_1 \times \{1\})$. Applying Proposition 5.1, $\Gamma_{O_1} \cap (O_1 \times \{1\})$ is a lattice in $O_1 \times L$, and thus, $\Gamma_{O_1} \cap (O_1 \times \{1\})$ is a lattice in $O_1 \times \{1\}$. On the other hand, $\pi_2 \upharpoonright_{\Gamma_{O_1}}$ is injective, hence the intersection $\Gamma_{O_1} \cap (O_1 \times \{1\})$ is trivial. The trivial group is then a lattice in $O_1 \times \{1\}$, which implies that O_1 is compact. □

Notice that if $\Gamma \leq G_1 \times G_2$ is a lattice such that $\overline{\pi_1(\Gamma)} = G_1$, then $\mathrm{Ker}(\pi_2 \upharpoonright_\Gamma)$ is a discrete normal subgroup of G_1, so it is contained in $\mathrm{QZ}(G_1)$. In particular, if G_1 has a trivial quasicenter, then the projection of Γ to G_2 is injective. The following partial converse was observed in a conversation with Marc Burger.

Lemma 5.11 *Let $\Gamma \leq G_1 \times G_2$ be a lattice whose projection to G_1 has dense image. Assume that $G_1 \times G_2$ is totally disconnected, and that at least one of the following conditions hold:*

(1) *Γ is cocompact, G_2 is compactly generated, and every cocompact lattice in G_2 has a discrete centralizer in G_2.*

(2) *G_2 has Kazhdan's property* (T), *and every lattice in G_2 has a discrete centralizer.*

If $\pi_2 \upharpoonright_\Gamma \colon \Gamma \to G_2$ is injective, then $\mathrm{QZ}(G_1)$ is locally elliptic.

In particular, if G_1 has trivial locally elliptic radical (eg if G_1 has trivial amenable radical), *then $\mathrm{QZ}(G_1) = 1$ if and only if the projection $\pi_2 \upharpoonright_\Gamma \colon \Gamma \to G_2$ is injective.*

Proof Suppose that $\mathrm{QZ}(G_1)$ is not locally elliptic. Then there exists a finite subset $\Sigma \subset \mathrm{QZ}(G_1)$ such that $\overline{\langle \Sigma \rangle}$ is not compact. By the definition of the quasicenter, the centralizer $C_{G_1}(\Sigma)$ is open, so it contains a compact open subgroup U. The normalizer $N_{G_1}(U)$ thus contains the noncompact closed group $\overline{\langle \Sigma \rangle}$. Applying Lemma 5.10, the projection $\pi_2 \upharpoonright_\Gamma \colon \Gamma \to G_2$ is not injective.

Suppose now that G_1 has trivial locally elliptic radical. If $\pi_2 \upharpoonright_\Gamma \colon \Gamma \to G_2$ is injective, then $\mathrm{QZ}(G_1)$ is locally elliptic, hence trivial by assumption. The converse part is straightforward and was observed above. □

Combining some of the previous results, we obtain the following criterion. It is good to keep in mind that, by Corollary 5.3, in a locally compact group with discrete amenable radical, the centralizer of every lattice is discrete.

Proposition 5.12 *Let $\Gamma \leq G_1 \times G_2$ be a lattice. Assume that the following hold:*

- $\overline{\pi_1(\Gamma)} = G_1$.
- $C_{G_1}(\mathrm{Res}(G_1)) = \{1\}$.
- *G_2 is compactly generated and totally disconnected, and every cocompact lattice in G_2 has a discrete centralizer in G_2.*
- *Γ is cocompact, or G_2 has Kazhdan's property* (T) *and is such that every lattice has a discrete centralizer.*

If a compact open subgroup of G_1 has a noncompact normalizer in G_1, then the projection of Γ to G_2 is not injective, and Γ is not residually Noetherian (in particular not residually finite).

Proof If a compact open subgroup of G_1 has a noncompact normalizer, then the restriction $\pi_2 \restriction_\Gamma \colon \Gamma \to G_2$ is not injective by Lemma 5.10. Moreover, the hypotheses of Lemma 5.8 are all fulfilled, so Γ is not residually Noetherian. □

Theorem 5.13 *Let $n \geq 2$, and let $G = G_1 \times \cdots \times G_n$ be a product of nondiscrete compactly generated totally disconnected locally compact groups with a trivial amenable radical. Suppose that for each i, every open normal subgroup of G_i is of finite index. Let $\Gamma \leq G$ be a lattice such that $\overline{\pi_i(\Gamma)} = G_i$ for all i. If Γ is not cocompact, we assume in addition that G has Kazhdan's property* (T). *If Γ is residually Noetherian, then* $\mathrm{QZ}(G) = \{1\}$, *and for all i, every compact open subgroup of G_i has a compact normalizer in G_i.*

Proof Fix $i \in \{1, \ldots, n\}$. Set $H_1 := G_i$ and $H_2 := \prod_{j \neq i} G_j$. Every open normal subgroup of H_1 is of finite index, so $C_{H_1}(\mathrm{Res}(H_1)) = \{1\}$ by Corollary 5.7. Moreover H_2 has a trivial amenable radical, so that every lattice in H_2 has a trivial centralizer by Corollary 5.3. Therefore $\Gamma \leq H_1 \times H_2$ is a lattice fulfilling all the hypotheses of Proposition 5.12. It follows that every compact open subgroup of $H_1 = G_i$ has a compact normalizer in G_i. Moreover, the projection of Γ to H_2 is injective by Lemma 5.8, so $\mathrm{QZ}(H_1) = \mathrm{QZ}(G_i) = \{1\}$ by Lemma 5.11. Appealing to Lemma 5.5, we deduce that $\mathrm{QZ}(G) = \{1\}$. □

The following related result describes a similar property under a stronger hypothesis of irreducibility on the lattice Γ.

Theorem 5.14 *Let $G = G_1 \times \cdots \times G_n$ be a product of nondiscrete locally compact groups with $n \geq 2$ and let $\Gamma \leq G$ be a lattice such that for each i, the projection of Γ to $\prod_{j \neq i} G_j$ has dense image.*

(i) *Assume that $\mathrm{C}_{G_i}(\mathrm{Res}(G_i)) = \{1\}$ for all i. If Γ is residually Noetherian, then the restriction $\pi_i \restriction_\Gamma \colon \Gamma \to G_i$ is injective for all i.*

(ii) *Assume that G is compactly generated, totally disconnected, with a trivial amenable radical. If Γ is not cocompact, assume in addition that G has Kazhdan's property* (T)*. Then $\mathrm{QZ}(G) = \{1\}$ if and only if the projection of Γ to G_i is injective for all i.*

Proof Let $i \in \{1, \ldots, n\}$ and set $H := \prod_{j \neq i} G_j$.

(i) By hypothesis, Γ is a lattice with dense projections in the product $H \times G_i$. In view of Lemma 5.5, we additionally have $\mathrm{C}_H(\mathrm{Res}(H)) = \{1\}$. Therefore, $\pi_i \restriction_\Gamma \colon \Gamma \to G_i$ is injective by Lemma 5.8.

(ii) By assumption, H and G_i both have a trivial amenable radical. By Corollary 5.3, every lattice in G_i has a trivial centralizer. It follows from Lemma 5.11 that $\mathrm{QZ}(H)$ is trivial if and only if the projection of Γ to G_i is injective. The required assertion now follows from Lemma 5.5. □

6 Trees and lattices in products

The goal of this section is to apply our abstract results on lattices in product groups to the geometric setting of groups acting on trees. We first identify a local criterion controlling the normalizers of compact open subgroups.

6.1 Normalizers of compact open subgroups

Proposition 6.1 *Let T be a locally finite tree with more than two ends and $G \leq \mathrm{Aut}(T)$ be a closed unimodular subgroup acting cocompactly. For each vertex $v \in VT$ and each edge $e \in E(v)$, suppose that the stabilizer $G_{(e)}$ fixes an edge $f \in E(v)$ different from e. Then there is an edge $e \in ET$ whose stabilizer $G_{(e)}$ has a noncompact normalizer.*

Proof By induction, we build a geodesic edge path $(f_n)_{n \geq 0}$ such that $G_{(f_n)} \leq G_{(f_{n+1})}$. As the base and successor cases are the same, suppose we have built our sequence up to f_k. Noting that $G_{(\bar{f}_k)} = G_{(f_k)}$, our hypothesis ensures that there exists $f_{k+1} \in E(t(f_k))$ different from $\overline{f_k}$ such that $G_{(f_k)}$ fixes f_{k+1}. That is to say, $G_{(f_k)} \leq G_{(f_{k+1})}$. This completes our inductive construction.

Since G acts cocompactly on T, the collection of edges $\{f_n\}_{n\geq 0}$ is covered by finitely many edge orbits. In particular, there is an infinite subset $I \subset \mathbb{N}$ such that $\{f_n\}_{n\in I}$ is contained in the same orbit. For $m \leq n$ with $m, n \in I$, the compact open subgroups $G_{(f_n)}$ and $G_{(f_m)}$ are conjugate. Since G is unimodular, we conclude that $G_{(f_n)} = G_{(f_m)}$. For $n \in I$, the normalizer of $G_{(f_n)}$ therefore contains elements mapping $o(f_n)$ arbitrarily far away from itself. The normalizer is thus noncompact. □

The hypothesis of Proposition 6.1 is satisfied in the following situation.

Corollary 6.2 *Let T be a locally finite tree with more than two ends and $G \leq \mathrm{Aut}(T)$ be a closed unimodular subgroup acting cocompactly on T. Suppose that for each vertex $v \in VT$, the local action of $G_{(v)}$ on $E(v)$ is nilpotent and does not have a unique fixed point. Then G has a compact open subgroup with a noncompact normalizer.*

Proof Let F be the natural image of $G_{(v)}$ in $\mathrm{Sym}(E(v))$, so F is nilpotent by hypothesis. In view of Lemma 5.10, it suffices to show that for each $e \in E(v)$, there exists $f \in E(v)$ different from e fixed by $F_{(e)}$. If e is not a fixed point of F in $E(v)$, then $F_{(e)}$ is a proper subgroup of F. Since F is nilpotent, it follows that $F_{(e)}$ is properly contained in its normalizer. Taking $g \in F$ normalizing $F_{(e)}$ without fixing e, $f := g(e)$ is a fixed point of $F_{(e)}$ different from e. If e is a fixed point of F, then by hypothesis there exists $f \in E(v)$ different from e which is also fixed by F. The conclusion now follows from Proposition 6.1. □

We pause to note a supplementary result in the same vein, which applies in particular to all Burger–Mozes groups $U_c(F)$ with F nilpotent.

Proposition 6.3 *Let T be a locally finite tree with more than two ends and $G \leq \mathrm{Aut}(T)$ be a closed unimodular subgroup acting vertex-transitively on T. Suppose that for each vertex $v \in VT$, the local action of $G_{(v)}$ on $E(v)$ is nilpotent. Then G has a compact open subgroup with a noncompact normalizer.*

Proof Let $v \in VT$ and $F \leq \mathrm{Sym}(E(v))$ denote the local action of $G_{(v)}$ at v. If F does not have a unique fixed point, then we may apply Corollary 6.2, since G is vertex-transitive, and the required conclusion follows. We assume henceforth that F has a unique fixed point, say e.

By Proposition 2.8(ii), there exists a regular, but not necessarily legal, coloring c of T such that G is contained as a closed subgroup in $U_c(F)$. Moreover, since G is unimodular and vertex-transitive, we deduce from Lemma 2.9 that $c(e) = c(\overline{e})$, where e is the unique edge fixed by the local action.

Let now $w \in VT$ be an arbitrary vertex and $e_w \in E(w)$ be the unique edge with $c(e_w) = c(e)$. For any edge $f \in E(w)$ different from e_w, the edge-stabilizer $G_{(f)}$ fixes an edge $f' \in E(w)$ which is different from both f and e_w. Proceeding as in the proof of Proposition 6.1, we can construct inductively a geodesic edge path $(f_n)_{n\geq 0}$ such that $G_{(f_n)} \leq G_{(f_{n+1})}$ with $c(f_n) \neq c(e) \neq c(\overline{f_n})$ for all n. The end of the proof is identical to that of Proposition 6.1. □

6.2 Application to lattices in products of trees

We now obtain the following criterion on the local action ensuring that some lattices are not residually finite.

Corollary 6.4 *Let T be a locally finite leafless tree such that $\mathrm{Aut}(T)$ acts cocompactly, let H be a compactly generated totally disconnected locally compact group with a trivial amenable radical, and let $\Gamma \leq \mathrm{Aut}(T) \times H$ be a cocompact lattice. Assume that at least one of the following conditions is satisfied:*

1. *For all $v \in VT$ and $e \in E(v)$, the stabilizer $\Gamma_{(e)}$ fixes an edge $f \in E(v)$ different from e.* (For example, the natural image of $\Gamma_{(v)}$ in $\mathrm{Sym}(E(v))$ is nilpotent and without a unique fixed point.)
2. *The Γ–action on T is vertex-transitive, and for every $v \in VT$, the local action of Γ at v is nilpotent.*

If the projection of Γ to H is nondiscrete, then Γ is not residually Noetherian, hence not residually finite.

Proof Let $G_1 \leq \mathrm{Aut}(T)$ denote the closure of the projection of Γ. The group G_1 acts cocompactly on T, since Γ is cocompact, hence G_1 acts minimally. Additionally, G_1 is unimodular because $G_1 \times H$ contains a lattice. In particular, G_1 does not fix an end of T. It follows from Proposition 2.3 that every nontrivial normal subgroup of G_1 has a trivial centralizer and is nonamenable. By Corollary 5.3, every lattice in H has a trivial centralizer. Since the projection of Γ to H is nondiscrete by hypotheses, it follows from Lemma 5.9 that G_1 is nondiscrete.

By Proposition 6.1 or Proposition 6.3, either of the hypotheses (1) or (2) implies the existence of a compact open subgroup $K \leq G_1$ whose normalizer in G_1 is noncompact. Invoking Proposition 5.12, we deduce that Γ is not residually Noetherian. □

In the special case where T is the 4–regular tree, $H = \mathrm{Aut}(T)$ and the local action of Γ on T is $C_2 \times C_2$, we recover [13, Lemma 9.4].

Proof of Corollary 1.4 It follows from [27, Theorem 5.3] that Γ contains an element which fixes a vertex in T_6 but whose image in $\mathrm{Aut}(T_6)$ generates an infinite group; see [26, Section II.4] and [21, Proposition 9]. In particular, the projection of Γ to $\mathrm{Aut}(T_6)$ is nondiscrete. On the other hand, the image of Γ in $\mathrm{Aut}(T_4)$ is vertex-transitive, and one verifies that it acts without inversion. Therefore, if the local action of Γ on T_4 is not a 2–group, it has exactly two orbits of respective sizes 1 and 3. This implies that the closure of the image of Γ in $\mathrm{Aut}(T_4)$ is a strictly ascending HNN extension, hence it is not unimodular, which is absurd since Γ is a lattice.

An alternative argument consists in computing the local action of Γ on T_4 directly from the presentation: it is easily verified to be $C_2 \times C_2$ acting on four points. This has been done by D Rattaggi in [20]: the Wise lattice is Example 2.36, and its local action is recorded in the table on page 280 in Section C.5.

The hypotheses of Corollary 6.4 are fulfilled, and the conclusion follows. □

Corollary 6.4 applies to numerous other lattices in products of trees. We mention the following example, which is remarkable by the conciseness of its presentation. Its irreducibility is due to Janzen and Wise [14]; its nonresidual finiteness has been observed independently in [3] and provides a positive answer to [14, Question 10].

Corollary 6.5 *The group*

$$\Gamma = \langle a, b, x, y \mid axay, ax^{-1}by^{-1}, ay^{-1}b^{-1}x^{-1}, bxb^{-1}y^{-1} \rangle$$

is a nonresidually finite vertex-transitive cocompact lattice in $\mathrm{Aut}(T_4) \times \mathrm{Aut}(T_4)$.

Proof By [14, Theorem 3], the group Γ is a discrete subgroup of $\mathrm{Aut}(T_4) \times \mathrm{Aut}(T_4)$ acting simply transitively on the vertices of $T_4 \times T_4$. Moreover [14, Theorem 3] ensures that the projection of Γ to both factors in the product $\mathrm{Aut}(T_4) \times \mathrm{Aut}(T_4)$ is nondiscrete. The local action of Γ on both factors has been computed by Rattaggi in [20, Table C.4 on page 278]. In order to identify the local actions of Γ in that table, one should compare it with the table in Section 7 of [15]. The presentation of the group Γ is the entry denoted 2×2.40 in the latter table. That table indicates that the abelianization of Γ is $C_2 \times C_2 \times C_3$. With that piece of information at hand, one sees from [20, Table C.4 on page 278] that the local action of Γ on the two tree factors is given by the alternating group Alt(4) and the dihedral group D_8. The latter being nilpotent, the nonresidual finiteness of Γ follows from Corollary 6.4. □

Remark 6.6 Since the local action of Γ on one of the tree factors is nilpotent, it follows from Corollary 1.2 that Γ is virtually indicable. This can actually be checked directly from the presentation: indeed Γ maps onto the infinite dihedral group $D_\infty = \langle s, t \mid s^2, t^2 \rangle$ via the assignments $a \mapsto s$, $b \mapsto s$, $x \mapsto sts$ and $y \mapsto t$.

As a final illustration, we mention the group

$$\Lambda = \langle a, b, c, x, y, z \mid a^2, b^2, c^2, x^2, y^2, z^2, axax, ayay, azbz, bxbx, bycy, cxcz \rangle$$

considered by N Radu [18, Proposition 5.4], who observes that Λ is a nonresidually finite vertex-transitive cocompact lattice in $\mathrm{Aut}(T_3) \times \mathrm{Aut}(T_3)$. As mentioned by Radu, the fact that Λ is not residually finite can be also deduced from Corollary 6.4.

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