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Classifying matchbox manifolds

ALEX CLARK
STEVEN HURDER
OLGA LUKINA

Matchbox manifolds are foliated spaces with totally disconnected transversals. Two matchbox manifolds which are homeomorphic have return equivalent dynamics, so that invariants of return equivalence can be applied to distinguish nonhomeomorphic matchbox manifolds. In this work we study the problem of showing the converse implication: when does return equivalence imply homeomorphism? For the class of weak solenoidal matchbox manifolds, we show that if the base manifolds satisfy a strong form of the Borel conjecture, then return equivalence for the dynamics of their foliations implies the total spaces are homeomorphic. In particular, we show that two equicontinuous $\mathbb{T}^n$–like matchbox manifolds of the same dimension are homeomorphic if and only if their corresponding restricted pseudogroups are return equivalent. At the same time, we show that these results cannot be extended to include the "*adic* surfaces", which are a class of weak solenoids fibering over a closed surface of genus 2.

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1 Introduction

A matchbox manifold is a compact, connected metrizable space $\mathfrak{M}$, equipped with a decomposition into leaves of constant dimension, so that the pair $(\mathfrak{M}, \mathcal{F})$ is a foliated space as defined by Candel and Conlon [9] and Moore and Schochet [36], for which the local transversals to the foliation are totally disconnected. In particular, the leaves of $\mathcal{F}$ are the path-connected components of $\mathfrak{M}$. A matchbox manifold with 2–dimensional leaves is a lamination by surfaces in the sense of Ghys [25] and Lyubich and Minsky [34]. The "solenoidal spaces" of Sullivan [44] and Verjovsky [45] are examples of matchbox manifolds. The dynamical and topological properties of matchbox manifolds have been studied in a series of works by the authors [12; 13; 14].

Matchbox manifolds arise naturally as exceptional minimal sets for foliations of compact manifolds (for example see Hurder [29; 30]); as the tiling spaces associated to repetitive, aperiodic tilings of Euclidean space $\mathbb{R}^n$ which have finite local complexity (for example

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see Anderson and Putnam [3] and Sadun [41; 42]); and they appear naturally in the study of group representation theory and index theory for leafwise elliptic operators for foliations, as discussed in the books [9; 36]. The classification problem for matchbox manifolds asks for invariants which distinguish their homeomorphism types. For example, in the study of aperiodic tilings and their invariants, the cohomology and K–theory groups of their associated tiling spaces have been calculated in many instances, as for example by Anderson and Putman [3], Barge and Swanson [5], Barge and Sadun [4], Clark and Hunton [10] and Forrest, Hunton and Kellendonk [24].

A matchbox manifold $(\mathfrak{M}, \mathcal{F})$ is also a type of dynamical system, as discussed in [30], for example. A homeomorphism between matchbox manifolds preserves the leaves, as they are the path-connected components of $\mathfrak{M}$, and thus many dynamical properties of $\mathcal{F}$ are invariants of the homeomorphism class of $\mathfrak{M}$. For example, the foliation $\mathcal{F}$ is said to be *minimal* if each leaf $L \subset \mathfrak{M}$ is dense, and this property is clearly a homeomorphism invariant. For a clopen transversal W of $\mathcal{F}$, the dynamical properties of a minimal foliation $\mathcal{F}$ are determined by the pseudogroup $\mathcal{G}_W$ of local holonomy maps acting on the transversal W. *Return equivalence* of pseudogroup actions on Cantor spaces is the analog of the notion of *Morita equivalence* for groupoids associated to smooth foliations of compact manifolds, as discussed for example by Haefliger [27; 28]. One then has the following result, whose proof follows along the same method as for the case of smooth foliations:

Theorem 1.1 *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be minimal matchbox manifolds. Suppose that there exists a homeomorphism $h\colon \mathfrak{M}_1 \to \mathfrak{M}_2$; then the holonomy pseudogroup actions associated to $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent.*

Now consider $\mathfrak{M}_1$ and $\mathfrak{M}_2$ which are minimal matchbox manifolds whose holonomy pseudogroups are return equivalent. That is, assume there exist clopen transversals W_1 to $\mathfrak{M}_1$ and W_2 to $\mathfrak{M}_2$, and a homeomorphism $h\colon W_1 \to W_2$ which conjugates the restricted holonomy actions. It is natural to ask for assumptions on $\mathfrak{M}_1$ and $\mathfrak{M}_2$ which are sufficient to guarantee that the transverse map h extends to a homeomorphism $H\colon \mathfrak{M}_1 \to \mathfrak{M}_2$. In the case of 1–dimensional flows, there is the following result of Aarts and Oversteegen [2, Theorem 17]:

Theorem 1.2 *Two orientable, minimal, 1–dimensional matchbox manifolds are homeomorphic if and only if they are return equivalent.*

Since any nonorientable, minimal, matchbox manifold admits an orientable double cover, this implies that the local dynamics determines the global topology in dimension one. For a matchbox manifold with leaves of dimension greater than one, the question whether there exists a converse to Theorem 1.1 is much more subtle. Julien and Sadun studied in [32] the homeomorphism classification for the tiling spaces associated to aperiodic tilings of the Euclidean space $\mathbb{R}^n$, and the relation to return equivalence for the associated pseudogroups.

In this work, we consider the converse to Theorem 1.1 when $\mathfrak{M}$ is homeomorphic to a weak solenoid. A *weak solenoid* is defined as the inverse limit of an infinite sequence of proper finite covering maps of a closed compact manifold, called the base of the solenoid. The properties of weak solenoids are recalled in Section 2. In particular, a weak solenoid is homeomorphic to the suspension of a minimal equicontinuous action of a finitely generated group on a Cantor set, called the global monodromy for the solenoid. In Section 3, the problem of showing that a pair of weak solenoids which are return equivalent are also homeomorphic is reduced to showing that they have presentations with homeomorphic base manifolds and conjugate global holonomy actions.

There is a special class of solenoidal spaces where the converse to Theorem 1.1 can be proved without further assumptions. We say that $\mathcal{S}_{\mathcal{P}}$ is a *toroidal solenoid* if it is defined by a presentation $\mathcal{P}$ as in (1), where each of the manifolds M_ℓ is homeomorphic to the n–torus $\mathbb{T}^n$. The toroidal solenoids arise as the minimal sets for smooth foliations, as shown by Clark and Hurder [11]. For $n \geq 2$, we have the following generalization of Theorem 1.1:

Theorem 1.3 *Suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic to toroidal solenoids of the same dimension n. Then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic if and only if the holonomy pseudogroup actions associated to $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent.*

For the toroidal solenoids with base dimension $n = 1$, the homeomorphism type of $\mathcal{S}_{\mathcal{P}}$ is determined by the asymptotic class of a sequence of integers $\{m_\ell \mid \ell > 0\}$, the covering indices, as shown by Bing [6] and McCord [35, Section 2], and see also Block and Keesling [7, Corollary 2.6]. Moreover, Aarts and Fokkink showed in [1, Section 3] that the asymptotic class of the sequence of covering indices $\{m_\ell \mid \ell > 0\}$ is determined by the return equivalence class of the flow. This result will be discussed further in Section 5 below.

For the toroidal solenoids with base dimension $n \geq 2$, the results of Giordano, Putnam and Skau in [26], and Cortez and Medynets in [15], provide complete invariants of the return equivalence class of minimal equicontinuous free $\mathbb{Z}^n$ actions on Cantor sets. Their invariants, combined with the conclusion of Theorem 1.3, yield a classification of toroidal solenoids up to homeomorphism.

In Section 5 below, we introduce the *adic* surfaces, which are 2–dimensional weak solenoids, and give examples of return equivalent adic surfaces which are nonhomeomorphic. For nontoroidal weak solenoids of dimension greater than one, it is necessary to impose geometric conditions which rule out the examples such as given in Section 5, in order to obtain a converse to Theorem 1.1.

The first condition we impose is that there exists a leaf for the foliation which is simply connected. Secondly, we impose topological restrictions on the base manifolds, in order that the homeomorphism types of their proper coverings are determined by their fundamental groups.

Recall that a finite CW–complex Y is *aspherical* if it is connected and its universal covering space is contractible. Let $\mathcal{A}$ denote the collection of CW–complexes which are aspherical. Also recall that the *Borel conjecture* is that if Y_1 and Y_2 are homotopy equivalent, aspherical closed manifolds, then a homotopy equivalence between Y_1 and Y_2 is homotopic to a homeomorphism between Y_1 and Y_2. The Borel conjecture has been proven for many classes of aspherical manifolds:

- The torus $\mathbb{T}^n$ for all $n \geq 1$.
- All infra-nilmanifolds.
- Closed Riemannian manifolds Y with negative sectional curvatures.
- Closed Riemannian manifolds Y of dimension $n \neq 3, 4$ with nonpositive sectional curvatures.

A compact connected manifold Y is an *infra-nilmanifold* if its universal cover $\widetilde{Y}$ is contractible, and the fundamental group of M has a nilpotent subgroup with finite index.

The above list is not exhaustive. The history and current status of the Borel conjecture is discussed in the surveys of Davis [18] and Lück [33]. We introduce the notion of a *strongly Borel* manifold.

Definition 1.4 A collection $\mathcal{A}_B$ of closed manifolds is called *Borel* if it satisfies the conditions

(1) each $Y \in \mathcal{A}_B$ is aspherical,

(2) any closed manifold X homotopy equivalent to some $Y \in \mathcal{A}_B$ is homeomorphic to Y, and

(3) if $Y \in \mathcal{A}_B$, then any finite covering space of Y is also in $\mathcal{A}_B$.

We say that a closed manifold Y is *strongly Borel* if the collection $\mathcal{A}_Y \equiv \langle Y \rangle$ of all finite covers of Y forms a Borel collection.

Each class of manifolds in the above list is strongly Borel. Here is our second main result:

Theorem 1.5 *Let $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ be weak solenoids for which the base manifolds M_0 of the presentation $\mathcal{P}$ and N_0 of the presentation $\mathcal{Q}$ are both strongly Borel closed manifolds of the same dimension. Assume that the foliations on $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ each contain a leaf which is simply connected. Then $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ are homeomorphic if and only if the holonomy pseudogroup actions associated to $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ are return equivalent.*

The requirement that there exists a simply connected leaf implies that the global holonomy maps associated to each of these foliations are injective maps. This conclusion yields a connection between return equivalence for the foliations of $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ and the homotopy types of the approximating manifolds in the presentations $\mathcal{P}$ and $\mathcal{Q}$. This requirement need not be imposed for the case of $Y = \mathbb{T}^n$ in Theorem 1.3, due to the algebraic properties of $\mathbb{Z}^n$. We also note that the injectivity of the global holonomy maps implies that the fundamental groups $\pi_1(M_0, x_0)$ and $\pi_1(N_0, y_0)$ are residually finite.

A key aspect of the hypotheses in Theorems 1.3 and 1.5 is that the domains of the return equivalence can be taken to have arbitrarily small diameter. Consequently, invariants of return equivalence developed to distinguish actions should have an asymptotic nature, in that they are defined for arbitrarily small transversals.

A homeomorphism between matchbox manifolds induces a quasi-isometry between the leaves of the respective foliations, equipped with the induced metrics. It is a classical result of Plante [37] that the quasi-isometry class of a leaf is determined by its intersection with any transversal, and thus provides a general invariant of asymptotic return equivalence. For example, bounds on the growth rates of the leaves are return equivalence invariants. This observation was used by Dyer, Hurder and Lukina [20] to give growth restrictions on the leaves which imply that the weak solenoid is a homogeneous continuum.

The *asymptotic discriminant* for an equicontinuous minimal Cantor action was defined by Hurder and Lukina [31], and is an invariant of the return equivalence class of the action, essentially by its definition. It thus provides an invariant of the homeomorphism class of the weak solenoid. Using this asymptotic invariant, the constructions of examples of wild solenoids in [31, Section 9] were shown to yield uncountable collections of nonhomeomorphic weak solenoids, all with the same compact base manifold whose fundamental group is a higher-rank lattice, and in particular is highly nonabelian.

2 Standard forms for weak solenoids

Weak solenoids were first introduced by McCord [35], and we recall here the definitions and some of their properties as developed by Schori [43], Rogers and Tollefson [38; 40] and Fokkink and Oversteegen [23]. We then recall the "odometer representation" of a weak solenoid as the suspension of a (nonabelian) group odometer (or subodometer) action.

A *presentation* (for a weak solenoid) is a collection

$$\mathcal{P} = \{p_{\ell+1}\colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\}, \tag{1}$$

where each M_ℓ is a connected compact manifold of dimension n, and each *bonding* map $p_{\ell+1}$ is a proper covering map of finite index. The weak solenoid $\mathcal{S}_\mathcal{P}$ is the inverse limit associated to the presentation $\mathcal{P}$,

$$\mathcal{S}_\mathcal{P} \equiv \varprojlim \{p_{\ell+1}\colon M_{\ell+1} \to M_\ell\} \subset \prod_{\ell \geq 0} M_\ell. \tag{2}$$

By definition, for a sequence $\{x_\ell \in M_\ell \mid \ell \geq 0\}$, we have

$$x = (x_\ell) \equiv (x_0, x_1, \dots) \in \mathcal{S}_\mathcal{P} \iff p_\ell(x_\ell) = x_{\ell-1} \quad \text{for all } \ell \geq 1.$$

The set $\mathcal{S}_\mathcal{P}$ is given the relative (or Tychonoff) topology induced from the product topology. Then $\mathcal{S}_\mathcal{P}$ is compact and connected. McCord showed in [35] that the space $\mathcal{S}_\mathcal{P}$ has a local product structure, and moreover we have:

Proposition 2.1 *Let $\mathcal{P}$ be a presentation with base space M_0 of dimension $n \geq 0$, and let $\mathcal{S}_\mathcal{P}$ be the associated weak solenoid. Then $\mathcal{S}_\mathcal{P}$ is a matchbox manifold of dimension n, and the leaves of the foliation $\mathcal{F}_\mathcal{S}$ are the path-connected components of $\mathcal{S}_\mathcal{P}$.*

Associated to a presentation $\mathcal{P}$ is a sequence of proper surjective maps

(3)
$$q_\ell = p_1 \circ \cdots \circ p_{\ell-1} \circ p_\ell \colon M_\ell \to M_0.$$

For each $\ell > 1$, projection onto the ℓ^{th} factor in the product $\prod_{\ell \geq 0} M_\ell$ in (2) yields a fibration map, denoted by $\Pi_\ell \colon \mathcal{S}_\mathcal{P} \to M_\ell$, for which $\Pi_0 = q_\ell \circ \Pi_\ell \colon \mathcal{S}_\mathcal{P} \to M_0$.

Fix a choice of a basepoint $x_0 \in M_0$ and let $\mathfrak{X}_0 = \Pi_0^{-1}(x_0)$ be the fiber over x_0. Then $\mathfrak{X}_0$ is a Cantor set by the assumption that the fibers of each map p_ℓ have cardinality at least 2.

Choose a basepoint $x \in \mathfrak{X}_0$, and for $\ell \geq 1$, define basepoints $x_\ell = \Pi_\ell(x) \in M_\ell$. Then let

(4)
$$G_\ell^x = \operatorname{image}\{(q_\ell)_\# \colon \pi_1(M_\ell, x_\ell) \to G_0\}$$

denote the image of the induced map $(q_\ell)_\#$ on fundamental groups. Associated to the presentation $\mathcal{P}$ and basepoint $x \in \mathfrak{X}_0$ we thus obtain a descending chain of subgroups of finite index,

(5)
$$\mathcal{G}^x \equiv \{G_\ell^x\}_{\ell \geq 0} = \{G_0 = G_0^x \supset G_1^x \supset G_2^x \supset \cdots \supset G_\ell^x \supset \cdots\}.$$

Each quotient $X_\ell^x = G_0/G_\ell^x$ is a finite set equipped with a left G_0–action, and the natural surjections $X_{\ell+1}^x \to X_\ell^x$ commute with the action of G_0. Thus, the inverse limit

(6)
$$X_\infty^x = \varprojlim\{p_{\ell+1} \colon X_{\ell+1}^x \to X_\ell^x\} \subset \prod_{\ell \geq 0} X_\ell^x$$

is a G_0–space. Give X_∞^x the relative topology induced from the product (Tychonoff) topology on the space $\prod_{\ell \geq 0} X_\ell^x$, so that X_∞^x is a totally disconnected perfect compact set, so is a Cantor space.

Note that the subgroups G_ℓ^x in (4) X_∞^x are not assumed be normal in G_0, and thus X_∞^x is not a profinite group in general, without some form of "normality" assumptions on the subgroups in the chain $\mathcal{G}^x$. The question of what assumptions are necessary for the limit X_∞^x to be a profinite group was first raised by Rogers and Tollefson [38], and further analyzed by Fokkink and Oversteegen in [23]. The subsequent work by Dyer, Hurder and Lukina in [19] characterized the necessary normality condition in terms of the discriminant invariant of the chain G_ℓ^x.

A sequence $(g_\ell) \subset G_0$ such that $g_\ell G_\ell^x = g_{\ell+1} G_\ell^x$ for all $\ell \geq 0$ determines a point $(g_\ell G_\ell^x) \in X_\infty^x$. Let $e \in G_0$ denote the identity element; then the sequence $e_0 = (eG_\ell^x)$

is the *standard basepoint* of X_∞^x. The action $\Phi_x\colon G_0 \times X_\infty^x \to X_\infty^x$ is given by coordinatewise multiplication, $\Phi_x(g)(g_\ell G_\ell^x) = (g g_\ell G_\ell^x)$.

We then have the standard observation:

Lemma 2.2 *$\Phi_x\colon G_0 \times X_\infty^x \to X_\infty^x$ defines an equicontinuous Cantor minimal system $(X_\infty^x, G_0, \Phi_x)$.*

When X_∞^x has the structure of a profinite group, the action $\Phi_x\colon G_0 \times X_\infty^x \to X_\infty^x$ is called an *odometer* by Cortez and Petite in [16], and when X_∞^x is simply a Cantor space they call the action a *subodometer*. If the group G_0 is abelian, then X_∞^x is a profinite abelian group, and, more generally, if the chain (5) consists of normal subgroups of G_0, then X_∞^x is a profinite group. For simplicity, we will call all of these equicontinuous minimal actions by the nomenclature "odometers".

Recall that $\Pi_\ell\colon \mathcal{S}_\mathcal{P} \to M_\ell$ is a fibration for each $\ell \geq 0$, and so the set $\mathfrak{X}_\ell^x = \Pi_\ell^{-1}(x_\ell)$ is a clopen subset of $\mathfrak{X}_0$. From the relation $q_{\ell+1} \circ \Pi_{\ell+1} = \Pi_\ell$ we have that $\mathfrak{X}_{\ell+1}^x \subset \mathfrak{X}_\ell^x$, so we obtain a nested chain of clopen subsets $\{\mathfrak{X}_{\ell+1}^x \subset \mathfrak{X}_\ell^x \mid \ell \geq 0\}$. Moreover, by the definition of the topology on the inverse limit $\mathcal{S}_\mathcal{P}$, the intersection of these sets is the chosen basepoint $x \in \mathfrak{X}_0$.

The global monodromy action $\Phi_\mathcal{F}\colon G_0 \times \mathfrak{X}_0 \to \mathfrak{X}_0$ is then defined as follows. Given a point $y \in \mathfrak{X}^x$, let $L_y \subset \mathcal{S}_\mathcal{P}$ be the leaf containing y. The restriction $\Pi_0\colon L_y \to M_0$ is a covering map, so given a closed path $\sigma\colon [0,1] \to M_0$ with basepoint x_0, there is a unique leafwise path σ_y in L_y with initial point y and terminal point $\sigma_y(1) \in \mathcal{S}_\mathcal{P}$. The terminal point $\sigma_y(1)$ depends only on the basepoint-preserving homotopy class of the path σ. Given $g \in G_0$ and $y \in \mathfrak{X}_0$ choose a closed path σ^g in M_0 representing g, choose a lift σ_y^g as above, then set $\Phi_\mathcal{F}(g)(y) = \sigma_y^g(1)$. This yields a well-defined group action of G_0 on the Cantor space $\mathfrak{X}_0$.

The subgroup $G_\ell^x \subset G_0 = \pi_1(M_0, x_0)$ is represented by closed paths in M_0 with basepoint x_0 and which admit a lift for the covering $q_\ell\colon M_\ell \to M_0$ to a closed path with endpoint x_ℓ. It follows that for the leaf $L_x \subset \mathcal{S}_\mathcal{P}$ containing $x \in \mathfrak{X}_0$, we can also characterize G_ℓ^x as the subgroup represented by those closed paths which admit a lift to L_x, start at x and terminate at a point in $L_x \cap \mathfrak{X}_\ell^x$. Thus, we have

$$G_\ell^x = \{g \in G_0 \mid \Phi_\mathcal{F}(g)(\mathfrak{X}_\ell^x) = \mathfrak{X}_\ell^x\}. \tag{7}$$

That is, the action $\Phi_\mathcal{F}$ of g fixes the set $\mathfrak{X}_\ell^x$, possibly permuting points within this subset.

Let $g \in G_0$ represent the coset $[g]_\ell \in G_0/G_\ell$. It follows from (7) that the image $\mathfrak{X}_\ell^{x,g} = \Phi_{\mathcal{F}}(g)(\mathfrak{X}_\ell^x)$ of $\mathfrak{X}_\ell^x$ under the action of g either coincides with $\mathfrak{X}_\ell^x$ or it is disjoint from $\mathfrak{X}_\ell^x$. Thus, the collection $\{\mathfrak{X}_\ell^{x,g}\}_{g \in G}$ is a finite collection of disjoint clopen sets which cover $\mathfrak{X}_0$. Moreover, for all $\ell' > \ell > 0$, the collection of clopen sets $\{\mathfrak{X}_{\ell'}^{x,g} \mid [g]_{\ell'} = gG_{\ell'}^x \in G_\ell^x/G_{\ell'}^x\}$ is a finite partition of X_ℓ^x.

Given $y \in \mathfrak{X}_0$ there exists a unique $(g_\ell G_\ell) \in X_\infty^x$ such that $y = \bigcap_{\ell \geq 0} \mathfrak{X}_\ell^{x,g_\ell}$. Define $\sigma_x \colon \mathfrak{X}_0 \to X_\infty^x$ by $\sigma_x(y) = (g_\ell G_\ell)$. The map σ_x is surjective, bijective and continuous, hence a homeomorphism. Define $\tau_x = \sigma_x^{-1} \colon X_\infty^x \to \mathfrak{X}_0$, so that $\tau_x(e_0) = x$. The map τ_x can be viewed as "coordinates" on $\mathfrak{X}_0$ centered at the chosen basepoint $x \in \mathfrak{X}_0$. It follows from the construction of τ_x that it commutes with the left G_0–actions $\Phi_{\mathcal{F}}$ on $\mathfrak{X}_0$ and Φ_x on X_∞^x.

The group chain (5) and the homeomorphism τ_x depend on the choice of a point $x \in \mathfrak{X}_0$. For a different basepoint $y \in \mathfrak{X}_0$ in the fiber over x_0, for each $\ell > 0$ there exists $g_\ell \in G_0$ such that $y \in \mathfrak{X}_\ell^y \equiv \Phi_{\mathcal{F}}(g_\ell)(\mathfrak{X}_\ell^x)$, and hence $y = \bigcap_{\ell \geq 0} \mathfrak{X}_\ell^y$. Then for each $\ell > 0$, define $G_\ell^y = g_\ell G_\ell^x g_\ell^{-1}$ which consists of elements of G_0 that leave the set $\mathfrak{X}_\ell^y$ invariant. Let $\mathcal{G}^y = \{G_\ell^y \mid \ell \geq 0\}$ be the resulting group chain, with corresponding inverse limit space X_∞^y. Then the map $\tau_y \colon X_\infty^y \to \mathfrak{X}_0$ gives coordinates on $\mathfrak{X}_0$ centered at the chosen basepoint $y \in \mathfrak{X}_0$.

The composition $\tau_y \circ \tau_x^{-1} \colon X_\infty^x \to X_\infty^y$ gives a topological conjugacy between the minimal Cantor actions $(X_\infty^x, G_0, \Phi_x)$ and $(X_\infty^y, G_0, \Phi_y)$, and the composition $\tau_y \circ \tau_x^{-1}$ can be viewed as a "change of coordinates". Properties of the minimal Cantor action $(X_\infty^x, G_0, \Phi_x)$ which are independent of the choice of these coordinates are thus properties of the topological type of $\mathcal{S}_{\mathcal{P}}$.

The group chains $\mathcal{G}^y$ and $\mathcal{G}^x$ are said to be *conjugate chains*. This notion forms an equivalence relation on group chains which was introduced by Fokkink and Oversteegen [23]. The properties of this equivalence relation were studied in depth in [19; 21].

The map $\tau_x \colon X_\infty^x \to \mathfrak{X}_0$ is used to give the "odometer model" for the solenoid $\mathcal{S}_{\mathcal{P}}$. Let $\widetilde{M}_0$ denote the universal covering of the compact manifold M_0, and let $(X_\infty^x, G_0, \Phi_x)$ be the minimal Cantor system associated to the presentation $\mathcal{P}$ and the choice of a basepoint $x \in \mathfrak{X}_0$. Associated to the left action Φ_x of G_0 on X_∞^x is a suspension space

(8) $$\mathfrak{M}_\Phi = \widetilde{M}_0 \times X_\infty^x/(z \cdot g^{-1}, y) \sim (z, \Phi_x(g)(y)) \quad \text{for } z \in \widetilde{M}_0,\ g \in G_0,\ y \in X_\infty^x$$

which is a minimal matchbox manifold. This construction is a generalization of a standard technique for constructing smooth foliations, as discussed in [8; 9] for example.

Moreover, the suspension space $\mathfrak{M}_\Phi$ of a minimal equicontinuous action φ has an inverse limit presentation, where all of the bonding maps between the coverings $M_\ell \to M_0$ are derived from the universal covering map $\widetilde{\pi}\colon \widetilde{M}_0 \to M_0$. The following result is given in [12], and its proof is a consequence of the lifting property for maps between coverings:

Theorem 2.3 *Let $\mathcal{S}_\mathcal{P}$ be a weak solenoid with base space M_0. Then the suspension of the map τ_x yields a foliated homeomorphism $\tau_x^*\colon \mathfrak{M}_\Phi \to \mathcal{S}_\mathcal{P}$.*

Corollary 2.4 *The homeomorphism type of a weak solenoid $\mathcal{S}_\mathcal{P}$ is completely determined by the base manifold M_0 and the associated minimal Cantor system $(X_\infty^x, G_0, \Phi_x)$.*

We conclude this discussion of some basic geometry of weak solenoids, by recalling some properties of the holonomy groups of the foliations of weak solenoids. First, recall a basic result of Epstein, Millet and Tischler [22]:

Theorem 2.5 *Let $(\mathfrak{X}, G, \Phi)$ be a given action, and suppose that $\mathfrak{X}$ is a Baire space. Then the union of all $x \in \mathfrak{X}$ such that the germinal holonomy group $\mathrm{Germ}(\Phi, x)$ at x is trivial forms a G_δ subset of $\mathfrak{X}$.*

The main result in [22] is stated in terms of the germinal holonomy groups of leaves of a foliation, but an inspection of the proof shows that it applies directly to a general action $(\mathfrak{X}, G, \Phi)$.

We conclude by introducing the following important notion:

Definition 2.6 The *kernel* of the group chain $\mathcal{G}^x = \{G_\ell^x\}_{\ell \geq 0}$ is the subgroup $K(\mathcal{G}^x) = \bigcap_{\ell \geq 0} G_\ell^x$.

For a weak solenoid $\mathcal{S}_\mathcal{P}$ with choice of a basepoint $x_0 \in M_0$ and fiber $\mathfrak{X}_0 = \Pi_0^{-1}(x_0)$, the kernel subgroup $K(\mathcal{G}^x) \subset G_0$ may depend on the choice of the basepoint $x \in \mathfrak{X}_0$. The dependence of $K(\mathcal{G}^x)$ on x is a natural aspect of the dynamics of the foliation $\mathcal{F}_\mathcal{S}$ on $\mathcal{S}_\mathcal{P}$, when $K(\mathcal{G}^x)$ is interpreted in terms of the topology of the leaves of $\mathcal{F}_\mathcal{S}$ as follows.

The map $\tau_x^*\colon \mathfrak{M}_\Phi \to \mathcal{S}_\mathcal{P}$ of Theorem 2.3 sends the quotient space $\widetilde{M}/K(\mathcal{G}^x)$ to the leaf $L_x \subset \mathcal{S}_\mathcal{P}$ through $x \in \mathfrak{X}_0$ in $\mathcal{S}_\mathcal{P}$, and so $K(\mathcal{G}^x)$ is naturally identified with the fundamental group $\pi_1(L_x, x)$. The global holonomy homomorphism $\Phi_{\mathcal{F},x}\colon \pi_1(L_x, x) \to \mathrm{Homeo}(\mathfrak{X}_0, x)$ of the leaf L_x in the suspension foliation $\mathcal{F}_\mathcal{S}$ of $\mathcal{S}_\mathcal{P}$ is then conjugate to the left action, $\Phi_0\colon K(\mathcal{G}^x) \to \mathrm{Homeo}(X_\infty^x, e_0)$.

From the point of view of foliation theory, the leaves of $\mathcal{F}_\mathcal{S}$ with holonomy are a "small" set by the proof of Theorem 2.5. There always exists leaves without holonomy, while there may exist leaves with holonomy, and so the fundamental groups of the leaves may vary accordingly. This aspect of the foliation dynamics of weak solenoids is discussed further in [21, Section 4.2].

3 Return equivalence

The conclusion of Theorem 2.3 is that a weak solenoid is homeomorphic to a suspension space (8) of an equicontinuous action on a Cantor space. In this section, we consider the notion of return equivalence between such suspension spaces.

Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be a minimal action on a Cantor space $\mathfrak{X}$. In order to give a precise definition of return equivalence, we introduce the pseudo⋆group associated to the action φ. A more general discussion of pseudo⋆groups can be found in [30; 31, Section 2.4].

For each $g \in G$ and open subset $U \subset \mathfrak{X}$, let $\varphi^U(g)\colon U \to V = \varphi(g)(U)$ denote the restricted homeomorphism. Then the pseudo⋆group associated to φ is the collection of maps

$$\Psi^*(\varphi, \mathfrak{X}) \equiv \{\varphi^U(g) \mid U \subset \mathfrak{X} \text{ open},\ g \in G\}. \tag{9}$$

The collection $\Psi^*(\varphi, \mathfrak{X})$ is not a pseudogroup, as it does not satisfy the "gluing" condition on maps, but $\Psi^*(\varphi, \mathfrak{X})$ does generate the usual pseudogroup $\Psi(\varphi, \mathfrak{X})$ associated to the action φ on $\mathfrak{X}$.

Given an open subset $W \subset \mathfrak{X}$, define the restriction of $\Psi^*(\varphi, \mathfrak{X})$ to W,

$$\Psi^*(\varphi, W) = \{\varphi^U(g) \mid U \subset W \text{ open},\ g \in G,\ \varphi(g)(U) \subset W\}.$$

Definition 3.1 Let $\varphi_i\colon G_i \times \mathfrak{X}_i \to \mathfrak{X}_i$ be minimal actions on Cantor spaces $\mathfrak{X}_i$ for $i = 1, 2$. Then φ_1 and φ_2 are return equivalent if there exist nonempty open sets $W_1 \subset \mathfrak{X}_1$ and $W_2 \subset \mathfrak{X}_2$, and a homeomorphism $h\colon W_1 \to W_2$ which conjugates the restricted pseudo⋆group $\Psi^*(\varphi_1, W_1)$ with the restricted pseudo⋆group $\Psi^*(\varphi_2, W_2)$.

It is an exercise to show that minimal suspension spaces $\mathfrak{M}_{\varphi_1}$ and $\mathfrak{M}_{\varphi_2}$ are return equivalent as foliated spaces if and only if their associated global monodromy actions satisfy Definition 3.1.

We next introduce a notion which especially pertains to equicontinuous Cantor actions.

Definition 3.2 Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be an action on a Cantor space $\mathfrak{X}$. A nonempty clopen subset $U \subset \mathfrak{X}$ is *adapted* to the action φ if, for any $g \in G$, $\varphi(g)(U) \cap U \neq \varnothing$ implies that $\varphi(g)(U) = U$. It follows that

$$G_U = \{g \in G \mid \varphi(g)(U) \cap U \neq \varnothing\} \tag{10}$$

is a subgroup of G.

Remark 3.3 For the action $\Phi_x\colon G_0 \times X_\infty^x \to X_\infty^x$ of Lemma 2.2, for each $\ell \geq 0$, the set $U = \mathfrak{X}_\ell^x$ is adapted with $G_U = G_\ell^x$ as defined in (7). Note that if $V \subset U \subset \mathfrak{X}$ are both adapted to an action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$, with associated groups G_V and G_U, then we have $G_V = \{g \in G_U \mid \varphi(g)(V) = (V)\}$. Moreover, if there exists a descending chain of clopen adapted sets $\{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ whose intersection is a point, then it is an exercise to show that the minimal action φ is equicontinuous. On the other hand, it is also easy to construct examples of actions which are not equicontinuous but admit a proper adapted clopen subset $U \subset \mathfrak{X}$. For example, consider any minimal Cantor action $\varphi_U\colon G_U \times U \to U$, choose a nontrivial finite group H and set $G = H \times G_U$, then extend the action φ_U on U to $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ acting factorwise on the product space $\mathfrak{X} = H \times U$.

We next establish two technical lemmas which are key for the proofs of Theorems 1.3 and 1.5.

Lemma 3.4 *Let $\varphi_i\colon G_i \times \mathfrak{X}_i \to \mathfrak{X}_i$ be minimal actions on Cantor spaces $\mathfrak{X}_i$ for $i = 1, 2$, and suppose there exists nonempty open sets $W_i \subset \mathfrak{X}_i$ and a homeomorphism $h\colon W_1 \to W_2$ which conjugates the restricted pseudo⋆groups $\Psi^*(\varphi_1, W_1)$ and $\Psi^*(\varphi_2, W_2)$. Then a clopen subset $U_1 \subset W_1$ is adapted to the action φ_1 if and only if $U_2 = h(U_1) \subset W_2$ is adapted to the action φ_2.*

Proof We show that U_2 is adapted to the action of φ_2. The reverse implication follows similarly.

First note that U_1 is an open subset of W_1 and h is a homeomorphism, hence U_2 is an open subset of W_2 in the relative topology on $\mathfrak{X}_2$ hence is an open subset of $\mathfrak{X}_2$. Also,

U_2 is compact as U_1 is compact and all spaces are Hausdorff, thus U_2 is a clopen subset of $\mathfrak{X}_2$.

Let $g_2 \in G_2$ satisfy $\varphi_2(g_2)(U_2) \cap U_2 \neq \varnothing$. Let $h^*\colon \Psi^*(\varphi_2, W_2) \to \Psi^*(\varphi_1, W_1)$ be the map induced by $h\colon W_1 \to W_2$ on the restricted pseudo⋆groups. By assumption, this map is an isomorphism, and in particular $h^*(\varphi_2^{U_2}(g_2)) \in \Psi^*(\varphi_1, W_1)$. Hence, there exists $g_1 \in G_1$ such that $\varphi_1^{U_1}(g_1) = h^*(\varphi_2^{U_2}(g_2))$. Thus, $\varphi_1(g_1)(U_1) \cap U_1 \neq \varnothing$. As U_1 is adapted to the action of φ_1 this implies that $\varphi_1(g_1)(U_1) = U_1$, which implies that $\varphi_2(g_2)(U_2) \cap U_2 = U_2$, as was to be shown. □

Lemma 3.5 *Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be a minimal action on a Cantor space $\mathfrak{X}$, and $U \subset \mathfrak{X}$ a clopen subset adapted to the action. Then the collection $\mathcal{S}_U \equiv \{\varphi(g)(U) \mid g \in G\}$ forms a finite disjoint clopen partition of $\mathfrak{X}$.*

Proof We first show that the images form a disjoint partition. Suppose that for $g_1, g_2 \in G$ we have $\varphi(g_1)(U) \cap \varphi(g_2)(U) \neq \varnothing$. Then $\varphi(g_2^{-1} g_1)(U) \cap U \neq \varnothing$, hence $\varphi(g_2^{-1} g_1)(U) = U$. It follows that $\varphi(g_1)(U) = \varphi(g_2)(U)$. Each image $\varphi(g_1)(U)$ is a clopen subset, and $\mathfrak{X}$ is compact, so there are only a finite number of disjoint images, which completes the proof. □

Assume that $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ is a minimal action on a Cantor space $\mathfrak{X}$, and $U \subset \mathfrak{X}$ a clopen subset adapted to the action. Let $p_U\colon \mathfrak{X} \to \mathcal{S}_U$ be the natural map to the elements of the partition of $\mathfrak{X}$, which exists by Lemma 3.5. Identify the collection $\mathcal{S}_U$ with the quotient set G/G_U via the map $q_U(\varphi(g)(U)) = gG_U \in G/G_U$; then the composition $\pi_U = q_U \circ p_U\colon \mathfrak{X} \to G/G_U$ is G–equivariant.

Given an action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$, we next construct the suspension foliated space for the action. Let M be a compact manifold without boundary, with a basepoint $x_0 \in M$, and let $G = \pi_1(M, x_0)$ denote its fundamental group based at x_0. Let $\tilde{\pi}\colon \widetilde{M} \to M$ denote the universal covering space of M, defined by endpoint-fixed homotopy classes of paths in M with initial point x_0. Then G acts on $\widetilde{M}$ on the right by deck transformations. Define the quotient foliated space

$$(11) \qquad \mathfrak{M}_\varphi = (\widetilde{M} \times \mathfrak{X})/\{(x \cdot \gamma, w) \sim (z, \varphi(\gamma) \cdot w)\}, \quad z \in \widetilde{M},\ w \in \mathfrak{X},\ \gamma \in G.$$

Let $\pi\colon \mathfrak{M}_\varphi \to M$ be the map induced by the projection $\tilde{\pi}\colon \widetilde{M} \times \mathfrak{X} \to \widetilde{M}$ onto the first factor.

Now assume that the action φ admits a proper adapted clopen subset $U \subset \mathfrak{X}$. Then we define

$$(12) \quad M_U = (\widetilde{M} \times G/G_U)/\{(x \cdot g, w) \sim (z, g \cdot w)\}, \quad z \in \widetilde{M},\ w \in G/G_U,\ g \in G.$$

Note that M_U is naturally identified with the finite covering space $\widetilde{M}/G_U$ of M associated to the subgroup $G_U \subset G$. Let $x_U \in M_U$ be the basepoint associated with the identity coset of G/G_U.

The quotient map $\pi_U \colon \mathfrak{X} \to G/G_U$ induces a quotient map $\Pi_U \colon \mathfrak{M}_\varphi \to M_U$ of suspension spaces, with $U = \Pi_U^{-1}(x_U) \subset \mathfrak{X}$, and there is a commutative diagram

$$(13) \qquad \begin{array}{ccc} \mathfrak{M}_\varphi & & \\ {\scriptstyle \pi}\downarrow & \searrow^{\Pi_U} & \\ M & \xleftarrow[\pi_{G_U}]{} & M_U \end{array}$$

Note that the above construction applies to any minimal action with a proper adapted clopen subset. If $U = \mathfrak{X}^x_\ell$ for an odometer action $\varphi = \Phi_x \colon G_0 \times X^x_\infty \to X^x_\infty$ and $\ell > 0$, then $G_U = G^x_\ell$ as in (7) and the map fibration Π_U is the same as the fibration Π_ℓ defined following (3).

We can now give a result which is a key observation for the proofs of Theorems 1.3 and 1.5. For $i = 1, 2$, let $\varphi_i \colon G_i \times \mathfrak{X}_i \to \mathfrak{X}_i$ be a minimal action on the Cantor space $\mathfrak{X}_i$. Let M_i be a compact manifold without boundary, with basepoint $x_i \in M_i$ and $G_i = \pi_1(M_i, x_i)$ its fundamental group based at x_i. Assume that the actions φ_1 and φ_2 are return equivalent, so there exist open sets $W_i \subset \mathfrak{X}_i$ and a homeomorphism $h \colon W_1 \to W_2$ which conjugates the restricted pseudo⋆group $\Psi^*(\varphi_1, W_1)$ with the restricted pseudo⋆group $\Psi^*(\varphi_2, W_2)$.

Let $U_1 \subset W_1$ be a clopen subset which is adapted to the action φ_1; then by Lemma 3.4 the image $U_2 = h(U_1)$ is a clopen subset adapted to the action φ_2. For $i = 1, 2$, let

$$G_{U_i} = \{g \in G_i \mid \varphi_i(g)(U_i) = U_i\} \subset G_i$$

be the stabilizer group of U_i for the action φ_i.

The action φ_i induces a homomorphism $\varphi_{U_i} \colon G_{U_i} \to \Lambda_i \subset \mathrm{Homeo}(U_i)$ onto a subgroup Λ_i. Then the inverse of the restriction $h_{U_1} \colon U_1 \to U_2$ induces an isomorphism $\lambda_h \colon \Lambda_1 \to \Lambda_2$.

Let $\pi_{G_{U_i}}\colon M_{U_i} \to M_i$ be the finite covering associated to G_{U_i} with basepoint $x_{U_i} \in M_{U_i}$ over x_i. A homeomorphism $f\colon M_{U_1} \to M_{U_2}$ is said to *realize* λ_h if the following diagram commutes:

$$\begin{array}{ccc} \pi_1(M_{U_1}, x_{U_1}) = G_{U_1} & \xrightarrow{f_\#} & G_{U_2} = \pi_1(M_{U_2}, x_{U_2}) \\ \varphi_{U_1}\downarrow & & \downarrow\varphi_{U_2} \\ \Lambda_1 & \xrightarrow{\lambda_h} & \Lambda_2 \end{array} \tag{14}$$

By (13) we can represent $\mathfrak{M}_i$ as a suspension space over M_{U_i} with basepoint fiber U_i and monodromy action $\varphi_{U_i}\colon G_{U_i} \to \mathrm{Homeo}(U_i)$. Let $\widetilde{f}\colon \widetilde{M}_{U_1} \to \widetilde{M}_{U_2}$ denote the lift of f to the universal covering spaces. Then the product map

$$\widetilde{f} \times h\colon \widetilde{M}_1 \times U_1 \to \widetilde{M}_{U_2} \times U_2 \tag{15}$$

is a homeomorphism, and intertwines the diagonal actions of G_1 and G_2, so descends to a homeomorphism between $\mathfrak{M}_{\varphi_1}$ and $\mathfrak{M}_{\varphi_2}$. We have thus shown:

Proposition 3.6 *Suppose there exists a homeomorphism $f\colon M_{U_1} \to M_{U_2}$ which realizes the isomorphism $\lambda_h\colon \Lambda_1 \to \Lambda_2$ between the groups of fiber automorphisms induced by return equivalence. Then the suspension spaces $\mathfrak{M}_{\varphi_1}$ and $\mathfrak{M}_{\varphi_2}$ are homeomorphic.*

4 Proofs of main theorems

In this section, we use Proposition 3.6 to obtain proofs of Theorems 1.3 and 1.5. For $i = 1, 2$, let $\mathfrak{M}_i$ be a matchbox manifold homeomorphic to a weak solenoid $\mathcal{S}_{\mathcal{P}_i}$ defined by a presentation

$$\mathcal{P}_i = \{p_{i,\ell+1}\colon M_{i,\ell+1} \to M_{i,\ell} \mid \ell \geq 0\}, \tag{16}$$

where the base manifolds $M_{1,0}$ and $M_{2,0}$ both have dimension $n \geq 1$. Let $\Pi_{\mathcal{P}_i}\colon \mathcal{S}_{\mathcal{P}_i} \to M_{i,0}$ denote the projection onto the base manifold.

Let $x_{i,0} \in M_{i,0}$ be a basepoint, let $G_{i,0} = \pi_1(M_{i,0}, x_{i,0})$ and set $\mathfrak{X}_{\mathcal{P}_i} = \Pi_{\mathcal{P}_i}^{-1}(x_{i,0})$.

The assumption that the holonomy pseudogroups defined by the foliations on $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent implies that the foliations of $\mathcal{S}_{\mathcal{P}_1}$ and $\mathcal{S}_{\mathcal{P}_2}$ are return equivalent. This in turn implies that the global monodromy actions

$$\Phi_{\mathcal{P}_1}\colon G_{1,0} \times \mathfrak{X}_1 \to \mathfrak{X}_1, \quad \Phi_{\mathcal{P}_2}\colon G_{2,0} \times \mathfrak{X}_2 \to \mathfrak{X}_2$$

are return equivalent in the sense of Definition 3.1. That is, there exist open sets $W_1 \subset \mathfrak{X}_1$ and $W_2 \subset \mathfrak{X}_2$ and a homeomorphism $h\colon W_1 \to W_2$ which conjugates the restricted pseudo⋆group $\Psi^*(\Phi_{\mathcal{P}_1}, W_1)$ with the restricted pseudo⋆group $\Psi^*(\Phi_{\mathcal{P}_2}, W_2)$.

4.1 Odometer models

Assume we are given weak solenoids $\mathcal{S}_{\mathcal{P}_1}$ and $\mathcal{S}_{\mathcal{P}_2}$. Then, as shown in Theorem 2.3, we can assume that the weak solenoids $\mathcal{S}_{\mathcal{P}_i}$ are homeomorphic to the suspension of odometer actions as in (8). To fix notation, recall the construction of the odometer actions. Choose a basepoint $x \in W_1 \subset \mathfrak{X}_1$, and set $y = h(x) \in W_2 \subset \mathfrak{X}_2$. Then form the group chains corresponding to the presentations $\mathcal{P}_1$ at x and $\mathcal{P}_2$ at y:

$$\mathcal{G}^x_{\mathcal{P}_1} \equiv \{G^x_{1,\ell}\}_{\ell\geq 0} = \{G_{1,0} \supset G^x_{1,1} \supset G^x_{1,2} \supset \cdots \supset G^x_{1,\ell} \supset \cdots\}, \tag{17}$$

$$\mathcal{G}^y_{\mathcal{P}_2} \equiv \{G^y_{2,\ell}\}_{\ell\geq 0} = \{G_{2,0} \supset G^y_{2,1} \supset G^y_{2,2} \supset \cdots \supset G^y_{2,\ell} \supset \cdots\}. \tag{18}$$

Let $\Phi_1\colon G_{1,0} \times X_{1,\infty} \to X_{1,\infty}$ be the odometer formed from the chain $\mathcal{G}^x_{\mathcal{P}_1}$ and let $\tau_{1,x}\colon X_{1,\infty} \to \mathfrak{X}_{\mathcal{P}_1}$ be the $G_{1,0}$–equivariant homeomorphism constructed in Section 2. Then we have $\tau_{1,x}(e_{1,0}) = x$, where $e_{1,0} = (eG^x_{1,\ell})$ is the basepoint of $X_{1,\infty}$. Moreover, recall from (7) that for $\ell > 0$ we have

$$G^x_{1,\ell} = \{g \in G_{1,0} \mid \Phi_{\mathcal{P}_1}(g)(\mathfrak{X}_{i,\ell}) = \mathfrak{X}_{i,\ell}\}.$$

Similarly, let $\Phi_2\colon G_{2,0} \times X_{2,\infty} \to X_{2,\infty}$ be the odometer formed from the chain $\mathcal{G}^y_{\mathcal{P}_2}$ and let $\tau_{2,y}\colon X_{2,\infty} \to \mathfrak{X}_{\mathcal{P}_2}$ be the corresponding $G_{2,0}$–equivariant homeomorphism with $\tau_{2,y}(e_{2,0}) = y$.

The preimage $\tau^{-1}_{1,x}(\mathfrak{X}^x_{1,\ell})$ is identified with the clopen set

$$U_{1,\ell} = \{(g_k G^x_{1,k}) \mid k \geq 0, g_0 = g_1 = \cdots = g_\ell \in G^x_{1,\ell}\} \subset X_{1,\infty}. \tag{19}$$

The collection $\{\mathfrak{X}^x_{1,\ell} \mid \ell > 0\}$ is a neighborhood basis around the basepoint $x \in W_1$, so there exists $\ell_1 > 0$ such that $U_{1,\ell} \subset \tau^{-1}_{1,x}(W_1)$ for $\ell \geq \ell_1$. Set $U_1 = U_{1,\ell_1}$; then the clopen subset U_1 is adapted to the action of Φ_1 with stabilizer subgroup $G_{U_1} = G^x_{\ell_1}$ by Remark 3.3. Thus, the action Φ_1 induces an epimorphism $\Phi_{U_1}\colon G_{U_1} \to \Lambda_1 \subset \mathrm{Homeo}(U_1)$.

The image $h \circ \tau_1(U_1) \subset \mathfrak{X}_2$ is a clopen subset adapted to the action of $\Phi_{\mathcal{P}_2}$ by Lemma 3.4. Set $U_2 = \tau_2^{-1} \circ h \circ \tau_1(U_1) \subset X_{2,\infty}$, which is a clopen set adapted to the action Φ_2. Let $G_{U_2} \subset G_{2,0}$ be the stabilizer group of U_2. Then the action Φ_2 induces an epimorphism $\Phi_{U_2}\colon G_{U_2} \to \Lambda_2 \subset \mathrm{Homeo}(U_2)$. Moreover, the homeomorphism $\tau_2^{-1} \circ h \circ \tau_1\colon U_1 \to U_2$ induces an isomorphism $\lambda_h\colon \Lambda_1 \to \Lambda_2$.

Remark 4.1 Before continuing with the proofs of the main theorems, we recall an aspect of the equivalence of weak solenoids from [23] and which is discussed in detail in [19]. The basepoint $e_{2,0}$ is in V, so there exists $\ell_2 > 0$ such that $V_{2,\ell} \subset V$ for $\ell \geq \ell_2$, where $V_{2,\ell}$ is defined as in (19). For the action Φ_2 the group $G_{2,\ell}$ stabilizes the clopen set $V_{2,\ell}$ and hence also stabilizes V. However, it need not be the case that G_V is equal to one of the subgroups $G_{2,\ell}$. It is only possible to conclude that there exists some $\ell \geq \ell_2$ for which $G_{2,\ell} \subset G_V$. This corresponds to the fact that homeomorphic weak solenoids are defined by group chains which are equivalent in the sense of [19; 23], which is to say that their group chains are interlaced up to isomorphism.

By Lemma 3.5, the collection $\mathcal{S}_2 \equiv \{\Phi_2(g)(U_2) \mid g \in G_{2,0}\}$ is a clopen partition of $X_{2,\infty}$. We will apply Proposition 3.6 to show that the suspension spaces $\mathfrak{M}_{\Phi_1}$ and $\mathfrak{M}_{\Phi_2}$ are homeomorphic. First, we must construct a map of fundamental groups $f_*\colon G_{U_1} \to G_{U_2}$ so that the diagram (14) is satisfied, and then construct a homeomorphism $f\colon M_{U_1} \to M_{U_2}$ which induces the map f_*.

4.2 Proof of Theorem 1.3

For $i = 1, 2$, we are given that $\mathcal{S}_{\mathcal{P}_i}$ is a toroidal solenoid whose base has dimension n, so $M_{i,0} = \mathbb{T}^n$ and hence $G_{i,0} \cong \mathbb{Z}^n$. The manifold M_{U_i} is a covering of $M_{i,0}$, hence is also a torus, with fundamental group which we identify with $\mathbb{Z}^n$. Introduce the subgroups

$$K_i = \ker\{\Phi_{U_i}\colon G_{U_i} \to \Lambda_i \subset \mathrm{Homeo}(U_i)\} \subset \mathbb{Z}^n. \tag{20}$$

Each K_i is a free abelian subgroup with rank $0 \leq r_i < n$, and there is a commutative diagram

$$\begin{array}{ccccc} K_1 & \hookrightarrow & G_{U_1} & \xrightarrow{\Phi_{U_1}} & \Lambda_1 \\ & & {\scriptstyle f_*}\downarrow & & \cong\downarrow{\scriptstyle \lambda_h} \\ K_2 & \hookrightarrow & G_{U_2} & \xrightarrow{\Phi_{U_2}} & \Lambda_2 \end{array} \tag{21}$$

Lemma 4.2 *There exists a map* $f_*\colon G_{U_1} \to G_{U_2}$ *such that the diagram* (21) *commutes.*

Proof This follows because $G_{U_1} \cong G_{U_2} \cong \mathbb{Z}^n$ are free abelian groups, hence projective $\mathbb{Z}$–modules. We give the details of the construction of the map f_*. Let $\{a_1, \ldots, a_d\} \subset \Lambda_1$ be a minimal set of generators for Λ_1; then $\{\lambda_h(a_1), \ldots, \lambda_h(a_d)\} \subset \Lambda_2$ is a minimal set of generators for Λ_2.

Choose $\{g_1, \ldots, g_d\} \subset G_{U_1}$ so that $a_i = \Phi_{U_1}(g_i)$ for $1 \leq i \leq d$. The kernel K_1 is free abelian, so we can extend this set to a basis $\{g_1, \ldots, g_n\}$ for G_{U_1}, where $\Phi_{U_1}(g_i)$ is the identity for $d < i \leq n$.

Choose elements $\{g'_1, \ldots, g'_d\} \subset G_{U_2}$ so that $\lambda_h(a_i) = \Phi_{U_2}(g'_i)$ for $1 \leq i \leq d$. Note that both K_1 and K_2 are free abelian of rank $n-d$, so we can extend this set to a basis $\{g'_1, \ldots, g'_n\}$ for G_{U_2}, where $\Phi_{U_2}(g'_i)$ is the identity for $d < i \leq n$.

Define the group isomorphism $f_*\colon G_{U_1} \to G_{U_2}$ by specifying $f_*(g_i) = g'_i$ for $1 \leq i \leq n$. Then the diagram (21) commutes by our choices of these bases. □

Finally, to complete the proof of Theorem 1.3, observe that f_* extends to a linear map $\widehat{f}_*\colon \mathbb{R}^n \to \mathbb{R}^n$, and so induces a diffeomorphism of the quotient spaces $f\colon \mathbb{T}^n \to \mathbb{T}^n$. Then the hypotheses of Proposition 3.6 are satisfied.

4.3 Proof of Theorem 1.5

The proof of Theorem 1.5 uses the geometric hypotheses on the foliations of the weak solenoids $\mathcal{S}_{\mathcal{P}_i}$ to show the existence of the map f_* such that the diagram (21) commutes, in place of the group extension arguments in the proof of Lemma 4.2. In particular, we assume that the foliations on $\mathcal{S}_{\mathcal{P}_1}$ and $\mathcal{S}_{\mathcal{P}_2}$ each contain a dense leaf which is simply connected. By the results of Section 4.1, we can assume that $\mathcal{S}_{\mathcal{P}_1}$ and $\mathcal{S}_{\mathcal{P}_2}$ are represented as suspensions of odometer actions, and thus it suffices to show that the hypotheses of Proposition 3.6 are satisfied.

We assume that the odometer actions $\Phi_i\colon G_{i,0} \times X_{i,\infty} \to X_\infty$ are return equivalent for $i = 1, 2$, and that open subsets $W_i \subset X_{i,0}$ are chosen so that the restricted pseudo⋆group $\Psi^*(\Phi_1, W_1)$ is conjugate to the restricted pseudo⋆group $\Psi^*(\Phi_2, W_2)$. Then let $U_i \subset W_i$ be chosen as above, with a homeomorphism $h\colon U_1 \to U_2$ conjugating the restricted actions $\Phi_{U_i}\colon G_{U_i} \to \Lambda_i \subset \mathrm{Homeo}(U_i)$.

Let $K_i \subset G_{U_i}$ denote the kernel of the map Φ_{U_i}, and for $z \in U_i$ define

$$K_i(z) = \{g \in G_{U_i} \mid \Phi_{U_i}(g)(z) = z\}. \tag{22}$$

Observe that $K_i \subset K_i(z)$ for all $z \in U_i$.

By the definition (11) of the suspension space $\mathfrak{M}_{\Phi_{U_i}}$, the leaf $L_z \subset \mathfrak{M}_{\Phi_{U_i}}$ defined by the point z is homeomorphic to the covering $\widetilde{M}_i / K_i(z) \to M_i$. By assumption, for each $i = 1, 2$ there exists $z \in U_i$ such that L_z is simply connected, which implies that

$K_i(z)$ is the trivial group, which implies that the kernel K_i is also the trivial group. Thus, the map $\Phi_{U_i}\colon G_{U_i} \to \Lambda_i$ is an isomorphism. Define the map

$$f_* \equiv \Phi_{U_2}^{-1} \circ \lambda_h \circ \Phi_{U_1}\colon G_{U_1} \to G_{U_2}, \tag{23}$$

which is an isomorphism such that the diagram (21) commutes.

By the hypotheses of Theorem 1.5 the manifolds M_1 and M_2 are both strongly Borel, hence their finite coverings M_{U_1} and M_{U_2} satisfy the Borel conjecture. The map f_* induces a homotopy equivalence between them, as both have contractible universal covering spaces. Then, by the solution of the Borel conjecture for these spaces, there exists a homeomorphism $f\colon M_{U_1} \to M_{U_2}$ which induces the map f_* on their fundamental groups. This completes the proof of Theorem 1.5.

Remark 4.3 The choice of the clopen set U_i in the above proofs can be chosen to have arbitrarily small diameter, and hence the degree of the corresponding covering map $\pi_{U_i}\colon M_{U_i} \to M_i$ in (13) can be chosen to be arbitrarily large. As remarked in [18], the homeomorphism f that is obtained from the solutions of the Borel conjecture can be assumed to be smooth for a sufficiently large finite covering. It follows that the homeomorphism $h\colon \mathcal{S}_{\mathcal{P}_1} \to \mathcal{S}_{\mathcal{P}_2}$ obtained from Proposition 3.6 can be chosen to be smooth along leaves.

5 Examples and counterexamples

In this section, we give several examples to illustrate the necessity of the hypotheses of Theorem 1.5. We first recall a classical result, the classification of Vietoris solenoids of dimension one. We then consider extensions of this construction to solenoids with dimension $n \geq 2$ and give examples of solenoids which are return equivalent but not homeomorphic. These examples are essentially the simplest possible constructions. Many other variants on their construction are clearly possible, especially for solenoids of dimensions greater than two, as briefly discussed in Section 5.3.

5.1 Vietoris solenoids

A *Vietoris solenoid* [17; 46] is a 1–dimensional solenoid $\mathcal{S}_{\mathcal{P}}$, where each M_ℓ is a circle, and each $p_\ell\colon \mathbb{S}^1 \to \mathbb{S}^1$ in the presentation $\mathcal{P}$ is an orientation-preserving covering map of degree $m_\ell \geq 2$. Let $\vec{m} = \{m_1, m_2, \dots\}$ be the list of covering degrees for $\mathcal{P}$. Then $\mathcal{S}_{\mathcal{P}}$ is also called an $\vec{m}$–adic solenoid of dimension one, and denoted by $\mathcal{S}(\vec{m})$.

Let $\vec{m} = \{m_\ell \mid \ell \geq 1\}$ denote a sequence of positive integers with each $m_i \geq 2$. Set $m_0 = 1$, then define the profinite group

$$
\begin{aligned}
(24)\quad \mathfrak{G}_{\vec{m}} &\stackrel{\text{def}}{=} \varprojlim\{q_{\ell+1}\colon \mathbb{Z}/m_1\cdots m_{\ell+1}\mathbb{Z} \to \mathbb{Z}/m_0 m_1 \cdots m_\ell\mathbb{Z} \mid \ell \geq 1\} \\
&= \varprojlim\{\mathbb{Z}/\mathbb{Z} \xleftarrow{m_1} \mathbb{Z}/m_1\mathbb{Z} \xleftarrow{m_2} \mathbb{Z}/m_1 m_2\mathbb{Z} \xleftarrow{m_3} \mathbb{Z}/m_1 m_2 m_3\mathbb{Z} \xleftarrow{m_4} \cdots\},
\end{aligned}
$$

where $q_{\ell+1}$ is the quotient map of degree $m_{\ell+1}$. Each of the profinite groups $\mathfrak{G}_{\vec{m}}$ contains a copy of $\mathbb{Z}$ embedded as a dense subgroup by $z \to ([z]_0, [z]_1, \ldots, [z]_k, \ldots)$, where $[z]_k$ corresponds to the class of z in the quotient group $\mathbb{Z}/m_0 \cdots m_k\mathbb{Z}$. There is a homeomorphism $a_{\vec{m}}\colon \mathfrak{G}_{\vec{m}} \to \mathfrak{G}_{\vec{m}}$ given by "addition of 1" in each finite factor group. The resulting action of $\mathbb{Z}$ is denoted by $\Phi_{\vec{m}}\colon \mathbb{Z} \times \mathfrak{G}_{\vec{m}} \to \mathfrak{G}_{\vec{m}}$. The dynamics of $a_{\vec{m}}$ acting on $\mathfrak{G}_{\vec{m}}$ is referred to as an *adding machine*, or equivalently as a (classical) *odometer*. We then have the standard result:

Proposition 5.1 *The Vietoris solenoid $\mathcal{S}(\vec{m})$ is homeomorphic to the suspension $\mathfrak{M}_{\Phi_{\vec{m}}}$ of the odometer action $\Phi_{\vec{m}}$ with base manifold $M_0 = \mathbb{S}^1$.*

Two Vietoris solenoids $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ are homeomorphic if and only if their presentations $\mathcal{P}$ and $\mathcal{Q}$ yield group chains as in (5) which are equivalent. As all of these are chains of subgroups of the fundamental group $\mathbb{Z}$ of $\mathbb{S}^1$, the equivalence problem for these chains reduces to giving conditions on the sequences of integer covering degrees in $\mathcal{P}$ and $\mathcal{Q}$ which imply equivalence of the chains. There are two invariants of sequences which arise in the classification problem. First, consider the function which counts the total number of occurrences of a given prime in the sequence of integers $\vec{m}$.

Definition 5.2 Given a sequence of positive integers $\vec{m}$ as above, let $C_{\vec{m}}$ denote the function from the set of prime numbers to the set of extended natural numbers $\{0, 1, 2, \ldots, \infty\}$ given by

$$C_{\vec{m}}(p) = \sum_{1}^{\infty} m_i(p),$$

where $m_i(p)$ is the power of the prime p in the prime factorization of m_i.

That is, $C_{\vec{m}}(p) = k$ means that the prime p occurs a total of k times in the prime factorization of the integers in the sequence $\vec{m}$.

Theorem 5.3 *The Vietoris solenoids $\mathcal{S}(\vec{m})$ and $\mathcal{S}(\vec{n})$ are homeomorphic* as bundles over the base manifold $\mathbb{S}^1$ *if and only if $C_{\vec{m}}(p) = C_{\vec{n}}(p)$ for all primes p.*

Next, we recall the notion of "tail equivalence" on sequences. This notion was introduced by Bing in [6], and plays a basic role in the study of return equivalence for Vietoris solenoids in [1].

Definition 5.4 Two infinite sets of integers, $\vec{m} = \{m_\ell \mid \ell \geq 1\}$ and $\vec{n} = \{n_\ell \mid \ell \geq 1\}$, are said to be *tail equivalent*, and we write $\vec{m} \sim_t \vec{n}$, if there exist cofinite subsequences $\vec{m}_* \subset \vec{m}$ and $\vec{n}_* \subset \vec{n}$ which are in bijective correspondence.

The following observation is a direct consequence of Definitions 5.2 and 5.4:

Lemma 5.5 *Two sequences of integers $\vec{m}$ and $\vec{n}$ as above are tail equivalent if and only if the following two conditions hold:*

(1) *for all but finitely many primes p, $C_{\vec{m}}(p) = C_{\vec{n}}(p)$, and*

(2) *for all primes p, $C_{\vec{m}}(p) = \infty$ if and only if $C_{\vec{n}}(p) = \infty$.*

The classification of Vietoris solenoids up to homeomorphism by Bing [6] and McCord [35] and the study of return equivalence by Aarts and Fokkink in [1] yields:

Theorem 5.6 [35; 1] *The Vietoris solenoids $\mathcal{S}(\vec{m})$ and $\mathcal{S}(\vec{n})$ are homeomorphic if and only if they are return equivalent, if and only if $\vec{m}$ and $\vec{n}$ are tail equivalent.*

5.2 $\vec{m}$–adic solenoids of dimension two

Let Σ_g be a closed surface of genus $g \geq 1$ which is obtained by attaching g torus handles $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ to the 2–sphere $\mathbb{S}^2$. For example, Σ_1 is homeomorphic to the 2–torus $\mathbb{T}^2$. Pick a basepoint $x_0 \in \Sigma_g$ and let $G_0 = \pi_1(\Sigma_g, x_0)$ be the fundamental group. Choose an epimorphism $a\colon G_0 \to \mathbb{Z}$, which corresponds to a nontrivial class $[a] \in H^1(\Sigma_g; \mathbb{Z})$ in integral homology.

Let $\vec{m} = \{m_\ell \mid \ell \geq 1\}$ denote a sequence of integers with each $m_i \geq 2$, and form the profinite $\vec{m}$–adic group $\mathfrak{G}_{\vec{m}}$ as in (24). Let $\Phi_{\vec{m}}$ denote the odometer action of $\mathbb{Z}$ described above. Extend this to an action of G_0,

$$(25) \qquad \Phi^a_{\vec{m}}\colon G_0 \times \mathfrak{G}_{\vec{m}} \to \mathfrak{G}_{\vec{m}}, \qquad \Phi^a_{\vec{m}}(g)(x) = \Phi_{\vec{m}}(a(g))(x), \quad g \in G_0,\ x \in \mathfrak{G}_{\vec{m}}.$$

Definition 5.7 The $\vec{m}$–adic surface $\mathfrak{M}(\Sigma_g, a, \vec{m})$ is the suspension space (11) associated to the action $\Phi^a_{\vec{m}}$ with base Σ_g.

We note a consequence of the construction of $\mathfrak{M}(\Sigma_g, a, \vec{m})$, which follows immediately from the fact that the action $\Phi^a_{\vec{m}}$ is induced from the action $\Phi_{\vec{m}}$ and the results of [1]:

Proposition 5.8 *Given closed orientable surfaces* Σ_{g_1} *and* Σ_{g_2} *of genus* $g_i \geq 1$ *for* $i = 1, 2$, *epimorphisms* $a_i \colon G_{i,0} \to \mathbb{Z}$ *and sequences* $\vec{m}$ *and* $\vec{n}$, *then* $\mathfrak{M}(\Sigma_{g_1}, a_1, \vec{m})$ *is return equivalent to* $\mathfrak{M}(\Sigma_{g_2}, a_2, \vec{n})$ *if and only if* $\vec{m}$ *and* $\vec{n}$ *are tail equivalent.*

Finally, we consider the problem, given adic surfaces $\mathfrak{M}(\Sigma_{g_1}, a_1, \vec{m})$ and $\mathfrak{M}(\Sigma_{g_2}, a_2, \vec{n})$ such that $\vec{m}$ is tail equivalent to $\vec{n}$, when are they homeomorphic as matchbox manifolds? First, consider the case of genus $g_1 = g_2 = 1$, so that $\Sigma_{g_1} = \Sigma_{g_2} = \mathbb{T}^2$. Then Theorem 1.3 and Proposition 5.8 yield:

Theorem 5.9 *The adic surfaces* $\mathfrak{M}(\mathbb{T}^2, a_1, \vec{m})$ *and* $\mathfrak{M}(\mathbb{T}^2, a_2, \vec{n})$ *are homeomorphic if and only if* $\vec{m}$ *and* $\vec{n}$ *are tail equivalent.*

For the general case of adic surfaces where at least one base manifold has higher genus, we next give examples of weak solenoids which are return equivalent but not homeomorphic. Note that in these examples, their base manifolds are compact surfaces, hence are strongly Borel, but all their leaves have nontrivial fundamental groups, so the hypotheses of Theorem 1.5 are not satisfied.

Theorem 5.10 *Let* $\mathfrak{M}_1 = \mathfrak{M}(\Sigma_{g_1}, a_1, \vec{m})$ *and* $\mathfrak{M}_2 = \mathfrak{M}(\Sigma_{g_2}, a_2, \vec{n})$ *be adic surfaces.*

(1) *If* $g_1 > 1$ *and* $g_2 = 1$, *then* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *are never homeomorphic.*

(2) *If* $g_1 = g_2 > 1$ *and* $a_1 = a_2$, *then* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *are homeomorphic if and only if* $C_{\vec{m}} = C_{\vec{n}}$.

(3) *If* $g_1 = g_2 > 1$ *and* $a_1 = a_2$, *then there exists* $\vec{m} \sim_{\mathfrak{t}} \vec{n}$ *but* $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$.

Proof First, recall that the Euler characteristic of the closed surface Σ_g of genus $g \geq 1$ has Euler characteristic $\chi(\Sigma_g) = 2 - 2g$, and the Euler characteristic is multiplicative for coverings. That is, if Σ'_g is a k–fold covering of Σ_g then $\chi(\Sigma'_g) = k \cdot \chi(\Sigma_g)$. In particular, for $g > 1$, a proper covering Σ'_g of Σ_g is never homeomorphic to Σ_g.

Next, each of the spaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is homeomorphic to an inverse limit as in (2):

$$\mathfrak{M}_1 = \mathfrak{M}(\Sigma_{g_1}, a_1, \vec{m}) \cong \varprojlim \{ f_{\ell+1} \colon M_{\ell+1} \to M_\ell \}, \tag{26}$$

$$\mathfrak{M}_2 = \mathfrak{M}(\Sigma_{g_2}, a_2, \vec{n}) \cong \varprojlim \{ g_{\ell+1} \colon N_{\ell+1} \to N_\ell \}, \tag{27}$$

where $M_0 = \Sigma_{g_1}$ and $N_0 = \Sigma_{g_2}$. For $\ell > 0$, let m_ℓ denote the degree of the covering map f_ℓ and let n_ℓ denote the degree of the covering map g_ℓ.

Now assume there is a homeomorphism $H \colon \mathfrak{M}_1 \to \mathfrak{M}_2$. By the results of Rogers and Tollefson in [39; 40], the map H is homotopic to a homeomorphism $\widehat{H}$ which is induced by a map between the inverse limit representations of $\mathfrak{M}_1$ in (26) and of $\mathfrak{M}_2$ in (27). Such a map has the following form:

There exists an increasing integer-valued function $k \to \ell_k$ for $k \geq 0$ and continuous onto maps $\widetilde{H}_k\colon M_{\ell_k} \to N_k$ where the collection of maps $\{\widetilde{H}_k \mid k \geq k_0\}$ form a commutative diagram

$$
\begin{array}{ccccccccc}
M_{\ell_0} & \xleftarrow{f_{\ell_0}^{\ell_1}} & M_{\ell_1} & \longleftarrow \cdots \longleftarrow & M_{\ell_k} & \xleftarrow{f_{\ell_k}^{\ell_{k+1}}} & M_{\ell_{k+1}} & \longleftarrow & \cdots \\
\widetilde{H}_0 \downarrow & & \downarrow \widetilde{H}_1 & & \downarrow \widetilde{H}_k & & \downarrow \widetilde{H}_{k+1} & & \\
N_0 & \xleftarrow[g_1]{} & N_1 & \longleftarrow \cdots \longleftarrow & N_k & \xleftarrow[g_k]{} & N_{k+1} & \longleftarrow & \cdots
\end{array} \tag{28}
$$

where the f_k and g_k are the bonding maps in the inverse limit representations (26) and (27), and $f_{\ell_k}^{\ell_{k+1}} = f_{\ell_k+1} \circ \cdots \circ f_{\ell_{k+1}}$ denotes the composition of bonding maps.

All of the horizontal maps in the diagram (28) are covering maps by construction. Moreover, as the spaces M_k and N_k are closed surfaces, we can assume that all of the vertical maps in (28) are also covering maps. Thus, the Euler classes of all surfaces there are related by the covering degrees of the maps. For example, $\chi(M_{\ell_k}) = d_k \cdot \chi(N_k)$, where d_k is the covering degree of $\widetilde{H}_k$.

To show (1) we assume that a homeomorphism H exists, and so we have diagram (28) as above. Observe that $g_2 = 1$ implies that $\chi(\Sigma_2) = \chi(\mathbb{T}^2) = 0$, hence $\chi(N_k) = 0$ for all $k \geq 0$. Then, as $d_k \geq 1$ for all k, we obtain $\chi(M_{\ell_k}) = 0$. But this contradicts the assumption that $g_1 > 1$, hence $\chi(M_{\ell_k}) < 0$ as M_{ℓ_k} is a covering of Σ_1 which has $\chi(\Sigma_1) < 0$. Thus, $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$.

To show (2) first assume that $C_{\vec{m}}(p) = C_{\vec{n}}(p)$ for all primes p. Then the odometer actions $\Phi_{\vec{m}}\colon \mathbb{Z} \times \mathfrak{G}_{\vec{m}} \to \mathfrak{G}_{\vec{m}}$ and $\Phi_{\vec{n}}\colon \mathbb{Z} \times \mathfrak{G}_{\vec{n}} \to \mathfrak{G}_{\vec{n}}$ are conjugate by an automorphism $\theta\colon \mathfrak{G}_{\vec{m}} \to \mathfrak{G}_{\vec{n}}$. Then, by Proposition 3.6, the suspension spaces $\mathfrak{M}(\Sigma_{g_1}, a_1, \vec{m})$ and $\mathfrak{M}_2 = \mathfrak{M}(\Sigma_{g_1}, a_1, \vec{n})$ are homeomorphic.

To show the converse in (2) assume that a homeomorphism H exists, and suppose that for some prime p we have $C_{\vec{m}}(p) \neq C_{\vec{n}}(p)$. We assume without loss of generality that $C_{\vec{m}}(p) < C_{\vec{n}}(p)$. If otherwise, then reverse the roles of $\vec{m}$ and $\vec{n}$ and consider the homeomorphism H^{-1}. Then, as $\chi(\Sigma_1) = \chi(\Sigma_2)$, for sufficiently large k the prime factorization of the Euler characteristic $\chi(M_{\ell_k})$ contains a lower power of p than the prime factorization of $\chi(N_k)$. But this contradicts the fact that $\chi(M_{\ell_k}) = d_k \cdot \chi(N_k)$, where d_k is the covering degree of $\widetilde{H}_k$.

Finally, to show (3) let $\Sigma = \Sigma_{g_1} = \Sigma_{g_2}$, where $g = g_1 = g_2 > 1$. It suffices to choose $\vec{m}$ and $\vec{n}$ such that $\vec{m} \sim_{\mathrm{t}} \vec{n}$, but $C_{\vec{m}} \neq C_{\vec{n}}$. It then follows from (2) that $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$. Pick a prime $p_1 \geq 3$ and let $\vec{m}$ be any sequence such that $C_{\vec{m}}(p_1) = 0$. Then define $\vec{n}$ by setting $n_1 = p_1$ and $n_{k+1} = m_k$ for all $k \geq 1$.

Note that $C_{\vec{m}}(p_1) = 0 \neq 1 = C_{\vec{n}}(p_1)$, so $C_{\vec{m}}(p) \neq C_{\vec{n}}(p)$ is satisfied. But clearly $\vec{m} \sim_{\mathrm{t}} \vec{n}$, so the adic surfaces $\mathfrak{M}(\Sigma_g, a_1, \vec{m})$ and $\mathfrak{M}(\Sigma_g, a_1, \vec{n})$ are return equivalent by Proposition 5.8, but are not homeomorphic by part (2) above. □

5.3 $\vec{m}$–adic solenoids of higher dimension

Observe that the requirements on the base manifold Σ used in the proofs of (2) and (3) of Theorem 5.10 are that:

(1) Σ is a strongly Borel manifold, so that the maps $\widetilde{H}_k$ can be assumed to be coverings;

(2) the fundamental group $G_0 = \pi_1(\Sigma, x)$ admits an epimorphism onto $\mathbb{Z}$, or equivalently that $H^1(\Sigma; \mathbb{Z})$ contains a copy of $\mathbb{Z}$;

(3) the Euler characteristic of Σ is nonzero.

Thus, the proof of parts (2) and (3) of Theorem 5.10 can be applied almost verbatim to show:

Theorem 5.11 *Let M be a closed manifold of dimension $n \geq 3$. Assume that M is strongly Borel, that $H^1(M; \mathbb{Z})$ has rank at least 1, and that the Euler characteristic of M is nonzero. Let $\mathfrak{M}_1 = \mathfrak{M}(M, a, \vec{m})$ and $\mathfrak{M}_2 = \mathfrak{M}(M, a, \vec{n})$ be the corresponding adic solenoids, where $a\colon \pi_1(M, x) \to \mathbb{Z}$ is an epimorphism. Then we have:*

(1) *$\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic if and only if $C_{\vec{m}} = C_{\vec{n}}$.*

(2) *There exist $\vec{m} \sim_{\mathrm{t}} \vec{n}$ with $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$.*

Finally, we comment on the requirement in Theorem 1.5 that the base manifolds be strongly Borel. Let M be a closed n–manifold with $n \geq 5$. Suppose that M satisfies the conditions of Theorem 5.11.

Let $N = M \# \mathbb{S}^2 \times \mathbb{S}^{n-2}$ be the closed n–manifold obtained by attaching the handle $\mathbb{S}^2 \times \mathbb{S}^{n-2}$. Then $\pi_1(M, x) \cong \pi_1(N, x)$, where we choose the basepoint $x \in M$ disjoint from the disk along which the handle is attached.

Form the adic solenoids $\mathfrak{M}_1 = \mathfrak{M}(M, a, \vec{m})$ and $\mathfrak{M}_2 = \mathfrak{M}(N, a, \vec{m})$ as before, but with bases M and N. Then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent, as in fact they have conjugate global monodromy actions. On the other hand, all leaves in $\mathfrak{M}_1$ have trivial higher homotopy groups, while all leaves in $\mathfrak{M}_2$ have nontrivial higher homotopy groups. Thus, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ can not be homeomorphic.

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Department of Mathematics, University of Leicester
Leicester, United Kingdom

Department of Mathematics, University of Illinois at Chicago
Chicago, IL, United States

Department of Mathematics, University of Illinois at Chicago
Chicago, IL, United States

alex.clark@leicester.ac.uk, hurder@uic.edu, lukina@uic.edu

Proposed: Steve Ferry Received: 7 November 2013
Seconded: Leonid Polterovich, Yasha Eliashberg Revised: 22 October 2017