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Topology of automorphism groups of parabolic geometries

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We prove for the automorphism group of an arbitrary parabolic geometry that the C^0– and C^∞–topologies coincide, and the group admits the structure of a Lie group in this topology. We further show that this automorphism group is closed in the homeomorphism group of the underlying manifold.

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1 Introduction

It is well known that the automorphism group of a rigid geometric structure is a Lie group. In fact, as there are multiple notions of rigid geometric structures, such as G–structures of finite type, Gromov rigid geometric structures or Cartan geometries, the property that the local automorphisms form a Lie pseudogroup is sometimes taken as an informal definition of rigidity for a geometric structure.

There remains, however, some ambiguity about the topology in which this transformation group is Lie. It is a subgroup of $\mathrm{Diff}(M)$, assuming the underlying structure is smooth, so one may ask whether it admits the structure of a Lie group in the C^∞–, C^m– for some positive integer m, or even the compact–open topology. A related interesting question is whether the automorphism group is closed in $\mathrm{Homeo}(M)$.

Theorems of Ruh [14] and Sternberg [17, Corollary VII.4.2] state that, if H is the automorphism group of a G–structure of finite type of order m, then H is a Lie group in the C^m–topology on $\mathrm{Diff}^{m+1}(M)$. Gromov proved a similar result in [5, Corollary 1.5.B] for a smooth Gromov-m–rigid geometric structure. In the case of a smooth Riemannian metric (M,g), the results above yield a Lie group structure for the C^1–topology on the isometry group $\mathrm{Isom}(M,g)$.

The classical theorems of Myers and Steenrod [11], however, say that in this Riemannian case the C^0– and C^m–topologies coincide on $\mathrm{Isom}(M,g)$ for all m. Nomizu [12] proved the same for the group of affine transformations of a connection (under an assumption of geodesic completeness, which can be removed). The essence of the proof

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is that exponential coordinates locally convert affine transformations to linear maps, and a sequence of linear transformations converging C^0 automatically converges C^∞.

This article is concerned with the topology of local automorphisms of parabolic geometries (see Section 1.2 below for the general definition). These form a rich class of differential-geometric structures which behave differently from Riemannian metrics in the sense that their automorphisms can have strong dynamics, so, for example, a convergent sequence of automorphisms need not limit to a homeomorphism. Parabolic geometries do not determine a connection; without the exponential map, it is no longer clear that a C^0–limit of smooth automorphisms should be smooth.

1.1 Statement of main results

We first briefly survey some results for specific parabolic geometries, which will be generalized by our main theorem. We remark that the first two theorems below, of Ferrand and Schoen, are proved by geometric-analytic techniques that are quite specific to the structures in question.

- In the course of proving the Lichnerowicz conjecture on Riemannian conformal automorphism groups, Ferrand showed, using techniques of quasiconformal analysis, that if a homeomorphism f is a C^0–limit of smooth conformal maps, then f is also smooth and conformal [9; 1].
- Schoen [15] reproved Ferrand's result above, and extended it to strictly pseudoconvex CR structures. His proof uses scalar curvature and the conformal Laplace operator in the conformal case, and the analogous Webster scalar curvature and pseudoconformal subelliptic operator in the CR setting.
- In [3], the first author proved for conformal pseudo-Riemannian structures that if a sequence of smooth local conformal transformations converges C^0, then it converges C^∞. His approach is very different from the analytic techniques of [9; 15]: he uses the Cartan connection associated to these structures and the dynamics of the action on null geodesics.

We prove a generalization of the results recounted above to local automorphisms of arbitrary parabolic geometries. Parabolic geometries are a broad family of geometric structures which nonetheless admit an extensive general theory. Well-known examples include the conformal semi-Riemannian structures and strictly pseudoconvex CR structures mentioned above, as well as more general nondegenerate CR structures, projective structures and so-called path geometries, which encode ODEs (see Čap and Slovák [19] for a comprehensive reference). Definitions 1.4 and 1.5 below explain precisely what is meant by *parabolic geometry* and *automorphism/automorphic immersion*. An automorphic immersion can be informally defined as a differentiable immersion $f\colon U \to M$, where $U \subset M$ is an open set, which preserves the Cartan geometry $\mathcal{C}$ on M. When $U = M$ and f is also a diffeomorphism, f is said to be an automorphism of $(M, \mathcal{C})$. The set of automorphisms is a group that will be denoted by $\mathrm{Aut}(M, \mathcal{C})$. Our main results can then be stated as follows:

Theorem 1.1 *Let $(M, \mathcal{C})$ be a smooth parabolic geometry. Let $f_k\colon U \to M$ be a sequence of automorphic immersions of $(M, \mathcal{C})$ converging in the C^0–topology on U to a map h. Then h is smooth and $f_k \to h$ also in the C^∞–topology.*

In Section 3.3 we will also prove the following:

Theorem 1.2 *Let $(M, \mathcal{C})$ be a smooth parabolic geometry. Then $\mathrm{Aut}(M, \mathcal{C})$ is a Lie transformation group in the compact–open topology. Moreover, $\mathrm{Aut}(M, \mathcal{C})$ is closed in $\mathrm{Homeo}(M)$ for this topology.*

1.2 Definitions

Parabolic geometries are most conveniently defined in terms of Cartan geometries. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and $P < G$ a closed subgroup. We will assume throughout the article that the pair (G, P) is *effective*, meaning G acts faithfully on G/P. A noneffective pair can always be replaced by an effective one, with the same quotient space G/P (see Sharpe [16]).

Definition 1.3 A *Cartan geometry* $\mathcal{C}$ on a manifold M, with model space $X = G/P$, comprises $(\widehat{M}, \omega)$, where $\pi\colon \widehat{M} \to M$ is a principal P–bundle and ω is a $\mathfrak{g}$–valued one-form on $\widehat{M}$ satisfying:

- For all $\hat{x} \in \widehat{M}$, $\omega_{\hat{x}}\colon T_{\hat{x}}\widehat{M} \to \mathfrak{g}$ is a linear isomorphism.
- For all $g \in P$, $R_g^*\omega = (\mathrm{Ad}\, g)^{-1} \circ \omega$, where R_g denotes the right translation by g on $\widehat{M}$.
- For all $X \in \mathfrak{p}$, $\omega(X^\ddagger) \equiv X$, where $X^\ddagger(\hat{x}) = \frac{d}{ds}\big|_0 \hat{x}.e^{sX}$.

The basic example of a Cartan geometry modeled on $X = G/P$ is the *flat* geometry on X comprising (G, ω_G), where ω_G is the Maurer–Cartan form.

Definition 1.4 A *parabolic geometry* is a Cartan geometry modeled on $X = G/P$, where G is a semisimple Lie group with finite center and without compact local factors and $P < G$ is a parabolic subgroup.

Our notion of parabolic subgroup is the standard one, which will be recalled in Section 2.5.1.

Essentially all classical rigid geometric structures correspond to a canonical Cartan geometry. The process of canonically associating a Cartan geometry is called the *equivalence problem* for a given geometric structure (see [16] for examples). Parabolic geometries admit a uniform solution of the equivalence problem, in which each corresponds to a type of "filtered manifold" (barring one exception, projective structures); see Čap and Slovák [19, Section 3.1] and Tanaka [18].

Definition 1.5 For $(M,\mathcal{C})$ a smooth Cartan geometry with $\mathcal{C}=(\widehat{M},\omega)$, an *automorphism* is $f \in \mathrm{Diff}(M)$ which lifts to a bundle automorphism $\hat{f}$ of $\widehat{M}$ satisfying $\hat{f}^*\omega=\omega$. The group of automorphisms is denoted by $\mathrm{Aut}(M,\mathcal{C})$.

For an open subset $U \subseteq M$, a smooth immersion $f\colon U \to M$ is an *automorphic immersion* of $(M,\mathcal{C})$ if it lifts to a bundle map $\hat{f}\colon \hat{U}=\pi^{-1}(U) \to \widehat{M}$ satisfying $\hat{f}^*\omega=\omega|_{\hat{U}}$.

As (G,P) is effective, the elements $f \in \mathrm{Aut}(M,\mathcal{C})$ correspond bijectively to their lifts $\hat{f}$ to $\widehat{M}$ satisfying $\hat{f}^*\omega=\omega$, and similarly for automorphic immersions (see Melnick [10, Proposition 3.6]).

1.3 Lie topology on the automorphism group

For $\mathcal{C}=(\widehat{M},\omega)$ a smooth Cartan geometry on M, the group $\mathrm{Aut}(M,\mathcal{C})$ can be endowed with the structure of a Lie transformation group as follows (we refer to the definition in Palais [13, Chapter IV] of *Lie transformation group*). The Cartan connection defines a framing $\mathcal{P}$ of $\widehat{M}$, the pullback by ω of any basis in $\mathfrak{g}$. The automorphisms of a framing form a Lie transformation group; more precisely:

Theorem 1.6 (S Kobayashi [8, Theorem I.3.2]) *Let N be a smooth, connected manifold with a smooth framing $\mathcal{P}$.*

(1) $\mathrm{Aut}(\mathcal{P}) < \mathrm{Diff}(N)$ *admits the structure of a Lie transformation group.*

(2) *For $m=0,\dots,\infty$, the C^m–topology on $\mathrm{Aut}(\mathcal{P})$ coincides with the Lie topology.*

(3) *A sequence $f_k \in \mathrm{Aut}(\mathcal{P})$ converges in the Lie topology if and only if there exists $z \in N$ such that $f_k(z)$ converges in N.*

Denote by $\widehat{\mathrm{Aut}}(M,\mathcal{C})$ the group of bundle automorphisms of $\widehat{M}$ preserving ω. This is a C^∞–closed subgroup of $\mathrm{Aut}(\widehat{M},\mathcal{P})$, so it is closed in the Lie topology and inherits the structure of a Lie transformation group. The isomorphism $\widehat{\mathrm{Aut}}(M,\mathcal{C}) \cong \mathrm{Aut}(M,\mathcal{C})$ then provides the latter with the structure of a Lie group, in fact of a Lie transformation group of M. The underlying topology on $\mathrm{Aut}(M,\mathcal{C})$, the pullback of the C^∞–topology on $\widehat{\mathrm{Aut}}(M,\mathcal{C})$, will henceforth be referred to as the *Lie topology*. For $U \subset M$, the automorphic immersions defined on U admit a similarly defined topology, which we will also call the Lie topology.

Recall that the Lie topology on $\mathrm{Aut}(M,\mathcal{C})$, as well as all C^m–topologies, are second countable. A sequence (f_k) of automorphic immersions of $(M,\mathcal{C})$ converges in the Lie topology if and only if the lifted sequence $(\widehat{f}_k)$ converges C^∞. Thus, if (f_k) converges for the Lie topology to an automorphic immersion, then it does for the C^∞–topology on M. In cases where $\widehat{M}$ is a subbundle of the r–frames of M, and $\widehat{f}_k$ are the corresponding natural lifts of f_k, then C^∞–convergence of (f_k) on M conversely implies convergence in the Lie topology. Such is the case for many parabolic geometries, but this property in general is unclear. Our proofs will go via the Lie topology on $\mathrm{Aut}(M,\mathcal{C})$, thus showing that it coincides with all C^m–topologies for $m = 0, \ldots, \infty$, and similarly for automorphic immersions of $(M,\mathcal{C})$.

1.4 Structure of the proof

A sequence (f_k) of automorphic immersions converging in the C^0–topology gives rise to a *holonomy sequence* (p_k) in P. The action of (p_k) on G/P reflects many features of the action of (f_k) on M. Section 2 contains the definition of holonomy sequences and their equicontinuity properties relative to those of (f_k). In Section 3, we translate the problem to a statement about holonomy sequences on G/P. The proof of this statement, Theorem 3.1, proceeds by induction on $\mathrm{rk}_{\mathbb{R}}\, G$. The base case, $\mathrm{rk}_{\mathbb{R}}\, G = 1$, is recalled from Frances [2] in Section 4. The task for the remainder of the paper is, given a holonomy sequence (p_k) not conforming to the conclusion of Theorem 3.1, to find an invariant lower-rank subvariety of G/P on which (p_k) exhibits the same behavior, thus contradicting the induction hypothesis. Section 5 develops tools for identifying such a lower-rank subvariety, corresponding to certain manipulations on the root spaces of $\mathfrak{g}$. In Section 6, we apply these tools to complete the induction step.

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2 Holonomy and equicontinuity with respect to segments

Let $(M, \mathcal{C})$ be a Cartan geometry modeled on $X = G/P$, not necessarily parabolic.

Definition 2.1 A sequence $f_k \colon U \to M$ of automorphic immersions of $(M, \mathcal{C})$ is *equicontinuous at* $x \in U$ if there exists $y \in M$ such that for any $x_k \to x$ in U, $f_k(x_k) \to y$.

If $f_k \colon U \to M$ converges C^0, then (f_k) is clearly equicontinuous at every point of U. The following theorem says that, conversely, equicontinuity *at a single point* implies local C^0–convergence, at least for parabolic geometries.

Theorem 2.2 *Let $(M, \mathcal{C})$ be a smooth parabolic geometry and (f_k) a sequence of automorphic immersions equicontinuous at $x \in M$. Then there exists an open neighborhood U of x on which a subsequence of (f_k) converges C^∞ to a smooth map h.*

Note that Theorem 2.2 implies Theorem 1.1.

2.1 Holonomy sequences

Let $f_k \colon U \to M$ be a sequence of automorphic immersions of $(M, \mathcal{C})$ which is equicontinuous at $x \in U$, with lifts $\hat{f}_k \colon \hat{U} \to \widehat{M}$. Associated to (f_k) is a holonomy sequence (p_k) in P, whose behavior around the basepoint $o = [P] \in G/P$ reflects much of the local behavior of f_k around x.

Definition 2.3 Let $x_k \to x$ in U. A sequence (p_k) of P is a *holonomy sequence of* (f_k) *along* (x_k) when there exist $\hat{x}_k \in \pi^{-1}(x_k)$ such that $\{\hat{x}_k\}_{k \in \mathbb{N}}$ and $\{\hat{y}_k\} = \{\hat{f}_k(\hat{x}_k).p_k^{-1}\}_{k \in \mathbb{N}}$ are bounded in $\widehat{M}$. A *holonomy sequence of* (f_k) *at* x is any holonomy sequence along some sequence $x_k \to x$.

We will denote by $\mathcal{H}ol(x)$ the set of all holonomy sequences of (f_k) at x. Equicontinuity of (f_k) at x ensures that $\mathcal{H}ol(x)$ is nonempty. Indeed, given $y \in M$ such that $f_k(x) \to y$, choose any $\hat{x} \in \pi^{-1}(x)$ and $\hat{y} \in \pi^{-1}(y)$. Then there exists a sequence (p_k) in P such that $\hat{f}_k(\hat{x}).p_k^{-1} \to \hat{y}$, so $(p_k) \in \mathcal{H}ol(x)$.

2.2 Equicontinuity with respect to segments

Equicontinuity of a sequence (f_k) at x will have strong consequences on the local behavior of its holonomy sequences around the basepoint $o \in G/P$. A useful notion to capture this local behavior is *equicontinuity with respect to segments*. An *unparametrized segment* in G/P is a set of the form $[\xi] = \{e^{t\xi}.o \mid t \in [0,1]\}$ for some $\xi \in \mathfrak{g}$. Note that distinct $\xi, \eta \in \mathfrak{g}$ may define the same unparametrized segment.

We fix a Riemannian metric in a fixed neighborhood of o in X, with respect to which we will measure the length of segments $[\xi]$ in this neighborhood, and denote the results by $L([\xi])$.

Definition 2.4 A sequence (p_k) in P is *equicontinuous with respect to segments* if, whenever a sequence of segments $[\xi_k]$ satisfies $L([\xi_k]) \to 0$ and $p_k.[\xi_k] = [\eta_k]$, every cluster value of (η_k) in $\mathfrak{g}$ is in $\mathfrak{p}$.

Observe that the condition $L([\xi_k]) \to 0$, hence the very notion of equicontinuity with respect to segments, does not depend on the choice of Riemannian metric, since any two are bi-Lipschitz equivalent in a neighborhood of o.

2.3 Relation of equicontinuity and equicontinuity with respect to segments

Proposition 2.5 *Let $(M, \mathcal{C})$ be a Cartan geometry and $f_k \colon U \to M$ a sequence of automorphic immersions of $(M, \mathcal{C})$. If (f_k) is equicontinuous at $x \in U$, then every holonomy sequence $(p_k) \in \mathcal{H}ol(x)$ is equicontinuous with respect to segments.*

The proof will use the development of curves $\gamma \colon [0,1] \to \widehat{M}$, a notion which we now recall. Given such a smooth curve γ, the equation $\omega_G(\tilde{\gamma}'(s)) = \omega(\gamma'(s))$, where ω_G is the Maurer–Cartan form of G, defines an ODE on G. The solution $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = \mathrm{id}$ will be called the *development of* γ.

The Cartan connection also yields an *exponential map* on $\widehat{M}$: any u in $\mathfrak{g}$ defines the ω–constant vector field $U^{\ddagger}$ on $\widehat{M}$ by $\omega(U^{\ddagger}) \equiv u$; denote by $\{\varphi^t_{U^{\ddagger}}\}$ the corresponding local flow. Observe that whenever $u \in \mathfrak{p}$, the flow $\{\varphi^t_{U^{\ddagger}}\}$ is globally defined and corresponds to right multiplication by e^{tu} in the bundle $\widehat{M}$ (by the third axiom in Definition 1.3). The exponential map at $\hat{x} \in \widehat{M}$ is defined in a neighborhood $\mathcal{U} = \mathcal{U}_{\hat{x}}$ of the origin in $\mathfrak{g}$ by

$$u \mapsto \exp(\hat{x}, u) := \varphi^1_{U^{\ddagger}}.\hat{x}.$$

Shrinking $\mathcal{U}$ if necessary makes the exponential map at $\hat{x}$ a diffeomorphism onto a neighborhood of $\hat{x}$ in $\widehat{M}$. For $u \in \mathcal{U} \subset \mathfrak{g}$, we will denote the exponential of u at $\hat{x}$ in M by $\exp(\hat{x}, u)$, and the exponential in the Lie group G by e^u.

It is easy to see that whenever $\hat{f}\colon \widehat{M} \to \widehat{M}$ is the lift of an automorphic immersion of M,

$$\exp(\hat{x}, u) = \exp(\hat{f}(\hat{x}), u).$$

The P–equivariance property of ω leads to a corresponding equivariance property for the exponential map, for all $p \in P$,

$$\exp(\hat{x}, u).p^{-1} = \exp(\hat{x}.p^{-1}, (\operatorname{Ad} p).u). \tag{1}$$

Last, we recall the following crucial reparametrization lemma:

Lemma 2.6 [4, Proposition 4.3] *Let $\gamma, \alpha\colon [0,1] \to \widehat{M}$ be smooth curves, with $\gamma(0) = \alpha(0)$, and let $q\colon [0,1] \to P$ be a smooth map satisfying $q(0) = \mathrm{id}$.*

1. *Assume that for the developments $\tilde{\gamma}$ and $\tilde{\alpha}$, the relation $\tilde{\gamma}(s) = \tilde{\alpha}(s).q(s)$ holds in G for every $s \in [0,1]$. Then $\gamma(s) = \alpha(s).q(s)$ holds in $\widehat{M}$.*
2. *In particular, if $u, v \in \mathfrak{g}$ and if there exists a smooth $a\colon [0,1] \to [0,1]$, with $a(0) = 0$ and $a(1) = 1$, such that*

$$e^{su} = e^{a(s)v} q(s) \quad \text{for all } s \in [0,1],$$

then, for every $\hat{y} \in \widehat{M}$ such that $\exp(\hat{y}, u)$ or $\exp(\hat{y}, v)$ is defined,

$$\exp(\hat{y}, u) = \exp(\hat{y}, v).q(1).$$

Proof of Proposition 2.5 Assume for a contradiction that (f_k) is equicontinuous at x, but that some holonomy sequence (p_k) of (f_k) at x does not act equicontinuously with respect to segments. Then $\hat{y}_k = \hat{f}_k(\hat{x}_k).p_k^{-1}$ is bounded for a bounded sequence $(\hat{x}_k)$ projecting to $x_k \to x$. After passing to a subsequence, we can assume $\hat{x}_k \to \hat{x}$ and $\hat{y}_k \to \hat{y}$.

Since (p_k) is not equicontinuous with respect to segments, passing again to a subsequence, there exists a sequence of segments $[\xi_k]$, with $L([\xi_k]) \to 0$, as well as a sequence (η_k) in $\mathfrak{g}$ converging to $\eta_\infty \notin \mathfrak{p}$, such that, for all k,

$$p_k.[\xi_k] = [\eta_k]. \tag{2}$$

This condition can be expressed by the relation, valid for all $s \in [0,1]$,

$$e^{s \operatorname{Ad}(p_k)(\xi_k)} = e^{\varphi_k(s)\eta_k}.p_k(s).$$

Here, $p_k\colon [0,1] \to P$ with $p_k(0) = 0$ denotes a smooth path and $\varphi_k\colon [0,1] \to [0,1]$ a nondecreasing diffeomorphism. Given $\lambda > 0$ arbitrary small, let $0 < \lambda_k < 1$ be such that $\varphi_k(\lambda_k) = \lambda$ for all k. Then write

$$e^{s\,\mathrm{Ad}(p_k)(\lambda_k\xi_k)} = e^{(\varphi_k(\lambda_k s)/\varphi_k(\lambda_k))\varphi_k(\lambda_k)\eta_k}.p_k(\lambda_k s). \tag{3}$$

Note that $L([\lambda_k\xi_k]) \to 0$. Thus, for λ sufficiently small, we can replace ξ_k and η_k by $\lambda_k\xi_k$ and $\varphi_k(\lambda_k)\eta_k$, so that (2) holds, with the extra property that $\exp(\hat{y}_k, \eta_k)$ is defined for all $k \in \mathbb{N}$, and η_∞ is in an injectivity domain of the map $u \mapsto \exp(\hat{y}, u)$. In particular, if we call $y := \pi(\hat{y})$, the fact that $\eta_\infty \notin \mathfrak{p}$ implies, shrinking λ again if necessary, $\pi(\exp(\hat{y}, \eta_\infty)) \neq y$.

The next step is to show that $\pi(\exp(\hat{x}_k, \xi_k))$ is defined for k large enough, and converges to x. To this aim, define a left-invariant Riemannian metric ρ_G on G by left-translating any scalar product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, and a corresponding Riemannian metric ρ on $\widehat{M}$, with

$$\rho(u, v) := \langle \omega(u), \omega(v) \rangle.$$

By the definition of ρ, if γ is a curve in $\widehat{M}$ and $\tilde{\gamma}$ its development in G, then $L_{\rho_G}(\tilde{\gamma}) = L_\rho(\gamma)$. Fix $\epsilon > 0$ small enough that for all $k \in \mathbb{N}$, the ρ–ball $B(\hat{x}_k, \epsilon)$ of center $\hat{x}_k$ and radius ϵ has compact closure in $\widehat{M}$.

Now consider the curve $s \mapsto e^{s\xi_k}$. We fix Σ a small submanifold of G containing 1_G, which is transverse to the fibers of $\pi_X\colon G \to X = G/P$ and such that the restriction of π_X to Σ yields a diffeomorphism $\psi\colon \Sigma \to U$, where U is a neighborhood of o in X. For k large enough, there exists a smooth $q_k\colon [0,1] \to P$, with $q_k(0) = \mathrm{id}$, such that $\alpha_k(s) = e^{s\xi_k}.q_k(s)$ is contained in Σ. Of course, $\psi(\alpha_k([0,1])) = [\xi_k]$. Two Riemannian metrics on Σ are always locally bi-Lipschitz equivalent, hence there exist $C_1, C_2 > 0$ such that, for k large enough,

$$C_1 L([\xi_k]) \leq L_{\rho_G}(\alpha_k) \leq C_2 L([\xi_k]).$$

We infer that $L_{\rho_G}(\alpha_k) \to 0$; in particular, for $k \geq k_0$, $L_{\rho_G}(\alpha_k) < \epsilon$. Now consider, for each $k \geq k_0$, the first-order ODE on $\widehat{M}$

$$\omega(\beta_k') = \alpha_k' \tag{4}$$

with initial condition $\beta_k(0) = \hat{x}_k$. If $[0, \tau_k^*)$, is a maximal interval of definition for $s \mapsto \exp(\hat{x}, s\xi_k)$ then, for all k, $\beta_k(s) := \exp(\hat{x}_k, s\xi_k).q_k(s)$ for $s \in [0, \tau_k^*)$ is a maximal solution of our ODE, by Lemma 2.6. By the definition of L_ρ, we have $L_\rho(\beta_k) = L_{\rho_G}(\alpha_k)$. If $\tau_k^* \leq 1$, the inequality $L_{\rho_G}(\alpha_k) < \epsilon$ implies that β_k is included in the relatively compact set $B(\hat{x}_k, \epsilon)$; this contradicts the maximality of τ_k^*. We thus infer $\tau_k^* > 1$, which ensures that $\beta_k(1)$, hence $\exp(\hat{x}_k, \xi_k) = \beta_k(1).q_k(1)^{-1}$ is defined. Moreover, $L_\rho(\beta_k) = L_{\rho_G}(\alpha_k) \to 0$, so $\beta_k(1) \to \hat{x}$. Projecting to M gives $\pi(\exp(\hat{x}, \xi_k)) \to x$.

Now Lemma 2.6, combined with equation (3) above says that, for all $k \geq k_0$,

$$f_k(\exp(\hat{x}_k, \xi_k).p_k^{-1}) = \exp(\hat{y}_k, \mathrm{Ad}(p_k)\xi_k) = \exp(\hat{y}_k, \eta_k).p_k(1).$$

Projecting this relation on M, we obtain

$$\widehat{f_k}(\pi(\exp(\hat{x}_k, \xi_k))) = \pi(\exp(\hat{y}_k, \eta_k)).$$

After possibly passing to a subsequence, the right-hand term converges to

$$\pi(\exp(\hat{y}, \eta_\infty)) \neq y,$$

while we just showed $\pi(\exp(\hat{x}_k, \xi_k)) \to x$; this yields the desired contradiction with the equicontinuity of (f_k) at x. □

2.4 Vertical and transverse perturbations of holonomy sequences

Proposition 2.5 translates equicontinuity of (f_k) at x to a property of sequences in $\mathcal{H}ol(x)$, which are in turn sequences of P acting on $X = G/P$. In this section we define several operations on sequences in P which preserve $\mathcal{H}ol(x)$.

Holonomy sequences involve many choices: of (x_k), $(\hat{x}_k)$ and $(\hat{y}_k) = (\hat{f}(\hat{x}_k)p_k^{-1})$, in the notation of Definition 2.3. The right and left *vertical perturbations* of (p_k) correspond to other possible choices of $(\hat{x}_k)$ and $(\hat{y}_k)$, respectively.

Definition 2.7 Let (p_k) be a sequence in P. A *vertical perturbation of* (p_k) is a sequence $q_k = l_k p_k m_k$ where (l_k) and (m_k) are two bounded sequences in P.

Transverse perturbations of (p_k) correspond roughly to other possible choices of (x_k) converging to x.

Definition 2.8 For (p_k) a sequence of P, a sequence (q_k) of P is said to be a *transverse perturbation of* (p_k) when there exist two sequences (ξ_k) and (η_k) in $\mathfrak{g} \backslash \mathfrak{p}$ such that:

(1) $q_k = e^{-\eta_k} p_k e^{\xi_k}$.

(2) The sequences (ξ_k) and (η_k) both converge to 0.

(3) For every $s \in \mathbb{R}$, $e^{-s\eta_k} p_k e^{s\xi_k}$ belongs to P.

The other choice of (x_k) in this case is $\pi(\exp(\hat{x}_k, \xi_k))$, as will be seen in the proof below.

Lemma 2.9 *Let $(M, \mathcal{C})$ be a Cartan geometry and let $f_k \colon U \to M$ be a sequence of automorphic immersions. For any $x \in U$, the set of holonomy sequences $\mathcal{H}ol(x)$ is stable by vertical and transverse perturbations.*

Proof We consider (p_k) a sequence belonging to $\mathcal{H}ol(x)$. By definition, there exists $(\hat{x}_k)$ a bounded sequence in $\widehat{M}$ such that $\hat{y}_k = \hat{f}_k(\hat{x}_k).p_k^{-1}$ is bounded and the projection x_k on M converges to x.

Assume that (q_k) is obtained from (p_k) by vertical perturbation, namely there exist bounded sequences (l_k) and (m_k) in P such that $q_k = l_k p_k m_k$. Then $(\hat{x}_k.m_k)$ is bounded in $\widehat{M}$, and still projects on (x_k). Moreover,

$$\hat{f}_k(\hat{x}_k.m_k)q_k^{-1} = \hat{y}_k.l_k^{-1}$$

is still bounded in $\widehat{M}$. It follows that (q_k) is a holonomy sequence at x.

We now handle the case of a transverse perturbation $q_k = e^{-\eta_k} p_k e^{\xi_k}$. The sequence $(\hat{x}_k)$ is bounded and $\xi_k \to 0$, hence $(\hat{z}_k) = (\exp(\hat{x}_k, \xi_k))$ is bounded in $\widehat{M}$, too; moreover, $\pi(\hat{z}_k)$ converges to x. It remains to show that $\hat{f}_k(\hat{z}_k).q_k^{-1}$ is bounded in M. Write this expression as $\hat{f}_k(\hat{z}_k).p_k^{-1}.p_k q_k^{-1}$. By the equivariance (1) of the exponential map,

$$\hat{f}_k(\hat{z}_k).p_k^{-1} = \exp(\hat{f}_k(\hat{x}_k).p_k^{-1}, \mathrm{Ad}(p_k)\xi_k).$$

Point (2) in the definition of transverse perturbation says that $q_k(s) = e^{-s\eta_k} p_k e^{s\xi_k}$ belongs to P for all $s \in \mathbb{R}$. Thus,

$$e^{s\,\mathrm{Ad}(p_k)\xi_k} = e^{s\eta_k} q_k(s) p_k^{-1},$$

where $s \mapsto q_k(s)p_k^{-1}$ is a smooth path in P passing through id when $s=0$. Lemma 2.6 then implies

$$\exp(\hat{f}_k(\hat{x}_k).p_k^{-1}, \mathrm{Ad}(p_k)\xi_k) = \exp(\hat{y}_k, \eta_k).q_k p_k^{-1}.$$

Right translation by $p_k q_k^{-1}$ gives $\hat{f}_k(\hat{z}_k).q_k^{-1} = \exp(\hat{y}_k, \eta_k)$. This expression is bounded, because $(\hat{y}_k)$ is a bounded sequence, and η_k tends to zero by definition of a transverse perturbation. □

2.5 Admissible operations

In this section, we specialize to $X = G/P$ a parabolic model space, and define some operations on holonomy sequences specific to parabolic geometries. We first introduce some notation in $\mathfrak{g}$.

2.5.1 Notation in $\mathfrak{g}$ Let G be semisimple with no compact local factors and with finite center. We denote by Θ a Cartan involution of the semisimple Lie algebra $\mathfrak{g}$. Associated to Θ, we choose a Cartan subspace $\mathfrak{a}$, and $\Phi = \{\alpha_1, \dots, \alpha_r\}$ a set of simple roots. The positive and negative roots are denoted by Φ^+ and Φ^-, respectively. The usual decomposition of the Lie algebra $\mathfrak{g}$ into root spaces is

$$\mathfrak{g} = \sum_{\alpha \in \Phi^-} \mathfrak{g}_\alpha \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha.$$

Recall that the Lie algebra $\mathfrak{m}$ is centralized by $\mathfrak{a}$, and lies in the Lie algebra $\mathfrak{k}$ comprising the $+1$–eigenspace of the Cartan involution Θ.

We will denote by $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) the sum $\sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ $\big($resp. $\sum_{\alpha \in \Phi^-} \mathfrak{g}_\alpha\big)$.

The minimal parabolic subalgebra of $\mathfrak{g}$ is $\mathfrak{p}_{\min} = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^+$. A general *parabolic subalgebra* $\mathfrak{p}$ is one containing $\mathfrak{p}_{\min}$, and is obtained as follows (up to conjugacy in G): there exists $\Lambda \subsetneq \Phi$, possibly empty, such that

$$\mathfrak{p}_\Lambda = \sum_{\alpha \in \Lambda^+} \mathfrak{g}_{-\alpha} \oplus \mathfrak{p}_{\min},$$

where Λ^+ is the set of roots in Φ^+ which are in the span of Λ. A *parabolic subgroup* of G is any Lie subgroup $P_\Lambda < G$ with Lie algebra $\mathfrak{p}_\Lambda$ for some Λ. We will sometimes denote this group simply by P when Λ is understood.

We denote by $\mathfrak{n}_\Lambda^+$ the nilpotent radical of $\mathfrak{p}$, which equals $\sum_{\alpha \in (\Lambda^+)^c} \mathfrak{g}_\alpha$. Here $(\Lambda^+)^c$ stands for the positive roots, written as linear combinations of roots in Φ involving at least one root which is not in Λ. Notice that $\mathfrak{n}_\Lambda^+$ is an ideal of $\mathfrak{n}^+$ and of $\mathfrak{p}$. Finally, we call $\mathfrak{h}_\Lambda$ the Lie algebra $\mathfrak{h}_\Lambda = \mathfrak{a} \ltimes \mathfrak{n}_\Lambda^+$.

We denote by A, N_Λ^+ and H_Λ the connected Lie subgroups of G with Lie algebras $\mathfrak{a}$, $\mathfrak{n}_\Lambda^+$ and $\mathfrak{h}_\Lambda$, respectively; they are all subgroups of P_Λ.

2.5.2 Reduced holonomy sequences A sequence (p_k) in P will be called *reduced* when it is a sequence of H_Λ.

Lemma 2.10 *Any sequence (p_k) in $P = P_\Lambda$ can be converted by left and right vertical perturbation to $(q_k) \in H_\Lambda$.*

Proof Consider the Levi decomposition of $P_\Lambda = S_\Lambda \ltimes N_\Lambda^+$, where S_Λ is the connected reductive subgroup of G with Lie algebra spanned by $\mathfrak{a}$ and the positive and negative root spaces of Λ^+. Write $p_k = s_k n_k$ according to this decomposition. As S_Λ is reductive, it admits a KAK decomposition, according to which $s_k = l'_k a_k l_k$, with $a_k \in A = \exp(\mathfrak{a})$ and $l_k, l'_k \in K$. As G has finite center, K is contained in a maximal compact subgroup of G and is a maximal compact subgroup of S_Λ. Then $p_k = l'_k a_k n'_k l_k$, where $n'_k = l_k^{-1} n_k l_k \in N_\Lambda^+$. Now $q_k = a_k n'_k$ is the desired reduced sequence. □

2.5.3 Weyl reflections

For $X = G/P$ parabolic, these are transformations of holonomy sequences in H_Λ, which will be useful in our proof.

For any root α, the *Weyl reflection* is $\rho_\alpha \colon \mathfrak{a}^* \to \mathfrak{a}^*$, with

$$\rho_\alpha(\xi) = \xi - \frac{2\langle \alpha, \xi \rangle}{\langle \alpha, \alpha \rangle}\alpha, \quad \xi \in \mathfrak{a}^*.$$

Recall that for α positive, ρ_α preserves $\Phi^+ \backslash \{\alpha\}$ and $\Phi^- \backslash \{-\alpha\}$, assuming 2α is not a root (in which case ρ_α preserves $\Phi^+ \backslash \{\alpha, 2\alpha\}$ and $\Phi^- \backslash \{-\alpha, -2\alpha\}$). Recall that whenever ξ is a root, $A_{\alpha\xi} = 2\langle \alpha, \xi \rangle / \langle \alpha, \alpha \rangle$ is an integer.

For any root α, there exists $k_\alpha \in G$ such that $\mathrm{Ad}(k_\alpha)$ preserves $\mathfrak{a}$ and the action of $\mathrm{Ad}(k_\alpha)$ on $\mathfrak{a}^*$ coincides with that of ρ_α (see [7, Proposition 6.52c]). In the sequel, we will denote by r_α any automorphism of G such that the action induced on $\mathfrak{g}$ preserves $\mathfrak{a}$ and sends every root space $\mathfrak{g}_\beta$ to the corresponding $\mathfrak{g}_{\rho_\alpha(\beta)}$; for instance, r_α could be conjugacy by k_α.

Let $\alpha \in \Lambda^+$. If a root β is a linear combination with integer coefficients of roots in Λ, then so is $\rho_\alpha(\beta)$; thus, ρ_α preserves $\Lambda^+ \cup -\Lambda^+$. As ρ_α sends all positive roots except multiples of α to positive roots, it also preserves $\Phi^+ \backslash \Lambda^+ = (\Lambda^+)^c$. We conclude that for every $\alpha \in \Lambda^+$, an automorphism r_α preserves the connected subgroups A and N_Λ^+ and the identity component P_Λ^0; in particular, it sends sequences (p_k) in H_Λ to $r_\alpha(p_k)$ in H_Λ. Note that, in general, P_Λ may not be invariant by r_α.

2.5.4 Definition of admissible operations, perturbations

Definition 2.11 Let $X = G/P$ be a parabolic variety with $P = P_\Lambda$. For (p_k) a sequence of P, an *elementary admissible operation* on (p_k) is of one of the three following types:

(1) A vertical perturbation of (p_k).

(2) A transverse perturbation of (p_k).

(3) For (p_k) in H_Λ, a Weyl reflection r_α applied to (p_k), with $\alpha \in \Lambda^+$.

An *admissible perturbation* of a sequence (p_k) in P is a sequence (q_k) which is obtained from (p_k) by finitely many elementary admissible operations.

Note that the result of an admissible perturbation of a sequence (p_k) of P is always in P. Weyl reflections are only allowed on sequences of H_Λ, which must be kept in mind when applying successive admissible operations.

We conclude this section with an important remark about Weyl reflections. We observed at the end of the last paragraph that a Weyl reflection r_α always coincides with the conjugacy by some element $k_\alpha \in G$. We also observed that r_α preserves the identity component P^0 of P, so that actually k_α belongs to $\mathrm{Nor}_G(P^0)$, the normalizer of P^0 in G. This normalizer $\mathrm{Nor}_G(P^0)$ has Lie algebra $\mathfrak{p}$ (see [19, Lemma 3.1.3 and Corollary 3.2.1(4)]), so that the inclusion $P \leq \mathrm{Nor}_G(P^0)$ holds. Observe that, in general, these groups need not coincide. However, when $P = \mathrm{Nor}_G(P^0)$, any Weyl reflection $r_\alpha(p_k)$ is actually *a vertical perturbation of* (p_k). We thus get a straightforward rephrasing of Lemma 2.9, namely:

Lemma 2.12 *Let $(M, \mathcal{C})$ be a parabolic geometry modeled on $X = G/P$, where $P = \mathrm{Nor}_G(P^0)$. Let $x \in M$, and let (f_k) be a sequence of automorphic immersions which is equicontinuous at x. Then, if (p_k) is in $\mathcal{H}ol(x)$, any admissible perturbation of (p_k) is in $\mathcal{H}ol(x)$.*

The case of equality, $P = \mathrm{Nor}_G(P^0)$, will thus be technically more convenient, since it means that Weyl reflections on holonomy sequences again yield holonomy sequences. It is explained in Section 3.2 why this equality may be assumed.

3 Translation of the main theorem to the model space

Via the holonomy sequences associated to an equicontinuous sequence (f_k) of automorphic immersions, we can translate Theorem 2.2 to an assertion about sequences of H_Λ acting equicontinuously with respect to segments on X.

Theorem 3.1 *Let $X = G/P$ be a parabolic variety with $P = P_\Lambda$. Given a sequence $(a_k n_k)$ of H_Λ which, together with all of its admissible perturbations, acts equicontinuously with respect to segments on X, the factor (n_k) is bounded.*

Theorem 3.1 is proved in Sections 4, 5 and 6.

3.1 Derivation of Theorem 2.2 from Theorem 3.1

Given a sequence (f_k) of automorphic immersions as in the statement of Theorem 2.2, let (p_k) be a holonomy sequence of (f_k) at x. We can assume by Lemmas 2.9 and 2.10 that $p_k \in H_\Lambda$ for all k.

We will first deduce Theorem 2.2 *under the extra assumption that P equals* $\mathrm{Nor}_G(P^0)$. Section 3.2 explains how to dispense with this assumption.

Proposition 2.5 ensures that (p_k) acts equicontinuously with respect to segments on X. Lemma 2.12 says that in fact every admissible perturbation of (p_k) does (under our assumption $P = \mathrm{Nor}_G(P^0)$). Now the hypotheses of Theorem 3.1 are satisfied. The conclusion implies that (a_k) is a right vertical perturbation of (p_k), which by Lemma 2.9 also belongs to $\mathcal{H}ol(x)$. The action of $\mathrm{Ad}(a_k)$ on $\mathfrak{g}$ preserves the subalgebra $\mathfrak{n}^-$; denote by L_k the endomorphism $\mathrm{Ad}(a_k)|_{\mathfrak{n}^-}$.

Lemma 3.2 *The sequence (L_k) is bounded in* $\mathrm{End}(\mathfrak{n}^-)$.

Proof The representation of $\mathrm{Ad}(a_k)$ on $\mathfrak{n}^-$ is diagonalizable with eigenvalues

$$(\lambda_1(k), \dots, \lambda_s(k)).$$

Assume for a contradiction that L_k is unbounded; we may assume that $\lambda_1(k)$ is unbounded and, after passing to a subsequence, that $|\lambda_1(k)| \to \infty$. Taking a subsequence also allows us to assume that, in $\widehat{M}$, the sequence $\hat{y}_k = f_k(\hat{x}_k).p_k^{-1}$ converges to $\hat{y}$.

For each k, let η_k be in the $\lambda_1(k)$–eigenspace of L_k such that $\eta_k \to \eta_\infty \neq 0$; these can moreover be chosen in the injectivity domain of $\exp_{\hat{y}_k}$, and such that η_∞ is in the injectivity domain of $\exp_{\hat{y}}$. Set $\xi_k := \eta_k/\lambda_1(k)$. Because $\xi_k \to 0$, the exponential $\exp(\hat{x}_k, \xi_k)$ is defined for sufficiently large k, and satisfies

$$f_k(\exp(\hat{x}_k, \xi_k)).a_k^{-1} = \exp(\hat{y}_k, \eta_k).$$

Projecting to M gives a contradiction to the equicontinuity of (f_k) at x, namely $\pi(\exp(\hat{x}_k, \xi_k)) \to x$, while $\pi(\exp(\hat{y}_k, \eta_k)) \to \pi(\exp(\hat{y}, \eta_\infty)) \neq \pi(\hat{y})$. □

Now, again passing to a subsequence of (f_k), we may assume that L_k tends to some $L \in \mathrm{End}(\mathfrak{n}^-)$. Let $K \subset \widehat{M}$ be a compact set containing both sequences $(\hat{x}_k)$ and $(\hat{y}_k)$, and let $\mathcal{U}$ and $\mathcal{V}$ be relatively compact neighborhoods of 0 in $\mathfrak{n}^-$, such that:

(1) $L_k(\overline{\mathcal{U}}) \subset \mathcal{V}$ for every $k \in \mathbb{N}$.

(2) For every $\hat{z} \in K$, the map $\Phi_{\hat{z}} \colon u \mapsto \pi(\exp(\hat{z}, u))$ is defined on $\overline{\mathcal{U}}$ and $\overline{\mathcal{V}}$, and is a diffeomorphism from $\mathcal{U}$ and $\mathcal{V}$ onto their respective images.

There exists an open neighborhood U of x such that $U \subseteq \Phi_{\hat{z}}(\mathcal{U})$ for $\hat{z} \in K$ close enough to $\hat{x}$. Then define the smooth map $h \colon U \to M$ by $h = \Phi_{\hat{y}} \circ L \circ \Phi_{\hat{x}}^{-1}$. Because L_k converges smoothly to L, and since on U, for k large enough,

$$f_k = \Phi_{\hat{y}_k} \circ L_k \circ \Phi_{\hat{x}_k}^{-1},$$

(f_k) converges smoothly to h on U. Thus, Theorem 2.2 is proved.

3.2 Justification of the assumption $P = \mathrm{Nor}_G(P^0)$

Let (f_k) be a sequence of automorphic immersions as in Theorem 2.2. In general, $P \leq \mathrm{Nor}_G(P^0)$, and they have the same Lie algebra, as remarked above (again, see [19, Lemma 3.1.3 and Corollary 3.2.1(4)]). Thus, $P' = \mathrm{Nor}_G(P^0)$ is an isogenous supergroup of P. The following lemma gives a general procedure for inducing a Cartan geometry modeled on G/P to one modeled on G/P', with respect to which the automorphism group behaves nicely.

Lemma 3.3 *Let $\mathcal{C} = (\widehat{M}, \omega)$ be a Cartan geometry on the manifold M, modeled on $X = G/P$. Let $P' < G$ be a closed subgroup, with $P \leq P'$ and $(P')^0 = P^0$. Then there exists a Cartan geometry $\mathcal{C}' = (\widehat{M}', \omega')$ on the manifold M, modeled on $X' = G/P'$, such that:*

(1) *Every automorphic immersion of $(M, \mathcal{C})$ is an automorphic immersion of $(M, \mathcal{C}')$.*

(2) *The corresponding inclusion of* $\mathrm{Aut}(M, \mathcal{C})$ *into* $\mathrm{Aut}(M, \mathcal{C}')$ *is a homeomorphism onto a closed subgroup with respect to the Lie topologies on each.*

Proof The bundle $\widehat{M}'$ is obtained as the quotient $\widehat{M} \times_P P'$, where P acts diagonally by $p.(\hat{x}, q) = (\hat{x}.p^{-1}, pq)$, freely and properly. There is an obvious commuting right P'–action on $\mathcal{M} = \widehat{M} \times P'$, which descends to $\widehat{M}'$, making it a P'–principal bundle over M.

To construct the Cartan connection on $\widehat{M}'$, we first build a one-form $\widetilde{\omega} \in \Omega^1(\mathcal{M}, \mathfrak{g})$. For $(\xi, u) \in T_{(\hat{x},q)}\mathcal{M}$, let

$$\widetilde{\omega}_{(\hat{x},q)}(\xi, u) := \mathrm{Ad}(q^{-1})\omega_{\hat{x}}(\xi) + (\omega_{P'})_q(u),$$

where $\omega_{P'}$ is the Maurer–Cartan form of P'. It is readily checked that $\widetilde{\omega}$ satisfies the equivariance relation $(R_p)^*\widetilde{\omega} = \mathrm{Ad}(p^{-1}) \circ \widetilde{\omega}$ for every $p \in P'$, and that it is invariant under the diagonal action of P on $\mathcal{M}$. Moreover,

$$\widetilde{\omega}_{(\hat{x},q)}(T_{\hat{x}}\widehat{M} \times \{0\}) = \mathrm{Ad}(q^{-1}) \circ \omega_{\hat{x}}(T_{\hat{x}}\widehat{M}) = \mathfrak{g},$$

showing that $\widetilde{\omega}\colon T\mathcal{M} \to \mathfrak{g}$ is onto at each point.

For $X \in \mathfrak{p}$, let $X^{\ddagger} \in \mathcal{X}(\widehat{M})$ be as in Definition 1.3, and let γ be the curve

$$\gamma(t) = e^{tX}.(\hat{x}, q) = (\hat{x}.e^{-tX}, e^{tX}q).$$

Then

$$\widetilde{\omega}(\gamma'(t)) = \mathrm{Ad}(q^{-1}) \circ \omega_{\hat{x}}(-X^{\ddagger}) + \mathrm{Ad}(q^{-1})X = 0$$

since $\omega(X^{\ddagger}) \equiv X$. Hence, the kernel of $\widetilde{\omega}_{(\hat{x},q)}$ contains the tangent space to the P–orbits on $\mathcal{M}$; by a dimension argument, these spaces are equal. We infer that $\widetilde{\omega}$ induces a 1–form $\omega' \in \Omega^1(\widehat{M}', \mathfrak{g})$, which is the desired Cartan connection on $\widehat{M}'$.

We prove point (1) for $f \in \mathrm{Aut}(M, \mathcal{C})$. The argument for automorphic immersions is similar. Let $\hat{f}$ be the lift of f to $\widehat{M}$, and define $\tilde{f}\colon \mathcal{M} \to \mathcal{M}$ by $\tilde{f}(\hat{x}, q) = (\hat{f}(\hat{x}), q)$. The P–equivariance of $\hat{f}$ gives the equivariance relation $p.\tilde{f}(\hat{x}, q) = \tilde{f}(p.(\hat{x}, q))$; obviously, $\tilde{f}((\hat{x}, q).p') = \tilde{f}(\hat{x}, q).p'$ for every $p' \in P'$. Thus, $\tilde{f}$ induces a bundle morphism $\hat{f}'$ of $\widehat{M}'$ covering f.

To prove that $f \in \mathrm{Aut}(M, \mathcal{C}')$, it remains to check that $\hat{f}'$ preserves ω'. To this end, we compute $\tilde{f}^*\widetilde{\omega}$ and show that it coincides with $\widetilde{\omega}$:

$$\widetilde{\omega}_{(\hat{f}(\hat{x}),q)}(D_{\hat{x}}\hat{f}(\xi), u) = \mathrm{Ad}(q^{-1}) \circ \omega_{\hat{f}(\hat{x})}(D_{\hat{x}}\hat{f}(\xi)) + (\omega_{P'})_q(u)$$

but $\omega_{\hat{f}(\hat{x})}(D_{\hat{x}}\hat{f}(\xi)) = \omega_{\hat{x}}(\xi)$ because $f \in \mathrm{Aut}(M, \mathcal{C})$. Finally,

$$\widetilde{\omega}_{(\hat{f}(\hat{x}),q)}(D_{\hat{x}}\hat{f}(\xi), u) = \mathrm{Ad}(q^{-1})\omega_{\hat{x}}(\xi) + (\omega_{P'})_q(u) = \widetilde{\omega}_{(\hat{x},q)}(\xi, u),$$

as desired, so (1) is proved.

There is a natural P–equivariant, proper embedding $j\colon (\widehat{M}, \omega) \to (\widehat{M}', \omega')$ defined by $j(\hat{x}) := [(\hat{x}, e)]$, the P–orbit in $\mathcal{M}$ of $(\hat{x}, e)$. For $f \in \mathrm{Aut}(M, \mathcal{C})$ with respective lifts $\hat{f}$ and $\hat{f}'$ to $\widehat{M}$ and $\widehat{M}'$, we have $j \circ \hat{f} = \hat{f}' \circ j$.

Now consider a sequence $f_k \in \mathrm{Aut}(M, \mathcal{C})$ converging for the Lie topology of $\mathrm{Aut}(M, \mathcal{C}')$ to an automorphism f. By Kobayashi's theorem (Theorem 1.6), the sequence of lifts $\hat{f}_k'$ converges in the C^∞–topology of $\widehat{M}'$ to a diffeomorphism $\hat{f}'$, which clearly preserves ω'. Properness of j implies that $j(\widehat{M})$ is closed. Then $\hat{f}'$ preserves $j(\widehat{M})$, because every $\hat{f}_k$ does. Thus, $\hat{f}_k = j^{-1} \circ \hat{f}_k' \circ j$ converges smoothly on $\widehat{M}$ to $\hat{f} := j^{-1} \circ \hat{f}' \circ j$, which preserves ω and covers f. It follows that $f \in \mathrm{Aut}(M, \mathcal{C})$, and, by Kobayashi's theorem, $f_k \to f$ in the Lie topology of $\mathrm{Aut}(M, \mathcal{C})$. We conclude moreover that $\mathrm{Aut}(M, \mathcal{C})$ is closed in the Lie topology of $\mathrm{Aut}(M, \mathcal{C}')$.

Conversely, given $f_k \to f$ in the Lie topology of $\mathrm{Aut}(M, \mathcal{C})$, with $f \in \mathrm{Aut}(M, \mathcal{C})$, the lifts $\hat{f}_k \to \hat{f}$ smoothly on $\widehat{M}$. These correspond, as in the proof of (1), to automorphisms $\hat{f}_k'$ and $\hat{f}'$ of $(\widehat{M}', \omega')$ with $\hat{f}_k' \to \hat{f}'$ on $j(\widehat{M})$. For any $\hat{y} \in \widehat{M}'$, there exists $p' \in P'$ such that $\hat{y}.p' \in j(\widehat{M})$. It follows by Theorem 1.6(3) that $\hat{f}_k' \to \hat{f}'$ smoothly on each connected component of $\widehat{M}'$; in other words, $f_k \to f$ holds in the Lie topology of $\mathrm{Aut}(M, \mathcal{C}')$. Thus, $\mathrm{Aut}(M, \mathcal{C}) \hookrightarrow \mathrm{Aut}(M, \mathcal{C}')$ is a homeomorphism onto its image with respect to the Lie topologies on each group. □

Now, given a sequence (f_k) as in Theorem 2.2, Lemma 3.3 with $P' = \mathrm{Nor}(P^0)$ allows us to consider (f_k) as a sequence of automorphic immersions of $(M, \mathcal{C}')$, modeled on $X' = G/P'$. The proof of Section 3.1 says that (f_k) converges smoothly on M to a smooth map h. We have thus shown that Theorem 3.1 implies Theorem 2.2.

3.3 Derivation of Theorem 1.2

Let $f_k \in \mathrm{Aut}(M, \mathcal{C})$ converge to $h \in \mathrm{Homeo}(M)$ in the C^0–topology. The aim is to show that $h \in \mathrm{Aut}(M, \mathcal{C})$, and $f_k \to h$ in the Lie topology on $\mathrm{Aut}(M, \mathcal{C})$.

By Lemma 3.3(2), we may assume that the model space G/P satisfies $P = \mathrm{Nor}_G(P^0)$. As in Section 3.1, (f_k) admits a holonomy sequence $a_k \in A$ at any $x \in M$, such that $L_k = \mathrm{Ad}(a_k)|_{\mathfrak{n}^-}$ is bounded in $\mathrm{End}(\mathfrak{n}^-)$. Moreover, in the notation of Section 3.1, there is a neighborhood U of x such that for any accumulation point L of (L_k) in $\mathrm{End}(\mathfrak{n}^-)$, a subsequence of (f_k) converges to $\Phi_{\hat{y}} \circ L \circ \Phi_{\hat{x}}^{-1}$ on U. Then $L|_U = \Phi_{\hat{y}}^{-1} \circ h \circ \Phi_{\hat{x}}$, so $L_k \to L$. Because h is a homeomorphism, L is injective around 0, hence $L \in \mathrm{GL}(\mathfrak{n}^-)$. As a consequence, (a_k) converges in P.

Now we have $\hat{f}_k(\hat{x}_k).a_k^{-1} = \hat{y}_k \to \hat{y}$ with (a_k) also converging, so $f_k(\hat{x}_k)$ tends to some point $\hat{z}$. As $\hat{x}_k \to \hat{x}$, for sufficiently large k, $\hat{x} = \exp(\hat{x}_k, \xi_k)$, with $\xi_k \to 0$ in $\mathfrak{g}$. Now $\hat{f}_k(\hat{x}) = \exp(\hat{f}_k(\hat{x}_k), \xi_k)$, so $f_k(\hat{x}) \to \hat{z}$. By Theorem 1.6(3), $\hat{f}_k$ and the inverses $\widehat{f}_k^{-1}$ both converge C^∞ on $\widehat{M}$ to smooth maps $\widehat{f}$ and $\widehat{g}$, which obviously satisfy $\widehat{f} \circ \widehat{g} = \mathrm{id}$. It is easy to see that $\widehat{f}$ is a bundle automorphism of $\widehat{M}$ preserving ω. It lifts h, hence $h \in \mathrm{Aut}(M, \mathcal{C})$. Finally, because $\widehat{f}_k \to \widehat{f}$ smoothly on $\widehat{M}$, Theorem 1.6(2) gives that $f_k \to h$ in the Lie topology on $\mathrm{Aut}(M, \mathcal{C})$.

4 Proof of Theorem 3.1 in rank one

Our proof of Theorem 3.1 will proceed by induction on $\mathrm{rk}_{\mathbb{R}}(G)$. The essential arguments for the base case, $\mathrm{rk}_{\mathbb{R}}(G) = 1$, are in the paper [2] by the first author. For the convenience of the reader, the proof is presented here in a manner consistent with our terminology and notation. Theorem 3.1 in this rank-one case will actually be a consequence of the following proposition.

Proposition 4.1 *Let $X = G/P$ be a parabolic space, with $\mathrm{rk}_{\mathbb{R}}(G) = 1$. If $p_k = a_k n_k$ is a sequence of $A \ltimes N^+$ such that (n_k) is unbounded, then (p_k) does not act equicontinuously with respect to segments.*

Recall the notation of Section 2.5.1. The rank-one Lie algebra can be decomposed as a vector space direct sum of subalgebras $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^+$. The Lie algebra $\mathfrak{n}^-$ (resp. $\mathfrak{n}^+$) is abelian if $\mathfrak{g} = \mathfrak{o}(1, n)$, and nilpotent of index 2, with center of respective dimension 1, 3 and 7 if $\mathfrak{g}$ is $\mathfrak{su}(1, n)$, $\mathfrak{sp}(1, n)$ or $\mathfrak{f}_4^{-20}$. In all cases, $\mathfrak{z}^-$ (resp. $\mathfrak{z}^+$) will denote the center of $\mathfrak{n}^-$ (resp. $\mathfrak{n}^+$). The nonequicontinuity will be observed on a restricted class of segments, namely those $[\xi]$ with

$$\xi \in \boldsymbol{Q} = \{\mathrm{Ad}(p)u \mid u \in \mathfrak{z}^-,\ p \in P\}.$$

This set of segments will be denoted by $[\boldsymbol{Q}]$ and corresponds to conformal circles when $\mathfrak{g} = \mathfrak{o}(1, n)$, and to chains and their generalizations in the other rank-one models. We will adopt the notation $\dot{\boldsymbol{Q}}$ (resp. $[\dot{\boldsymbol{Q}}]$) for $\boldsymbol{Q} \setminus \{0\}$ (resp. $[\boldsymbol{Q}] \setminus \{[o]\}$).

We now recall two results from [2] regarding these distinguished segments.

Lemma 4.2 [2, Lemme 2] *Let $([\alpha_k])$ be a sequence of segments in $[\boldsymbol{Q}]$. If $[\alpha_k]$ tends to $[o]$ for the Hausdorff topology, then $L([\alpha_k]) \to 0$.*

Lemma 4.3 [2, Proposition 1(ii)] *There exists a continuous section $s\colon [\dot{\boldsymbol{Q}}] \to \dot{\boldsymbol{Q}}$. In other words, if a sequence of segments $([\alpha_k])$ tends to a segment $[\beta] \neq [o]$, there is a convergent sequence (ξ_k) in $\mathfrak{g}$ such that $[\alpha_k] = [\xi_k]$.*

By these two lemmas, if we can find a sequence of segments $[\alpha_k]$ in $[\dot{\boldsymbol{Q}}]$, tending to $[o]$, such that $p_k.[\alpha_k]$ tends to $[\beta] \in [\dot{\boldsymbol{Q}}]$ (maybe considering a subsequence), then (p_k) does not act equicontinuously with respect to segments.

The group A has exactly two fixed points on X, namely o and another point ν. To better understand the action of P on $[\boldsymbol{Q}]$, it is convenient to work in the chart $\rho\colon \mathfrak{n}^+ \to X \setminus \{o\}$ given by $\rho(x) = e^x.\nu$. In this chart, elements of P act as affine transformations, and segments $[\alpha] \in [\dot{\boldsymbol{Q}}]$ coincide with half-lines $[x, u) = \{x + tu \mid t \in \mathbb{R}\}$, where $x \in \mathfrak{n}^+$ and u is a unit vector in $\mathfrak{z}^+$ (for any given norm in $\mathfrak{g}$ which is invariant by the Cartan involution). More precisely, the action of A in the chart ρ is linear, and is equivalent to the adjoint action on $\mathfrak{n}^+$, and the action of an element $n = e^\xi$ with $\xi \in \mathfrak{n}^+$ is given, by the Baker–Campbell–Hausdorff formula, by $x \mapsto \bigl(\mathrm{Id} + \frac{1}{2}\operatorname{ad}\xi\bigr)(x) + \xi$ for all $x \in \mathfrak{n}^+$.

Now, let us write $n_k = e^{v_k}$. By assumption, (v_k) is an unbounded sequence in $\mathfrak{n}^+$. We claim there is an unbounded sequence (x_k) in $\mathfrak{n}^+$ such that

$$x_k + \tfrac{1}{2}[v_k, x_k] + v_k = 0. \tag{5}$$

To see this, decompose $\mathfrak{n}^+$ as a direct sum $\mathfrak{n}^+ = \mathfrak{h} \oplus \mathfrak{z}^+$ (observe that $\mathfrak{h} = \{0\}$ when $\mathfrak{g} = \mathfrak{o}(1, n)$). Split equation (5) into two equations in $\mathfrak{h}$ and $\mathfrak{z}$, namely

$$\overline{x}_k + \overline{v}_k = 0,$$

where $\overline{x}_k$ and $\overline{v}_k$ are the components of x_k and v_k on $\mathfrak{h}$, respectively, and

$$\widetilde{x}_k + \tfrac{1}{2}[\overline{v}_k, \overline{x}_k] + \widetilde{v}_k = 0,$$

where $\widetilde{x}_k$ and $\widetilde{v}_k$ are the components of x_k and v_k on $\mathfrak{z}^+$. If $(\overline{v}_k)$ is unbounded, then so is $(\overline{x}_k)$, and the same is true for (x_k). If $(\overline{v}_k)$ is bounded, then $(\widetilde{v}_k)$ is unbounded because (v_k) is unbounded. This forces $(\widetilde{x}_k)$ to be unbounded.

We can now conclude the proof of Proposition 4.1. Since $a_k n_k(x_k) = 0$, then for ξ of norm 1 in $\mathfrak{z}^+$, the sequence of segments $[x_k, \xi)$ is mapped to $[0, \xi)$ by (p_k). Now, after taking a subsequence, $x_k/|x_k|$ tends to ξ_∞. Thus, for $\xi \neq -\xi_\infty$, the sequence of half-lines $[x_k, \xi)$ goes to infinity in the chart ρ, which means that the corresponding sequence of segments $[\alpha_k]$ tends to $[o]$ in X. On the other hand, $p_k([\alpha_k])$ is equal to a constant segment $[\alpha] \neq [o]$, and the nonequicontinuity of (p_k) with respect to segments follows.

5 Tools for the induction step: sliding along root spaces

The proof in the previous section for $\mathrm{rk}_{\mathbb{R}}(G) = 1$ relies heavily on the fact that the action of P on the complement of its fixed point $o \in G/P$ is by affine transformations. In higher rank, the P–action on G/P is a compactification of an affine action, but no longer a one-point compactification. This difference creates significantly more complexity, which motivates our choice to prove Theorem 3.1 by induction rather than directly in arbitrary rank.

The tools developed in this section build on those of Sections 2.4 and 2.5, with the purpose of simplifying holonomy sequences.

5.1 Essential range of (p_k)

The group exponential of G restricts to a diffeomorphism of $\mathfrak{a}$ onto A by definition. Moreover, $\mathrm{Ad}\, N^+_\Lambda$ is unipotent, and $Z(G) \cap N^+_\Lambda = 1$, so N^+_Λ is simply connected; thus, exp restricts to a diffeomorphism $\mathfrak{n}^+_\Lambda \to N^+_\Lambda$.

Fix an ordering $\alpha_1 > \cdots > \alpha_r$ of Φ, and endow Φ^+ with the lexicographical ordering. Then we obtain exponential coordinates $\ln a = (Z^1, \ldots, Z^r)$ on A and $\ln n = Y = (Y^\alpha)_{\alpha \in (\Lambda^+)^c}$, where Y^α is a vector in $\mathfrak{g}_\alpha$, on N^+_Λ.

Proposition 5.1 *Let $p_k = a_k n_k \in H_\Lambda$ with exponential coordinates $((Z^i_k), (Y^\alpha_k))$. Then, up to vertical perturbation of (p_k), we may assume each component sequence (Y^α_k) is either trivial or unbounded.*

Proof The group N^+_Λ is nilpotent; write the lower central series

$$N^+_\Lambda = N^{(0)} \rhd N^{(1)} \rhd \cdots \rhd N^{(d)} \rhd \mathrm{id}.$$

Each $\mathfrak{n}^{(i)}/\mathfrak{n}^{(i+1)}$ is abelian and can be spanned by a direct sum of certain root spaces; denote the corresponding set of roots by $\Sigma^{(i)}$. Let $\Pi \subset (\Lambda^+)^c$ be the set of roots α with (Y^α_k) bounded. We first multiply p_k on the right by $e^{-Y^\alpha_k}$ for all $\alpha \in \Pi \cap \Sigma^{(0)}$, in any order. The Baker–Campbell–Hausdorff formula implies that the resulting exponential coordinates $((Y')^\alpha_k)$ are trivial or bounded for all $\alpha \in \Pi \cap \Sigma^{(0)}$. Then proceed sequentially through $\Pi \cap \Sigma^{(i)}$ for $i = 1, \ldots, d$ to obtain (p'_k) satisfying the conclusion of the proposition. □

We remark that (Z^i_k) can also be assumed trivial or bounded by a similar argument, which is not given because this fact is not needed below.

Definition 5.2 Let $p_k = a_k n_k \in H_\Lambda$ with exponential coordinates $((Z_k^i), (Y_k^\alpha))$. The *essential range* of (p_k), denoted by $\mathrm{ER}(p_k)$, is the set of roots $\lambda \in (\Lambda^+)^c$ for which the component sequence (Y_k^λ) is unbounded.

5.2 Transverse and vertical sliding along root spaces

In our proof by induction on the rank of G, the goal will be, given a sequence (p_k) in H_Λ, to obtain roots in the essential range of (p_k) that belong to a lower-rank subspace of the span of Φ. More precisely, given $\lambda \in \mathrm{ER}(p_k)$ such that λ has nontrivial component on some $\alpha \in (\Lambda^+)^c$, we will perform admissible perturbations on (p_k) to obtain a new sequence $(q_k) \subset H_\Lambda$ with $\lambda - \alpha \in \mathrm{ER}(q_k)$. Such a manipulation is possible only under some circumstances, which are enunciated in Propositions 5.5 and 5.6 below. First, the following proposition holds the basic Lie-algebraic calculations that make our "sliding along $\mathfrak{g}_{-\alpha}$" procedure work:

Proposition 5.3 *Assume that $\alpha, \nu, \nu+\alpha \in \Phi^+$. Given a sequence (Y_k) in $\mathfrak{n}^+$ with $(Y_k^{\nu+\alpha})$ unbounded, there exists $\xi_k \to 0$ in $\mathfrak{g}_{-\alpha}$ such that*

(1) $[\xi_k, Y_k^{\nu+\alpha}] = [\xi_k, Y_k]^\nu$ *is unbounded;*

(2) $(\mathrm{Ad}(e^{\xi_k})Y_k)^\nu$ *is unbounded.*

Proof The bilinear map $\mathfrak{g}_{-\alpha} \times \mathfrak{g}_{\nu+\alpha} \to \mathfrak{g}_\nu$ induced by the bracket is nondegenerate; we recall the proof of this fact for real semisimple Lie algebras. Denote by B the Killing form on $\mathfrak{g}$, Θ the Cartan involution as in Section 2.5.1 and $H_{\nu+\alpha} \in \mathfrak{a}$ the dual with respect to B of $\nu+\alpha$. Then, given $Y \in \mathfrak{g}_{\nu+\alpha}$ nonzero, $[\Theta(Y), Y] = B(\Theta(Y), Y) H_{\nu+\alpha}$. Rescaling Y if necessary, the vectors Y, $\Theta(Y)$ and $[\Theta(Y), Y] = H$ form an $\mathfrak{sl}_2$–triple. Consider $V = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{-\alpha+k(\nu+\alpha)}$, which is an $\mathfrak{sl}_2$–module. If $[\mathfrak{g}_{-\alpha}, Y]$ were zero, then $V' = \bigoplus_{k\le 0} \mathfrak{g}_{-\alpha+k(\nu+\alpha)}$ would be a submodule with highest weight $-\alpha(H)$, which implies $\alpha(H) < 0$. On the other hand, V/V' is also an $\mathfrak{sl}_2$–module with lowest weight $\nu(H) = -\alpha(H) + (\nu+\alpha)(H) > 0$, which is impossible.

Given $Y \in \mathfrak{g}_{\nu+\alpha}$ with $|Y| = 1$ (for any norm on $\mathfrak{g}$), let

$$m(Y) = \max_{X \in \mathfrak{g}_{-\alpha},\, |X|=1} |[X, Y]| > 0.$$

Then $\inf_{Y \in \mathfrak{g}_{\nu+\alpha},\, |Y|=1} m(Y) \ge c > 0$. In particular, there exist $\xi_k \in \mathfrak{g}_{-\alpha}$ with $|\xi_k| = 1$ such that

$$|[\xi_k, Y_k]^\nu| = |[\xi_k, Y_k^{\nu+\alpha}]| = m\left(\frac{Y_k^{\nu+\alpha}}{|Y_k^{\nu+\alpha}|}\right) |Y_k^{\nu+\alpha}| \ge c|Y_k^{\nu+\alpha}|$$

is unbounded. Observe that replacing ξ_k by $\xi_k/|Y_k^{\nu+\alpha}|^{1/2}$ gives the same conclusion with the extra property that $\xi_k \to 0$. Now (1) is proved.

The conjugates in (2) are given, for some $m \in \mathbb{N}$, by

$$\mathrm{Ad}(e^{\xi_k})Y_k = Y'_k = \sum_{j=0}^{m} \frac{1}{j!}(\mathrm{ad}\,\xi_k)^j(Y_k).$$

After replacing ξ_k with $s\xi_k$, the ν components are

$$Y_k'^{\nu} = \sum_{j=0}^{m} \frac{s^j}{j!}(\mathrm{ad}\,\xi_k)^j(Y_k^{\nu+j\alpha}).$$

From (1), the ν components of the terms corresponding to $j=1$ form an unbounded sequence. The following lemma shows that replacing ξ_k by $s\xi_k$, with a suitable $s \in (0,1]$, makes the components $(Y_k'^{\nu})$ unbounded too. □

Lemma 5.4 *Let $(u_0(k)), \dots, (u_m(k))$ be m sequences in a finite-dimensional vector space V. Assume that one of the sequences $(u_j(k))$ is unbounded. Then, for a suitable choice of $s \in (0,1]$, the sequence $u_0(k) + su_1(k) + s^2u_2(k) + \cdots + s^m u_m(k)$ is unbounded.*

Proof There exist $m+1$ values of s in $(0,1]$, say $s_0, \dots, s_m$, such that the vectors $v_i = (1, s_i, \dots, s_i^m)$ form a basis of $\mathbb{R}^{m+1}$. Let $|\cdot|$ be any norm on V. Then, on the vector space of linear maps $\mathcal{L}(\mathbb{R}^{m+1}, V)$, we have two norms,

$$\|f\|_1 = \sup_{|v|=1} |f(v)|$$

and

$$\|f\|_2 = \max_{i=0,\dots,m} |f(v_i)|.$$

If f_k denotes the linear map $(\lambda_0, \dots, \lambda_m) \mapsto \lambda_0 u_0(k) + \cdots + \lambda_m u_m(k)$, then the sequence $(\|f_k\|_1)_{k\in\mathbb{N}}$ is unbounded (which is the case under the hypothesis of the lemma) if and only if $(\|f_k\|_2)_{k\in\mathbb{N}}$ is unbounded. The lemma follows. □

Define $\Phi^+_{\max} \subset \Phi^+$ to be the subset comprising the positive roots in which all $\alpha_i \in \Phi$ occur with a positive coefficient. Observe that this set is nonempty only when G is simple.

Proposition 5.5 (transverse sliding) *Let $p_k = a_k n_k \in H_\Lambda$ with $\mathrm{ER}(p_k) \subseteq \Phi^+_{\max}$, and assume (p_k) and all its admissible perturbations act equicontinuously with respect to segments on G/P. Let $\alpha \in (\Lambda^+)^c$ be such that for all $\lambda \in \mathrm{ER}(p_k)$, for all $l \geq 0$, if $\lambda - l\alpha$ is a root, then it belongs to $(\Lambda^+)^c$. Suppose $\alpha + \nu \in \mathrm{ER}(p_k)$ for some $\nu \in \Phi^+$. Then vertical and transverse perturbation of (p_k) yields $q_k = a_k n'_k \in H_\Lambda$ such that $\nu \in \mathrm{ER}(q_k)$.*

Proof If (Y_k^ν) is unbounded, there is nothing to do. By Proposition 5.1, we may assume after a vertical perturbation that (Y_k^ν) is trivial for all k for all $\lambda \notin \mathrm{ER}(p_k)$, in particular for ν. Let $x_k = e^{\xi_k}$ for $\xi_k \to 0$ in $\mathfrak{g}_{-\alpha}$. Then, for some $m \in \mathbb{N}$,

$$\mathrm{Ad}(x_k^{-1}) Y_k = Y'_k = Y_k + \sum_{j=1}^{m} \frac{(-1)^j}{j!} (\mathrm{ad}\, \xi_k)^j (Y_k).$$

By our hypotheses, $Y'_k \in \mathfrak{n}^+_\Lambda$, hence $n'_k = e^{Y'_k} \in P$ and $a_k n'_k \in H_\Lambda$. By Proposition 5.3, we can choose $\xi_k \to 0$ in $\mathfrak{g}_{-\alpha}$ such that the sequence $(Y_k'^{\nu})$ is unbounded.

We have the relation

$$a_k n_k e^{\xi_k} = e^{\mathrm{Ad}(a_k)\xi_k} a_k n'_k.$$

We wish to show that $\mathrm{Ad}(a_k)\xi_k \to 0$. The action of $\mathrm{Ad}(a_k)$ on $\mathfrak{g}_{-\alpha}$ is scalar multiplication by $\lambda_k = e^{-\alpha(Z_k)}$, where $Z_k = \ln a_k$, so it is enough to show that $\lambda_k \leq C$, for some constant $C \in \mathbb{R}$. If this were not the case, then, up to taking a subsequence, there would be $\zeta_k \to 0$ in $\mathfrak{g}_{-\alpha}$ with $\mathrm{Ad}(a_k)\zeta_k \to \zeta_\infty \neq 0$. For the product

$$p_k e^{\zeta_k} = e^{\mathrm{Ad}(a_k)\zeta_k} a_k e^{-\zeta_k} n_k e^{\zeta_k},$$

we know from above that $a_k e^{-\zeta_k} n_k e^{\zeta_k} \in P$. Thus, $p_k.[\zeta_k] = [\mathrm{Ad}(a_k)\zeta_k] \to [\zeta_\infty]$, while $L([\zeta_k]) \to 0$, which contradicts the fact that (p_k) acts equicontinuously with respect to segments.

Now let $\eta_k = \mathrm{Ad}(a_k)\xi_k$, which tends to 0. It is easy to verify that

$$e^{-s\eta_k} p_k e^{s\xi_k} \in P \quad \text{for all } s \in \mathbb{R}.$$

Thus, $q_k = a_k n'_k$ is a transverse perturbation of (p_k) according to Definition 2.8, and, because $(Y_k'^{\nu})$ is unbounded, it has $\nu \in \mathrm{ER}(q_k)$, as desired. □

Proposition 5.6 (vertical sliding) *Let $\nu \in (\Lambda^+)^c$ and $\alpha \in \Lambda^+$. Let $p_k = a_k n_k \in H_\Lambda$ with $\alpha(Z_k) \geq M > -\infty$ ($\alpha(Z_k) \leq M < \infty$). If $\nu + \alpha \in \mathrm{ER}(p_k)$ (or $\nu - \alpha \in \mathrm{ER}(p_k)$), then left and right vertical perturbation of (p_k) yields $q_k = a_k n'_k \in H_\Lambda$ such that $\nu \in \mathrm{ER}(q_k)$.*

Proof We can assume after vertical perturbation that $Y_k^\nu \equiv 0$. We apply Proposition 5.3 to obtain $\xi_k \to 0$ in $\mathfrak{g}_{-\alpha}$ such that $(Y_k'^\nu)$ is unbounded, where

$$Y_k' = \mathrm{Ad}(x_k^{-1}) Y_k = Y_k + \sum_{j=1}^{m} \frac{(-1)^j}{j!} (\mathrm{ad}\, \xi_k)^j (Y_k)$$

for some $m \in \mathbb{N}$, with $x_k = e^{\xi_k}$. In this case, $Y_k \in \mathfrak{n}_\Lambda^+$ and $\alpha \in \Lambda^+$ together imply that $(\mathrm{ad}\, \xi_k)^j (Y_k) \in \mathfrak{n}_\Lambda^+$ for all $j \in \mathbb{N}$. Thus, $Y_k' \in \mathfrak{n}_\Lambda^+$.

Let $n_k' = e^{Y_k'}$. The lower bound on $\alpha(Z_k)$ implies $(\mathrm{Ad}\, a_k)(\xi_k) \to 0$, so

$$e^{-\mathrm{Ad}(a_k)\xi_k} a_k n_k e^{\xi_k} = a_k n_k'$$

is obtained by left and right vertical perturbation from (p_k).

The proof for $\alpha(Z_k) \le M < \infty$ and $Y_k^{\nu-\alpha}$ unbounded is similar. □

5.3 Algebraic proposition to reduce rank

Using the tools developed so far in this section, we will now state the algebraic proposition that drives our induction step. The next section contains the geometric interpretation of this result, and explains how to prove Theorem 3.1 by induction on $\mathrm{rk}_{\mathbb{R}}\, G$.

Proposition 5.7 *Let $(p_k) = (a_k n_k)$ be a sequence of H_Λ with (n_k) unbounded. Assume that (p_k), together with all its admissible perturbations, acts equicontinuously with respect to segments. Then an admissible perturbation of (p_k) yields (q_k) such that $\mathrm{ER}(q_k)$ contains a root in $(\Lambda^+)^c \backslash \Phi_{\max}^+$.*

The proof of this proposition is given in Sections 6.3 and 6.4 below.

6 Proof of Theorem 3.1 by induction on rank

The first half of this section gives the proof of Theorem 3.1 from Proposition 5.7. The second half gives the proof of Proposition 5.7.

6.1 Invariant parabolic subvarieties

Let $X = G/P$ with G semisimple of real rank r and P a parabolic subgroup with a Lie algebra $\mathfrak{p} = \mathfrak{p}_\Lambda$, $\Lambda \subsetneq \Phi$. Let $V \subset X$ be a parabolic subvariety through the basepoint o. (These will be defined precisely below.) If (p_k) acts equicontinuously with respect to segments on X and preserves V, then clearly it is equicontinuous with respect to segments on V. The strategy for our induction argument is to find (p_k)–invariant $V \subset X$ of rank less than r.

Recall the notation introduced in Section 2.5.1, and denote by B the Killing form on $\mathfrak{g}$. Given a subset $\Psi \subset \Phi$, let $\mathfrak{a}_0$ and $\mathfrak{m}_0$ be the ideals of $\mathfrak{a}$ and $\mathfrak{m}$, respectively, commuting with $\bigoplus_{\alpha \in \Psi^+} \mathfrak{g}_\alpha$. Let $\mathfrak{a}_\Psi = \mathfrak{a}_0^\perp$ and $\mathfrak{m}_\Psi = \mathfrak{m}_0^\perp$, where the orthogonal is taken with respect to the scalar product $\langle X, Y \rangle = -B(X, \Theta Y)$. We obtain a subalgebra of $\mathfrak{g}$,

$$\mathfrak{g}_\Psi = \sum_{\alpha \in \Psi^-} \mathfrak{g}_\alpha \oplus \mathfrak{a}_\Psi \oplus \mathfrak{m}_\Psi \oplus \sum_{\alpha \in \Psi^+} \mathfrak{g}_\alpha.$$

It is easy to check that $\mathfrak{g}_\Psi$ is Θ–invariant, hence reductive, and has trivial center. It follows that $\mathfrak{g}_\Psi$ is semisimple.

The corresponding connected subgroup $G_\Psi < G$ is closed. Indeed, $\mathrm{ad}(\mathfrak{g}_\Psi)$ is a semisimple subalgebra of $\mathrm{End}(\mathfrak{g})$, hence is an algebraic subalgebra (see [6, Theorem 3.2, page 112]). For G'_Ψ the corresponding Zariski closed subgroup of $\mathrm{GL}(\mathfrak{g})$, the group $\mathrm{Ad}^{-1}(G'_\Psi)$ is closed in G, and so is its identity component G_Ψ.

A minimal parabolic of G_Ψ is contained in $P_{\min}$. The stabilizer of o in G_Ψ contains $P_{\min} \cap G_\Psi$ and is algebraic, hence is a parabolic subgroup of G_Ψ, denoted by Q_Ψ. The orbit $G_\Psi.o$ is a *parabolic subvariety* $V_\Psi \cong G_\Psi / Q_\Psi$, nontrivial provided $\Psi \not\subset \Lambda$, and of rank less than r.

Proposition 6.1 *Let $p_k = a_k n_k \in H_\Lambda$ and let $((Z_k^i), (Y_k^\alpha))$ be the exponential coordinates of p_k. Then, for any $\Psi \subset \Phi$, the variety $V_\Psi \subset X$ is invariant by (p_k). If $Z_k^i = 0$ for all $\alpha_i \in \Psi$, then a_k acts trivially on V_Ψ; if $Y_k^\alpha = 0$ for all $\alpha \in \Psi^+ \cap (\Lambda^+)^c$, then n_k is trivial on V_Ψ.*

Proof Let $\xi \in \sum_{\alpha \in \Psi^+} \mathfrak{g}_{-\alpha}$ and $x = e^\xi$.

Given (Z_k) as in the hypotheses above, $\alpha(Z_k) \equiv 0$ for all $\alpha \in \Psi^+$. Thus, $\mathrm{ad}(\xi) Z_k = 0$ and $\mathrm{Ad}(x) Z_k = Z_k$ for all k. Thus, $a_k x.o = x a_k.o = x.o$, and a_k acts trivially on V_Ψ.

Now let $Y \in \mathfrak{n}_\Lambda^+$ with $Y^\alpha = 0$ for all $\alpha \in \Psi^+$. Write

$$\mathrm{Ad}(x) Y = Y' = Y + \sum_{k=1}^{m} \frac{(-1)^k}{k!} (\mathrm{ad}\, \xi)^k (Y).$$

Note that $Y'^{\lambda} = 0$ unless $\lambda = \mu + \nu$, with μ a sum with negative integral coefficients of elements of Ψ and ν in $(\Psi^+)^c$; in particular, $\mu + \nu$ has positive coefficient on some simple root of $\Phi \backslash \Psi$. In this case, λ is a positive root, so $Y' \in \mathfrak{n}^+$, and $e^{Y'} \in P$. Thus, $e^Y x.o = xe^{Y'}.o = x.o$, and e^Y is trivial on V_Ψ.

The above calculation with $Y \in \sum_{\alpha \in \Psi^+} \mathfrak{g}_\alpha$ shows that V_Ψ is invariant by e^Y; it is easy to see that A leaves V_Ψ invariant. For invariance under a general sequence $p_k = a_k n_k$ in H_Λ, we can use the following basic lemma, the proof of which we leave to the reader:

Lemma 6.2 *Let N be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let $\mathfrak{n}_0$ be an ideal of $\mathfrak{n}$, and let Y and Y_0 be elements of $\mathfrak{n}$ and $\mathfrak{n}_0$. Then there exists $Y_0' \in \mathfrak{n}_0$ such that*

$$e^{Y+Y_0} = e^Y e^{Y_0'}.$$

This lemma lets us write $n_k = e^{W_k} e^{U_k}$ with $W_k \in \sum_{\alpha \in \Psi^+} \mathfrak{g}_\alpha$ and $U_k \in \sum_{(\Psi^+)^c} \mathfrak{g}_\alpha$. We can then conclude because each factor a_k, e^{W_k} and e^{U_k} preserves V_Ψ. $\square$

The unipotent radical of Q_Ψ is $N^+_{\Psi,\Lambda} < N^+_\Lambda$ with Lie algebra

$$\mathfrak{n}^+_{\Psi,\Lambda} = \bigoplus_{\alpha \in \Psi^+ \backslash \Lambda^+} \mathfrak{g}_\alpha.$$

The analogue of H_Λ in G_Ψ is $H_{\Psi,\Lambda} = A_\Psi \ltimes N^+_{\Psi,\Lambda}$. Note that

$$N^+_\Lambda = N^+_{\Psi,\Lambda} \cdot (N^+_\Psi \cap N^+_\Lambda),$$

and that the second factor is normal in H_Λ. We will also need below the decomposition $A = A_\Psi \cdot A_{\Phi \backslash \Psi}$.

6.2 The induction step

Suppose that Theorem 3.1 holds for all parabolic models G/P of real rank at most $r-1$. We will prove using Proposition 5.7 that it holds for all models of real rank r. Let $X = G/P_\Lambda$ of rank r be given, and let (p_k) be a sequence of H_Λ which, together with all its admissible perturbations, acts equicontinuously with respect to segments. The aim is to show that (n_k) is bounded. If not, then Proposition 5.7 gives, after an admissible perturbation, (q_k) with $\mathrm{ER}(q_k)$ containing a root $\lambda \in (\Lambda^+)^c \backslash \Phi^+_{\max}$.

There is a proper subset Ψ of Φ such that $\lambda \in \Psi^+$. It cannot be that Ψ is contained in Λ, because $\lambda \in (\Lambda^+)^c$. Now $q_k \in H_\Lambda$ preserves V_Ψ by Proposition 6.1; denote the restriction by (q'_k), which is a sequence of Q_Ψ, and let $a'_k n'_k$ be the decomposition into components on A_Ψ and $N^+_{\Psi,\Lambda}$, respectively. Because $\lambda \in \mathrm{ER}(q_k)$, it follows that (n'_k) is unbounded.

As $\mathrm{rk}_{\mathbb{R}}\, G_\Psi \leq r-1$, the induction hypothesis yields a contradiction *provided that all admissible perturbations of (q'_k) in G_Ψ act equicontinuously with respect to segments on V_Ψ*. Admissible perturbation in G_Ψ means more precisely that vertical and transverse perturbations are as in Section 2.4 with $\mathfrak{g}_\Psi$ in place of $\mathfrak{g}$, and Q_Ψ in place of P, and Weyl reflections are done with respect to roots α in $(\Psi \cap \Lambda)^+$. The following lemma ensures that (q'_k) satisfies the hypotheses of Theorem 3.1 and allows us to apply our induction hypothesis:

Lemma 6.3 *Let $X = G/P_\Lambda$ be a parabolic variety, and (q_k) be a sequence of H_Λ. Assume that (q_k) preserves a parabolic subvariety V_Ψ on which it restricts to (q'_k). If every admissible perturbation of (q_k) acts equicontinuously with respect to segments in X, then every admissible perturbation of (q'_k) in G_Ψ acts equicontinuously with respect to segments in V_Ψ.*

Proof We will prove that any admissible perturbation of the sequence (q'_k) in G_Ψ can be obtained by an admissible perturbation of (q_k), restricted to V_Ψ. Assume that (p'_k) is obtained from (q'_k) by an admissible perturbation in G_Ψ. We seek an admissible perturbation (p_k) of (q_k), such that p_k preserves V_Ψ, and the restriction of p_k to V_Ψ is precisely p'_k. Existence of such (p_k) can be checked for each of the three kinds of admissible perturbations in G_Ψ:

(1) Vertical perturbation There are bounded sequences (l_k) and (m_k) in Q_Ψ such that $p'_k = l_k q'_k m_k$ on V_Ψ. Because $Q_\Psi < P$, the desired vertical perturbation of (q_k) in G is simply $(p_k) = (l_k q_k m_k)$.

(2) Transverse perturbation In this case, write $p'_k = e^{-\eta_k} q'_k e^{\xi_k}$, where (η_k) and (ξ_k) are two sequences of $\mathfrak{g}_\Psi \setminus \mathfrak{q}_\Psi$ tending to 0. As these are also sequences of $\mathfrak{g} \setminus \mathfrak{p}$, we can set $p_k = e^{-\eta_k} q_k e^{\xi_k}$; we will show that this is a transverse perturbation in G.

Let $x \in V_\Psi$. Observe that, because $\xi_k, \eta_k \in \mathfrak{g}_\Psi$,

$$e^{-s\eta_k} q_k e^{s\xi_k}.x = e^{-s\eta_k} q'_k e^{s\xi_k}.x \quad \text{for all } s \in \mathbb{R};$$

thus, $e^{-s\eta_k} q_k e^{s\xi_k}$ preserves V_Ψ and acts on it by $e^{-s\eta_k} q'_k e^{s\xi_k}$. Taking $x = o$ gives $e^{-s\eta_k} q_k e^{s\xi_k}.o = e^{-s\eta_k} q'_k e^{s\xi_k}.o = o$, because the latter is in Q_Ψ for all s. This proves $e^{-s\eta_k} q_k e^{s\xi_k} \in P$ for all $s \in \mathbb{R}$, and p_k is a transverse perturbation of q_k.

(3) Weyl reflection Let $r_\alpha \in \mathrm{Aut}(G_\Psi)$ realize the Weyl reflection ρ_α for $\alpha \in (\Psi \cap \Lambda)^+$. Decompose, using Lemma 6.2,

$$q_k = a_k n_k = a_k'' a_k' n_k' n_k'',$$

where $a_k' \in A_\Psi$, $n_k' \in N^+_{\Psi,\Lambda}$, $a_k'' \in A_{\Phi\setminus\Psi}$ and $n_k'' \in (N^+_\Psi \cap N^+_\Lambda)$. By Proposition 6.1, both a_k'' and n_k'' are in the kernel of the restriction to V_Ψ, so we can write $q_k' = a_k' n_k'$.

Now let $\tilde{r}_\alpha$ be an automorphism of G effecting ρ_α on $\mathfrak{a}^*$. Because $\alpha \in (\Psi \cap \Lambda)^+$, the derivative of $\tilde{r}_\alpha$ preserves the Lie algebras $\mathfrak{a}_\Psi$, $\mathfrak{a}_{\Phi\setminus\Psi}$, $\mathfrak{n}^+_{\Psi,\Lambda}$ and $(\mathfrak{n}^+_\Psi \cap \mathfrak{n}^+_\Lambda)$, so $\tilde{r}_\alpha$ preserves the corresponding connected subgroups in G. Thus, $\tilde{r}_\alpha(q_k') = r_\alpha(q_k')$, and

$$\tilde{r}_\alpha(q_k) = \tilde{r}_\alpha(a_k'') r_\alpha(q_k') \tilde{r}_\alpha(n_k'')$$

preserves V_Ψ and restricts on it to $r_\alpha(q_k')$, as desired. □

The proof by induction of Theorem 3.1 is now complete, once we prove Proposition 5.7.

6.3 Proof of Proposition 5.7 (assuming the root system of $\mathfrak{g}$ is not of type G_2)

Proposition 5.7 is vacuously true if the set $\Phi^+_{\max}$ is empty. Thus, we assume from now on that G *is a simple Lie group*.

Let $(p_k) = (a_k n_k)$ be a sequence of H_Λ with (n_k) unbounded. That means $\mathrm{ER}(p_k) \subseteq (\Lambda^+)^c$ is nonempty. If it contains a root not in $\Phi^+_{\max}$, then there is nothing to show, so we suppose that $\mathrm{ER}(p_k) \subseteq \Phi^+_{\max}$. Define the *degree* of $\alpha \in \Phi^+$ to be the sum of the coefficients in the unique expression of α as a positive integral linear combination of roots in Φ.

Let $Y_k = \ln n_k$. By Proposition 5.1, we may assume $Y_k^\lambda \equiv 0$ for $\lambda \notin \mathrm{ER}(p_k)$. To prove that an admissible perturbation of (p_k) results in (q_k) with $\mathrm{ER}(q_k)$ not contained in $\Phi^+_{\max}$, we will show that for any $\lambda \in \mathrm{ER}(p_k)$ of minimal degree, there is a sequence of admissible operations resulting in $\lambda' \in \mathrm{ER}(q_k)$ with the degree of λ' strictly lower than the degree of λ.

Let $\lambda \in \mathrm{ER}(p_k) \subseteq \Phi^+_{\max}$ be of minimal degree. There is some $\alpha \in \Phi$ with $\langle \alpha, \lambda \rangle > 0$; otherwise, λ would be in the negative of the Weyl chamber spanned by Φ, contradicting that it is a positive root. For such α,

$$A_{\alpha\lambda} = \frac{2\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} > 0.$$

Case $\alpha \in \Lambda$ In this case, the Weyl reflection $\rho_\alpha \in W_\Lambda$ yields

$$\rho_\alpha(\lambda) = \lambda' = \lambda - A_{\alpha\lambda}\alpha \in (\Lambda^+)^c$$

of smaller degree. The admissible operation r_α yields $q_k \in H_\Lambda$ with $\lambda' \in \mathrm{ER}(q_k)$.

Case $\alpha \in \Phi \backslash \Lambda$ Note that $\nu = \lambda - \alpha \in \Phi^+$, because $\lambda - A_{\alpha\lambda}\alpha \in \Phi^+$, and strings are unbroken.

If $P = P_\Lambda$ is not a maximal parabolic with $\Lambda = \Phi\backslash\{\alpha\}$, then (p_k), α and ν satisfy the hypotheses of Proposition 5.5, which thus gives another holonomy sequence (q_k) with $\nu = \lambda - \alpha \in \mathrm{ER}(q_k)$, which has lower degree than λ.

Now suppose P is a maximal parabolic, with $\Lambda = \Phi\backslash\{\alpha\}$. Every root in $\mathrm{ER}(p_k)$ has the form $\lambda_i = m_i\alpha + \mu_i$, where $m_i \geq 1$, and μ_i is in the positive integral span of Λ. If none of the μ_i is a root, then again the hypotheses of Proposition 5.5 are satisfied, so, as above, there is a holonomy sequence (q_k) with $\lambda - \alpha \in \mathrm{ER}(q_k)$.

Thus, we may assume that μ_i is a root for some i.

Lemma 6.4 *Let $P_\Lambda < G$ be a maximal parabolic with $\Lambda = \Phi\backslash\{\alpha\}$. If $m\alpha + \mu \in \Phi^+_{\max}$ for $m \geq 1$ and $\mu \in \Lambda^+$, then α is a valence-one vertex of the Dynkin graph of $\mathfrak{g}$ — that is, $A_{\alpha\beta} \neq 0$ for exactly one element $\beta \in \Lambda$.*

Proof The root μ belongs to some basis of simple roots, and the Weyl group W acts transitively on such sets (see [7, Theorem 2.6.3]), which means there is $\rho \in W$ sending some $\alpha_i \in \Phi$ to μ. This ρ is moreover a product $\rho_{i_\ell} \cdots \rho_{i_1}$ of Weyl reflections. Let $\mu_0 = \alpha_i$ and μ_j be the result after performing j reflections. Then one can see that at each step, μ_j is a positive root, comprised of simple roots that form a connected subset of the Dynkin graph. If ρ_{i_j} is the reflection at the j^{th} step, then α_{i_j} is connected to exactly one of the simple roots appearing in μ_{j-1} because the Dynkin diagram is a tree, and it adds a positive multiple of α_{i_j} to make μ_j.

We conclude that the elements of Φ appearing in the decomposition of μ correspond to a connected subset of the Dynkin graph. These are precisely the elements of $\Lambda = \Phi\backslash\{\alpha\}$. As the Dynkin graph is a connected tree, the conclusion follows. □

Let $\beta \in \Lambda$ with $A_{\alpha\beta} \neq 0$. Write $\lambda_i = \lambda' = m'\alpha + \mu'$, where $\mu' \in \Lambda^+_{\max}$, and let $c' \in \mathbb{Z}^+$ be the coefficient of β in μ'. The product

$$A_{\alpha\mu'}A_{\mu'\alpha} = \frac{(c')^2 A_{\alpha\beta}A_{\beta\alpha}\langle\beta,\beta\rangle}{\langle\mu',\mu'\rangle} \in \{1,2,3\}.$$

(Although our root system is not necessarily reduced, the value 4 could only occur for $\mu' = 2\alpha$ or $\alpha = 2\mu'$, neither of which is the case.) First suppose the Dynkin diagram has no double or triple edges, so the root system of $\mathfrak{g}$ is A_r, D_r, E_6, E_7 or E_8. Then all roots of Φ^+ have the same length and $A_{\alpha\beta} A_{\beta\alpha} = 1$. In this case, $A_{\alpha\mu'} A_{\mu'\alpha} = (c')^2$, so $c' = 1$ and $A_{\alpha\mu'} = -1 = A_{\mu'\alpha}$. The α–string of μ' comprises μ' and $\mu' + \alpha$. Hence $m' = 1$ and $\lambda' = \mu' + \alpha$. The μ'–string of α comprises α and λ'. Now $\rho_{\mu'}(\lambda') = \alpha$, so the Weyl reflection $r_{\mu'}(p_k)$ is an admissible perturbation resulting in (q_k) with $\alpha \in \mathrm{ER}(q_k)$.

Under the assumption that $\mathfrak{g}$ is not of type G_2, there are no triple bonds in the Dynkin diagram of $\mathfrak{g}$, so it remains to consider the root systems with double bonds: B_r, BC_r, C_r and F_4. Let λ with $A_{\alpha\lambda} > 0$ as above be of minimal degree in $\mathrm{ER}(p_k)$. Write $\lambda = m\alpha + \mu$, where μ — not necessarily a root — is a positive integral combination of elements of Λ, and let $c \in \mathbb{Z}^+$ be the coefficient of β in μ. Because $\lambda - A_{\alpha\lambda}\alpha$ is a positive root,

$$0 < A_{\alpha\lambda} = 2m + cA_{\alpha\beta} \leq m. \tag{6}$$

Write $\Phi = \{\gamma_1, \dots, \gamma_r\}$, numbered from left to right in the Dynkin diagram, where we follow the ordering of [7]. We have $\alpha = \gamma_1$ or γ_r.

Type B_r or BC_r For B_r, the set $\Phi^+_{\max}$ comprises, for $i = 2, \dots, r$,

$$\lambda_1 = \gamma_1 + \cdots + \gamma_r, \quad \lambda_i = \lambda_1 + \gamma_i + \cdots + \gamma_r.$$

If α is the short root, γ_r, then $A_{\alpha\beta} = -2$. The possibility $m = 1$ is incompatible with (6). If $m = 2$, then the same inequality implies $c = 1$, so $\lambda = \lambda_r$. If $r > 2$, then $\rho_{\gamma_1}(\lambda)$ has lower degree, so a Weyl reflection r_{γ_1} is an admissible perturbation with the desired effect. Otherwise, $r = 2$ and $\lambda = \beta + 2\alpha$. In this case, as λ is an element of $\mathrm{ER}(p_k)$ of minimal degree, $\mathrm{ER}(p_k) = \{\lambda\}$. There is a rank-one subvariety $V_\lambda \subset X$ left invariant by (p_k) and on which it restricts to $(a'_k n_k)$ with (n_k) unbounded. Proposition 4.1 leads to a contradiction.

If α is the long root γ_1, then $m = 1$ and $A_{\alpha\beta} = -1$, so (6) implies $c = 1$. Then $\lambda = \lambda_1$ or $r > 2$. In the first case, $\mu = \gamma_2 + \cdots + \gamma_r \in \Lambda^+$ is a short root with $A_{\mu\alpha} = -2$. Proposition 5.6 permits vertical sliding along $-\mu$, resulting in (q_k) with $\alpha \in \mathrm{ER}(q_k)$, or along μ, resulting in (q_k) with $\alpha + 2\mu \in \mathrm{ER}(q_k)$. In the latter case, the Weyl reflection $\rho_\mu(\alpha + 2\mu) = \alpha$, so r_μ leads to the desired conclusion. Otherwise, $\lambda = \lambda_i$ for $2 < i \leq r$; in this case, Weyl reflection in $\gamma_i \in \Lambda$ results in (q_k) with a minimal element of $\mathrm{ER}(q_k)$ of lower degree.

In BC_r, the set $\Phi^+_{\max}$ comprises $\{\lambda_i : 1 \leq i \leq r\}$ from above, together with $2\lambda_1$. If $\lambda = 2\lambda_1$, then $\mathrm{ER}(p_k) = \{\lambda\}$; in this case, restricting to the rank-one subvariety V_λ yields a contradiction to Proposition 4.1.

Type C_r The set $\Phi^+_{\max}$ comprises, for $i = 1, \dots, r-1$,

$$\lambda_r = \gamma_1 + \cdots + \gamma_r, \quad \lambda_i = \lambda_r + \gamma_i + \cdots + \gamma_{r-1}.$$

If α equals the long root, γ_r, then $A_{\alpha\beta} = -1$ and $m = 1$. The inequality (6) gives $c = 1$ and $\lambda = \lambda_r$. If $r > 2$, then $A_{\gamma_1\lambda} = 1$, and $\rho_{\gamma_1}(\lambda)$ is a root of lower degree. The remaining possibility is $r = 2$ with $\mathrm{ER}(p_k) = \{\alpha+\beta, \alpha+2\beta\}$ or simply $\{\alpha+\beta\}$. In the first case, the Weyl reflection r_β results in (q_k) with $\alpha \in \mathrm{ER}(q_k)$. In the second case, we again apply Proposition 4.1.

When α equals the short root γ_1, we first consider $\lambda = \lambda_i$ for $i \neq 1$. The Weyl reflection $\rho_{\gamma_i}(\lambda)$ has lower degree. If $\lambda = \lambda_1$, then $\mathrm{ER}(p_k) = \{\lambda\}$, so Proposition 4.1 completes the proof.

Type F_4 The roots in $\Phi^+_{\max}$, in terms of the basis $\{\gamma_i\}$, are [7, Appendix C]

$$(1,1,1,1), \quad (1,1,2,1), \quad (1,1,2,2), \quad (1,2,2,1), \quad (1,2,2,2),$$
$$(1,2,3,1), \quad (1,2,3,2), \quad (1,2,4,2), \quad (1,3,4,2), \quad (2,3,4,2).$$

Recall that $\mathrm{ER}(p_k)$ contains $\lambda' = m'\alpha + \mu'$ with μ' a root in $\Lambda^+_{\max}$. The roots of $\Lambda^+_{\max}$ correspond to those of C_3 when α equals the long root γ_1 and B_3 when α equals the short root γ_4. In the first case the possibilities are

$$\lambda' \in \Lambda' = \{(1,1,1,1), (1,1,2,1), (1,1,2,2)\}.$$

The maximum degree in Λ' is 6. As all other roots of $\Phi^+_{\max}$ have degree at least 6, we may assume $\lambda \in \mathrm{ER}(p_k)$ of minimal degree belongs to Λ'. If

$$\mathrm{ER}(p_k) = \{\gamma_1 + \gamma_2 + 2\gamma_3 + 2\gamma_4\},$$

then we can invoke Proposition 4.1. Otherwise, a Weyl reflection in γ_4 or γ_3 reduces the degree of λ and yields a new holonomy sequence (q_k) with an element of lower degree in $\mathrm{ER}(q_k)$.

In the second case, Λ' contains the roots listed above, together with

$$(1,2,2,1), \quad (1,2,2,2).$$

Now the maximal degree in Λ' is 7, and all other roots of $\Phi^+_{\max}$ have degree at least 7, so we may again assume $\lambda \in \Lambda'$. A Weyl reflection in γ_1 or γ_2 will reduce the degree of any $\lambda \in \Lambda'$, giving the desired conclusion in this case.

6.4 Proof of Proposition 5.7 for G_2

Assume $\mathfrak{g}$ is of type G_2, and write $\Phi = \{\alpha, \beta\}$ with $|\alpha| \leq |\beta|$. Then

$$\Phi^+_{\max} = \{\alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}.$$

Assume first that $\Lambda = \{\alpha\}$, so $A_{\alpha\beta} = -3$. Given $\lambda \in \mathrm{ER}(p_k)$ of minimal degree, the goal is to find an admissible perturbation (q_k) with $\beta \in \mathrm{ER}(q_k)$. As in the previous section (but with the roles of α and β switched), we can assume that $A_{\beta\lambda} > 0$. The two possibilities for λ are thus $3\alpha+2\beta$ or $\alpha+\beta$. In the first case, λ is the only element of $\mathrm{ER}(p_k)$, so we can conclude using Proposition 4.1 as in the cases of C_2 and B_2. In the second case, we apply Proposition 5.6. We can assume, after passing to a subsequence, that $\alpha(Z_k)$ is bounded either below or above. If it is bounded below, then a vertical sliding on (p_k) yields (q_k) with $\beta \in \mathrm{ER}(q_k)$, as desired. If $\alpha(Z_k)$ is bounded above, then vertical slidings give $3\alpha+\beta$ in $\mathrm{ER}(q_k)$. Then the Weyl reflection r_α on (q_k) gives (s_k) with $\beta \in \mathrm{ER}(s_k)$.

Now consider $\Lambda = \{\beta\}$, so $A_{\beta\alpha} = -1$. The condition $A_{\alpha\lambda} > 0$ leaves the possibilities $2\alpha+\beta$ or $3\alpha+\beta$ for λ. Unfortunately, the tools used above don't help in either of these cases. The solution is to slide along $-\alpha$, although it does not satisfy the hypotheses of Proposition 5.5.

Let $S \cong Z(S)S_0$ be the reductive complement in a Levi decomposition of P_β, where S_0 is simple of rank one. The group S admits a KAK decomposition, where $A = \exp(\mathfrak{a})$ as defined above, and K is a maximal compact subgroup of S_0. Write N^+_β for the unipotent radical of P_β. The decomposition of the corresponding Lie algebra $\mathfrak{n}^+_\beta$ into irreducible subspaces under $\mathrm{Ad}(S)$ is $E_1 \oplus E_2 \oplus E_3$, where $E_1 = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+\beta}$, $E_2 = \mathfrak{g}_{2\alpha+\beta}$ and $E_3 = \mathfrak{g}_{3\alpha+\beta} \oplus \mathfrak{g}_{3\alpha+2\beta}$. This decomposition can be seen from the fact that $\mathfrak{s}$ is contained in the sum of root spaces $\mathfrak{g}_{-\beta} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\beta$.

Recall that $p_k = a_k n_k$ with $Y^\nu_k \equiv 0$ if $\nu \notin \mathrm{ER}(p_k)$. Let $\xi_k \to 0$ in $\mathfrak{g}_{-\alpha}$ and $x_k = e^{\xi_k}$, and set

$$q_k = e^{-\mathrm{Ad}(a_k)\xi_k} p_k e^{\xi_k} = a_k x_k^{-1} n_k x_k.$$

Just as in the proof of Proposition 5.5, $\mathrm{Ad}(a_k)\xi_k \to 0$ and (q_k) is a transverse perturbation of (p_k); it is in particular a sequence in P, although it may not be in H_β. More precisely, $x_k^{-1} n_k x_k \in N^+$, which can be deduced from the formula

$$\mathrm{Ad}(x_k^{-1}) Y_k = Y_k + \sum_{j=1}^{m} \frac{(-1)^j}{j!} (\mathrm{ad}\, \xi_k)^j (Y_k)$$

with $Y_k = \ln n_k$. Using Lemma 6.2, write $q_k = a_k u_k n_k''$ with $a_k u_k \in S$ and $n_k'' \in N_\beta^+$. Proposition 5.3 gives that $\lambda - \alpha \in \mathrm{ER}(n_k'')$. Performing this transverse sliding twice if necessary, depending on λ, we arrive at $\alpha + \beta \in \mathrm{ER}(n_k'')$.

Next, let $l_k' a_k' l_k$ be the KAK decomposition of $a_k u_k$ in S. Finally, set

$$q_k' = a_k' n_k', \quad \text{where } n_k' = l_k^{-1} n_k'' l_k.$$

Note that $a_k' \in A$ and $n_k' \in N_\beta^+$, so $q_k' \in H_\beta$. Clearly (q_k') is a vertical perturbation of (q_k), so it is an admissible perturbation of (p_k). The conjugation by l_k on N_β^+ preserves the subspace $E_1 = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+\beta}$, so $\mathrm{ER}(q_k')$ contains α or $\alpha + \beta$. If it only contains $\alpha + \beta$, then we perform a Weyl reflection r_β to finally obtain an admissible perturbation (q_k'') of (p_k) with $\alpha \in \mathrm{ER}(q_k'')$.

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