Operads of genus zero curves and the Grothendieck–Teichmüller group

PEDRO BOAVIDA DE BRITO
GEOFFROY HOREL
MARCY ROBERTSON

We show that the group of homotopy automorphisms of the profinite completion of the genus zero surface operad is isomorphic to the (profinite) Grothendieck–Teichmüller group. Using a result of Drummond-Cole, we deduce that the Grothendieck–Teichmüller group acts nontrivially on $\overline{\mathcal{M}}_{0,\bullet+1}$, the operad of *stable* curves of genus zero. As a second application, we give an alternative proof that the framed little 2–disks operad is formal.

1 Introduction

The moduli space of genus g curves with n marked points, $\mathcal{M}_{g,n}$, is defined over $\mathbb{Q}$ and as such its geometric fundamental group

$$\pi_1^{\mathrm{geom}}(\mathcal{M}_{g,n}) := \pi_1^{\mathrm{et}}(\overline{\mathbb{Q}} \times_{\mathbb{Q}} \mathcal{M}_{g,n})$$

has an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In [16], Grothendieck proposed to study $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via its action on the geometric fundamental groups of *all* the stacks $\mathcal{M}_{g,n}$ and the natural maps relating these for various g and n. This collection he called the "Teichmüller tower".

A motivation for this idea was a theorem of Belyi's (see [3]), which implies that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the geometric fundamental group $\pi_1^{\mathrm{geom}}(\mathcal{M}_{0,4}) \cong \widehat{F}_2$ is faithful. It follows that the absolute Galois group acts faithfully on the whole Teichmüller tower and it is an open question whether there are other automorphisms. An appealing aspect of this program is that the Teichmüller tower is a purely topological object since the geometric fundamental group of $\mathcal{M}_{g,n}$ is also the profinite completion of the mapping class group $\Gamma_{g,n}$ of a genus g surface with n marked points. In this way, Grothendieck's proposal creates a remarkable and unexpected bridge between number theory and low-dimensional topology.

At the genus zero level, Ihara showed that the image of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the geometric fundamental group of $\mathcal{M}_{0,4}$ lies in an explicitly defined profinite group $\widehat{\mathrm{GT}}$, constructed by Drinfeld and called the Grothendieck–Teichmüller group. It is a longstanding problem to determine whether this injection $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathrm{GT}}$ is an isomorphism. Given this, and granting the hypothetical relation between the absolute Galois group and the Teichmüller tower, one may also wonder whether there is a relation between $\widehat{\mathrm{GT}}$ and this tower.

In this paper, we show that $\widehat{\mathrm{GT}}$ is the group of homotopy automorphisms of the genus zero Teichmüller tower. We do so for an *operadic* definition of the *genus zero Teichmüller tower*, which encodes the natural relations between curves, as observed by Grothendieck. To define the genus zero Teichmüller tower, we replace *marked points* by *boundary components*. More precisely, we replace the group $\Gamma_{0,n}$ by Γ_0^n, the mapping class group of a genus zero Riemann surface with n boundary components. This is not a drastic change since there is a short exact sequence

$$1 \to \mathbb{Z}^n \to \Gamma_0^n \to \Gamma_{0,n} \to 1$$

obtained by collapsing boundary components to points. The advantage, however, is that the collection of spaces $\{B\Gamma_0^{n+1}\}_{n \geq 0}$ supports a much richer algebraic structure because of this small change. Indeed, two Riemann surfaces with boundary can be glued together along one of their boundary components. This gives us composition maps

$$B\Gamma_0^{n+1} \times B\Gamma_0^{m+1} \to B\Gamma_0^{n+m}.$$

If we replace the group Γ_0^{n+1} by a certain homotopy-equivalent groupoid $\mathcal{S}(n)$, we obtain the structure of an operad on the collection of spaces $\{B\mathcal{S}(n)\}$. This means that the composition maps satisfy associativity, Σ–equivariance and unital conditions. This object is denoted by $\mathcal{M}$ and called the genus zero surface operad (we refer the reader to Definition 6.5 for a more precise definition). It is a suboperad of the operad constructed by Tillmann [29, Definition 2.3] and it is equivalent to the classical framed little 2–disks operad (Proposition 6.8). Our first main theorem (Theorem 8.4) can then be stated as follows:

Theorem *The group $\widehat{\mathrm{GT}}$ is isomorphic to the group of homotopy automorphisms of the profinite completion of the genus zero surface operad.*

Besides the gluing along boundary components, there are also natural maps $\Gamma_0^{n+1} \to \Gamma_0^n$ for $n \geq 0$, corresponding to filling in boundary components (ie extending diffeomorphisms to the missing disks by the identity). We prove in Section 11 a variant

of the above theorem, in which we incorporate these operations to the genus zero surface operad. This does not affect the conclusion and the group of automorphisms remains $\widehat{GT}$.

The theorem implies that there is a faithful action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the profinite completion of the genus zero surface operad. If we look at this result aritywise, this is not particularly surprising as the group $\widehat{\Gamma}_0^n$ has an obvious action of $\widehat{GT}$ that is faithful when n is at least 3. The difficulty in this theorem is to show that this action of $\widehat{GT}$ is compatible with the operad structure and accounts for all the operad automorphisms. A corollary of this result is a new proof of the formality of the operad $\mathcal{M}$, or, equivalently, the operad of framed little disks (see Section 9).

The compatibility of the $\widehat{GT}$ action with the operad structure is somewhat related to a result of Hatcher, Lochak and Schneps [18] that a certain subgroup of $\widehat{GT}$ acts on the collection of the profinite completion of the pure mapping class groups $\Gamma_{g,n}^m$ of a genus g Riemann surface with n punctures and m boundary components. They show that this action is compatible with certain geometric operations relating these groups: it preserves conjugacy classes of Dehn twists along closed embedded curves and it is compatible with the operation of extending a diffeomorphism on a subsurface by the identity.

When dealing with profinite completions, an important observation is that the profinite completion of an operad in spaces is no longer an operad, but rather an operad "up to homotopy". This technical detail requires the use of ∞–operads and is the topic of Sections 4 and 5. Another key step in the proof is to replace the genus zero surface operad by a homotopy-equivalent one: the parenthesized ribbon braid operad. This operad has a very combinatorial description that makes computing automorphisms more practical.

Returning to $\mathcal{M}_{0,n}$, the gluing of curves along marked points creates a nodal singularity and so the collection $\{\mathcal{M}_{0,n}\}$ does not form an operad. This can be fixed if we allow curves with singularities, ie if we replace the schemes $\mathcal{M}_{0,n}$ by their compactifications $\overline{\mathcal{M}}_{0,n}$, the moduli spaces of *stable* genus zero curves with n marked points. Indeed, the collection of spaces $\{\overline{\mathcal{M}}_{0,n+1}\}$ has the structure of an operad; the composition maps

$$\overline{\mathcal{M}}_{0,n+1} \times \overline{\mathcal{M}}_{0,m+1} \to \overline{\mathcal{M}}_{0,n+m}$$

are obtained by gluing curves (possibly with nodal singularities) along marked points.

The genus zero surface operad $\mathcal{M}$ maps to the operad $\{\overline{\mathcal{M}}_{0,\bullet+1}\}$. By a theorem of Drummond-Cole, the latter can be seen as an operadic quotient of the former by

homotopically killing the circle in arity 1. Our second main theorem (Theorem 10.3) is the following:

Theorem *The action of $\widehat{\mathrm{GT}}$ on the profinite completion of the genus zero surface operad extends to an action of $\widehat{\mathrm{GT}}$ on the profinite completion of $\overline{\mathcal{M}}_{0,\bullet+1}$. Moreover, this action is nontrivial.*

We point out that the complex analytic spaces underlying the schemes $\overline{\mathcal{M}}_{0,n}$ are simply connected, and so the geometric fundamental groups of these schemes are trivial. This deviates from most of the literature, where $\widehat{\mathrm{GT}}$ actions are constructed on schemes whose associated complex analytic spaces are $K(\pi, 1)$'s.

Finally, there is also a standard action of the absolute Galois group of $\mathbb{Q}$ on the profinite completion of the operad $\overline{\mathcal{M}}_{0,\bullet+1}$, coming from the fact that this operad can be obtained as the geometric étale homotopy type of an operad in $\mathbb{Q}$–schemes. It seems plausible that the $\widehat{\mathrm{GT}}$ action that we construct coincides with the action of the absolute Galois group of $\mathbb{Q}$ restricted along the injection $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathrm{GT}}$.

Acknowledgements

2 Homotopical recollections

This background section serves as a brief overview of the homotopical constructions that we use in the paper. Throughout we use the language of Quillen model categories, and take [19] as our standard reference. We also make a mild but essential use of the vantage point of ∞–categories (via relative categories, recalled below). Throughout we use the term *space* to mean simplicial set.

2.1 Relative categories and derived mapping spaces

A *relative category* is a pair (C, W) where C is a category and W is a wide subcategory of C whose arrows we will call *weak equivalences* in C. A *relative functor*

$$F \colon (C, W) \to (D, W')$$

is a functor $F\colon \boldsymbol{C} \to \boldsymbol{D}$ such that $F(\boldsymbol{W}) \subset \boldsymbol{W}'$. The *homotopy category* of $\boldsymbol{C}$, denoted by $\operatorname{Ho}\boldsymbol{C}$, is the category obtained from $\boldsymbol{C}$ by formally inverting the maps in $\boldsymbol{W}$.

The homotopy category of $\boldsymbol{C}$ does not capture all of the higher-order homotopical information contained in the relative category. As a homotopical enhancement for $\operatorname{Ho}\boldsymbol{C}$, Dwyer and Kan [10] constructed a simplicial category $L(\boldsymbol{C}, \boldsymbol{W})$, together with a natural embedding $\boldsymbol{C} \to L(\boldsymbol{C}, \boldsymbol{W})$, with the property that the category of components of $L(\boldsymbol{C}, \boldsymbol{W})$ (ie the category obtained by applying π_0 to the morphism spaces of $L(\boldsymbol{C}, \boldsymbol{W})$) agrees with $\operatorname{Ho}\boldsymbol{C}$.

The simplicial category $L(\boldsymbol{C}, \boldsymbol{W})$ has the same objects as $\boldsymbol{C}$ and for any two objects X and Y in $\boldsymbol{C}$ a space of maps $\mathbb{R}\operatorname{Map}_{\boldsymbol{C}}(X, Y)$. We will write $\mathbb{R}\operatorname{Map}(X, Y)$ if $\boldsymbol{C}$ is understood. One of the important features of $L(\boldsymbol{C}, \boldsymbol{W})$ is that its morphism spaces are homotopically meaningful; that is, for any Y in $\boldsymbol{C}$ and weak equivalence $X \to X'$, the induced maps

$$\mathbb{R}\operatorname{Map}(X', Y) \to \mathbb{R}\operatorname{Map}(X, Y)$$

and

$$\mathbb{R}\operatorname{Map}(Y, X) \to \mathbb{R}\operatorname{Map}(Y, X')$$

are weak equivalences of spaces. Calculating $\mathbb{R}\operatorname{Map}$ for an arbitrary relative category is often not a feasible task. The situation simplifies if the relative category comes equipped with extra structure. If $\boldsymbol{C}$ has the extra structure of a simplicial model category, then we have the following:

Theorem 2.1 [11, Corollary 4.7] *Let $\boldsymbol{C}$ be a simplicial model category, and let X and Y be objects in $\boldsymbol{C}$ such that X is cofibrant and Y is fibrant. Denote by $\operatorname{Map}_{\boldsymbol{C}}(X, Y)$ the space of maps in $\boldsymbol{C}$ coming from the simplicial structure. Then $\mathbb{R}\operatorname{Map}_{\boldsymbol{C}}(X, Y)$ is related to $\operatorname{Map}_{\boldsymbol{C}}(X, Y)$ by a natural zigzag of weak equivalences.*

2.2 Adjunctions

A simplicial Quillen pair (L, R) between simplicial model categories $\boldsymbol{C}$ and $\boldsymbol{D}$ gives rise to a homotopy adjunction

$$\mathbb{L}L\colon L(\boldsymbol{C}, \boldsymbol{W}) \rightleftarrows L(\boldsymbol{D}, \boldsymbol{W}') \colon \mathbb{R}R$$

of simplicially enriched categories in the sense that there is a weak equivalence of spaces

$$\mathbb{R}\operatorname{Map}(\mathbb{L}L(X), Y) \simeq \mathbb{R}\operatorname{Map}(X, \mathbb{R}R(Y)),$$

natural in $X \in C$ and $Y \in D$, where $\mathbb{L}$ and $\mathbb{R}$ denote, respectively, the left and right derived functor constructions. That is to say, given a cofibrant replacement $X_c \xrightarrow{\sim} X$ of X and a fibrant replacement $Y \xrightarrow{\sim} Y_f$, we obtain a weak equivalence

$$\mathrm{Map}_D(L(X_c), Y_f) \simeq \mathrm{Map}_C(X_c, R(Y_f))$$

of nonderived mapping spaces.

2.3 Spaces and groupoids

We write S for the category of simplicial sets with its usual Kan–Quillen simplicial model structure. We write G for the category of groupoids with a model structure in which weak equivalences are equivalences of categories, cofibrations are morphisms that are injective on objects, and fibrations are isofibrations. Isofibrations are those functors which have the right-lifting property against the map $[0] \to E$, where $[0]$ denotes the trivial category with a single object 0 and E denotes the groupoid with two objects 0 and 1 and exactly two nonidentity morphisms $0 \to 1$ and $1 \to 0$. In particular, every object in G is both fibrant and cofibrant. For details, see [1, Section 5].

The relationship between S and G is classical: the classifying space functor

$$B: G \to S$$

has a left adjoint π which assigns to a space X its fundamental groupoid πX. The model category structure on G is simplicial with mapping space given by

$$\mathrm{Map}(C, D) := \mathrm{Map}(BC, BD)$$

for every pair of groupoids C and D. The pair (π, B) then forms a simplicial Quillen pair. The classifying space functor preserves and reflects all weak equivalences and fibrations and is homotopically fully faithful in the sense that the natural map

$$\mathbb{R}\mathrm{Map}(C, D) \to \mathbb{R}\mathrm{Map}(BC, BD)$$

is a weak equivalence for every pair of groupoids C and D.

2.4 Operads

A *symmetric sequence* in spaces is a sequence of spaces $\{P(n)\}_{n \geq 0}$, in which each space $P(n)$ is equipped with an action of the symmetric group Σ_n. An *operad* in spaces P is a symmetric sequence $\{P(n)\}_{n \geq 0}$ together with composition maps

$$\circ_i: P(n) \times P(m) \to P(n + m - 1)$$

for $i \in \{1, \dots, m\}$ which are compatible with the symmetric group actions and subject to associativity and unit axioms. A map of operads is a map of symmetric sequences which preserves the operadic structure. Substitution of the word *space* for the word *groupoid* gives us the notion of an operad in groupoids.

We denote by $\mathbf{Op}(S)$ and $\mathbf{Op}(G)$ the category of operads in S and G, respectively. These categories are equipped with simplicial model category structures in which weak equivalences and fibrations are defined levelwise. More explicitly, a map of operads $f: \mathsf{P} \to \mathsf{Q}$ is a weak equivalence (respectively, fibration) if $f(n): \mathsf{P}(n) \to \mathsf{Q}(n)$ is a weak equivalence (respectively, fibration) for each nonnegative integer n. For more details, consult [4, Theorem 3.1].

Both the classifying space functor B and the fundamental groupoid functor π preserve products, and so induce an adjunction

$$\pi: \mathbf{Op}(S) \rightleftarrows \mathbf{Op}(G) :B$$

by levelwise application. It follows from [5, Theorem 4.7] that this is a simplicial Quillen adjunction. Moreover, B is homotopically fully faithful.

The functor which to a groupoid G associates its set of objects $\mathrm{Ob}(G)$ is product-preserving, and hence induces a functor Ob from the category of operads in groupoids to the category of operads in sets.

An operad P is fibrant if each object $\mathsf{P}(n)$ is fibrant. In practice, it is more difficult to tell if an operad is cofibrant. However, for operads in groupoids we have the following useful criterion:

Proposition 2.2 [20, Proposition 6.8] *The cofibrations in $\mathbf{Op}(G)$ are morphisms of operads in groupoids $f: \mathsf{P} \to \mathsf{Q}$ with the property that $\mathrm{Ob}(f)$ has the left lifting property with respect to operad maps which are levelwise surjective. In particular, any operad P in groupoids with the property that $\mathrm{Ob}(\mathsf{P})$ is free as an operad in $\mathbf{Set}$ is cofibrant in $\mathbf{Op}(G)$.*

3 Profinite completion

Given any small category C, the associated *category of pro-objects* in C (also called the pro-category of C), $\mathrm{Pro}(C)$, is obtained by freely adding all cofiltered limits to C. Formally, the opposite of $\mathrm{Pro}(C)$ is the full subcategory of the category of functors from C to $\mathbf{Set}$ spanned by those which are filtered colimits of representables. If C

has finite limits, then the opposite of $\mathrm{Pro}(C)$ is also equivalent to the category of finite limit-preserving functors from C to **Set**.

Alternatively, $\mathrm{Pro}(C)$ is the category whose objects are pairs (I, X) where I is a cofiltered category and $X = \{X_i\}_{i \in I}$ is a diagram $I \to C$. Morphisms are defined as

$$\mathrm{Hom}_{\mathrm{Pro}(C)}(\{X\}_i, \{Y\}_j) = \lim_{j \in J} \mathrm{colim}_{i \in I} \mathrm{Hom}_C(X_i, Y_j).$$

Clearly, C embeds fully faithfully in $\mathrm{Pro}(C)$.

Example 3.1 Let Fin be the category of finite sets. The category of profinite sets $\widehat{\mathbf{Set}} := \mathrm{Pro}(\mathrm{Fin})$ is the associated pro-category. The category of profinite sets is equivalent to the category of compact, totally disconnected Hausdorff spaces and continuous maps. There is an adjunction

$$\widehat{(-)} \colon \mathbf{Set} \rightleftarrows \widehat{\mathbf{Set}} \colon |-|,$$

where the right adjoint sends a diagram to its limit in **Set**. The left adjoint sends a set X to the diagram $R \mapsto X/R$, where R runs over all equivalence relations on X with finitely many equivalence classes. Another description of $\widehat{X}$ is as the finite limit-preserving functor $\mathrm{Fin} \to \mathbf{Set}$ which sends a finite set F to $\mathrm{Hom}(X, F)$.

Example 3.2 The category of profinite groups $\widehat{\mathbf{Grp}}$ is the category of pro-objects in the category of finite groups. This category is equivalent to the category of group objects in $\widehat{\mathbf{Set}}$ (see eg [22, page 237]). In other words, the category of profinite groups is the category of topological groups whose underlying topological space is a totally disconnected, compact Hausdorff space. There exists an adjunction

$$\widehat{(-)} \colon \mathbf{Grp} \rightleftarrows \widehat{\mathbf{Grp}} \colon |-|,$$

where the right adjoint sends a profinite group to the underlying discrete group. The left adjoint $\widehat{(-)}$ is called *profinite completion*. It sends a group G to the inverse limit of the diagram

$$N \to G/N,$$

where N runs over normal subgroups of G with finite index and G/N is given the discrete topology.

Definition 3.3 We say that a groupoid A is *finite* if it has finitely many morphisms (and so also finitely many objects). The category of all finite groupoids will be denoted

by $f\mathbf{G}$. The associated pro-category is called the category of *profinite groupoids*. It will be denoted by $\widehat{\mathbf{G}} := \mathrm{Pro}(f\mathbf{G})$.

Definition 3.4 Let A be a profinite groupoid. Let S be any finite set and G be any finite group; then we define:

- $H^0(A, S) := \mathrm{Hom}_{\widehat{\mathbf{G}}}(A, S)$.
- $Z^1(A, G) := \mathrm{Hom}_{\widehat{\mathbf{G}}}(A, *//G)$, where $*//G$ denotes the groupoid with a unique object whose group of automorphisms is the group G.
- $B^1(A, G) := \mathrm{Hom}_{\widehat{\mathbf{G}}}(A, G)$, where G denotes the group G seen as a discrete groupoid (ie with only identity morphisms).
- $H^1(A, G) := Z^1(A, G)/B^1(A, G)$, where the quotient is taken with respect to a certain right action of the group $B^1(A, G)$ on $Z^1(A, G)$ (see [20, Definition 4.1] for more details).

The following two results are proved in [20].

Theorem 3.5 *The category $\widehat{\mathbf{G}}$ admits a left proper, cocombinatorial model structure in which a map $A \to B$ is a weak equivalence in $\widehat{\mathbf{G}}$ if*

(1) *for all finite sets S, $H^0(B, S) \to H^0(A, S)$ is an isomorphism, and*

(2) *for all finite groups G, $H^1(B, G) \to H^1(A, G)$ is an isomorphism.*

The cofibrations are the maps which are monomorphisms on objects.

As with groups, there exists an adjunction

$$(1) \qquad\qquad (\widehat{-}) \colon \mathbf{G} \rightleftarrows \widehat{\mathbf{G}} \colon |{-}|$$

in which the right adjoint sends a profinite groupoid, seen as diagram of groupoids, to its limit in $\mathbf{G}$. The left adjoint $(\widehat{-}) \colon \mathbf{G} \to \widehat{\mathbf{G}}$ is called *profinite completion*.

Proposition 3.6 [20, Proposition 4.22] *The profinite completion functor*

$$(\widehat{-}) \colon \mathbf{G} \to \widehat{\mathbf{G}}$$

is a left Quillen functor.

Remark 3.7 As a left adjoint, the profinite completion functor should not be expected to preserve limits. However, it does preserve certain products. More precisely, suppose A and B are two groupoids with finitely many objects. Then $\widehat{A \times B}$ is isomorphic to $\widehat{A} \times \widehat{B}$. This fact appears in [20, Proposition 4.23].

Recall that S denotes the category of simplicial sets. We always consider this category as a simplicial model category with the Kan–Quillen model structure. We denote the underlying relative category by $\mathcal{S}$.

The definition of a profinite space and the homotopy theory of such is more involved than that of profinite groupoids. We begin by describing an ∞–categorical incarnation of the category of profinite spaces. A space is said to be π–*finite* if it has finitely many components and finitely many nontrivial homotopy groups, each of which is finite. The category $\mathcal{S}_{\pi\text{–fin}}$ of π–finite spaces forms a relative subcategory of $\mathcal{S}$. It has finite homotopy limits since a homotopy pullback of π–finite spaces is π–finite. We can thus form $\mathrm{Pro}(\mathcal{S}_{\pi\text{–fin}})$, which is an ∞–category with all limits. The pro-category of an ∞–category is defined similarly to the 1–categorical case but replacing **Set** with $\mathcal{S}$ (see eg around [23, 7.1.6.1]). We view $\mathrm{Pro}(\mathcal{S}_{\pi\text{–fin}})$ as a homotopical enhancement of Example 3.1 and call it the ∞–*category of profinite spaces*.

A presentation of this ∞–category as a model category is given by Quick [26]. Denote the category of simplicial objects in profinite sets $\mathrm{Fun}(\Delta^{\mathrm{op}}, \widehat{\mathbf{Set}})$ by $\widehat{S}$. Quick equips the category $\widehat{S}$ with a model structure in which the cofibrations are the monomorphisms and the weak equivalences are those maps that induce isomorphisms on π_0, π_1 and on twisted cohomology with finite coefficients. This model structure is cocombinatorial. It is proved in [2, Corollary 7.4.8] that its underlying ∞–category models the ∞–category of profinite spaces.

The adjunction $\mathbf{Set} \rightleftarrows \widehat{\mathbf{Set}}$ between sets and profinite sets induces an adjunction

$$(2) \qquad\qquad (\widehat{-})\colon S \rightleftarrows \widehat{S} :\!|-|,$$

which Quick shows to be a Quillen adjunction in [26, Proposition 2.28].

The following definition is due to Serre:

Definition 3.8 A discrete group G is said to be *good* if for any finite abelian group M equipped with a G action, the map $G \to \widehat{G}$ induces an isomorphism

$$H^i(\widehat{G}, M) \xrightarrow{\ \cong\ } H^i(G, M).$$

Proposition 3.9 *Let X and Y be two connected spaces whose homotopy groups are good. Then the map $\widehat{X \times Y} \to \widehat{X} \times \widehat{Y}$ is a weak equivalence of profinite spaces.*

Proof The profinite completion of a finite product of groups is the product of the profinite completions, and so the profinite completion of $\pi_i(X \times Y)$ is isomorphic to

$$\widehat{\pi_i X} \times \widehat{\pi_i Y}.$$

For a connected space Z whose homotopy groups are good, the profinite completion of $\pi_i Z$ is isomorphic to $\pi_i \hat{Z}$ by [27, Theorem 3.14]. The hypothesis holds for X and Y by assumption, and also for the product $X \times Y$ since a product of good groups is a good group. We conclude that the map $\widehat{X \times Y} \to \hat{X} \times \hat{Y}$ is an isomorphism on homotopy groups.

In order to finish this proof it suffices to prove that weak equivalences in $\hat{S}$ are detected on homotopy groups. This is claimed without proof in the last paragraph of the second section of [27], and thus we give a quick proof here. In [2, Theorem 7.4.7], it is shown that there is a Quillen equivalence

$$\Psi \colon L\mathrm{Pro}(S) \rightleftarrows \hat{S} \colon \Phi,$$

where $L\mathrm{Pro}(S)$ is a certain Bousfield localization of Isaksen's model category on $\mathrm{Pro}(S)$ (defined in [21]). Since weak equivalences in $\mathrm{Pro}(S)$ are detected on homotopy groups, it follows that the same is true in $\hat{S}$. $\qquad\square$

Remark 3.10 The proposition also holds if X and Y have finitely many path components. However, it does not hold when the set of components is infinite. As a counterexample, let us take X and Y to be the discrete space $\mathbb{N}$; we claim that the map

$$\widehat{\mathbb{N} \times \mathbb{N}} \to \hat{\mathbb{N}} \times \hat{\mathbb{N}}$$

is not an isomorphism. Indeed, let us consider the map $f \colon \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ which sends (x, y) to 1 if $x > y$ and 0 otherwise. Then this map extends to a map $\widehat{\mathbb{N} \times \mathbb{N}}$ by the universal property of the profinite completion. However, it is an easy exercise to check that, if P and Q are two equivalence relations on $\mathbb{N}$ that are such that the quotients $\mathbb{N}/P$ and $\mathbb{N}/Q$ are finite, then there cannot exist a map g that makes the triangle

$$
\begin{array}{ccc}
\mathbb{N} \times \mathbb{N} & \longrightarrow & (\mathbb{N}/P) \times (\mathbb{N}/Q) \\
 & {}_{f}\searrow & \downarrow {}^{g} \\
 & & \{0, 1\}
\end{array}
$$

commute. In fact, one can even show that if there exists such a g, then P and Q must both be the finest equivalence relation on $\mathbb{N}$ (ie the one for which the equivalence classes are singletons).

4 ∞–operads

The profinite completion of a product of spaces is not in general isomorphic to the product of the profinite completions. However, in favorable cases, as in Proposition 3.9, the comparison map is a weak equivalence. For this reason — and under the assumptions of Proposition 3.9 — the profinite completion of an operad does not yield an operad but rather an ∞–operad. We explain this in detail below, after giving a meaning to the term ∞–operad which suits our purpose.

We use the dendroidal category Ω from [24]. Objects of Ω are finite rooted trees. Each such tree $T \in \Omega$ generates a colored operad $\Omega(T)$ which has the set of edges of T for colors and operations are generated by the vertices of T. A morphism in Ω from S to T is defined as an operad map from $\Omega(S)$ to $\Omega(T)$. (For more details, see [24].)

Some objects in Ω are given special mention and notation: the tree with no vertices (ie consisting of a single edge, the root) is denoted by η; the tree with a single vertex and $n + 1$ edges is called the nth corolla and is denoted by C_n. For a tree T and vertex v with $|v|$ input edges, there is an inclusion $C_{|v|} \hookrightarrow T$ which selects the edges connected to v; this gives rise to a map $X_T \to X_{C_{|v|}}$ for X a contravariant functor on Ω. A *dendroidal object* in C is a contravariant functor X from Ω to C. The category of dendroidal objects is denoted by $\mathbf{d}C$.

Definition 4.1 Let C be a model category. An ∞–*operad* in C is a dendroidal object X such that the map $X_\eta \to *$ is a weak equivalence and, for every tree T, the Segal map

$$X_T \to \prod_{v \in T} (X_{C_{|v|}})_f$$

induced by the inclusion of corollas in T, is a weak equivalence in C. Here, the product runs over all vertices of T, $|v|$ denotes the set of inputs at v, and the subscript f denotes a fibrant replacement in C.

A couple of comments on the definition above: The first condition should be interpreted as the requirement that the dendroidal object X has a single color. The role of the fibrant replacements is to guarantee that the product is the homotopy product. In many cases, like in the category of spaces or groupoids, every finite product is a homotopy product hence such fibrant replacements are not needed.

Given a model category C, the category of dendroidal objects $\mathbf{d}C$ admits a model category structure in which a map $X \to Y$ is a weak equivalence if $X_T \to Y_T$ is a

weak equivalence in C for every tree T. We say such a weak equivalence is given levelwise.

Definition 4.2 The relative category of infinity operads $\mathbf{Op}_\infty(C)$ is the full relative subcategory of $\mathbf{d}C$ spanned by ∞–operads (Definition 4.1) and the levelwise weak equivalences between them.

By definition, given two ∞–operads X and Y, the derived mapping space in $\mathbf{Op}_\infty$ is simply the derived mapping space $\mathbb{R}\mathrm{Map}(X, Y)$ computed in $\mathbf{d}C$.

When C is the category of spaces or groupoids, an ∞–operad will be a fibrant object in the left Bousfield localization of $\mathbf{d}C$ at the Segal maps

$$\coprod_{v\in T} \Omega(-, C_{|v|}) \to \Omega(-, T)$$

for every tree T, and at the map $\varnothing \to \Omega(-, \eta)$. We denote this model structure by $L_S(\mathbf{d}C)$. (See also around Proposition 5.5 in [7] and [6, Proposition 4.3] for the monochromatic case.) If C is the category of profinite spaces, there are technical challenges to performing a similar left Bousfield localization. This stems from the fact that the model structure on profinite spaces is not cofibrantly generated but rather fibrantly generated. The definition of $\mathbf{Op}_\infty(C)$ above allows us to circumvent this issue, while still being robust enough for our purposes.

4.1 The nerve of an operad

For a monochromatic operad P in C, one can associate a dendroidal object $N\mathsf{P}$, called the *nerve* of P, by declaring

$$(N\mathsf{P})_T := \prod_{v\in T} \mathsf{P}(|v|)$$

for every tree T. Note that the value of $N\mathsf{P}$ at η is a point. This is an ∞–operad if P is levelwise fibrant or if in C finite products and finite homotopy products agree.

The following theorem is essentially due to Cisinski and Moerdijk, slightly adjusted to our context.

Theorem 4.3 (Cisinski and Moerdijk) *Let C denote the category of spaces or groupoids. The nerve functor N is a right Quillen equivalence from the model structure on monochromatic operads in C to $L_S(\mathbf{d}C)$, the localization of the projective model structure on dendroidal objects in C where the fibrant objects are the ∞–operads.*

The following technical step will be used in the proof of Theorem 4.3:

Lemma 4.4 *Let C denote the category of spaces or the category of groupoids, and let $Z := \Omega(-, C_n) \times K$ be an object of $\mathbf{d}C$ with K a space (or a groupoid) and C_n a corolla. Let $F(K)$ denote the free operad on the (nonsymmetric) sequence with K in degree n and empty otherwise. Then the map*

$$Z \to NF(K)$$

is a weak equivalence in $L_S(\mathbf{d}C)$.

Proof We prove this for C the category of spaces. Since the nerve functor $B \colon \mathbf{G} \to \mathbf{S}$ commutes with N and F, the case of groupoids follows.

For a dendroidal space Z, the *reduction* Z_* is the dendroidal space given by the pushout

(3)
$$\begin{array}{ccc}
\Omega(-, \eta) \times Z_\eta & \longrightarrow & Z \\
\downarrow & & \downarrow \\
\Omega(-, \eta) & \longrightarrow & Z_*
\end{array}$$

where the top horizontal map is the adjoint to the identity. The map in the statement of the lemma factors as

(4)
$$Z \to Z_* \to NF(K).$$

According to [6, Proposition 4.4], the right-hand map is a weak equivalence in a Segal-type model structure obtained from $\mathbf{d}C_*$, the category of dendroidal objects X for which X_η is the terminal object of C. The inclusion of that model structure into $L_S(\mathbf{d}C)$ preserves all weak equivalences, hence the right-hand map is also a weak equivalence in $L_S(\mathbf{d}C)$.

It remains to show that the left-hand map of (4) is a weak equivalence. The top horizontal map of the square (3) is a Reedy cofibration of dendroidal spaces, therefore the square is a homotopy pushout. Moreover, the left-hand map in that square is a weak equivalence in $L_S(\mathbf{d}C)$, so the result follows. $\square$

Proof of Theorem 4.3 We take C to be the category of spaces. The case of groupoids is similar. The nerve functor N has a left adjoint τ. The value of τ on a representable $\Omega(-, T)$ is the free operad on a sequence $\{X(n)\}$ with $X(n)$ the set of vertices of T with n inputs. This prescription uniquely defines τ. The pair (τ, N) is Quillen since N

preserves fibrations and weak equivalences. Moreover, N detects weak equivalences, ie N is homotopy conservative. Therefore, to show that the pair is a Quillen equivalence, it is enough to show that the derived unit map

$$X \to N\mathbb{L}\tau(X)$$

is a weak equivalence for every dendroidal object X.

Lemma 4.4 shows that the derived unit map is a weak equivalence for dendroidal spaces of the form $\Omega(-, C) \times K$ for C a corolla and K a space. Using the Segal condition, we can deduce that this is also the case for dendroidal spaces of the form $\Omega(-, T) \times K$ for T an arbitrary tree. Given an arbitrary dendroidal space X, take a resolution $Z_\bullet \to X$ by representatives. That is, each Z_n is given by a coproduct of dendroidal spaces of the form $\Omega(-, T) \times K$, for T a tree, and the natural map

$$\underset{[n]\in\Delta^{\mathrm{op}}}{\mathrm{hocolim}}\, Z_n \to X$$

is a degreewise weak equivalence. The unit map $X \to N\mathbb{L}\tau X$ factors as

$$\underset{[n]\in\Delta^{\mathrm{op}}}{\mathrm{hocolim}}\, Z_n \to \underset{[n]\in\Delta^{\mathrm{op}}}{\mathrm{hocolim}}\, N\mathbb{L}\tau(Z_n) \to N\mathbb{L}\tau\big(\underset{[n]\in\Delta^{\mathrm{op}}}{\mathrm{hocolim}}\, Z_n\big).$$

We have already established that the first map is a weak equivalence. To see that the second map is a weak equivalence, we use the fact that taking homotopy colimits over Δ^{op} commutes with finite homotopy products and so the Segal condition is preserved. Hence, it is enough to show that the second map is a weak equivalence once we evaluate on corollas. This verification amounts to checking that the underlying map of sequences

$$\underset{[n]\in\Delta^{\mathrm{op}}}{\mathrm{hocolim}}\, \mathbb{L}\tau(Z_n) \to \mathbb{L}\tau\big(\underset{[n]\in\Delta^{\mathrm{op}}}{\mathrm{hocolim}}\, Z_n\big)$$

is a weak equivalence, which holds since $\mathbb{L}\tau$ preserves homotopy colimits. $\square$

5 Profinite completion of operads

The adjunctions (1) and (2) relating groupoids and profinite groupoids and the space version, give rise to simplicial Quillen adjunctions

$$\mathbf{d}S \rightleftarrows \mathbf{d}\widehat{S} \quad \text{and} \quad \mathbf{d}G \rightleftarrows \mathbf{d}\widehat{G},$$

where all the categories are equipped with model structures where weak equivalences are given levelwise.

Given an ∞–operad X in spaces or groupoids, write $\widehat{X}$ for the dendroidal object obtained by applying profinite completion levelwise. In general, $\widehat{X}$ is not an ∞–operad. If it is, then we have a weak equivalence

$$\mathbb{R}\mathrm{Map}(\widehat{X}, Y) \simeq \mathbb{R}\mathrm{Map}(X, |Y_f|),$$

natural in $Y \in \mathbf{Op}_\infty(\widehat{S})$, where $|Y_f|$ is the ∞–operad in spaces whose value at a tree T is $|(Y_T)_f|$, and f is a fibrant replacement functor in $\widehat{S}$. In these circumstances, we call $\widehat{X}$ the *profinite completion* of the ∞–operad X.

In general, it is reasonable to define the profinite completion of X as the ∞–operad characterized by the formula above. In other words, it is the ∞–operad in profinite spaces which corepresents the functor

$$Y \mapsto \mathbb{R}\mathrm{Map}(X, |Y_f|)$$

for $Y \in \mathbf{Op}_\infty(\widehat{S})$. At any rate, in the cases that we are interested in, the levelwise profinite completion always produces an ∞–operad, using the following observation:

Proposition 5.1 Let P *be an operad in spaces such that each* $\mathsf{P}(n)$ *has finitely many components and its homotopy groups are good. Then* $(N\mathsf{P})^\wedge$ *is an* ∞*–operad in profinite spaces.*

Proof This is an immediate consequence of [Proposition 3.9](). □

Remark 5.2 For operads in groupoids, the situation is nicer. Indeed, if P is an operad in groupoids in which each groupoid $\mathsf{P}(n)$ has finitely many objects, then $(N\mathsf{P})^\wedge$ is a strict operad in profinite groupoids by [Remark 3.7](). This applies for instance to the operads $\mathtt{PaB}$ and $\mathtt{PaRB}$ defined in the next section.

6 Braids and ribbon braids

The braid group on n strands, hereafter denoted by $\mathsf{B}(n)$, is the fundamental group of the space of unordered configurations of n points in the complex plane. This group has a preferred presentation with generators $\{\beta_i\}_{1 \le i \le n-1}$ which are subject to the so-called Artin relations

- $\beta_i \beta_j = \beta_j \beta_i$ if $|i - j| \ge 2$,
- $\beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1}.$

The braid and symmetric groups fit in a short exact sequence of groups

$$1 \to \mathsf{PB}(n) \to \mathsf{B}(n) \to \Sigma_n \to 1,$$

where $\mathrm{PB}(n)$ is the pure braid group on n strands. The pure braid group is also the fundamental group of the space of ordered configurations of points in the plane. In terms of generators, $\mathrm{B}(n) \to \Sigma_n$ sends the elementary braid $\boldsymbol{\beta}_i$ to the permutation $(i, i+1)$.

In this paper, we are also concerned with the ribbon versions of these two groups. The ribbon braid group, denoted by $\mathrm{RB}(n)$ is the fundamental group of the space of unordered configurations of n points in the plane, where each point is equipped with a choice of label in S^1. There is an obvious map $\mathrm{RB}(n) \to \mathrm{B}(n)$ that corresponds, at the space level, to forgetting the data of the label. This map is split surjective (a section exists at the space level by giving each point in the configuration a fixed label). There is a presentation of the ribbon braid group $\mathrm{RB}(n)$ that is compatible with the inclusion of $\mathrm{B}(n)$. It has generators $\boldsymbol{\beta}_i$ with i in $\{1, \ldots, n-1\}$ and $\boldsymbol{\tau}_j$ with j in $\{1, \ldots, n\}$ subject to the relations

- $\boldsymbol{\beta}_i \boldsymbol{\tau}_j = \boldsymbol{\tau}_j \boldsymbol{\beta}_i$ for $j \notin \{i, i+1\}$,
- $\boldsymbol{\beta}_i \boldsymbol{\tau}_{i+1} = \boldsymbol{\tau}_i \boldsymbol{\beta}_i$,
- $\boldsymbol{\tau}_i \boldsymbol{\tau}_j = \boldsymbol{\tau}_j \boldsymbol{\tau}_i$ if $i \neq j$,

as well as the Artin relations. Another way to think of this group is to let $\mathrm{B}(n)$ act on the left on $\mathbb{Z}^n$ by the composite

$$\mathrm{B}(n) \to \Sigma_n \to \mathrm{GL}_n(\mathbb{Z}),$$

where the second map is the map that sends a permutation to its permutation matrix. The reader can easily check from the above presentation that there is an isomorphism $\mathrm{RB}(n) \cong \mathrm{B}(n) \ltimes \mathbb{Z}^n$.

There is also a map $\mathrm{RB}(n) \to \Sigma_n$ that is given in terms of generators by sending $\boldsymbol{\beta}_i$ to $(i, i+1)$ and $\boldsymbol{\tau}_j$ to the identity. This map is surjective and its kernel is the pure ribbon braid group on n strands, denoted by $\mathrm{PRB}(n)$. This group is also the fundamental group of the space of ordered configurations of n points in the plane each equipped with a choice of label in S^1. This space splits as a product of the space of ordered configurations of n points in the plane with the space $(S^1)^n$. It follows that $\mathrm{PRB}(n)$ splits as $\mathrm{PB}(n) \times \mathbb{Z}^n$.

6.1 Colored (ribbon) braid operad

In this section, we describe two operads in groupoids, CoB and CoRB, which are central to the paper. They are models for the operad of little 2–disks and its framed version (in the variants without 0–arity operations).

We first recall the definition of the nonunital associative operad below.

Definition 6.1 The operad Σ is an operad in sets whose arity 0 term is the empty set and whose arity n term (for n positive) is the symmetric group Σ_n. Operadic composition $\circ_k \colon \Sigma_m \times \Sigma_n \to \Sigma_{m+n-1}$ consists of placing $\sigma \in \Sigma_n$ in position k of $\tau \in \Sigma_m$ and reindexing. For example, $(132) \circ_2 (12) = (1423)$.

Definition 6.2 The operad of colored braids $\mathsf{CoB} = \{\mathsf{CoB}(n)\}_{n \geq 0}$ consists of a collection of groupoids $\mathsf{CoB}(n)$ defined as follows:

- $\mathsf{CoB}(0)$ is the empty groupoid.

- For $n > 0$, the set of objects $\mathrm{Ob}(\mathsf{CoB}(n))$ is Σ_n.

- A morphism in $\mathsf{CoB}(n)$ from p to q is a braid $\alpha \in \mathsf{B}(n)$ whose associated permutation is $q^{-1} p$.

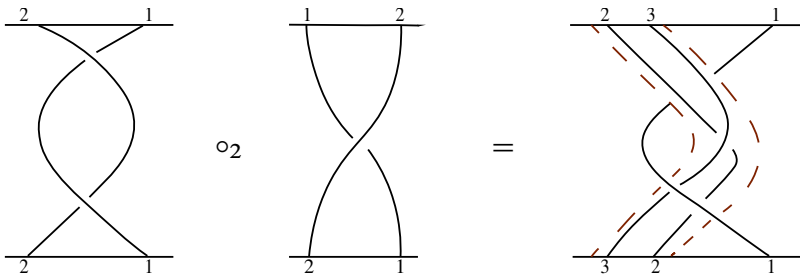The categorical composition in $\mathsf{CoB}(n)$,

$$\mathrm{Hom}_{\mathsf{CoB}(n)}(p, q) \times \mathrm{Hom}_{\mathsf{CoB}(n)}(q, t) \to \mathrm{Hom}_{\mathsf{CoB}(n)}(p, t),$$

is given by the concatenation operation of braids, inherited from the braid group. We write $a \cdot b$ for the *categorical composition* of a and b.
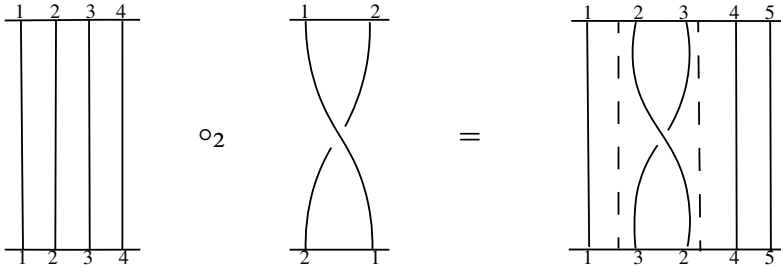
The operadic composition operation

$$\circ_k \colon \mathsf{CoB}(m) \times \mathsf{CoB}(n) \to \mathsf{CoB}(m + n - 1)$$

is defined as follows: On objects, it is given by operadic composition of permutations, as in the associative operad. On morphisms, it corresponds to replacing a chosen strand by a braid; given morphisms α in $\mathsf{CoB}(m)$ and β in $\mathsf{CoB}(n)$, the braid $\alpha \circ_k \beta$ is obtained by replacing the kth strand in α by the braid β as in the picture below:

Remark 6.3 Any morphism in $\mathsf{CoB}(n)$, ie a braid, has an expression as a categorical composition of elementary braids $\boldsymbol{\beta}_i$. Moreover, each elementary braid can be expressed as an operadic composition of an identity morphism (a trivial braid) and a morphism $\boldsymbol{\beta}$ in $\mathsf{CoB}(2)$, the nontrivial braid on two strands pictured below:

Therefore, any morphism in $\mathsf{CoB}(n)$ can be expressed as a categorical composition of morphisms obtained as operadic compositions of identities and $\boldsymbol{\beta}$.

The operad CoB has a ribbon version CoRB, which we now define.

Definition 6.4 [31, Example 1.2.9] The groupoid of colored ribbon braids on n strands $\mathsf{CoRB}(n)$ is the groupoid defined as follows:

- $\mathsf{CoRB}(0)$ is the empty groupoid.

- For $n > 0$, the set of objects $\mathrm{Ob}(\mathsf{CoRB}(n))$ is Σ_n.

- A morphism in $\mathsf{CoRB}(n)$ from p to q is a pair $(\gamma, [x_1, \dots, x_n])$ where $\gamma \in \mathrm{Hom}_{\mathsf{CoB}(n)}(p, q)$ and $x_i \in \mathbb{Z}$. We think of such a morphism as a braid equipped with the additional data of a twisting number x_i for each strand i.

Composition of morphisms in $\mathsf{CoRB}(n)$ is given by composing the morphisms in $\mathsf{CoB}(n)$ and adding the twisting numbers. The identity element is the trivial braid with no twists on each strand.

The sequence of groupoids $\mathsf{CoRB} = \{\mathsf{CoRB}(n)\}_{n \geq 0}$ forms an operad. On objects, it is the associative operad, as for CoB. To define the operadic composition on morphisms, we first introduce some notation: for a nonnegative integer m, we write $\boldsymbol{R}_m$ for the element $(\boldsymbol{\beta}_1 \dots \boldsymbol{\beta}_{m-1})^m$ in $\mathrm{PB}(m)$. Given morphisms $(\gamma, [x_1, \dots, x_n])$ and $(\alpha, [y_1, \dots, y_m])$ in $\mathsf{CoRB}(n)$ and $\mathsf{CoRB}(m)$, respectively, their operadic composition (at entry k) is the morphism

$$(\omega, [x_1, \dots, x_{k-1}, x_k + y_1, \dots, x_k + y_m, \dots, y_m])$$

in $\mathsf{CoRB}(n + m - 1)$ with

$$\omega := \gamma \circ_k ((\boldsymbol{R}_m)^{x_k} \cdot \alpha),$$

where $(\boldsymbol{R}_m)^{x_k}$ is the x_k–fold categorical composition of $\boldsymbol{R}_m$ considered as an automorphism in $\mathsf{CoB}(m)$ and $\circ_k$ is the operadic composition product in CoB. Since $\boldsymbol{R}_m$ is an element of the center of $\mathrm{PB}(m)$, the operation just defined is compatible with the groupoid structure. The operadic identity is given by the trivial braid with one strand and no twists.

Below is a picture of a special case of the operadic composition in CoRB, corresponding to $(\mathrm{id}, [1]) \circ (\mathrm{id}, [0, 0])$:
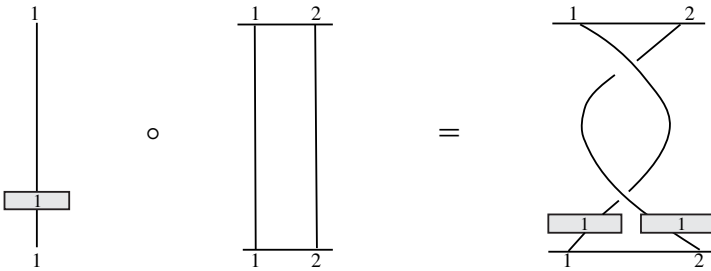
(When it is nonzero, we draw the twisting number of a strand in a gray box over that strand.)

Another example of an operadic composition in CoRB is pictured below:
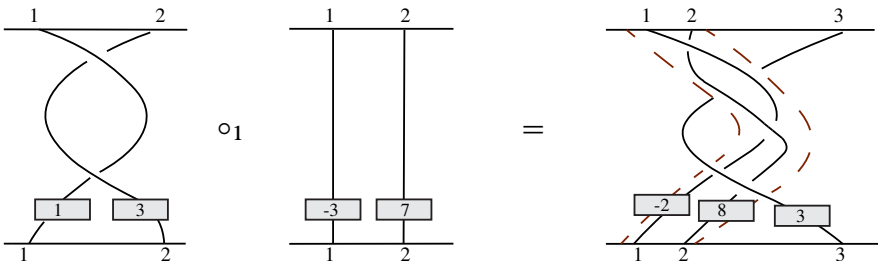
The operad CoRB is a model for the genus zero surface operad and, equivalently, the framed little 2–disks operad, whose definitions we now recall.

Let $F_{0,n+1}$ denote an oriented surface of genus zero and with $n + 1$ boundary components. We choose one of these boundary components and call it marked and we say the other n components are free. We also assume each boundary component ∂_i comes equipped with a collar, ie a neat embedding of $[0, \epsilon) \times S^1$ in a neighborhood

of ∂_i. The mapping class group Γ_0^{n+1} is the group of isotopy classes of orientation-preserving diffeomorphisms of $F_{0,n+1}$ which fix collars pointwise. By gluing the boundary components of surfaces one obtains composition maps, but these are not strictly associative nor unital. To fix the issue, we recall a construction of Tillmann [29, Construction 2.2], and a later improvement by Wahl [32, Section 3.1], which replaces the mapping class group Γ_0^{n+1} with an equivalent connected groupoid.

We begin by defining a groupoid $\mathcal{E}_{n+1}$ whose objects are surfaces with n free boundary components having a particular decomposition into standard pairs of pants P and standard disks D. By a standard pair of pants we mean a fixed pair of pants with (variable) collars at each boundary component and an assigned ordering of the free boundary components. A morphism in $\mathcal{E}_{n+1}$ is an isotopy class of diffeomorphisms that preserves the boundary collars and their ordering. By gluing the marked boundary component of a surface to the ith free boundary component of another surface, one obtains composition maps $\circ_i \colon \mathcal{E}_{n+1} \times \mathcal{E}_{m+1} \to \mathcal{E}_{n+m}$. These are associative, but still not unital. Moreover, there are now too many objects in $\mathcal{E}_{n+1}$, so Tillmann identifies two such surfaces if one can be obtained from the other by replacing a subsurface of the form $P \circ_1 P$ by a subsurface of the form $P \circ_2 P$. To have a strict unit, Tillmann then introduces a second identification by declaring two surfaces equal if one can be obtained from the other by replacing a subsurface of the form $P \circ_1 D$ or $P \circ_2 D$ by a circle. After making these two identifications, any object has a unique representative as a surface having no subsurfaces of the form $P \circ_2 P$, $P \circ_1 D$ or $P \circ_2 D$. For $n \neq 1$, let $\mathcal{S}_n$ denote the full subgroupoid of $\mathcal{E}_n$ spanned by these special surfaces. As explained by Wahl, it is a consequence of the Alexander trick that there is a canonical way to, given a morphism in $\mathcal{E}_n$, produce a morphism in $\mathcal{S}_n$ such that the resulting maps

$$\circ_i \colon \mathcal{S}_{n+1} \times \mathcal{S}_{m+1} \to \mathcal{E}_{n+1} \to \mathcal{S}_{n+m}$$

are associative. Now we can define the surface operad.

Definition 6.5 [29, Definition 2.3; 32, Section 3.1] The surface groupoids $\mathcal{S}_+(n)$ are defined by:

- $\mathcal{S}_+(0)$ is the trivial groupoid, whose only object is the standard disk D.

- $\mathcal{S}_+(1)$ is a groupoid with one object S^1 and with $\mathbb{Z}$ as morphisms (thought of as the Dehn twists around that circle).

- For $n \geq 2$, $\mathcal{S}_+(n)$ is the groupoid $\mathcal{S}_n$ defined above. Namely, an object in $\mathcal{S}_+(n)$ is a surface having no subsurfaces of the form $P \circ_2 P$, $P \circ_1 D$ or $P \circ_2 D$, together

with an ordering of the boundary components and a choice of collar around each. Morphisms are isotopy classes of orientation-preserving diffeomorphisms that fix the collars and their ordering.

The groupoids $\mathcal{S}_+(n)$ assemble into an operad. The operadic composition maps $\circ_i$ in $\mathcal{S}_+$ are induced from $\mathcal{E}$ as explained above. When no arity 1 operations are involved, this corresponds to the gluing of surfaces at the object-set level. Composition with arity 0 operations is essentially filling in a boundary component of a surface. Composition with arity 1 operations maps is more subtle. When $n > 1$, the composition map

$$\circ_i\colon \mathcal{S}_+(n) \times \mathcal{S}_+(1) \to \mathcal{S}_+(n)$$

sends a surface to itself, but changes the diffeomorphism on the collar of the relevant free boundary component by a Dehn twist. When $n = 1$, the composition corresponds to addition of integers.

The space $B\mathcal{S}_+(n)$ is homotopy-equivalent to $B\Gamma_0^{n+1}$ and we define the unital genus zero surface operad $\mathcal{M}_+$ to be the operad

$$\mathcal{M}_+(n) = B\mathcal{S}_+(n) \simeq B\Gamma_0^{n+1}.$$

The nonunital variant of $\mathcal{S}_+$ is defined as

$$\mathcal{S}(n) = \begin{cases} \varnothing & \text{for } n = 0, \\ \mathcal{S}_+(n) & \text{for all } n > 0, \end{cases}$$

and the genus zero surface operad $\mathcal{M}$ as

$$\mathcal{M}(n) = B\mathcal{S}(n) \simeq B\Gamma_0^{n+1}.$$

Remark 6.6 There exists a canonical isomorphism between the groupoids $\mathcal{S}(n)$ and $\mathsf{CoRB}(n)$ which extends to an isomorphism of operads. We alternatively could have defined the genus zero surface operad $\mathcal{M}$ as $B\mathsf{CoRB}$.

Definition 6.7 The framed little 2–disks operad FD is an operad given in positive arity n by the space of all smooth, orientation-preserving embeddings of the disjoint union of n disks into a single disk (see for example [13, page 20]). We define FD(0) to be the empty space.

By remembering where the center of each disk goes and the value of the derivative at each of those centers, the space FD(n) is homotopy-equivalent to the space of ordered configurations of n points in the disk each equipped with a label in S^1. This space is itself homotopy-equivalent to $B\mathsf{PRB}(n)$.

Proposition 6.8 [31, Proposition 1.3.14] *The classifying space of the operad* CoRB *is weakly equivalent to the operad* FD.

6.2 Parenthesized ribbons

The goal of this section is to give a cofibrant resolution of the operad CoRB, called the parenthesized ribbon braid operad and denoted by PaRB.

The operad of objects of PaRB is the free operad generated by a single operation in arity 2. We give a more concrete description below.

Definition 6.9 Let S be a finite set. We define the set of nonassociative monomials of length n, $\mathbb{M}_n S$, inductively as

- $\mathbb{M}_0 S = \varnothing$,
- $\mathbb{M}_1 S = S$, and
- $\mathbb{M}_n S = \coprod_{p+q=n} \mathbb{M}_p S \times \mathbb{M}_q S$.

Alternatively, $\mathbb{M}_n S$ is the set of rooted, binary (ie each vertex has exactly two incoming edges) planar trees with n leaves labeled by elements of S. A shorthand notation for elements $\mathbb{M}_n S$ is as parenthesized words in S. For example, for $S = \{a, b, c, d\}$, the expression $(a(db))(ba)$ represents an element in $\mathbb{M}_5 S$.

Definition 6.10 Let $\mathsf{M} = \{\mathsf{M}(n)\}$ be the symmetric sequence where $\mathsf{M}(n)$ is the subset of $\mathbb{M}_n\{1, \ldots, n\}$ consisting of the monomials in $\{1, \ldots, n\}$ where each element of the set occurs exactly once. The symmetric group Σ_n acts from the right on $\mathsf{M}(n)$ by permuting the elements of the set $\{1, \ldots, n\}$. The symmetric sequence M becomes an operad with operadic composition given by replacing letters by monomials (or grafting binary trees). For example, we have the composition

$$(1(34))(25) \circ_4 (13)2 = (1(3((46)5)))(27).$$

The operad M is called the magma operad.

There is an obvious operad map $u \colon \mathsf{M} \to \Sigma$ which forgets the parenthesization.

Definition 6.11 The operad of parenthesized braids PaB is the operad in groupoids defined as follows:

- The operad of objects is the magma operad, ie $\mathrm{Ob}(\mathsf{PaB}) = \mathsf{M}$.

- For each $n \geq 0$, the morphisms of the groupoid $\mathsf{PaB}(n)$ are morphisms in $\mathsf{CoB}(n)$:

$$\mathrm{Hom}_{\mathsf{PaB}(n)}(p, q) = \mathrm{Hom}_{\mathsf{CoB}(n)}(u(p), u(q)).$$

The collection of groupoids $\{\mathsf{PaB}(n)\}_{n \geq 0}$ forms an operad. On objects, it has the operad structure of M and on morphisms that of CoB.

There is also a ribbon version of PaB.

Definition 6.12 The operad of parenthesized ribbon braids PaRB is the operad in groupoids defined as follows:

- The operad of objects is the magma operad, ie $\mathrm{Ob}(\mathsf{PaRB}) = \mathsf{M}$.
- For each $n \geq 0$, the morphisms of the groupoid $\mathsf{PaRB}(n)$ are morphisms in $\mathsf{CoRB}(n)$:

$$\mathrm{Hom}_{\mathsf{PaRB}(n)}(p, q) = \mathrm{Hom}_{\mathsf{CoRB}(n)}(u(p), u(q)).$$

The collection of groupoids $\{\mathsf{PaRB}(n)\}_{n \geq 0}$ forms an operad in groupoids. On objects, it has the operad structure of M and on morphisms that of CoRB.

Recall that a map $\mathsf{P} \to \mathsf{Q}$ of operads in groupoids is a weak equivalence if, aritywise, $\mathsf{P}(n) \to \mathsf{Q}(n)$ is a weak equivalence of groupoids.

Lemma 6.13 *The forgetful map* $\mathsf{PaRB} \to \mathsf{CoRB}$ *is a weak equivalence of operads in groupoids. Therefore,* PaRB *is also a model for the genus zero surface operad.*

Proof The map is surjective on object sets for each arity n, and it is bijective on morphisms by construction. □

Corollary 6.14 PaRB *is a cofibrant replacement of* CoRB.

Proof From Proposition 2.2 we know that an operad P in groupoids is cofibrant if $\mathrm{Ob}(\mathsf{P})$ is free. The magma operad M (Definition 6.10) is free on one operation of arity 2, therefore the result follows. □

7 Operad maps out of PaRB

Throughout this section, P is a fixed operad in groupoids. For $\sigma \in \Sigma_m$ and x an object or morphism in $\mathsf{P}(m)$, we write σx for the action of σ on x.

Lemma 7.1 [12, Theorem 6.2.4] *The set of operad maps from* PaB *to* P *is identified with the set of triples* (m, β, α) *where* $m \in \mathrm{Ob}\,\mathsf{P}(2)$, β *is a morphism in* $\mathsf{P}(2)$ *from* m *to* σm, *where* $\sigma = (21)$ *is the nontrivial element in* Σ_2, *and* α *is a morphism in* $\mathsf{P}(3)$ *between* $m \circ_1 m$ *and* $m \circ_2 m$. *These triples are subject to the pentagon and hexagon relations spelled out below.*

The hexagon relations state that the diagrams

(5)

and

(6)
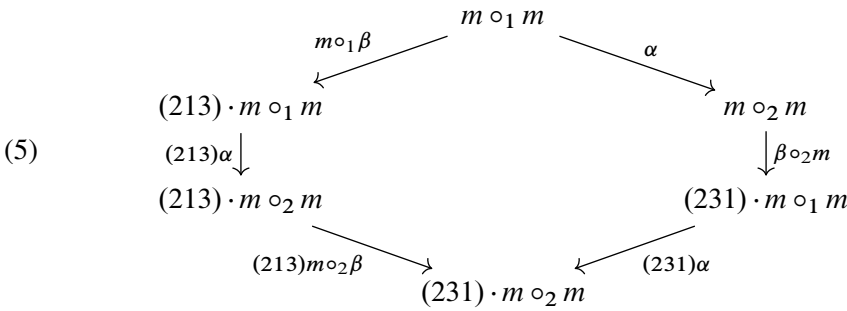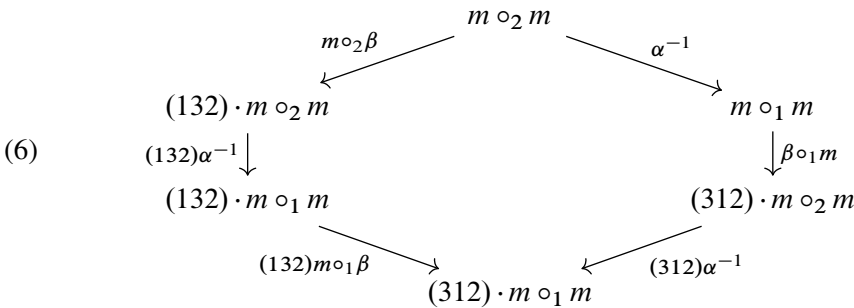
commute in $\mathsf{P}(3)$.

The pentagon relation states that the diagram

(7)

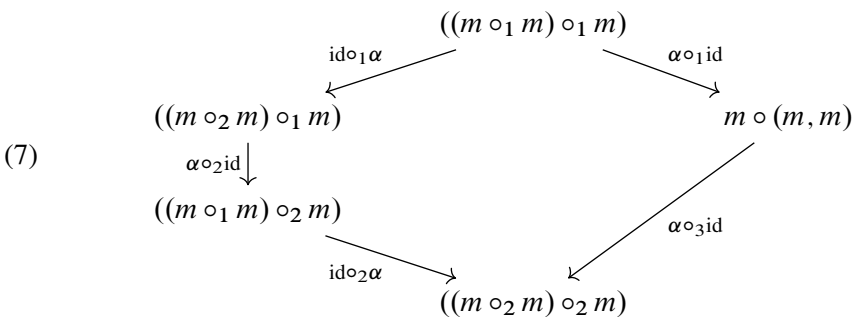
commutes in $\mathsf{P}(4)$.

Proof This is proved in detail in [12, Theorem 6.2.4]. One implication is easy: a map of operads from PaB to P determines such a triple (m, β, α), where m is the image of the object $(12) \in \mathsf{PaB}(2)$, and β and α (called the *braiding* and the *associator*, respectively) are the images of the morphisms pictured in Figure 1. The reverse implication involves a version of the coherence theorem of Mac Lane. □

In preparation for the definition of the Grothendieck–Teichmüller group, we recall some standard notations. Let Y be a profinite group and let α and β be elements of Y. Let f be an element of $\widehat{\mathbb{F}}_2$, the profinite completion of the free group on two generators x and y. Let $\sigma \colon \mathbb{F}_2 \to Y$ be a homomorphism defined by $\sigma(x) = \alpha$ and $\sigma(y) = \beta$. Then, we write $f(\alpha, \beta)$ for the image of f under σ. (By the universal property of profinite completion, to specify a map from a group G to a profinite group Y is equivalent to specifying a map from the profinite completion of G to Y.)

For $1 \le i < j \le n$, we follow common practice and denote by x_{ij} the element of the pure braid group $\mathsf{PB}(n)$ given by $(\beta_{j-1} \ldots \beta_{i+1})\beta_{j-1}^2(\beta_{j-1} \ldots \beta_{i+1})^{-1}$.

Definition 7.2 (Drinfeld) The *Grothendieck–Teichmüller monoid* $\underline{\widehat{\mathrm{GT}}}$ is the monoid of endomorphisms σ of $\widehat{\mathbb{F}}_2$ of the form

$$\sigma(x) = x^\lambda, \quad \sigma(y) = f^{-1}y^\lambda f$$

for some $(\lambda, f) \in \widehat{\mathbb{Z}} \times \widehat{\mathbb{F}}_2$ satisfying the following equations:

(I) $f(x, y)f(y, x) = 1$.

(II) $f(z, x)z^m f(y, z)y^m f(x, y)x^m = 1$, with $z = (xy)^{-1}$ and $m = \frac{1}{2}(\lambda - 1)$.

(III) $f(x_{12}, x_{23}x_{24})f(x_{13}x_{23}, x_{34}) = f(x_{23}, x_{34})f(x_{12}x_{13}, x_{24}x_{34})f(x_{12}, x_{23})$.

The first two equations hold in $\widehat{\mathbb{F}}_2$ and the last equation holds in $\widehat{\mathsf{PB}}(4)$.

The pair (λ, f) is uniquely determined by σ. This follows from equation (III), which guarantees that f belongs to the commutator subgroup of $\widehat{\mathbb{F}}_2$. The multiplication of two such pairs $(\lambda, f) \cdot (\mu, g)$ is given by

$$(\lambda\mu, f(gx^\mu g^{-1}, y^\mu) \cdot g).$$

The Grothendieck–Teichmüller group $\widehat{\mathrm{GT}}$ is the group of units of $\underline{\widehat{\mathrm{GT}}}$.

The following proposition translates the definition $\underline{\widehat{\mathrm{GT}}}$ into operadic language:

Proposition 7.3 *The monoid* $\underline{\widehat{\mathrm{GT}}}$ *is the monoid of endomorphisms of* $\widehat{\mathsf{PaB}}$ *fixing the objects.*

Proof (see [12, Propositions 11.1.3 and 11.3.4]) By Lemma 7.1, an endomorphism of $\widehat{\mathsf{PaB}}$ fixing the objects is uniquely specified by pair (β, α), where β is a morphism in $\widehat{\mathsf{PaB}}(2)$ from (12) to (21) and α is a morphism in $\widehat{\mathsf{PaB}}(3)$ from $(12)3$ to $1(23)$, ie $\beta \in \widehat{\mathbb{Z}}$ and $\alpha = (n, f) \in \widehat{\mathbb{Z}} \times \widehat{\mathbb{F}}_2$. The pair (β, α) is subject to the hexagon and pentagon relations. The hexagon relations force n to be 0 (see the proof of Proposition 11.5). Drinfeld shows in [8, Section 4] that equations (I) and (II) taken together are equivalent to both hexagon relations, and that equation (III) is equivalent to the pentagon relation. $\qquad\square$

The goal of this section is to prove the following lemma. As in Definition 6.4, we write $\cdot$ for categorical composition.

Lemma 7.4 *The set of operad maps from* PaRB *to* P *is identified with the set of pairs* (g, τ) *where* $g = (m, \beta, \alpha)$ *is an operad map from* PaB *to* P *and* τ *is a morphism in* $\mathsf{P}(1)$, *subject to the relation that the image of* (τ, id) *under the map*

$$\circ_1 \colon \operatorname{Mor}\mathsf{P}(1) \times \operatorname{Mor}\mathsf{P}(2) \to \operatorname{Mor}\mathsf{P}(2)$$

agrees with the categorical composition $\beta \cdot \sigma\beta \cdot (\mathrm{id} \circ (\tau, \tau))$, *where* $\mathrm{id} \circ (\tau, \tau)$ *is the image of* $(\mathrm{id}, \tau, \tau)$ *under the operadic composition map*

$$\operatorname{Mor}\mathsf{P}(2) \times \operatorname{Mor}\mathsf{P}(1) \times \operatorname{Mor}\mathsf{P}(1) \to \operatorname{Mor}\mathsf{P}(2).$$

Before we go into the proof, let us fix some notation. The elements τ, m, β and α will be the images in P of certain elements in PaRB, which we now describe. We will use boldface notation $\boldsymbol{\tau}$, $\boldsymbol{m}$, $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ for these elements in PaRB, and set $\boldsymbol{m} = (12)$ and $\boldsymbol{\tau}$ to be the morphism in $\mathsf{PaRB}(1)$ of the form $(\mathrm{id}, [1])$ pictured in Figure 1 along with the other elements.
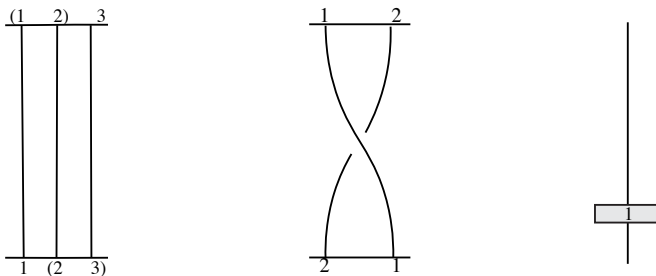

Figure 1: $\boldsymbol{\alpha}$ (left), $\boldsymbol{\beta}$ (center) and $\boldsymbol{\tau}$ (right)

For a sequence of integers $n_1, \ldots, n_k$, we write

$$[n_1, \ldots, n_k]$$

for the morphism in $\mathsf{PaRB}(k)$ given by the pair $(\mathrm{id}, [n_1, \ldots, n_k])$. (This is also the image of $(\mathrm{id}, \tau^{n_1}, \ldots, \tau^{n_k})$ under the operadic composition map

$$\mathrm{Mor}\,\mathsf{PaRB}(k) \times \mathrm{Mor}\,\mathsf{PaRB}(1) \times \cdots \times \mathrm{Mor}\,\mathsf{PaRB}(1) \to \mathrm{Mor}\,\mathsf{PaRB}(k),$$

where τ^m denotes the m–fold categorical composition of τ.) The reader should note that there is one such morphism for each object of $\mathsf{PaRB}(k)$, however, we do not include the object in the notation in order to keep it as light as possible.

Proof Starting with the map $g \colon \mathsf{PaB} \to \mathsf{P}$, we want to lift it to a map $f \colon \mathsf{PaRB} \to \mathsf{P}$ such that precomposition of f with the canonical map $\mathsf{PaB} \to \mathsf{PaRB}$ is g. We first define $f(k) \colon \mathsf{PaRB}(k) \to \mathsf{P}(k)$ for each k using g and τ, and then use the relation in the statement of the lemma to show that the definition is indeed a map of operads.

Firstly, it is enough to describe $f(k)$ on morphism sets; the definition extends to object sets via the source-target maps. Recall that a morphism in $\mathsf{PaRB}(k)$ is a pair $(\gamma, [n_1, \ldots, n_k])$, where γ is a morphism in $\mathsf{PaB}(k)$ and each n_i is an integer, and that the *categorical* composition in $\mathsf{PaRB}(k)$ separates the braid concatenation and the addition of twists on each strand. Therefore, any morphism $(\gamma, [n_1, \ldots, n_k])$ in $\mathsf{PaRB}(k)$ has a unique expression as a categorical composition of the form $(\gamma, [0, \ldots, 0]) \cdot (\mathrm{id}, [n_1, \ldots, n_k])$, which we abbreviate as $\gamma \cdot [n_1, \ldots, n_k]$.

Now we come to the definition of f. We declare $f(\tau) = \tau$,

$$f([n_1, \ldots, n_k]) = g(\mathrm{id}) \circ (\tau^{n_1}, \ldots, \tau^{n_k})$$

and

$$f(\gamma \cdot [n_1, \ldots, n_k]) = g(\gamma) \cdot f([n_1, \ldots, n_k]).$$

One easily checks that $f(k)$ is a map of groupoids for each k.

We now check that f is a map of operads. Let $\gamma \cdot [n_1, \ldots, n_k]$ and $\gamma' \cdot [m_1, \ldots, m_\ell]$ be elements in $\mathrm{Mor}\,\mathsf{PaRB}(k)$ and $\mathrm{Mor}\,\mathsf{PaRB}(\ell)$ respectively. We need to show that

(8) $\quad f(\gamma \cdot [n_1, \ldots, n_k] \circ_i \gamma' \cdot [m_1, \ldots, m_\ell]) = f(\gamma \cdot [n_1, \ldots, n_k]) \circ_i f(\gamma' \cdot [m_1, \ldots, m_\ell]).$

Writing $\gamma \cdot [n_1, \ldots, n_k] \circ_i \gamma' \cdot [m_1, \ldots, m_\ell]$ as a categorical composition of $\gamma \circ_i \gamma'$ and $[n_1, \ldots, n_k] \circ_i [m_1, \ldots, m_\ell]$, we can express the left-hand side of (8) as

$$g(\gamma \circ_i \gamma') \cdot f([n_1, \ldots, n_k] \circ_i [m_1, \ldots, m_\ell]).$$

On the other hand, using that P is an operad, the right-hand side of (8) is equal to

$$(g(\gamma) \circ_i g(\gamma')) \cdot (f([n_1, \ldots, n_k]) \circ_i f([m_1, \ldots, m_\ell])).$$

Now, since g is a map of operads we know that $g(\gamma \circ_i \gamma') = g(\gamma) \circ_i g(\gamma')$. Therefore, the equations (8) hold if and only if the equations

$$f([n_1, \ldots, n_k] \circ_i [m_1, \ldots, m_\ell]) = f([n_1, \ldots, n_k]) \circ_i f([m_1, \ldots, m_\ell])$$

hold. At this point, we remind the reader that $[n_1, \ldots, n_k]$ is notation for an automorphism of some unspecified object in $\mathsf{PaRB}(k)$; thus, the equation above is really a collection of equations, one for each choice of objects that makes the source and target of both sides coincide.

There are further reductions to be made. As a first step, by expressing $[n_1, \ldots, n_k]$ as $[n_1, \ldots, n_k] \cdot \mathrm{id}_k$ and $[m_1, \ldots, m_\ell]$ as $\mathrm{id}_\ell \cdot [m_1, \ldots, m_\ell]$, we split the problem into two:

$$(9) \qquad f(\mathrm{id}_k \circ_i [m_1, \ldots, m_\ell]) = f(\mathrm{id}_k) \circ_i f([m_1, \ldots, m_\ell])$$

and

$$(10) \qquad f([n_1, \ldots, n_k] \circ_i \mathrm{id}_\ell) = f([n_1, \ldots, n_k]) \circ_i f(\mathrm{id}_\ell),$$

where we use the notation id_k or id_ℓ to denote the identity of an unspecified object in arity k or ℓ. By definition of f, the equation (9) is always satisfied.

In order to check equation (10), we do a further reduction. Using that

$$[n_1, \ldots, n_k] = [n_1, 0, \ldots, 0] \cdot [0, n_2, 0, \ldots, 0] \cdots [0, \ldots, 0, n_k]$$

we may assume that, for a given i, n_j is zero for all $j \neq i$ and $n_i = 1$. Thus, equation (10) holds if and only if the equation

$$f([0, \ldots, 1, \ldots, 0] \circ_i \mathrm{id}_\ell) = f([0, \ldots, 1, \ldots, 0]) \circ_i f(\mathrm{id}_\ell),$$

holds, where 1 is in the ith position.

To proceed, we look at a generalization of equation (9): for any morphism $S \in \mathrm{Mor}\,\mathsf{PaRB}(\ell)$, we have that $f(\mathrm{id}_k \circ_i S) = f(\mathrm{id}_k) \circ_i f(S)$. Using this, and the fact that $[0, \ldots, 1, \ldots, 0] = \mathrm{id}_k \circ_i [1]$, we deduce that equation (10) holds if and only the equation

$$(11) \qquad f(\tau \circ \mathrm{id}_\ell) = f(\tau) \circ f(\mathrm{id}_\ell)$$

holds (recall that $[1] = \tau$).

By assumption, equation (11) holds when $\ell = 2$. Indeed, the relation $\tau \circ \mathrm{id}_2 = \beta \cdot \sigma\beta \cdot (\mathrm{id} \circ (\tau, \tau))$ holds in PaRB and, applying f, we obtain the identity

$$f(\tau \circ \mathrm{id}_2) = f\big(\beta \cdot \sigma\beta \cdot (\mathrm{id} \circ (\tau, \tau))\big).$$

By definition of f, the right-hand side equals $\beta \cdot \sigma\beta \cdot (\mathrm{id} \circ (\tau, \tau))$ and so, by hypothesis, it equals $\tau \circ \mathrm{id}_2$.

Now assume that equation (11) has been established (for a chosen ℓ). By associativity of operadic composition, we have equations

$$(\tau \circ \mathrm{id}_\ell) \circ_j \mathrm{id}_2 = \tau \circ \mathrm{id}_{\ell+1} \quad \text{and} \quad (\tau \circ \mathrm{id}_\ell) \circ_j \mathrm{id}_2 = \tau \circ \mathrm{id}_{\ell+1}.$$

Since $\mathrm{Ob}(\mathsf{PaRB})$ is generated by arity 2 operations, any object in $\mathsf{PaRB}(\ell+1)$ is in the image of at least one of the composition maps

$$\circ_j \colon \mathsf{PaRB}(\ell) \times \mathsf{PaRB}(2) \to \mathsf{PaRB}(\ell+1)$$

In particular, the equation

$$f(\tau \circ \mathrm{id}_{\ell+1}) = \tau \circ \mathrm{id}_{\ell+1}$$

holds if and only if, for each j, the equation

(12) $$f((\tau \circ \mathrm{id}_\ell) \circ_j \mathrm{id}_2) = (\tau \circ \mathrm{id}_\ell) \circ_j \mathrm{id}_2$$

holds. But as we have already observed, the equation $f(\tau \circ \mathrm{id}_2) = \tau \circ \mathrm{id}_2$ holds. If we reverse all the steps that let us reduce equation (8) to equation (11), we see that (8) holds when $\ell = 2$. In particular, equation (12) holds for each j. □

7.1 Endomorphisms of PaRB fixing the objects

We write $\mathrm{End}_0(\mathsf{PaRB})$ and $\mathrm{End}_0(\mathsf{PaB})$ for the set of endomorphisms of PaRB and PaB which are the identity on objects.

Definition 7.5 The operad of twists T is the operad in groupoids which in degree n is the group $\mathbb{Z}^n$ seen as a groupoid with a unique object. The operadic composition

$$\circ_i \colon \mathsf{T}(n) \times \mathsf{T}(m) \to \mathsf{T}(n+m-1)$$

is given by the formula

$$(a_1, \ldots, a_n) \circ_i (b_1, \ldots, b_m) := (a_1, \ldots, a_{i-1}, a_i + b_1, \ldots, a_i + b_m, a_{i+1}, \ldots, a_n).$$

Note that there is a trivial morphism from any operad in groupoids to T which sends any morphism to $(0, \ldots, 0)$.

Lemma 7.6 *The trivial morphism is the only morphism from* PaB *to* T.

Proof By Lemma 7.1, such a morphism is determined by where it sends β and α. Let us call the image of these two morphisms $(x, y) \in \mathbb{Z}^2$ and $(a, b, c) \in \mathbb{Z}^3$, respectively. The pentagon relation gives us

$$(2a, 2b + a, c + 2b, 2c) = (2a, a + b, b + c, 2c),$$

which implies that $b = 0$. The two hexagon relations force a, c, x and y to all be 0. □

Proposition 7.7 *The following holds:*

(1) *Any endomorphism of* PaRB *fixing the objects has to preserve* PaB $\subset$ PaRB.

(2) *The induced map*
$$\mathrm{End}_0(\mathsf{PaRB}) \to \mathrm{End}_0(\mathsf{PaB})$$
is an isomorphism.

Proof Using Lemma 7.4 and the notation therein, an endomorphism of PaRB fixing the objects is uniquely specified by a pair (g, τ), where $g = (m, \beta, \alpha)$ represents a map from PaB to PaRB fixing the objects and τ is a morphism in PaRB(1). We claim that the map g has to send PaB to PaB. Indeed, there is a nontrivial map PaRB $\to$ T that sends a ribbon braid to the list of its twists. We can form the composite

$$\mathsf{PaB} \xrightarrow{\ g\ } \mathsf{PaRB} \to \mathsf{T}$$

and according to Lemma 7.6 such a map has to be the trivial map. Therefore, g factors through the inclusion PaB $\to$ PaRB (viewing PaB as the suboperad whose morphisms have no twists). This proves (1).

By taking the underlying braiding together with the number of twists on each strand, the set $\mathrm{Mor}_{\mathsf{PaRB}(2)}(12, 12)$ is identified with $\mathrm{Mor}_{\mathsf{PaB}(2)}(12, 12) \times \mathbb{Z} \times \mathbb{Z}$, ie $2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Recall that the set of morphisms in PaB(2) from (12) to (21) agrees with $\mathbb{Z}$, the braid group on two strands. The morphism $\beta \cdot \sigma\beta$ is thus given by a triple $(2\beta_1, \beta_2, \beta_3)$ in $2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and τ is given by a single integer. By part (1), β_2 and β_3 have to be zero. The relation

(13) $$\tau \circ \mathrm{id} = \beta \cdot \sigma\beta \cdot (\mathrm{id} \circ (\tau, \tau)),$$

which holds in the set of morphisms on $\mathsf{PaRB}(2)$ from (12) to (12), may then be expressed as a relation

$$(2\tau, \tau, \tau) = (2\beta_1, 0, 0) \cdot (0, \tau, \tau).$$

Therefore, $\beta_1 = \tau$. Hence we can construct a map $\mathrm{End}_0(\mathsf{PaB}) \to \mathrm{End}_0(\mathsf{PaRB})$ sending g to (g, β_1) which is an inverse to the restriction map. $\qquad\square$

7.2 Endomorphisms of PaRB up to homotopy

The category of operads in groupoids is cotensored over groupoids. It follows that we can define a homotopy between two maps of operads in groupoids. We denote by E the groupoid completion of $[1] = \{0 < 1\}$ and by s and t the two maps $[0] \to E$. For $f, g \colon \mathsf{P} \to \mathsf{Q}$ two maps of operads, a homotopy between f and g is a map $H \colon \mathsf{P} \to \mathsf{Q}^E$ such that when we postcompose with the two maps $\mathsf{Q}^E \to \mathsf{Q}$ induced by s and t, we recover f and g. The relation "being homotopic" is an equivalence relation between maps from P to Q and this equivalence relation is compatible with composition of morphisms. It follows that for P an operad in groupoids, the set of endomorphisms of P up to homotopy gets a monoid structure. We denote that monoid by $\mathrm{HoEnd}(\mathsf{P})$.

Proposition 7.8 *The composition*

$$\mathrm{End}_0(\mathsf{PaRB}) \to \mathrm{End}(\mathsf{PaRB}) \to \mathrm{HoEnd}(\mathsf{PaRB})$$

is an isomorphism.

Proof The surjectivity of this map can be translated by saying that any endomorphism of PaRB is homotopic to one that fixes the objects. This can be proven exactly as in [20, Theorem 7.8].

Now, we prove injectivity. We define $\mathrm{HoEnd}_0(\mathsf{P}) \subset \mathrm{HoEnd}(\mathsf{P})$ to be those homotopy classes that contain an endomorphism fixing the objects. We denote by $\mathrm{End}_0(\mathsf{PRB}(3))$ the monoid of endomorphisms of $\mathsf{PRB}(3)$ that preserve the subgroup $\mathsf{PB}(3)$ and by $\mathrm{HoEnd}_0(\mathsf{PRB}(3))$ the monoid of endomorphisms of $\mathsf{PRB}(3)$ that preserve the subgroup $\mathsf{PB}(3)$ modulo homotopies. We construct a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{End}_0(\mathsf{PaRB}) & \longrightarrow & \mathrm{HoEnd}_0(\mathsf{PaRB}) & \xrightarrow{\ f\ } & \mathrm{HoEnd}_0(\mathsf{PRB}(3)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{End}_0(\mathsf{PaB}) & \longrightarrow & \mathrm{HoEnd}_0(\mathsf{PaB}) & \xrightarrow[\ g\ ]{} & \mathrm{HoEnd}(\mathsf{PB}(3))
\end{array}
$$

The map labeled f is induced by the restriction map

$$\mathrm{End}_0(\mathsf{PaRB}) \to \mathrm{End}_0(\mathsf{PaRB}(3)) \to \mathrm{End}_0(\mathsf{PRB}(3)),$$

where the first map is the restriction to arity 3 and the second map is the restriction to an object in $\mathsf{PaRB}(3)$. The map labeled g is defined analogously.

The right-hand vertical map is obtained by restriction to the subgroup $\mathsf{PB}(3)$. This is well defined, as we now explain. Let u and v be two endomorphisms of $\mathsf{PRB}(3) \cong \mathsf{PB}(3) \times \mathbb{Z}^3$ fixing the subgroup $\mathsf{PB}(3) \subset \mathsf{PRB}(3)$ and such that there exists an element h in $\mathsf{PRB}(3)$ such that $u(x) = h^{-1}v(x)h$. Since the subgroup $\mathbb{Z}^3$ in $\mathsf{PRB}(3)$ is contained in the center, we may assume without loss of generality that h lies in $\mathsf{PB}(3)$ and we deduce that the restrictions of u and v to $\mathsf{PB}(3)$ are conjugate. The commutativity of the diagram is immediate.

By [20, Proposition 7.7], the lower horizontal composite is injective. By our Proposition 7.7, the left vertical arrow is an isomorphism. It follows that the map $\mathrm{End}_0(\mathsf{PaRB}) \to \mathrm{HoEnd}_0(\mathsf{PaRB})$ is injective. Since the map

$$\mathrm{End}_0(\mathsf{PaRB}) \to \mathrm{HoEnd}(\mathsf{PaRB})$$

factors through $\mathrm{HoEnd}_0(\mathsf{PaRB})$, we are done. $\qquad\qquad\square$

8 The main theorem

Propositions 7.7 and 7.8 have profinite variants, which we state below. The proofs are similar.

Proposition 8.1 *The map*

$$\mathrm{End}_0(\widehat{\mathsf{PaRB}}) \to \mathrm{End}_0(\widehat{\mathsf{PaB}})$$

and the composite

$$\mathrm{End}_0(\widehat{\mathsf{PaRB}}) \to \mathrm{End}(\widehat{\mathsf{PaRB}}) \to \mathrm{HoEnd}(\widehat{\mathsf{PaRB}})$$

are isomorphisms of monoids.

By Proposition 7.3, the monoid $\mathrm{End}_0(\widehat{\mathsf{PaB}})$ is isomorphic to $\widehat{\mathrm{GT}}$, the Grothendieck–Teichmüller monoid; it follows that

$$(14) \qquad\qquad\qquad \widehat{\mathrm{GT}} \cong \mathrm{HoEnd}(\widehat{\mathsf{PaRB}}).$$

Proposition 8.2 *The monoid* $\mathrm{HoEnd}(\widehat{\mathsf{PaRB}})$ *is isomorphic to the monoid of path components of* $\mathbb{R}\mathrm{Map}(N\,\widehat{\mathsf{PaRB}}, N\,\widehat{\mathsf{PaRB}})$.

Proof We claim that the statement holds for any operad P in groupoids which, like PaRB, is cofibrant and such that $\mathsf{P}(n)$ has finitely many objects for each n. For such an operad, the set $\mathrm{HoEnd}(\widehat{\mathsf{P}})$ is identified with the set of path components of

$$\mathrm{Map}(\mathsf{P}, |\widehat{\mathsf{P}}|),$$

where Map refers to the mapping space in the category of operads in groupoids. Since the dendroidal nerve functor is homotopically fully faithful, the map

$$\mathbb{R}\mathrm{Map}(\mathsf{P}, |\widehat{\mathsf{P}}|) \to \mathbb{R}\mathrm{Map}(N\mathsf{P}, N|\widehat{\mathsf{P}}|)$$

is a weak equivalence of spaces. The dendroidal space $N|\widehat{\mathsf{P}}|$ is an ∞–operad by Remark 5.2. The right-hand mapping space is identified with

$$\mathbb{R}\mathrm{Map}(\widehat{N\mathsf{P}}, N\widehat{\mathsf{P}})$$

since $|-|$ and N commute and the profinite completion functor agrees with its left (and right) derived functor since it preserves all weak equivalences. Moreover, $\widehat{N\mathsf{P}} \cong N\widehat{\mathsf{P}}$ since completion of groupoids with finitely many objects commutes with products (Remark 5.2). □

Given an operad in (profinite) groupoids G, we let BG denote the operad in (profinite) spaces obtained via the classifying space construction. There is a natural map

$$(15) \qquad\qquad (BN\mathsf{PaRB})^{\wedge} \to BN\widehat{\mathsf{PaRB}},$$

where the left-hand side is an alternative notation for the profinite completion of $BN\mathsf{PaRB}$. This map is given as the adjoint of the composite

$$BN\mathsf{PaRB} \to BN|\widehat{\mathsf{PaRB}}| \xrightarrow{\;\cong\;} |BN\widehat{\mathsf{PaRB}}|,$$

where the first map is the unit of the adjunction between operads in groupoids and operads in profinite groupoids.

Lemma 8.3 *The map* (15) *is a weak equivalence.*

Proof The pure ribbon braid groups $\mathrm{PRB}(n)$ are good since they split as a product of good groups $\mathrm{PB}(n) \times \mathbb{Z}^n$ and so we can apply [20, Corollaries 5.11 and 5.12]. Thus, by Proposition 5.1, both sides are ∞–operads in profinite spaces. It suffices to prove that the map is a weak equivalence on corollas, ie that

$$(B\mathrm{PRB}(n))^{\wedge} \to B\widehat{\mathrm{PRB}(n)}$$

is a weak equivalence for every n. This follows again from the fact that the pure ribbon braid groups are good. □

Putting it all together, we obtain:

Theorem 8.4 *There is an isomorphism*

$$\widehat{\mathrm{GT}} \cong \pi_0 \, \mathbb{R}\mathrm{End}(\widehat{\mathcal{M}}),$$

where $\mathcal{M}$ denotes the version of the genus zero surface operad without 0–arity operations.

Proof By Lemma 6.13, $\mathbb{R}\mathrm{End}(\widehat{\mathcal{M}})$ is weakly equivalent to $\mathbb{R}\mathrm{End}((BN\mathrm{PaRB})^\wedge)$. The latter is weakly equivalent to $\mathbb{R}\mathrm{End}(BN\widehat{\mathrm{PaRB}})$ by Lemma 8.3. The classifying space functor B induces a homotopically fully faithful functor from ∞–operads in profinite groupoids to ∞–operads in profinite spaces. In particular, the map

$$\mathbb{R}\mathrm{Map}(N\widehat{\mathrm{PaRB}}, N\widehat{\mathrm{PaRB}}) \to \mathbb{R}\mathrm{Map}(BN\widehat{\mathrm{PaRB}}, BN\widehat{\mathrm{PaRB}})$$

is a weak equivalence of spaces. By Proposition 8.2 and the isomorphism (14), the monoid of path components of the source is isomorphic to $\widehat{\mathrm{GT}}$. □

9 Formality of the genus zero surface operad

It has been proved independently by Severa [30] and Giansiracusa and Salvatore [15] that the framed little disks operad is rationally formal. That is to say, there is a zigzag of quasi-isomorphisms of dg–operads between $C_*(\mathrm{FD}, \mathbb{Q})$ and its homology, seen as a dg–operad with zero differential. In this section, we exploit the action of the Grothendieck–Teichmüller group on the profinite completion of the genus zero surface operad in order to give an alternative proof of the formality of $\mathcal{M}$ and, equivalently, FD. The idea is to use the fact that there is a model for $C_*(\mathcal{M}, \mathbb{Q}_p)$ that is computed using the profinite completion of $\mathcal{M}$ and thus inherits a $\widehat{\mathrm{GT}}$ action. This large supply of automorphisms on the chains on $\mathcal{M}$ allows us to apply a formality criterion introduced by Guillen, Navarro, Pascual and Roig [17].

In preparation for our proof, we introduce a notation. For $X = \lim_i X_i$ a pro-simplicial set and R a commutative ring, we denote by $C^\bullet(X, R)$ the cosimplicial R–module given by the formula

$$C^\bullet(X, R) := \operatorname*{colim}_i C^\bullet(X_i, R).$$

There is a Künneth isomorphism at the level of cosimplicial objects in the sense that there is a natural isomorphism

$$C^\bullet(X \times Y, R) \cong C^\bullet(X, R) \otimes_R C^\bullet(Y, R).$$

In particular, if Q is an operad in pro-simplicial sets, $C^\bullet(\mathsf{Q}, R)$ has the structure of a cosimplicial cooperad in R–modules.

Theorem 9.1 *The operad $\mathcal{M}$ is formal, that is, there exists a zigzag of quasi-isomorphisms of dg–operads in $\mathbb{Q}$–vector spaces,*

$$C_*(\mathcal{M}, \mathbb{Q}) \leftarrow X \rightarrow H_*(\mathcal{M}, \mathbb{Q}).$$

Proof   We follow the strategy of [25]. First, by [17, Theorem 6.2.1], it suffices to prove that $C_*(\mathcal{M}, \mathbb{Q}_p)$ is formal as a dg–operad in $\mathbb{Q}_p$–vector spaces. For any positive integer n, there is a quasi-isomorphism of cosimplicial cooperads in $\mathbb{Z}/p^n$–modules,

$$C^\bullet(\mathcal{M}, \mathbb{Z}/p^n) \simeq C^\bullet(B\mathsf{PaRB}, \mathbb{Z}/p^n) \simeq C^\bullet(B\widehat{\mathsf{PaRB}}, \mathbb{Z}/p^n).$$

Taking the limit over n (which in this case is a homotopy limit since the transition maps are surjections), we get a quasi-isomorphism of cosimplicial cooperads,

$$\lim_n C^\bullet(\mathcal{M}, \mathbb{Z}/p^n) \simeq \lim_n C^\bullet(B\widehat{\mathsf{PaRB}}, \mathbb{Z}/p^n).$$

We also claim that the map

$$C^\bullet(\mathcal{M}, \mathbb{Z}_p) \rightarrow \lim_n C^\bullet(\mathcal{M}, \mathbb{Z}/p^n)$$

is a quasi-isomorphism, as can be seen from Milnor's short exact sequence and the fact that the cohomology of the spaces $\mathcal{M}(n)$ is finitely generated, which implies that the Mittag-Leffler condition holds. Tensoring with $\mathbb{Q}_p$, we get a quasi-isomorphism of cosimplicial cooperads over $\mathbb{Q}_p$,

$$C^\bullet(\mathcal{M}, \mathbb{Q}_p) \simeq (\lim_n C^\bullet(B\widehat{\mathsf{PaRB}}, \mathbb{Z}/p^n)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

After dualizing, the universal coefficient theorem, gives us a quasi-isomorphism of simplicial operads,

$$C_\bullet(\mathcal{M}, \mathbb{Q}_p) \simeq C^\bullet(\mathcal{M}, \mathbb{Q}_p)^\vee \simeq ((\lim_n C^\bullet(B\widehat{\mathsf{PaRB}}, \mathbb{Z}/p^n)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\vee.$$

We denote by P the underlying dg–operad of the simplicial operad on the right-hand side. Our goal is to show that the dg–operad P is formal. By our main theorem, P has

an action of the group $\widehat{GT}$. We claim that the induced map

$$\widehat{GT} \to \operatorname{Aut}(H_*(P))$$

factors as the composite of the cyclotomic character

$$\chi \colon \widehat{GT} \to \widehat{\mathbb{Z}}^\times \to \mathbb{Z}_p^\times$$

with the map

$$\mathbb{Z}_p^\times \to \operatorname{Aut}(H_*(P))$$

sending $u \in \mathbb{Z}_p^\times$ to the automorphism ϕ_u of $H_*(P)$ that acts as multiplication by u^n in homological degree n. Firstly, it is well known that $H_*(P)$ is the operad BV of Batalin–Vilkovisky algebras. It is generated by a commutative algebra product in arity 2 and degree 0 and an operator Δ in arity 1 and homological degree 1. Therefore, it suffices to prove the claim on these two homology groups. It is straightforward that $\widehat{GT}$ acts trivially on $H_0(P(2))$. The action of $\widehat{GT}$ on $B\,\widehat{\operatorname{PRB}(1)} \cong B\widehat{\mathbb{Z}}$ is given precisely by inducing the obvious action of $\widehat{\mathbb{Z}}^\times$ on $\widehat{\mathbb{Z}}$ along the projection $\widehat{GT} \to \widehat{\mathbb{Z}}^\times$ (see the proof of Proposition 7.7(2)). It follows that the action of $\widehat{GT}$ on $H_1(P)$ is the desired action. This proves the claim.

Now, we follow the strategy explained by Petersen [25, page 819, Proposition]. We pick an infinite-order unit u in $\mathbb{Z}_p$. Since the cyclotomic character map $\chi \colon \widehat{GT} \to \mathbb{Z}_p^\times$ is surjective, we can find an automorphism of P that induces the grading automorphism ϕ_u on the homology. $\qquad\square$

Remark 9.2 The surjectivity of $\chi \colon \widehat{GT} \to \mathbb{Z}_p^\times$ follows from the surjectivity of the cyclotomic character $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$. Contrary to Petersen's argument, the proof above does not rely on the existence of Drinfeld associators. In fact, Drinfeld gives a proof for the existence of associators using the surjectivity of this map [8, Proposition 5.3].

10 An action of GT on the operad of compactified moduli spaces

For $n \geq 3$, the moduli space $\mathcal{M}_{0,n}$ of compact complex algebraic curves of genus zero with n punctures is identified with the space of configurations of n distinct points on the complex projective line $\mathbb{CP}^1$ modulo the action of $\operatorname{PGL}_2(\mathbb{C})$. The Deligne–Knudsen–Mumford compactification of $\mathcal{M}_{0,n}$, denoted by $\overline{\mathcal{M}}_{0,n}$, is the space of isomorphism classes of stable n–punctured complex curves of genus zero. By convention, $\overline{\mathcal{M}}_{0,2} = *$.

The collection of moduli spaces $\overline{\mathcal{M}}_{0,\bullet+1} := \{\overline{\mathcal{M}}_{0,n+1}\}_{n\geq 1}$ forms an operad in spaces with no arity 0 term [14]. For a curve in $\overline{\mathcal{M}}_{0,n+1}$, we consider the first n points as inputs and the last point as the output. The symmetric group Σ_n acts on $\overline{\mathcal{M}}_{0,n+1}$ by permuting the labels of the inputs and leaving the output untouched. Operad composition

$$\circ_k \colon \overline{\mathcal{M}}_{0,n+1} \times \overline{\mathcal{M}}_{0,m+1} \to \overline{\mathcal{M}}_{0,n+m}$$

is given by attaching the output of $\overline{\mathcal{M}}_{0,m+1}$ to the kth input in $\overline{\mathcal{M}}_{0,n+1}$ and creating a new genus zero stable curve with one additional double point.

A theorem of Drummond-Cole proved in [9] relates the framed disks operad to the operad $\overline{\mathcal{M}}_{0,\bullet+1}$ via a homotopy pushout diagram

(16)
$$\begin{array}{ccc} S^1 & \longrightarrow & \mathsf{FD} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \overline{\mathcal{M}}_{0,\bullet+1} \end{array}$$

in the category of operads in spaces, where S^1 and $*$ denote the topological groups S^1 and $*$ seen as operads concentrated in arity 1 and the map $S^1 \to \mathsf{FD}$ is the inclusion of arity 1 operations. Given that the operad FD is homotopy-equivalent to $\mathcal{M}$ we can replace FD in the homotopy pushout square.

Proposition 10.1 *The homotopy groups of $\overline{\mathcal{M}}_{0,n}$ are good groups.*

Proof The spaces $\overline{\mathcal{M}}_{0,n}$ are simply connected, compact, complex manifolds. As such their homotopy groups are finitely generated abelian groups. More generally, any finitely generated abelian group is good (see for instance [28, page 5]). Indeed, such a group is a finite product of copies of $\mathbb{Z}$ and $\mathbb{Z}/n$ for various n. A finite group is automatically good and $\mathbb{Z}$ is good. The claim then follows from [20, Proposition 5.10]. □

Corollary 10.2 *The dendroidal profinite space $(N\overline{\mathcal{M}}_{0,\bullet+1})^{\wedge}$ is an ∞–operad.*

Proof This follows from the previous proposition and Proposition 5.1. □

The main goal of this section is to prove the following:

Theorem 10.3 *There exists an action of $\widehat{\mathrm{GT}}$ on the ∞–operad $(N\overline{\mathcal{M}}_{0,\bullet+1})^{\wedge}$ that makes the map*

$$(N\mathcal{M})^{\wedge} \to (N\overline{\mathcal{M}}_{0,\bullet+1})^{\wedge}$$

into a $\widehat{\mathrm{GT}}$–equivariant map.

Proof The functor N is a right Quillen equivalence (Theorem 4.3) and hence it preserves homotopy pushout squares. Therefore, the square

$$\begin{array}{ccc} NS^1 & \longrightarrow & N\mathcal{M} \\ \downarrow & & \downarrow \\ * & \longrightarrow & N\overline{\mathcal{M}}_{0,\bullet+1} \end{array}$$

is a pushout square of ∞–operads in spaces. Applying the profinite completion functor levelwise, we obtain a square of dendroidal objects in profinite spaces. In fact, each term of this new square is an ∞–operad in profinite spaces, by Proposition 5.1. We claim that this square of ∞–operads is a pushout square in the ∞–category (relative category) of ∞–operads in profinite spaces. This is a consequence of the following formal observation. Given a pushout square in the ∞–category of ∞–operads in spaces, consider the resulting square obtained by applying profinite completion levelwise. Then this square is a pushout in the ∞–category of ∞–operads in profinite spaces if each of its terms is an ∞–operad. To see this, one can use the fact that the hypothetical pushout has the correct universal property in the ∞–category of ∞–operads in profinite spaces.

We can now prove the statement of the theorem. The top horizontal map of the square is the inclusion of arity 1 operations. It follows that the action of $\widehat{\mathrm{GT}}$ on $(N\mathcal{M})^\wedge$ restricts to an action of $(NS^1)^\wedge$ in such a way that this map becomes a $\widehat{\mathrm{GT}}$–equivariant map. On the other hand, the map $(NS^1)^\wedge \to *$ is obviously $\widehat{\mathrm{GT}}$–equivariant for the trivial action on $*$. It follows that the ∞–operad $(N\overline{\mathcal{M}}_{0,\bullet+1})^\wedge$ inherits a $\widehat{\mathrm{GT}}$ action that makes the square $\widehat{\mathrm{GT}}$–equivariant. $\qquad\square$

We now want to prove that the action constructed in the previous theorem is nontrivial. In order to do so, we will prove that this action is nontrivial after application of $H_*(-, \mathbb{Q}_p)$. First we need to explain what we mean by $H_*(X, \mathbb{Q}_p)$ when X is a profinite space.

Construction 10.4 We have explained in the previous section how to construct a cosimplicial $\mathbb{Z}/p^n$–module $C^\bullet(X, \mathbb{Z}/p^n)$. We can then define $C^*(X, \mathbb{Z}/p^n)$ as the associated cochain complex. Define the chain complex

$$D_*(X, \mathbb{Q}_p) := \left(\left(\lim_n C^*(X, \mathbb{Z}/p^n)\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right)^\vee.$$

As in the proof of Theorem 9.1, one can show that $D_*(\widehat{Y}, \mathbb{Q}_p)$ is naturally quasi-isomorphic to $C_*(Y, \mathbb{Q}_p)$ when Y is a space with finitely generated homology. We

denote the homology of $D_*(X, \mathbb{Q}_p)$ by $H_*(X, \mathbb{Q}_p)$. Since $C^*(-, \mathbb{Z}/p^n)$ sends homotopy colimits to homotopy limits, we deduce that $D_*(-, \mathbb{Q}_p)$ preserves homotopy colimits.

To make explicit the action of $\widehat{\mathrm{GT}}$ on $H_*(\widehat{\mathcal{M}}, \mathbb{Q}_p)$ we use the p–adic cyclotomic character $\chi_p \colon \widehat{\mathrm{GT}} \to \hat{\mathbb{Z}}^\times \to \mathbb{Z}_p^\times$.

Proposition 10.5 *Let g be an element of $\widehat{\mathrm{GT}}$. Then the action of g on the vector space $H_i(\widehat{\mathcal{M}(n)}, \mathbb{Q}_p)$ is given by multiplication by $\chi_p(g)^i$.*

Proof This vector space is isomorphic as a $\widehat{\mathrm{GT}}$–representation to the vector space $H_i(\mathrm{P}(n))$ appearing in the proof of Theorem 9.1. The desired statement can be found in that proof. □

Proposition 10.6 *The action of $\widehat{\mathrm{GT}}$ on $(\overline{\mathcal{M}}_{0,\bullet+1})^\wedge$ is nontrivial.*

Proof A standard argument with simplicial model categories applied to the homotopy pushout square (16) tells us that the operad $\overline{\mathcal{M}}_{0,\bullet+1}$ is the homotopy colimit of the simplicial diagram

$$[n] \mapsto \mathcal{M} \sqcup (S^1)^{\sqcup n} \sqcup *,$$

where $\sqcup$ denotes the coproduct in the category of operads. Applying the dendroidal nerve functor followed by the profinite completion we get a simplicial diagram

$$[n] \mapsto N(\mathcal{M} \sqcup (S^1)^{\sqcup n} \sqcup *)^\wedge$$

in the relative category $\mathbf{d}\hat{S}$ whose homotopy colimit computes $(N\overline{\mathcal{M}}_{0,\bullet+1})^\wedge$. Indeed, since the category Δ is sifted, this homotopy colimit coincides with the homotopy colimit computed in $\mathbf{Op}_\infty(\hat{S})$.

Evaluating at the corolla C_n, we get a simplicial profinite space

$$[n] \mapsto N(\mathcal{M} \sqcup (S^1)^{\sqcup n} \sqcup *)^\wedge_{C_n}$$

whose homotopy colimit is $(\overline{\mathcal{M}}_{0,n+1})^\wedge$. We can hit this diagram with the functor $D_*(-, \mathbb{Q}_p)$ constructed in Construction 10.4 and we get a simplicial chain complex

$$[n] \mapsto D_*(N(\mathcal{M} \sqcup (S^1)^{\sqcup n} \sqcup *)^\wedge_{C_n}, \mathbb{Q}_p)$$

whose homotopy colimit is $D_*((\overline{\mathcal{M}}_{0,n+1})^\wedge, \mathbb{Q}_p)$. This simplicial diagram has an action of $\widehat{\mathrm{GT}}$ that induces the action of $\widehat{\mathrm{GT}}$ on $D_*((\overline{\mathcal{M}}_{0,n+1})^\wedge, \mathbb{Q}_p)$ constructed in

Theorem 10.3. We thus get a $\widehat{\mathrm{GT}}$–equivariant spectral sequence of the form

$$E_{s,t}^1 = H_t((* \sqcup (S^1)^{\sqcup s} \sqcup \mathcal{M})(n), \mathbb{Q}_p) \Rightarrow H_{s+t}(\overline{\mathcal{M}}_{0,n+1}, \mathbb{Q}_p).$$

By the Künneth isomorphism, and Proposition 10.5, we deduce that the action of $g \in \widehat{\mathrm{GT}}$ on $E_{s,t}^1$ is given by multiplication by $\chi_p(g)^t$. This implies that the same is true for $E_{s,t}^r$ for all r. Since the differentials must commute with the $\widehat{\mathrm{GT}}$ action and d^r has degree $(-r, r-1)$, we see that only d^1 can be nonzero.

Now we study the behavior of the line $t = 0$ of this spectral sequence. We can do that by comparing it to the similar spectral sequence, denoted by $E_{s,t}^{\prime r}$, that computes the pushout square of simplicial operads

$$
\begin{array}{ccc}
* & \longrightarrow & \mathrm{Com} \\
\downarrow & & \downarrow \\
* & \longrightarrow & \mathrm{Com}
\end{array}
$$

We have a map of spectral sequences $E \to E'$ that comes from a map between the two pushout squares. This map is an isomorphism on the line $t = 0$ at the E^1 page. The spectral sequence E' is very easy to understand: the differential d^1 kills everything except the generator in degree $(0, 0)$ and no further differentials can occur. Therefore, the same patterns must occur on the 0th line of the spectral sequence E.

We have said that E collapses at the E^2 page. Using the observation that $E_{s,0}^2$ is trivial for $s > 0$, we deduce that for positive k the vector space $H_k((\overline{\mathcal{M}}_{0,n+1})^\wedge, \mathbb{Q}_p)$ has a filtration which is compatible with the $\widehat{\mathrm{GT}}$ action and that the action of $\widehat{\mathrm{GT}}$ on the associated graded splits as a direct sum of representations that are nontrivial. $\qquad \square$

Remark 10.7 As pointed out to us by the referee, it would be interesting to know whether the kernel of the cyclotomic character acts nontrivially on $(\overline{\mathcal{M}}_{0,\bullet+1})^\wedge$. The above proof does not shed light on this question.

11 Unital case

In this section, we allow operads to have *nonempty* space of 0–arity operations. For us the most relevant such operad is $\mathcal{M}_+$, the variant of the operad of framed little 2–disks where $\mathcal{M}_+(0) = *$. The essential difference between $\mathcal{M}$ and $\mathcal{M}_+$ is that in the latter we include the operation of filling in boundary components. While this is a substantial difference, we show that it does not affect the monoid of derived endomorphisms:

Theorem 11.1 $$\widehat{\mathrm{GT}} \cong \pi_0 \, \mathbb{R}\mathrm{End}(\widehat{N\mathcal{M}_+}).$$

There are two relevant unital variants of the operad PaRB. The first one is an operad PaRB$_+$ which coincides with PaRB in positive arities and is a point in arity 0. The operadic composition maps $\circ_i$ of the form

$$\mathrm{PaRB}_+(n) \times \mathrm{PaRB}_+(0) \to \mathrm{PaRB}_+(n-1)$$

are given (on morphism sets) by removing the ith strand. We do not expect the operad PaRB$_+$ to be cofibrant in any reasonable sense. So we introduce a second operad, denoted by PaRB$_\star$, which is cofibrant as a monochromatic operad in groupoids and is equivalent to PaRB$_+$. In detail, the operad Ob PaRB$_\star$ of objects of PaRB$_\star$ is the free operad on the (nonsymmetric) sequence which is a point in degree 0 and 2 and empty otherwise. We think of an element in PaRB$_\star(n)$ as a parenthesization of a word $a_1 \dots a_\ell$ where each $i \in \{1, \dots, n\}$ occurs exactly once as one of the a_j and all the other symbols are labeled $\ast$. (Alternatively, that element may be regarded as a tree with n leaves where the vertices have either two inputs or no inputs.) For example, $((\ast(1\ast))((\ast\ast)(32)))$ is an element in Ob PaRB$_\star(3)$.

There is a canonical map of operads $u \colon \mathrm{Ob}\,\mathrm{PaRB}_\star \to \mathrm{Ob}\,\mathrm{PaRB}_+$ that drops the symbols $\ast$ (and the appropriate parentheses). For instance, the element in the example just above is sent to $(1(32))$ via this map.

Given two objects x and y in PaRB$_\star$, the set of morphisms in PaRB$_\star$ from x to y is by definition the set of morphisms in PaRB$_+$ from $u(x)$ to $u(y)$. This defines the operad Mor PaRB$_\star$ and thus the operad PaRB$_\star$ together with a map $v \colon \mathrm{PaRB}_\star \to \mathrm{PaRB}_+$.

Proposition 11.2   *The map v is a cofibrant replacement of* PaRB$_+$ *in* $\mathbf{Op}(G)$.

Proof   This map is clearly a levelwise weak equivalence. Moreover, since the operad Ob PaRB$_\star$ is freely generated by an operation in degree 0 and an operation in degree 2, we can apply Proposition 2.2 and conclude that PaRB$_\star$ is cofibrant.   □

The following two lemmas are variations on Lemmas 7.1 and 7.4.

Lemma 11.3   *Let* P *be an operad in groupoids with* $\mathrm{P}(0) = \ast$ *and* $\mathrm{Ob}\,\mathrm{P}(1) = \ast$. *The set of operad maps from* PaB$_+$ *to* P *is identified with the set of maps* $g = (m, \beta, \alpha) \colon \mathrm{PaB} \to \mathrm{P}$ *subject to the relation* $\alpha \circ_i \mathrm{id}_\ast = \mathrm{id}_m$ *for* $i = 1, 2, 3$, *where* $\mathrm{id}_\ast$ *denotes the identity element of* $\ast \in \mathrm{P}(0)$.

Proof   See [12, 6.2.4(c)].   □

Lemma 11.4 *Let* P *be an operad in groupoids with* $P(0) = *$. *The set of operad maps from* PaRB_+ *to* P *is identified with the set of pairs* (g, τ) *where* $g = (m, \beta, \alpha)$ *is an operad map from* PaB_+ *to* P *and* τ *is a morphism in* $P(1)$, *subject to the relation that the operadic composition* $\tau \circ_1 \mathrm{id}$ *agrees with the categorical composition* $\beta \cdot \sigma\beta \cdot (\mathrm{id} \circ (\tau, \tau))$.

Proof The proof of [Lemma 7.4](#) applies verbatim, but one extra check needs to be completed, namely that

$$f([n_1, \ldots, n_k] \circ_i \mathrm{id}_*) = f([n_1, \ldots, n_k]) \circ_i f(\mathrm{id}_*),$$

where id_* denotes the identity map of the (unique) element in arity 0. This can be reduced to checking that $f([1] \circ \mathrm{id}_*) = f([1]) \circ_i f(\mathrm{id}_*)$, which holds automatically because $P(0) = *$. $\square$

Proposition 11.5 *The restriction map* $\mathrm{End}_0(\mathsf{PaRB}_+) \to \mathrm{End}_0(\mathsf{PaRB})$ *is an isomorphism.*

Proof Exactly as in [Proposition 7.7](#), we can prove that any endomorphism of PaRB_+ fixing the objects has to restrict to an endomorphism of the suboperad PaB_+. This is by definition the suboperad with the same objects but only those morphisms that have trivial twists. We thus get a commutative square of restriction maps

$$
\begin{array}{ccc}
\mathrm{End}_0(\mathsf{PaRB}_+) & \longrightarrow & \mathrm{End}_0(\mathsf{PaB}_+) \\
\downarrow & & \downarrow \\
\mathrm{End}_0(\mathsf{PaRB}) & \longrightarrow & \mathrm{End}_0(\mathsf{PaB})
\end{array}
$$

in which the lower horizontal map is an isomorphism by [Proposition 7.7](#) and the top horizontal map is an isomorphism by [Lemma 11.4](#). In order to see that the right-hand vertical map is an isomorphism, it is enough, by [Lemma 11.3](#), to show that for any map $g \colon \mathsf{PaB} \to \mathsf{PaB}_+$ fixing the objects, the equation $\alpha \circ_i \mathrm{id}_* = \mathrm{id}_m$ automatically holds. This equation boils down to the condition that the image of α, viewed as an element in the pure braid group on three strands, under the map $\partial_i \colon PB(3) \to PB(2) \cong \mathbb{Z}$ which forgets the ith strand for each $i = 1, 2, 3$, is zero. We can use one of the hexagon relations to deduce this. We explain this for $i = 2$ (the middle strand); the other cases can be treated similarly. Recall that $\beta \in \mathsf{PaB}_+(2)$ is by definition the image of $\boldsymbol{\beta} \in \mathsf{PaB}(2)$ under g. The hexagon relation [(5)](#) reads

$$(17) \qquad (m \circ_1 \beta) \cdot (213)\alpha \cdot (213)(m \circ_2 \beta) = \alpha \cdot (\beta \circ_2 m) \cdot (231)\alpha$$

in $\mathsf{PB}(3)$. (In this equation, m is shorthand for id_m.) By inspection, we have that

$$\partial_2(m \circ_1 \beta) = 0, \quad \partial_2((213)m \circ_2 \beta) = \beta \quad \text{and} \quad \partial_2(\beta \circ_2 m) = \beta$$

in $\mathsf{PB}(2)$. (To verify these it may be helpful to draw a picture.) Write n for $\partial_2(\alpha) = \partial_2((231)\alpha)$. Now apply ∂_2 to both sides of (17) to deduce that $0 + n + \beta = n + \beta + n$. Hence, $\partial_2(\alpha) = 0$, as claimed. $\qquad\square$

Corollary 11.6 *The action of $\widehat{\mathrm{GT}}$ on PaRB extends to an action on PaRB_+ and the induced map*

$$\widehat{\mathrm{GT}} \to \mathrm{End}_0(\mathsf{PaRB}_+)$$

is an isomorphism.

In analogy with previous notation, we write $\mathrm{Hom}_0(\mathsf{PaRB}_\star, \mathsf{PaRB}_+)$ for the set of operad maps which induce the map $u\colon \mathrm{Ob}\,\mathsf{PaRB}_\star \to \mathrm{Ob}\,\mathsf{PaRB}_+$ on objects.

Lemma 11.7 *The map $\mathrm{End}_0(\mathsf{PaRB}_+) \to \mathrm{Hom}_0(\mathsf{PaRB}_\star, \mathsf{PaRB}_+)$ is an isomorphism.*

Proof This map is injective since the map $v\colon \mathsf{PaRB}_\star \to \mathsf{PaRB}_+$ is an epimorphism in the category of operads in groupoids.

Before proving surjectivity, we start by making the observation that the groupoid $\mathsf{PaRB}_+(n)$ sits naturally inside $\mathsf{PaRB}_\star(n)$ as the full subgroupoid spanned by those objects that do not have the symbol $\star$. One should observe that these maps do not assemble into a map of operads $\mathsf{PaRB}_+ \to \mathsf{PaRB}_\star$. We will use these maps implicitly to see morphisms in PaRB_+ as morphisms in $\mathsf{PaRB}_\star$ when needed.

In order to prove the surjectivity of the map under consideration, we first observe that its image is the set of operad maps $f\colon \mathsf{PaRB}_\star \to \mathsf{PaRB}_+$ inducing the map u on objects and with the property that $fx = fvx$ for any morphism x in $\mathsf{PaRB}_\star$. Now, we observe that any morphism x in $\mathsf{PaRB}_\star$ can be written as a composition $a \cdot vx \cdot b$, where a and b are morphisms in $\mathsf{PaB}_\star \subset \mathsf{PaRB}_\star$ that are such that ua and ub are identity maps. Hence, it suffices to prove that $f(a)$ and $f(b)$ are identity morphisms. But, one can prove exactly as in Proposition 7.7 that f has to restrict to a map of operads $\mathsf{PaB}_\star \to \mathsf{PaB}_+$. Moreover, by [20, Lemma 6.6], any map of operads $\mathsf{PaB}_\star \to \mathsf{PaB}_+$ lies in the image of the map

$$\mathrm{Hom}_0(\mathsf{PaB}_+, \mathsf{PaB}_+) \to \mathrm{Hom}_0(\mathsf{PaB}_\star, \mathsf{PaB}_+).$$

Therefore, $f(a)$ and $f(b)$ are identity morphisms, as desired. $\qquad\square$

Lemma 11.8 *The map*

$$\mathrm{Hom_0}(\mathsf{PaRB}_\star, \mathsf{PaRB}_+) \to \mathrm{HoHom}(\mathsf{PaRB}_\star, \mathsf{PaRB}_+)$$

is an isomorphism.

Proof Denote the map under consideration by α. We begin by proving injectivity. Unlike the inclusion of PaRB_+ in $\mathsf{PaRB}_\star$, the inclusion of PaRB in $\mathsf{PaRB}_\star$ is a map of operads. Hence, by precomposition, we obtain a map

$$\mathrm{HoHom}(\mathsf{PaRB}_\star, \mathsf{PaRB}_+) \to \mathrm{HoHom}(\mathsf{PaRB}, \mathsf{PaRB}_+) \xrightarrow{\cong} \mathrm{HoEnd}(\mathsf{PaRB}),$$

which we call θ. Using Lemma 11.7 and Proposition 11.5, the map $\theta \circ \alpha$ is identified with the obvious map

$$\mathrm{End_0}(\mathsf{PaRB}) \to \mathrm{HoEnd}(\mathsf{PaRB}),$$

which is an isomorphism by Proposition 7.8. It follows that α is injective.

In order to prove surjectivity, we have to show that any morphism f from $\mathsf{PaRB}_\star$ to PaRB_+ is homotopic to one which induces the map υ on objects. Since $\mathrm{Ob}\,\mathsf{PaRB}_\star$ is freely generated by $*$ and (12) in degrees 0 and 2, and since $\mathsf{PaRB}_+(0)$ is a point, $\mathrm{Ob}\,f$ is determined by the image of (12). This image is either (12), in which case $\mathrm{Ob}\,f = \upsilon$, or (21). In the second case, a homotopy of f with the required property can be constructed as in [20, Theorem 7.8]. □

Proof of Theorem 11.1 We first have an isomorphism

$$\pi_0\,\mathbb{R}\mathrm{End}(\widehat{N\mathcal{M}_+}) \simeq \pi_0\,\mathbb{R}\mathrm{End}(BN\widehat{\mathsf{PaRB}_+}),$$

which comes from the fact that the map

$$(B\mathsf{PaRB}_+(n))^\wedge \to B(\mathsf{PaRB}_+(n))^\wedge$$

is an equivalence (by goodness of the pure ribbon braid groups as in Lemma 8.3). By full faithfulness of the functor B, we are reduced to proving that the action of $\widehat{\underline{\mathrm{GT}}}$ on PaRB_+ induces an isomorphism

$$\widehat{\underline{\mathrm{GT}}} \cong \pi_0\,\mathbb{R}\mathrm{End}(N\widehat{\mathsf{PaRB}_+}).$$

Corollary 11.6 and Lemmas 11.7 and 11.8 also hold if we replace PaRB and its variants by their profinite completion. Hence, we deduce an isomorphism

$$\widehat{\underline{\mathrm{GT}}} \cong \mathrm{HoHom}(\widehat{\mathsf{PaRB}_\star}, \widehat{\mathsf{PaRB}_+}).$$

By adjunction, this gives us an isomorphism

$$\widehat{\mathrm{GT}} \cong \mathrm{HoHom}(\mathrm{PaRB}_\star, |\widehat{\mathrm{PaRB}}_+|) \cong \pi_0 \, \mathbb{R}\mathrm{Map}(\mathrm{PaRB}_\star, |\widehat{\mathrm{PaRB}}_+|),$$

where the second isomorphism comes from the fact that $\mathrm{PaRB}_\star$ is cofibrant. Since $\mathrm{PaRB}_\star \to \mathrm{PaRB}_+$ is an equivalence, we see that the action of $\widehat{\mathrm{GT}}$ on PaRB_+ induces an isomorphism

$$\widehat{\mathrm{GT}} \cong \pi_0 \, \mathbb{R}\mathrm{Map}(\mathrm{PaRB}_+, \widehat{\mathrm{PaRB}}_+).$$

But we have an isomorphism

$$\pi_0 \, \mathbb{R}\mathrm{Map}(\mathrm{PaRB}_+, |\widehat{\mathrm{PaRB}}_+|) \cong \pi_0 \, \mathbb{R}\mathrm{Map}(N\mathrm{PaRB}+, |N\widehat{\mathrm{PaRB}}_+|),$$

coming from the fact that $N \colon \mathbf{Op}G \to \mathbf{Op}_\infty G$ is fully faithful, and an isomorphism

$$\pi_0 \, \mathbb{R}\mathrm{Map}(N\mathrm{PaRB}_+, |N\widehat{\mathrm{PaRB}}_+|) \cong \pi_0 \, \mathbb{R}\mathrm{Map}((N\mathrm{PaRB}_+)^\wedge, (N\mathrm{PaRB}_+)^\wedge)$$

by the derived adjunction between ∞–operads in profinite groupoids and ∞–operads in groupoids. Putting everything together, we deduce the desired result. $\qquad \square$

References

[1] **D W Anderson**, *Fibrations and geometric realizations*, Bull. Amer. Math. Soc. 84 (1978) 765–788 MR

[2] **I Barnea**, **Y Harpaz**, **G Horel**, *Pro-categories in homotopy theory*, Algebr. Geom. Topol. 17 (2017) 567–643 MR

[3] **G V Belyĭ**, *On Galois extensions of a maximal cyclotomic field*, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979) 267–276 MR In Russian; translated in Math. USSR-Izv. 14 (1980) 247–256

[4] **C Berger**, **I Moerdijk**, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. 78 (2003) 805–831 MR

[5] **C Berger**, **I Moerdijk**, *Resolution of coloured operads and rectification of homotopy algebras*, from "Categories in algebra, geometry and mathematical physics" (A Davydov, M Batanin, M Johnson, S Lack, A Neeman, editors), Contemp. Math. 431, Amer. Math. Soc., Providence, RI (2007) 31–58 MR

[6] **J E Bergner**, **P Hackney**, *Group actions on Segal operads*, Israel J. Math. 202 (2014) 423–460 MR

[7] **D-C Cisinski**, **I Moerdijk**, *Dendroidal Segal spaces and ∞–operads*, J. Topol. 6 (2013) 675–704 MR

[8] **V G Drinfeld**, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* Gal($\overline{\mathbb{Q}}/\mathbb{Q}$), Algebra i Analiz 2 (1990) 149–181 MR In Russian; translated in Leningrad Math. J. 2 (1991) 829–860

[9] **G C Drummond-Cole**, *Homotopically trivializing the circle in the framed little disks*, J. Topol. 7 (2014) 641–676 MR

[10] **W G Dwyer**, **D M Kan**, *Calculating simplicial localizations*, J. Pure Appl. Algebra 18 (1980) 17–35 MR

[11] **W G Dwyer**, **D M Kan**, *Function complexes in homotopical algebra*, Topology 19 (1980) 427–440 MR

[12] **B Fresse**, *Homotopy of operads and Grothendieck–Teichmüller groups, I: The algebraic theory and its topological background*, Mathematical Surveys and Monographs 217, Amer. Math. Soc., Providence, RI (2017) MR

[13] **E Getzler**, *Batalin–Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. 159 (1994) 265–285 MR

[14] **E Getzler**, *Operads and moduli spaces of genus* 0 *Riemann surfaces*, from "The moduli space of curves" (R Dijkgraaf, C Faber, G van der Geer, editors), Progr. Math. 129, Birkhäuser, Boston, MA (1995) 199–230 MR

[15] **J Giansiracusa**, **P Salvatore**, *Formality of the framed little* 2*–discs operad and semidirect products*, from "Homotopy theory of function spaces and related topics" (Y Félix, G Lupton, S B Smith, editors), Contemp. Math. 519, Amer. Math. Soc., Providence, RI (2010) 115–121 MR

[16] **A Grothendieck**, *Esquisse d'un programme*, from "Geometric Galois actions, I" (L Schneps, P Lochak, editors), London Math. Soc. Lecture Note Ser. 242, Cambridge Univ. Press (1997) 5–48 MR Translated in ibid. 243–283

[17] **F Guillén Santos**, **V Navarro**, **P Pascual**, **A Roig**, *Moduli spaces and formal operads*, Duke Math. J. 129 (2005) 291–335 MR

[18] **A Hatcher**, **P Lochak**, **L Schneps**, *On the Teichmüller tower of mapping class groups*, J. Reine Angew. Math. 521 (2000) 1–24 MR

[19] **P S Hirschhorn**, *Model categories and their localizations*, Mathematical Surveys and Monographs 99, Amer. Math. Soc., Providence, RI (2003) MR

[20] **G Horel**, *Profinite completion of operads and the Grothendieck–Teichmüller group*, Adv. Math. 321 (2017) 326–390 MR

[21] **D C Isaksen**, *A model structure on the category of pro-simplicial sets*, Trans. Amer. Math. Soc. 353 (2001) 2805–2841 MR

[22] **P T Johnstone**, *Stone spaces*, Cambridge Studies in Advanced Mathematics 3, Cambridge Univ. Press (1982) MR

[23] **J Lurie**, *Higher topos theory*, Annals of Mathematics Studies 170, Princeton Univ. Press (2009) MR

[24] **I Moerdijk**, **I Weiss**, *Dendroidal sets*, Algebr. Geom. Topol. 7 (2007) 1441–1470 MR

[25] **D Petersen**, *Minimal models, GT–action and formality of the little disk operad*, Selecta Math. 20 (2014) 817–822 MR

[26] **G Quick**, *Profinite homotopy theory*, Doc. Math. 13 (2008) 585–612 MR

[27] **G Quick**, *Some remarks on profinite completion of spaces*, from "Galois–Teichmüller theory and arithmetic geometry" (H Nakamura, F Pop, L Schneps, A Tamagawa, editors), Adv. Stud. Pure Math. 63, Math. Soc. Japan, Tokyo (2012) 413–448 MR

[28] **D Sullivan**, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. 100 (1974) 1–79 MR

[29] **U Tillmann**, *Higher genus surface operad detects infinite loop spaces*, Math. Ann. 317 (2000) 613–628 MR

[30] **P Ševera**, *Formality of the chain operad of framed little disks*, Lett. Math. Phys. 93 (2010) 29–35 MR

[31] **N Wahl**, *Ribbon braids and related operads*, PhD thesis, Oxford University (2001) Available at http://eprints.maths.ox.ac.uk/43/

[32] **N Wahl**, *Infinite loop space structure(s) on the stable mapping class group*, Topology 43 (2004) 343–368 MR

Instituto Superior Técnico, Universidade de Lisboa
Lisboa, Portugal

LAGA, Institut Galilée, Université Paris 13
Villetaneuse, France

School of Mathematics and Statistics, The University of Melbourne
Melbourne, Victoria, Australia

pedrobbrito@tecnico.ulisboa.pt, horel@math.univ-paris13.fr,
marcy.robertson@unimelb.edu.au