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Towards a quantum Lefschetz hyperplane theorem in all genera

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An effective algorithm of determining Gromov–Witten invariants of smooth hypersurfaces in any genus (subject to a degree bound) from Gromov–Witten invariants of the ambient space is proposed.

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0 Introduction

0.1 A brief history

Let $D \subset X$ be a smooth hypersurface in a smooth projective variety. One important question in Gromov–Witten theory is to determine all genus Gromov–Witten invariants of D by the invariants of X. An example is the quintic hypersurface in $\mathbb{P}^4$. In genus 0, this relationship enters A Givental's mirror theorem (for example in [8]) as a key step to compute genus-0 invariants for the quintic 3–folds, or more generally semipositive complete intersections in the projective spaces. This genus zero relationship has been further generalized into the quantum Lefschetz hyperplane theorem by Lee [18], Kim [14] and Coates and Givental [5], relating genus-0 Gromov–Witten theory of an ample hypersurface to the (twisted) theory of the ambient space. One ingredient in this genus zero quantum Lefschetz hyperplane theorem is the functoriality of virtual classes based on a classical idea in enumerative geometry and proven by Kim, Kresch and Pantev [15] and Kontsevich [17].

In high genus, a naïve generalization of the main result in [15] is not valid. That is, the Gromov–Witten theories of hypersurfaces are very different from the twisted theories of the ambient spaces. A simple example is a quartic curve in a quartic surface. Although there is no completely satisfactory interpretation of the quantum Lefschetz in higher genus, there have been various approaches, most of which put a special focus on the quintic hypersurface of $\mathbb{P}^4$. In genus 1, there is work by Kim and Lho [16]

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and Zinger [23] among others. In higher genus there is an ongoing *mixed spin P–field* approach pioneered by H-L Chang, J Li, W-P Li and C-C Liu eg in [4; 3]. More recently, while this article was under preparation, S Guo, F Janda and Y Ruan [10] announced a new method which determines higher-genus invariants of quintic 3–folds via twisted invariants of the ambient space $\mathbb{P}^4$ plus some "effective invariants" which are finite for a fixed genus. In the same paper the authors used this in genus 2 to prove the BCOV holomorphic anomaly conjecture of Bershadsky, Cecotti, Ooguri and Vafa [2], and we expect it to be useful in higher genus. There is other interesting work on quintic 3–folds, including Guo and Ross [11; 12] and many others. In general, the question remains: To what degree can the Gromov–Witten invariants of a hypersurface of all genera be determined by those of the ambient space?

In this paper, we would like to propose a different method of computing the Gromov–Witten invariants of a hypersurface. Although our current algorithm is subjected to a degree bound (1), similar bounds also explicitly or inexplicitly appear in other works. For example, Wu [22, Conjecture 1.5] proposed exactly the same bound in determining quintic invariants. Meanwhile, in the work of [10], $2g-2 \geq 5d$ appears as a condition of their effective invariants which are needed in computing the quintic invariants. We hope that this different approach provides a new angle and might be of independent interest in Gromov–Witten theory.

0.2 Weak form of the quantum Lefschetz hyperplane theorem

Let $D \subset X$ be a smooth hypersurface in a smooth projective variety X. Let $\ell \in \mathrm{NE}(D)$ be an effective curve class in D. We are only concerned about the cohomology insertions that are restrictions of classes in X. We make the following definition:

Definition 0.1 Consider the morphism $H^*(X) \to H^*(D)$ given by the restriction under the inclusion of D into X. Denote its image by

$$H^*(D)_X = \mathrm{im}(H^*(X) \to H^*(D)).$$

We also adopt the following convention, which is consistent with all previous results on the quantum Lefschetz hyperplane theorem.

Convention 1 When *GW invariants of D* are mentioned, we always refer to only invariants whose cohomology insertions lie in $H^*(D)_X$.

Now we are ready to present some consequences of the results in Section 1.2.

Theorem 0.2 *When D is an ample hypersurface in X, the (descendant) Gromov–Witten invariants of D for a fixed genus g_0 (of any degree) are determined by GW invariants of X and a* finite number *of GW invariants of D with $g \le g_0$ and*

$$(\ell, D) \le 2g - 2. \tag{1}$$

When D is an ample Calabi–Yau three-dimensional hypersurface in X, all descendant Gromov–Witten invariants of fixed genus g and effective curve class $\ell \in \mathrm{NE}(D)$ can be expressed in terms of (descendant) Gromov–Witten invariants of X and a finite number of invariants of D,

$$\langle - \rangle^D_{g',0,\ell'} \quad \text{for } g' \le g,\ \ell' < \ell \text{ and } (\ell', D) \le 2g' - 2,$$

where $\langle - \rangle^D_{g',0,\ell'}$ is the genus-g', 0–pointed GW invariants on D with class ℓ'. In particular, when $D = Q$ is the quintic threefold in $\mathbb{P}^4$ and $g \le 3$, the GW invariants of D are completely determined.

This can be considered as a weak form of a *quantum Lefschetz hyperplane theorem in higher genus*. Previous results are all in genus zero — see Lee [18], Kim [14] and Coates and Givental [5] — and genus one — see Zinger [23]. The main technique in the proof of the above theorem is the virtual localization in an auxiliary space $\mathfrak{X}$ (introduced in Section 2.1).

Remark 0.3 Let us point out a few takeaways from our method.

(a) It confirms that, in principle, the traditional localization method of moduli of stable maps is powerful enough to effectively determine Gromov–Witten invariants of a hypersurface assuming the knowledge of low-degree invariants (finitely many degrees if the hypersurface is ample) for each genus.

(b) The localization formula can be naturally reorganized according to Givental's technique as in [9]. Very briefly, certain genus zero generating functions can be defined and used to organize the recursions.

For example, since localization is also available in the symplectic geometric definition of Gromov–Witten theory, (a) indicates that we have similar relations in the symplectic setting. It might be possible that these results together with some deep symplectic geometric results (eg Ionel and Parker [13]) can give us more insights in the algebraic setting.

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1 Statements of main results

1.1 Twisted Gromov–Witten invariants

We recall the definitions and set up the notation of the twisted Gromov–Witten invariants. Let Y be a smooth projective variety, E be a vector bundle over Y and $\overline{\mathcal{M}}_{g,n}(Y,\beta)$ be the moduli stack of stable maps from genus-g, n–pointed curves to Y with class $\beta \in \mathrm{NE}(Y)$ in the Mori cone of curves. Let

$$ft_{n+1}\colon \overline{\mathcal{M}}_{g,n+1}(Y,\beta) \to \overline{\mathcal{M}}_{g,n}(Y,\beta)$$

be the map forgetting the last marked point. It gives rise to a universal family over $\overline{\mathcal{M}}_{g,n}(Y,\beta)$. Furthermore, the evaluation map of the last marked point

$$\mathrm{ev}_{n+1}\colon \overline{\mathcal{M}}_{g,n+1}(Y,\beta) \to Y$$

serves as the universal stable map over $\overline{\mathcal{M}}_{g,n}(Y,\beta)$.

Given the setting, there is a $\mathbb{C}^*$–action on E by scaling the fibers. Let λ be the corresponding equivariant parameter. By projectivity of Y, there exists a two-term complex of vector bundles over $\overline{\mathcal{M}}_{g,n}(Y,\beta)$

$$0 \to E^0_{g,n,\beta} \to E^1_{g,n,\beta} \to 0$$

representing the element $R(ft_{n+1})_*\mathrm{ev}^*_{n+1}E \in D^b(\overline{\mathcal{M}}_{g,n}(Y,\beta))$. Denote by $E_{g,n,\beta}$ the two-term complex $[E^0_{g,n,\beta} \to E^1_{g,n,\beta}]$, on which there is a $\mathbb{C}^*$–action induced from the action on E. It is easy to see that the equivariant Euler class of $E_{g,n,\beta}$

$$e_{\mathbb{C}^*}(E_{g,n,\beta}) := \frac{e_{\mathbb{C}^*}(E^0_{g,n,\beta})}{e_{\mathbb{C}^*}(E^1_{g,n,\beta})} \in H^*(\overline{\mathcal{M}}_{g,n}(X,\beta)) \otimes_{\mathbb{C}} \mathbb{C}[\lambda,\lambda^{-1}]$$

is well defined.

The twisted Gromov–Witten invariants needed for this paper are of the form

$$\langle \psi^{k_1}\alpha_1,\dots,\psi^{k_n}\alpha_n\rangle^{X,E}_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \frac{1}{e_{\mathbb{C}^*}(E_{g,n,\beta})} \cup \prod_{i=1}^n \psi_i^{k_i}\,\mathrm{ev}_i^*\alpha_i.$$

This is an element in $\mathbb{C}[\lambda,\lambda^{-1}]$.

Convention 2 We distinguish two types of the equivariant twisted invariants. When the $\mathbb{C}^*$ acts on the fibers on E by "positive" scaling, sending a vector v to λv, we write E^+ for the equivariant bundle and use the notation

$$\langle\cdots\rangle^{X,E^+}_{g,n,\beta}$$

for the corresponding equivariant twisted invariant. Similarly, $\langle\cdots\rangle^{X,E^-}_{g,n,\beta}$ stands for invariants with inverse scaling action $v \mapsto \lambda^{-1}v$.

1.2 Quantum Lefschetz

Let $D \subset X$ be a smooth hypersurface in a smooth projective variety X. Let

$$\mathrm{vdim}_D = \mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(D,\ell)) = (1-g)(\dim(D)-3) + (\ell,-K_D) + n.$$

In $N_1(D), N_1(X)$, we say $\ell \geq \ell'$ if $\ell-\ell'$ is effective. Here we abuse notation and use ℓ and ℓ' as curve classes both in D and in X via pushforward. By the classical Lefschetz hyperplane theorem, when $\dim D > 2$, $H_2(D) = H_2(X)$ and not much information is lost by this identification. Recall the notation in Definition 0.1.

Theorem 1.1 *Let $g \geq 0$ and $\ell \in N_1(D)$ be such that $2g-2 < (\ell,D)$. Let $\alpha_1,\dots,\alpha_n \in H^*(D)_X$ be cohomology classes and $a_1,\dots,a_n \in \mathbb{N}$ be nonnegative integers such that $\sum_{i=1}^n(a_i+\deg(\alpha_i)) = \mathrm{vdim}_D$. There is a formula expressing the Gromov–Witten invariant*

$$\langle \psi^{a_1}\alpha_1,\dots,\psi^{a_n}\alpha_n\rangle^D_{g,n,\ell}$$

in terms of the following:

- *Invariants of the form* $\langle \psi^{k_1}\alpha'_1, \dots, \psi^{k_{n'}}\alpha'_{n'}\rangle^{D,\mathcal{O}(D)^+\oplus\mathcal{O}^-}_{g',n',\ell'}$*, where* $g' \le g$ *and* $\ell' < \ell$*.*
- *Invariants of the form* $\langle \psi^{k_1}\alpha'_1, \dots, \psi^{k_{n'}}\alpha'_{n'}\rangle^{X,\mathcal{O}(-D)^-}_{g',n',\ell'}$*, where* $g' \le g$ *and* $\ell' \le \ell$*.*
- *At most one invariant of the form* $\langle \psi^{k_1}\alpha'_1, \dots, \psi^{k_n}\alpha'_n\rangle^{X,\mathcal{O}^+}_{g,n,\ell}$*.*

The proof of this theorem will occupy the next section. We first explicate some consequences here. Recall we adopt Convention 1.

Corollary 1.2 *When D is an ample hypersurface in X, the (descendant) Gromov–Witten invariants of D for a fixed genus g_0 (of any degree) are determined by GW invariants of X and a* finite number *of GW invariants of D with $g \le g_0$ and $(\ell, D) \le 2g-2$.*

Proof By Theorem 1.1, if $(\ell, D) > 2g-2$, it can be expressed in terms of the twisted invariants on X and the twisted invariants on D of lower inductive order. By the quantum Riemann–Roch theorem in [5], the twisted invariants can be expressed in terms of ordinary invariants of the same or lower inductive order. At the end of the recursive process, the genus and degree bounds will be reached. Note that any insertion of 1 can be removed by the string equation, and we assume henceforth no such insertion. Now, given g and ℓ, the virtual dimension $\mathrm{vdim}_D - n$ above is fixed. Therefore, there are only finite number of choices of insertions for a given n. Indeed, when $n \gg 0$, most of the insertions must be of $\deg_{\mathbb{C}} = 1$, which can be removed by divisor and dilaton equations. Therefore, only a finite number of GW invariants are needed. □

This can be considered as a weak form of *quantum Lefschetz hyperplane theorem in higher genus*.

When D is a Calabi–Yau threefold, the relation between twisted and untwisted invariants in D is especially simple.

Lemma 1.3 *Suppose D is a Calabi–Yau threefold; the twisted invariants*

$$\langle \psi^{a_1}\sigma_1, \dots, \psi^{a_n}\sigma_n\rangle^{D,\mathcal{O}^-\oplus\mathcal{O}(D)^+}_{g,n,\ell}$$

can be determined by the invariant $\langle - \rangle^D_{g,0,\ell}$

Proof Since the virtual dimension of $\overline{\mathcal{M}}_{g,n}(D,\ell)$ is n, it follows from the string equation, dilaton equation, divisor equation and the projection formula that

$$\langle \psi^{a_1}\sigma_1, \dots, \psi^{a_n}\sigma_n\rangle^{D,\mathcal{O}^-\oplus\mathcal{O}(D)^+}_{g,n,\ell} = C_{g,n,\ell}(a,\sigma)\langle - \rangle^{D,\mathcal{O}^-\oplus\mathcal{O}(D)^+}_{g,0,\ell},$$

where $C_{g,n,\ell}(a,\sigma)$ is a constant depending on g, n, ℓ, a_i and σ_i via the above equations. The explicit formula is given in Section 1.3.

It is not difficult to see that

$$\big(e_{\mathbb{C}^*}\big((\mathcal{O}\oplus\mathcal{O}(D))_{g,0,\ell}\big)\big)^{-1} = (-1)^{1-g}\lambda^{2g-2-(\ell,D)}(1+O(\lambda^{-1})).$$

Since $\dim([\overline{\mathcal{M}}_{g,0}(D,\ell)]^{\mathrm{vir}})=0$ already, the only summand that contributes nontrivially to the integral is the leading term $(-1)^{1-g}\lambda^{2g-2-(\ell,D)}$. As a result,

$$\langle - \rangle^{D,\mathcal{O}^-\oplus\mathcal{O}(D)^+}_{g,0,\ell} = (-1)^{1-g}\langle - \rangle^{D}_{g,0,\ell}\lambda^{2g-2-(\ell,D)}. \tag{2}$$

This completes the proof. □

Note that the constant $C_{g,n,\ell}$ can be made explicit. We present it in the Section 1.3. In this (CY3) case, Corollary 1.2 has an especially simple form.

Corollary 1.4 *When D is an ample* CY3 *hypersurface in X, all descendant Gromov–Witten invariants of fixed (g,n,ℓ) with $2g-2<(\ell,D)$ in D can be expressed in terms of GW invariants of X and a finite number of GW invariants of D,*

$$\langle - \rangle^{D}_{g',0,\ell'}, \quad \text{for } g'\le g,\ \ell'<\ell \text{ and } (\ell',D)\le 2g'-2.$$

In particular, when $D=Q$ is the quintic threefold in $\mathbb{P}^4$ and $g\le 3$, the GW invariants of D are completely determined.

Proof The first part follows directly from Lemma 1.3. For the quintic case, $(\ell,D)\ge 5$ for $\ell\ne 0$. and hence $2g-2<(\ell,D)$ for $g\le 3$. Therefore, the GW invariants of D are completely determined by those of $\mathbb{P}^4$, which in turn are completely determined. For example, by results of Givental [9], all higher-genus invariants of $\mathbb{P}^4$ are determined by genus-zero invariants. In genus zero, the J–function of $\mathbb{P}^4$ is known [8] and a reconstruction theorem in [19] completely determines all genus-zero (descendant) invariants from the J–function. □

Remark 1.5 Corollaries 1.2 and 1.4 are reminiscent of the holomorphic anomaly conjecture of Bershadsky, Cecotti, Ooguri and Vafa [1; 2]. The first step towards linking these two will be a generating function formulation alluded to in Remark 0.3(b) Indeed, the recent remarkable result by Guo, Janda and Ruan [10], posted while this paper was in preparation, accomplished the genus-two case via a different method.

The main ingredient in formulating and proving Theorem 1.1 is the virtual localization on the moduli of stable maps to an auxiliary $(\dim(X)+1)$–dimensional space $\mathfrak{X}$, defined in Section 2.1.

1.3 Multiple point functions of Calabi–Yau threefolds

Now let D be any Calabi–Yau threefold. As mentioned above, the virtual dimension of $\overline{\mathcal{M}}_{g,n}(D,\ell)$ is n. It follows that the cohomology class insertions must be divisors. In this case, there is a closed formula for the multiple point function in terms of 0–pointed invariants. Let H be any divisor in D. (In the context of GW theory, one often has $H=\sum_i t^i h_i$, where $\{h_i\}$ is a basis of $H^2(D)$ and $\{t^i\}$ the dual coordinates.)

Proposition 1.6 *Let D be a Calabi–Yau threefold. Fix integers $N,n,m\geq 0$ such that $N\geq n+m$. Let*

$$\sum_{\substack{k_1,\dots,k_m,\\ l_1,\dots,l_n}} \langle H\psi^{k_1},\dots,H\psi^{k_m},\psi^{l_1},\dots,\psi^{l_n},1,\dots,1\rangle^D_{g,N,d\ell}\prod_{i=1}^m x_i^{k_i}\prod_{j=1}^n y_j^{l_j}$$
$$=\sum_{p=0}^n\biggl(\prod_{r=1}^p(2g+m+r-4)\biggr)\sigma_p(y_1,\dots,y_n)\biggl(\sum_{i=1}^m x_i+\sum_{j=1}^n y_j\biggr)^{N-m-p}\biggl(\int_\ell H\biggr)^m N_{g,\ell},$$

where

$$\sigma_p(y_1,\dots,y_n)=\sum_{1\leq j_1<\dots<j_p\leq n} y_{j_1}\cdots y_{j_p}$$

is the p^{th} elementary symmetric polynomial, $\int_\ell H=(\ell,H)$ the pairing and

$$N_{g,\ell}:=\langle - \rangle^D_{g,0,\ell}.$$

Proof One can check that both sides satisfy the string, dilaton and divisor equations, which then uniquely determines the function up to $N_{g,\ell}$. □

The above formula can be used to organize invariants of the form

$$\biggl\langle \frac{A+BH}{w_1-\psi},\dots,\frac{A+BH}{w_n-\psi}\biggr\rangle_{g,n,\ell},$$

which occur frequently in the virtual localization.

Corollary 1.7 *We have*

$$\left\langle \frac{t_0 1+t_1 H}{1-z_1\psi}, \dots, \frac{t_0 1+t_1 H}{1-z_n\psi} \right\rangle^D_{g,n,d} = N_{g,d} \sum_{p+q+m=n} t_0^{p+q} \left(\int_\ell H\right)^m t_1^m \frac{(2g+m+p-4)!}{(2g+m-4)!} \cdot \frac{(m+q)!}{m!q!} \sigma_p(z_1,\dots,z_n) \times \left(\sum_{i=1}^n z_i\right)^q,$$

where m, p *and* q *are nonnegative integers in the above formula, and, by definition,*

$$\frac{(2g+m+p-4)!}{(2g+m-4)!} = \left(\prod_{r=1}^p (2g+m+r-4)\right).$$

2 Virtual localization on the master space

In this section, Theorem 1.1 is proved.

2.1 The master space and its fixed loci

2.1.1 Compactified deformation to the normal cone Define $\mathfrak{X}$ to be the *compactified deformation to the normal cone* as follows. Consider the product $X \times \mathbb{P}^1$. Pick two distinct points on $\mathbb{P}^1$ and call them 0 and ∞; then

$$\mathfrak{X} = Bl_{D\times\{0\}} X \times \mathbb{P}^1.$$

There is a birational morphism $\mathfrak{X} \to X \times \mathbb{P}^1$, and it can be composed with projections to X and $\mathbb{P}^1$. We denote the first composition by $p\colon \mathfrak{X} \to X$ and the second by $\pi\colon \mathfrak{X} \to \mathbb{P}^1$. The fiber

$$\mathfrak{X}_0 := \pi^{-1}(\{0\}) \cong X \cup_D \mathbb{P}_D(\mathcal{O} \oplus \mathcal{O}(D))$$

is the union of X and $\mathbb{P}_D(\mathcal{O} \oplus \mathcal{O}(D))$. These two pieces glue transversally along the hypersurface $D \subset X$ and the section

$$D \cong \mathbb{P}_D(\mathcal{O}(D)) \subset \mathbb{P}_D(\mathcal{O} \oplus \mathcal{O}(D)).$$

The following subvarieties of $\mathfrak{X}$ will be used frequently:

- $X_\infty := \pi^{-1}(\{\infty\}) \cong X$.
- X_0 is the irreducible component of $\mathfrak{X}_0$ which is isomorphic to X.
- $D_0 := \mathbb{P}_D(\mathcal{O}) \subset \mathbb{P}_D(\mathcal{O} \oplus \mathcal{O}(D)) \subset \mathfrak{X}_0$.

2.1.2 Fixed loci on $\mathfrak{X}$ One can put $\mathbb{C}^*$ actions on the base $\mathbb{P}^1$ fixing 0 and ∞. There are different choices and we need to fix one throughout this paper. We adopt the following convention:

Convention 3 The $\mathbb{C}^*$ action acts on the tangent space of $0 \in \mathbb{P}^1$ with weight -1.

It induces a $\mathbb{C}^*$ action on $X \times \mathbb{P}^1$ by acting trivially on the first factor X. Since $\mathfrak{X}$ is the blow-up of a fixed locus, *there is an induced* $\mathbb{C}^*$ *action on* $\mathfrak{X}$. This action acts trivially on X, but scales on the fibers of the $\mathbb{P}^1$ fibration $\mathbb{P}_D(\mathcal{O} \oplus \mathcal{O}(D))$.

Under this $\mathbb{C}^*$ action, the fixed loci are

(a) D_0,

(b) X_0,

(c) X_∞,

and their normal bundles are

(a) $N_{D_0/\mathfrak{X}} = \mathcal{O}_D(D) \oplus \mathcal{O}_D$ with the induced $\mathbb{C}^*$ action of character 1 on $\mathcal{O}_D(D)$ factor and of character -1 on $\mathcal{O}_D$ factor;

(b) $N_{X_0/\mathfrak{X}} = \mathcal{O}_X(-D)$ with the induced $\mathbb{C}^*$ action of character -1 on the fibers;

(c) $N_{X_\infty/\mathfrak{X}} = \mathcal{O}_X$ with the induced $\mathbb{C}^*$ action of character 1 on the fibers.

We introduce the following notation for curve classes:

Definition 2.1 Let $\gamma', \gamma \in \mathrm{NE}(\mathfrak{X})$ be such that

- γ denotes the pushforward of the fiber class in the $\mathbb{P}^1$ bundle $\mathbb{P}_D(\mathcal{O} \oplus \mathcal{O}(D))$;
- γ' denotes the class of the strict transform of $\{p\} \times \mathbb{P}^1 \subset X \times \mathbb{P}^1$, where $p \notin D$.

Definition 2.2
- Let $i_{X_0}\colon X_0 \to \mathfrak{X}$ be the inclusion of X_0.
- Let $i_{D_0}\colon D_0 \to \mathfrak{X}$ be the inclusion of D_0.

Lemma 2.3
$$N_1(\mathfrak{X}) = (i_{X_0})_* N_1(X) \oplus \mathbb{Z}\gamma \oplus \mathbb{Z}\gamma'.$$

The proof is straightforward.

2.2 The decorated graphs

We recall the general framework of virtual localization. Let Y be a smooth projective variety admitting an action by a torus $T = (\mathbb{C}^*)^m$. It induces an action of T on $\overline{\mathcal{M}}_{g,n}(Y, \beta)$ and on its perfect obstruction theory. Let $\overline{\mathcal{M}}_\alpha$ be the connected components of the fixed loci $\overline{\mathcal{M}}_{g,n}(Y,\beta)^T$ labeled by α with the inclusion $i_\alpha\colon \overline{\mathcal{M}}_\alpha \hookrightarrow \overline{\mathcal{M}}_{g,n}(Y,\beta)$. The virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(Y,\beta)]^{\mathrm{vir}}$ can be written as

$$[\overline{\mathcal{M}}_{g,n}(Y,\beta)]^{\mathrm{vir}} = \sum_\alpha (i_\alpha)_* \frac{[\overline{\mathcal{M}}_\alpha]^{\mathrm{vir}}}{e_T(N_\alpha^{\mathrm{vir}})},$$

where $[\overline{\mathcal{M}}_\alpha]^{\mathrm{vir}}$ is constructed from the fixed part of the restriction of the perfect obstruction theory of $\overline{\mathcal{M}}_{g,n}(Y,\beta)$, and the virtual normal bundle N_α^{vir} is the moving part of the two-term complex in the perfect obstruction theory of $\overline{\mathcal{M}}_{g,n}(Y,\beta)$ restricted to $\overline{\mathcal{M}}_\alpha$.

In the following we apply the above localization to the case $Y=\mathfrak{X}$. Moreover, we adopt the following convention:

Convention 4 We only consider $\overline{\mathcal{M}}_{g,n}(\mathfrak{X},\beta)$ such that $\beta \in (i_{X_0})_* N_1(X) \oplus \mathbb{N}\gamma$ throughout the rest of the paper, with the only exception in Section 2.5.

Consider $T=\mathbb{C}^*$ acting on X as described in Section 2.1.1. By analyzing the summands of the virtual localization formula, one can index the fixed loci of $\overline{\mathcal{M}}_{g,n}(\mathfrak{X},\beta)$ by the *decorated graphs* defined below:

Definition 2.4 A *decorated graph* $\Gamma = (\Gamma, \vec{p}, \vec{\beta}, \vec{s}, \vec{\chi})$ for a genus-g, n–pointed, degree β $\mathbb{C}^*$–invariant stable map consists of the following data:

- Γ a finite connected graph, $V(\Gamma)$ the set of vertices and $E(\Gamma)$ the set of edges;
- $F(\Gamma) = \{(e,v) \in E(\Gamma) \times V(\Gamma) \mid v \text{ incident to } e\}$ the set of flags;
- the label map $\vec{p}\colon V(\Gamma) \to \{D_0, X_0, X_\infty\}$;
- the degree map $\vec{\beta}\colon E(\Gamma) \cup V(\Gamma) \to \mathrm{NE}(\mathfrak{X})$;
- the marking map $\vec{s}\colon \{1,2,\dots,n\} \to V(\Gamma)$ for $n>0$;
- the genus map $\vec{g}\colon V(\Gamma) \to \mathbb{Z}_{\geq 0}$.

They are required to satisfy the following conditions:

- $V(\Gamma)$, $E(\Gamma)$ and $F(\Gamma)$ determine a connected graph.
- $\sum_{e\in E(\Gamma)} \vec{\beta}(e) + \sum_{v\in V(\Gamma)} \vec{\beta}(v) = \beta$.
- $\sum_{v\in V(\Gamma)} \vec{g}(v) + h^1(|\Gamma|) = g$, where $h^1(|\Gamma|)$ is the "number of loops" of the graph Γ.

To simplify the notation, we will sometimes denote $\vec{p}(v)$, $\vec{\beta}(v)$, $\vec{\beta}(e)$, $\vec{s}(i)$ and $\vec{g}(v)$ by p_v, β_v, β_e, s_i and g_v, respectively.

Let $f\colon (C, x_1, \dots, x_n) \to \mathfrak{X}$ be a $\mathbb{C}^*$ invariant stable map. We can associate a decorated graph Γ to f as follows:

Vertices

- The connected components in $f^{-1}(\mathfrak{X}^T)$ are either curves or points. Assign a vertex v to a connected component $\mathfrak{c}_v$ in $f^{-1}(\mathfrak{X}^T)$.
- Define $p_v = D_0$, X_0 or X_∞ depending on whether $f(\mathfrak{c}_v) \subset D_0$, X_0 or X_∞, respectively.
- When $\mathfrak{c}_v$ is a curve, define $\beta_v = f_*[\mathfrak{c}_v] \in N_1(\mathfrak{X})$. When $\mathfrak{c}_v$ is a point, define $\beta_v = 0$.
- When $\mathfrak{c}_v$ is a curve, define g_v to be the genus of $\mathfrak{c}_v$. When it is a point, define $g_v = 0$.
- When the i^{th} marking lies in the component $\mathfrak{c}_v$, we define $s_i = v$.

Edges

- Assign each component of $C - \bigcup_{v \in V(\Gamma)} \mathfrak{c}_v$ an edge e. Let $\mathfrak{c}_e$ be the closure of the corresponding component.
- Write $\beta_e = f_*[\mathfrak{c}_e] \in N_1(\mathfrak{X})$. Due to Convention 4, β_e is a multiple of γ. Write k_e for the integer such that $\beta_e = k_e \gamma$.

Remark 2.5 If the numerical class of $f_*([C])$ had nontrivial coefficient on ℓ', the assignment of the graph would have involved a balancing condition on the nodes. See [6] or [21] for details. This general case is not needed in this paper.

Definition 2.6 For our convenience, we introduce the following notation:

- E_v denotes the set of edges that are incident to v.
- $\mathrm{val}(v) = |E_v|$ denotes the valence of v.
- n_v is the number of markings on the vertex v.

Lemma 2.7 *Suppose $f_*([C]) = \ell \in (i_{D_0})_* N_1(D)$. If there is one $v \in V(\Gamma)$ such that $p_v = X_\infty$, then Γ is a graph with a single vertex and without any edge. The same conclusion holds if there is one i such that $p_{s_i} = X_\infty$. In particular, each graph we consider in the paper is either a trivial one over X_∞, or a bipartite graph with vertex components over D_0 and X_0.*

Proof This is due to the fact that the curve class β_Γ has no horizontal l' components and is connected. $\square$

The fixed loci of $\overline{\mathcal{M}}_{g,n}(\mathfrak{X},\beta)$ can be grouped by the decorated graphs. Denote by $\overline{\mathcal{M}}_\Gamma$ the union of the fixed components parametrizing stable maps corresponding to Γ and by N_Γ^{vir} the virtual normal bundle of $\overline{\mathcal{M}}_\Gamma$. The next goal is to make the localization residue for each $\overline{\mathcal{M}}_\Gamma$ more explicit. A few more definitions are introduced below, partially following [20, Definition 53].

Definition 2.8 A vertex $v \in V(\Gamma)$ is called *stable* if $2g_v - 2 + \mathrm{val}(v) + n_v > 0$. Let $V^S(\Gamma)$ be the set of stable vertices in $V(\Gamma)$. Let

$$\begin{aligned} V^1(\Gamma) &= \{v \in V(\Gamma) \mid g_v = 0,\ \mathrm{val}(v) = 1,\ n_v = 0\}, \\ V^{1,1}(\Gamma) &= \{v \in V(\Gamma) \mid g_v = 0,\ \mathrm{val}(v) = n_v = 1\}, \\ V^2(\Gamma) &= \{v \in V(\Gamma) \mid g_v = 0,\ \mathrm{val}(v) = 2,\ n_v = 0\}. \end{aligned}$$

The union of $V^1(\Gamma)$, $V^{1,1}(\Gamma)$ and $V^2(\Gamma)$ is the set of *unstable* vertices.

Definition 2.9 Define an equivalence relation $\sim$ on the set $E(\Gamma)$ by setting $e_1 \sim e_2$ if there is a $v \in V^2(\Gamma)$ such that $e_1, e_2 \in E_v$. Let $\bar{E}(\Gamma) := E/{\sim}$.

One easily sees that a class $[e] \in \bar{E}(\Gamma)$ consists of a chain of edges, say $e_1, e_2, \ldots, e_m$, such that e_i and e_{i+1} intersect at a $v_i \in V^2(\Gamma)$. There are also two vertices $v_0 \in e_1$ and $v_m \in e_m$ such that $v_0, v_m \notin V^2(\Gamma)$.

Definition 2.10 Define $V_{[e]}^{\mathrm{in}} = \{v_1, \ldots, v_{m-1}\}$ and $V_{[e]}^{\mathrm{end}} = \{v_0, v_m\}$.

Definition 2.11 Define $\bar{E}^{\mathrm{leg}}(\Gamma)$ to be the set of edge classes $[e] \in \bar{E}(\Gamma)$ such that $V_{[e]}^{\mathrm{end}} \bigcap V^1(\Gamma) \neq \varnothing$ or $V_{[e]}^{\mathrm{end}} \bigcap V^{1,1}(\Gamma) \neq \varnothing$.

Definition 2.12 $V^{D_0}(\Gamma) := \{v \in V(\Gamma) \mid p_v = D_0\}$ and $V^P(\Gamma) := \{v \in V(\Gamma) \mid p_v = X_0\}$.

Definition 2.13 Define $V_{[e]}^{D_0,S} = V^{D_0}(\Gamma) \cap \bigl(V_{[e]}^{\mathrm{in}} \cup (V_{[e]}^{\mathrm{end}} \cap V^S(\Gamma))\bigr)$. In other words, they are all the D_0 vertices on the chain $[e]$ except for unstable ones at the two ends. Similarly, we define $V_{[e]}^{X_0,S} = V^{X_0}(\Gamma) \cap \bigl(V_{[e]}^{\mathrm{in}} \cup (V_{[e]}^{\mathrm{end}} \cap V^S(\Gamma))\bigr)$.

Definitions 2.8–2.13 are artificially introduced mostly because we want to label certain terms in the virtual localization formula later.

2.3 Recursion

In this subsection we derive the recursion formula which yields Theorem 1.1. The idea is to consider equivariant Gromov–Witten invariants on X which on the one hand vanish for dimension reasons, and on the other hand provide relations among invariants on D and (twisted) invariants on X thanks to the localization formula. Here we note that each fixed locus in $\mathfrak{X}$ is isomorphic to either D or X. Our claim is that, by carefully choosing the ingredients, one can obtain a set of relations which completely determines invariants on D from those on X.

Let $\ell \in \mathrm{NE}(D)$ be an effective curve class. Recall that

$$\mathrm{vdim}_D = \mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(D,\ell)) = (1-g)(\dim(D)-3) + (\ell, -K_D) + n.$$

We can similarly denote

$$\mathrm{vdim}_{\mathfrak{X}} = \mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(\mathfrak{X}, (i_{D_0})_*\ell)) = (1-g)(\dim(\mathfrak{X})-3) + (\ell, -K_D) + (\ell, D) + n.$$

Let $\alpha_1, \dots, \alpha_n \in H^*(D)_X$ be cohomology classes and $a_1, \dots, a_n \in \mathbb{N}$ be nonnegative integers such that $\sum_{i=1}^n (a_i + \deg(\alpha_i)) = \mathrm{vdim}_D$. By pulling back from X, there are liftings of cohomology classes $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_n \in H^*_{\mathbb{C}^*}(\mathfrak{X})$ such that $i^*_{D_0}\widetilde{\alpha}_i = \alpha_i$. Fix such a lifting and consider the invariant

$$\langle \psi^{a_1}\widetilde{\alpha}_1, \dots, \psi^{a_n}\widetilde{\alpha}_n \rangle^{\mathfrak{X}}_{g,n,(i_{D_0})_*\ell}. \tag{3}$$

For virtual dimension reasons, we have:

Lemma 2.14 *If $2(g-1) < (\ell, D)$, then $\langle \psi^{a_1}\widetilde{\alpha}_1, \dots, \psi^{a_n}\widetilde{\alpha}_n \rangle^{\mathfrak{X}}_{g,n,(i_{D_0})_*\ell} = 0$.*

The virtual localization formula states that

$$\langle \psi^{a_1}\widetilde{\alpha}_1, \dots, \psi^{a_n}\widetilde{\alpha}_n \rangle^{\mathfrak{X}}_{g,n,(i_{D_0})_*\ell} = \sum_{\Gamma} \mathrm{Cont}_\Gamma, \tag{4}$$

where Cont_Γ is the localization residue associated with the fixed locus M_Γ whose closed points are invariant stable maps assigned corresponding to the graph Γ. We can separate graphs with 1 vertex and graphs with more than 1 vertex:

$$\langle \psi^{a_1}\widetilde{\alpha}_1, \dots, \psi^{a_n}\widetilde{\alpha}_n \rangle^{\mathfrak{X}}_{g,n,(i_{D_0})_*\ell} = \sum_{|V(\Gamma)|=1} \mathrm{Cont}_\Gamma + \sum_{|V(\Gamma)|>1} \mathrm{Cont}_\Gamma.$$

This formula can be made more explicit as follows:

$$(5)\quad \langle \psi^{a_1}\widetilde{\alpha}_1, \dots, \psi^{a_n}\widetilde{\alpha}_n \rangle^{\mathfrak{X}}_{g,n,(i_{D_0})_*\ell} = \langle \psi^{a_1}\alpha_1, \dots, \psi^{a_n}\alpha_n \rangle^{D,\mathcal{O}^-\oplus\mathcal{O}(D)^+}_{g,n,\ell} + \langle \psi^{a_1}\widetilde{\alpha}_1|_{X_\infty}, \dots, \psi^{a_n}\widetilde{\alpha}_n|_{X_\infty} \rangle^{X,\mathcal{O}^+}_{g,n,\ell} + \sum_{|V(\Gamma)|>1} \frac{1}{\mathrm{Aut}(\Gamma)} \Bigg(\sum_{\{i_{[e]}\}_{[e]\in\overline{E}(\Gamma)\setminus\overline{E}^{\mathrm{leg}}(\Gamma)}} \prod_{v\in V^S(\Gamma)} \langle \cdots \rangle^{p_v,\mathcal{E}_v}_{g_v,\mathrm{val}(v),\beta_v} \Bigg),$$

where the sum ranges over all decorated graphs of genus g and degree ℓ with n markings. A few explanations about the notation are in order:

(a) In "$\langle \cdots \rangle^{p_v,\mathcal{E}_v}_{g_v,\mathrm{val}(v),\beta_v}$", when $p_v = D_0$, the twisting bundle $\mathcal{E}_v = \mathcal{O}^- \oplus \mathcal{O}(D)^+$. When $p_v = X_0$, $\mathcal{E}_v = \mathcal{O}(-D)^-$. Then p_v won't be X_∞ in this summation by Lemma 2.7.

(b) Let $\{T_i\}$ be a $\mathbb{C}$–basis of $H^*(D;\mathbb{C})$ and $\{T^i\}$ the dual basis. In the summation, each $i_{[e]}$ in $\{i_{[e]}\}_{e\in\overline{E}(\Gamma)-\overline{E}^{\mathrm{leg}}(\Gamma)}$ determines an element $T_{i_{[e]}}$ in the basis $\{T_i\}$. And we run over all basis for each edge class in $\overline{E}(\Gamma) - \overline{E}^{\mathrm{leg}}(\Gamma)$.

(c) $\langle \cdots \rangle^{p_v,\mathcal{E}_v}_{g_v,\mathrm{val}(v),\beta_v}$ is formulated according to the following rules. For the i^{th} marking, the class $\psi^{a_i}\alpha_i|_{p_{s_i}}$ is inserted into $\langle \cdots \rangle^{p_{s_i},\mathcal{E}_{s_i}}_{g_{s_i},\mathrm{val}(s_i),\beta_{s_i}}$ as long as $s_i \notin V^{1,1}(\Gamma)$. For each $[e] \in \overline{E}(\Gamma)$, insertions are added according to the rules below.

Notation

- For any $[e] \in \overline{E}(\Gamma)$, let v_+ and v_- be the two vertices in V_e^{end} (whichever are v_+ and v_- is arbitrary).
- Define ι_+ to be the inclusion $\iota_+\colon D \to X$ if $p_{v_+} = X_0$, or the identity map $\iota_+\colon D \to D$ if $p_{v_+} = D_0$. Then ι_- is defined in the same way according to p_{v_-}.
- Let e_+ and e_- be the edges in the class $[e]$ that contain v_+ and v_-, respectively.
- For a vertex v, define δ_v to be 1 if p_v is D_0, and -1 if it is X_0.

Rules For each class $[e] \in \overline{E}(\Gamma)$, an insertion is added into the $\langle \cdots \rangle^{p_{v_+},\mathcal{E}_{v_+}}_{g_{v_+},\mathrm{val}(v_+),\beta_{v_+}}$ and $\langle \cdots \rangle^{p_{v_-},\mathcal{E}_{v_-}}_{g_{v_-},\mathrm{val}(v_-),\beta_{v_-}}$ factors for each v_+ and v_- as long as it is a stable vertex. These insertions are explicitly described below:

- Suppose one of v_+ and v_- is in $V^1(\Gamma)$. Say $v_- \in V^1(\Gamma)$. Then

$$(\iota_+)_*\bigg(\frac{1}{k_{e_-}} \cdot \frac{\mathrm{Edge}(\Gamma,[e])}{\delta_{v_+}(\lambda + D)/k_{e_+} - \psi} \bigg)$$

is inserted into the summand $\langle\cdots\rangle^{p_{v_+},\mathcal{E}_{v_+}}_{g_{v_+},\mathrm{val}(v_+),\beta_{v_+}}$.

- Suppose one of v_+ and v_- is in $V^{1,1}(\Gamma)$. Say $v_-\in V^1(\Gamma)$ and the marking on v_- is the i^{th} marking of Γ. Then

$$(\iota_+)_*\left(\frac{(\delta_{v_-}(\lambda+D)/k_{e_-})^{a_i}\alpha_i|_{p_{s_i}}}{\delta_{v_-}(\lambda+D)}\cdot\frac{\mathrm{Edge}(\Gamma,[e])}{\delta_{v_+}(\lambda+D)/k_{e_+}-\psi}\right)$$

is inserted into the summand $\langle\cdots\rangle^{p_{v_+},\mathcal{E}_{v_+}}_{g_{v_+},\mathrm{val}(v_+),\beta_{v_+}}$.

- Suppose otherwise. An insertion

$$(\iota_+)_*\left(\frac{\mathrm{Edge}(\Gamma,[e])T_{i_{[e]}}}{\delta_{v_+}(\lambda+D)/k_{e_+}-\psi}\right)$$

should be placed in the summand $\langle\cdots\rangle^{p_{v_+},\mathcal{E}_{v_+}}_{g_{v_+},\mathrm{val}(v_+),\beta_{v_+}}$. In the meantime an insertion

$$(\iota_-)_*\left(\frac{T^{i_{[e]}}}{\delta_{v_-}(\lambda+D)/k_{e_-}-\psi}\right)$$

should be placed in the $\langle\cdots\rangle^{p_{v_-},\mathcal{E}_{v_+}}_{g_{v_-},\mathrm{val}(v_-),\beta_{v_-}}$ summand.

This $\mathrm{Edge}(\Gamma,[e])$ can be computed in our case:

$$\mathrm{Edge}(\Gamma,[e])=\frac{1}{k_e}\frac{(-\lambda)^{|V^{D_0,S}_{[e]}|}D^{|V^{X_0,S}_{[e]}|}}{\prod_{v\in V^{\mathrm{in}}_{[e]}}\left(\sum_{e'\in E_v}\delta_v(\lambda+D)^2/k_{e'}\right)}\times\prod_{e\in[e]}\frac{\prod_{m=1}^{k_e-1}(-\lambda+(m/k_e)(\lambda+D))}{(-1)^{k_e-1}\left[\prod_{m=1}^{k_e-1}(m/k_e)(\lambda+D)\right]^2}.$$

2.3.1 Conclusion of proof When $|V(\Gamma)|=1$, the two possible graphs are $p_v=D_0$ and $p_v=X_\infty$ where v is the unique element in $V(\Gamma)$. These two cases correspond to the two summands in the first line of the right-hand side of (5),

$$\langle\psi^{a_1}\alpha_1,\dots,\psi^{a_n}\alpha_n\rangle^{D,\mathcal{O}^-\oplus\mathcal{O}(D)^+}_{g,n,\ell}+\langle\psi^{a_1}\tilde{\alpha}_1|_{X_\infty},\dots,\psi^{a_n}\tilde{\alpha}_n|_{X_\infty}\rangle^{X,\mathcal{O}^+}_{g,n,\ell}.$$

To obtain the form of Theorem 1.1, we notice that

$$\langle\psi^{a_1}\alpha_1,\dots,\psi^{a_n}\alpha_n\rangle^{D,\mathcal{O}^-\oplus\mathcal{O}(D)^+}_{g,n,\ell}=(-1)^{1-g}\langle\psi^{a_1}\alpha_1,\dots,\psi^{a_n}\alpha_n\rangle^{D}_{g,n,\ell}\lambda^{-2+2g-(\ell,D)}.$$

The reason is the following. As part of the assumption, we already have

$$\sum_{i=1}^n(a_i+\deg(\alpha_i))=\mathrm{vdim}_D.$$

We know the twisting satisfies

$$\frac{1}{e_{\mathbb{C}^*}(\mathcal{O}_{g,n,\ell} \oplus \mathcal{O}(D)_{g,n,\ell})} = \lambda^{-2+2g-(\ell,D)} + \cdots .$$

But there is no room for the lower-degree terms to come into play. In Equation (5), one can easily verify that all other terms are lower-order terms, as described in Theorem 1.1. This concludes the proof of Theorem 1.1.

2.4 An example

Let D be $\mathbb{P}^1$ and X be $\mathbb{P}^2$. D embeds into X as a line. Consider ℓ to be the degree 1 class in D. Applying (5), we get

$$\langle - \rangle^{D}_{0,0,1} \lambda^{-3} = \langle - \rangle^{\mathbb{P}^2, \mathcal{O}^+}_{0,0,1} + \text{(lower order terms)}. \tag{6}$$

Denote the hyperplane class in $\mathbb{P}^2$ by H. One easily finds out that the lower-order terms are of the form

$$\left\langle \frac{H}{-\lambda - \psi - H} \right\rangle^{\mathbb{P}^2, \mathcal{O}(-1)^-}_{0,1,1}. \tag{7}$$

Expand the geometric series as

$$\frac{H}{-\lambda - \psi - H} = -\frac{H}{\lambda}\left(1 - \frac{\psi + H}{\lambda} + \left(\frac{\psi + H}{\lambda}\right)^2 + \cdots\right).$$

By counting the virtual dimension, the only nonzero term is

$$-\left\langle \frac{H(\psi + H)^2}{\lambda^3} \right\rangle^{\mathbb{P}^2, \mathcal{O}(-1)^-}_{0,1,1}.$$

One expands to get

$$-\left\langle \frac{H\psi^2 + 2H^2\psi}{\lambda^3} \right\rangle^{\mathbb{P}^2, \mathcal{O}(-1)^-}_{0,1,1}.$$

Since $h^0(\mathbb{P}^1, \mathcal{O}(-1)) = h^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, the above is nothing but an untwisted invariant. One expands to get

$$-\langle H\psi^2 \rangle^{\mathbb{P}^2}_{0,1,1} \lambda^{-3} - 2\langle H^2\psi \rangle^{\mathbb{P}^2}_{0,1,1} \lambda^{-3}.$$

Writing out the small J–function for $\mathbb{P}^2$, one gets

$$\langle H\psi^2 \rangle^{\mathbb{P}^2}_{0,1,1} = -3, \quad \langle H^2\psi \rangle^{\mathbb{P}^2}_{0,1,1} = 1.$$

Plugging everything into (6), one computes $\langle - \rangle^{D}_{0,0,1} = 1$, which is exactly what we expect.

2.5 A remark on the degree bound

This subsection summarizes joint work with Ernst Schulte-Geers.

As mentioned in the introduction, our results on the quantum Lefschetz are subjected to a degree bound (1) which occurs due to dimensional consideration in Lemma 2.14. One might be hopeful that the degree bound can be removed if we allow more general curve classes in the localization calculation in this section. We worked on the quintic threefold case and obtained negative results, which are briefly presented below. The interested reader can find more details in [7, Appendix].

Let D be the quintic 3–fold in $X = \mathbb{P}^4$. A general curve class in $\mathfrak{X}$ can be written as $\beta = dl + K\gamma + d_3\gamma'$, where l here is the line class in $\mathbb{P}^4$. We first consider the case $\beta = dl + K\gamma$. In this case, we have $\operatorname{vdim}(\overline{\mathcal{M}}_{g,0}(\mathfrak{X}, dl + K\gamma)) = 5d + 2 - 2g + K$. Therefore, when $5d + 2 - 2g + K > 0$,

$$\int_{[\overline{\mathcal{M}}_{g,0}(\mathfrak{X}, l+K\gamma)]^{\mathrm{vir}}} 1 = 0.$$

When g is fixed and K is sufficiently large, the vanishing will hold for any $d \geq 0$. Now we evaluate it using localization, and we focus on the leading term involving $\langle - \rangle^D_{g,0,d}$.

Definition 2.15 Let $\mathcal{G}$ be the set of degree $dl + K\gamma$, genus-g, 0–marked graphs with a vertex of degree d and genus g over D_0 (see Section 2.1.2).

Similar to (4), we have

$$0 = \langle - \rangle^{\mathfrak{X}}_{g,0,dl+K\gamma} = \sum_{\Gamma \in \mathcal{G}} \mathrm{Cont}_\Gamma + \sum_{\Gamma \notin \mathcal{G}} \mathrm{Cont}_\Gamma. \tag{8}$$

In this, $\sum_{\Gamma \in \mathcal{G}} \mathrm{Cont}_\Gamma$ can be explicitly evaluated via Proposition 1.6:

Theorem 2.16 *We have*

$$\sum_{\Gamma \in \mathcal{G}} \mathrm{Cont}_\Gamma = [(1-z)^{2g-2-5d}]_{z^K} \langle - \rangle^D_{g,0,dl} \lambda^{2g-2-5d-K},$$

where $[f(z)]_{z^K}$ is the coefficient of z^K in $f(z)$.

This follows from [7, Theorem A.7]. As a result, when $2g - 2 - 5d - K < 0$, the *leading coefficient* is 0. Hence, this does not give additional recursion relation outside the degree bounds.

Finally, for a general curve class $\beta = dl + K\gamma + d_3\gamma'$, one can perform similar calculations. Let C_{g,d,K,d_3} be the leading coefficient coming from $\sum_{\Gamma\in\mathcal{G}} \mathrm{Cont}_\Gamma$. Extensive numerical checks suggest that

$$C_{g,d,K,d_3} = (-1)^{d_3}\binom{K+d_3}{K} C_{g,d,K+d_3,0}.$$

Therefore, the leading coefficient also vanishes outside the degree bounds and no new recursion is obtained.

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