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(Log-)epiperimetric inequality and regularity over smooth cones for almost area-minimizing currents

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We prove a new logarithmic epiperimetric inequality for multiplicity-one stationary cones with isolated singularity by flowing any given trace in the radial direction along appropriately chosen directions. In contrast to previous epiperimetric inequalities for minimal surfaces (eg work of Reifenberg, Taylor and White), we need no a priori assumptions on the structure of the cone (eg integrability). If the cone is integrable (not only through rotations), we recover the classical epiperimetric inequality. As a consequence we deduce a new regularity result for almost area-minimizing currents at singular points where at least one blowup is a multiplicity-one cone with isolated singularity. This result is similar to the one for stationary varifolds of Leon Simon (1983), but independent from it since almost-minimizers do not satisfy any equation.

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1 Introduction

In this paper we prove a new (log-)epiperimetric inequality for multiplicity-one smooth minimal cones. To give the precise statement, we recall the notion of spherical graph over a cone and of integrability. Let $\boldsymbol{C} \subset \mathbb{R}^{n+k}$ be a multiplicity-one stationary cone and suppose that $\Sigma := \boldsymbol{C} \cap \partial B_1$ is a smooth embedded compact submanifold of ∂B_1. Given a function $u \in C^{1,\alpha}(\boldsymbol{C}, \boldsymbol{C}^\perp)$, where $\boldsymbol{C}^\perp$ denotes the normal bundle over $\boldsymbol{C}$, we define its *spherical graph* over $\boldsymbol{C}$, in polar coordinates, and its *renormalized volume* to be, respectively,

$$\boldsymbol{G}_{\boldsymbol{C}}(u) := \left\{ r\,\frac{r\theta + u(r,\theta)}{\sqrt{r^2 + |u(r,\theta)|^2}} : r\theta \in \boldsymbol{C} \right\} \quad \text{and} \quad \mathcal{A}_{\boldsymbol{C}}(u) := \mathcal{H}^n(\boldsymbol{G}_{\boldsymbol{C}}(u)) - \mathcal{H}^n(\boldsymbol{C} \cap B_1).$$

Given a cone $\boldsymbol{C}$, we say that $\boldsymbol{C}$ is *integrable* if every Jacobi field on $\boldsymbol{C}$ is generated by a one-parameter family of minimal cones, that is, if for every 1–homogeneous

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solution, ϕ, of the second variation $\delta^2 \mathcal{A}_{\boldsymbol{C}}(0)$, there exists a 1–parameter family $(\Phi_t)_{|t|<1}$ of diffeomorphisms such that $\Phi_0 = \mathrm{Id}$, $d\Phi_t/dt = \phi(\Phi_t)$ and

$$\Phi_t(\boldsymbol{C}) \text{ is a minimal cone with } \mathrm{Sing}(\Phi_t(\boldsymbol{C})) = \{0\} \text{ for every } |t| < 1. \tag{1-1}$$

The (log-)epiperimetric inequality then says, roughly, that stationary cones are quantitatively isolated (as measured by $\mathcal{A}_{\boldsymbol{C}}$) in the space of cones:

Theorem 1.1 ((log-)epiperimetric inequality for multiplicity-one smooth cones) *Let $\boldsymbol{C} \subset \mathbb{R}^{n+k}$ be an n–dimensional multiplicity-one stationary cone. There exist constants $\varepsilon, \delta > 0$ and $\gamma \in [0,1)$ depending on the dimension and $\boldsymbol{C}$ such that the following holds. Let $c \in C^{1,\alpha}(\Sigma, \boldsymbol{C}^\perp)$ be such that $\|c\|_{C^{1,\alpha}} \le \delta$; then there exists a function $h \in H^1(\boldsymbol{C} \cap B_1, \boldsymbol{C}^\perp)$ such that $h|_{\partial B_1} = c$ and*

$$\mathcal{A}_{\boldsymbol{C}}(h) \le (1 - \varepsilon |\mathcal{A}_{\boldsymbol{C}}(z)|^\gamma) \mathcal{A}_{\boldsymbol{C}}(z), \tag{1-2}$$

*where $z(x) := |x| c(x/|x|)$ is the one-homogeneous extension of c. If the cone $\boldsymbol{C}$ is **integrable**, then we can take $\gamma = 0$.*

An epiperimetric inequality (ie (1-2) with $\gamma = 0$) was first proven for regular points in the celebrated work of Reifenberg [10], and later extended to branch points of 2–dimensional area-minimizing currents by White [16] and to singular points of 2–dimensional area-minimizing flat chains modulo 3 and $(\boldsymbol{M}, \varepsilon, \delta)$–minimizers by Taylor [14; 15]. In all these situations, the admissible blowups are cones which are *integrable through rotations* (see Remark 1.3). However, there exist cones with isolated singularities which are not integrable and for which the rate of blowup has logarithmic decay (see Nagura [9] and Adams and Simon [1, Remarks 5.3 and 5.4]). Since a (classical) epiperimetric inequality implies an exponential rate of decay, we cannot hope that (1-2) with $\gamma = 0$ holds for all cones. Instead we prove what is called a (log-)epiperimetric inequality, that is, (1-2), with $\gamma \in [0,1)$.

We remark that (log-)epiperimetric inequalities were introduced by the second and third authors with Maria Colombo in the context of the obstacle and thin-obstacle problems [5; 4]; however, the proof in that setting is substantially different (and simpler). The proof of Theorem 1.1 bears more similarity to our recent work on isolated singularities of the Alt–Caffarelli functional [7]. This method seems to be very flexible and we hope to apply it to other problems (for example Yang–Mills) and to the more difficult case of higher-order singularities.

Remark 1.2 The final steps of the proof of Theorem 1.1 are inspired by the beautiful work of Simon [12], where the author proved uniqueness of blowup at singularities of stationary varifolds in which at least one blowup is a multiplicity-one cone with isolated singularity. A similar approach for generic singularities of the mean curvature flow, but with an entirely new proof of an infinite-dimensional Łojasiewicz inequality, has recently been given by Colding and Minicozzi (see [3]). However, our approach doesn't need the surface to satisfy any PDE and is purely variational, thus allowing us to deal with almost-minimizers.

Remark 1.3 Recall that a cone $\boldsymbol{C}$ is *integrable through rotation* if the family $(\Phi_t)_{|t|<1}$ in (1-1) is given by $\Phi_t = \exp(tA)$, where A is any fixed $n \times n$ skew symmetric matrix. We observe that a simple modification of White's proof of the epiperimetric inequality for 2–dimensional area-minimizing cones (see [16]) would establish an epiperimetric inequality for multiplicity-one cones with isolated singularity that are integrable through rotations. However, our proof of Theorem 1.1 is different than that of White [16] and Taylor [15; 14], and allows us to assume the more general notion of integrability (1-1), under which no epiperimetric inequality exists in the literature. In particular, this allows us to give an alternative proof of the beautiful work of Allard and Almgren [2].

As a consequence of Theorem 1.1, we prove a new uniqueness of the blowup result for almost area-minimizing currents. This result is similar to the one of Simon for stationary varifolds (see [12]), however, as mentioned above, the two results are independent from each other since stationarity and almost-minimality are independent properties. We use here standard notations for integral currents (see for instance Simon [13]).

Definition 1.4 (almost-minimizers) An n–dimensional integer rectifiable current T in $\mathbb{R}^{n+k}$ is *almost (area-)minimizing* if for every $x_0 \in \operatorname{spt}(\partial T)$ there are constants $C_0, r_0, \alpha_0 > 0$ such that

$$\|T\|(B_r(x_0)) \le \|T + \partial S\|(B_r(x_0)) + C_0 r^{n+\alpha_0} \tag{1-3}$$

for all $0 < r < r_0$ and all integral $(n+1)$–dimensional currents S supported in $B_r(x_0)$.

For any given integer rectifiable current $R \in \boldsymbol{I}_n(\mathbb{R}^{n+k})$ we define the flat norm of R to be

$$\mathcal{F}(R) := \inf\{\boldsymbol{M}(Z) + \boldsymbol{M}(W) : Z \in \boldsymbol{I}_n,\ W \in \boldsymbol{I}_{n+1}, Z + \partial W = R\}.$$

Theorem 1.5 (uniqueness of smooth tangent cone for almost-minimizers) *Let $T \in \boldsymbol{I}_n$ be an almost area-minimizing current and let $x_0 \in \operatorname{spt}(T)$. Suppose that there exists a* *multiplicity-one area-minimizing cone $\boldsymbol{C}$ such that $\boldsymbol{C} \cap \partial B_1$ is a smooth embedded orientable submanifold of ∂B_1 and $\boldsymbol{C}$ is a blowup of T at x_0. Then $\boldsymbol{C}$ is the* ***unique blowup*** *of T at x_0 and there exist constants $\gamma \in (0,1)$ and $C, r_0 > 0$, depending on $\boldsymbol{C}$ and n, such that*

$$\mathcal{F}((T-\boldsymbol{C}) \llcorner B_r) \le C\Big(-\log\Big(\frac{r}{r_0}\Big)\Big)^{(\gamma-1)/2\gamma}, \quad 0 < r < r_0, \tag{1-4}$$

$$\operatorname{dist}\big(\operatorname{spt}(T \llcorner \boldsymbol{B}_r(x)), \boldsymbol{C}\big) \le C\Big(-\log\Big(\frac{r}{r_0}\Big)\Big)^{(\gamma-1)/2\gamma}, \quad 0 < r < r_0. \tag{1-5}$$

If the cone $\boldsymbol{C}$ is integrable, then the above logarithms can be replaced by powers of r/r_0.

Similar results for almost area-minimizers are the one of Taylor [15] and of the second author together with De Lellis and Spadaro [6]. However, there are two additional difficulties in our situations. First of all, our epiperimetric inequality is logarithmic and not a classical one, since the cone is not assumed to be integrable. Secondly, in both [15; 6] the admissible blowups are rotations of a fixed cone, so that one can assume, through a simple compactness argument, that (1-2) holds at every scale. However, we do not require this to be the case, and in fact we ask for only one of the possible blowups to have the required structure.

We also stress that the combined works of Allard–Almgren and Simon (eg [2; 12]) prove the analogue of Theorem 1.5 for multiplicity-one stationary varifolds. However, their proofs do not apply to almost-minimizers as they require a PDE to be satisfied. Moreover, our approach unifies the situations of integrability and nonintegrability of the cone; this relationship is investigated in Section 2.4.

The following corollary is a consequence of Theorem 1.5, since in codimension 1 the multiplicity-one assumption on the blowup is always guaranteed:

Corollary 1.6 (uniqueness for 7–dimensional hypersurfaces) *Suppose that $T \in \boldsymbol{I}_7(U)$ is almost area-minimizing in an open set $U \subset N$, where N is a C^2 orientable smooth manifold of dimension 8 with $(\overline{N} \setminus N) \cap U = \varnothing$. Then T has a unique tangent cone at every point and is locally $C^{1,\log}$ diffeomorphic to it.*

1.1 Idea of the proof of Theorem 1.1

Let z be the function of Theorem 1.1, that is, the one-homogenous extension of the trace c. We need to construct a competitor function h whose volume is smaller than that of z. Our first step is a slicing lemma (Lemma 2.2), which says that for every $g \in C^{1,\alpha}(\boldsymbol{C}, \boldsymbol{C}^{\perp})$ we have

$$\text{(1-6)}\quad \mathcal{A}_{\boldsymbol{C}}(rg) - \mathcal{A}_{\boldsymbol{C}}(rc) \le \int_0^1 (\mathcal{A}_\Sigma(g) - \mathcal{A}_\Sigma(c)) r^{n-1}\,dr + C \underbrace{\int_0^1 \int_\Sigma |\partial_r g|^2 \, d\mathcal{H}^{n-1}\, r^{n+1}\,dr}_{=:E_r},$$

where $\mathcal{A}_\Sigma$ is the renormalized area on the sphere defined in (2-1). In order to gain in the first term, we build h by "flowing" c along r, so that the area of its spherical slices is decreasing. To choose good directions for the flow we use the Jacobi operator for $\mathcal{A}_\Sigma$, which we denote by $\delta^2\mathcal{A}_\Sigma$. This is an operator with compact resolvent, therefore we can decompose c as

$$c = c_{\boldsymbol{K}} + c_+ + c_-,$$

where $c_{\boldsymbol{K}}$ is the projection of c on the kernel of $\delta^2\mathcal{A}_\Sigma$, c_- is the projection on the index of $\delta^2\mathcal{A}_\Sigma$ and c_+ is the projection on the positive eigenspaces of $\delta^2\mathcal{A}_\Sigma$. Since Σ is stationary in the sphere (being the trace of a stationary cone), the positive directions increase the volume of Σ at second order, and so we want to move c towards zero in these directions, while the negative directions decrease it, and so we don't want to move them. In general, we cannot assume that any of $c_{\boldsymbol{K}}$, c_+ or c_- is zero, but to better explain the argument, let us address the two opposing cases, when $c_{\boldsymbol{K}} = 0$ and when $c_+ + c_- = 0$.

If $c_{\boldsymbol{K}} = 0$, we define

$$h(r,\theta) := r\eta_+(r)c_+(\theta) + rc_-(\theta),$$

for a suitably chosen function η_+, with $\eta_+' = \varepsilon$. Then, using (1-6), we have

$$\mathcal{A}_{\boldsymbol{C}}(h) - (1-\varepsilon)\mathcal{A}_{\boldsymbol{C}}(z) \le (\varepsilon(-\lambda_+ + \lambda_-) + C\|c\|_{C^{1,\alpha}(\Sigma)} + C\varepsilon^2)\|c\|^2_{H^1(\Sigma)} < 0,$$

where $\lambda_+ > 0$ is the smallest positive eigenvalue of $\delta^2\mathcal{A}_\Sigma$ and $\lambda_- < 0$ the biggest negative eigenvalue, and ε depends only on the dimension and the spectral gap, and so on $\boldsymbol{C}$. Note that the first term on the right-hand side above comes from our choice of η_+ and the aforementioned properties of the positive and negative eigenspaces of $\delta^2\mathcal{A}_\Sigma$. The second term on the right-hand side comes from the Taylor expansion of the area, while the third bounds the radial error coming from (1-6).

When $\boldsymbol{C}$ is integrable through rotations we can take $c_{\boldsymbol{K}} = 0$ by a simple reparametrization (using for instance the implicit function theorem as in White [16]). In the more general setting of integrability, we can also take $c_K = 0$, but we must use a slightly more complicated Lyapunov–Schmidt reduction and the analyticity of the area functional over graphs (see Section 2.4).

If $c = c_K$, we cannot hope to gain to second order as above. Instead, following Simon [12], we consider the function $A(\mu_1, \dots, \mu_l) := \mathcal{A}_\Sigma(\mu_1\phi_1 + \cdots + \mu_l\phi_l)$, where $\phi_1, \dots, \phi_l$ are the Jacobi fields of Σ and $l := \dim\ker(\delta^2\mathcal{A}_\Sigma(0)) < \infty$. To decrease this quantity we let the coordinates $\mu = (\mu_1, \dots, \mu_l)$ flow according to the negative gradient flow of A (that is, a finite-dimensional mean curvature flow) in the following way:

$$\begin{cases} \mu'(t) := -\nabla A(\mu(t))/|\nabla A(\mu(t))|, \\ \mu(0) = \mu^0 = \text{coordinates of } c_K, \end{cases} \tag{1-7}$$

and we define

$$h(r,\theta) := r\sum_{j=1}^{\ell} \mu_j(\eta(r))\phi_j(\theta).$$

Clearly the function $r \mapsto A(\mu(r))$ is decreasing, but to make it quantitative we use the Łojasiewicz inequality to deduce that for some $\gamma \in (0,1)$ and constant $C_{\mathbf{C}} > 0$ we have

$$A(\mu(\eta(r))) - (1-\varepsilon)A(\mu^0) \le -(C_{\mathbf{C}}\eta(r) - \varepsilon A(\mu^0)^\gamma)A(\mu^0)^{1-\gamma}.$$

If we choose η and ε cleverly (both proportional to a small constant times $A(\mu^0)^{1-\gamma}$), then the gain above will be larger than the radial error caused by the flow, which according to (1-6) is proportional to $[\eta'(r)]^2$. This in turn will imply the logarithmic epiperimetric inequality (1-2). □

Organization of the paper

The paper is divided in two parts: In the first part we give the proof of Theorem 1.1. In the second, we show how to use Theorem 1.1 to deduce Theorem 1.5. Finally, in the appendix (for the sake of completeness), we compute the Taylor expansion of the area for spherical graphs and construct the Lyapunov–Schmidt reduction. Let us point out that the proof of Theorem 1.1 requires no familiarity with the language of currents. However, when we apply the epiperimetric inequality to obtain regularity, we will use some theorems and notations which are standard in the literature. For an introduction to currents and their relevant properties, see [13].

2 The (log-)epiperimetric inequality via deformations along positive directions and gradient flow

In this section we first recall some basic notations and facts about the area functional for spherical graphs. After that we give the proof of Theorem 1.1.

2.1 Preliminaries

Let $\Sigma := \boldsymbol{C} \cap \partial B_1$ be a smooth embedded submanifold of ∂B_1. Given a function $u \in C^1(\Sigma, \boldsymbol{C}^\perp)$, we define its spherical graph over Σ and its (renormalized) volume to be, respectively,

$$\boldsymbol{G}_\Sigma(u) := \left\{ \frac{\theta + u(\theta)}{\sqrt{1+|u(\theta)|^2}} : \theta \in \Sigma \right\}, \quad \mathcal{A}_\Sigma(u) := \mathcal{H}^{n-1}(\boldsymbol{G}_\Sigma(u)) - \mathcal{H}^{n-1}(\Sigma). \tag{2-1}$$

Next we recall the Euler–Lagrange and Jacobi operators of these functionals.

Lemma 2.1 (first and second variations of area) *Let $g \in C^{1,\alpha}(\Sigma, \boldsymbol{C}^\perp)$, with Σ a closed smooth minimal surface in $\mathbb{S}^{n+k-1}$, then we have the Taylor expansion formula*

$$\left| \mathcal{A}_\Sigma(g) - \frac{1}{2} \int_\Sigma \left(|(Dg)^\perp|^2 - \sum_{i,j=1}^{n} (B(\tau_i, \tau_j) \cdot g)^2 - (n-1)|g|^2 \right) d\mathcal{H}^{n-1} \right| \leq C \|g\|_{C^{1,\alpha}(\Sigma, \boldsymbol{C}^\perp)} \|g\|^2_{H^1(\Sigma, \boldsymbol{C}^\perp)}, \tag{2-2}$$

where $|(Dg)^\perp|^2 = \sum_{i=1}^{n-1} |(D_i g)^\perp|^2$, B is the second fundamental form of Σ and $(Dg)^\perp$ is the projection of Dg on the normal bundle of Σ in the sphere.

Although standard, we give the proof of this lemma in Appendix A for the reader's convenience. In the codimension one case, that is, when $g = \zeta\nu$, with ν normal to Σ in the sphere, the bound above becomes

$$\left| \mathcal{A}_\Sigma(g) - \frac{1}{2} \int_\Sigma \left(|\nabla\zeta|^2 - (|B|^2 - (n-1))|\zeta|^2 \right) d\mathcal{H}^{n-1} \right| \leq C \|\zeta\|_{C^{1,\alpha}(\Sigma, \boldsymbol{C}^\perp)} \|\zeta\|^2_{H^1(\Sigma, \boldsymbol{C}^\perp)},$$

since $D_i(\zeta\nu)^\perp = D_i\zeta\,\nu + \zeta D_i\nu = D_i\zeta\,\nu$, because $D_i\nu \in T\Sigma$.

2.2 Slicing lemma

In this section we estimate the difference between the area of a general graph and a cone, by bounding the additional radial error. Although simple, this lemma is the starting point of our proof, as it suggests how to modify the trace.

Lemma 2.2 (slicing lemma) *For every function* $g = g(r,\theta) \in C^{1,\alpha}(\boldsymbol{C}, \boldsymbol{C}^{\perp})$,

$$\mathcal{A}_{\boldsymbol{C}}(rg) \le \int_0^1 \mathcal{A}_{\Sigma}(g(r,\cdot)) r^{n-1}\, dr + C\big(1+\sup_{r\in(0,1)} \|g(r,\cdot)\|_{C^{1,\alpha}(\Sigma,\boldsymbol{C}^{\perp})}\big) \int_0^1 \int_{\Sigma} |\partial_r g|^2 \, d\mathcal{H}^{n-1}\, r^{n+1}\, dr. \tag{2-3}$$

In particular, if $g(r,\theta) = c(\theta)$, *then we have*

$$\mathcal{A}_{\boldsymbol{C}}(rc) = \frac{1}{n}\mathcal{A}_{\Sigma}(c). \tag{2-4}$$

Proof Consider the function $G(r,\theta) := r(\theta + g(r,\theta))/\sqrt{1+g^2(r,\theta)}$. We can compute

$$\mathcal{A}_{\boldsymbol{C}}(rg) := \int_0^1 \int_{\Sigma} \Big| D_r G \wedge \frac{1}{r} D_\theta G \Big| \, d\theta\, r^{n-1}\, dr - \mathcal{H}^n(\boldsymbol{C} \cap B_1).$$

In particular, notice that if $g(r,\theta) = c(\theta)$, then we have $|G| = r$, so that

$$1 = D_r|G| = \frac{G}{|G|} \cdot D_r G \quad \text{and} \quad |D_r G| = 1.$$

Using again $|G| = r$ and the first equality above, which implies that $D_r G = G/|G|$, we deduce that

$$0 = D_\theta |G| = \frac{G}{|G|} \cdot D_\theta G = D_r G \cdot D_\theta G,$$

so that $|D_r G \wedge D_\theta G| = |D_r G||D_\theta G| = |D_\theta G|$. From this and the fact that $r^{-1} D_\theta G$ is independent of r, we deduce the well-known formula

$$\mathcal{A}_{\boldsymbol{C}}(rc) = \frac{1}{n} \int_{\Sigma} \frac{1}{r} |D_\theta G| \, d\theta - \frac{1}{n} \mathcal{H}^{n-1}(\Sigma) = \frac{1}{n}(\mathcal{H}^{n-1}(\boldsymbol{G}_{\Sigma}(c)) - \mathcal{H}^{n-1}(\Sigma)) = \frac{1}{n}\mathcal{A}_{\Sigma}(c).$$

Now assume g has no special structure; we can estimate

$$\mathcal{A}_{\boldsymbol{C}}(rg) \le \int_0^1 \int_{\Sigma} |D_r G| \Big| \frac{1}{r} D_\theta G \Big| \, d\theta\, r^{n-1}\, dr - \int_0^1 \mathcal{H}^{n-1}(\Sigma)\, r^{n-1}\, dr. \tag{2-5}$$

A simple computation gives

$$D_r G = \frac{1}{(1+|g|^2)^{3/2}} \big(\theta(1+|g|^2 - rg\cdot\partial_r g) + g(1+|g|^2 - rg\cdot\partial_r g) + r\partial_r g(1+|g|^2)\big),$$

so that, using the orthogonality between θ and each of g and $\partial_r g$, where the second follows by the fact that $\boldsymbol{C}$ is a cone, we deduce

$$\begin{aligned}|D_r G| &= \frac{\sqrt{(1+|g|^2 - rg\cdot\partial_r g)(1+|g|^2+rg\cdot\partial_r g)+(r\partial_r g)^2(1+|g|^2)^2}}{(1+|g|^2)} \\ &= \frac{\sqrt{(1+|g|^2)^2-(rg\cdot\partial_r g)^2+(r\partial_r g)^2(1+|g|^2)^2}}{(1+|g|^2)} \\ &\le \sqrt{1+\frac{(r\partial_r g)^2}{1+|g|^2}} \le 1+r^2(\partial_r g)^2.\end{aligned}$$

Using this bound in (2-5), together with $r^{-1}|D_\theta G| \le C(1+\|g\|_{C^{1,\alpha}})$ and the fact that $D_\theta G$ is the Jacobian of the graph in the θ variables, concludes the proof. □

2.3 Proof of Theorem 1.1

As outlined above, Lemma 2.2 suggests that to construct the competitor function, h, we should change c radially by flowing in the directions that decrease the volume $\mathcal{A}_\Sigma$, that is, by decreasing radially the directions corresponding to positive eigenvalues of $\delta^2\mathcal{A}_\Sigma$ (and leaving alone those directions which correspond to negative eigenvalues). However, there is the possibility that $c\in\ker(\delta^2\mathcal{A})$ (or merely that "most of" c is in the kernel), in which case we cannot hope to gain at second order and must use the analyticity of the area functional to construct a finite-dimensional gradient flow to achieve the desired gain.

We begin constructing the competitor function $h\in H^1(\boldsymbol{C},\boldsymbol{C}^\perp)\cap C^{1,\alpha}(\Sigma,\boldsymbol{C}^\perp)$. Let $K=\ker\delta^2\mathcal{A}_\Sigma(0)\subset L^2(\Sigma,\boldsymbol{C}^\perp)$, where the second variation of $\mathcal{A}_\Sigma(0)$ is the self-adjoint operator with compact resolvent defined by

$$\delta^2\mathcal{A}_\Sigma(0)[\zeta,\cdot] := -\Delta_\Sigma^\perp\zeta - \sum_{i,j=1}^{n-1}(B(\tau_i,\tau_j)\cdot\zeta)B(\tau_i,\tau_j)-(n-1)\zeta \qquad \text{for every } \zeta\in C^2(\Sigma,\boldsymbol{C}^\perp).$$

This is a system of equations, with as many equations as the dimension of the normal bundle of Σ in the sphere. Let $\Upsilon\in C^\omega(K,K^\perp)$, where $K^\perp$ is the orthogonal complement of K inside $L^2(\Sigma,\boldsymbol{C}^\perp)$, be the operator given by the Lyapunov–Schmidt reduction in Appendix B, and write the trace c as

$$c = P_K c + P_{K^\perp}c = P_K c+\Upsilon(P_K c)+(P_{K^\perp}c-\Upsilon(P_K c)) \equiv P_K c+\Upsilon(P_K c)+c_\Upsilon^\perp,$$

where P_K and $P_{K^\perp}$ are the projections on K and $K^\perp$, respectively. By the spectral theory for operators with compact resolvent, we know that there exists an orthonormal basis $\{\phi_j\}_{j=1}^\infty$ of $H^1(\Sigma, \boldsymbol{C}^\perp)$ and numbers $\{\lambda_j\}_{j=1}^\infty$ accumulating at $+\infty$ such that

$$\delta^2\mathcal{A}_\Sigma(0)[\phi_j,\cdot] = \lambda_j\phi_j \quad \text{for every } j \in \mathbb{N},$$

where each eigenvalue has finite multiplicity. In particular, we set $\ell := \dim K$ and K is spanned by the eigenfunctions $\phi_1, \dots, \phi_l$. Then we can decompose, up to relabelling,

$$c_\Upsilon^\perp := \sum_{\{j\,|\,\lambda_j<0\}} c_j\phi_j + \sum_{\{j\,|\,\lambda_j>0\}} c_j\phi_j =: c_-^\perp + c_+^\perp, \quad P_K(c) := \sum_{\{j\,|\,\lambda_j=0\}} \mu_j^0\phi_j = \sum_{j=1}^{\ell} \mu_j^0\phi_j.$$

We then define the competitor function, h, by

$$rh(r\theta) := r\biggl(\sum_{j=1}^{\ell}\mu_j(\eta(r))\phi_j(\theta) + \Upsilon\biggl(\sum_{j=1}^{\ell}\mu_j(\eta(r))\phi_j(\theta)\biggr) + c_-^\perp(\theta) + \eta_+(r)c_+^\perp(\theta)\biggr), \tag{2-6}$$

where $\mu(\eta(r)) := \bigl(\mu_1(\eta(r)), \dots, \mu_\ell(\eta(r))\bigr)$ is the vector field defined by the renormalized gradient flow

$$\begin{aligned} \mu'(t) &= \begin{cases} -\nabla A(\mu(t))/|\nabla A(\mu(t))| & \text{if } A(\mu(t)) > \tfrac{1}{2}A(\mu(0)), \\ 0 & \text{otherwise,} \end{cases} \\ \mu(0) &= (\mu_1^0, \dots, \mu_l^0) =: \mu^0, \end{aligned} \tag{2-7}$$

and $A(\mu) := \mathcal{A}_\Sigma\bigl(\sum_{\{j\,|\,\lambda_j=0\}}\mu_j\phi_j + \Upsilon\bigl(\sum_{\{j\,|\,\lambda_j=0\}}\mu_j\phi_j\bigr)\bigr)$ is an analytic function from $\mathbb{R}^\ell$ to $\mathbb{R}$. Note that if $A(\mu(0)) \le 0$ then the flow is constant throughout. We will show below, using the Łojasiewicz inequality, that this flow is well defined for all times.

The two cut-off functions, η_+ and η, are chosen to be

$$\eta_+(r) := 1 - (1-r)\frac{\varepsilon}{n+2} \quad \text{and} \quad \eta(r) := \varepsilon_A A(\mu^0)^{1-\gamma}C(1-r),$$

where ε, ε_A, C and γ will be chosen later in the proof, depending only on Σ, and so on $\boldsymbol{C}$. Notice that $h(1,\cdot) = c(\cdot)$, so the first property required of our competitor is satisfied. Also note that $h \in C^{1,\alpha}(\boldsymbol{C}, \boldsymbol{C}^\perp)$ as each $\phi_j \in C^{1,\alpha}(\Sigma, \boldsymbol{C}^\perp)$ (by elliptic regularity) and $\Upsilon(\mu) \in C^{1,\alpha}(\Sigma, \boldsymbol{C}^\perp)$ (see Lemma B.1).

Thus we can use Lemma 2.2, and estimate

$$\text{(2-8)}\quad \mathcal{A}_{\boldsymbol{C}}(rh)-(1-\varepsilon)\mathcal{A}_{\boldsymbol{C}}(rc) \le \int_0^1 \big(\mathcal{A}_\Sigma(h(r,\cdot))-(1-\varepsilon)\mathcal{A}_\Sigma(c)\big)r^{n-1}\,dr + \underbrace{C\big(1+\sup_{r\in(0,1)}\|h(r,\cdot)\|_{C^{1,\alpha}(\Sigma,\boldsymbol{C}^\perp)}\big)\int_0^1\int_\Sigma |\partial_r h|^2\,d\mathcal{H}^{n-1}\,r^{n+1}\,dr}_{=:E_r}.$$

By the definition of h (and (B-3)) we have that $\sup_{r\in(0,1)}\|h(r,\cdot)\|_{C^{1,\alpha}(\Sigma,\boldsymbol{C}^\perp)} \le 5\|c\|^\gamma_{C^{1,\alpha}(\Sigma,\boldsymbol{C}^\perp)} \le 1$ (for more details see (2-13) and the discussion below) and, moreover,

$$\int_0^1\int_\Sigma |\partial_r h|^2\,d\mathcal{H}^{n-1}\,r^{n+1}\,dr \le 2\int_0^1 r^{n+1}\big((\eta_+'(r))^2\|c_+^\perp\|^2+(\eta'(r))^2(1+\|\delta\Upsilon(\mu)[\mu']\|_\infty^2)\big)\,dr \le C\int_0^1 r^{n+1}\big(\varepsilon^2\|c_\Upsilon^\perp\|^2_{H^1(\Sigma,\boldsymbol{C}^\perp)}+(\eta'(r))^2\big)\,dr,$$

where in the second inequality we used (B-3) to estimate $\|\delta\Upsilon(\mu)[\mu']\|_\infty \le C\|\mu'\| = C$. It follows that

$$\text{(2-9)}\qquad |E_r| \le C(\varepsilon^2\|c_\Upsilon^\perp\|^2_{H^1}+\varepsilon_A^2 A(\mu^0)^{2-2\gamma}).$$

For the main term, we split the estimate in two parts:

$$\mathcal{A}_\Sigma(h)-(1-\varepsilon)\mathcal{A}_\Sigma(c) = \underbrace{\big(\mathcal{A}_\Sigma(h)-\mathcal{A}_\Sigma(\mu+\Upsilon(\mu))\big)-(1-\varepsilon)\big(\mathcal{A}_\Sigma(c)-\mathcal{A}_\Sigma(P_K c+\Upsilon(P_K c))\big)}_{=:E^\perp} + \underbrace{\mathcal{A}_\Sigma(\mu+\Upsilon(\mu))-(1-\varepsilon)\mathcal{A}_\Sigma(P_K c+\Upsilon(P_K c))}_{=:E^T}.$$

For the first part, denoting by $h_\Upsilon^\perp := h-(\mu+\Upsilon(\mu))$, we have, by a simple Taylor expansion,

$$\text{(2-10)}\quad E^\perp = \delta\mathcal{A}_\Sigma(\mu+\Upsilon(\mu))[h_\Upsilon^\perp]+\delta^2\mathcal{A}_\Sigma(\mu+\Upsilon(\mu)+sh_\Upsilon^\perp)[h_\Upsilon^\perp,h_\Upsilon^\perp] -(1-\varepsilon)\big(\delta\mathcal{A}_\Sigma(\mu^0+\Upsilon(\mu^0))[c_\Upsilon^\perp] + \delta^2\mathcal{A}_\Sigma(\mu^0+\Upsilon(\mu^0)+tc_\Upsilon^\perp)[c_\Upsilon^\perp,c_\Upsilon^\perp]\big)$$

$$\le \delta^2 \mathcal{A}_\Sigma(\mu + \Upsilon(\mu) + s h_\Upsilon^\perp) \times [h_\Upsilon^\perp, h_\Upsilon^\perp] - (1-\varepsilon)\delta^2 \mathcal{A}_\Sigma(\mu^0 + \Upsilon(\mu^0) + t c_\Upsilon^\perp)[c_\Upsilon^\perp, c_\Upsilon^\perp],$$

where $s, t \in (0,1)$ and the second inequality holds thanks to (B-2) and the fact that $h_\Upsilon^\perp, c_\Upsilon^\perp \in K^\perp$. Using Lemma 2.1, we easily see that

$$|\delta^2 \mathcal{A}_\Sigma(f)[\zeta,\zeta] - \delta^2 \mathcal{A}_\Sigma(0)[\zeta,\zeta]| = \left| \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{A}_\Sigma(f + t\zeta) - \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{A}_\Sigma(t\zeta) \right| \le C \|f\|_{C^{1,\alpha}(\Sigma, \boldsymbol{C}^\perp)} \|\zeta\|^2_{H^1(\Sigma,\boldsymbol{C}^\perp)}$$

with

$$\delta^2 \mathcal{A}_\Sigma(0)[\zeta,\zeta] = \frac{1}{2} \int_\Sigma \left(|(Dg)^\perp|^2 - \sum_{i,j=1}^n (B(\tau_i,\tau_j)\cdot g)^2 - (n-1)|g|^2 \right) d\mathcal{H}^{n-1}.$$

Using this estimate in (2-10) and the fact that $|\eta|, |\eta_+| \le 1$, we deduce

$$\begin{aligned} E^\perp &\le \delta^2 \mathcal{A}_\Sigma(0)[c_-^\perp + \eta_+ c_+^\perp, c_-^\perp + \eta_+ c_+^\perp] - (1-\varepsilon)\delta^2 \mathcal{A}_\Sigma(0)[c_-^\perp + c_+^\perp, c_-^\perp + c_+^\perp] \\ &\quad + C\big(\|\mu + \Upsilon(\mu) + s h_\Upsilon^\perp\|_{C^{1,\alpha}} + \|\mu^0 + \Upsilon(\mu^0) + t c_\Upsilon^\perp\|_{C^{1,\alpha}}\big) \|c_\Upsilon^\perp\|^2_{H^1(\Sigma,\boldsymbol{C}^\perp)} \\ &\le \varepsilon \delta^2 \mathcal{A}_\Sigma(0)[c_-^\perp, c_-^\perp] + (\eta_+^2 - (1-\varepsilon))\delta^2 \mathcal{A}_\Sigma(0)[c_+^\perp, c_+^\perp] \\ &\quad + C\big(2\|c_\Upsilon^\perp\|_{C^{1,\alpha}} + \|\mu\|_{C^{1,\alpha}} + \|\mu^0\|_{C^{1,\alpha}}\big) \|c_\Upsilon^\perp\|^2_{H^1(\Sigma,\boldsymbol{C}^\perp)}. \end{aligned} \tag{2-11}$$

Integrating this error in the radii and recalling that, by our choice of η_+, we have $\int_0^1 (\eta_+^2(r) - (1-\varepsilon)) r^{n-1}\, dr \le -\varepsilon$, we conclude

$$\begin{aligned} \int_0^1 E^\perp r^{n-1}\, dr &\le \varepsilon \max_{\lambda_j < 0} \lambda_j \|c_-^\perp\|^2_{H^1(\Sigma,\boldsymbol{C}^\perp)} - \varepsilon \min_{\lambda_j > 0} \lambda_j \|c_+^\perp\|^2_{H^1(\Sigma,\boldsymbol{C}^\perp)} \\ &\quad + C(\|c_\Upsilon^\perp\|_{C^{1,\alpha}} + \|\mu\|_{C^{1,\alpha}} + \|\mu^0\|_{C^{1,\alpha}}) \|c_\Upsilon^\perp\|^2_{H^1(\Sigma,\boldsymbol{C}^\perp)} \\ &\le -\big(C_{\boldsymbol{C}}\varepsilon - C(\|c_\Upsilon^\perp\|_{C^{1,\alpha}} + \|\mu\|_{C^{1,\alpha}} + \|\mu^0\|_{C^{1,\alpha}})\big) \|c_\Upsilon^\perp\|^2_{H^1(\Sigma,\boldsymbol{C}^\perp)}, \end{aligned} \tag{2-12}$$

where $C_{\boldsymbol{C}} > 0$ is a strictly positive constant depending only on the spectral gap between 0 and the other eigenvalues of $\delta^2 \mathcal{A}_\Sigma(0)$, that is, depending only on Σ and so on $\boldsymbol{C}$. Note in the first line of (2-12), we use the H^1 norm as opposed to the more natural L^2 norm. However, if ϕ_j is a eigenfunction of $\delta^2 \mathcal{A}_\Sigma(0)$ associated to a nonzero eigenvalue, λ_j, then it is an easy computation to see that $\|\nabla \phi_j\|_{L^2} \le c\|\phi_j\|_{L^2}$ with a constant c that depends on $|\lambda_j|^{-1}$. Thus, for $c_\pm^\perp$ the norms are comparable with a constant depending on the gap in the spectrum between the zero and the eigenvalues above and below it.

Noticing that, by the definition of η, we have

$$
\begin{aligned}
|\mu(\eta(r))-\mu^0| &\le \int_0^{\eta(r)} |\mu'(t)|\,dt \le |\eta(r)| \le C\varepsilon_A A(\mu^0)^{1-\gamma}, \\
\left|\frac{d}{dr}\mu(\eta(r))\right| &\le |\mu'(\eta(r))|\,|\eta'(r)| \le C\varepsilon_A A(\mu^0)^{1-\gamma}.
\end{aligned} \tag{2-13}
$$

These estimates, combined with elliptic regularity applied to $\phi_j \in K$, allow us to bound $\|\mu\|_{C^{1,\alpha}} \le C\varepsilon_A A(\mu^0)^{1-\gamma} + \|\mu^0\|_{C^{1,\alpha}} \le 2\|\mu^0\|_{C^{1,\alpha}}^{1-\gamma}$ (for $\varepsilon_A > 0$ sufficiently small but depending only on $\boldsymbol{C}$).

By choosing $\|c\|_{C^{1,\alpha}}$ (which is bigger than $\|\mu^0\|_{C^{1,\alpha}}$ and $\|c_\Upsilon^\perp\|_{C^{1,\alpha}}$) sufficiently small, depending only on $C_{\boldsymbol{C}}$, we conclude

$$
\int_0^1 E^\perp r^{n-1}\,dr \le -C_{\boldsymbol{C}}\varepsilon \|c_\Upsilon^\perp\|^2_{H^1(\Sigma,\boldsymbol{C}^\perp)}. \tag{2-14}
$$

Next we estimate E^T. If $A(\mu^0) \le 0$, then $E^T \le 0$ trivially. When $A(\mu^0) > 0$, we recall the Łojasiewicz inequality (see [8]) for the analytic function A, which says there exists a neighborhood U of 0 and constants $\gamma \in \left(0, \frac{1}{2}\right]$ and $c > 0$ (which depend on the cone $\boldsymbol{C}$) such that

$$
|A(\tau)-A(0)|^{1-\gamma} = |A(\tau)|^{1-\gamma} \le c|\nabla A(\tau)| \quad \text{for all } \tau \in U. \tag{2-15}
$$

The inequality (2-15) implies that $|\nabla A(\mu(t))| > 0$ whenever $A(\mu(t)) > \frac{1}{2}A(\mu^0) > 0$, and thus the flow given by (2-7) is well defined for all times.

We can estimate

$$
\begin{aligned}
A(\mu(t)) - A(\mu^0) &= \int_0^t \frac{d}{d\tau}A(\mu(\tau))\,d\tau = \int_0^t \nabla A(\mu(\tau))\cdot\mu'(\tau)\,d\tau \\
&= -\int_0^t |\nabla A(\mu(\tau))|\,d\tau \le 0,
\end{aligned} \tag{2-16}
$$

so that the function $t \mapsto A((\mu(t))$ is nonincreasing, and therefore there exists a first time $t_1 > 0$ such that

$$
\begin{cases} A(\mu(t)) \ge \frac{1}{2}A(\mu^0) & \text{if } 0 \le t \le t_1, \\ A(\mu(t)) = \frac{1}{2}A(\mu^0) & \text{if } t \ge t_1. \end{cases}
$$

If $\eta(r) \le t_1$ then we have

$$
\begin{aligned}
E^T &= A(\mu(\eta(r))) - A(\mu^0) + \varepsilon A(\mu^0) \\
&\le -\int_0^{\eta(r)} |\nabla A(\mu(\tau))|\,d\tau + \varepsilon A(\mu^0)
\end{aligned} \tag{2-17}
$$

$$
\begin{aligned}
&\le -C_{\boldsymbol{C}} \int_0^{\eta(r)} |A(\mu(\tau))|^{1-\gamma}\, d\tau + \varepsilon A(\mu^0) \\
&\le -C_{\boldsymbol{C}} A(\mu(\eta(r))^{1-\gamma} \eta(r) + \varepsilon A(\mu^0) \\
&\le -\frac{C_{\boldsymbol{C}}}{2^{1-\gamma}} |A(\mu^0)|^{1-\gamma} \eta(r) + \varepsilon A(\mu^0) \\
&= -(\widetilde{C}_{\boldsymbol{C}} \eta(r) - \varepsilon A(\mu^0)^{\gamma}) A(\mu^0)^{1-\gamma},
\end{aligned}
$$

where in the first inequality we used (2-16) and in the second inequality we use the Łojasiewicz inequality, (2-15), for the analytic function A at the point 0 (it is here that the constant depends on $\boldsymbol{C}$). Finally, in the third inequality we use the monotonicity of A and in the fourth we use $\eta(r) \le t_1$.

If $\eta(r) > t_1$, then

$$
\begin{aligned}
E^T &= A(\mu(\eta(r))) - (1-\varepsilon) A(\mu^0) < -\left(\tfrac{1}{2} - \varepsilon\right) A(\mu^0) \\
&< -(C_{\boldsymbol{C}} \eta(r) - \varepsilon A(\mu^0)^{\gamma}) A(\mu^0)^{1-\gamma},
\end{aligned}
\tag{2-18}
$$

where the last inequality follows since $|\eta| \le C \varepsilon_A A(\mu^0)^{1-\gamma} < \frac{1}{2}$ as long as μ^0 is small enough.

We now have two cases:

Case 1 $A(\mu^0) < \tau^2 \|c^{\perp}\|^2_{H^1(\Sigma, \boldsymbol{C}^{\perp})}$ for some $\tau > 0$ small but universal (ie depending only on $\boldsymbol{C}$ and n, but not on c). Note this includes when $A(\mu^0) \le 0$. In this scenario, let $\eta \equiv 0$ (ie $\varepsilon_A \equiv 0$), so that $E^T = \varepsilon A(\mu^0)$, and combine (2-8), (2-9) and (2-14), to deduce that

$$
\begin{aligned}
\mathcal{A}_{\boldsymbol{C}}(rh) - (1-\varepsilon)\mathcal{A}_{\boldsymbol{C}}(rc) &\le -C_{\boldsymbol{C}} \varepsilon \|c^{\perp}_{\Upsilon}\|^2_{H^1} + (\varepsilon A(\mu^0) + \varepsilon^2 \|c^{\perp}_{\Upsilon}\|^2_{H^1}) \\
&\le -(C_{\boldsymbol{C}} - \tau - \varepsilon)\varepsilon \|c^{\perp}_{\Upsilon}\|^2_{H^1} < 0
\end{aligned}
\tag{2-19}
$$

for a proper choice of $\varepsilon > 0$ and $\tau > 0$ small enough depending only on n and $\boldsymbol{C}$.

Case 2 Otherwise, we choose $\varepsilon = \varepsilon_A A(\mu^0)^{1-\gamma}$ for some $\varepsilon_A > 0$ small, depending only on n and $\boldsymbol{C}$. Using (2-17) and (2-18) we can estimate

$$
\begin{aligned}
\int_0^1 E^T r^{d-1}\, dr &\le -A(\mu^0)^{1-\gamma} \int_0^1 (C_{\boldsymbol{C}} \eta(r) - \varepsilon A(\mu^0)^{\gamma}) r^{d-1}\, dr \\
&= -\varepsilon_A A(\mu^0)^{2-2\gamma} \int_0^1 (C_{\boldsymbol{C}} C(1-r) - A(\mu^0)^{\gamma}) r^{d-1}\, dr \\
&\le -C_{\boldsymbol{C}} \varepsilon_A A(\mu^0)^{2-2\gamma}.
\end{aligned}
\tag{2-20}
$$

Then, using (2-20) together with (2-8), (2-9) and (2-14), we deduce

$$\begin{aligned}\mathcal{A}_{\boldsymbol{C}}(rh)-(1-\varepsilon)\mathcal{A}_{\boldsymbol{C}}(rc) &\le \underbrace{-C_{\boldsymbol{C}}\varepsilon\|c_\Upsilon^\perp\|_{H^1}^2}_{E^\perp}\underbrace{-C_{\boldsymbol{C}}\varepsilon_A A(\mu^0)^{2-2\gamma}}_{E^T}\underbrace{+C(\varepsilon^2\|c_\Upsilon^\perp\|_{H^1}^2+\varepsilon_A^2A(\mu^0)^{2-2\gamma})}_{E_r}\\ &\le -(C_{\boldsymbol{C}}\varepsilon-\varepsilon^2)\|c_\Upsilon^\perp\|_{H^1}^2-(C_{\boldsymbol{C}}\varepsilon_A+C\varepsilon_A^2)A(\mu^0)^{2-2\gamma}<0,\end{aligned} \tag{2-21}$$

since we are in the case $A(\mu^0)>0$ and by choosing ε_A small enough depending only on n and $\boldsymbol{C}$. Moreover, since we are in the case $A(\mu^0)^{1/2}\ge\tau\|c^\perp\|_{H^1(\Sigma,\boldsymbol{C}^\perp)}$, we can use Lemma 2.2 to write

$$\begin{aligned}\mathcal{A}_{\boldsymbol{C}}(rc)=\frac{1}{n}\mathcal{A}_\Sigma(c)&=\frac{1}{n}\big(\mathcal{A}_\Sigma(c)-\mathcal{A}(P_Kc+\Upsilon(P_Kc))+\mathcal{A}(P_Kc+\Upsilon(P_Kc))\big)\\ &\le C_{\boldsymbol{C}}\|c_\Upsilon^\perp\|_{H^1}^2+A(\mu^0)\le(C_{\boldsymbol{C}}\tau^{-1}+1)A(\mu(0)),\end{aligned} \tag{2-22}$$

where in the first inequality we used Lemma 2.1. Finally, combining (2-21) and (2-22) we conclude

$$\mathcal{A}_{\boldsymbol{C}}(rh)-(1-\varepsilon_A(\mathcal{A}_{\boldsymbol{C}}(rc))^{1-\gamma})\mathcal{A}_{\boldsymbol{C}}(rc)<0. \tag{2-23}$$

Combining the two previous cases concludes the proof. □

2.4 The integrability case

To finish the proof of the epiperimetric inequality for integrable cones we need the following lemma, which is based on the analyticity of A and whose proof can be found also in [1].

Lemma 2.3 (constant area on the kernel) *A cone $\boldsymbol{C}$ is integrable if and only if $A(\mu)=A(0)=0$ in a neighborhood of 0.*

Using this lemma it is immediate to see that if $\boldsymbol{C}$ is integrable then we always fall in Case 1 of the proof of Theorem 1.1, so that we have (1-2) with $\gamma=0$.

Proof of Lemma 2.3 The integrability condition (1-1) is equivalent to

$$\forall\phi\in\ker\delta^2\mathcal{A}_\Sigma(0)\ \ \exists(\Psi_s)_{s\in(0,1)}\subset C^2(\Sigma,\boldsymbol{C}^\perp)\quad \begin{cases}\lim_{s\to0}\Psi_s=0,\\ \delta\mathcal{A}_\Sigma(\Psi_s)=0 \quad\text{for } s\in(0,1),\\ d\Psi_s/ds|_{s=0}=\lim_{s\to0}\Psi_s/s=\phi.\end{cases} \tag{2-24}$$

Assume (2-24) holds, and recall the definition $A(\mu) = \mathcal{A}_\Sigma(\mu + \Upsilon(\mu))$. If $A \equiv 0$ in a neighborhood of zero then we are done. Otherwise, we can write $A(\mu) = A_p(\mu) + A_R(\mu)$, where $A_p \not\equiv 0$, $A_p(\lambda\mu) = \lambda^p A(\mu)$ for $\lambda > 0$ and $A_R(\mu)$ is the sum of homogeneous polynomials of degrees $\geq p+1$ (here, again, we use the analyticity of A). Note there exists some $\phi \in \ker \delta^2 \mathcal{A}_\Sigma(0)$ such that $\nabla A_p(\phi) \neq 0$; let Ψ_s be the one-parameter family of critical points that is generated by ϕ (as in (2-24)).

As Ψ_s is a critical point, Lemma B.1 allows us to write $\Psi_s = \phi_s + \Upsilon(\phi_s)$, where $\phi_s \in K$ and $\phi_s/s \to \phi$ as $s \downarrow 0$. We compute

$$0 = \delta\mathcal{A}_\Sigma(\Psi_s) = \nabla A(\phi_s) = \nabla A_p(\phi_s) + \nabla A_R(\phi_s) = s^{p-1}\nabla A(\phi) + o(s^{p-1}).$$

Divide the above by s^{p-1} and let $s \downarrow 0$ to obtain a contradiction to $\nabla A_p(\phi) \neq 0$.

In the other direction assume that $A \equiv 0$ in a neighborhood of 0. This implies that $\nabla A \equiv 0$ in a (perhaps slightly smaller) neighborhood of 0. Therefore, for any $\mu \in \ker \delta^2 \mathcal{A}_\Sigma(0)$, letting $\Psi_s = s\mu + \Upsilon(s\mu)$ and recalling (B-3) establishes (2-24). □

3 Almost area-minimizing currents and applications

In this section we apply the (log-)epiperimetric inequality of Theorem 1.1 to deduce Theorem 1.5. As mentioned in the introduction, for the classical epiperimetric inequality this has been done in [6] by De Lellis, Spadaro and the second author. Here, however, the strategy is slightly different since we do not know that every blowup is of the same type (ie our uniqueness result not only determines a rotation, but actually prevents the formation of additional singularities).

3.1 Technical preliminaries

We start by recalling the following well-known proposition, whose proof can be found in [6, Proposition 2.1].

Proposition 3.1 (almost monotonicity [6, Proposition 2.1]) *Let $T \in \boldsymbol{I}_n(\mathbb{R}^{n+k})$ be an almost-minimizer and $x \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$. There are constants $C, \bar{r}, \alpha_0 > 0$ such that*

$$\int_{B_r(x)\setminus B_s(x)} \frac{|(z-x)^\perp|^2}{|z-x|^{n+2}} \, d\|T\|(z) \leq C\left(\frac{\|T\|(B_r(x))}{\omega_n r^n} - \frac{\|T\|(B_s(x))}{\omega_n s^n} + r^{\alpha_0}\right) \tag{3-1}$$

for all $0 < s < r < \overline{r}$ (in (3-1), $(z-x)^\perp$ denotes the projection of the vector $z-x$ on the orthogonal complement of the approximate tangent to T at z). *In particular, the function* $r \to \|T\|(B_r(x))/(\omega_n r^n) + C r^{\alpha_0}$ *is nondecreasing.*

Using (3-1) together with the almost-minimizing property, it is easy to see that the same blowup analysis holds for almost-minimizing and minimizing currents. That is, we can consider the blowup sequence of T at x defined by $(\iota_{x,r})_\sharp T$, where the map $\iota_{x,r}$ is given by $\mathbb{R}^{n+k} \ni y \mapsto (y-x)/r \in \mathbb{R}^{n+k}$. Recall that an area-minimizing cone S is an integral area-minimizing current such that $(\iota_{0,r})_\sharp S = S$ for every $r > 0$. Then, by the almost monotonicity of $\|T_{x,r}\|$, $T_{x,r} \to S$ up to subsequences, with S an area-minimizing cone. Furthermore, by the almost minimality of T, the convergence is strong, ie their difference goes to zero in the flat norm, the support of $T_{x,r}$ converges to the support of S in the Hausdorff distance sense and the mass of $T_{x,r}$ converges to that of S.

Below, we will continue to denote by $\boldsymbol{C}$ an arbitrary multiplicity-1 area-minimizing cone; occasionally we abuse notation slightly and identify the cone with its support. Moreover, T will be almost area-minimizing with parameters r_0 and α_0 (see Definition 1.4). Finally, we set

$$\Theta_M(T,x) := \lim_{r\to 0} \frac{\|T\|(B_r(x))}{\omega_n r^n} \quad \text{and} \quad \Theta_{\boldsymbol{C}} := \|\boldsymbol{C}\|(B_1) = \frac{\|\boldsymbol{C}\|(B_r)}{\omega_n r^n}.$$

We first prove a standard parametrization lemma over a multiplicity-1 cone.

Proposition 3.2 (spherical parametrization from a cone) *Let* $\tau, \varepsilon \in \big(0, \frac{1}{4}\big)$, $\boldsymbol{C}$ *be a multiplicity-*1 *area-minimizing cone, and* $T \in \boldsymbol{I}_n$ *be an almost area-minimizing current with* $\Theta_M(T,0) = \Theta_{\boldsymbol{C}}$. *There are constants* $\delta_1, \eta, r_1 > 0$ *(which depend on* τ, ε, *the almost-minimizing parameters* C, α_0 *and* r_0 *and the dimension,* n, *and codimension,* k*) such that if* $r < r_1$, *and*

$$\frac{\|T\|(B_{4r})}{4r^n\omega_n} - \Theta_M(0) \le \eta \quad \textit{and} \quad \mathcal{F}(\partial((T_{2r} - \boldsymbol{C}) \llcorner B_1)) \le \eta, \tag{3-2}$$

then there exists $u \in C^{1,\alpha}(\boldsymbol{C} \cap B_r \setminus B_{\tau r}, \boldsymbol{C}^\perp)$ *such that*

$$T \llcorner (B_r \setminus B_{\tau r}) = \boldsymbol{G}_{\boldsymbol{C}}(u) \quad \textit{and} \quad \sup_{\boldsymbol{C}\cap(B_r\setminus B_{\tau r})} \sum_{j=0}^{3} |D^j u| \le \varepsilon. \tag{3-3}$$

Proof Arguing by contradiction, we assume there exist sequences of almost area-minimizing currents $(T^k)_k$, all with the same constants $r_0, C, \alpha_0 > 0$, and radii $(r_k)_k$, with $r_k \to 0$, such that, if we consider $R_k := T^k_{r_k}$, then

$$\frac{\|R_k\|(B_4)}{4^n\omega_n} - \Theta_{\boldsymbol{C}} \le \frac{1}{k} \quad \text{and} \quad \mathcal{F}\big(\partial((R_k - \boldsymbol{C}) \llcorner B_2)\big) \le \frac{1}{k}. \tag{3-4}$$

Notice that, by the first inequality above, we have a uniform bound for $\|R_k\|(B_4)$, so that, up to subsequences, $R_k \to V$ in B_4. By the same uniform bound and the usual slicing theorem, passing to a subsequence there is a radius $\rho_0 \in (2,4)$ such that $\boldsymbol{M}\big(\partial((R_k - V) \llcorner B_{\rho_0})\big)$ is uniformly bounded. On the other hand, $R_k - V$ converges to 0 in the sense of currents and hence, by [13, Theorem 31.2], $\mathcal{F}((R_k - V) \llcorner B_{\rho_0}) \to 0$. This means, for all $\rho \le \rho_0$, that there are integral currents H_k and G_k (depending on ρ) with $\boldsymbol{M}(H_k) + \boldsymbol{M}(G_k) \to 0$ such that

$$(R_k - V) \llcorner B_\rho = \partial H_k + G_k.$$

Taking the boundary of the latter identity, we conclude that $\partial G_k = \partial((R_k - V) \llcorner B_\rho)$. Now, rescaling the almost minimality property of T_k, we conclude that

$$\|R_k\|(B_\rho) \le \|V\|(B_\rho) + \boldsymbol{M}(G_k) + C\rho^{\alpha_0} r_k^{\alpha_0}.$$

Since $(\boldsymbol{M}(G_k) + r_k) \downarrow 0$, we infer

$$\limsup_{k\to\infty} \|R_k\|(B_\rho) \le \|V\|(B_\rho).$$

On the other hand, $R_k \to V$ in B_1, so we also have

$$\|V\|(B_\rho) \le \liminf_{k\to\infty} \|R_k\|(B_\rho).$$

Using the almost-monotonicity identity (3-1) and passing to the limit in k, we conclude by a standard argument that $V \llcorner B_2$ is a cone. Passing to the limit in the second inequality of (3-4) we get $\partial(V \llcorner B_2) = \partial(\boldsymbol{C} \llcorner B_2)$. Since both V and $\boldsymbol{C}$ are integral cones, we deduce that $V = \boldsymbol{C}$. Finally, since $\boldsymbol{C}$ has multiplicity 1 and is smooth away from 0, and R_k converges to $\boldsymbol{C}$ by Allard's theorem for almost area-minimizing currents (see for instance [11]), we get a contradiction. □

Since the previous lemma gives graphicality in the interior of the ball, before we can prove Theorem 1.5 we need a way to transfer small excess in B_1 to small excess in B_ρ for some $\rho \in \left(\frac{1}{2}, 1\right)$.

Lemma 3.3 (tangent cones at comparable scales) *Let T be an almost area-minimizing integral current. Then, for all $\varepsilon_2 > 0$, there exists $\delta_2 = \delta_2(\varepsilon_2, C, \alpha_0, r_0) > 0$ such* *that for all* $0 < 2r < \delta_2$ *and all* $\rho \in [r, 2r]$ *we have*

$$\mathcal{F}(\partial((T_\rho - T_r) \llcorner B_1)) < \varepsilon_2. \tag{3-5}$$

Proof We argue by contradiction. Assume there are sequences $r_n \downarrow 0$ and $\rho_n \downarrow 0$ with $\rho_n \in \left[\frac{1}{2} r_n, r_n\right]$ and such that

$$\mathcal{F}\big(\partial((T_{\rho_n} - T_{r_n}) \llcorner B_1)\big) \geq \varepsilon_2.$$

As $1 \leq r_n/\rho_n \leq 2$ for every $n \in \mathbb{N}$, we can assume (passing to subsequences) $0 < L = \lim_n r_n/\rho_n < \infty$. We then compute

$$\mathcal{F}\Big(\partial\Big(\big(V - \lim_{n\to\infty} T_{r_n}\big) \llcorner B_1\Big)\Big) = \mathcal{F}\Big(\lim_{n\to\infty} (\iota_L)_\sharp \partial((V - T_{\rho_n}) \llcorner B_1)\Big) = 0,$$

where V is the tangent cone associated with the sequence $(\rho_n)_n$ and we used the fact that V is a cone. It follows that both sequences T_{r_n} and T_{ρ_n} approach the same tangent cone V, and by the triangle inequality we get a contradiction for n sufficiently big. □

3.2 Proof of Theorem 1.5

Assume that $x_0 = 0$ and recall that $r_0 > 0$ is given by Definition 1.4. We divide the proof into several steps.

Step 1 ((log-)epiperimetric inequality) Assume that for every $0 < s < r < r_0$, there exists a c with small $C^{1,\alpha}$ norm such that $\partial(T_r \llcorner B_1) = \boldsymbol{G}_\Sigma(c)$. By Theorem 1.1 there exist $\varepsilon, C, \gamma > 0$, with $\gamma \in [0, 1)$ and $h \in H^1(\boldsymbol{C}, \boldsymbol{C}^\perp)$ such that

$$\mathcal{A}_{\boldsymbol{C}}(h) \leq (1 - \varepsilon |\mathcal{A}_{\boldsymbol{C}}(rc)|^\gamma) \mathcal{A}_{\boldsymbol{C}}(rc).$$

Set $f(r) := \|T\|(B_r) - \Theta_{\boldsymbol{C}} r^n$ and recall that, since $r \mapsto \|T\|(B_r)$ is monotone, the function f is differentiable a.e. and its distributional derivative is a measure. Its absolutely continuous part coincides a.e. with the classical differential and its singular part is nonnegative, so that

$$r^n \mathcal{A}_{\boldsymbol{C}}(rc) = \|0 \times \partial(T \llcorner B_r)\|(B_r) - \Theta_{\boldsymbol{C}} r^n \leq \frac{r}{n} f'(r).$$

Using the almost minimality of T and the previous two inequalities, we get

$$\begin{aligned} f(r) &\leq r^n \mathcal{A}_{\boldsymbol{C}}(h) + C r^{n+\alpha} \leq (1 - \varepsilon |\mathcal{A}_{\boldsymbol{C}}(rc)|^\gamma) \frac{r}{n} f'(r) + C r^{n+\alpha} \\ &\leq (1 - \varepsilon |e(r)|^\gamma) \frac{r}{n} f'(r) + C r^{n+\alpha}, \end{aligned}$$

where $e(r) := f(r)/r^n$. Rearranging this inequality and dividing it by r^{n+1}, we get

$$e'(r) = \left(\frac{f'(r)}{r^n} - f(r)\frac{n}{r^{n+1}}\right) \geq n\varepsilon \frac{|e(r)|^{1+\gamma}}{r(1-\varepsilon|e(r)|^\gamma)} - C\frac{1}{r^{1-\alpha}} \geq n\varepsilon\frac{|e(r)|^{1+\gamma}}{r} - C\frac{1}{r^{1-\alpha}}. \tag{3-6}$$

We now define $\widetilde{e}(r) = e(r) + 2\alpha^{-1}Cr^\alpha$ and we notice that, from the previous inequality and since $a^{1+\gamma} + b^{1+\gamma} \geq 2^{-\gamma}(a+b)^{1+\gamma}$, for any $a, b \geq 0$,

$$\widetilde{e}'(r) \geq \frac{n\varepsilon}{r}|e(r)|^{1+\gamma} + \frac{C}{r^{1-\alpha}} \geq \frac{n\varepsilon}{r}[|e(r)| + Cr^{\alpha/(1+\gamma)}]^{1+\gamma} \geq \frac{n\varepsilon}{r}[e(r) + Cr^{\alpha/(1+\gamma)}]^{1+\gamma}.$$

Note that by the almost-minimality of T, $\widetilde{e}(r) \geq 0$, so that for r sufficiently small, the previous inequality implies that

$$\widetilde{e}'(r) \geq \frac{n\varepsilon}{r}\widetilde{e}(r)^{1+\gamma}. \tag{3-7}$$

From this inequality we obtain that

$$\frac{d}{dr}\left(\frac{-1}{\gamma\widetilde{e}(r)^\gamma} - n\varepsilon\log r\right) = \frac{1}{\widetilde{e}(r)^{1+\gamma}}\widetilde{e}'(r) - \frac{n\varepsilon}{r} \geq 0$$

and this in turn implies that $-\widetilde{e}(r)^{-\gamma} - n\varepsilon\gamma\log r$ is an increasing function of r, namely that $e(r)$ decays as

$$e(r) + 2\alpha^{-1}Cr^\alpha \leq \widetilde{e}(r) \leq (\widetilde{e}(r_0)^{-\gamma} + n\varepsilon\gamma\log r_0 - n\varepsilon\gamma\log r)^{-1/\gamma} \leq \left(-n\varepsilon\gamma\log\left(\frac{r}{r_0}\right)\right)^{-1/\gamma},$$

which for r_0 sufficiently small implies

$$e(r) \leq 2\left(-n\varepsilon\gamma\log\left(\frac{r}{r_0}\right)\right)^{-1/\gamma}, \quad s < r < r_0. \tag{3-8}$$

Step 2 (decay of the flat norm) Under the same assumptions as Step 1, consider the map $F(x) := x/|x|$ and radii $0 < s \leq r < r_0$. By the area formula,

$$\boldsymbol{M}\big(F_\sharp(T \llcorner (B_r \setminus B_s))\big) \leq \int_{B_r\setminus B_s} \frac{|x^\perp|}{|x|^{n+1}}\, d\|T\| \tag{3-9}$$

$$\leq \left(\int_{B_r \setminus B_s} \frac{|x^\perp|^2}{|x|^{n+2}} \, d\|T\|\right)^{\frac{1}{2}} \underbrace{\left(\int_{B_r \setminus B_s} \frac{1}{|x|^n} \, d\|T\|\right)^{\frac{1}{2}}}_{I_2}$$

$$\leq (e(r) - e(s) + C_1 t^{\alpha_0})^{1/2} I_2,$$

where the last inequality is by (3-1). We estimate I_2 using monotonicity:

$$\text{(3-10)} \qquad \begin{aligned} I_2^2 &\leq \int_{(B_r \setminus B_s) \cap C} \frac{1}{|x|^n} \, d\|T\| = \int_s^r \frac{1}{t^n} \frac{d}{dt}(\|T\|(B_t)) \, dt \\ &\leq \frac{\|T\|(B_r)}{r^n} - \frac{\|T\|(B_s)}{s^n} + n \int_s^r \frac{1}{t} \frac{\|T\|(B_t)}{t^n} \, dt \\ &\leq \left(1 + n \log\left(\frac{r}{s}\right)\right)\left(\frac{\|T\|(B_r)}{r^n} + C r^{\alpha_0}\right). \end{aligned}$$

In particular, we conclude that

$$\text{(3-11)} \quad \boldsymbol{M}\big(F_\sharp(T \llcorner (B_r \setminus B_s))\big) \leq C(\log r - \log s)(e(r) - e(s) + C_1 t^{\alpha_0})^{1/2} \quad \text{for all } 0 < s \leq r < r_0$$

Let $0 < s^{1/2} < r^{1/2} < r_0$ be such that $s/r_0 \in [2^{-2^{i+1}}, 2^{-2^i})$ and $t/r_0 \in [2^{-2^{j+1}}, 2^{-2^j})$ for some $j \leq i$; then, applying the previous estimate to the exponentially dyadic decomposition, we obtain

$$\text{(3-12)} \quad \begin{aligned} &\boldsymbol{M}\big(F_\sharp(T \llcorner (B_t \setminus B_s))\big) \\ &\leq \boldsymbol{M}\big(F_\sharp(T \llcorner (B_t \setminus B_{2^{-2^{j+1}} r_0}))\big) + \boldsymbol{M}\big(F_\sharp(T \llcorner (B_{2^{-2^i} r_0} \setminus B_s))\big) \\ &\qquad + \sum_{k=j+1}^{i-1} \boldsymbol{M}\big(F_\sharp(T \llcorner (B_{2^{-2^{k+1}} r_0} \setminus B_{2^{-2^k} r_0}))\big) \\ &\leq C \sum_{k=j}^{i} (\log(2^{-2^k}) - \log(2^{-2^{k+1}}))^{1/2} (e(2^{-2^k} r_0) - e(2^{-2^{k+1}} r_0))^{1/2} \\ &\leq C \sum_{k=j}^{i} 2^{k/2} e(2^{-2^k} r_0)^{1/2} \leq C \sum_{k=j}^{i} 2^{(1-1/\gamma)k/2} \\ &\leq C 2^{(1-1/\gamma)j/2} \leq C\left(-\log\left(\frac{t}{r_0}\right)\right)^{(\gamma-1)/(2\gamma)}, \end{aligned}$$

where C is a dimensional constant that may vary from line to line. Since

$$\partial F_\sharp(T \llcorner (B_r \setminus B_s)) = \partial(T_r \llcorner B_1) - \partial(T_s \llcorner B_1)$$

for a.e. $0 < s < r$, from the definition of $\mathcal{F}$ we get, by (3-12),

$$\mathcal{F}\big(\partial((T_r - T_s)\llcorner B_1)\big) \le C\left(-\log\left(\frac{r}{r_0}\right)\right)^{(\gamma-1)/(2\gamma)}. \tag{3-13}$$

Step 3 (uniqueness of tangent cone) Let $\delta = \delta(\boldsymbol{C}) > 0$ be the constant of the epiperimetric inequality, Theorem 1.1, and let $\delta_1 = \delta_1\big(\delta_0, \frac{1}{4}, \boldsymbol{C}\big) > 0$ be the constant of Proposition 3.2 with $\tau = \frac{1}{4}$. Moreover, let $\delta_2 = \delta_2(\varepsilon_2) > 0$ be the constant of Lemma 3.3. Thanks to the assumption that $\boldsymbol{C}$ is a blowup of T at 0, we can choose $\varepsilon_2 = \varepsilon_2(\boldsymbol{C}) > 0$ and $r = r(\boldsymbol{C}) > 0$, with $r < \min\{\delta_2, \delta_1\}$, in such a way that

$$(\|T_{4r}\|(B_1) - \Theta_{\boldsymbol{C}}) + C\left(-\log\left(\frac{r}{r_0}\right)\right)^{(\gamma-1)/(2\gamma)} + \varepsilon_2 + \mathcal{F}\big(\partial((T_{2r} - \boldsymbol{C})\llcorner B_1)\big) \le \eta, \tag{3-14}$$

where $\eta > 0$ is the constant of Proposition 3.2, and C and γ are constants depending only on $\boldsymbol{C}$ chosen as in (3-13). Notice that by Proposition 3.2, the assumptions of Steps 1 and 2 are satisfied, with $t = r$ and $s = \frac{1}{4}r$, so that by (3-13) we get

$$\mathcal{F}(\partial((T_r - T_{r/4})\llcorner B_1)) \le C\left(-\log\left(\frac{r}{r_0}\right)\right)^{(\gamma-1)/(2\gamma)}.$$

Thanks to our choice (3-14) and Lemma 3.3, we can then apply Theorem 1.1 at the scales $[2^{-2^2} r_0, 2^{-2} r_0]$, and, proceeding inductively in this way to establish (3-13) on exponentially dyadic scales, we conclude that the blowup is unique.

Step 4 (proof of (1-4) and (1-5)) The proofs of (1-4) and (1-5) are analogous to [6, Theorem 3.1] using (3-12) instead of the power rate given by (3.13) there.

The whole proof when $\gamma = 0$ (ie the classical epiperimetric inequality) follows similarly, but we get a power instead of logarithmic rate of convergence. For details, see [6]. □

3.3 Proof of Corollary 1.6

We start by observing that thanks to the decomposition lemma [13, Corollary 3.16], we can decompose $T = \sum_{j=-\infty}^{\infty} \partial[\![U_j]\!]$, with each $\partial[\![U_j]\!]$ almost area-minimizing. It follows that if $\boldsymbol{C}$ is a blowup of $\partial[\![U_J]\!]$ at $x_0 \in \operatorname{spt} T$, then $\boldsymbol{C}$ is either a multiplicity-1 plane or a multiplicity-1 cone with $\boldsymbol{C} \cap \partial B_1$ a smooth embedded submanifold of ∂B_1. If we can prove that each $\partial[\![U_J]\!]$ is almost area-minimizing in some $\mathbb{R}^{n+k}$, the conclusion then follows by Theorem 1.5.

To see this it is enough to prove that T is almost area-minimizing in $\mathbb{R}^{n+k}$, where k is chosen so that by Nash's theorem we can isometrically embed N in $\mathbb{R}^{n+k}$. Indeed, consider $x \in N$ and a ball $B_r(x) \subset \mathbb{R}^{n+k}$. If $\bar{r}$ is sufficiently small, there is a well-defined C^1 orthogonal projection $\boldsymbol{p}\colon B_{\bar r}(x) \to N$ with the property that $\mathrm{Lip}(\boldsymbol{p}) \le 1 + CAr$, where C is a geometric constant and A denotes the L^∞ norm of the second fundamental form of N. Consider T area-minimizing in N and assume $\bar r < \mathrm{dist}(x, \mathrm{spt}(\partial T))$. Let $r \le \bar r$ and $S \in \boldsymbol{I}_{n+1}(\mathbb{R}^{n+k})$ be such that $\mathrm{spt}(S) \subset B_r(x)$. We set $W := T + \partial S$. If $\|W\|(B_r(x)) \ge \|T\|(B_r(x))$, there is nothing to prove; otherwise, by the standard monotonicity formula we have $\|W\|(B_r(x)) \le \|T\|(B_r(x)) \le Cr^n$. Then $W' := \boldsymbol{p}_\sharp W$ is an admissible competitor for the almost-minimality property of T and we have

$$\begin{aligned}\|T\|(B_r(x)) &\le \|W'\|(B_r(x)) + Cr^{n+\alpha_0} \le (\mathrm{Lip}(\boldsymbol{p}))^n \|W\|(B_r(x)) \\ &\le \|W\|(B_r(x)) + Cr^{n+\min\{1,\alpha_0\}}.\end{aligned}$$

□

Appendix A The Taylor expansion of the area of a spherical graph

In this section, for the reader's convenience, we prove Lemma 2.1.

Proof of Lemma 2.1 Let $H(r,\theta) := (\theta + g(r,\theta))/\sqrt{1+|g(r,\theta)|^2}$, and let $(\tau_i)_{i=1}^{n-1}$ be an orthonormal basis of $T\Sigma$. We observe that

$$D_{\tau_i} H = \frac{1}{(1+|g|^2)^{3/2}}((\tau_i + D_i g)(1+|g|^2) - (\theta+g)(g\cdot D_i g))$$

and we consider the $(n-1)\times(n-1)$ matrix M with entries

$$\begin{aligned} M^{ij} &:= D_{\tau_i} H \cdot D_{\tau_j} H \\ &= \frac{1}{(1+|g|^2)}\Big(\delta^{ij} + \underbrace{\tau_i\cdot D_j g + \tau_j \cdot D_i g + D_i g\cdot D_j g - \frac{1}{(1+|g|^2)}(g\cdot D_i g)(g\cdot D_j g)}_{A^{ij}}\Big),\end{aligned}$$

where we used the orthogonality of τ_i, g and θ. Next, using the formula $\det(I+A) = 1 + \mathrm{trace}(A) + \frac12(\mathrm{trace}(A))^2 - \frac12\,\mathrm{trace}(A^2) + O(|A|^3)$ and recalling that

$$\tau_i\cdot D_j g = \nabla_j(\tau_i\cdot g) - D_j\tau_i\cdot g = -B(\tau_i,\tau_j)\cdot g \quad\text{and}\quad \mathrm{tr}(A) = -2H_\Sigma\cdot g + |Dg^T|^2.$$

with B the second fundamental form of Σ, we deduce that

$$\det M = \frac{1}{(1+|g|^2)^{n-1}}\Big(1 - 2H_\Sigma\cdot g + |(Dg)^\perp|^2 + 2|H_\Sigma\cdot g|^2 \\ - \sum_{i,j=1}^{n}(\tau_i\cdot D_j g)(\tau_j\cdot D_i g) + P_1(g, Dg)\Big)$$

$$= 1+|(Dg)^{\perp}|^2-\sum_{i,j=1}^{n}(B(\tau_i,\tau_j)\cdot g)^2-(n-1)|g|^2+P_1(g,Dg),$$

where $|P_1(s,t)| \le C(st^2+ts^2)$, and in the last inequality we used the fact, since Σ is the spherical cross-section of a stationary cone, $H_\Sigma=(n-1)\theta$, so that $H_\Sigma\cdot g=0$. Now, using the fact that $\sqrt{1+t}=1+\frac{1}{2}t+P_2(t)$, with $|P_2(t)|\le Ct^3$, we conclude

$$|D_\theta H|=\sqrt{\det M}=1+\frac{1}{2}\Big(|(Dg)^{\perp}|^2-\sum_{i,j=1}^{n}(B(\tau_i,\tau_j)\cdot g)^2-(n-1)|g|^2\Big)+P_3(g,Dg)$$

with $|P_3(s,t)|\le C(st^2+ts^2)$. In conclusion we have

$$\text{(A-1)}\quad \Big|\mathcal{A}_\Sigma(g)-\frac{1}{2}\int_\Sigma\Big(|(Dg)^{\perp}|^2-\sum_{i,j=1}^{n}(B(\tau_i,\tau_j)\cdot g)^2-(n-1)|g|^2\Big)\,d\mathcal{H}^{n-1}\Big|$$

$$=\Big|\int_\Sigma\Big[1+\frac{1}{2}\Big(|(Dg)^{\perp}|^2-\sum_{i,j=1}^{n}(B(\tau_i,\tau_j)\cdot g)^2-(n-1)|g|^2\Big)+P(g,Dg)-1\Big]$$

$$-\frac{1}{2}\Big(|(Dg)^{\perp}|^2-\sum_{i,j=1}^{n}(B(\tau_i,\tau_j)\cdot g)^2-(n-1)|g|^2\Big)\,d\mathcal{H}^{n-1}\Big|$$

$$\le\int_\Sigma|P_3(g,Dg)|\,d\mathcal{H}^{n-1}\le C\|g\|_{C^{1,\alpha}(\Sigma,\boldsymbol{C}^\perp)}\|g\|^2_{H^1(\Sigma,\boldsymbol{C}^\perp)},$$

which is (2-2). □

Appendix B Lyapunov–Schmidt reduction for the area functional

We prove the following lemma, which is a modification of the reduction in [12]. First we need some notation; let $K:=\ker\delta^2\mathcal{A}_\Sigma(0)$ and $\ell:=\dim K$, which is finite by spectral theory (as $\delta^2\mathcal{A}_\Sigma(0)$ has compact resolvent). Let P_K be the projection of $L^2(\Sigma;\boldsymbol{C}^\perp)$ onto K and similarly $P_{K^\perp}$ the projection onto $K^\perp$.

Lemma B.1 *There exists a neighborhood U of 0 in $C^{2,\alpha}(\Sigma;\boldsymbol{C}^\perp)$ and an analytic map $\Upsilon\colon K\to K^\perp\subset C^2(\Sigma;\boldsymbol{C}^\perp)$ such that*

$$\text{(B-1)}\qquad \Upsilon(0)=0 \quad\textit{and}\quad \delta\Upsilon(0)=0,$$

and, in addition,

$$\text{(B-2)}\qquad \begin{cases} P_{K^\perp}\big(\delta\mathcal{A}_\Sigma(\zeta+\Upsilon(\zeta))\big)=0 & \text{for all } \zeta\in K\cap U,\\ P_K\big(\delta\mathcal{A}_\Sigma(\zeta+\Upsilon(\zeta))\big)=\nabla A(\zeta) & \text{for all } \zeta\in K\cap U,\end{cases}$$

where $A(\zeta)=\mathcal{A}(\zeta+\Upsilon(\zeta))$ *for every* $\zeta\in K\cap U$. *Furthermore, the critical points of* $\mathcal{A}$ *inside* U *are given by*

$$\mathcal{C}:=\{\zeta+\Upsilon(\zeta)\mid\zeta\in U\cap K\quad\text{and}\quad\nabla A(\zeta)=0\},$$

which is an analytic subvariety of the ℓ*–dimensional manifold given by*

$$\mathcal{M}:=\{\zeta+\Upsilon(\zeta)\mid\zeta\in U\cap K\}.$$

Finally, for all $\zeta,\eta\in U\cap K$, *there is a constant* $C<\infty$ *such that*

$$\text{(B-3)}\qquad \|\delta\Upsilon(\zeta)[\eta]\|_{C^{2,\alpha}}\le C\|\eta\|_{C^{0,\alpha}}.$$

Proof Define the operator

$$\mathcal{N}(\zeta):=P_{K^\perp}\delta\mathcal{A}_\Sigma(\zeta)+P_K\zeta\colon L^2(\Sigma;\boldsymbol{C}^\perp)\to L^2(\Sigma;\boldsymbol{C}^\perp).$$

Since $\boldsymbol{C}$ is a critical point for $\mathcal{A}_\Sigma$ we see that $\mathcal{N}(0)=0$. Furthermore,

$$\delta\mathcal{N}(0)[\zeta]=\frac{d}{dt}\mathcal{N}(t\zeta)|_{t=0}=P_{K^\perp}\delta^2\mathcal{A}_\Sigma(0)[\zeta,-]+P_K\zeta.$$

In particular, $\delta\mathcal{N}(0)$ has trivial kernel. Then Schauder estimates (applied to $-\Delta_\Sigma^\perp-B^TB-(n-1)+P_K$) imply that $\delta\mathcal{N}(0)$ is an isomorphism (in a neighborhood of zero) from $C^{2,\alpha}(\Sigma,\boldsymbol{C}^\perp)$ to $C^{0,\alpha}(\Sigma,\boldsymbol{C}^\perp)$.

We apply the inverse function theorem to $\mathcal{N}$ in this neighborhood, producing the map $\Psi:=\mathcal{N}^{-1}$, which is a bijection from a neighborhood of 0, $W\subset C^{0,\alpha}(\Sigma;\boldsymbol{C}^\perp)$, to U, a neighborhood of 0 in $C^{2,\alpha}(\Sigma;\boldsymbol{C}^\perp)$.

We claim our desired map is given by $\Upsilon:=P_{K^\perp}\circ\Psi\colon K\to K^\perp$. In particular, for $\zeta\in K$ we have $\Psi(\zeta)=\zeta+\Upsilon(\zeta)$. The first conclusion of (B-1) is trivial as $\Upsilon(0)=\Upsilon(\mathcal{N}(0))=P_{K^\perp}\big(\Psi(\mathcal{N}(0))\big)=0$.

To check (B-2), we first notice that

$$\text{(B-4)}\qquad \zeta=\mathcal{N}(\Psi(\zeta))=P_{K^\perp}\delta\mathcal{A}_\Sigma(\Psi(\zeta))+P_K\Psi(\zeta).$$

Applying P_K or $P_{K^\perp}$ to both sides of that equation we get

$$P_K\zeta=P_K\Psi(\zeta),\quad\text{and}\quad P_{K^\perp}\zeta=P_{K^\perp}\delta\mathcal{A}(\Psi(\zeta)).$$

Plugging the first identity into the second we obtain

$$P_{K^\perp}\zeta = P_{K^\perp}\delta\mathcal{A}(P_K\zeta + \Upsilon(\zeta)),$$

which implies, for $\zeta \in K \cap U$, that

$$0 = P_{K^\perp}\delta\mathcal{A}(\zeta + \Upsilon(\zeta)).$$

To prove the second line of (B-2), we compute, for any $\eta \in K$,

$$\langle \nabla A(\zeta), \eta\rangle = \delta\mathcal{A}_\Sigma(\zeta + \Upsilon(\zeta))[\eta + \delta\Upsilon(\zeta)[\eta]] = \delta\mathcal{A}_\Sigma(\zeta + \Upsilon(\zeta))[\eta],$$

which implies the second claim of (B-2) (as $\eta \in K$ is arbitrary). The second inequality above follows from the fact that $\delta\Upsilon(\zeta)[\eta] \in K^\perp$ (as the image of Υ is in $K^\perp$) and then from the first line of (B-2).

To see that all critical points are given by $\zeta + \Upsilon(\zeta)$ we turn to (B-4). Let η be an arbitrary critical point of $\mathcal{A}_\Sigma$, in a neighborhood of zero. We write $\eta = \Psi(\zeta)$, and (B-4) reads $\zeta = P_K\eta$, which implies

$$\eta = P_K\eta + P_{K^\perp}\eta = \zeta + P_{K^\perp}\Psi(\zeta) = \zeta + \Upsilon(\zeta),$$

as desired (the condition on ∇A follows trivially from (B-2)).

Finally, to prove (B-3) we write

$$\eta = \delta\mathcal{N}(\Psi(\zeta))[\delta\Psi(\zeta)[\eta]] = P_K\delta\Psi(\zeta)[\eta] + P_{K^\perp}\delta^2\mathcal{A}_\Sigma(\Psi(\zeta))[\delta\Psi(\zeta)[\eta], -],$$

which implies

$$\begin{aligned} &P_K\delta\Psi(\zeta)[\eta] + P_{K^\perp}\delta^2\mathcal{A}(0)[\delta\Psi(\zeta)[\eta], -] \\ &\qquad = \eta + P_{K^\perp}\big(\delta^2\mathcal{A}_\Sigma(0) - \delta^2\mathcal{A}_\Sigma(\Psi(\zeta))\big)[\delta\Psi(\zeta)[\eta], -]. \end{aligned}$$

When you apply P_K to both sides of the above equation you get $P_K\delta\Psi(\zeta)[\eta] = P_K\eta = \eta$. Applying $P_{K^\perp}$ to both sides and taking $C^{0,\alpha}$ norms yields the more complicated

$$\begin{aligned} \|\delta\Upsilon(\zeta)[\eta]\|_{C^{2,\alpha}} &\le \|P_{K^\perp}\delta^2\mathcal{A}(0)[\delta\Psi(\zeta)[\eta], -]\|_{C^{0,\alpha}} \\ &\le \|P_{K^\perp}\big(\delta^2\mathcal{A}_\Sigma(0) - \delta^2\mathcal{A}_\Sigma(\Psi(\zeta))\big)[\delta\Psi(\zeta)[\eta], -]\|_{C^{0,\alpha}} \\ &\le \varepsilon\|\delta\Psi(\zeta)[\eta]\|_{C^{2,\alpha}}, \end{aligned}$$

where $\varepsilon > 0$ is a constant which can be taken arbitrarily small with the size of the neighborhood U. Note the first inequality above follows from Schauder estimates on the operator $-\Delta_\Sigma^\perp - B^TB - (n-1) + P_K$.

Writing $\delta\Psi(\zeta)[\eta] = \delta\Upsilon(\zeta)[\eta] + P_K\delta\Psi(\zeta)[\eta]$, we have

$$\|\delta\Upsilon(\zeta)[\eta]\|_{C^{2,\alpha}} \le C\|P_K\delta\Psi(\zeta)[\eta]\|_{C^{2,\alpha}} \simeq \|P_K\delta\Psi(\zeta)[\eta]\|_{C^{0,\alpha}},$$

as P_K is a finite-dimensional projection (so all norms are equivalent). Recalling the above observation, that $P_K\delta\Psi(\zeta)[\eta] = \eta$, finishes the proof. □

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References

[1] **D Adams**, **L Simon**, *Rates of asymptotic convergence near isolated singularities of geometric extrema*, Indiana Univ. Math. J. 37 (1988) 225–254 MR

[2] **W K Allard**, **F J Almgren, Jr**, *On the radial behavior of minimal surfaces and the uniqueness of their tangent cones*, Ann. of Math. 113 (1981) 215–265 MR

[3] **T H Colding**, **W P Minicozzi, II**, *Uniqueness of blowups and Łojasiewicz inequalities*, Ann. of Math. 182 (2015) 221–285 MR

[4] **M Colombo**, **L Spolaor**, **B Velichkov**, *Direct epiperimetric inequalities for the thin obstacle problem and applications*, preprint (2017) arXiv

[5] **M Colombo**, **L Spolaor**, **B Velichkov**, *A logarithmic epiperimetric inequality for the obstacle problem*, Geom. Funct. Anal. 28 (2018) 1029–1061 MR

[6] **C De Lellis**, **E Spadaro**, **L Spolaor**, *Uniqueness of tangent cones for two-dimensional almost-minimizing currents*, Comm. Pure Appl. Math. 70 (2017) 1402–1421 MR

[7] **M Engelstein**, **L Spolaor**, **B Velichkov**, *Uniqueness of the blow-up at isolated singularities for the Alt–Caffarelli functional*, preprint (2018) arXiv

[8] **S Łojasiewicz**, *Ensembles semi-analytiques*, IHES, Bures-sur-Yvette, France (1965)

[9] **T Nagura**, *On the Jacobi differential operators associated to minimal isometric immersions of symmetric spaces into spheres, III*, Osaka J. Math. 19 (1982) 241–281 MR

[10] **E R Reifenberg**, *An epiperimetric inequality related to the analyticity of minimal surfaces*, Ann. of Math. 80 (1964) 1–14 MR

[11] **R Schoen**, **L Simon**, *A new proof of the regularity theorem for rectifiable currents which minimize parametric elliptic functionals*, Indiana Univ. Math. J. 31 (1982) 415–434 MR

[12] **L Simon**, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. 118 (1983) 525–571 MR

[13] **L Simon**, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University 3, Australian National University, Canberra (1983) MR

[14] **J E Taylor**, *Regularity of the singular sets of two-dimensional area-minimizing flat chains modulo* 3 *in* R^3, Invent. Math. 22 (1973) 119–159 MR

[15] **J E Taylor**, *The structure of singularities in solutions to ellipsoidal variational problems with constraints in* R^3, Ann. of Math. 103 (1976) 541–546 MR

[16] **B White**, *Tangent cones to two-dimensional area-minimizing integral currents are unique*, Duke Math. J. 50 (1983) 143–160 MR

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