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Gauge theory on Aloff–Wallach spaces

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For gauge groups $U(1)$ and SO(3) we classify invariant G_2–instantons for homogeneous coclosed G_2–structures on Aloff–Wallach spaces $X_{k,l}$. As a consequence, we give examples where G_2–instantons can be used to distinguish between different strictly nearly parallel G_2–structures on the same Aloff–Wallach space. In addition to this, we find that while certain G_2–instantons exist for the strictly nearly parallel G_2–structure on $X_{1,1}$, no such G_2–instantons exist for the 3–Sasakian one. As a further consequence of the classification, we produce examples of some other interesting phenomena, such as irreducible G_2–instantons that, as the structure varies, merge into the same reducible and obstructed one and G_2–instantons on nearly parallel G_2–manifolds that are not locally energy-minimizing.

53C07, 53C29, 53C38, 57R57

1 Introduction

A 3–form φ on an oriented 7–dimensional manifold X^7 is called a G_2–structure if it takes values in a certain open subbundle $\Lambda^3_+ \subset \Lambda^3$. Such 3–forms φ determine (in a nonlinear way) a Riemannian metric g_φ. In the case when the holonomy of g_φ lies inside the exceptional Lie group G_2, the pair (X^7, φ) is called a G_2–manifold, or equivalently φ is said to be torsion free. A G_2–instanton is a solution to a gauge theoretical equation that can be written in an oriented 7–dimensional manifold X^7 equipped with a G_2–structure φ. Even though G_2–instantons have been part of the mathematical literature for over 30 years now (see Corrigan, Devchand, Fairlie and Nuyts [12]), it was only in the past few years that the first nontrivial examples appeared, namely from Sá Earp and Walpuski [25; 26; 27], Clarke [11] and Lotay and Oliveira [22; 24]. This and recent interest in G_2–instantons is mostly due to the suggestion by Donaldson, Segal and Thomas [14; 15] that it may be possible to use G_2–instantons to construct an enumerative invariant of G_2–manifolds. However, adding to the scarcity of examples there are substantial difficulties in constructing such an invariant. In fact, it is conceivable that in order to overcome some of these difficulties

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one may need to consider G_2–structures that are not torsion free. Indeed, there is a larger class of G_2–structures, other than just the torsion-free class, with respect to which the G_2–instanton equation still lies in an elliptic complex. All of this leads us to investigate G_2–instantons for these more general G_2–structures. For example, one may ask to what extent G_2–instantons are persistent under deformations of the G_2–structure. In this paper we classify homogeneous (invariant) G_2–instantons on an infinite family of 7–manifolds admitting many such G_2–structures. As a consequence we find many examples of new phenomena and are able to investigate what happens to the G_2–instantons when the G_2–structure varies.

1.1 Preliminaries

Let (X^7, φ) be a compact, oriented, 7–manifold equipped with a G_2–structure φ. Let g_φ be the induced Riemannian metric, $*_\varphi$ the associated Hodge star, and ψ the 4–form $*_\varphi \varphi$. If G is a compact, semisimple Lie group and $P \to X$ is a principal G–bundle, a connection A on P is called a G_2–instanton if

$$F_A \wedge \psi = 0, \tag{1-1}$$

where F_A denotes the curvature of A. When the G_2–structure is coclosed, ie $d\psi = 0$, the G_2–instanton equation lies in an elliptic complex and we shall restrict to this case. The torsion-free G_2–structures correspond to the special case when φ is harmonic. One other special class of coclosed G_2–structures are the so-called nearly parallel ones, for which $d\varphi = \lambda\psi$ for some $\lambda \neq 0$. If φ is nearly parallel, then g_φ is Einstein with positive scalar curvature. Another perspective on nearly parallel G_2–structures is that they are exactly those G_2–structures for which the metric cone $(\mathbb{R}^+ \times X^7, g_C = dr^2 + r^2 g_\varphi)$ has holonomy contained in $\mathrm{Spin}(7)$.

One other interesting class of connections on a principal bundle over an oriented Riemannian manifold are the Yang–Mills connections. These are defined as the critical points of the Yang–Mills energy

$$E(A) = \frac{1}{2}\int_X |F_A|^2,$$

where we use an Ad–invariant inner product to compute the norm $|F_A|$. If the G_2–structure is either torsion free or nearly parallel, then G_2–instantons are also Yang–Mills connections. Moreover, in the torsion-free case a simple computation (see (2-4)) shows that any G_2–instanton actually minimizes the Yang–Mills energy.

1.2 Summary of the main results

The Aloff–Wallach space $X_{k,l}$ is defined as the quotient of $\mathrm{SU}(3)$ by a $U(1)$ subgroup, whose embedding in $\mathrm{SU}(3)$ is determined by two integers k and l. On each $X_{k,l}$ we consider a real 4–dimensional family $\mathcal{C}$ of G_2–structures, which contains exactly two nearly parallel G_2–structures. As proved by Cabrera, Monar and Swann [9], for most[1] k and l this family completely exhausts all homogeneous, coclosed G_2–structures. In fact, for $k \neq l$, $k \neq 2l$, $l \neq -2k$, the two nearly parallel G_2–structures are strict, meaning that the holonomy of the cone metric $g_C = dr^2 + r^2 g_\varphi$ on $\mathbb{R}^+ \times X_{k,l}$ is exactly $\mathrm{Spin}(7)$. These and other facts regarding the geometry of Aloff–Wallach spaces are recalled, with more detail, in Section 3. In Section 3.2, we classify invariant connections on each $X_{k,l}$. These results are then used in Section 4 to investigate G_2–instantons on the Aloff–Wallach spaces $X_{k,l}$, for $k \neq l$, $k \neq 2l$, $l \neq -2k$. The remaining cases are analyzed separately in Section 5. We now summarize the main results of those sections starting with the more general situation. In Section 4.2 we classify invariant abelian G_2–instantons with respect to all $\varphi \in \mathcal{C}$; see Theorem 42. Here we only state a corollary, which is proved in the third item of Remark 43:

Theorem 1 *Let $k \neq l$, $k \neq 2l$, $l \neq -2k$. For the generic $\varphi \in \mathcal{C}$ there is a unique invariant G_2–instanton on any homogeneous complex line bundle over $X_{k,l}$. However, for any such k and l, there do exist $\varphi \in \mathcal{C}$ so that any such bundle has a 1–parameter family of invariant G_2–instantons.*

Then, in Section 4.3, we focus on invariant G_2–instantons with gauge group $\mathrm{SO}(3)$. Any homogeneous $\mathrm{SO}(3)$–bundle on $X_{k,l}$ can be constructed as

$$P_{\lambda_n} = \mathrm{SU}(3) \times_{U(1)_{k,l},\lambda_n} \mathrm{SO}(3),$$

where $\lambda_n\colon U(1)_{k,l} \to \mathrm{SO}(3)$ is a group homomorphism and the integer $n \in \mathbb{Z}$ denotes the degree of the induced map between maximal tori. We construct explicit maps $\sigma_i\colon \mathcal{C} \to \mathbb{R}$, for $i = 1, 2, 3$, whose significance is given in Theorem 44. Below we give a summarized version of that result, when combined with Theorem 46.

Theorem 2 *Let $k \neq l$, $k \neq 2l$, $l \neq -2k$, and let φ be a homogeneous coclosed G_2–structure on $X_{k,l}$. Then invariant and irreducible G_2–instantons on P_{λ_n} with respect to φ exist if and only if one of the following holds:*

[1] $k \neq \pm l$, $k \neq 0$, $l \neq 0$, $k \neq 2l$, $l \neq -2k$.

(1) $n = k - l$ *and* $\sigma_1(\varphi) > 0$,

(2) $n = 2l + k$ *and* $\sigma_2(\varphi) > 0$,

(3) $n = -l - 2k$ *and* $\sigma_3(\varphi) > 0$.

Moreover, if $\{\varphi(s)\}_{s\in\mathbb{R}} \subset \mathcal{C}$ *is a continuous family of* G_2*–structures with* $\{\sigma_1(\varphi(s))\}_{s\in\mathbb{R}}$ *crossing zero once from above, then as* $\sigma_1(\varphi(s)) \searrow 0$*, two irreducible* G_2*–instantons on* P_{k-l} *merge and become the same reducible and obstructed* G_2*–instanton for* $\sigma_1(\varphi(s)) \leq 0$*. Similar statements hold for* σ_2 *and* σ_3*.*

To better visualize the content of the last part of this theorem we refer the reader to Examples 48 and 49, together with their respectively accompanying Figures 1 and 2. Recall that for $k \neq l$, $k \neq 2l$, $l \neq -2k$, the Aloff–Wallach space $X_{k,l}$ admits two strictly nearly parallel G_2–structures. As an application of Theorem 2, in Section 4.4, we use G_2–instantons to distinguish these for many values of k and l. Here we will simply state the following:

Corollary 3 *The are many examples of* k *and* l *as in Theorem 2 such that the two inequivalent strictly nearly parallel* G_2*–structures on* $X_{k,l}$ *always admit invariant and irreducible* G_2*–instantons, but on topologically different* SO(3)*–bundles.*

In Section 4.6 we consider a particular example, namely $X_{1,-1}$. As one other application of Theorem 2, we show in Section 4.6.1 that $X_{1,-1}$ admits nonabelian, irreducible G_2–instantons for a strictly nearly parallel G_2–structure. These G_2–instantons are also Yang–Mills, as the G_2–structure is nearly parallel, but contrary to the torsion-free case we show in Section 4.6.2 that they are not energy-minimizing (not even locally). We refer the reader to Figure 3 for a contour plot of the invariant Yang–Mills functional. The results quoted above can be combined into the following:

Theorem 4 *There is a strictly nearly parallel* G_2*–structure* φ *on* $X_{1,-1}$ *such that:*

- *For gauge group* SO(3)*, there is an irreducible* G_2*–instanton* A *with respect to* φ*.*
- *As a Yang–Mills connection,* A *is not locally energy-minimizing.*

We now turn to the case when either $k = l$ or $k = 2l$ or $l = -2k$, which was excluded from the previous results. Using the action of the Weyl group of SU(3), and up to coverings, we may assume $k = l = 1$, so that we are working on $X_{1,1}$. This case is analyzed in Section 5. As already remarked before, on $X_{1,1}$ the G_2–structures we consider, ie those in $\mathcal{C}$, are not all the homogeneous, coclosed ones. Nevertheless, $\mathcal{C}$ does contain nearly parallel G_2–structures, inducing two different metrics, one of which is 3–Sasakian and the other strictly nearly parallel. There is however, one other homogeneous nearly parallel G_2–structure not in $\mathcal{C}$, which a Sasaki–Einstein metric. Our first result for $X_{1,1}$ is Theorem 62, which classifies invariant abelian G_2–instantons with respect to the $\varphi \in \mathcal{C}$. The statement is similar to the case $k \neq l$ in Theorem 1. As in that case, the generic φ admits a unique invariant G_2–instanton on any line bundle, however there do exist $\varphi \in \mathcal{C}$ so that the space of invariant G_2–instantons on any complex line bundle is 3–dimensional. In fact, this can be interpreted in light of a more general phenomenon explained in Proposition 17. Then, in Theorem 64, we consider $\mathrm{SO}(3)$–bundles over $X_{1,1}$, and for all $\varphi \in \mathcal{C}$ classify irreducible invariant G_2–instantons on them. The statement is however very similar to that of Theorem 2 and we shall omit it in this introduction. Instead, we state here Corollary 73, which is a direct application of that result. Its content is that the existence of invariant G_2–instantons, with gauge group $\mathrm{SO}(3)$, distinguishes between the G_2–structures inducing the 3–Sasakian and the strictly nearly parallel metrics.

Theorem 5 *Let φ^{ts} and φ^{np} be respectively the G_2–structures inducing the 3–Sasakian and the strictly nearly parallel metrics on $X_{1,1}$. Then there are no irreducible invariant G_2–instantons with gauge group $\mathrm{SO}(3)$ for φ^{ts}, but such G_2–instantons do exist for φ^{np}.*

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2 Gauge theory and coclosed G_2–structures

2.1 Background

We begin, in Section 2.1.1, with some basic facts about G_2–structures[2] and their torsion. In Section 2.1.2 we recall some background on G_2–gauge theory. In particular, we identify the coclosed G_2–structures, ie those for which $d\psi = 0$, as the ones for which the G_2–instanton equation lies in an elliptic complex. Then, in Section 2.1.3, we derive

[2] See [8] for more on this and other aspects of G_2–structures.

some general results on the deformation theory of G_2–instantons. These will be used to give an abstract result, Proposition 13, yielding a criterion for when a G_2–structure has the property that any circle bundle processes a G_2–instanton. As a consequence, in Corollary 14 this result is applied in the strictly nearly parallel setting.

2.1.1 Coclosed G_2–structures

Torsion of a G_2–structure Fernández and Gray first classified the torsion of G_2–structures in [16] by decomposing $\nabla\varphi$ into irreducible G_2–representations. The components of $d\varphi$ and $d\psi = d(*\varphi)$ can then be written in terms of those of $\nabla\varphi$. What is nontrivial, but easily checked using the representation theory of G_2, is that the converse is also true. Recall that the 2–forms and 3–forms decompose into irreducible G_2–representations as $\Lambda^2 \cong \Lambda^2_7 \oplus \Lambda^2_{14}$ and $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, where the subscript denotes the dimension of the representation. The Hodge star is an isomorphism of representations and so induces isomorphic decompositions in Λ^4 and Λ^5. Using these decompositions the Fernández–Gray classification can be recast as follows. Given a G_2–structure φ, we have

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3 \quad \text{and} \quad d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi$$

for some uniquely determined $\tau_0 \in \Omega^0(X)$, $\tau_1 \in \Omega^1(X)$, $\tau_2 \in \Omega^2_{14}(X)$ and $\tau_3 \in \Omega^3_{27}(X)$. Of special interest to us will be the case when the G_2–structure is coclosed, ie when $d\psi = d(*\varphi) = 0$. Then $\tau_1 = \tau_2 = 0$ and $d\varphi = \tau_0\psi + *\tau_3$.

For future reference we shall use π_i, for $i = 1, 7, 14, 27$, to denote the projection onto an i–dimensional irreducible representation. For example, if ω is a 2–form we shall denote by $\pi_7(\omega)$ the component of $\omega \in \Lambda^2_7$.

Nearly parallel G_2–structures We now turn to the definition of nearly parallel G_2–structures. Given a closed, oriented, 7–manifold (X^7, φ) equipped with a G_2–structure, its metric cone $(\mathbb{R}^+ \times X^7, g_C = dr^2 + r^2 g_\varphi)$ comes equipped with a Spin(7)–structure determined by $\Omega = r^3\, dr \wedge \varphi + r^4\psi$. From the Riemannian holonomy point of view, if g_C is nonsymmetric its holonomy is one of the groups in the ascending chain

$$\{1\} \subset \mathrm{Sp}(2) \subset \mathrm{SU}(4) \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8).$$

Equivalently, thinking of G_2 as the group stabilizing a nonvanishing spinor in seven dimensions, the groups above are possible stabilizers of spinors in eight dimensions and each is determined by the number of linearly independent spinors fixed. In the language of spinors, the condition that the holonomy reduces to one of the groups above is then that the respective spinors are parallel. Given a metric g on X^7, the cone metric $g_C = dr^2 + r^2 g$ has holonomy contained in $\mathrm{Spin}(7)$ if and only if there is a compatible G_2–structure φ such that the 4–form $\Omega = r^3\, dr \wedge \varphi + r^4 \psi$ is closed. That is the case if and only if $d\varphi = 4\psi$, which up to scaling and changing the orientation can be written as

$$d\varphi = \lambda\psi, \tag{2-1}$$

for some $\lambda \in \mathbb{R}\backslash\{0\}$.

Definition 6 A Riemannian manifold (X^7, g_φ) is said to be nearly parallel if, after possibly scaling the metric g_φ and changing the orientation, the holonomy of the metric cone satisfies $\mathrm{Hol}(g_C) \subseteq \mathrm{Spin}(7)$. A metric g_φ is said to be 3–Sasakian, Sasaki–Einstein, or strictly nearly parallel if, again after possibly scaling the metric g_φ and changing the orientation, $\mathrm{Hol}(g_C)$ is $\mathrm{Sp}(2)$, $\mathrm{SU}(4)$, or $\mathrm{Spin}(7)$, respectively. A G_2–structure φ is said to be nearly parallel, 3–Sasakian, Sasaki–Einstein, or strictly nearly parallel if the induced metric g_φ is nearly parallel, 3–Sasakian, Sasaki–Einstein, or strictly nearly parallel, respectively.

Equivalently, nearly parallel G_2–structures are exactly those satisfying (2-1). Notice that, as ψ is exact, (2-1) implies $d\psi = 0$ so that φ is coclosed, meaning that, from the point of view of torsion of G_2–structures, τ_1, τ_2 and τ_3 all vanish and $\tau_0 = \lambda$ is the only nonzero component. As τ_0 is the torsion component living in the smallest irreducible representation, this is the sense in which we think of nearly parallel G_2–structures as close to being parallel.

Remark 7 In fact, if we require that $d\psi = 0$ separately and allow λ to vanish, then (2-1) also includes the torsion-free case. This shall be useful as some arguments used for nearly parallel G_2–structures also work in the torsion-free case.

In [18], the authors classify homogeneous nearly parallel G_2–manifolds, and give a construction of strictly nearly parallel G_2–structures starting from 3–Sasakian manifolds. We shall recall and use this construction in Section 2.2.

2.1.2 Gauge theory Let G be a compact semisimple Lie group and P a principal G–bundle over a manifold X, equipped with a G_2–structure φ. Recall that a connection A

on P is called a G_2–instanton if $F_A \wedge \psi = 0$, equivalently if $\pi_7(F_A) = 0$, or if the following analogue of anti-self-duality holds:

$$*F_A = -F_A \wedge \varphi. \tag{2-2}$$

On the other hand, a connection A is said to be Yang–Mills if it is a critical point of the Yang–Mills energy

$$E(A) = \frac{1}{2}\int_X |F_A|^2 \,\mathrm{dvol}_g, \tag{2-3}$$

and so satisfies the Yang–Mills equation $d_A^* F_A = 0$, which together with the Bianchi identity $d_A F_A = 0$ forms a second-order elliptic system for the connection (up to gauge). G_2–instantons satisfy a first-order equation which in this generality need not imply they are Yang–Mills connections. Nevertheless we have the following folklore result, which in the nearly parallel case is due to Harland and Nölle [19].

Proposition 8 [19] *If the G_2–structure is either parallel or nearly parallel, then any G_2–instanton is a Yang–Mills connection.*

Proof If the G_2–structure is either parallel or nearly parallel, $d\psi = 0$ and $d\varphi = \lambda\psi$ for some $\lambda \in \mathbb{R}$, as in Remark 7. Then, if A is a G_2–instanton, $*F_A = F_A \wedge \varphi$ and so

$$d_A{*}F_A = d_A(F_A \wedge \varphi) = \lambda F_A \wedge \psi = 0,$$

where in the last equality we use the Bianchi identity and $d\varphi = \lambda\psi$. □

The Yang–Mills energy can be equivalently written as

$$E(A) = -\frac{1}{2}\int_X \langle F_A \wedge F_A\rangle \wedge \varphi + \frac{1}{2}\|F_A \wedge \psi\|_{L^2}^2. \tag{2-4}$$

In particular, if φ is torsion free, then the first term is topological and G_2–instantons minimize the Yang–Mills energy. It is then a natural question to ask if the same must hold for nearly parallel G_2–structures. We shall show in Example 28 that is not the case, by providing an example of a nearly parallel G_2–structure, together with a G_2–instanton which is unstable as a Yang–Mills connection.

Remark 9 The variation of the Yang–Mills functional at a connection A is

$$\delta^2 E_A(a) = \frac{d^2}{ds^2}\bigg|_{s=0} E(A+sa) = \int_X |d_A a|^2 - \langle [a \wedge a], F_A\rangle, \tag{2-5}$$

and so we may instead think of the second-order operator $H = d_A^* d_A a - *[a \wedge *F_A]$.

When the G_2–structure φ is coclosed the G_2–instanton equation lies on the elliptic complex

$$\Omega^0(X,\mathfrak{g}_P)\xrightarrow{-d_A\cdot}\Omega^1(X,\mathfrak{g}_P)\xrightarrow{d_A\cdot\wedge\psi}\Omega^6(X,\mathfrak{g}_P)\xrightarrow{d_A}\Omega^7(X,\mathfrak{g}_P). \tag{2-6}$$

Hence, in the coclosed case the G_2–instanton equation is elliptic modulo gauge (rather than overdetermined). From now on we shall suppose this is the case.

Remark 10 (1) The reason the G_2–instanton equation is consistent in the torsion-free case can be interpreted as follows. The G_2–monopole equation

$$*\nabla_A\Phi = F_A\wedge\psi$$

is always elliptic modulo gauge. Moreover, if φ is coclosed, then the monopole equation, $d\psi=0$, and the Bianchi identity, $d_AF_A=0$, give $\Delta_A\Phi=0$. We can then compute $\Delta|\Phi|^2=-2|\nabla_A\Phi|^2\leq 0$, and the maximum principle implies that $|\Phi|^2$ is constant. Then $|\nabla_A\Phi|^2$ must vanish, and the monopole equation reduces to the G_2–instanton equation. Furthermore, the fact that $\nabla_A\Phi=0$ implies that if $\Phi\neq 0$, and G is semisimple, then A must be reducible.

(2) If the G_2–structure φ is not coclosed one may ask questions similar to those answered in this paper, but for G_2–monopoles rather than G_2–instantons.

In particular, if (X,φ) is a compact irreducible G_2–manifold, ie the holonomy of the metric g_φ induced by φ is equal to G_2, any harmonic 2–form can be shown to be of type Λ^2_{14} and so if $F\in\Omega^2(X)$ is harmonic and has integer periods, it defines the curvature of a connection on a line bundle whose first Chern class is $[F]/2\pi i$. Still in the torsion-free case, Thomas Walpuski [26; 27], using the results of [25], constructed the only known examples of nonabelian G_2–instantons on compact, irreducible, G_2–manifolds. There are also examples in the noncompact case; see [11; 24; 22].

2.1.3 Deformation theory and abelian G_2–instantons The main idea for this approach to the deformation theory comes from Remark 10. This suggests that given a coclosed G_2–structure, instead of studying the deformation theory of an irreducible G_2–instanton A we may instead study that of a G_2–monopole (A,Φ) with $\Phi=0$. Before restricting to that case suppose for now that $\Phi\neq 0$. Then the relevant elliptic complex is

$$\Omega^0(X,\mathfrak{g}_P)\xrightarrow{d_1}\Omega^1(X,\mathfrak{g}_P)\oplus\Omega^0(X,\mathfrak{g}_P)\xrightarrow{d_2}\Omega^1(X,\mathfrak{g}_P), \tag{2-7}$$

with $d_1(\phi) = (-d_A\phi, [\phi, \Phi])$ and $d_2(a, \phi) = *(d_A a \wedge \psi) - [a, \Phi] - d_A\phi$. Equivalently, we can consider the elliptic operator

$$d_1^* \oplus d_2\colon \Omega^1(X, \mathfrak{g}_P) \oplus \Omega^0(X, \mathfrak{g}_P) \to \Omega^1(X, \mathfrak{g}_P) \oplus \Omega^0(X, \mathfrak{g}_P)$$

given by

$$(d_1^* \oplus d_2)(a, \phi) = (*(d_A a \wedge \psi) - d_A\phi, -d_A^* a) + ([\Phi, a], [\Phi, \phi]),$$

which is self-adjoint when φ is coclosed. The following result, which is a consequence of Remark 10, shows that in the coclosed case any infinitesimal monopole deformation of a G_2–instanton is actually an infinitesimal instanton deformation. This fully justifies studying the deformation theory of the complex (2-7).

Proposition 11 *Let A be an irreducible G_2–instanton with respect to a coclosed G_2–structure on a closed manifold. Then, if $(a, \phi) \in \ker(d_2)$, where d_2 is the operator associated with $(A, 0)$, we have $\phi = 0$.*

Proof Let $(a, \phi) \in \ker(d_2)$. Then $d_A\phi = *(d_A a \wedge \psi)$ and $d_A^* a = 0$. Combining these and using that ψ is closed, we compute

$$d_A^* d_A \phi = -{*d_A}(d_A a \wedge \psi) = -{*[F_A \wedge a]} \wedge \psi.$$

This vanishes as A is a G_2–instanton and so $F_A \wedge \psi = 0$. Then taking the inner product with ϕ gives $d_A\phi = 0$ and so ϕ must vanish as A is assumed to be irreducible. □

Next we shall study the operator $d_1^* \oplus d_2$ for the trivial connection $A = d$. It will be used later to give an existence result for G_2–instantons in the abelian case.

Lemma 12 *Let L be the operator*

$$L\colon L^{2,1}(\Lambda^0 \oplus \Lambda^1) \to L^2(\Lambda^0 \oplus \Lambda^1),$$

given by $L(f, a) = (-d^*a, -df + *(da \wedge \psi))$. Its cokernel can be identified with those $(g, b) \in \Omega^0(X) \oplus \Omega^1(X)$ such that g is constant and b is a coclosed 1–form satisfying $d(b \wedge \psi) = 0$.

In particular, if (X, φ) has the property that there are no coclosed 1–forms b such that $d(b \wedge \psi) = 0$, then L is surjective onto $\Omega_0^0(X) \oplus \Omega^1(X)$, where $\Omega_0^0(X)$ denotes the functions with zero average on X.

Proof We shall identify the cokernel of L with the kernel of its formal adjoint L^*, using the L^2 inner product. Then one computes $L^*(g,b) = (-d^*b, -dg + *d(b\wedge\psi))$, and so

$$LL^*(g,b) = (\Delta g, dd^*b) + \big(0, *\big(d(*d(b\wedge\psi))\wedge\psi\big)\big).$$

By taking the L^2 inner product with (g,b) and using Stokes' theorem we obtain

$$\begin{aligned}\langle (g,b), LL^*(g,b)\rangle_{L^2} &= \|dg\|^2_{L^2} + \|d^*b\|^2_{L^2} + \big\langle b, *\big(d(*d(b\wedge\psi))\wedge\psi\big)\big\rangle_{L^2}\\ &= \|dg\|^2_{L^2} + \|d^*b\|^2_{L^2} + \int_X b\wedge d(*d(b\wedge\psi))\wedge\psi\\ &= \|dg\|^2_{L^2} + \|d^*b\|^2_{L^2} + \int_X d(b\wedge\psi)\wedge *d(b\wedge\psi)\\ &= \|dg\|^2_{L^2} + \|d^*b\|^2_{L^2} + \|d(b\wedge\psi)\|^2_{L^2}.\end{aligned}$$

Hence if (g,b) is in the kernel of L^*, then also $LL^*(g,b) = 0$ and the computation above shows that $dg = d^*b = d(b\wedge\psi) = 0$. □

The next result gives a criterion for an abstract construction of abelian G_2–instantons.

Proposition 13 *If (X,φ) has no nonzero coclosed 1–forms b such that $d(b\wedge\psi) = 0$ and B is a complex line bundle over X, then there is a monopole (ϕ, A) on B.*

Moreover, if φ is coclosed, then any such monopole is actually a G_2–instanton and it is unique.

Proof We start with any connection A_0 on B and look for $(\phi,a)\in\Omega^0(X)\oplus\Omega^1(X)$ such that (ϕ, A_0+a) solves the monopole equation $d\phi = *(F_{A_0+a}\wedge\psi)$. This can be rewritten in the form

$$-d\phi + *(d_{A_0}a\wedge\psi) = -*(F_{A_0}\wedge\psi),$$

and so, together with the gauge-fixing condition $-d^*_{A_0}a = 0$, it suffices to solve the equation $L(\phi,a) = (0, -*(F_{A_0}\wedge\psi))$. Since 0 certainly has vanishing average, by Lemma 12 this right-hand side lies in the image of the operator L and we can find (ϕ,a) such that (ϕ, A_0+a) is a monopole on B.

The fact that in the coclosed case the monopoles are actually instantons follows from the discussion in Remark 10. The uniqueness follows from the fact that in this case the operator L is formally self-adjoint. However, since once restricted to $\Omega^0_0(X)\oplus\Omega^1(X)$ it has no kernel, it is an isomorphism from $L^{2,1}$ to L^2. □

As a particular example of how to apply the previous result we shall now consider the strictly nearly parallel case.

Corollary 14 *Let (X,φ) be a nearly parallel G_2–manifold. For any $\alpha \in H^2(X,\mathbb{Z})$, there is a unique G_2–instanton on the complex line bundle B with $c_1(B)=\alpha$.*

Proof We start by showing that in the nearly parallel case we are in the setup of Proposition 13. Suppose $b \in \Omega^1(X)$ is such that $d^*b=0$ and $d(b\wedge\psi)=0$. First notice that in this case ψ is exact, so the second equation can be written $db\wedge\psi=0$. This shows that $3d^7b=*(*(db\wedge\psi)\wedge\psi)=0$, which we can rewrite as $0=3d^7b=db-*(db\wedge\varphi)$. Hence, taking d^* of this equation, we find

$$0=3d^*d^7b=d^*db-*(db\wedge d\varphi)=d^*db-\lambda(*(db\wedge\psi))=d^*db,$$

where we have used that $d\varphi=\lambda\psi$ and $db\wedge\psi=0$ by hypothesis. Putting this together with $d^*b=0$, we conclude that $\Delta b=0$ and so b is a harmonic 1–form. However, nearly parallel G_2–structures are Einstein with positive constant, and so have positive Ricci curvature. It then follows from the Bochner formula and Myers theorem that $b=0$. We are then in position to apply Proposition 13 and conclude that there is a G_2–instanton on any line bundle over X. □

Remark 15 (1) One may wonder if the previous corollary extends from nearly parallel to a more general class of G_2–structures. We will see in the second bullet of Theorem 67 examples of coclosed G_2–structures where we do not have uniqueness of abelian G_2–instantons. See also the second item in Remark 68.

(2) The previous proof works equally well for torsion-free, irreducible G_2–manifolds, ie those with holonomy equal to G_2. In that case, $\lambda=0$ and $\mathrm{Ric}=0$, but the irreducibility shows that there are no harmonic 1–forms.

(3) In fact, the previous corollary has the following consequence. Any harmonic 2–form on a strictly nearly parallel G_2–manifold must lie on Λ^2_{14}. As proved by Lorenzo Foscolo [17, Theorem 3.23], a similar result holds for nearly Kähler manifolds.

2.1.4 S^1–invariant G_2–instantons In Section 3 we will be interested in studying G_2–instantons that are invariant under the action of a group which acts transitively. Here we make a detour into $U(1)$–invariant G_2–instantons, on $U(1)$–invariant G_2–structures. We include this section so we can refer to its main computation in the proof of Proposition 57. Let V be the infinitesimal generator of a $U(1)$–action preserving a coclosed G_2–structure, ie $\mathcal{L}_V\varphi=0$ and so $\mathcal{L}_V\psi=0$ as well. Now let $\eta\in\Omega^1(X^7)$ be the unique connection form on the circle bundle $X^7\to M^6=X^7/S^1$ such that $\eta(V)=1$ and $\eta|_{V^\perp}=0$. Then the equation $\mathcal{L}_V\psi=0$, together with $d\psi=0$, shows that both $\iota_V\psi$ and $\psi-\eta\wedge\iota_V\psi$ are V–basic, and so are pulled back from M^6. We may then write

$$\psi=-\eta\wedge\Omega_1+\tau,$$

where Ω_1 and τ are $-\iota_V\psi$ and $\psi-\eta\wedge\iota_V\psi$, respectively. Moreover, the equations $\mathcal{L}_V\psi=0$ and $d\psi=0$ further imply

$$d\Omega_1=0 \quad \text{and} \quad d\eta\wedge\Omega_1=d\tau.$$

In fact, since $\psi=*\varphi$ is the 4–form associated with the G_2–structure φ, there must further exist V–semibasic forms $\omega\in\Omega^2(X)$ and $\Omega_2\in\Omega^3(X)$ such that $\varphi=\eta\wedge\omega+\Omega_2$ and $\tau=\frac{1}{2}\omega^2$. In the setting we will be interested in, all the relevant principal bundles P over X can actually be regarded as bundles pulled back from M. Hence, if A is a connection on P over X and a' a connection pulled back from M to X, we have that $A-a'\in\Omega^0(X,\Lambda^1\otimes\mathfrak{g}_P)$. Then, using the splitting $\Lambda^1=\langle\eta\rangle\oplus\langle\eta\rangle^\perp$, we can write $A-a'=a''+\phi\otimes\eta$, where $a''\in\Omega^0(X,\langle\eta\rangle^\perp\otimes\mathfrak{g}_P)$ and $\phi\in\Omega^0(X,\mathfrak{g}_P)$. Defining now $a=a'+a''$, the connection A may written as $A=a+\phi\otimes\eta$. Its curvature may then be computed to be $F_A=F_a+d_a\phi\wedge\eta+\phi\otimes d\eta$, and $F_a=F_a^\perp-\mathcal{L}_Va\wedge\eta$ with $F_a^\perp$ semibasic. However, as the connection is assumed to be invariant under the action generated by V, $\mathcal{L}_Va=0$ and $F_a=F_a^\perp$ is actually V–basic. We then compute

$$\begin{aligned}F_A\wedge\psi&=(F_a+d_a\phi\wedge\eta+\phi\otimes d\eta)\wedge(-\eta\wedge\Omega_1+\tau)\\&=-\eta\wedge(F_a\wedge\Omega_1+\phi\otimes d\eta\wedge\Omega_1+d_a\phi\wedge\tau)+(F_a+\phi\otimes d\eta)\wedge\tau,\end{aligned}$$

and so the G_2–instanton equation amounts to

$$(F_a+\phi\otimes d\eta)\wedge\Omega_1+d_a\phi\wedge\tau=0 \quad \text{and} \quad (F_a+\phi\otimes d\eta)\wedge\tau=0. \tag{2-8}$$

2.2 Examples from 3–Sasakian geometry

We start this subsection with a brief discussion of 3–Sasakian geometry, following the nice review paper [6]. Then, starting from a 3–Sasakian manifold, we construct a family of coclosed G_2–structures containing a strictly nearly parallel structure, and give some existence results for G_2–instantons; see Propositions 17, 18, and 22.

A 3–Sasakian 7–manifold may be equivalently defined as a Riemannian 7–manifold (X^7, g_7) equipped with a 3–orthonormal vector field $\{\xi_i\}_{i=1}^3$ satisfying $[\xi_i, \xi_j] = \epsilon_{ijk}\xi_k$. Any 3–Sasakian X is quasiregular in the sense that the vector fields $\{\xi_i\}_{i=1}^3$ generate a locally free SU(2)–action. The space of leaves Z^4, equipped with the Riemannian metric g_Z such that $\pi\colon X^7 \to Z^4$ is an orbifold Riemannian submersion, has the structure of a self-dual, Einstein orbifold with scalar curvature $s > 0$. Let g_7 be the 3–Sasakian metric on X^7 and regard $\pi\colon X^7 \to Z^4$ as an SU(2)– or SO(3)–(orbi)bundle of frames of $\Lambda^2_- Z^4$. The Levi-Civita connection of Z^4 equips it with a connection $\eta = \eta_i \otimes T_i \in \Omega^1(X^7, \mathfrak{so}(3))$, where the T_i form a standard basis of $\mathfrak{so}(3)$ satisfying $[T_i, T_j] = 2\epsilon_{ijk}T_k$. This has the property that the η–horizontal forms ω_i defined by

$$F_\eta = d\eta + \frac{1}{2}[\eta \wedge \eta] = \frac{s}{24}\omega_i \otimes T_i$$

form an orthogonal basis of $(\Lambda^2_- \ker(\eta), g_7|_{\ker(\eta)})$ with $|\omega_i| = \sqrt{2}$ and $s \in \mathbb{R}^+$. We further remark that the metric g_7 can be written as

$$g_7 = \eta^i \otimes \eta^i + \pi^* g_Z.$$

Remark 16 To make a connection with the holonomy point of view used in Definition 6 we remark that the 2–forms $\varpi_i = r\,dr \wedge \eta_i + \frac{1}{2}r^2\,d\eta_i$ equip the cone $(\mathbb{R}^+_r \times X, g_C = dr^2 + r^2 g_7)$ with a compatible, torsion-free Sp(2)–structure.

The strictly nearly parallel G_2–structure φ constructed in [18] determines a Riemannian metric g_φ which is a squash of the 3–Sasakian metric g_7. We shall consider the 1–parameter family of G_2–structures $\{\varphi_t\}_{t\in\mathbb{R}\setminus 0}$ such that

$$\varphi_t = t^3\eta_1 \wedge \eta_2 \wedge \eta_3 + t\frac{s}{48}(\eta_1 \wedge \omega_1 + \eta_2 \wedge \omega_2 + \eta_3 \wedge \omega_3), \tag{2-9}$$

which determines $g_{\varphi_t} = t^2(\eta_1^2 + \eta_2^2 + \eta_3^2) + \pi^* g_Z$ and

$$\psi_t = \frac{1}{6}\left(\frac{s}{48}\right)^2 \omega_i \wedge \omega_i + t^2\frac{s}{48}(\eta_1 \wedge \eta_2 \wedge \omega_3 + \eta_2 \wedge \eta_3 \wedge \omega_1 + \eta_3 \wedge \eta_1 \wedge \omega_2).$$

Recall that, up to scaling, the condition that φ_t be nearly parallel can be written as $d\varphi_t = \lambda\psi_t$ for some constant $\lambda > 0$. In our case we can easily compute from $\frac{s}{24}\omega_i = d\eta^i + \epsilon_{ijk}\eta^j \wedge \eta^k$ that

$$\begin{aligned} d\varphi_t = t(t^2+1)\frac{s}{24}&(\eta_1 \wedge \eta_2 \wedge \omega_3 + \eta_2 \wedge \eta_3 \wedge \omega_1 + \eta_3 \wedge \eta_1 \wedge \omega_2) \\ &+ 2t\left(\frac{s}{48}\right)^2(\omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3). \end{aligned}$$

Then the equation $d\varphi_t = \lambda\psi_t$ becomes the system $12t = \lambda$ and $t^2 + 1 = 2\lambda t$, which has the solutions $t = \frac{1}{\sqrt{5}}$, $\lambda = \frac{12}{\sqrt{5}}$ and $t = -\frac{1}{\sqrt{5}}$, $\lambda = -\frac{12}{\sqrt{5}}$. Note that we can scale λ by scaling the metric and change the sign of λ by changing the orientation. Conversely, it is possible to show that given a positive Einstein, anti-self-dual orbifold (Z, g_Z) there is an SO(3)– or SU(2)–bundle $\pi\colon X^7 \to Z$ equipped with a 3–Sasakian structure [6], so having a strictly nearly parallel G_2–structure as above. We further remark that this converse statement may however produce nonsmooth X^7. We are now in position to give some examples of G_2–instantons, starting first with SU(2)–invariant instantons and then with S^1–invariant examples.

Proposition 17 *For any $b_1, b_2, b_3 \in \mathbb{R}$ the 1–form $\eta = b_1\eta_1 + b_2\eta_2 + b_3\eta_3$ equips the trivial complex line bundle over X^7 with a G_2–instanton with respect to $\varphi_{1/\sqrt{2}}$.*

Moreover, if L is a complex line bundle over X^7 admitting a G_2–instanton with respect to $\varphi_{1/\sqrt{2}}$, then L actually has a real 3–parameter family of G_2–instantons.

Proof The connection $\eta = b_1\eta_1 + b_2\eta_2 + b_3\eta_3$ is not only S^1–invariant but also SU(2)–invariant. Its curvature is $d\eta$ and to show that $d\eta \wedge \psi_{1/\sqrt{2}} = 0$ it is enough to show that $d\eta_1 \wedge \psi_{1/\sqrt{2}} = 0$. The $d\eta_2$ and $d\eta_3$ equations are dealt with similarly. So we compute

$$\begin{aligned} d\eta_1 \wedge \psi_t &= \left(\frac{s}{24}\omega_1 - 2\eta_{23}\right) \wedge \left(\frac{1}{6}\left(\frac{s}{48}\right)^2 \omega_i \wedge \omega_i + t^2 \frac{s}{48}(\eta_{23} \wedge \omega_1 + \cdots)\right) \\ &= 2\left(\frac{s}{48}\right)^2 \left(t^2 - \frac{1}{2}\right) \eta_{23} \wedge \omega_1 \wedge \omega_1, \end{aligned}$$

which vanishes if and only if $t = \frac{1}{\sqrt{2}}$.

The second part of the theorem follows immediately from the fact that the G_2–instanton equation is linear in the abelian case. □

Proposition 18 *Let A be a self-dual connection on a bundle over a positive, self-dual, Einstein orbifold (Z, g_Z). Then, for all $t > 0$, the G_2–structure φ_t is coclosed and:*

- *π^*A is a G_2–instanton on X^7 with respect to φ_t. In particular, π^*A is a G_2–instanton for the strictly nearly parallel G_2–structure $\varphi_{1/\sqrt{5}}$.*
- *π^*A is Yang–Mills with respect to φ_t.*

Proof The fact that the G_2–structure φ_t is coclosed for any $t > 0$ follows from computing that $d\psi_t = 0$. This follows easily from the fact that $\eta_1 \wedge \eta_2 \wedge \eta_3$ is closed (in fact exact) and that each $\omega_i \wedge \omega_i$ is closed as well, since $d\omega_i = 2\epsilon_{ijk}\omega_j \wedge \eta_k$ and $\omega_i \wedge \omega_j = 0$ for $i \neq j$. This shows that φ_t is coclosed.

We start by proving the first bullet in the statement, ie that A pulls back to a G_2–instanton. Let F_A denote the curvature of A, which is self-dual by hypothesis. Hence, as π is a Riemannian submersion with respect to all g_{φ_t}, $\pi^* F_A \wedge \omega_i = 0$ for $i = 1, 2, 3$. It is then easy to check that $\pi^* F_A \wedge \psi_t = 0$.

Now we prove the second bullet in the proposition. To ease notation denote by A the pullback of such a self-dual connection. Then F_A takes values in $\Lambda^2_+ \otimes \mathfrak{g}_P$ and we compute

$$d_A(*_{g_{\varphi_t}} F_A) = d_A(F_A \wedge \varphi_t) = F_A \wedge d\varphi_t. \tag{2-10}$$

However, $d\varphi_t = t^3 d(\eta_{123}) + t\, d\eta_i \wedge \omega_i + t\eta_i \wedge d\omega_i$ and it is easy to check that $d(\eta_{123}) = \omega_1 \wedge \eta_{23} + \mathrm{cp}$ and $\eta_i \wedge d\omega_i = 2(\eta_{13} \wedge \omega_2 - \eta_{12} \wedge \omega_3) + \mathrm{cp}$, where cp denotes cyclic permutations. Putting all these together we have

$$d\varphi_t = t\omega_i \wedge \omega_i + (t^2 - 6t)(\omega_1 \wedge \eta_{23} + \omega_2 \wedge \eta_{31} + \omega_3 \wedge \eta_{12}).$$

As F_A is self-dual, $F_A \wedge \omega_i = 0$, and hence, inserting $d\varphi_t$ into (2-10), we conclude that $d_A(*F_A) = 0$ and A is Yang–Mills. □

Remark 19 One may also consider the G_2–structures obtained by scaling differently each of the η_i, while keeping them orthonormal, ie

$$\varphi_{a,b,c} = abc\eta_1 \wedge \eta_2 \wedge \eta_3 + a\eta_1 \wedge \omega_1 + b\eta_2 \wedge \omega_2 + c\eta_3 \wedge \omega_3.$$

It is easy to check that any such G_2–structure is coclosed if and only if $a = b = c$.

We now change the point of view on (X^7, g_7) equipped with its 3–Sasakian structure, and regard it as a Sasakian manifold with respect to any of the Reeb vector fields $\xi_q = q_1\xi_1 + q_2\xi_2 + q_3\xi_3$, for a unit quaternion $q = q_1 i + q_2 j + q_3 k \in \mathrm{Im}(\mathbb{H})$. In fact, the resulting Sasakian manifold is always quasiregular and does not depend on q. Take $\xi = \xi_1$ for example, ie (X^7, ξ_1, g_7), then the leaf space $\left(Y^6, \omega_{\mathrm{KE}} = \frac{1}{2} d\eta_1\right)$ is a Kähler–Einstein Fano orbifold. In fact Y^6 is the twistor space associated with the quaternionic Kähler structure on Z. Moreover, Y is smooth if and only if Z is. In fact, the twistor space also comes equipped with a nearly Kähler structure; see [23]. The next result relates this nearly Kähler structure with the G_2–structure $\varphi_{1/\sqrt{2}}$ on X. We came across it after a conversation with Mark Haskins, so it may be known to experts. However, we were unable to locate a reference.

Proposition 20 *Let (X^7, g^7) be a 3–Sasakian manifold. Then $(\iota_{\xi_1/t}\varphi_t, -\iota_{\xi_1/t}\psi_t)$ are basic with respect to ξ_1 and equip the twistor space with a nearly Kähler structure if and only if $t = \pm\frac{1}{\sqrt{2}}$.*

Proof The forms $\omega = \iota_{\xi_1/t}\varphi_t$, $\Omega_1 = -\iota_{\xi_1/t}\psi_t$ and $\Omega_2 = \varphi_t - t\eta_1 \wedge \iota_{\xi_1/t}\varphi_t$ are all basic with respect to ξ_1 and so they are the pullback of forms on the twistor Y. We denote these also by ω, Ω_1 and Ω_2, respectively, and we must check these equip Y^6 with a nearly Kähler structure. Back in X^7 these can be written as

$$\omega = t^2\eta_{23} + \frac{s}{48}\omega_1, \quad \Omega_1 = \frac{st}{48}(\eta_2\wedge\omega_3 - \eta_3\wedge\omega_2), \quad \Omega_2 = -\frac{st}{48}(\eta_2\wedge\omega_2 + \eta_3\wedge\omega_3).$$

Then we compute that $d\omega = -3\lambda\Omega_1$ and $d\Omega_2 = 2\lambda\omega_1^2$ for some λ if and only if $t = \pm\frac{1}{\sqrt{2}}$, in which case $\lambda = \mp\sqrt{2}$ and so (ω, Ω_1) does equip Y^6 with a nearly Kähler structure. □

Remark 21 In particular, using the notation introduced in the proof of the previous proposition, we can recover the G_2–structure φ_t by

$$\varphi_t = t\eta_1 \wedge \omega + \Omega_2 \quad \text{and} \quad \psi_t = -t\eta_1 \wedge \Omega_1 + \tfrac{1}{2}\omega^2.$$

As a consequence, we have:

Proposition 22 *Let A be a pseudo-Hermitian Yang–Mills (pHYM) connection for the nearly Kähler structure (ω, Ω_1) on Y^6. Then its pullback is a G_2–instanton with respect to $\varphi_{1/\sqrt{2}}$.*

Proof If A is pHYM, its curvature F satisfies $F \wedge \omega^2 = 0 = F \wedge \Omega_1$. Then, writing $\varphi_{1/\sqrt{2}}$ in terms of (ω, Ω_1) as in Remark 21, we have $F \wedge \psi_{1/\sqrt{2}} = 0$ and so A is a G_2–instanton with respect to $\varphi_{1/\sqrt{2}}$. □

Remark 23 (1) Every nearly parallel G_2–manifold carries a metric-compatible connection A, in the tangent bundle whose holonomy is in G_2. Therefore, by the Ambrose–Singer theorem, F_A takes values in $\Lambda^2 \otimes \mathfrak{g}_2$. This connection is metric-compatible and has antisymmetric torsion, and then one can show that F_A takes values in $S^2(\Lambda^2)$; see Proposition 3.1 in [19] for example. Putting all this together we see that actually F_A takes values in $S^2(\Lambda^2_{14})$, as $\mathfrak{g}_2 \cong \Lambda^2_{14}$, and so is a G_2–instanton.

(2) A similar statement to Proposition 22 holds for the pullback of an HYM connection on Y^6 with respect to its Kähler–Einstein structure $\omega_{\mathrm{KE}} = \frac{1}{2}\,d\eta_1$. Namely, the pullback of such an HYM connection yields a G_2–instanton for φ^{ts}.

2.3 Deformation theory revisited

In this subsection we shall restrict to the case where (X^7,φ) is a nearly parallel G_2–manifold and prove some rigidity results regarding G_2–instantons on them. Then, in Section 2.3.2, we prove that on nearly parallel manifolds there are G_2–instantons which are not locally energy-minimizing. Recall, from formula (2-4), that the analogous statement for torsion-free G_2–structures is always false.

2.3.1 Rigidity The fact that nearly parallel manifolds are Einstein with positive Einstein constant gives some hope of obtaining higher regularity for the moduli space of G_2–instantons than on torsion-free G_2–manifolds. In this direction we have:

Proposition 24 *Let (X^7,φ) be a nearly parallel G_2–manifold and A be a G_2–instanton with the property that all the eigenvalues of the endomorphism of $\Omega^1(X)$ given by $b\mapsto -14(*[*(F_A^{14})\wedge b])$ are smaller than s_φ, where $s_\varphi>0$ is the scalar curvature of g_φ. Then A is rigid as a G_2–instanton and $(A,0)$ unobstructed as a monopole. Moreover, if A is irreducible, then $(A,0)$ is also rigid as a monopole.*

Proof Let A be a connection as in the statement. Then we shall consider the operators d_1 and d_2 from the complex (2-7), associated with $(A,0)$. As φ is coclosed these can be written as

$$d_2(a,\phi)=*(d_Aa\wedge\psi)-d_A\phi \quad\text{and}\quad d_2^*b=(*d_Ab\wedge\psi,-d_A^*b),$$

while $d_1(\psi)=(-d_A\psi,0)$ and $d_1^*(a,\phi)=-d_A^*a$. Then the operator $d_1^*\oplus d_2$ which controls the deformation theory of the G_2–instanton equation is

$$(d_1^*\oplus d_2)(a,\phi)=(*(d_Aa\wedge\psi)-d_A\phi,-d_A^*a),$$

which is self-adjoint. In order to study its infinitesimal deformations we must therefore study its kernel. So let A be as in the statement and $(a,\phi)\in\ker(d_1^*\oplus d_2)$. Then $*(d_Aa\wedge\psi)=d_A\phi$ and $d_A^*a=0$, and moreover as φ is coclosed we have that

$$\begin{aligned}0&=(d_1^*\oplus d_2)^2(a,\phi)\\&=(\Delta_A\phi+*([F_A\wedge a]\wedge\psi),*d_A(*(d_Aa\wedge\psi)\wedge\psi)+d_Ad_A^*a).\end{aligned}$$

Then, if A is an irreducible G_2–instanton, the first entry gives $\Delta_A\phi=0$. Hence, taking the inner product with ϕ and integrating by parts we get $d_A\phi=0$. From the second entry above and using that $d\varphi = \lambda\psi$ we compute

$$\begin{aligned} 0 &= 3{*}d_A{*}d_A^7 a + d_A d_A^* a \\ &= {*}d_A{*}(d_A a - {*}(d_A a \wedge \varphi)) + d_A d_A^* a \\ &= \Delta_A a - {*}([F_A \wedge a] \wedge \varphi) + \lambda({*}(d_A a \wedge \psi)) \\ &= \Delta_A a - {*}([F_A \wedge a] \wedge \varphi), \end{aligned}$$

where in the last equality we used that ${*}(d_A a \wedge \psi) = d_A \phi = 0$. Putting this together with the Weitzenböck formula $\Delta_A a = \nabla_A^* \nabla_A a + {*}[{*}F_A \wedge a] + \mathrm{Ric}(a)$, we obtain

$$\nabla_A^* \nabla_A a + {*}[({*}F_A + F_A \wedge \varphi) \wedge a] + \mathrm{Ric}(a) = 0.$$

As $F_A \wedge \varphi = -2({*}F_A^7) + {*}F_A^{14}$, and g_φ is Einstein with positive scalar curvature $s_\varphi > 0$, ie $\mathrm{Ric} = \frac{s}{7}\,\mathrm{id}$, we have

$$\nabla_A^* \nabla_A a + {*}[{*}(2F_A^{14} - F_A^7) \wedge a] + \frac{s_\varphi}{7} a = 0.$$

If A is as in the hypothesis of the statement, then taking the inner product with b, the sum of the last two terms is positive and so we have

$$\|\nabla_A a\|_{L^2}^2 + \mu \|a\|_{L^2}^2 \leq 0,$$

for some $\mu > 0$. We conclude that a must vanish identically and as we have already seen $d_A \phi = 0$. Hence, any infinitesimal monopole deformation of $(A, 0)$ is of the form $(0, \phi)$ for some ϕ satisfying $d_A \phi = 0$. These can obviously be integrated as the path $\{(A, t\phi)\}_{t \in \mathbb{R}}$ and so this is a purely monopole deformation which keeps the connection A the same G_2–instanton.

Exactly the same proof shows that d_2 is surjective (by showing that $\ker(d_2^*) = 0$), proving that $(A, 0)$ is unobstructed as a monopole. Moreover, if A is irreducible, then $d_A \phi = 0$ implies that ϕ must vanish and so $(A, 0)$ is also rigid as a monopole. □

Corollary 25 *Let (X^7, φ) be a nearly parallel G_2–manifold. Then*

(1) *abelian G_2–instantons are rigid;*

(2) *flat connections are rigid as G_2–instantons.*

One may wonder if the rigidity of abelian G_2–instantons extends from strictly nearly parallel G_2–structures to a more general class, say coclosed ones. We will see a counterexample to this in the second bullet of Theorem 67; see also the second item in Remark 68.

We shall now comment on the relation of Proposition 24 to the G_2–instantons we constructed earlier in this section.

Remark 26 (1) Through Corollary 14 we know that there is a unique G_2–instanton on every complex line bundle L over a nearly parallel G_2–manifold. This actually supersedes Corollary 25.

(2) A similar result to Corollary 25 holds for nearly Kähler manifolds; see Theorem 1 in [10]. In fact, also in that case any complex line bundle admits a unique pseudo-Hermitian Yang–Mills connection. See Theorem 3.23 and Remark 3.25 in [17].

It is also possible to find examples of G_2–instantons on strictly nearly parallel G_2–manifolds for which Proposition 24 does not apply:

Example 27 Consider a self-dual, Einstein 4–orbifold (Z, g_Z), with positive scalar curvature admitting a family of self-dual connections (eg $\mathbb{S}^4$). Then, by Proposition 18, these connections lift to a family of G_2–instantons for a strictly nearly parallel G_2–structure constructed on the principal SO(3)–bundle associated with $\Lambda^2_- Z$. Therefore, in this case G_2–instantons have nontrivial moduli and so the hypothesis in Proposition 24 must fail.

2.3.2 Yang–Mills unstable G_2–instantons Let A be a G_2–instanton for a nearly parallel G_2–structure φ such that $d\varphi = \lambda\psi$. We have seen, in Proposition 8, that such G_2–instantons are actually Yang–Mills connections. Moreover, (2-4) and the subsequent discussion show that in the torsion-free case a G_2–instanton minimizes the Yang–Mills energy. That need not be the case for strictly nearly parallel G_2–structures as we now show with a counterexample.

Example 28 Equip the 7–dimensional sphere, $\mathbb{S}^7$, with the nearly parallel G_2–structure φ^{ts} induced from the 3–Sasakian one, as in Remark 19. Then $g_{\varphi^{\mathrm{ts}}}$ is the round metric. Now consider the Hopf bundle $\pi_H\colon \mathbb{S}^7 \to \mathbb{S}^4$. A verbatim repetition of the proof of Proposition 18 shows that the pullback, via π_H, of a self-dual connection on $\mathbb{S}^4$ is also a G_2–instanton with respect to φ^{ts}. Hence, if A is the pullback of a charge 1 self-dual connection on $\mathbb{S}^4$, it is a G_2–instanton for φ^{ts}. As $d\varphi^{\mathrm{ts}} = 4\psi^{\mathrm{ts}}$, we have that A is also a (nonflat) Yang–Mills connection. However, it is shown in [5] that any nonflat Yang–Mills connection on S^n, where $n > 4$, is Yang–Mills unstable.

Remark 29 We have also proved in Proposition 18 that the pullback of a Yang–Mills connection on a quaternion-Kähler manifold is both a G_2–instanton and a Yang–Mills connection, with respect to any of the G_2–structures φ_t, for $t > 0$. Hence, the example above also works also for any φ_t with t in a neighborhood of 1.

3 Aloff–Wallach spaces

We begin, in Section 3.1, by summarizing some facts about the geometry of homogeneous, coclosed G_2–structures on Aloff–Wallach spaces. Then in Section 3.2 we determine all the invariant connections on homogeneous SO(3)–bundles over the Aloff–Wallach spaces and use them in Sections 4 and 5 to classify invariant G_2–instantons on the Aloff–Wallach spaces. As a consequence, we discover that G_2–instantons can distinguish between different strictly nearly parallel G_2–structures on the same Aloff–Wallach space. We also produce examples of some interesting phenomena, for instance, irreducible G_2–instantons that merge into the same reducible G_2–instanton as the G_2–structure varies. This particular phenomenon was expected to occur, but these are the first examples. In Section 4.6 we shall also give examples of G_2–instantons for a nearly parallel G_2–structure in $X_{1,-1}$. Some of these are then shown to not be locally energy-minimizing. In fact, they are saddles of the invariant Yang–Mills functional. Further, in Section 5.3 we show that the existence of G_2–instantons distinguishes between a 3–Sasakian and a strictly nearly parallel G_2–structure on $X_{1,1}$.

3.1 Geometry of coclosed G_2–structures

Let $k, l \in \mathbb{Z}$, and let $U(1)_{k,l}$ be a circle subgroup of $\mathrm{SU}(3)$ consisting of elements of the form

$$\begin{pmatrix} e^{ik\theta} & 0 & 0 \\ 0 & e^{il\theta} & 0 \\ 0 & 0 & e^{im\theta} \end{pmatrix},$$

where $k+l+m=0$. The Aloff–Wallach space $X_{k,l} = \mathrm{SU}(3)/U(1)_{k,l}$ is the quotient of $\mathrm{SU}(3)$ by this circle subgroup. We shall now recall some basic facts about the geometry and topology of the Aloff–Wallach spaces. Aloff–Wallach spaces inherited their name from [1], where they were shown to admit homogeneous metrics with positive curvature, for $klm \neq 0$ (see also page 18 of the survey paper [30]). Later, Wang showed in [29] that Aloff–Wallach spaces admit homogeneous Einstein metrics with positive scalar curvature, not all of which are the ones considered by Aloff and Wallach. In [4, page 116], the authors show that each $X_{k,l}$ admits at least two homogeneous Einstein metrics. The authors further show, that for $X_{k,k}$ (and those related to it through the action of the Weyl group of $\mathrm{SU}(3)$; see Remark 34) one of these is 3–Sasakian and the other strictly nearly parallel, while on the other $X_{k,l}$ they are both strictly nearly parallel. As a side remark, we mention that there are examples of different pairs (k,l) such that the corresponding Aloff–Wallach spaces are homeomorphic, but not diffeomorphic [21].

Regarding coclosed G_2–structures, Aloff–Wallach spaces were shown to admit a real 4–dimensional family of homogeneous, coclosed G_2–structures as described in [9]. We now give details of this family of homogeneous coclosed G_2–structures on $X_{k,l}$. Let

$$s = \frac{\sqrt{k^2+l^2+m^2}}{\sqrt{6}},$$

and write the canonical left-invariant form on $\mathrm{SU}(3)$ as

$$\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{i}{s}\big(\frac{k}{\sqrt{3}}\eta + \frac{l-m}{3}\omega_4\big) & \omega_1 + i\omega_5 & -\omega_3 + i\omega_7 \\ -\omega_1 + i\omega_5 & \frac{i}{s}\big(\frac{l}{\sqrt{3}}\eta + \frac{m-k}{3}\omega_4\big) & \omega_2 + i\omega_6 \\ \omega_3 + i\omega_7 & -\omega_2 + i\omega_7 & \frac{i}{s}\big(\frac{m}{\sqrt{3}}\eta + \frac{k-l}{3}\omega_4\big) \end{pmatrix}.$$

Let $(\{e_i\}_{i=1}^7, H)$ be the vector fields dual to $(\{\omega_i\}_{i=1}^7, \eta)$, using the $\mathrm{SU}(3)$–invariant metric $\omega_1^2+\omega_2^2+\omega_3^2+\omega_4^2+\omega_5^2+\omega_6^2+\omega_7^2+\eta^2$. Then $\sqrt{6}sH$ is the infinitesimal generator of the $\mathfrak{u}(1)_{k,l}$–action.

Let A, B, C and D be nonzero constants. The G_2–structures under consideration are given by

$$\varphi = ABC(\omega_{123} - \omega_{167} + \omega_{257} - \omega_{356}) - D\omega_4 \wedge (A^2\omega_{15} + B^2\omega_{26} + C^2\omega_{37}). \tag{3-1}$$

The metric g_φ and the 4–form $\psi = *_\varphi \varphi$ associated to the G_2–structure are

$$\begin{aligned} g_\varphi &= A^2(\omega_1^2+\omega_5^2) + B^2(\omega_2^2+\omega_6^2) + C^2(\omega_3^2+\omega_7^2) + D^2\omega_4^2, \\ \psi &= ABCD(\omega_{4567} - \omega_{2345} + \omega_{1346} - \omega_{1247}) + B^2C^2\omega_{2367} + A^2C^2\omega_{1357} \\ &\qquad + A^2B^2\omega_{1256}. \end{aligned}$$

Here we fixed the orientation induced by the volume form $\mathrm{vol}_\varphi = 7A^2B^2C^2D\omega_{1234567}$. Also, notice that this family of G_2–structures is, up to scaling, only 3–dimensional. The exterior derivatives of the $\{\omega_i\}_{i=1}^7$ and η may be computed using the Maurer–Cartan formula $d\mu = -\mu \wedge \mu$. Here we use these formulas to compute the exterior derivatives of φ and ψ, to get information about the torsion of these G_2–structures. We find

$$\begin{aligned}\sqrt{2}\,d\varphi = {} & D(A^2+B^2+C^2)(-\omega_{4567}+\omega_{2345}-\omega_{1346}+\omega_{1247}) \\ & + \left(\frac{D}{s}(kA^2+mB^2)-4ABC\right)\omega_{1256} + \left(\frac{D}{s}(lB^2+kC^2)-4ABC\right)\omega_{2367} \\ & + \left(\frac{D}{s}(mC^2+lA^2)-4ABC\right)\omega_{1357}, \\ d\psi = {} & 0.\end{aligned}$$

From these we can extract the torsion component τ_0:

$$\frac{7}{\sqrt{2}}\tau_0 = -4\left(\frac{A}{BC}+\frac{B}{AC}+\frac{C}{AB}\right)+\frac{D}{s}\left(\frac{l}{C^2}+\frac{k}{B^2}+\frac{m}{A^2}\right).$$

Definition 30 Let $\mathcal{C}$ denote the spaces of G_2–structures of the form (3-1).

Lemma 31 *Let $k \neq \pm l$, $l \neq \pm m$, $m \neq k$. Then the space of homogeneous coclosed G_2–structures $\mathcal{C}$ may be identified with $(\mathbb{R}^+)^2 \times (\mathbb{R}\backslash\{0\})^2$. Moreover, given $(A,B,C,D) \in \mathcal{C}$, the corresponding G_2–structure can be written as in* (3-1).

Proof It follows from the analysis in [9] that for $k \neq \pm l$, $l \neq \pm m$, $m \neq k$, any homogeneous, coclosed G_2–structure is one of those considered above. These are precisely those with $s' = 0$, in that reference. Now notice that the G_2–structures (3-1) are parametrized by $(A,B,C,D) \in (\mathbb{R}\backslash\{0\})^4$ minus the coordinate hyperplanes. Moreover, (3-1) stays invariant by any of the following maps: $(A,B) \mapsto (-A,-B)$, $(B,C) \mapsto (-B,-C)$ and $(A,C) \mapsto (-A,-C)$. These discrete symmetries give rise to a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$–action on $(\mathbb{R}\backslash\{0\})^4$, generated by the first two symmetries. Hence, the G_2–structures in (3-1) are parametrized by $(\mathbb{R}\backslash\{0\})^4/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Taking a fundamental domain for the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$–action we may equally well regard the space of G_2–structures as in (3-1) as $\mathbb{R}^+_A \times \mathbb{R}^+_B \times (\mathbb{R}_C\backslash\{0\}) \times (\mathbb{R}_D\backslash\{0\})$. □

Remark 32 (1) Up to a cover, and the action of the Weyl group (see Remark 34), the restrictions in the lemma above can be simply written as $(k,l) \notin \{(1,1),(1,-1)\}$.

(2) In the case when $(k,l) \in \{(1,1),(1,-1)\}$ we will continue to use $\mathcal{C}$ to denote the G_2–structures as in (3-1). However, in that case there are homogeneous coclosed G_2–structures that can not be written as in (3-1) and so are not in $\mathcal{C}$.

(3) We know that $\tau_1 = \tau_2 = 0$ because the G_2–structure is coclosed, and we can compute τ_3 by $\tau_3 = *(d\varphi - \tau_0\psi)$.

A G_2–structure of the form (3-1) is nearly parallel, ie $d\varphi = \lambda\psi$, when (A, B, C, D) satisfy

$$
\begin{aligned}
A^2 + B^2 + C^2 + \sqrt{2}\lambda ABC &= 0, \\
D(kA^2 + mB^2) - 4sABC - \sqrt{2}\lambda s A^2 B^2 &= 0, \\
D(lB^2 + kC^2) - 4sABC - \sqrt{2}\lambda s B^2 C^2 &= 0, \\
D(lA^2 + mC^2) - 4sABC - \sqrt{2}\lambda s A^2 C^2 &= 0.
\end{aligned} \tag{3-2}
$$

By fixing an orientation we can suppose that $\lambda > 0$. Then, in [9], it is shown that for $k \neq \pm l$, $l \neq \pm m$, $m \neq \pm l$, the system (3-2) admits precisely eight solutions. Moreover, up to the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ alluded to in the proof of Lemma 31, these eight solutions give only two nonequivalent solutions $\varphi \in \mathcal{C}$, which are in fact strictly nearly parallel. The following result completely determines the connected component in $\mathcal{C}$ in which each of these structures lives.

Lemma 33 *Let $k \neq \pm l$, $l \neq \pm m$, $m \neq \pm l$, and let $\varphi^{\mathrm{np}_1}, \varphi^{\mathrm{np}_2} \in \mathcal{C}$ denote the two strictly nearly parallel G_2–structures. Then $C(\varphi^{\mathrm{np}_1})$ and $C(\varphi^{\mathrm{np}_2})$ have the same sign, while those of $D(\varphi^{\mathrm{np}_1})$ and $D(\varphi^{\mathrm{np}_2})$ are opposite. Moreover, $\mathrm{sign}(C)$ is constrained by $\lambda C < 0$ and determines the orientation.*

Proof Fix an orientation and suppose that $\lambda > 0$. Then the first equation in (3-2) implies that ABC must be negative for any such φ. On the other hand, it follows from the analysis in the bottom of page 413 in [9] that the two solutions have different signs of $ABCD$ and so they must in fact have different signs of D. Choosing $\varphi \in \mathcal{C}$, we have $A > 0$ and $B > 0$, so we must also have $C < 0$ (as $ABC < 0$), which then implies each of the solutions has a different sign of D. □

Remark 34 (1) The Weyl group of $\mathrm{SU}(3)$ moves the $U(1)_{k,l}$ subgroup inducing an action in the set of Aloff–Wallach spaces. In fact, this action is generated by $X_{k,l} \mapsto X_{l,k}$ and $X_{k,l} \mapsto X_{k,m}$, which can be combined to generate the order 3 element $\sigma\colon X_{k,l} \to X_{l,m}$, ie cyclic permutations of (k, l, m). Hence, up to coverings and this action, there is no loss in supposing that k and l are coprime and that $k \geq 0$ and $-l \leq k \leq 2l$.

(2) Consider the $U(2)$–subgroup of $\mathrm{SU}(3)$ generated by the image of the homomorphism $\mathrm{SU}(2) \times U(1) \to \mathrm{SU}(3)$ given by

$$(A, e^{i\theta}) \mapsto \mathrm{diag}(Ae^{i\theta}, \det(Ae^{i\theta})^{-1}).$$

As $\mathbb{CP}^2 \cong \mathrm{SU}(3)/U(2)$, we obtain a canonical fibration

$$\pi_1\colon X_{k,l} \to \mathbb{CP}^2,$$

whose fibers one can check to be the lens spaces $U(2)/U(1)_{k,l} \cong S^3/\mathbb{Z}_{|k+l|}$, if $k+l \neq 0$, or $S^1 \times S^2$, if $k+l=0$. In fact, using the order 3 element σ, we may obtain two more fibrations $\pi_2 = \pi_1 \circ \sigma$ and $\pi_3 = \pi_1 \circ \sigma^2$ of $X_{k,l}$ over $\mathbb{CP}^2$. At least two of these have fibers $S^3/\mathbb{Z}_p$ for a nonzero $p \in \{|k|, |l|, |m|\}$.

3.2 Invariant connections

Given a Lie group G, a principal G–bundle P over $X_{k,l} = \mathrm{SU}(3)/U(1)_{k,l}$ is said to be SU(3)–homogeneous (or just homogeneous) if there is a lift of the SU(3)–action on $X_{k,l}$ to the total space which commutes with the right G–action on P. In general, homogeneous SO(3)–principal bundles over $X_{k,l}$ are determined by their isotropy homomorphisms $\lambda_n\colon U(1) \to \mathrm{SO}(3)$, and are constructed via

$$P_n = \mathrm{SU}(3) \times_{(U(1)_{k,l},\lambda_n)} \mathrm{SO}(3),$$

where the possible group homomorphisms λ_n are parametrized by $n \in \mathbb{Z}$. Explicitly we can think of SO(3) as $\mathrm{SU}(2)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts via multiplication by minus the identity matrix, $-\mathbf{1}$, then λ_n is given by

$$\lambda_n(\theta) = \begin{pmatrix} e^{i(n/2)\theta} & 0 \\ 0 & e^{-i(n/2)\theta} \end{pmatrix} \quad (\mathrm{mod}\ -\mathbf{1}).$$

Definition 35 Let $\{T_1, T_2, T_3\}$ be a basis for $\mathfrak{su}(2)$ such that $[T_i, T_j] = 2\epsilon_{ijk} T_k$. Then the canonical invariant connection on P_n is

$$A_c^n = \frac{n}{2} \frac{\eta}{\sqrt{6}s} \otimes T_1.$$

Using the Maurer–Cartan equations, the curvature of the canonical invariant connection A_c^n is found to be

$$F_c^n = -\frac{n}{12s^2}((k-l)\omega_{15} + (l-m)\omega_{26} + (m-k)\omega_{37}).$$

Wang's theorem [28] classifies invariant connections on homogeneous bundles. In our situation, Wang's theorem says that SU(3)–invariant connections on P_n are in bijection with morphisms of $U(1)$–representations

$$\Lambda\colon (\mathfrak{m}, \mathrm{Ad}) \to (\mathfrak{so}(3), \mathrm{Ad} \circ \lambda_n),$$

where $\mathfrak{m}$ is the $U(1)_{k,l}$–Ad complement to $\langle H\rangle$ in $\mathfrak{su}(3)$. If (k,l) is not in the Weyl orbit of $(1,1)$ and $n \neq 0$, these split into irreducible real representations as

$$\begin{aligned}\mathfrak{m} &= \langle X_1, X_5\rangle_{k-l} \oplus \langle X_2, X_6\rangle_{l-m} \oplus \langle X_3, X_7\rangle_{m-k} \oplus \langle X_4\rangle,\\ \mathfrak{so}(3) &= \langle T_1\rangle \oplus \langle T_2, T_3\rangle_n,\end{aligned}$$

where the weight of each 2–dimensional irreducible representation is indicated by a subscript. It will be useful to use the notation $V_1 = \langle X_1, X_5\rangle$, $V_2 = \langle X_2, X_6\rangle$ and $V_3 = \langle X_3, X_7\rangle$ (these are simply the real root spaces of $\mathfrak{su}(3)$). Applying Schur's lemma and Wang's theorem [28] we have:

Lemma 36 $((k,l) \neq (1,1))$ *Let $A^n \in \Omega^1(\mathrm{SU}(3), \mathfrak{so}(3))$ be the connection 1–form of an invariant connection on P_n over $X_{k,l}$, for (k,l) not in the Weyl orbit of $(1,1)$. Then it is left-invariant and can be written as $A^n = A^n_c + (A^n - A^n_c)$, where $(A - A^n_c) \in \mathfrak{m}^* \otimes \mathfrak{so}(3)$, extended as a left-invariant 1–form with values in $\mathfrak{so}(3)$, is given by*

$$A - A^n_c = a_1\psi_1 + a_2\psi_2 + a_3\psi_3 + b\omega_4 \otimes T_1.$$

Here the ψ_i denote isomorphisms $\psi_i\colon V_i \xrightarrow{\sim} \langle T_2, T_3\rangle$ with $|\psi| \in \{0,1\}$ with respect to the fixed basis, and the $a_i, b \in \mathbb{R}$ are constants. Moreover, each a_i must vanish if the weight of V_i is not equal to n, ie

$$\begin{aligned} a_1 &= 0 \quad \text{if } n \neq k-l,\\ a_2 &= 0 \quad \text{if } n \neq l-m,\\ a_3 &= 0 \quad \text{if } n \neq m-k.\end{aligned}$$

Remark 37 (1) The order 3 element of the Weyl group W permutes the different roots and so the different root spaces. In particular, there is no loss in considering the Aloff–Wallach spaces up to the action of W. Hence, in the previous lemma when we consider the case $k \neq l$, it is implicit that also $l \neq m$ or $m \neq k$.

(2) Since it is not possible to have $k-l = l-m = m-k = n$ without forcing $k = l = m = n = 0$, we must have $a_1a_2a_3 = 0$. This splits us into seven cases to be analyzed below.

Lemma 38 $((k,l) = (1,1))$ *Let $A^n \in \Omega^1(\mathrm{SU}(3), \mathfrak{so}(3))$ be the connection 1–form of an invariant connection on P_n over $X_{1,1}$. Then it is left-invariant and can be written as $A^n = A^n_c + (A^n - A^n_c)$, where $(A - A^n_c) \in \mathfrak{m}^* \otimes \mathfrak{so}(3)$, extended as a left-invariant 1–form with values in $\mathfrak{so}(3)$, is given by*

$$A - A^n_c = a_1\chi + a_2\psi_2 + a_3\psi_3.$$

Here the ψ_i denote isomorphisms $\psi_i\colon V_i \xrightarrow{\sim} \langle T_2, T_3\rangle$ with $|\psi| = 1$ with respect to the fixed basis, and $\chi\colon \langle X_1, X_5, X_4\rangle \to \mathfrak{so}(3)$ denotes a linear map, which in the case $n \neq 0$ must take values in $\langle T_1\rangle \subset \mathfrak{so}(3)$. Moreover,

$$a_2 = 0 \quad \text{if } n \neq 3,$$
$$a_3 = 0 \quad \text{if } n \neq -3.$$

Proof The proof in this case is similar and we simply give the main steps. As before the proof amounts to using Wang's theorem [28] to find the invariant connections. One must split the corresponding representations into irreducibles as

$$\mathfrak{m} = \langle X_1\rangle \oplus \langle X_5\rangle \oplus \langle X_4\rangle \oplus \langle X_2, X_6\rangle_3 \oplus \langle X_3, X_7\rangle_{-3},$$
$$\mathfrak{so}(3) = \begin{cases} \langle T_1\rangle \oplus \langle T_2, T_3\rangle_n & \text{if } n \neq 0, \\ \langle T_1\rangle \oplus \langle T_2\rangle \oplus \langle T_3\rangle & \text{if } n = 0. \end{cases}$$

Then the conclusion follows from a similar application of Schur's lemma. □

3.2.1 Case splitting, for $k \neq l$ We shall now consider the case when $X_{k,l}$ is such that (k,l) is not in the Weyl orbit of $(1,1)$; the other case will be investigated separately. Here we use Lemma 36 in order to write down all the possible connection 1–forms, up to invariant gauge transformations. We shall analyze the different cases corresponding to the different values of n.

Case 0 $(n \neq k-l,\ l-m,\ m-k)$ In this case $a_1 = a_2 = a_3 = 0$ and so every connection is reducible, with

$$A^n = \left(\frac{n}{2}\frac{\eta}{\sqrt{6}s} + b\omega_4\right) \otimes T_1.$$

Case 1 $(n = k-l)$ In this case $a_2 = a_3 = 0$ and we may use our gauge freedom to write the isomorphism $\psi_1\colon V_1 \xrightarrow{\sim} \langle T_2, T_3\rangle$ as $\psi_1 = \omega_1 \otimes T_2 + \omega_5 \otimes T_3$. Then

$$A^{k-l} = \left(\frac{k-l}{2}\frac{\eta}{\sqrt{6}s} + b\omega_4\right) \otimes T_1 + a_1(\omega_1 \otimes T_2 + \omega_5 \otimes T_3).$$

Case 2 $(n = l-m)$ Now we must have $a_1 = a_3 = 0$ and as in Case 1 we may use our gauge freedom to fix the form of ψ_2. We can write the connection form as

$$A^{l-m} = \left(\frac{l-m}{2}\frac{\eta}{\sqrt{6}s} + b\omega_4\right) \otimes T_1 + a_2(\omega_2 \otimes T_2 + \omega_6 \otimes T_3).$$

Case 3 $(n = m - k)$ Similarly, in this case $a_1 = a_2 = 0$ and we can write the connection form as

$$A^{m-k} = \left(\frac{m-k}{2}\frac{\eta}{\sqrt{6}s} + b\omega_4\right) \otimes T_1 + a_3(\omega_3 \otimes T_2 + \omega_7 \otimes T_3).$$

Case 4 $(n = m - k = l - m$, ie $n = l = -k)$ In this case $a_1 = 0$ and we exhaust our gauge freedom in fixing $\psi_2 = \omega_2 \otimes T_2 + \omega_6 \otimes T_3$, so that

$$\psi_3 = \omega_3 \otimes (\cos(\beta)T_2 + \sin(\beta)T_3) + \omega_7 \otimes (-\sin(\beta)T_2 + \cos(\beta)T_3)$$

is dependent on an angle parameter β. The connection form is

$$\begin{aligned} A^l = \left(\frac{l}{2}\frac{\eta}{\sqrt{6}s} + b\omega_4\right) &\otimes T_1 + a_2(\omega_2 \otimes T_2 + \omega_6 \otimes T_3) \\ &+ a_3\big(\omega_3 \otimes (\cos(\beta)T_2 + \sin(\beta)T_3) + \omega_7 \otimes (-\sin(\beta)T_2 + \cos(\beta)T_3)\big). \end{aligned}$$

Case 5 $(n = l - m = k - l$, ie $n = k = -m)$ This is similar to Case 4, but with $a_2 = 0$. The connection form is

$$\begin{aligned} A^k = \left(\frac{k}{2}\frac{\eta}{\sqrt{6}s} + b\omega_4\right) &\otimes T_1 + a_1(\omega_1 \otimes T_2 + \omega_5 \otimes T_3) \\ &+ a_3\big(\omega_3 \otimes (\cos(\beta)T_2 + \sin(\beta)T_3) + \omega_7 \otimes (-\sin(\beta)T_2 + \cos(\beta)T_3)\big). \end{aligned}$$

Case 6 $(a_3 = 0$ and $n = k - l = m - k$, so that $n = m = -l)$ This is similar to Cases 4 and 5, except that we use α for the angle parameter. The connection form is

$$\begin{aligned} A^m = \left(\frac{m}{2}\frac{\eta}{\sqrt{6}s} + b\omega_4\right) &\otimes T_1 + a_1(\omega_1 \otimes T_2 + \omega_5 \otimes T_3) \\ &+ a_2\big(\omega_2 \otimes (\cos(\alpha)T_2 + \sin(\alpha)T_3) + \omega_6 \otimes (-\sin(\alpha)T_2 + \cos(\alpha)T_3)\big). \end{aligned}$$

3.2.2 Case splitting, for $k = l = 1$ Now we use Lemma 38 to write down the possible connection 1–forms for an invariant connection on P_n over $X_{1,1}$, splitting into cases depending on the value of n.

Case 0 $(n \neq 3, -3, 0)$ In this case,

$$A^n = \left(\frac{n}{2}\frac{\eta}{\sqrt{6}} + b\omega_4 + a_1\omega_1 + a_5\omega_5\right) \otimes T_1,$$

where $a_1, a_5, b \in \mathbb{R}$.

Case 1 $(n = 0)$ In this case,

$$A^0 = \omega_1 \otimes c_1 + \omega_4 \otimes c_4 + \omega_5 \otimes c_5,$$

where $c_1, c_4, c_5 \in \mathfrak{so}(3)$.

Case 2 $(n = 3)$ In this case,

$$A^{-3} = \left(\frac{3}{2}\frac{\eta}{\sqrt{6}} + b\omega_4 + a_1\omega_1 + a_5\omega_5\right) \otimes T_1 + a_2(\omega_2 \otimes T_2 + \omega_6 \otimes T_3),$$

where $a_1, a_2, a_5, b \in \mathbb{R}$.

Case 3 $(n = -3)$ In this case.

$$A^3 = \left(-\frac{3}{2}\frac{\eta}{\sqrt{6}} + b\omega_4 + a_1\omega_1 + a_5\omega_5\right) \otimes T_1 + a_3(\omega_3 \otimes T_2 + \omega_7 \otimes T_3),$$

where $a_1, a_3, a_5, b \in \mathbb{R}$.

3.2.3 Topology of the homogenous bundles P_n Recall from the beginning of Section 3.2 that given a group homomorphism $\lambda_n \colon U(1) \to \mathrm{SO}(3)$ we may construct the homogeneous bundle

$$P_n = \mathrm{SU}(3) \times_{(U(1)_{k,l}, \lambda_n)} \mathrm{SO}(3)$$

over $X_{k,l}$. In this section we compute the first Pontryagin and second Stiefel–Whitney classes of the associated vector bundle E_n with respect to standard action of SO(3) on $\mathbb{R}^3$. To compute its characteristic classes it will be convenient to use a lift of E_n to a $\mathrm{Spin}^c(3)$=$U(2)$–bundle W_n. Then the adjoint bundle $\mathfrak{g}_{W_n}$ of W_n splits as $\mathfrak{g}_{W_n} \cong \underline{\mathbb{R}} \oplus E_n$, where $\underline{\mathbb{R}}$ denotes the trivial bundle. We can then compute the characteristics of E_n via the Chern classes of W_n as

$$w_2(E_n) = c_1(W_n) \pmod 2 \qquad \text{and} \qquad p_1(E_n) = c_1(W_n)^2 - 4c_2(W_n).$$

To state the result we recall some facts about the cohomology ring of $X_{k,l}$ [21], namely, that $H^2(X_{k,l}, \mathbb{Z}) \cong \mathbb{Z}$ and that the square of its generator is the generator of $H^4(X_{k,l}, \mathbb{Z}) \cong \mathbb{Z}_{k^2+l^2+kl}$. We now state and prove:

Lemma 39 *The associated homogeneous* SO(3)*–bundle* E_n *has*

$$w_2(E_n) = n \pmod 2 \qquad \textit{and} \qquad p_1(E_n) = n^2 \pmod{k^2 + kl + l^2}.$$

Proof The first step towards the computation is to notice that, for any $n \in \mathbb{Z}$, there is actually a homogeneous lift of P_n to a $\mathrm{Spin}^c(3) = U(2)$–bundle. To see this, we make the identification $\mathrm{SU}(2) \times U(1)/\mathbb{Z}_2 \cong U(2)$ by the isomorphism $[(A, e^{i\theta})] \mapsto \mathrm{diag}(e^{i\theta}, e^{i\theta})A$, and it is easy to see that there is a group homomorphism $\tau\colon U(2) \to \mathrm{SO}(3)$ which is simply $\tau([A, e^{i\theta}]) = A \in \mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{SO}(3)$.

Remark 40 One other way to describe this is by considering the adjoint action of $U(2)$ on its Lie algebra. This decomposes as $\mathfrak{u}(2) = \mathbb{R} \oplus \mathfrak{so}(3)$, and $U(2)$ acts on $\mathfrak{so}(3) \cong \mathbb{R}^3$ via $\mathrm{SO}(3)$.

Then the bundle P_n can be homogeneously lifted to a $U(2)$–bundle if and only if there is a group homomorphism $\mu_n\colon U(1) \to U(2)$ such that $\lambda_n = \tau \circ \mu_n$. That is indeed the case, as we can simply check that

$$\mu_n(e^{i\theta}) = \left[\begin{pmatrix} e^{in\theta/2} & 0 \\ 0 & e^{-in\theta/2} \end{pmatrix}, e^{in\theta/2}\right] \in \mathrm{SU}(2) \times U(1)/\mathbb{Z}_2$$

does the job. Then the canonical invariant connection on $W_n = \mathrm{SU}(3) \times_{(U(1)_{k,l}, \mu_n)} U(2)$ is $A_c^n = n\eta/(\sqrt{6}s) \otimes \mathrm{diag}(i, 0)$ and its curvature $F_c^n = n\, d\eta/(\sqrt{6}s) \otimes \mathrm{diag}(i, 0)$. Then $c_1(W_n) = [i\,\mathrm{tr}(F_n^c)] = -n[d\eta]/(\sqrt{6}s)$ with $[d\eta]/(\sqrt{6}s)$ being the generator of $H^2(X_{k,l}, \mathbb{Z})$, and so $w_2(E_n) = n \pmod 2$. We now turn to the computation of $p_1(E_n)$, which besides $c_1(W_n)$ also requires $c_2(W_n)$, which we can check to be zero using the formula $\frac{1}{2}[\mathrm{tr}(F_n^c \wedge F_n^c) - \mathrm{tr}(F_n^c)^2]$. Therefore, we conclude that $p_1(E_n) = n^2 \in \mathbb{Z}_{k^2+l^2+kl}$, finishing the proof of Lemma 39. □

A short computation also yields:

Corollary 41 *Let* $n_1 = k - l$, $n_2 = l - m$, $n_3 = m - k$. *Then*

$$\begin{aligned} w_2(E_{n_1}) &= k - l \pmod 2, & p_1(E_{n_1}) &= -3kl \pmod{k^2 + kl + l^2}, \\ w_2(E_{n_2}) &= k \pmod 2, & p_1(E_{n_2}) &= -3k^2 \pmod{k^2 + kl + l^2}, \\ w_2(E_{n_3}) &= l \pmod 2, & p_1(E_{n_3}) &= -3l^2 \pmod{k^2 + kl + l^2}. \end{aligned}$$

4 Gauge theory on $X_{k,l}$, with $(k, l) \neq (1, 1)$

This section is concerned with stating and proving the main results of our paper, namely Theorems 42 and 44, which classify all invariant G_2–instantons with gauge groups $U(1)$ and SO(3), for any G_2–structure $\varphi \in \mathcal{C}$ as in Definition 30. Recall that, as proved in [9], for $k \neq \pm l$, $l \neq \pm m$, $m \neq \pm k$, these are in fact all the homogeneous coclosed G_2–structures on $X_{k,l}$. Then, in Theorem 46, we use the classification to show that in any Aloff–Wallach space as above, there are irreducible G_2–instantons, with gauge group SO(3), which as the G_2–structure varies merge into the same reducible and obstructed G_2–instanton. This phenomenon was expected to be possible and Theorem 46 gives plenty of explicit examples; see for instance Examples 48 and 49, together with their accompanying Figures 1 and 2, representing the merge of the G_2–instantons. As a consequence of Theorem 44 we give in Section 4.4 examples of Aloff–Wallach spaces where G_2–instantons can be used to distinguish between the two inequivalent strictly nearly parallel G_2–structures. More precisely, we show that in these examples there always exist invariant and irreducible G_2–instantons, which however live on topologically different SO(3)–bundles.

In Section 4.6, we fix $(k,l) = (1,-1)$ and a nearly parallel G_2–structure on $X_{1,-1}$. After finding the corresponding invariant G_2–instantons we show that any irreducible such G_2–instanton is not a local minimum of the Yang–Mills functional. In fact, they are saddles of the invariant Yang–Mills functional.

4.1 G_2–instantons

Before stating the main results we introduce some quantities which will simplify the notation later on:

$$\Gamma = A^2B^2(m-k) + A^2C^2(l-m) + B^2C^2(k-l),$$
$$\Delta = A^2B^2 l + A^2C^2 k + B^2C^2 m.$$

Note that for a given Aloff–Wallach space $X_{k,l}$ each of these quantities only depends on the G_2–structure (3-1) and varies continuously with it.

4.2 Abelian case

We start below by stating the result classifying G_2–instantons with gauge group $U(1)$. In this abelian case, some particular examples of the instantons appearing in our classification are already present in [20, Equation (3.29)]. In this case, the possible homogeneous bundles are parametrized by $n \in \mathbb{Z}$, which denotes the degree of the homomorphism $\lambda_n \colon U(1)_{k,l} \to U(1)$ used to constructed the bundle $Q_n = \mathrm{SU}(3) \times_{(U(1)_{k,l},\lambda_n)} U(1)$.

Theorem 42 (abelian case) *Let $(k,l) \neq (1,1)$ and A be a G_2–instanton on a line bundle over $X_{k,l}$ equipped with the G_2–structure* (3-1). *Then one of the following holds:*

(1) $\Delta \neq 0$, *in which case there is a unique G_2–instanton in any homogeneous line bundle. For instance, if A lives on the bundle associated with λ_n, its connection* 1*–form is*
$$A = \frac{n}{2}\left(\frac{1}{\sqrt{6}s}\eta + \frac{\Gamma}{3\sqrt{2}s\Delta}\omega_4\right).$$

(2) $\Delta = 0$, *but $\Gamma \neq 0$, in which case A lives in the trivial homogenous bundle* (ie that associated with λ_0), *and A is simply one of the* 1*–forms $b\omega_4$, for some $b \in \mathbb{R}$.*

(3) $\Delta = 0$ *and $\Gamma = 0$, in which case there is a real* 1*–parameter family of such instantons on any homogeneous line bundle.*

Proof Any abelian G_2–instanton can also be interpreted as a reducible SU(2)–instanton. Hence, we can use the formula for the connection in the previous section. More precisely, for the instanton to be reducible we must have $a_1 = a_2 = a_3 = 0$, so
$$A^n = \frac{n}{2\sqrt{6}s}\eta + b\omega_4.$$
Its curvature is
$$F^n = F_c^n + b\, d\omega_4,$$
where
$$F_c^n = -\frac{n}{12s^2}((k-l)\omega_{15} + (l-m)\omega_{26} + (m-k)\omega_{37}),$$
$$d\omega_4 = \frac{1}{\sqrt{2}s}(m\omega_{15} + k\omega_{26} + l\omega_{37}).$$
Then we write $\psi = -D\omega_4 \wedge \Omega_2 + \frac{1}{2}\omega^2$, with Ω_2 and ω^2 the pullbacks of differential forms on the flag manifold $\mathbb{F}_2 = \mathrm{SU}(3)/T^2$, and determined by this relation. As in Section 2.1.4, more precisely, (2-8), we compute that the G_2–instanton equation reduces to the equations
$$(F_c^n + b\, d\omega_4) \wedge \Omega_2 = 0 \quad \text{and} \quad (F_c^n + b\, d\omega_4) \wedge \omega^2 = 0.$$
It is easy to check that $F_c^n \wedge \Omega_2 = 0 = d\omega_4 \wedge \Omega_2$ always. We are, therefore, reduced to the second equation, which turns into
$$-n\Gamma + 6\sqrt{2}s\Delta b = 0,$$

where Γ and Δ are as in the beginning of this section. In particular we see that $F_c^n \wedge \Omega_2 = 0$ if and only if $\Gamma = 0$ and $d\omega_4 \wedge \Omega_2 = 0$ if and only if $\Delta = 0$. Therefore, if $\Delta \neq 0$ there is exactly one SU(3)–invariant instanton, whose connection form is

$$A^n = \frac{n}{2}\left(\frac{1}{\sqrt{6}s}\eta + \frac{\Gamma}{3\sqrt{2}s\Delta}\omega_4\right) \otimes T_1.$$

However, if $\Delta = 0$ there are no instantons unless $n\Gamma = 0$ as well, in which case there is a 1–parameter family of instantons as we can chose b arbitrarily. □

A few remarks are in order, related to how the existence of invariant abelian G_2–instantons varies with the G_2–structure.

Remark 43 (1) For a fixed Aloff–Wallach space $X_{k,l}$ both Δ and Γ vary smoothly with the G_2–structure, and generically $\Delta \neq 0$. Note that $\Delta = 0$ defines a hypersurface in the space of coclosed homogeneous G_2–structures.

(2) Suppose that we vary the G_2–structure always keeping $\Gamma \neq 0$, but crossing the hypersurface defined by $\Delta = 0$. We see that the instantons on the bundles Q_n, for $n \neq 0$, "disappear" when $\Delta = 0$ and "reappear" on the other side of the hypersurface.

(3) For any (k,l) it is easy to find examples where the situation $\Delta = 0 = \Gamma$ occurs. These equations, ie $\Delta = 0$ and $\Gamma = 0$, can also be written as

$$A^2(B^2 - C^2)l = B^2(C^2 - A^2)k,$$
$$C^2(A^2 - B^2)(l-k) = A^2(B^2 - C^2)(k+l).$$

For example, it is easy to see that any G_2–structure having $A^2 = B^2 = C^2$ satisfies these equations.

(4) The conditions $\Delta = 0$ and $\Gamma = 0$ are independent of scaling the metric as expected.

(5) Both Γ and Δ are independent of D. This can be understood directly from the proof, as follows. Recall that (ω, Ω_2) induces an SU(3)–structure on the flag $\mathbb{F}_2 = \mathrm{SU}(3)/T^2$. Then it follows from the proof of Proposition 57 that

$$F_c^n \wedge \Omega_2 = 0 = d\omega_4 \wedge \Omega_2$$

always. Notice that both F_c^n and $d\omega_4$ are the pullback of 2–forms from $\mathbb{F}_2$. Hence $F_c^n \wedge \Omega_2$ and $d\omega_4 \wedge \Omega_2$ measure the components of these 2–forms in $\Lambda^{2,0}$ with respect to the complex structure on $\mathbb{F}_2$ induced by Ω_2. In particular, the canonical connection A_c^n, which is induced from a connection on $\mathbb{F}_2$, is always pseudoholomorphic.

Furthermore, the proof also shows that $F_c^n \wedge \omega^2$ and $d\omega_4 \wedge \omega^2$ are proportional to Γ and Δ respectively. Given that F_c^n and $d\omega_4$, as 2–forms on $\mathbb{F}_2$, are of type $(1,1)$, the constants Γ and Δ measure the components of these 2–forms along ω. Thus, A_c^n is pHYM with respect to (ω, Ω_2) if and only if $\Gamma = 0$.

(6) Any abelian connection can be written as a direct sum of connections with gauge group $U(1)$, so there is no loss of generality in working with gauge group $U(1)$ when investigating abelian connections.

4.3 Nonabelian case

In this section we prove Theorem 44, which classifies invariant and irreducible G_2–instantons on SO(3)–bundles, with respect to the G_2–structures $\varphi \in \mathcal{C}$ on the $X_{k,l}$, for $k \neq \pm l$, $l \neq \pm m$, $m \neq \pm k$. Recall that in these cases, the G_2–structures in $\mathcal{C}$ are in fact all the homogeneous coclosed G_2–structures on $X_{k,l}$. Then we prove Theorem 46, which yields examples of irreducible G_2–instantons that, as the G_2–structure varies, merge into the same reducible and obstructed G_2–instanton (see also Examples 48 and 49).

The reason for focusing our attention on irreducible G_2–instantons is that any reducible one is already taken into consideration by Theorem 42. Recall, from the previous section, that the homogenous SO(3)–bundles are also parametrized by an integer $n \in \mathbb{Z}$ and we denote them by P_n.

Theorem 44 (nonabelian case) *Let $(k,l) \neq (1,1)$ and $X_{k,l}$ be an Aloff–Wallach space equipped with one of the G_2–structures φ in* (3-1) *and $n \in \mathbb{Z}$. Then irreducible and invariant G_2–instantons on P_n exist if and only if:*

(1) $n = k - l$ *and* $\sigma_1(\varphi) = 3\big(m/2 - s(AD)/(BC)\big)\Delta + ((k-l)/2)\Gamma > 0$, *in which case the instantons have* $a_2 = a_3 = 0$,
$$a_1^2 = \frac{1}{12B^2C^2s^2}\left(3\left(\frac{m}{2} - s\frac{AD}{BC}\right)\Delta + \frac{k-l}{2}\Gamma\right) \quad \text{and} \quad b = \frac{1}{\sqrt{2}}\left(-\frac{m}{2s} + \frac{AD}{BC}\right);$$

(2) $n = l - m$ *and* $\sigma_2(\varphi) = 3\big(k/2 - s(BD)/(AC)\big)\Delta + ((l-m)/2)\Gamma > 0$, *in which case the instantons have* $a_1 = a_3 = 0$,
$$a_2^2 = \frac{1}{12A^2C^2s^2}\left(3\left(\frac{k}{2} - s\frac{BD}{AC}\right)\Delta + \frac{l-m}{2}\Gamma\right) \quad \text{and} \quad b = \frac{1}{\sqrt{2}}\left(-\frac{k}{2s} + \frac{BD}{AC}\right);$$

(3) $n = m-k$ and $\sigma_3(\varphi) = 3\big(l/2 - s(CD)/(AB)\big)\Delta + ((m-k)/2)\Gamma > 0$, *in which case the instantons have* $a_1 = a_2 = 0$,

$$a_3^2 = \frac{1}{12B^2A^2s^2}\left(3\left(\frac{l}{2} - s\frac{CD}{AB}\right)\Delta + \frac{m-k}{2}\Gamma\right) \quad \text{and} \quad b = \frac{1}{\sqrt{2}}\left(-\frac{l}{2s} + \frac{CD}{AB}\right).$$

Proof Let A^n be an irreducible, invariant G_2–instanton on P_n over $X_{k,l}$. In order to compute the instanton equations we must compute the curvature F^n first. This may be found by the formula

$$F^n = F_c^n + d_{A_c^n}(A^n - A_c^n) + \tfrac{1}{2}[A^n - A_c^n, A^n - A_c^n],$$

and the Maurer–Cartan equations. Our strategy for finding instantons will be simply to solve the equations $F^n \wedge \psi = 0$ for the a_i and b in each of the cases listed above.

Case 0 $(n \neq k-l,\ l-m,\ m-k)$ Here $a_1 = a_2 = a_3 = 0$, so A^n is always reducible and we immediately deduce that for A to be irreducible we are reduced to one of the items in the statement. We also remark that the G_2–instantons arising from this case are precisely those from Theorem 42.

Case 1 $(n = k-l)$ Here $a_2 = a_3 = 0$, and

$$A^{k-l} = \left(\frac{k-l}{2\sqrt{6}s}\eta + b\omega_4\right) \otimes T_1 + a_1(\omega_1 \otimes T_2 + \omega_5 \otimes T_3),$$

whose curvature F^{l-k} is

$$\begin{aligned}\frac{1}{12s^2}\Big(&\big(-(k-l)^2 + 6\sqrt{2}smb + 24s^2a_1^2\big)\omega_{15} + \big(-(k-l)(l-m) + 6\sqrt{2}skb\big)\omega_{26} \\ &+ \big(-(k-l)(m-k) + 6\sqrt{2}slb\big)\omega_{37}\Big) \otimes T_1 \\ &+ \frac{a_1}{\sqrt{2}}\left(\omega_{23} - \omega_{67} - \left(\frac{m}{s} + 2\sqrt{2}b\right)\omega_{45}\right) \otimes T_2 \\ &+ \frac{a_1}{\sqrt{2}}\left(-\omega_{27} + \omega_{36} - \left(\frac{m}{s} + 2\sqrt{2}b\right)\omega_{14}\right) \otimes T_3.\end{aligned}$$

The equations resulting from $F^{k-l} \wedge \psi = 0$ are

$$\begin{aligned}6\sqrt{2}s\Delta b + 24B^2C^2s^2a_1^2 - (k-l)\Gamma &= 0,\\ a_1BC(2ADs - BC(2\sqrt{2}sb + m)) &= 0.\end{aligned}$$

Hence, if $a_1 = 0$ we obtain the same reducible instanton as in Case 0 and Theorem 42, while if $a_1 \neq 0$, the solutions satisfy

$$a_1^2 = \frac{1}{12B^2C^2s^2}\left(3\left(\frac{m}{2} - s\frac{AD}{BC}\right)\Delta + \frac{k-l}{2}\Gamma\right),$$
$$b = \frac{1}{\sqrt{2}}\left(-\frac{m}{2s} + \frac{AD}{BC}\right).$$

Therefore, in this case the existence of SU(3)–invariant irreducible instantons depends on the sign of $\sigma_1 = 3\big(m/2 - s(AD)/(BC)\big)\Delta + ((k-l)/2)\Gamma$.

Case 2 $(n = l - m)$ As this case is very similar to Case 1, we will omit the details. We must have $a_1 = a_3 = 0$ and if $a_2 \neq 0$, solutions to $F^{l-m} \wedge \psi = 0$ must satisfy

$$a_2^2 = \frac{1}{12A^2C^2s^2}\left(3\left(\frac{k}{2} - s\frac{BD}{AC}\right)\Delta + \frac{l-m}{2}\Gamma\right),$$
$$b = \frac{1}{\sqrt{2}}\left(-\frac{k}{2s} + \frac{BD}{AC}\right).$$

The sign of $\sigma_2 = 3\big(k/2 - s(BD)/(AC)\big)\Delta + ((l-m)/2)\Gamma$ determines whether solutions exist.

Case 3 $(n = m - k)$ Again, we will omit the details. Now $a_1 = a_2 = 0$ and if $a_3 \neq 0$, the equation $F^{m-k} \wedge \psi = 0$ gives

$$a_3^2 = \frac{1}{12B^2A^2s^2}\left(3\left(\frac{l}{2} - s\frac{CD}{AB}\right)\Delta + \frac{m-k}{2}\Gamma\right),$$
$$b = \frac{1}{\sqrt{2}}\left(-\frac{l}{2s} + \frac{CD}{AB}\right).$$

The sign of $\sigma_3 = 3\big(l/2 - s(CD)/(AB)\big)\Delta + ((m-k)/2)\Gamma$ determines whether solutions exist.

Case 4 $(n = m - k = l - m$, and so $n = l = -k)$ Recall that in this case we have an angle parameter β. Then the equation $F^l \wedge \psi = 0$ becomes

$$6\sqrt{2}s\Delta b + 24A^2s^2(B^2a_3^2 + C^2a_2^2) - l\Gamma = 0,$$
$$a_2(2BDs + AC(-2\sqrt{2}sb + l)) = 0,$$
$$a_3\sin(\beta)(2CDs - AB(2\sqrt{2}sb + l)) = 0,$$
$$a_3\cos(\beta)(2CDs - AB(2\sqrt{2}sb + l)) = 0.$$

Squaring and summing the last two equations we are left with

$$a_3(2CDs - AB(2\sqrt{2}sb + l)) = 0.$$

This together with the second equation then implies that either $a_3 = 0$ or $a_2 = 0$, in which case we can then use an invariant gauge transformation to set $\beta = 0$. We have then reduced this case to Cases 2 and 3 above. In particular, the existence of G_2–instantons is determined by the signs of σ_3 and σ_2 (note that here we have $l = -k$).

Cases 5 and 6 These cases exhibit the same phenomena as in the last one and so reduce to Cases 1, 2 and 4 above. □

Remark 45 Fix $X_{k,l}$ and the bundle P_{k-l}. Then Theorem 44(1) shows that for a G_2–structure φ such that $\sigma_1(\varphi) > 0$ there are two irreducible G_2–instantons. In addition, we also have a reducible G_2–instanton given by Theorem 42 (with $n = k - l$). Varying φ so that $\sigma_1(\varphi) \searrow 0$, the two irreducible, invariant G_2–instantons existent when $\sigma_1 > 0$ merge with the reducible abelian G_2–instanton from Theorem 42. Indeed, it is easy to check that if $\sigma_1 = 0$ (and $\Delta \neq 0$) then $a_1 = 0$ and $b = n\Gamma/(6\sqrt{2}s\Delta)$. We shall see below that the resulting G_2–instanton is obstructed. From Theorem 44(2)–(3), a similar phenomena holds on the bundles P_{l-m} and P_{m-k}.

Theorem 46 *Let $n = k - l$, and suppose $\{\varphi(s)\}_{s\in\mathbb{R}}$ is a continuous family of homogeneous, coclosed G_2–structures such that $\sigma_1(\varphi(s)) > 0$ for $s < 0$ and $\sigma_1(\varphi(s)) < 0$ for $s > 0$. Then, as $s \nearrow 0$, the two irreducible G_2–instantons on P_n from Theorem 44 merge and become the same reducible and obstructed G_2–instanton when $s \geq 0$.*

Proof Recall that an invariant connection on P_{k-l} can be written as

$$A = A_c^{k-l} + b\omega_4 \otimes T_1 + a_1(\omega_1 \otimes T_2 + \omega_5 \otimes T_3).$$

Similarly, an invariant 1–form with values in the adjoint bundle can be written as

$$a = x\omega_4 \otimes T_1 + y(\omega_1 \otimes T_2 + \omega_5 \otimes T_3),$$

for some $x, y \in \mathbb{R}$. Using these it is easy to compute

$$\begin{aligned} d_A a = (x\, d\omega_4 + 4a_1 y\omega_1 \wedge \omega_5) \otimes T_1 \\ + \left(y\left(d\omega_1 - \frac{k-l}{\sqrt{6}s}\eta \wedge \omega_5\right) + 2(by + xa_1)\omega_5 \wedge \omega_4\right) \otimes T_2 \\ + \left(y\left(d\omega_5 + \frac{k-l}{\sqrt{6}s}\eta \wedge \omega_1\right) - 2(by + xa_1)\omega_1 \wedge \omega_4\right) \otimes T_3. \end{aligned}$$

We are now ready to find the invariant Lie algebra valued 1–forms a which lie in the cokernel of the deformation operator of the G_2–instanton equation $L(\cdot) = *(d_A \cdot \wedge \psi)$.

As the G_2–structure is coclosed L is self-adjoint and we can identify the cokernel with its own kernel. Hence $a \in \ker(L)$ if and only if $d_A a \wedge \psi = 0$, which we compute to be equivalent to

$$\sqrt{2}\Delta x + 8B^2C^2 s a_1 y = 0, \tag{4-1}$$

$$-4BCsa_1 x + \left(\sqrt{2}\left(2\frac{AD}{BC}s - m\right) - 4sb\right)BCy = 0. \tag{4-2}$$

Hence, there is a nonzero solution (x, y) if and only if the linear operator in the left-hand side of (4-1)–(4-2) is not invertible, ie its determinant vanishes:

$$32B^3C^3s^2a_1^2 + (4ADs - 2BCm - 4\sqrt{2}BCbs)\Delta = 0. \tag{4-3}$$

Inserting into (4-3) the formulas in Theorem 44 for the reducible instantons when $n = k - l$ we obtain

$$\tfrac{8}{3}BCs^2\sigma_1 = 0,$$

which holds if and only if $\sigma_1 = 0$. We have thus proved that as the instantons from Theorem 44 on P_{k-l} merge, when $\sigma_1 = 0$ they become reducible and obstructed before disappearing. □

Remark 47 A similar statement to Theorem 46 holds for $n = l - m$ and $n = m - k$, with σ_1 replaced by σ_2 and σ_3 respectively.

Here are two examples of this phenomenon.

Example 48 On the Aloff–Wallach space $X_{1,-1}$ consider the G_2–structures given by $B = 1$, $C = 1$ and $D = 1$ with A allowed to vary freely in order to make σ_1 change sign. Then, as A varies, the condition for irreducible G_2–instantons on P_2 to exist is that $\sigma_1(\varphi) = 2(1 - A^2)$ be positive, which happens if and only if $A^2 < 1$. See Figure 1 for a plot of a_1 (the "irreducible part" of the connections) as A varies. There one can clearly see that the irreducible G_2–instantons merge into the same reducible and obstructed (by Theorem 46) G_2–instanton.

Example 49 Similarly we consider G_2–instantons on P_6 over $X_{1,-5}$, equipped with the G_2–structures having $B = C = D = 1$. In this case the existence of irreducible G_2–instantons is controlled by the positivity of $\sigma_1(\varphi) = (A^2 - 1)(12\sqrt{7}A - 42)$, which is positive if and only if $A^2 < 1$ or $A > \frac{\sqrt{7}}{2}$. Figure 2 illustrates the two irreducible G_2–instantons merging into the same reducible and obstructed G_2–instanton.

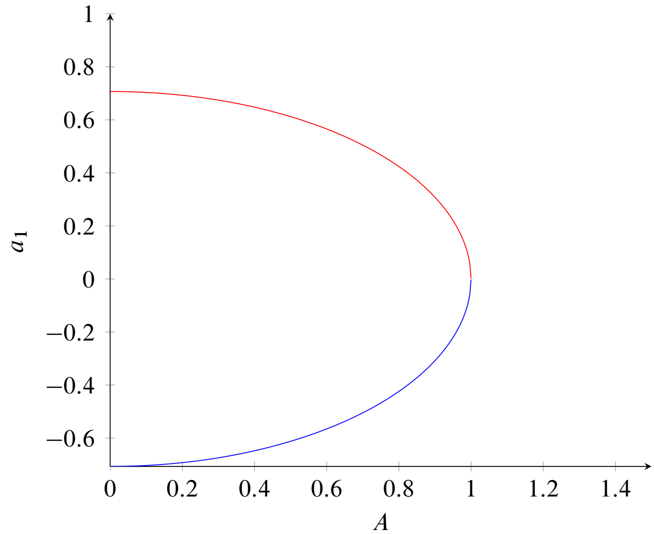

Figure 1: Instantons on P_2 over $X_{1,-1}$

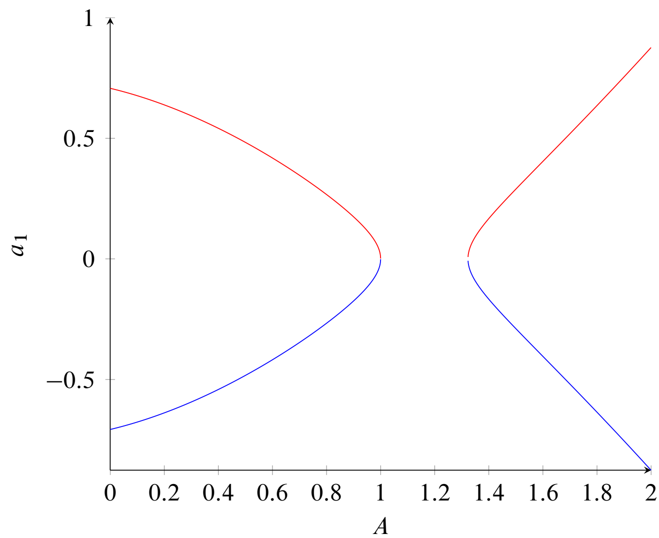

Figure 2: Instantons on P_6 over $X_{1,-5}$

Remark 50 The phenomenon described above can be interpreted as the G_2 analogue of a family of stable holomorphic bundles in a Kähler manifold, that become polystable as either the Kähler metric or the complex structure varies; see for example [2] and [3].[3]

[3] We thank Mark Stern for these references.

4.4 Distinguishing strictly nearly parallel structures

Suppose that $k \neq \pm l$, $l \neq \pm m$, $m \neq \pm k$. As remarked in Section 3, it is shown in [9] that the system (3-2) yields two inequivalent solutions $\varphi^{\mathrm{np}_1}, \varphi^{\mathrm{np}_2} \in \mathcal{C}$, which are strictly nearly parallel. In this section we will give examples of $X_{k,l}$ where the G_2–instantons can be used to distinguish between φ^{np_1} and φ^{np_2}. More precisely, we shall prove that in many examples of k and l the structures φ^{np_1} and φ^{np_2} do admit invariant and irreducible G_2–instantons with gauge group $\mathrm{SO}(3)$. However, the G_2–instantons live on topologically different $\mathrm{SO}(3)$–bundles.

To fix notation, let φ^+ denote the solution of (3-2) that satisfies $C(\varphi^+) > 0$ and $D(\varphi^+) > 0$, and let φ^- denote the solution satisfying $C(\varphi^-) > 0$ and $D(\varphi^-) < 0$. Let $A^{\pm}, \dots, D^{\pm}$ denote the parameters determining the nearly parallel G_2–structures $\varphi^{\pm}$. While it is possible to solve equations (3-2) symbolically, the resulting formulas are extremely unwieldy, so we will instead just give approximations.

Example 51 $(k = 1, l = 2)$ On $X_{1,2}$,

$$A^+ = 2.822, \quad B^+ = 2.296, \quad C^+ = 1.797, \quad D^+ = 2.496,$$
$$\sigma_1(\varphi^+) = -694.918, \quad \sigma_2(\varphi^+) = -357.130, \quad \sigma_3(\varphi^+) = 102.969,$$

while

$$A^- = 1.699, \quad B^- = 2.639, \quad C^- = 2.720, \quad D^- = -1.727,$$
$$\sigma_1(\varphi^-) = 257.213, \quad \sigma_2(\varphi^-) = -623.289, \quad \sigma_3(\varphi^-) = -676.142.$$

Hence, Theorem 44 implies that for φ^+, irreducible, invariant G_2–instantons exist only on the bundle P_{-4}, while for φ^-, irreducible, invariant G_2–instantons exist only on the bundle P_{-1}. These bundles are topologically distinct: indeed using the formulas from Corollary 41 we find that $w_2(E_{-4}) = 0 \pmod 2$ and $p_1(E_{-4}) = 2 \pmod 7$, while $w_2(E_{-1}) = 1 \pmod 2$ and $p_1(E_{-1}) = 1 \pmod 7$.

Example 52 $(k = 1, l = 3)$ On $X_{1,3}$,

$$A^+ = 2.813, \quad B^+ = 2.385, \quad C^+ = 1.760, \quad D^+ = 2.304,$$
$$\sigma_1(\varphi^+) = -1304.737, \quad \sigma_2(\varphi^+) = -794.177, \quad \sigma_3(\varphi^+) = 286.314,$$

while

$$A^- = 1.702, \quad B^- = 2.615, \quad C^- = 2.737, \quad D^- = -1.764,$$
$$\sigma_1(\varphi^-) = 468.212, \quad \sigma_2(\varphi^-) = -1124.808, \quad \sigma_3(\varphi^-) = -1272.289.$$

Hence, for φ^+, irreducible, invariant G_2–instantons exist only on the bundle P_{-5}, while for φ^-, irreducible, invariant G_2–instantons exist only on the bundle P_{-2}. The bundles are topologically distinct: $w_2(E_{-5}) = 1 \pmod 2$ and $p_1(E_{-5}) = 12 \pmod{13}$, while $w_2(E_{-2}) = 0 \pmod 2$ and $p_1(E_{-2}) = 4 \pmod{13}$.

Example 53 $(k = 1,\ l = 4)$ On $X_{1,4}$,

$$A^+ = 2.806, \quad B^+ = 2.425, \quad C^+ = 1.746, \quad D^+ = 2.208,$$
$$\sigma_1(\varphi^+) = -2113.761, \quad \sigma_2(\varphi^+) = -1378.207, \quad \sigma_3(\varphi^+) = 526.442,$$

while

$$A^- = 1.011, \quad B^- = 2.425, \quad C^- = 1.746, \quad D^- = -1.792,$$
$$\sigma_1(\varphi^-) = 349.253, \quad \sigma_2(\varphi^-) = -1593.714, \quad \sigma_3(\varphi^-) = -823.167.$$

Hence, for φ^+, irreducible, invariant G_2–instantons exist only on the bundle P_{-6}, while for φ^-, irreducible, invariant G_2–instantons exist only on the bundle P_{-3}. The bundles are topologically distinct: $w_2(E_{-6}) = 0 \pmod 2$ and $p_1(E_{-6}) = 15 \pmod{21}$, while $w_2(E_{-3}) = 1 \pmod 2$ and $p_1(E_{-3}) = 9 \pmod{21}$.

Example 54 $(k = 2,\ l = 3)$ On $X_{2,3}$,

$$A^+ = 2.827, \quad B^+ = 2.197, \quad C^+ = 1.848, \quad D^+ = 2.668,$$
$$\sigma_1(\varphi^+) = -1857.936, \quad \sigma_2(\varphi^+) = -753.703, \quad \sigma_3(\varphi^+) = 107.336,$$

while

$$A^- = 1.698, \quad B^- = 2.658, \quad C^- = 2.707, \quad D^- = -1.708,$$
$$\sigma_1(\varphi^-) = 705.209, \quad \sigma_2(\varphi^-) = -1726.540, \quad \sigma_3(\varphi^-) = -1812.541.$$

Hence, for φ^+, irreducible, invariant G_2–instantons exist only on the bundle P_{-7}, while for φ^-, irreducible, invariant G_2–instantons exist only on the bundle P_{-1}. The bundles are topologically distinct: $w_2(E_{-7}) = 1 \pmod 2$ and $p_1(E_{-7}) = 11 \pmod{19}$, while $w_2(E_{-1}) = 1 \pmod 2$ and $p_1(E_{-1}) = 1 \pmod{19}$.

Example 55 $(k = 2,\ l = 11)$ On $X_{2,11}$,

$$A^+ = 2.800, \quad B^+ = 2.456, \quad C^+ = 1.736, \quad D^+ = 2.132,$$
$$\sigma_1(\varphi^+) = -14809.573, \quad \sigma_2(\varphi^+) = -10158.191, \quad \sigma_3(\varphi^+) = 4009.812,$$

while

$$A^- = 1.706, B^- = 2.584, C^- = 2.755, E^- = -1.823,$$
$$\sigma_1(\varphi^-) = 5116.368, \quad \sigma_2(\varphi^-) = -12243.994, \quad \sigma_3(\varphi^-) = -14559.716.$$

Hence, for φ^+, irreducible, invariant G_2–instantons exist only on the bundle P_{-15}, while for φ^-, irreducible, invariant G_2–instantons exist only on the bundle P_{-9}. The bundles are topologically distinct: $w_2(E_{-15}) = 1 \pmod 2$ and $p_1(E_{-15}) = 78 \pmod{147}$, while $w_2(E_{-9}) = 1 \pmod 2$ and $p_1(E_{-9}) = 81 \pmod{147}$.

Remark 56 We did not find any Aloff–Wallach space for which one of the strictly nearly parallel G_2–structures does not admit irreducible, invariant G_2–instantons with gauge group SO(3).

4.5 Yang–Mills connections

It is interesting to consider the question: what conditions on a G_2–structure ensure that a G_2–instanton is a Yang–Mills connection? Proposition 8 says that this is the case for parallel and nearly parallel G_2–structures. In this section we shall characterize the homogeneous coclosed G_2–structures $\varphi \in \mathcal{C}$ for which an abelian G_2–instanton is a critical point for the Yang–Mills energy.

Proposition 57 *Equip $X_{k,l}$ with a G_2–structure* (3-1) *such that $\Delta \neq 0$. Let A^n be the unique G_2–instanton on the line bundle associated with λ_n. Then A is a critical point for the Yang–Mills energy if and only if the G_2–structure satisfies*

$$A^2B^2(A^2 - B^2)l + A^2C^2(C^2 - A^2)k + B^2C^2(B^2 - C^2)m = 0. \tag{4-4}$$

Proof From the proof of Theorem 42 we have

$$A^n = \frac{n}{2}\left(\frac{1}{\sqrt{6}s}\eta + \frac{\Gamma}{3\sqrt{2}s\Delta}\omega_4\right) \otimes T_1.$$

The Yang–Mills energy for an invariant abelian connection

$$A^n = \left(\frac{n}{2\sqrt{6}s}\eta + b\omega_4\right) \otimes T_1$$

is

$$E(b) = \frac{1}{144s^4}\left(\frac{1}{A^4}(6\sqrt{2}bms - n(k-l))^2 + \frac{1}{B^4}(6\sqrt{2}bms - n(l-m))^2 + \frac{1}{C^4}(6\sqrt{2}bms - n(m-k))^2\right).$$

Then we require that at the G_2–instanton, ie when $b = n\Gamma/(6\sqrt{2}s\Delta)$, there be a critical point of $E(b)$, which immediately yields (4-4).

For completeness we also remark that, in general, the critical points of E have

$$b = -\frac{n}{6\sqrt{2}s}\frac{A^4B^4l(k-m) + A^4C^4k(m-l) + B^4C^4m(l-k)}{A^4B^4l^2 + A^4C^4k^2 + B^4C^4m^2}. \qquad \square$$

Remark 58 (1) If $\Delta = 0$ then only one of the G_2–instantons in the 1–parameter family described in Theorem 42 is a critical point for the Yang–Mills energy.

(2) For a given $X_{k,l}$ condition (4-4) describes a hypersurface in the space of homogeneous coclosed G_2–structures, containing the nearly parallel G_2–structures.

(3) One can carry out a similar analysis to determine conditions on the G_2–structure so that the irreducible G_2–instantons described in Theorem 44 are Yang–Mills. The space of such G_2–structures is cut out in $\mathcal{C}$ by two real algebraic equations.

4.6 For a nearly parallel structure on $X_{1,-1}$

We shall now see an example of a nearly parallel G_2–structure on an Aloff–Wallach space, namely $X_{1,-1}$, for which instantons do exist and do not minimize the Yang–Mills–Higgs energy.

4.6.1 G_2–instantons The precise statement we shall prove in this section is:

Theorem 59 *Let φ be the nearly parallel G_2–structure on $X_{1,-1}$.*

(1) *For each n, there is a unique, invariant, G_2–instanton on the line bundle $L_n = \mathrm{SU}(3) \times_{U(1)_{1,-1},\rho_n} \mathbb{C}$.*

(2) *Let A be an irreducible and invariant G_2–instanton, with gauge group $\mathrm{SO}(3)$ on $X_{1,-1}$. Then A lives on the bundle P_{-1}. Moreover, such instantons do exist.*

The rest of this section is dedicated to proving this result. First we must obtain the strictly nearly parallel G_2–structure on $X_{1,-1}$. This is of the form (3-1), with

$$A = -4\sqrt{\tfrac{2}{5}}, \quad B = \tfrac{4}{15}\sqrt{75 + 15\sqrt{5}}, \quad C = -\tfrac{4}{15}\sqrt{75 - 15\sqrt{5}}, \quad D = -\tfrac{16}{45}\sqrt{30},$$

as a straightforward computation shows. We shall now compute G_2–instantons for this structure, starting with abelian ones on the bundles $L_n = \mathrm{SU}(3) \times_{\lambda_n} \mathbb{C}$. The invariant connections are of the form $\frac{n}{2}\eta + a_4\omega_4$ and the G_2–instanton equation $\left(\frac{n}{2}\, d\eta + a_4\, d\omega_4\right) \wedge \psi = 0$ gives

$$\tfrac{256}{135}\sqrt{6}\big(\sqrt{3}n - \tfrac{18}{\sqrt{5}}a_4\big)\omega_{1234567} = 0.$$

Hence, we must have $a_4 = n\frac{\sqrt{15}}{18}$ and the resulting G_2–instanton has curvature

$$F = \tfrac{1}{\sqrt{2}}n\big(-\omega_{15} + \tfrac{1}{2}(\omega_{26} + \omega_{37}) + \tfrac{\sqrt{5}}{6}(\omega_{26} - \omega_{37})\big).$$

We turn now to nonabelian G_2–instantons, namely those with gauge group SO(3) that we constructed before. We start by considering the case $n = k - l = 2$, ie instantons on the bundle on $P_2 = \mathrm{SU}(3) \times_{\lambda_2} \mathrm{SO}(3)$. Inserting the A, B, C and D associated with the nearly parallel G_2–structure into our general formula one can check that the quantity inside the square root is negative and so there are no invariant, irreducible, G_2–instantons on P_2. In fact, to be a little more explicit we shall explain all the steps underlying that computation in this case. First, the more general invariant connection on P_2 has $a_2 = a_3 = 0$ and so is of the form

$$A = \big(\tfrac{1}{\sqrt{6}}\eta + a_4\omega_4\big) \otimes T_1 + a_1(\omega_1 \otimes T_2 + \omega_5 \otimes T_3).$$

We compute its curvature F_A as before and set $F_A \wedge \psi = 0$, which yields the equations

(4-5) $$3\sqrt{30}a_4 - 20a_1^2 - 5 = 0,$$

(4-6) $$a_1(2\sqrt{2} + \sqrt{15}a_4) = 0.$$

From (4-6) we see that either $a_1 = 0$, in which case the connection is reducible, or $a_4 = -2\sqrt{\frac{2}{15}}$. Inserting this into (4-5) we then have to solve $-20a_1^2 - 17 = 0$, which has no real solutions. Alternatively we could have just found that $\sigma_1 = -\frac{14336}{225}$, whose being negative shows that there are no irreducible instantons on P_2.

We analyze now the case when $n = l - m$ or $n = m - k$ as in both these cases we have $n = -1$. In this case an invariant connection must have $a_1 = 0$, while a_2 and a_3 can be nonzero. However, as we have seen in our analysis of the general case, the G_2–instanton equations imply that at least one of these vanishes. In fact, after inserting the values of A, B, C and D into the formulas of Theorem 44, we can check that $\sigma_2 < 0$ and $\sigma_3 > 0$. Hence, there are irreducible G_2–instantons and any such has $a_2 = 0$,

$$a_3 = \pm\tfrac{1}{6}\sqrt{-21 + 12\sqrt{5}} \quad \text{and} \quad a_4 = -\tfrac{\sqrt{6}}{36}(4\sqrt{5} - 13).$$

The quantities appearing inside the square root are positive and so these solutions do correspond to genuine G_2–instantons for the nearly parallel G_2–structure on $X_{1,-1}$. For completeness we write the curvature of such an instanton in the usual way with

$$F_1 = \tfrac{1}{2}\omega_{15} + \big(\tfrac{5}{6} - \tfrac{1}{3}\sqrt{5}\big)\omega_{26} + \big(-\tfrac{5}{2} + \sqrt{5}\big)\omega_{37},$$
$$F_2 = \pm\tfrac{\sqrt{2}}{12}\sqrt{-21+12\sqrt{5}}\big(\omega_{12} - \omega_{56} + \tfrac{4\sqrt{3}}{9}(-1+\sqrt{5})\omega_{47}\big),$$
$$F_3 = \pm\tfrac{\sqrt{2}}{12}\sqrt{-21+12\sqrt{5}}\big(-\omega_{16} + \omega_{25} + \tfrac{4\sqrt{3}}{9}(-1+\sqrt{5})\omega_{34}\big).$$

4.6.2 Yang–Mills unstable G_2–instantons Let A be a G_2–instanton for a nearly parallel or torsion-free G_2–structure φ, ie such that $d\varphi = \lambda\psi$ for $\lambda \in \mathbb{R}$. We have seen, in Proposition 8, that such G_2–instantons are actually Yang–Mills connections. Moreover, (2-4) and the subsequent discussion show that in the torsion-free case a G_2–instanton minimizes the Yang–Mills energy, and so is Yang–Mills stable. That need not be the case for nearly parallel G_2–structures as we now show with a counterexample on the nearly parallel $X_{1,-1}$.

Proposition 60 *The irreducible G_2–instantons constructed in the second item of Theorem 59, over the nearly parallel $X_{1,-1}$, are unstable as Yang–Mills connections.*

Proof In order to demonstrate instability, it will be sufficient to consider the Yang–Mills energy only for invariant connections with $a_2 = 0$. We will denote a_3 simply by a. The Yang–Mills energy for the connection

$$A^{-1} = \left(-\frac{\eta}{2\sqrt{6}} + b\omega_4\right) \otimes T_1 + a(\omega_3 \otimes T_2 + \omega_7 \otimes T_3)$$

on P_{-1} is

$$E(a,b) = \tfrac{25}{4096} + \tfrac{15}{65536}(3-\sqrt{5})(12b-\sqrt{6})^2 + \tfrac{45}{32768}(3+\sqrt{5})(8a^2 - 2\sqrt{6}b - 1)^2 + \tfrac{15}{1024}a^2(5-\sqrt{5}) + \tfrac{405}{65536}a^2(5+\sqrt{5})(4b-\sqrt{6})^2.$$

A routine calculation shows that, as expected from Proposition 8, the G_2–instantons at

$$a = \pm\tfrac{1}{6}\sqrt{-21+12\sqrt{5}} \quad \text{and} \quad b = -\tfrac{\sqrt{6}}{36}(4\sqrt{5} - 13)$$

are critical points for this energy. For both of these G_2–instantons the determinant and trace of the Hessian of $E(a,b)$ are

$$\det(\mathrm{Hess}(E)) = \tfrac{196425}{524288} - \tfrac{83025}{262144}\sqrt{5} < 0 \quad \text{and} \quad \mathrm{tr}(\mathrm{Hess}(E)) = \tfrac{735}{8192} + \tfrac{4155}{8192}\sqrt{5} > 0.$$

Thus they are critical points of index one, hence unstable as Yang–Mills connections. □

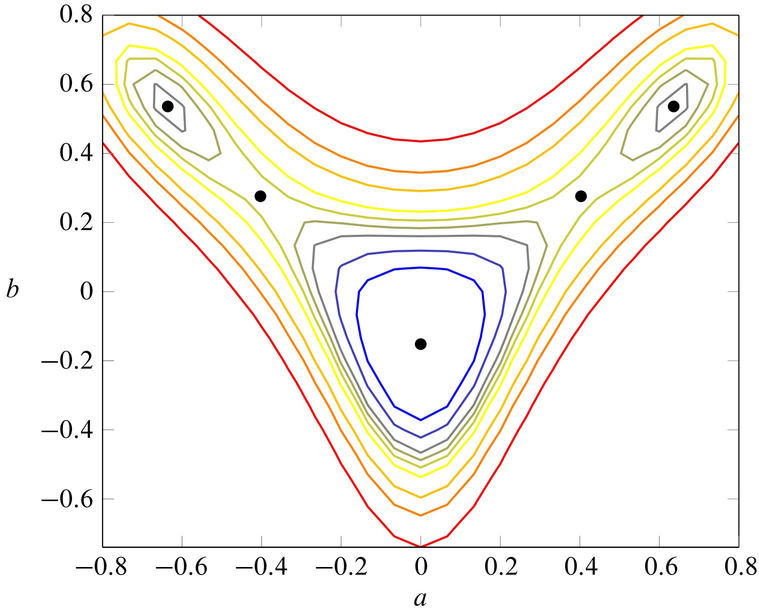

Figure 3: Level sets of the invariant Yang–Mills functional with $a_2 = 0$. One can see three local minima. The global minimum is on top and is a reducible G_2–instanton. There are also two saddles which lie on straight lines from the reducible G_2–instanton to the other local minima. Those saddle points correspond to the irreducible G_2–instantons.

Remark 61 (1) It is not difficult to check that the reducible G_2–instanton is the global minimum for the Yang–Mills energy among all invariant connections on the bundle P_{-1} (ie even when $a_2 \neq 0$).

(2) When restricting to the $a_2 = 0$ case there are three local minima of the Yang–Mills energy: the reducible G_2–instanton, and a pair of Yang–Mills connections that are not G_2–instantons; see Figure 3. The two irreducible G_2–instantons are the two saddles in that figure.

5 Gauge theory on $X_{1,1}$

In this section we study G_2–instantons on $X_{1,1}$, with respect to the G_2–structures (3-1). This case was excluded from the previous section, since here the existence result for invariant connections, Lemma 38, requires a separate analysis. We further remark that in the case $(k, l) = (1, 1)$, the form (3-1) for the G_2–structure does not yield the most general homogeneous coclosed G_2–structure. We start by proving Theorems 62 and 64, which are the analogues of Theorems 42 and 44, classifying abelian and nonabelian G_2–instantons on $X_{1,1}$. Then, in Theorem 65, we prove that the same phenomenon as in Theorem 46 occurs in the case of $X_{1,1}$. Namely, we prove that on $X_{1,1}$ there are irreducible invariant G_2–instantons, with gauge group $\mathrm{SO}(3)$, that as the G_2–structure varies merge into the same reducible and obstructed one.

Then, in Section 5.3, we specialize to a certain subfamily of G_2–structures in $\mathcal{C}$ and write down the explicit formulas for the G_2–instantons in this subfamily. The main results here are Theorems 67 and 69. In particular, this last one proves that there are two bundles (one of which is the trivial one) carrying irreducible G_2–instantons, with gauge group $\mathrm{SO}(3)$, for a continuous family of G_2–structures. Also, we prove in Theorem 71 that as the fibers of a projection $\pi\colon X_{1,1} \to \mathbb{CP}^2$ collapse, the irreducible G_2–instantons in the trivial bundle converge to the pullback of a connection from $\mathbb{CP}^2$. We also show this cannot be true for the G_2–instantons in the other bundle. Finally, in Corollary 73 we prove that while there are no invariant irreducible G_2–instantons with gauge group $\mathrm{SO}(3)$ for the 3–Sasakian structure on $X_{1,1}$, these do exist for the strictly nearly parallel one.

5.1 Abelian case

The following theorem is the analogue of Theorem 42, classifying invariant G_2–instantons on $X_{1,1}$ with gauge group $U(1)$. Note that for $(k,l)=(1,1)$,

$$\Gamma = 3A^2(C^2-B^2) \quad \text{and} \quad \Delta = A^2B^2 + A^2C^2 - 2B^2C^2.$$

Theorem 62 *Equip $X_{1,1}$ with the G_2–structure* (3-1). *Let A^n be an invariant G_2–instanton on the line bundle Q_n over $X_{1,1}$. Then:*

(1) *If $AD+BC \neq 0$, then one of the following holds:*

  (a) *$\Delta \neq 0$, in which case A^n is the unique G_2–instanton on Q_n. Its connection 1–form is*
$$A^n = \frac{n}{2}\left(\frac{1}{\sqrt{6}}\eta + \frac{\Gamma}{3\sqrt{2}\Delta}\omega_4\right).$$

  (b) *$\Delta = 0$, but $\Gamma \neq 0$, in which case $n=0$ and so A lives in the trivial homogenous bundle* (ie that associated with λ_0)*, and A^n is simply one of the 1–forms $b\omega_4$, for some $b \in \mathbb{R}$.*

  (c) *$\Delta = 0$ and $\Gamma = 0$, in which case there is a real 1–parameter family of such instantons on each Q_n.*

(2) *If $AD + BC = 0$, then one of the following holds:*

(a) $\Delta \neq 0$, *in which case there is a real 2–parameter family of such G_2–instantons on Q_n, and A^n is given by*

$$A^n = \frac{n}{2}\left(\frac{1}{\sqrt{6}}\eta + \frac{\Gamma}{3\sqrt{2}\Delta}\omega_4 + a_1\omega_1 + a_5\omega_5\right),$$

for some $a_1, a_5 \in \mathbb{R}$.

(b) $\Delta = 0$, *but $\Gamma \neq 0$, in which case $n = 0$ and so A lives in the trivial homogenous bundle* (ie that associated with λ_0), *and A is simply one of the 1–forms $b\omega_4 + a_1\omega_1 + a_5\omega_5$, for some $a_1, a_5, b \in \mathbb{R}$.*

(c) $\Delta = 0$ *and $\Gamma = 0$, in which case there is a real 3–parameter family of such instantons on each Q_n.*

Proof Any abelian G_2–instanton can be interpreted as a reducible SO(3)–instanton. Hence, we can use the formulas from Section 3.2.2 for the connection form

$$A^n = \frac{n}{2}\frac{\eta}{\sqrt{6}} + b\omega_4 + a_1\omega_1 + a_5\omega_5.$$

For this connection the 6–form $F^n \wedge \psi$ becomes

$$\sqrt{2}BC(AD + BC)(a_1\omega_{234567} + a_5\omega_{123467}) + \left(\tfrac{1}{\sqrt{2}}\Delta a_4 - \tfrac{1}{12}n\Gamma\right)\omega_{123567}.$$

If we equate this to zero, then the result follows from splitting into the various possible cases and simple algebraic manipulations. □

Remark 63 • The condition that $\Delta = 0 = \Gamma$ and $AD + BC = 0$ can occur. Take for example a G_2–structure with $A = B = C$ and $D = -A$. In this case there is a 3–parameter family of invariant G_2–instantons on any complex line bundle over $X_{1,1}$.

• The existence of this real 3–parameter family for these G_2–structures can be understood in light of Proposition 17.

5.2 Nonabelian case

Next we have the analogue of Theorem 44, classifying invariant, irreducible G_2–instantons over $X_{1,1}$ with gauge group $\mathrm{SU}(2)$.

Theorem 64 *Equip $X_{1,1}$ with the G_2–structure* (3-1). *Then invariant, irreducible G_2–instantons exist on the bundle P_{λ_n} if and only if:*

(1) *$n = 0$ and $-\Delta(1 + (AD)/(BC)) > 0$, in which case the G_2–instanton has connection 1–form*

$$A^0 = a_4\omega_4 \otimes T_1 + a_1(\omega_1 \otimes T_2 + \omega_5 \otimes T_3),$$

where the a_i satisfy

$$a_1^2 = \frac{-\Delta}{4B^2C^2}\left(1 + \frac{AD}{BC}\right) \quad \text{and} \quad a_4 = \frac{1}{\sqrt{2}}\left(1 + \frac{AD}{BC}\right);$$

(2) *$n = 3$ and $\sigma_2(\varphi) = 3\big(\frac{1}{2} - (BD)/(AC)\big)\Delta + \frac{3}{2}\Gamma > 0$, in which case $a_1 = a_5 = 0$,*

$$a_2^2 = \frac{1}{12A^2C^2}\left(3\left(\frac{1}{2} - \frac{BD}{AC}\right)\Delta + \frac{3}{2}\Gamma\right) \quad \text{and} \quad b = \frac{1}{\sqrt{2}}\left(-\frac{1}{2} + \frac{BD}{AC}\right);$$

(3) *$n = -3$ and $\sigma_3(\varphi) = 3\big(\frac{1}{2} - (CD)/(AB)\big)\Delta - \frac{3}{2}\Gamma > 0$, in which case $a_1 = a_5 = 0$,*

$$a_3^2 = \frac{1}{12A^2B^2}\left(3\left(\frac{1}{2} - \frac{CD}{AB}\right)\Delta - \frac{3}{2}\Gamma\right) \quad \text{and} \quad b = \frac{1}{\sqrt{2}}\left(-\frac{1}{2} + \frac{CD}{AB}\right).$$

Proof We follow the same strategy as in the proof of Theorem 44, splitting into the cases described above.

Case 0 $(n \neq 0, 3, -3)$ Here A^n is always reducible so there cannot be an invariant, irreducible instanton. We note that the reducible G_2–instantons arising from this case are exactly those appearing in Theorem 62.

Cases 2 and 3 $(n = 3, -3)$ These cases can be handled in the same way as the second and third items in Theorem 44, so we omit the details.

Case 1 $(n = 0)$ Here any invariant connection is simply a left-invariant, and $\mathrm{Ad}(U(1)_{1,1})$–invariant, 1–form with values in $\mathfrak{so}(3)$. We write it as

$$A^0 = \omega_1 \otimes c_1 + \omega_4 \otimes c_4 + \omega_5 \otimes c_5,$$

where $c_1, c_4, c_5 \in \mathfrak{so}(3)$. We compute the curvature of this connection using the formula $F^0 = dA^0 + \frac{1}{2}[A^0 \wedge A^0]$. This gives

$$F^0 = d\omega_1 \otimes c_1 + d\omega_4 \otimes c_4 + d\omega_5 \otimes c_5 + \omega_{14} \otimes [c_1, c_4] + \omega_{15} \otimes [c_1, c_5] + \omega_{45} \otimes [c_4, c_5].$$

The equation $F^0 \wedge \psi = 0$, after a small amount of simplification, yields

$$\begin{aligned} BC[c_1, c_4] &= -\sqrt{2}(AD + BC)c_5, \\ BC[c_4, c_5] &= -\sqrt{2}(AD + BC)c_1, \\ \sqrt{2}\, B^2C^2[c_1, c_5] &= -\Delta c_4. \end{aligned}$$

Bracketing the third equation with c_4 gives us $[[c_1, c_5], c_4] = 0$. We first assume that $[c_1, c_5] \neq 0$, which by the third equation implies $c_4 \neq 0$. This being the case, we may change gauge to require that

$$c_1 = r_1 T_2, \quad c_4 = r_4 T_1 \quad \text{and} \quad c_5 = r_5 T_3$$

for some nonzero real constants r_1, r_4 and r_5. With this choice, the system becomes

$$\begin{aligned} 2BC r_1 r_4 &= \sqrt{2}(AD + BC) r_5, \\ 2BC r_4 r_5 &= \sqrt{2}(AD + BC) r_1, \\ 2\sqrt{2} B^2C^2 r_1 r_5 &= -\Delta r_4. \end{aligned}$$

Since we have assumed that the r_i are nonzero, we must have $\Delta \neq 0$ and $AD + BC \neq 0$. The solutions to these equations are readily found to be

$$r_1^2 = \frac{-\Delta}{4B^2C^2}\left(1 + \frac{AD}{BC}\right), \quad r_5 = \pm r_1 \quad \text{and} \quad r_4 = \pm\frac{1}{\sqrt{2}}\left(1 + \frac{AD}{BC}\right),$$

which seems to yield four solutions, provided

$$-\left(1 + \frac{AD}{BC}\right) > 0.$$

However, the solutions differing only by the $\pm$ sign are gauge equivalent: we can change gauge to send T_1 to $-T_1$, and T_3 to $-T_3$. At this point we set $a_1 = r_1$ and $a_4 = r_4$ yielding the result in the statement.

If $[c_1, c_5] = 0$ then we may by change of gauge fix $c_1 = \lambda_1 T_1$ and $c_5 = \lambda_5 T_1$ for some (possibly zero) constants λ_1 and λ_5. Then, considering the first equation $BC[c_1, c_4] = -\sqrt{2}(AD + BC)c_5$, we must have $[c_1, c_4] = 0$. Therefore the connection is reducible, and the solutions will correspond to abelian G_2–instantons already described in Theorem 42. □

With exactly the same method as in Theorem 65 we can prove that when the G_2–instantons merge they become reducible and obstructed.

Theorem 65 *Let* $\{\varphi(s)\}_{s\in\mathbb{R}}$ *be a continuous family of* G_2*–structures as in* (3-1) *such that* $\sigma_1(\varphi(s)) > 0$ *for* $s < 0$ *and* $\sigma_1(\varphi(s)) < 0$ *for* $s > 0$*. Then, as* $s \nearrow 0$*, the two irreducible* G_2*–instantons on* P_{λ_0} *from Theorem 44 merge and become the same reducible and obstructed* G_2*–instanton when they disappear for* $s \le 0$*.*

Remark 66 A similar statement holds for the G_2–instantons on $P_{\lambda_\pm}$ with σ_1 replaced by σ_2 and σ_3 respectively.

5.3 An example of merging G_2–instantons on $X_{1,1}$

We may think of $\pi_1\colon X_{1,1} \to \mathbb{CP}^2$ as in Remark 34, ie as an SO(3)–bundle over $\mathbb{CP}^2$, which is a quaternion-Kähler 4–manifold (self-dual, Einstein) with positive scalar curvature. The discussion before Proposition 18, in Section 2.2, shows that $X_{1,1}$ carries two nearly parallel G_2–structures, one inducing a 3–Sasakian metric and the other inducing a strictly nearly parallel one. This last one will be contained in the family of G_2–structures we consider in this section. Proposition 18 gives some examples of G_2–instantons on $X_{1,1}$ by pulling back self-dual connections on $\mathbb{CP}^2$. In fact, on any line bundle over $\mathbb{CP}^2$ there is one such connection that is SU(3)–invariant, namely the canonical connection $\frac{n}{2\sqrt{6}}\eta$ on the degree n–bundle. In what follows we shall confirm this fact and we will also obtain other examples of G_2–instantons that are not pulled back from $\mathbb{CP}^2$.

In this subsection we will consider the G_2–structures in the family (3-1) that satisfy $C = B$ and $D = A$. This is, up to scaling, the 1–parameter family in the hypothesis of Proposition 18 with t proportional to A/B. For completeness we note that the G_2–structure in (3-1) gives

$$\psi = B^4\left(\omega_{2367} - \frac{A^2}{B^2}(\omega_{15}\wedge\Omega_1 + \omega_{45}\wedge\Omega_2 - \omega_{14}\wedge\Omega_3)\right),$$

where $\Omega_1 = \omega_{26} - \omega_{73}$, $\Omega_2 = \omega_{23} - \omega_{67}$ and $\Omega_3 = \omega_{27} - \omega_{36}$ form an orthonormal basis for the pullback of the space of anti-self-dual 2–forms on $\mathbb{CP}^2$. One can then check that this family contains one of the homogeneous nearly parallel G_2–structures on $X_{1,1}$. In fact, one can check that $A = -2\sqrt{2}/\lambda$ and $B = 2/\lambda$ satisfy $d\varphi = \lambda\psi$.

For the structures we are considering,

$$AD + BC = A^2 + B^2 \neq 0, \quad \Delta = 2B^2(A^2 - B^2) \quad \text{and} \quad \Gamma = 0,$$

and thus Theorem 62(1)(a) tells us that for $\Delta \neq 0$, ie $A^2 \neq B^2$, there is a unique G_2–instanton on Q_n. This has $b = 0$ and so is precisely the canonical invariant connection $\frac{n}{2\sqrt{6}}\eta$. Its curvature is

$$\frac{n}{2\sqrt{6}}\,d\eta = -\frac{n}{4}(\omega_{26} + \omega_{73}),$$

and as remarked before, is actually the pullback from $\mathbb{CP}^2$ of a self-dual 2–form. On the other hand, Theorem 62(1)(c) shows that when $A^2 = B^2$ there is a 1–parameter family of G_2–instantons, namely any of the connections $\frac{n}{2\sqrt{6}}\eta + b\omega_4$, for $b \in \mathbb{R}$. We state these conclusions as:

Theorem 67 *Let $A, B \in \mathbb{R}^+$ and equip $X_{1,1} = \mathrm{SU}(3)/U(1)_{1,1}$ with the G_2–structure*

$$\varphi_{A,B} = A^3\omega_{145} + AB^2(\omega_{123} - \omega_{167} + \omega_{257} - \omega_{356} - \omega_{426} - \omega_{437}).$$

If L is a complex line bundle over $X_{1,1}$ with $c_1(L) = n \in \mathbb{Z} \cong H^2(X_{1,1}, \mathbb{Z})$, then:

- *If $A^2 \neq B^2$, the canonical connection $\frac{n}{2\sqrt{6}}\eta$ is the unique invariant G_2–instanton on L.*
- *If $A^2 = B^2$, then the connections $\frac{n}{2\sqrt{6}}\eta + b\omega_4$ are G_2–instantons for any $b \in \mathbb{R}$. These are the unique invariant G_2–instantons on L.*

Remark 68 (1) The canonical connection $\frac{n}{2\sqrt{6}}\eta$ is the pullback of a self-dual connection on $\mathbb{CP}^2$. Therefore, the fact that it is a G_2–instanton with respect to $\varphi_{A,B}$ also follows from Proposition 18. Its uniqueness for the nearly parallel structure is also a consequence of Corollary 14, however uniqueness amongst invariant ones for other structures in the family $\{\varphi_{A,B}\}_{A \neq B}$ is not.

(2) The abelian instantons constructed for $A = B$ show that the uniqueness part of Corollary 14 does not extend from nearly parallel to general coclosed G_2–structures. In fact, not even the rigidity stated in Corollary 25 holds.

We turn now to invariant, irreducible, nonabelian G_2–instantons. We start with the case $n = k - l = 0$. Theorem 64 tells us that G_2–instantons on P_0 exist if and only if

$$-2B^2(A^2 - B^2)\left(1 + \frac{A^2}{B^2}\right) > 0,$$

or in other words if and only if $B^2 > A^2$. In this case we have

$$A^0 = a_4\omega_4 \otimes T_1 + a_1(\omega_1 \otimes T_2 + \omega_5 \otimes T_3),$$

where the a_i must satisfy

$$a_1 = \pm\sqrt{\frac{B^4 - A^4}{2B^4}} \quad \text{and} \quad a_4 = \frac{A^2 + B^2}{\sqrt{2}B^2}.$$

The curvature of these connections is

$$F = F_1 \otimes T_1 + F_2 \otimes T_2 + F_3 \otimes T_3,$$

with

$$F_1 = -\left(\frac{A^2}{B^2} + 1\right)\left(\frac{A^2}{B^2}\omega_{15} - \frac{1}{2}(\omega_{26} - \omega_{73})\right), \tag{5-1}$$

$$F_2 = \mp\sqrt{1 - \frac{A^4}{B^4}}\left(\frac{A^2}{B^2}\omega_{45} - \frac{1}{2}(\omega_{23} - \omega_{67})\right), \tag{5-2}$$

$$F_3 = \mp\sqrt{1 - \frac{A^4}{B^4}}\left(\frac{A^2}{B^2}\omega_{14} - \frac{1}{2}(\omega_{36} - \omega_{27})\right). \tag{5-3}$$

The other cases in which there exist nontrivial invariant connections are when $n = \pm 3$. Notice that P_3 and P_{-3} are interchanged by the automorphism of SU(3) given by $g \mapsto g^{-1}$. This automorphism preserves $U(1)_{1,1}$ and so descends to a diffeomorphism of $X_{1,1} = X_{-1,-1}$. We shall therefore consider only the case $n = 3$ where $a_1 = a_3 = 0$. Also in this case, our work above gives that there are irreducible, invariant G_2–instantons on P_3 (resp. P_{-3}) if and only if

$$\sigma_2 = \sigma_3 = 3B^2(B^2 - A^2) \geq 0,$$

ie $B^2 > A^2$. In that case we have

$$a_2 = \pm\frac{1}{2}\sqrt{-1 + \frac{B^2}{A^2}} \quad \text{and} \quad a_4 = \frac{1}{2\sqrt{2}},$$

and their curvature is such that

$$F_1 = -\frac{1}{2}\omega_{15} - \left(1 - \frac{B^2}{2A^2}\right)\omega_{26} + \omega_{37}, \tag{5-4}$$

$$F_2 = \mp\frac{1}{\sqrt{2}}\sqrt{-1 + \frac{B^2}{A^2}}\left(\omega_{46} + \frac{1}{2}(\omega_{13} - \omega_{57})\right), \tag{5-5}$$

$$F_3 = \pm\frac{1}{\sqrt{2}}\sqrt{-1 + \frac{B^2}{A^2}}\left(-\omega_{24} + \frac{1}{2}(\omega_{17} - \omega_{35})\right). \tag{5-6}$$

As before these are clearly irreducible and not pulled back from $\mathbb{CP}^2$ via π. We have thus proved:

Theorem 69 *For $A, B \in \mathbb{R}^+$, let $\varphi_{A,B}$ be the G_2–structure on $X_{1,1} = \mathrm{SU}(3)/U(1)_{1,1}$ from Theorem 67. Let ∇_A be an $\mathrm{SU}(3)$–invariant, irreducible G_2–instanton for $\varphi_{A,B}$, with gauge group $\mathrm{SO}(3)$. Then either:*

(1) *∇_A lives on P_0, the trivial $\mathrm{SO}(3)$–bundle over $X_{1,1}$, in which case the following hold:*

- *If $A < B$, then ∇_A is one of two G_2–instantons on P_0, having curvature as in equations* (5-1)–(5-3).
- *If $A \geq B$, there is no invariant, irreducible G_2–instanton on P_0.*

(2) *∇_A lives on one of the bundles P_3 or P_{-3}, in which case the following hold:*

- *If $A < B$, then ∇_A is one of two invariant, irreducible G_2–instantons on $P_{\pm 3}$. If ∇_A lives on P_3, its curvature is as in equations* (5-4)–(5-6).
- *If $A \geq B$, there is no invariant, irreducible G_2–instanton on either $P_{\pm 3}$.*

Remark 70

- Both in P_0 and P_3, the G_2–instantons $(\nabla_A)_{A,B}$ constructed above become abelian when $A = B$.
- None of the irreducible G_2–instantons on P_0 and P_3 constructed for $A < B$ is pulled back from $\mathbb{CP}^2$ and so none follows from Proposition 18.

The instantons on P_0 and P_3 constructed above are quite different. In fact, looking at the expressions for the curvature of these, we see that by metrically collapsing the fibers of $\pi\colon X_{1,1} \to \mathbb{CP}^2$ by sending A to 0, the instantons constructed on P_0 converge to the pullback of a connection on $\mathbb{CP}^2$. However, this property does not hold for those constructed on P_3. More precisely, we have:

Theorem 71 *Let $(\nabla_A)_{A,B}$ be the G_2–instanton associated with $\varphi_{A,B}$ on P_0. Then there is an $\mathrm{SO}(3)$–connection ∇ on $\mathbb{CP}^2$ such that as $A \to 0$, $(\nabla_A)_{A,B}$ converges uniformly with all its derivatives to $\pi^*\nabla$.*

Let $(\widetilde{\nabla}_A)_{A,B}$ be the G_2–instanton associated with $\varphi_{A,B}$ on P_3. There is no connection ∇ on $\mathbb{CP}^2$ such that $(\nabla_A)_{A,B} \to \pi^*\nabla$ uniformly with respect to $\varphi_{1,1}$ as $A \to 0$.

Proof Let $P = \mathrm{SU}(3) \times_{U(2),\lambda} \mathrm{SO}(3)$ be the bundle constructed from

$$\lambda\colon \mathrm{SU}(2) \times U(1)/\mathbb{Z}_2 \to \mathrm{SO}(3) \quad \text{with } \lambda(g, e^{i\theta}) = g \bmod -1.$$

The canonical invariant connection ∇ associated with this bundle is

$$\frac{1}{\sqrt{2}}(\omega_4 \otimes T_1 + \omega_1 \otimes T_2 + \omega_5 \otimes T_3) \in \Omega^1(\mathrm{SU}(3), \mathfrak{so}(3)).$$

Its curvature is $F = T_1 \otimes T_1 + F_2 \otimes T_2 + F_3 \otimes T_3$ such that

$$F_1 = \tfrac{1}{2}(\omega_{26} - \omega_{73}), \quad F_2 = \tfrac{1}{2}(\omega_{23} - \omega_{67}) \quad \text{and} \quad F_3 = -\tfrac{1}{2}(\omega_{27} - \omega_{36}),$$

so it is a anti-self-dual connection. In fact notice that the components F_1, F_2 and F_3 of the curvature pull back respectively to Ω_1, Ω_2 and Ω_3 on $X_{1,1}$. We now let $(\nabla_A)_{A,B}$ be our G_2–instanton on $\varphi_{A,B}$, which has connection 1–form

$$\frac{1}{\sqrt{2}}\left(\frac{A^2}{B^2} + 1\right)\omega_4 \otimes T_1 + \frac{1}{\sqrt{2}}\sqrt{1 - \frac{A^4}{B^4}}(\omega_1 \otimes T_2 + \omega_5 \otimes T_3),$$

seen as an element of $\Omega^1(\mathrm{SU}(3), \mathfrak{so}(3))$. Hence the difference of the two connections $a_{A,B} = (\nabla_A)_{A,B} - \pi^*\nabla$ is a $\frac{1}{2\sqrt{6}}\eta$–horizontal 1–form in $\mathrm{SU}(3)$ given by

$$a = \frac{1}{\sqrt{2}}\frac{A^2}{B^2}\omega_4 \otimes T_1 + \frac{1}{\sqrt{2}}\left(\sqrt{1 - \frac{A^4}{B^4}} - 1\right)(\omega_1 \otimes T_2 + \omega_5 \otimes T_3).$$

Using the fixed metric associated with the G_2–structure $\varphi_{1,1}$ to take norms we compute that for any $k \in \mathbb{Z}^+$,

$$\|a_{A,B}\|_{C^k} \le c_k \frac{A^2}{B^2},$$

for some positive constant c_k independent of A and B. Taking A to 0 we see that $a_{A,B}$ converges uniformly to 0 with all derivatives, proving the first assertion in the statement.

We turn now to the proof of the second assertion, namely, that the same phenomena cannot happen for the instantons we constructed on P_{λ_3}. If such a statement was to be true, the curvatures $\widetilde{F}_{A,B}$ of $(\widetilde{\nabla}_A)_{A,B}$ should converge to an $\mathfrak{so}(3)$–valued 2–form on $\mathrm{SU}(3)$ that is basic with respect to the projection $\mathrm{SU}(3) \to \mathbb{CP}^2$. Any linear combination V of the vector fields e_1, e_4 and e_5 is vertical with respect to this projection. Taking $V = e_1$ we have

$$\iota_{e_1}\widetilde{F}_{A,B} = \tfrac{1}{2}\omega_5 \otimes T_1 \mp \frac{1}{2\sqrt{2}}\sqrt{-1 + \frac{B^2}{A^2}}(\omega_3 \otimes T_2 - \omega_7 \otimes T_3)$$

and clearly $\lim_{A\to 0}\|\iota_{e_1}\widetilde{F}_{A,B}\|_{C^k} = +\infty$ for all $k \in \mathbb{N}_0$. Hence, $\widetilde{F}_{A,B}$ cannot converge to a basic form. □

Remark 72 The SO(3)–connection ∇ on $\mathbb{CP}^2$ appearing in the previous theorem is in fact anti-self-dual. However, we do not want to emphasize this fact too much, as it may be misleading. Indeed, we expect that in other similar situations the same phenomena can occur with the corresponding ∇ not being anti-self-dual.

There is one other homogeneous nearly parallel G_2–structure on $X_{1,1}$. In fact, the equations for homogeneous nearly parallel G_2–structures in the case $(k,l)=(1,1)$ yield eight solutions, which give rise to two different metrics. The solutions are completely determined by $C^2=B^2$, $D^2=A^2$ and these two cases:

- $A^2=2B^2$ and $ABCD>0$, which fits into the family just described and in which case the corresponding metric is 3–Sasakian.
- $A^2=2B^2/5$ and $ABCD<0$, and so the G_2–structure is obtained from the above through the squashing construction in Section 2.2. In this case, the corresponding metric is a strictly nearly parallel G_2–metric; see [18, Theorem 5.5].

Notice that Theorem 69 does not yield any irreducible G_2–instanton for the nearly parallel G_2–structure contained in the family we are analyzing, which is the one inducing the 3–Sasakian structure. However, as we shall now show, the theorem does yield irreducible G_2–instantons for the strictly nearly parallel structure.

Corollary 73
- *There are no irreducible, invariant G_2–instantons with gauge group* SO(3) *for the nearly parallel G_2–structure on $X_{1,1}$ inducing the* 3–*Sasakian metric.*
- *There are irreducible, invariant G_2–instantons with gauge group* SO(3) *for the strictly nearly parallel G_2–structure on $X_{1,1}$.*

Proof Any homogeneous nearly parallel G_2–structure on $X_{1,1}$ satisfies $A^2=D^2$ and $B^2=C^2$. There are two cases:

- $A^2=2B^2$ and $ABCD>0$. In fact, for $ABCD>0$ we compute

$$\sigma_1(\varphi)=6(B^4-A^4) \quad \text{and} \quad \sigma_2(\varphi)=\sigma_3(\varphi)=3B^2(B^2-A^2).$$

As the nearly parallel G_2–structure in this case has $A^2=2B^2>B^2$ we see that all σ_i, for $i=1,2,3$, are negative and so there are no G_2–instantons.

- $A^2=2B^2/5$ and $ABCD<0$. In this case we compute that for $ABCD<0$,

$$\sigma_1(\varphi)=6(A^2-B^2)^2 \quad \text{and} \quad \sigma_2(\varphi)=\sigma_3(\varphi)=9B^2(A^2-B^2).$$

The nearly parallel G_2–structure has $A^2 = 2B^2/5 < B^2$, so both σ_2 and σ_3 are negative. On the other hand σ_1 is positive and thus irreducible G_2–instantons on this nearly parallel G_2–structure do exist. Any such must live in the trivial bundle P_{λ_0}. □

Remark 74
- The previous result shows the G_2–structures inducing the 3–Sasakian and the strictly nearly parallel G_2–structures on $X_{1,1}$ can be distinguished by the existence of an irreducible, invariant G_2–instanton with gauge group $\mathrm{SO}(3)$.
- We further remark that we are not analyzing the most general homogeneous and coclosed G_2–structures on $X_{1,1}$. In fact, for $(k,l)=(1,1)$ there is a larger-dimensional family, containing in particular a nearly parallel G_2–structure whose associated metric is Sasaki–Einstein; see [7] and [9].
- G_2–instantons with gauge group $\mathrm{SU}(3)$ for the 3–Sasakian structure on $X_{1,1}$ have been considered in [20].

6 Questions for further work

The following are natural directions for further work:

(1) Similar methods can be used in many other cases where homogeneous G_2–structures exist. Of particular interest would be the cases admitting nearly parallel G_2–structures; see [18] for the classification of homogeneous nearly parallel G_2–manifolds.

(2) Carry on a general analysis of the following question: for which (k,l) do Theorems 42 and 44 provide irreducible G_2–instantons for the nearly parallel G_2–structures in $X_{k,l}$? We intend to address this in the future.

(3) Compute the Crowley–Nordström invariants [13] for the G_2–structures $\varphi \in \mathcal{C}$ and check if this distinguishes the two disconnected components in $\mathcal{C}$. If that is the case, then for $k \neq l$, $l \neq m$, $m \neq k$ these invariants can be used to distinguish the two strictly nearly parallel G_2–structures.

(4) Given a G_2–instanton A for a G_2–structure on $X_{k,l}$ such that A is also Yang–Mills, in which cases is A stable as a Yang–Mills connection? Here, it would be interesting to understand better how the answer to this question depends on the G_2–structure.

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