Orbifolds of n–dimensional defect TQFTs

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We introduce the notion of n–dimensional topological quantum field theory (TQFT) with defects as a symmetric monoidal functor on decorated stratified bordisms of dimension n. The familiar closed or open–closed TQFTs are special cases of defect TQFTs, and for $n = 2$ and $n = 3$ our general definition recovers what had previously been studied in the literature.

Our main construction is that of "generalised orbifolds" for any n–dimensional defect TQFT: Given a defect TQFT $\mathcal{Z}$, one obtains a new TQFT $\mathcal{Z}_{\mathcal{A}}$ by decorating the Poincaré duals of triangulated bordisms with certain algebraic data $\mathcal{A}$ and then evaluating with $\mathcal{Z}$. The orbifold datum $\mathcal{A}$ is constrained by demanding invariance under n–dimensional Pachner moves. This procedure generalises both state sum models and gauging of finite symmetry groups for any n. After developing the general theory, we focus on the case $n = 3$.

1 Introduction

Topological quantum field theory (TQFT) interrelates topology, higher categories and mathematical physics, with prominent ties also to algebra and geometry. In recent years the subject has enjoyed further attention via the study of ∞–categories and topological phases of matter.

The simplest "closed" case of a TQFT is a symmetric monoidal functor $\mathrm{Bord}_n \to \mathrm{Vect}_{\Bbbk}$ for some positive integer n, where Bord_n has closed smooth oriented $(n-1)$–dimensional manifolds as objects and diffeomorphism classes of bordisms as morphisms. To work towards classification results, to gain structural insight into their inner workings and interrelations, and to address questions in neighbouring fields such as knot theory or theoretical physics, it is desirable to enrich the basic notion of closed TQFTs. Two natural directions to do so are to "extend" and to add "defects".

Extended TQFTs in n dimensions are higher functors on higher bordism categories that involve $n-$ and $(n-1)$–dimensional manifolds as in Bord_n, but also manifolds with corners of lower dimension. Accordingly, for an extended TQFT also a higher symmetric monoidal target category has to be specified. On the other hand, the idea behind defect TQFTs $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ is to concentrate all the enriched structure into the source category $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ while staying within the realm of ordinary symmetric monoidal 1–categories. As will be explained in detail in Section 2, objects and morphisms in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ are $(n-1)$–dimensional oriented manifolds and their n–dimensional bordism classes, respectively, which come with a decomposition (or stratification) into j–dimensional submanifolds (or j–strata) for all $j \in \{0, 1, \ldots, n\}$. Furthermore, all strata are decorated by "defect data" from a given collection $\mathbb{D}$. These decorated strata can be thought of as "extended physical observables", of which boundary conditions are familiar special cases. One may argue that this approach is closer to the original motivation to axiomatise structures from physics, laying emphasis on the combinatorics of defect conditions. An n–category may then subsequently be extracted from the functor $\mathcal{Z}$ as an algebraic invariant (this has been worked out in detail for $n \in \{1, 2, 3\}$; see Davydov, Kong and Runkel [19] and Carqueville, Meusburger and Schaumann [11]).

Defects can be used to describe symmetries of a closed n–dimensional TQFT $\mathcal{Z}_{\mathrm{cl}}$: Given a representation ρ of a finite group G on $\mathcal{Z}_{\mathrm{cl}}$, it may be possible to find a family $\{\rho(g)\}_{g \in G}$ of $(n-1)$–dimensional defects whose "fusion" reproduces the product in G. The "superposition" $\mathcal{A}_G = \bigoplus_{g \in G} \rho(g)$ together with coherently chosen lower-dimensional defects (such a choice may be obstructed and is typically not unique) can be used to construct the G–orbifold theory $\mathcal{Z}_{\mathrm{cl}}^G$. This is done by "averaging" over the symmetry — a procedure implemented by evaluating $\mathcal{Z}$ on a network of $\mathcal{A}_G$–defects Poincaré dual to a triangulation of the bordisms. One should think of $\mathcal{A}_G$ and its lower-dimensional defects as an algebraic structure encoding the symmetry G and the necessary extra information needed to orbifold, or, in other words, gauge, that symmetry.

In 2 dimensions this is well understood, with the algebraic structure of $\mathcal{A}_G$ turning out to be that of a Δ–separable symmetric Frobenius algebra; see Fuchs, Runkel and Schweigert [23] and Carqueville and Runkel [13]. More recently G–crossed fusion categories have been studied in connection with G–actions on 3–dimensional TQFTs; see Etingof, Nikshych and Ostrik [20], Cui, Galindo, Plavnik and Wang [18] and Barkeshli, Bonderson, Cheng and Wang [3]. A study of orbifolds of n–dimensional TQFTs by finite groups G using the language of G–equivariant TQFTs rather than that of defects has been carried out by Schweigert and Woike [42].

Algebraic structures $\mathcal{A}$ like those above, describing defects in all dimensions (but which need not necessarily arise from group actions), are also at the centre of state sum constructions: special symmetric Frobenius algebras over a field $\Bbbk$ in 2 dimensions — see Bachas and Petropoulos [1] and Fukuma, Hosono and Kawai [27] — and spherical fusion categories over $\Bbbk$ in 3 dimensions; see Turaev and Viro [43] and Barrett and Westbury [6]. Here again one decorates the Poincaré dual of a triangulation (or a similar type of decomposition) with the data of $\mathcal{A}$ and then uses the algebraic structure to evaluate. Thus, it is natural to generalise the notion of orbifold to encompass any system of defects which can be used to decorate suitable decompositions of bordisms. For the resulting orbifold to be well defined, one must impose the condition that the evaluation with the TQFT functor is invariant under the specific choice of decomposition. Then, in particular, one finds that "state sum models are orbifolds of the trivial theory".

The idea of generalised orbifolds in 2 dimensions was first put forward by Fröhlich, Fuchs, Runkel and Schweigert [21] in the context of rational conformal field theory. Later, by Carqueville and Runkel [13], it was adapted to 2–dimensional defect TQFTs and developed into a Morita-type theory of Δ–separable symmetric Frobenius algebras and their bimodules internal to any pivotal 2–category. Out of this emerged a notion of "orbifold equivalence" which has since found applications in algebra, geometry and mathematical physics; see eg Carqueville, Ros Camacho and Runkel [12], Brunner, Carqueville and Plencner [8; 9] and Carqueville and Velez [17].

A good way to think of orbifolds in this generalised sense is the slogan:

> *Carry out a state sum construction with defects internal to a given n–dimensional quantum field theory.*

The present paper provides a way to make this idea precise and productive for TQFTs of any dimension n. It originally grew out of a desire to extend the theory of generalised

orbifolds from 2 to 3 dimensions, with applications to topological invariants, tensor categories, quantum computation and topological phases of matter in mind.

In this paper, our first contribution is to give a detailed definition of oriented defect TQFTs as symmetric monoidal functors[1]

$$(1\text{-}1) \qquad\qquad \mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$$

in any dimension n. The definition of the defect bordism category $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ (see Definitions 2.1 and 2.4) is inductive in n and controls the ways in which we allow defects, ie decorated strata in stratified bordisms, to meet in terms of iterated cones and cylinders over basic configurations. In particular, we will describe how the "defect data" $\mathbb{D}$ include label sets D_j to decorate j–strata for all $j \in \{0, 1, \ldots, n\}$. We then prove that n–dimension defect TQFTs themselves form a symmetric monoidal category $\mathrm{TQFT}_n^{\mathrm{def}}$; see Proposition 2.11.

In Sections 2.4 and 2.5 we develop the theory further and, in particular, consider decorations for point defects. We show that, without loss of generality, in a defect TQFT one can identify labels for 0–strata with states associated to the surrounding sphere that are invariant under certain automorphisms of the sphere, and we prove that such states form an algebra. In Section 2.5 we exponentiate invertible point defects with the Euler characteristic of the surrounding stratum to construct a refinement of a given defect TQFT, which we call the "Euler completion". The details of these subsections are, however, not required to understand Sections 3 and 4.

The other central notion which we introduce (in Definition 3.5) is that of an "orbifold datum" $\mathcal{A}$ for a given defect TQFT $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$. An orbifold datum consists of a collection of labels $\mathcal{A}_j \in D_j$ to decorate the j–strata of Poincaré duals of triangulated bordisms for $j \in \{1, \ldots, n\}$, as well as two labels $\mathcal{A}_0^+, \mathcal{A}_0^- \in D_0$ for 0–strata. The defining constraints on the orbifold datum $\mathcal{A}$ are precisely that evaluation with $\mathcal{Z}$ of $\mathcal{A}$–decorated bordisms is invariant under the choice of triangulation, ie under oriented versions of Pachner moves (which we recall in Section 3.1). Our main result (Theorem/Definition 3.10) is then the construction of the "$\mathcal{A}$–orbifold theory" $\mathcal{Z}_{\mathcal{A}}$:

Theorem *For every defect TQFT $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ and every orbifold datum $\mathcal{A}$ for $\mathcal{Z}$, there is an associated closed TQFT*

$$(1\text{-}2) \qquad\qquad \mathcal{Z}_{\mathcal{A}}\colon \mathrm{Bord}_n \to \mathrm{Vect}_{\Bbbk}.$$

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[1]In fact the definition of defect TQFT works for any symmetric monoidal target category, and, given the existence of certain limits, also all other constructions in this paper go through; see Remark 3.11.

After a brief discussion in Section 3.3 of how the above-mentioned Δ–separable symmetric Frobenius algebras for $n = 2$ fit into the general picture, we finally concentrate on the 3–dimensional case in Section 3.4. For $n = 3$, the number of defining constraints for an orbifold datum (ie the number of independent oriented Pachner moves) is already 30. We will introduce (Definition 3.13) the notion of a "special orbifold datum" which involves only ten conditions that can be checked more easily in practice. Consistent with the general (expected) relation between state sum models and orbifolds, a special orbifold datum may be characterised as "spherical fusion categories internal to Gray categories with duals" (which need not have units), as we discuss in Section 4.

We note that the main constructions in Sections 2 and 3 are exclusively in terms of ordinary symmetric monoidal categories and their functors. It is only in Section 4 that we discuss higher categorical formulations — which may prove worthwhile independently of their TQFT origin.

The only examples in the present paper are the invertible "Euler defect TQFTs" (Example 2.14). In follow-up work [16] we will study examples of orbifolds of 3–dimensional defect TQFTs, namely Turaev–Viro models as orbifolds of the trivial TQFT, and two different types of $\mathbb{Z}_2$–orbifold of the Reshetikhin–Turaev theory for $\widehat{\mathfrak{sl}}(2)_k$. Reshetikhin–Turaev TQFT with defects is developed in our companion paper [15].

2 Defect TQFTs

An n–dimensional defect TQFT is a symmetric monoidal functor from a category $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ of decorated stratified n–dimensional bordisms to $\mathrm{Vect}_{\Bbbk}$. To explain the details, we start in Section 2.1 by describing our conventions for stratifications. In Section 2.2 we define $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ in two steps: first without decorations (Definition 2.1) and then fully with decorations by "defect data" $\mathbb{D}$ (Definition 2.4). Then, in Section 2.3 we define n–dimensional defect TQFTs, show how they are themselves the objects of a symmetric monoidal category, and discuss the example of "Euler theories". Finally, in Sections 2.4 and 2.5 we define two completions of a given defect TQFT: passing to a maximal set of labels for point defects, and internalising the example of Euler theories.

2.1   Stratified bordisms

Defects are geometrically realised as a system of submanifolds of a bordism. To specify the types of allowed neighbourhoods for the submanifolds in a defect TQFT, we rely on the concept of stratifications, as discussed in [11] or [39] (where the term "decomposed space" is used).

By an n–dimensional *stratified manifold*[2] we mean an n–dimensional topological manifold M (with empty boundary), together with a filtration $M = F_n \supset F_{n-1} \supset \cdots \supset F_0 \supset F_{-1} = \varnothing$ such that $M_i := F_i \setminus F_{i-1}$ is a smooth i–dimensional submanifold of M for all $i \in \{0, 1, \ldots, n\}$. The connected components M_i^α of M_i are called i–*strata*, and we ask the sets of i–strata to be finite. Furthermore, for all strata M_i^α and M_j^β with $i < j$ and $M_i^\alpha \cap \overline{M}_j^\beta \neq \varnothing$, we demand $M_i^\alpha \subset \overline{M}_j^\beta$.

Every triangulated manifold (see Section 3.1) is a stratified manifold. Here, F_i consists of the union of all (closed) j–simplices for $j \leqslant i$, and M_i consists of the open i–simplices. The Poincaré dual of a triangulated manifold is typically not triangulated, but it continues to be stratified.

We are interested in *oriented stratified n–manifolds M*, where the manifold M is itself oriented. Furthermore, each i–stratum with $i < n$ is equipped with a choice of orientation, while the orientation of each n–stratum is taken to be the one induced from M. *Morphisms between oriented stratified manifolds* are continuous maps that restrict to smooth orientation-preserving maps between the strata; see eg [39, Chapter 1].

A *stratified manifold M with boundary ∂M* is an n–manifold M with boundary together with a filtration $M = F_n \supset F_{n-1} \supset \cdots \supset F_0 \supset F_{-1} = \varnothing$ as above such that the interior of M is a stratified manifold and each i–stratum is a submanifold whose boundary is empty or lies in the boundary of M, intersecting ∂M transversely. We also view ∂M a stratified manifold with the stratification induced from M. If M is oriented, we equip ∂M and all its strata with the induced orientations from the strata of M. For more details we refer to [11, Section 2.1]

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[2]In the literature it is common to say a stratified manifold is a topological space that is a manifold outside of certain singularities that are located on the strata and satisfy certain regularity conditions. These conditions specify the type of allowed singularity and the allowed adjacency conditions for all strata. The stratified manifolds we will consider are, however, such that the total space is still a topological manifold.

So the regularity conditions are only needed to specify adjacency conditions, and we note that what we call a "stratified manifold" in the present paper is only the nonsingular subclass of what is often called by that name in the literature.

Convention From now on we make the assumption that, unless specified otherwise, all manifolds and all stratified manifolds we consider are compact and oriented, possibly with boundary, while all maps between them are continuous and their restrictions to strata are smooth and orientation-preserving.

As an example we consider a stratified 3–manifold M where a small part of the stratification is shown here:

(2-1)
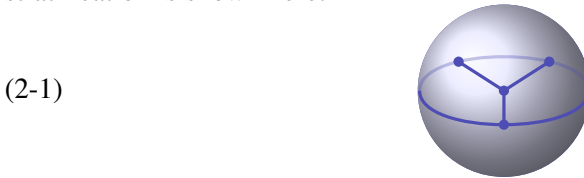
In this case there are two 3–strata (the two half-balls), five 2–strata (the two hemispheres and the three triangle-shaped regions), six 1–strata, and four 0–strata shown. Other 2–strata that are not shown might for example meet the equator from outside. Keeping the outer stratification in M, we obtain another stratification of M by exchanging the ball (2-1) with just one 2–stratum, three 1–strata, and three 0–strata:

(2-2)

Later in Section 3.4.2, local changes of stratifications which exchange (2-1) and (2-2) (called "bubble moves") will be important for us.

For any $n \in \mathbb{Z}_+$ one may now consider the category $\mathrm{Bord}_n^{\mathrm{strat}}$ of *stratified bordisms*. An object in $\mathrm{Bord}_n^{\mathrm{strat}}$ is a closed $(n-1)$–dimensional stratified manifold Σ (oriented and compact by the above convention). A morphism $\Sigma \to \Sigma'$ is an equivalence class of n–dimensional stratified manifolds M with parametrised boundary. We describe the parametrisation and the equivalence relation in turn. The boundary parametrisation is a germ (in $\varepsilon > 0$) of orientation-preserving embeddings

(2-3) $$\iota \colon (\Sigma \times [0, \varepsilon)) \sqcup (\Sigma' \times (-\varepsilon, 0]) \to M$$

(which are in particular continuous maps whose restrictions to strata are smooth and orientation-preserving by our convention) which map $(\Sigma \times \{0\})^{\mathrm{rev}} \sqcup (\Sigma' \times \{0\})$ onto ∂M; we will use ι to denote both the germ and a representative map. The operation $(-)^{\mathrm{rev}}$ reverses the orientation of all strata. Two such stratified manifolds (M, ι) and $(\widetilde{M}, \tilde{\iota})$ are equivalent if there is an isomorphism $f \colon M \to \widetilde{M}$ such that $f \circ \iota = \tilde{\iota}$ on $(\Sigma \times [0, \delta)) \sqcup (\Sigma' \times (-\delta, 0])$ for some small enough $\delta > 0$. Composition in $\mathrm{Bord}_n^{\mathrm{strat}}$ is

defined by choosing representatives, gluing along the boundary parametrisation and then taking the bordism class. Thus, composition is well defined, associative and unital. Further standard arguments show that $\text{Bord}_n^{\text{strat}}$ has a natural structure of a symmetric monoidal category.

2.2 Defect bordisms

2.2.1 Undecorated defect bordisms We think of defects in a defect TQFT as "combinatorial" in nature, meaning that for a defect confined to a given stratum Y it is only the distribution of strata in the immediate surroundings of Y that matters. We will impose regularity conditions by requiring the existence of certain local neighbourhoods (detailed below) for all strata.

Like in the case of a smooth manifold, we shall first specify the type of open stratified manifolds that we want to allow as local neighbourhoods and then define a defect bordism via an atlas of charts taking values in these neighbourhoods. The definition is inductive on the dimension n, and stratified manifolds whose underlying manifolds are standard n–spheres feature prominently in the induction step:

Definition 2.1 For all $n \in \mathbb{Z}_+$ we define three related structures recursively:

- the sets $\mathcal{N}_n$ of *local neighbourhoods for n–dimensional defect bordisms*,
- the symmetric monoidal *category of n–dimensional defect bordisms* $\text{Bord}_n^{\text{def}}$,
- the *set of defect n–spheres* $\text{Sphere}_n^{\text{def}}$.

For $n = 1$ the above data is fixed as follows:

- The set $\mathcal{N}_1$ consists of three open stratified 1–manifolds: the oriented interval $(-1, 1)$, oriented from -1 to 1, and the interval $(-1, 1)$ with the same orientation and an oriented 0–stratum at 0 which is either oriented $+$ or $-$:

(2-4)
$$\mathcal{N}_1 = \{ \underline{\hspace{3cm}} , \underline{\hspace{1.3cm}}^{+}\hspace{-0.2cm}\underline{\hspace{1.3cm}} , \underline{\hspace{1.3cm}}^{-}\hspace{-0.2cm}\underline{\hspace{1.3cm}} \}.$$

- $\text{Bord}_1^{\text{def}} := \text{Bord}_1^{\text{strat}}$.

- $\text{Sphere}_1^{\text{def}} := \{S \mid S \text{ is a stratified 1–manifold with underlying manifold } S^1\}$.

Now assume that $\mathcal{N}_n$, $\text{Bord}_n^{\text{def}}$ and $\text{Sphere}_n^{\text{def}}$ are defined for a given $n \geqslant 1$.

- The set $\mathcal{N}_{n+1}$ consists of all open stratified $(n+1)$–manifolds of two types.

One type is $X \times (-1, 1)$ for $X \in \mathcal{N}_n$ with orientations induced from the orientation of X taken together with the standard orientation of $(-1, 1)$. Each j–stratum in X produces a $(j+1)$–stratum in $X \times (-1, 1)$.[3] For $p \in X$ a 0–stratum with orientation $-$, the orientation of the 1–stratum $\{p\} \times (-1, 1)$ is obtained by reversing the orientation of the interval.

The other type of element in $\mathcal{N}_{n+1}$ is an open cone

$$(2\text{-}5) \qquad\qquad C(\Sigma) = (\Sigma \times [0, 1))/(\Sigma \times \{0\}),$$

where Σ is an element in $\mathrm{Sphere}_n^{\mathrm{def}}$. We identify $C(\Sigma)$ with the open $(n+1)$–ball $B^{n+1} \subset \mathbb{R}^{n+1}$, with $0 \in \mathbb{R}^{n+1}$ as cone point. It has a natural structure of a stratified manifold with the cone point as 0–stratum and with j–strata Y on Σ inducing $(j+1)$–strata $Y \times (0, 1)$ of $C(\Sigma)$. Each $\Sigma \in \mathrm{Sphere}_n^{\mathrm{def}}$ yields two elements in $\mathcal{N}_{n+1}$, one for either orientation of the 0–stratum at the cone point of (2-5).

• The symmetric monoidal category $\mathrm{Bord}_{n+1}^{\mathrm{def}}$ has as objects closed stratified n–manifolds equipped with a compatible system of charts mapping an open neighbourhood of each point isomorphically (as oriented stratified manifolds) to an element in $\mathcal{N}_n$. The morphisms of $\mathrm{Bord}_{n+1}^{\mathrm{def}}$ are those morphisms of $\mathrm{Bord}_{n+1}^{\mathrm{strat}}$ whose representatives have open neighbourhoods around interior points that are isomorphic an element of $\mathcal{N}_{n+1}$.

• The set $\mathrm{Sphere}_{n+1}^{\mathrm{def}}$ consists of all those stratified $(n+1)$–manifolds S whose underlying manifold is the standard $(n+1)$–sphere S^{n+1}, and such that the isomorphism class of S defines a morphism $[S]\colon \varnothing \to \varnothing$ in $\mathrm{Bord}_{n+1}^{\mathrm{def}}$.

Note that as a consequence of the recursive definition we have the following equivalent ways to think about defect n–spheres:

$$(2\text{-}6) \quad \mathrm{Sphere}_n^{\mathrm{def}} = \big\{ S \mid S \text{ is a stratified } n\text{–manifold with underlying manifold } S^n$$
$$\text{whose class is a morphism } [S]\colon \varnothing \to \varnothing \text{ in } \mathrm{Bord}_n^{\mathrm{def}} \big\}$$
$$= \{ S \in \mathrm{Bord}_{n+1}^{\mathrm{def}} \mid \text{the underlying manifold of } S \text{ is } S^n \}.$$

In the definition of the bordisms M in $\mathrm{Bord}_{n+1}^{\mathrm{def}}$ above it is enough to consider open neighbourhoods only of interior points. Points in ∂M then have compatible neighbourhoods coming from the parametrisation of ∂M with respect to source or target objects.

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[3]The convention for the orientation is chosen such that the orientation on X is the induced boundary orientation of $X \times [-1, 1)$.

Remark 2.2 Why is this the correct definition of local neighbourhoods for our purposes? Ultimately we are motivated by examples and the attitude that topological defects should be combinatorial objects. Still, there might be situations where one is led to considering different types of local neighbourhoods. At least our definition is closed under taking iterated cones: for $X \in \mathcal{N}_{n-k}$ we have that the interior of $C^k X$ is isomorphic to an element of $\mathcal{N}_n$, as the cone of an i–ball is topologically an $(i+1)$–ball. The resulting n–ball can also be realised as a neighbourhood of 0 in $CX \times (-1,1)^{k-1}$.

To illustrate Definition 2.1 we now go through the iteration for $n \in \{1, 2, 3\}$:

Example 2.3 (i) The sets $\mathcal{N}_1$, $\mathrm{Sphere}_1^{\mathrm{def}}$ and the category $\mathrm{Bord}_1^{\mathrm{def}}$ are directly given in the definition. An example of a morphism from $\varnothing$ to $\varnothing$ (which is thus also an element in $\mathrm{Sphere}_1^{\mathrm{def}}$) is

(2-7)

(ii) The set $\mathcal{N}_2$ consists of open neighbourhoods of two types. The first type are cylinders over the elements (2-4) in $\mathcal{N}_1$, resulting in the three elements of $\mathcal{N}_2$:

(2-8)
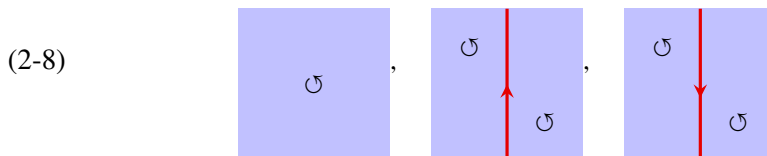
The second type of allowed neighbourhood is a cone over a circle in $\mathrm{Sphere}_1^{\mathrm{def}}$. Hence, there are infinitely many elements of $\mathcal{N}_2$ of this second type. Taking for example the circle (2-7) and the two possible orientations of the cone point, we arrive at:

(2-9)
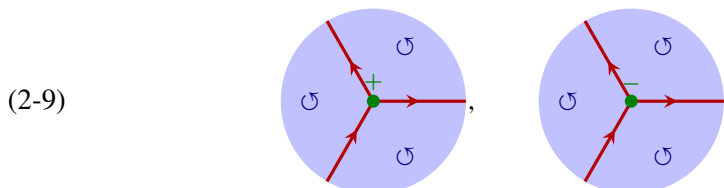
The category $\mathrm{Bord}_2^{\mathrm{def}}$ has stratified circles like (2-7) as objects. Morphisms of $\mathrm{Bord}_2^{\mathrm{def}}$ are those morphisms of $\mathrm{Bord}_2^{\mathrm{strat}}$ where each point has a local neighbourhood

in $\mathcal{N}_2$.[4] An example of an element of $\mathrm{Sphere}_2^{\mathrm{def}}$ is:

(2-10)

,

where the orientation of the 2–strata is counterclockwise on the front side of the sphere.

(iii) Next we describe the set $\mathcal{N}_3$. By taking a cylinder over the first cylinder in (2-8) we obtain the empty 3–ball (drawn as a cube)

(2-11)

.

This ball has the induced standard orientation from $\mathbb{R}^3$, as do all 3–strata of all other elements in $\mathcal{N}_3$. For the other two discs in (2-8) we get (the shading emphasises the opposite orientation)

(2-12)

,

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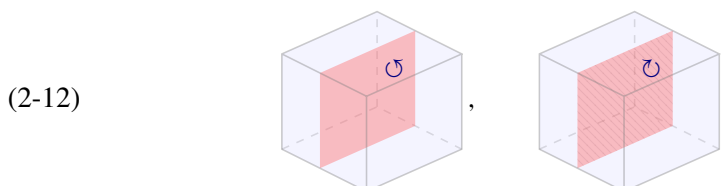
Taking cylinders over the cones in $\mathcal{N}_2$ we obtain

(2-13)

,
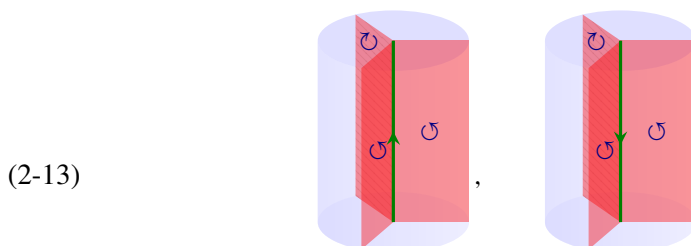
for the two discs in (2-9), respectively.

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[4]This situation illustrates our choice of the definition of stratified manifolds and their morphisms as given in Section 2.1, rather than requiring a smooth structure on the total space and smooth maps that respects the filtration: a smooth map has a differential at the 0–stratum in (2-9), and since the differential is linear, such smooth maps cannot relate local situations with arbitrary angles between the 1–strata meeting at the 0–stratum.

Finally, we have to consider cones over spheres in $\mathrm{Sphere}_2^{\mathrm{def}}$. For example, the cones over (2-10) (with the two possible orientations for the interior cone point) are the open stratified 3–balls

(2-14)
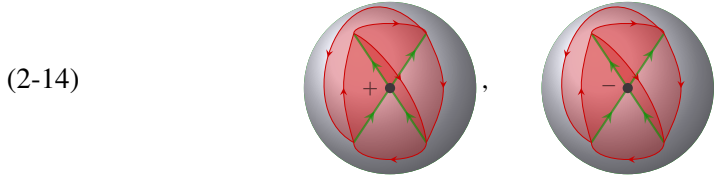
2.2.2 Decorated defect bordisms

Having specified the allowed geometric configurations for defects, we next turn to the sets of labels that defects carry. Our definition of "defect data" $\mathbb{D}$ in n dimensions and of the decorated bordism category $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ is again inductive.

Roughly, these definitions amount to the following. The defect labels for i–strata Y consist of a set D_i together with a map f_i that specifies the allowed labels for the adjacent strata. A local neighbourhood of Y has the form $B^{n-i} \times (-1, 1)^i$, and the relevant information about the adjacent strata is encoded in a decorated stratification of the sphere $S^{n-i-1} = \partial B^{n-i}$. More precisely, we must consider the stratified sphere only up to isomorphism. This leads to an equivalence class (with respect to isomorphisms of stratified manifolds) of decorated spheres, the set of which we will denote $[\mathrm{Sphere}_{n-i-1}^{\mathrm{def}}(\partial^{i+1}\mathbb{D})]$, and the map that specifies the allowed configuration for labels D_i is a map $f_i\colon D_i \to [\mathrm{Sphere}_{n-i-1}^{\mathrm{def}}(\partial^{i+1}\mathbb{D})]$.

The precise definition is as follows:

Definition 2.4 For all $n \in \mathbb{N}$ a set of n–*dimensional defect data*

(2-15) $$\mathbb{D}^n \equiv (D_n^n, D_{n-1}^n, \ldots, D_0^n; f_{n-1}^n, f_{n-2}^n, \ldots, f_0^n)$$

consists of sets D_k^n of k–*dimensional defect labels* together with the k–*dimensional adjacency maps* f_k^n out of $D_k^n \times \{\pm\}$, whose targets will be defined inductively below. Namely, for all $n \in \mathbb{N}$ we define two related structures recursively:

- $\mathcal{D}^n$, the *class of all n–dimensional defect data*, together with a *boundary map* $\partial\colon \mathcal{D}^n \to \mathcal{D}^{n-1}$ (for $n \geqslant 1$), and

- for each $\mathbb{D}^n \in \mathcal{D}^n$ and $n \geqslant 1$ the symmetric monoidal category $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^n)$ of n–*dimensional decorated defect bordisms* (while for $n = 0$ we define a set $\mathrm{Bord}_0^{\mathrm{def}}(\mathbb{D}^0)$).

We start the induction with $n = 0$:

- $\mathcal{D}^0 := \{(D_0^0) \mid D_0^0 \text{ is a nonempty set}\}$.
- For $\mathbb{D}^0 \in \mathcal{D}^0$, $\mathrm{Bord}_0^{\mathrm{def}}(\mathbb{D}^0)$ is the set whose elements are isomorphism classes of finite sets of oriented points labelled with elements in D_0^0. We take

$$(2\text{-}16) \qquad [\mathrm{Sphere}_0^{\mathrm{def}}(\mathbb{D}^0)] \subset \mathrm{Bord}_0^{\mathrm{def}}(\mathbb{D}^0)$$

to be the subset consisting of classes whose underlying point-set is an oriented 0–sphere (ie two points with opposite orientation).

Now assume that $\mathcal{D}^n$, $\partial\colon \mathcal{D}^n \to \mathcal{D}^{n-1}$ (for $n \geqslant 1$) and $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^n)$ have been defined up to and including a given $n \geqslant 0$. For $n \geqslant 1$ we set (compare to (2-6))

$$(2\text{-}17) \quad [\mathrm{Sphere}_n^{\mathrm{def}}(\mathbb{D}^n)]$$
$$:= \{[M] \in \mathrm{Hom}_{\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^n)}(\varnothing, \varnothing) \mid M \text{ has underlying manifold } S^n\}.$$

Then:

- The set $\mathcal{D}^{n+1}$ consists of sets

$$(2\text{-}18) \qquad \mathbb{D}^{n+1} \equiv (D_{n+1}^{n+1}, D_n^{n+1}, \dots, D_0^{n+1}; f_n^{n+1}, f_{n-1}^{n+1}, \dots, f_0^{n+1})$$

subject to two conditions:

  - $\partial(\mathbb{D}^{n+1}) := (D_{n+1}^{n+1}, D_n^{n+1}, \dots, D_1^{n+1}; f_n^{n+1}, f_{n-1}^{n+1}, \dots, f_1^{n+1})$ is an element of $\mathcal{D}^n$, ie omitting D_0^{n+1} and f_0^{n+1} from $\mathbb{D}^{n+1}$ gives a set of n–dimensional defect data. We thus obtain a map $\partial\colon \mathcal{D}^{n+1} \to \mathcal{D}^n$.
  - f_0^{n+1} is a map $D_0^{n+1} \times \{\pm\} \to [\mathrm{Sphere}_n^{\mathrm{def}}(\partial\mathbb{D}^{n+1})]$ such that

$$(2\text{-}19) \qquad f_0^{n+1}(\phi, -) = (f_0^{n+1}(\phi, +))^{\mathrm{rev}},$$

where for any $S \in [\mathrm{Sphere}_n^{\mathrm{def}}(\mathbb{D}^n)]$, S^{rev} denotes the bordism class with reversed orientation for all strata.

Iterating the boundary map we have that, for $0 \leqslant j \leqslant m$,

$$(2\text{-}20) \qquad \partial^j \mathbb{D}^m := (D_m^m, \dots, D_j^m; f_{m-1}^m, \dots, f_j^m) \in \mathcal{D}^{m-j},$$

and that the source and target of the adjacency map f_j^m are, for $j < m$,

$$(2\text{-}21) \qquad f_j^m\colon D_j^m \times \{\pm\} \to [\mathrm{Sphere}_{m-j-1}^{\mathrm{def}}(\partial^{j+1}\mathbb{D}^m)],$$

which satisfies the duality condition

$$(2\text{-}22) \qquad f_j^m(\phi, -) = (f_j^m(\phi, +))^{\mathrm{rev}}$$

for all $\phi \in D_j^m$.

- For $\mathbb{D}^{n+1} \in \mathcal{D}^{n+1}$, the category $\mathrm{Bord}_{n+1}^{\mathrm{def}}(\mathbb{D}^{n+1})$ is defined as follows:

  - Objects are those of $\mathrm{Bord}_{n+1}^{\mathrm{def}}$ together with a decoration by $\partial\mathbb{D}^{n+1}$ in such a way that their classes define bordisms $\varnothing \to \varnothing$ in $\mathrm{Bord}_n^{\mathrm{def}}(\partial\mathbb{D}^{n+1})$.

  - Morphisms are the morphisms Y of $\mathrm{Bord}_{n+1}^{\mathrm{def}}$ where each i–stratum Y_i^α is labelled by an element of D_i^{n+1}, subject to two requirements:

    (i) If the stratum meets the boundary of the bordism, it restricts to the decorations on the boundary.

    (ii) Let the stratum Y_i^α be labelled by $\phi \in D_i^{n+1}$ and let $\varepsilon = +$ for $i > 0$[5] and let $\varepsilon \in \{\pm\}$ be the orientation of Y_i^α for $i = 0$. Let X be a representative $(n-i)$–sphere of the class $f_i^{n+1}(\phi, \varepsilon) \in [\mathrm{Sphere}_{n-i}^{\mathrm{def}}(\partial^{i+1}\mathbb{D}^{n+1})]$. Let U be the stratified manifold given by the interior of $C(X) \times [-1, 1]^i$, equipped with the induced labelling by $\mathbb{D}^{n+1}$ (and $\{0\} \times (-1, 1)^i$ labelled by ϕ). Then each interior point of Y_i^α has a local neighbourhood isomorphic to U in a way compatible with the labelling.

Usually the dimension n of a set of defect data $\mathbb{D}^n$ will be clear from the context, in which case we will simply write $\mathbb{D}$, D_i and f_i for $\mathbb{D}^n$, D_i^n and f_i^n, respectively. Also, according to (2-22) it suffices to know the values $f_j^m(\phi, +)$ of the adjacency maps, so we will sometimes abbreviate

$$(2\text{-}23) \qquad\qquad f_j^m(\phi) := f_j^m(\phi, +).$$

The signs in (2-19) and (2-21) in particular ensure that a label $x \in D_i$ can occur both on the in- and outgoing boundary of a bordism. For $i = 1$, if a 1–stratum meets the boundary with both endpoints, the neighbourhood of one endpoint is determined by $\varepsilon = +$, of the other by $\varepsilon = -$.

Example 2.5 Let us spell out what the induction in Definition 2.4 amounts to for $n = 1$ and $n = 2$:

(i) A set of 1–dimensional defect data $\mathbb{D}$ consist of two sets D_0 and D_1. Elements in the set of spheres classes $[\mathrm{Sphere}_0^{\mathrm{def}}(\partial\mathbb{D})]$ consist of two oppositely oriented points decorated each with D_1. Thus, f_0 is a map

$$(2\text{-}24) \qquad\qquad f_0 \colon D_0 \times \{\pm\} \to D_1 \times D_1.$$

---

[5] One may equally well choose $\varepsilon = -$ for $i > 0$ as the open sets U below resulting from $\varepsilon = +$ and $\varepsilon = -$ will be related by an isomorphism which preserves the orientation of all strata.

Since reversing the orientation of the two points of the 0–sphere lies in the same class as keeping the orientation of the two points but exchanging their labels, condition (2-19) states that if $f_1(x, +) = (a, b)$ then $f_0(x, -) = (b, a)$.

It can be helpful to think of $x \in D_0$ as a "morphism" with source a and target b, that is, we define maps $s, t \colon D_0 \to D_1$ such that $f_1(x, +) = (s(x), t(x))$ (see Section 4 for more on this point of view).

$\mathrm{Bord}_1^{\mathrm{def}}(\mathbb{D})$ has as objects oriented points decorated with D_1. Morphisms are stratified 1–manifolds with 1–strata decorated with D_1, compatible with the boundary decorations, together with 0–strata p with orientation ε that are decorated with elements $\phi \in D_0$ such that for $\varepsilon = +$, $s(\phi)$ is the decoration on the incoming 1–stratum at p and $t(\phi)$ the decoration of the outgoing 1–stratum, while for $\varepsilon = -$ this order is reversed:

$$(2\text{-}25) \qquad \xrightarrow{} \overset{+}{\underset{s(\phi) \quad \phi \quad t(\phi)}{\bullet}} \xrightarrow{} \quad , \quad \xrightarrow{} \overset{-}{\underset{t(\phi) \quad \phi \quad s(\phi)}{\bullet}} \xrightarrow{}$$

(ii) A set of 2–dimensional defect data $\mathbb{D}$ consists of three sets D_0, D_1 and D_2 together with maps f_0 and f_1 as follows. We demand that $\partial\mathbb{D} = (D_2, D_1; f_1)$ forms a set of 1–dimensional defect data, ie the map

$$(2\text{-}26) \qquad f_1 \colon D_1 \times \{\pm\} \to D_2 \times D_2$$

is defined as in point (i) above.

The set $[\mathrm{Sphere}_1^{\mathrm{def}}(\partial\mathbb{D})]$ is already defined by the 1–dimensional case: it consists of (isotopy classes of) oriented stratified decorated circles where each 1–stratum is decorated with elements in D_2, and each 0–stratum with elements in D_1, such that the source and target of the decorations match with the map f_1. Possibly there are no 0–strata, in which case the circle is decorated just with an element in D_2.

By definition, the map

$$(2\text{-}27) \qquad f_0 \colon D_0 \times \{\pm\} \to [\mathrm{Sphere}_1^{\mathrm{def}}(\partial\mathbb{D})]$$

is subject to the duality condition (2-19). The elements of $[\mathrm{Sphere}_1^{\mathrm{def}}(\partial\mathbb{D})]$ are spheres considered up to isomorphisms of stratified manifolds, so in particular up to rotations. Describing the stratified circles combinatorially and restricting to positively oriented points, we recover the junction map j of [19, Equation (2.2)]: $f_0(x, +) = j(x)$, where

$$(2\text{-}28) \qquad j \colon D_0 \to D_2 \sqcup \bigsqcup_{m \in \mathbb{Z}_+} ((D_1 \times \{\pm\}) \times_{D_2} \cdots \times_{D_2} (D_1 \times \{\pm\}))/C_m$$

determines the neighbourhood of a D_0–decorated 0–stratum (with the first factor D_2 corresponding to the case of no 1–strata in the neighbourhood). The condition in the product on the right is that source and target of adjacent D_1–terms must agree, including the first and last term. C_m denotes the cyclic group with m elements, which acts naturally on the product of m elements.

We do not describe the case $n = 3$ in detail, since a combinatorial description of defect 2–spheres seems impracticable. Note, however, that the map (2-27) also appears as the folding map f_1^3 for defect 1–strata in 3–dimensional defect bordisms [11, Definition 2.6], just as (2-26) is a reincarnation of (2-24).

For any i–dimensional defect label $x \in D_i$ in $\mathbb{D}$, we can and do choose a representative $(n-i-1)$–sphere S_x such that the adjacency map f_i sends x to the class of S_x, that is, $f_i(x) = [S_x]$. From S_x we produce a decorated stratified n–ball by first taking the cone and then iterated cylinders. More precisely, we define the n–dimensional open x–*defect ball* as

$$(2\text{-}29) \qquad\qquad B_x := C(S_x) \times (-1, 1)^i \cap B^n.$$

For later use we note that its closure is

$$(2\text{-}30) \qquad\qquad \overline{B}_x = \overline{C(S_x) \times (-1, 1)^i} \cap \overline{B}^n,$$

and we denote its boundary by

$$(2\text{-}31) \qquad\qquad \Sigma_x = \partial \overline{B}_x.$$

The open ball B_x describes what the surroundings of an x–labelled i–stratum look like in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$, and it will be important to us in Sections 2.4 and 2.5.

2.2.3 Symmetric monoidal category of defect data
We now turn the sets of defect data $\mathcal{D}^n$ into symmetric monoidal categories. We start by adding maps of defect data as morphisms:

Definition 2.6 A *morphism of n–dimensional defect data* $h = (h_n, \ldots, h_0)\colon \mathbb{D} \to \mathbb{D}'$ is a collection of maps $h_i\colon D_i \to D_i'$ such that for all $i \in \{0, \ldots, n\}$ the squares

$$(2\text{-}32) \qquad
\begin{array}{ccc}
D_i & \xrightarrow{\ f_i\ } & [\mathrm{Sphere}_{n-i-1}^{\mathrm{def}}(\partial^{i+1}\mathbb{D})] \\[4pt]
{\scriptstyle h_i}\big\downarrow & & \big\downarrow{\scriptstyle h_*} \\[4pt]
D_i' & \xrightarrow[\ f_i'\ ]{} & [\mathrm{Sphere}_{n-i-1}^{\mathrm{def}}(\partial^{i+1}\mathbb{D}')]
\end{array}$$

commute, where $h_*\colon [\mathrm{Sphere}^{\mathrm{def}}_{n-i-1}(\partial^{i+1}\mathbb{D})] \to [\mathrm{Sphere}^{\mathrm{def}}_{n-i-1}(\partial^{i+1}\mathbb{D}')]$ is the map of decorated spheres induced by the maps h_i.

Together with the above notion of morphism, the obvious identity morphisms and composition, the sets of defect data $\mathcal{D}^n$ form a category for all $n \in \mathbb{Z}_+$.

The monoidal structure on $\mathcal{D}^n$ is slightly less straightforward, as illustrated by the following example. Let $\mathbb{D}, \mathbb{D}' \in \mathcal{D}^n$ and pick defect labels $a \in D_{n-2}$ and $a' \in D'_{n-2}$ of codimension-two defects. Suppose that $f^{\mathbb{D}}_{n-2}(a)$ is a 1–sphere with two 0–strata labelled by $x, y \in D_{n-1}$ for $x \neq y$. Similarly, $f^{\mathbb{D}'}_{n-2}(a')$ is a 1–sphere with two 0–strata labelled by $x', y' \in D'_{n-1}$. How many defect conditions should a and a' give rise to in the to-be-constructed tensor product $\mathbb{D} \otimes \mathbb{D}'$? One might guess that there should be one defect condition labelled by the pair (a, a'). However, there are now two distinct choices for what the neighbourhood of (a, a') should look like. One is the 1–sphere with 0–strata labelled (x, x') and (y, y'), and the other is the 1–sphere with 0–strata labelled (x, y') and (y, x'). Thus, in the tensor product $\mathbb{D} \otimes \mathbb{D}'$, we need two distinct defect labels corresponding to the pair (a, a'), which have to be distinguished by the adjacency maps $f^{\mathbb{D} \otimes \mathbb{D}'}_i$.

This example shows that the tensor product of defect data is more complicated than just the Cartesian product of the sets of defect labels. Instead, one has to take into account automorphisms of the defect sphere describing the surrounding arrangement of higher-dimensional defect strata. We now describe this construction in detail.

For $\mathbb{D}, \mathbb{D}' \in \mathcal{D}^n$ we define their tensor product $\mathbb{D} \otimes \mathbb{D}' \in \mathcal{D}^n$ to have component sets of the form

$$(2\text{-}33) \qquad (\mathbb{D} \otimes \mathbb{D}')_i = \{[x, x', S \xrightarrow{\varphi} U \xleftarrow{\varphi'} S']\}$$

for all $i \in \{0, 1, \ldots, n\}$. Here $x \in D_i$ and $x' \in D'_i$ are defect labels, $S \in f^{\mathbb{D}}_i(x)$ and $S' \in f^{\mathbb{D}'}_i(x')$ are decorated defect spheres, $U \in \mathrm{Sphere}^{\mathrm{def}}_{n-i-1}$ is an undecorated defect sphere, and $\varphi\colon S \to U$ and $\varphi'\colon S' \to U$ are isomorphisms of undecorated defect spheres. By definition, two such tuples

$$(2\text{-}34) \qquad (x, x', S \xrightarrow{\varphi} U \xleftarrow{\varphi'} S'), \quad (x, x', \widetilde{S} \xrightarrow{\widetilde{\varphi}} \widetilde{U} \xleftarrow{\widetilde{\varphi}'} \widetilde{S}')$$

represent the same equivalence class $[x, x', S \xrightarrow{\varphi} U \xleftarrow{\varphi'} S']$ if there are isomorphisms of decorated defect spheres $\sigma\colon S \to \widetilde{S}$ and $\sigma'\colon S' \to \widetilde{S}'$ and an isomorphism of

undecorated defect spheres $v\colon U \to \widetilde{U}$ such that the diagram

(2-35)

commutes. Furthermore, we define the adjacency maps $f_i^{\mathbb{D}\otimes\mathbb{D}'}$ by setting

$$(2\text{-}36)\quad f_i^{\mathbb{D}\otimes\mathbb{D}'}([x, x', S \xrightarrow{\varphi} U \xleftarrow{\varphi'} S']) := [U_{\varphi,\varphi'}] \in \mathrm{Sphere}_{n-i-1}^{\mathrm{def}}(\partial^{i+1}(\mathbb{D} \otimes \mathbb{D}')),$$

where $U_{\varphi,\varphi'}$ has underlying stratified manifold U, and the decoration of the strata U_j^α is obtained from the pair φ and φ'.

We illustrate the above definition by spelling it out for the motivating example above: Let $U \subset \mathbb{R}^2$ be the standard circle with two 0–strata at $(0, +1)$ and $(0, -1)$. For the defect circle S with $f_{n-2}^{\mathbb{D}}(a) = [S]$, we take U with the labels $x, y \in D_{n-1}$ at $(0, +1)$ and $(0, -1)$, respectively, while S' with $f_{n-2}^{\mathbb{D}'}(a') = [S']$ is U with $(0, +1)$, $(0, -1)$ decorated by $x', y' \in D'_{n-1}$. Then, in $\mathbb{D} \otimes \mathbb{D}'$ we have two inequivalent elements $(a, a', S \xrightarrow{\mathrm{id}} U \xleftarrow{\mathrm{id}} S')$ and $(a, a', S \xrightarrow{\mathrm{id}} U \xleftarrow{\pi} S')$, where π is rotation by π.

Given two morphisms $h\colon \mathbb{D} \to \mathbb{E}$ and $g\colon \mathbb{D}' \to \mathbb{E}'$ in $\mathcal{D}^n$, we define their tensor product $h \otimes g\colon \mathbb{D} \otimes \mathbb{D}' \to \mathbb{E} \otimes \mathbb{E}'$ by setting, for all $i \in \{0, 1, \ldots, n\}$,

$$(2\text{-}37)\quad (h\otimes g)_i([x, x', S \xrightarrow{\varphi} U \xleftarrow{\varphi'} S']) := [h_i(x), g_i(x'), h_*(S) \xrightarrow{\varphi} U \xleftarrow{\varphi'} g_*(S')].$$

Here, the induced maps h_* and g_* are as in Definition 2.6, and it follows that $(h \otimes g)_* \circ f_i^{\mathbb{D}\otimes\mathbb{D}'} = f_i^{\mathbb{E}\otimes\mathbb{E}'} \circ (h \otimes g)_i$, so that $h \otimes g$ really is a morphism in $\mathcal{D}^n$.

The monoidal unit in $\mathcal{D}^n$ is $\mathbb{D}^{\mathrm{id}}$, whose sets

$$(2\text{-}38)\qquad\qquad\qquad D_i^{\mathrm{id}} := [\mathrm{Sphere}_{n-i-1}^{\mathrm{def}}]$$

consist of all defect spheres for all $i \in \{0, 1, \ldots, n\}$, and whose adjacency maps are identities, $f_i^{\mathbb{D}^{\mathrm{id}}}([S]) = [S]$.

Finally, we equip $\mathcal{D}^n$ with the symmetric braiding given by the exchange of factors.

Lemma 2.7 $\mathcal{D}^n$ *is a symmetric monoidal category.*

Proof We show that for the tensor product given in (2-33), (2-37) and (2-38) there are natural unitors $\mathbb{D} \otimes \mathbb{D}^{\mathrm{id}} \cong \mathbb{D} \cong \mathbb{D}^{\mathrm{id}} \otimes \mathbb{D}$; the associator is straightforward to determine, and the pentagon and hexagon are then clear.

For any $\mathbb{D} \in \mathcal{D}^n$, the isomorphism $\rho_{\mathbb{D}} \colon \mathbb{D} \otimes \mathbb{D}^{\mathrm{id}} \to \mathbb{D}$ has components which project $[x, [\Sigma], S \xrightarrow{\varphi} U \xleftarrow{\varphi'} \Sigma] \in (\mathbb{D} \otimes \mathbb{D}^{\mathrm{id}})_i$ to $x \in D_i$. To give the action of $\rho_{\mathbb{D}}^{-1}$, we choose a representative $\widetilde{S} \in f_i(x)$, and we write $\widetilde{S}_{\circ} \in \mathrm{Sphere}^{\mathrm{def}}_{n-i-1}$ for $\widetilde{S}$ with its $\mathbb{D}$–decoration discarded. Then

$$(2\text{-}39) \qquad (\rho_{\mathbb{D}}^{-1})_i(x) := [x, [\widetilde{S}_{\circ}], \widetilde{S} \xrightarrow{\mathrm{id}} \widetilde{S}_{\circ} \xleftarrow{\mathrm{id}} \widetilde{S}_{\circ}].$$

Note that the right-hand side is independent of the choice of representative $\widetilde{S}$.

Clearly we have $\rho_{\mathbb{D}} \circ \rho_{\mathbb{D}}^{-1} = 1_{\mathbb{D}}$. To see that also $\rho_{\mathbb{D}}^{-1} \circ \rho_{\mathbb{D}}$ equals $1_{\mathbb{D} \otimes \mathbb{D}^{\mathrm{id}}}$, simply act with the former on $(x, [\Sigma], S \xrightarrow{\varphi} U \xleftarrow{\varphi'} \Sigma)$ and note that the following diagram commutes:

$$(2\text{-}40)$$

This concludes the proof. □

2.3 *n*–dimensional defect TQFTs

We will now define defect TQFTs in arbitrary dimension, describe maps between them, and discuss their tensor products. Unless specified otherwise, in this section n is any positive integer and $\mathbb{D}$ is a set of n–dimensional defect data.

2.3.1 Defect TQFTs and their morphisms

Definition 2.8 An *n–dimensional defect TQFT* with defect data $\mathbb{D}$ is a symmetric monoidal functor

$$(2\text{-}41) \qquad \mathcal{Z} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}.$$

The most basic example of a defect TQFT for fixed defect data $\mathbb{D}$ is the *identity TQFT*

$$(2\text{-}42) \qquad\qquad \mathbf{1}_{\mathbb{D}} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk},$$

which by definition maps all objects to $\Bbbk$, and all morphisms to $\mathrm{id}_{\Bbbk}$. Closed n–dimensional TQFTs, ie symmetric monoidal functors $\mathrm{Bord}_n \to \mathrm{Vect}_{\Bbbk}$, provide another class examples by viewing them as functors on $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^*)$, where the set of defect data $\mathbb{D}^*$ consists of the singleton set $D_n^* = \{*\}$ and $D_j^* = \varnothing$ for all $j \in \{0, 1, \ldots, n-1\}$.

Another class of examples of defect TQFTs comes about by "compactifying" higher-dimensional defect TQFTs. Indeed, thanks to the prominent role of the boundary map ∂ in Definition 2.4 of defect data, the following observation is immediate:

Lemma 2.9 *Let $\mathcal{Z}$ be an n–dimensional defect TQFT with defect data $\mathbb{D}$, and let M a closed k–manifold with $k < n$. Then the **compactification of $\mathcal{Z}$ along M** is an $(n-k)$–dimensional defect TQFT $\mathcal{Z}_M \colon \mathrm{Bord}_{n-k}^{\mathrm{def}}(\partial^k \mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ with*

$$(2\text{-}43) \qquad\qquad \mathcal{Z}_M(N) = \mathcal{Z}(N \times M)$$

on both objects and morphisms N in $\mathrm{Bord}_{n-k}^{\mathrm{def}}(\partial^k \mathbb{D})$; here the decorated stratification of N naturally induces the decorated stratification of $N \times M$.

Defect TQFTs in dimension n form the objects of a symmetric monoidal category $\mathrm{TQFT}_n^{\mathrm{def}}$, to which we turn next. We start by defining the morphisms between two defect TQFTs $\mathcal{Z} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ and $\mathcal{Z}' \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}') \to \mathrm{Vect}_{\Bbbk}$. If $\mathbb{D} = \mathbb{D}'$, one can in particular consider monoidal natural transformations $\mathcal{Z} \Rightarrow \mathcal{Z}'$. In general, one has to take into account maps of defect data $h \colon \mathbb{D} \to \mathbb{D}'$ as in Definition 2.6, which induce symmetric monoidal functors

$$(2\text{-}44) \qquad\qquad h_* \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}')$$

by applying the component maps h_i to the decoration of the strata of the objects and morphisms in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$. Hence, we define a *morphism* from $\mathcal{Z}$ to $\mathcal{Z}'$ to be a pair (h, φ) where $h \colon \mathbb{D} \to \mathbb{D}'$ is a map of defect data and $\varphi \colon \mathcal{Z} \Rightarrow \mathcal{Z}' \circ h_*$ is a monoidal natural transformation,

$$(2\text{-}45)$$

In the same way as for closed TQFTs without defects, one proves that φ in the above definition is necessarily invertible (see eg [14]); note however that this does not imply $\mathrm{TQFT}_n^{\mathrm{def}}$ is a groupoid as h_* need not have an inverse.

The composition of morphisms $(h,\varphi)\colon \mathcal{Z} \to \mathcal{Z}'$ and $(g,\psi)\colon \mathcal{Z}' \to \mathcal{Z}''$ is defined by composing h_* with g_* and pasting φ and ψ together:

(2-46)

$$
\begin{array}{ccc}
\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) & & \\
\Big\downarrow{\scriptstyle h_*} \quad {\scriptstyle\varphi}\!\!\diagup\!\!\diagup \quad {}^{\mathcal{Z}} & & \\
\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}') & \xrightarrow{\ \mathcal{Z}'\ } & \mathrm{Vect}_{\Bbbk} \\
\Big\downarrow{\scriptstyle g_*} \quad \diagup\!\!\diagup\,{\scriptstyle\psi} & & \Big\| \\
\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}'') & \xrightarrow[\ \mathcal{Z}''\ ]{} & \mathrm{Vect}_{\Bbbk}
\end{array}
$$

The identity morphism on $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ is $(\mathrm{id}_{\mathbb{D}}, \mathrm{id}_{\mathcal{Z}})$. Altogether, we have a category $\mathrm{TQFT}_n^{\mathrm{def}}$.

2.3.2 Symmetric monoidal category of defect TQFTs To describe the monoidal structure on $\mathrm{TQFT}_n^{\mathrm{def}}$, we have to produce symmetric monoidal functors on the category $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D} \otimes \mathbb{D}')$ from defect TQFTs $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ and $\mathcal{Z}'\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}') \to \mathrm{Vect}_{\Bbbk}$, where $\mathbb{D} \otimes \mathbb{D}'$ is the tensor product in $\mathcal{D}^n$ defined in Section 2.2. For an object or morphism M in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D} \otimes \mathbb{D}')$, we write $p_1(M)$ for the object or morphism in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ which is M but with the decorations from $\mathbb{D}'$ forgotten. Similarly, we write $p_2(M)$ for the object or morphism in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}')$ which is obtained by discarding the $\mathbb{D}$–decoration. This gives us a symmetric monoidal functor

(2-47) $\quad P\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D} \otimes \mathbb{D}') \to \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \times \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}'), \quad M \mapsto (p_1(M), p_2(M)).$

With this notation we have a natural notion of tensor product of defect TQFTs:

Definition 2.10 Let $\mathcal{Z}$ and $\mathcal{Z}'$ be two n–dimensional defect TQFTs with defect data $\mathbb{D}$ and $\mathbb{D}'$, respectively. Their *tensor product* is the defect TQFT

(2-48) $\quad \mathcal{Z} \otimes \mathcal{Z}'\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D} \otimes \mathbb{D}') \xrightarrow{\ P\ } \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \times \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}')$

$$\xrightarrow{\ \mathcal{Z} \times \mathcal{Z}'\ } \mathrm{Vect}_{\Bbbk} \times \mathrm{Vect}_{\Bbbk} \xrightarrow{\ \otimes_{\Bbbk}\ } \mathrm{Vect}_{\Bbbk}.$$

It follows that we have

(2-49) $$\qquad\qquad (\mathcal{Z} \otimes \mathcal{Z}')(M) = \mathcal{Z}(p_1(M)) \otimes_{\Bbbk} \mathcal{Z}'(p_2(M))$$

for objects and morphisms M in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D} \otimes \mathbb{D}')$. The unit for the tensor product is the identity defect TQFT

$$(2\text{-}50) \qquad\qquad\qquad \mathbf{1} := \mathbf{1}_{\mathbb{D}^{\mathrm{id}}},$$

which is the special case of (2-42) where $\mathbb{D} = \mathbb{D}^{\mathrm{id}}$ is the monoidal unit of $\mathcal{D}^n$ with $D_i^{\mathrm{id}} = [\mathrm{Sphere}_{n-i-1}^{\mathrm{def}}]$; see (2-38). This means that for each local neighbourhood there exists precisely one defect label in $\mathbb{D}^{\mathrm{id}}$, that is, the forgetful functor

$$(2\text{-}51) \qquad\qquad\qquad \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\mathrm{id}}) \to \mathrm{Bord}_n^{\mathrm{def}}$$

is an equivalence.

To describe the tensor product also on morphisms in $\mathrm{TQFT}_n^{\mathrm{def}}$, let us consider defect TQFTs

$$(2\text{-}52) \qquad \begin{aligned} \mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}, &\quad \mathcal{Z}'\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}') \to \mathrm{Vect}_{\Bbbk}, \\ \mathcal{Y}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{E}) \to \mathrm{Vect}_{\Bbbk}, &\quad \mathcal{Y}'\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{E}') \to \mathrm{Vect}_{\Bbbk}. \end{aligned}$$

For two morphisms $(g, \psi)\colon \mathcal{Y} \to \mathcal{Y}'$ and $(h, \varphi)\colon \mathcal{Z} \to \mathcal{Z}'$, their tensor product

$$(2\text{-}53) \qquad\qquad (g, \psi) \otimes (h, \varphi)\colon \mathcal{Y} \otimes \mathcal{Z} \to \mathcal{Y}' \otimes \mathcal{Z}'$$

is defined to be the commutative diagram

$$(2\text{-}54)$$

It is straightforward to verify that the identity defect TQFT $\mathbf{1}$ is the monoidal unit, and again unitors, associator and symmetric braiding are given by the obvious choice. To summarise:

Proposition 2.11 *The category $\mathrm{TQFT}_n^{\mathrm{def}}$ of n–dimensional defect TQFTs is symmetric monoidal.*

There is an equivalence relation on the set of objects in $\mathrm{TQFT}_n^{\mathrm{def}}$ which will be relevant for us in Sections 2.4 and 2.5:

Definition 2.12 Two TQFTs $\mathcal{Z} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ and $\mathcal{Z}' \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}') \to \mathrm{Vect}_{\Bbbk}$ are *equivalent*, $\mathcal{Z} \sim \mathcal{Z}'$, if there are morphisms $(h, \varphi) \colon \mathcal{Z} \rightleftarrows \mathcal{Z}' \colon (g, \psi)$ between them.

This equivalence relation is compatible with the monoidal structure on $\mathrm{TQFT}_n^{\mathrm{def}}$ in the sense that if $\mathcal{Z} \sim \mathcal{Z}'$ and $\mathcal{Y} \sim \mathcal{Y}'$, then also $\mathcal{Y} \otimes \mathcal{Z} \sim \mathcal{Y}' \otimes \mathcal{Z}'$. The morphisms witnessing the latter equivalence are constructed from those of the former equivalences and the tensor product defined in (2-54).

Remark 2.13 (i) If $\mathcal{Z}$ and $\mathcal{Z}'$ are equivalent as in Definition 2.12 then there are isomorphisms of symmetric monoidal functors $\mathcal{Z} \cong \mathcal{Z}' \circ h_*$ and $\mathcal{Z}' \cong \mathcal{Z} \circ g_*$ and thus also $\mathcal{Z} \cong \mathcal{Z} \circ g_* \circ h_*$ and $\mathcal{Z}' \cong \mathcal{Z}' \circ h_* \circ g_*$. In this sense equivalent TQFTs determine one another and thus deserve to be called equivalent.

 (ii) If $\mathcal{Z}$ and $\mathcal{Z}'$ are isomorphic in $\mathrm{TQFT}_n^{\mathrm{def}}$ then they are clearly also equivalent in the sense of Definition 2.12. The converse is not true: a morphism $(h, \phi) \colon \mathcal{Z} \to \mathcal{Z}'$ is an isomorphism if and only if h is invertible and in Sections 2.4 and 2.5 we will encounter examples of equivalent defect TQFTs with nonisomorphic sets of defect labels.

(iii) A natural mechanism to produce additional equivalences in a 1–category is to add a layer of 2–morphisms, thereby turning it into a 2–category, and then passing to its homotopy 1–category. In $\mathrm{TQFT}_n^{\mathrm{def}}$, for example, one could add a single 2–(iso)morphism between any two 1–morphisms. In the homotopy category, all 1–morphisms then lie in one equivalence class, and this produces Definition 2.12. Finding a more natural 2–categorical structure on defect TQFTs is a problem for future research.

An *invertible n–dimensional defect TQFT* is an invertible object $\mathcal{Z}$ in the monoidal category $\mathrm{TQFT}_n^{\mathrm{def}}$. This means that $\mathcal{Z}$ is invertible if and only if there exists an n–dimensional defect TQFT $\mathcal{Z}'$ and an isomorphism of defect TQFTs $\mathcal{Z} \otimes \mathcal{Z}' \cong \mathbf{1}$. A particular class of invertible defect TQFTs in any dimension n are "Euler defect TQFTs":

Example 2.14 Let us for the moment restrict the discussion to the field $\Bbbk = \mathbb{C}$ and later return to general fields. Recall that for a triangulated topological manifold M its

Euler characteristic $\chi(M)$ can be computed as the alternating sum of the number of k–simplices over all dimensions $k \leqslant \dim M$. From this characterisation it is immediate that if $M \circ_\Sigma N$ is the result of gluing two triangulated manifolds M and N along a common boundary Σ, then

$$(2\text{-}55) \qquad\qquad \chi(M \circ_\Sigma N) = \chi(M) + \chi(N) - \chi(\Sigma).$$

Following [40], for any $n \in \mathbb{Z}_+$ and $\kappa, \lambda \in \mathbb{C}$, one thus obtains a closed TQFT $\mathcal{Z}_{(\kappa,\lambda)} \colon \mathrm{Bord}_n \to \mathrm{Vect}_\mathbb{C}$ which maps all objects to $\mathbb{C}$, and on a bordism M with boundary $(\partial_{\mathrm{in}} M)^{\mathrm{rev}} \cup \partial_{\mathrm{out}} M$ we set

$$(2\text{-}56) \qquad \mathcal{Z}_{(\kappa,\lambda)}(M) = \exp\big(\kappa(\chi(M) - \lambda \cdot \chi(\partial_{\mathrm{in}} M) - (1 - \lambda) \cdot \chi(\partial_{\mathrm{out}} M))\big).$$

These exponents ensure that $\mathcal{Z}_{(\kappa,\lambda)}(M \circ_\Sigma N) = \mathcal{Z}_{(\kappa,\lambda)}(M) \cdot \mathcal{Z}_{(\kappa,\lambda)}(N)$, and one can verify that $\mathcal{Z}_{(\kappa,\lambda)} \cong \mathcal{Z}_{(\kappa,\lambda')}$ for all $\lambda, \lambda' \in \mathbb{C}$. Clearly, $\mathcal{Z}_{(\kappa,\lambda)}$ is an invertible closed TQFT with inverse $\mathcal{Z}_{(-\kappa,\lambda)}$.

For later convenience we prefer the symmetric choice $\lambda = \frac{1}{2}$, as this attaches the same weight to in- and outgoing boundaries and reduces the risk of confusion. To shorten notation and avoid lots of factors of two, we will use a rescaled version of the Euler character:

$$(2\text{-}57) \qquad\qquad \chi_{\mathrm{sym}}(M) := 2\chi(M) - \chi(\partial M),$$

so that $\mathcal{Z}_{(\kappa,1/2)}(M) = e^{\kappa \cdot \chi_{\mathrm{sym}}(M)/2}$. We will also return to the case of general fields $\mathbb{k}$ by replacing e^κ in the above expression by Ψ^2 for some $\Psi \in \mathbb{k}^\times$. Altogether, we define the closed *Euler theory* to be the TQFT

$$(2\text{-}58) \qquad\qquad \mathcal{Z}_\Psi^{\mathrm{eu}} \colon \mathrm{Bord}_n \to \mathrm{Vect}_\mathbb{k}, \quad \mathcal{Z}_\Psi^{\mathrm{eu}}(M) = \Psi^{\chi_{\mathrm{sym}}(M)}.$$

We can lift this example to an invertible n–dimensional defect TQFT

$$(2\text{-}59) \qquad\qquad \mathcal{Z}_\Psi^{\mathrm{eu}} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\mathrm{id}}) \to \mathrm{Vect}_\mathbb{k}$$

for any tuple $\Psi = (\psi_1, \ldots, \psi_n) \in (\mathbb{k}^\times)^n$ as follows. The defect data $\mathbb{D}^{\mathrm{id}}$ for $\mathcal{Z}_\Psi^{\mathrm{eu}}$ is the one for the monoidal unit $\mathbf{1}$ as in (2-38) (and hence by (2-51) we can also think of $\mathcal{Z}_\Psi^{\mathrm{eu}}$ as an unlabelled defect TQFT, ie a symmetric monoidal functor $\mathrm{Bord}_n^{\mathrm{def}} \to \mathrm{Vect}_\mathbb{k}$).

For all objects Σ we set $\mathcal{Z}_\Psi^{\mathrm{eu}}(\Sigma) = \mathbb{k}$. For a morphism M in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\mathrm{eu}})$ as before we write $M_j^{\alpha_j}$ for its j–strata, and we recall that $\partial M_j^{\alpha_j} = \partial M \cap M_j^{\alpha_j}$. Then we define the *Euler defect TQFT* on M as the natural generalisation of (2-58), by assigning a

weight $\psi_j^{\chi_{\mathrm{sym}}(M_j^{\alpha_j})}$ *to every stratum* $M_j^{\alpha_j}$ *of dimension* $j \geqslant 1$,

$$(2\text{-}60) \qquad \mathcal{Z}_{\Psi}^{\mathrm{eu}}(M) = \prod_{j=1}^{n} \prod_{\alpha_j} \psi_j^{\chi_{\mathrm{sym}}(M_j^{\alpha_j})}.$$

2.4 Point defects from states on spheres

Suppose we are given a defect TQFT $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ with some set D_0 of labels for point defects. If one "regularises" a point defect by replacing it by a small sphere around that point, one can interpret certain states in the vector space associated to that sphere as being located on the point. To allow for such an interpretation, the states need to satisfy an invariance condition given below.

In this section we define when a defect TQFT has a "complete set of point defects" (which we will call D_0–complete) and we will show that every defect TQFT factors through a D_0–complete one. For a D_0–complete theory we describe a multiplication of point defects in which one replaces two neighbouring point defects by a "fused" point defect.

We first need to define the notion of an "invariant state" in the state space of a defect sphere. Let $\Sigma \in \mathrm{Sphere}_{n-1}^{\mathrm{def}}(\partial\mathbb{D})$ be given. Write $\bar{B} := C(\Sigma) \subset \mathbb{R}^n$ for the closed unit ball given by the cone over Σ, as a decorated stratified manifold but without a label for its central 0–stratum. Note that $\partial\bar{B} = \Sigma$. Similarly, for $\varepsilon > 0$ write $B_{1+\varepsilon}$ for the open ball of radius $1 + \varepsilon$ which is given by the interior of the (now slightly larger) cone over Σ. By construction, $\bar{B} \subset B_{1+\varepsilon}$ as decorated stratified manifolds.

Define the set of embeddings

$$(2\text{-}61) \qquad\qquad\qquad \mathrm{Emb}(\bar{B})$$

of $\bar{B}$ into itself as follows: an element of $\mathrm{Emb}(\bar{B})$ is a germ (in $\varepsilon > 0$) of maps $f\colon B_{1+\varepsilon} \to \bar{B}$ of decorated stratified manifolds which are isomorphisms onto their images; as for boundary parametrisations, we will write f both for the germ and for a representative map. Composition of maps is independent of representatives and turns $\mathrm{Emb}(\bar{B})$ into a nonunital semigroup.

Let B be the interior of $\bar{B}$. Given an element $f \in \mathrm{Emb}(\bar{B})$, define a bordism $H_f\colon \Sigma \to \Sigma$ in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ as

$$(2\text{-}62) \qquad\qquad\qquad H_f := \bar{B} \setminus f(B),$$

where the outer boundary is the outgoing boundary and is parametrised by Σ via the identity map (with its canonical germ on $\Sigma \times (1-\varepsilon, 1]$, where the second coordinate is the radius), while the inner boundary is the ingoing boundary and is parametrised by Σ via the germ f restricted to $\Sigma \times [1, 1+\varepsilon)$. For $f, g \in \mathrm{Emb}(\bar{B})$, composition of bordisms is compatible with the semigroup structure on $\mathrm{Emb}(\bar{B})$:

$$(2\text{-}63) \qquad\qquad\qquad\qquad H_f \circ H_g = H_{f \circ g}$$

in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$. The *subspace of invariant states in* $\mathcal{Z}(\Sigma)$ is defined to be

$$(2\text{-}64) \qquad \mathcal{Y}_\Sigma := \{\psi \in \mathcal{Z}(\Sigma) \mid \mathcal{Z}(H_f)(\psi) = \psi \text{ for all } f \in \mathrm{Emb}(\bar{B})\}.$$

One reason to introduce $\mathcal{Y}_\Sigma$ is that point defects give rise to such invariant states, as proved in the next lemma. Conversely, if one tries to describe point defects by cutting out small balls and assigning states to the resulting boundary spheres, one has to make sure that the result is independent of the chosen boundary parametrisation. We will see in the proof of Proposition 2.17 how this is ensured by the above invariance condition.

Lemma 2.15 Let $x \in D_0$ be such that $f_0(x) = [\Sigma]$ and let $\bar{B}_x$ be the closed cone $C(\Sigma)$ as above, but with central point labelled x. Then $\bar{B}_x$ is a bordism $\varnothing \to \Sigma$ and we have[6]

$$(2\text{-}65) \qquad\qquad\qquad\qquad \mathcal{Z}(\bar{B}_x) \in \mathcal{Y}_\Sigma.$$

Proof We have $\mathcal{Z}(H_f) \circ \mathcal{Z}(\bar{B}_x) = \mathcal{Z}(H_f \circ \bar{B}_x)$. It is therefore enough to show that $H_f \circ \bar{B}_x$ is isomorphic to $\bar{B}_x$ as a decorated stratified manifold with parametrised boundary. But this is clear from the construction: the isomorphism $\varphi\colon H_f \circ \bar{B}_x \to \bar{B}_x$ is given by $\varphi|_{H_f} = \mathrm{id}\colon H_f \to H_f \subset \bar{B}_x$ and $\varphi|_{B_x} = f\colon B_x \to f(B_x) \subset \bar{B}_x$. $\square$

After these preparations we can introduce the notion of D_0–completeness. For $\Sigma \in \mathrm{Sphere}_{n-1}^{\mathrm{def}}(\partial\mathbb{D})$ define the subset

$$(2\text{-}66) \qquad\qquad\qquad D_0(\Sigma) := \{x \in D_0 \mid f_0(x) = [\Sigma]\},$$

that is, $D_0(\Sigma)$ contains all point defect labels that can be assigned to the 0–stratum in the cone $C(\Sigma)$. By Lemma 2.15 we obtain a map

$$(2\text{-}67) \qquad\qquad\qquad Y_\Sigma\colon D_0(\Sigma) \to \mathcal{Y}_\Sigma, \quad x \mapsto \mathcal{Z}(\bar{B}_x),$$

---

[6]Since $\bar{B}_x$ is a morphism from $\varnothing$ to Σ, $\mathcal{Z}(\bar{B}_x)$ is a linear map from $\Bbbk \to \mathcal{Z}(\Sigma)$. We will identify linear maps $\Bbbk \to \mathcal{Z}(\Sigma)$ with $\mathcal{Z}(\Sigma)$ by evaluating on $1 \in \Bbbk$. This is to avoid writing lots of "(1)", eg (2-65) would read $\mathcal{Z}(\bar{B}_x)(1) \in \mathcal{Y}_\Sigma$.

which assigns to a point defect label the corresponding state on the surrounding sphere. This map can be thought of as a variant of the state–field correspondence one has in conformal field theories.

Definition 2.16 A defect TQFT $\mathcal{Z} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_\Bbbk$ is called D_0–*complete* if for all $\Sigma \in \mathrm{Sphere}_{n-1}^{\mathrm{def}}(\partial\mathbb{D})$, the map (2-67) is a bijection of sets.

For a D_0–complete TQFT, the spaces $D_0(\Sigma)$ inherit the structure of a $\Bbbk$–vector space via the bijection (2-67), and we will use this vector space structure for $D_0(\Sigma)$ below.

The next proposition shows that working with D_0–complete theories is not a restriction, as every defect TQFT factors through such a theory.

Proposition 2.17 *For a given defect TQFT* $\mathcal{Z} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_\Bbbk$ *there exist defect data* $\mathbb{D}^\bullet$, *a map of defect data* $h \colon \mathbb{D} \to \mathbb{D}^\bullet$ *and a defect TQFT* $\mathcal{Z}^\bullet \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^\bullet) \to \mathrm{Vect}_\Bbbk$ *such that* $\mathcal{Z}^\bullet$ *is* D_0–*complete and*

$$(2\text{-}68) \qquad\qquad \mathcal{Z} = \mathcal{Z}^\bullet \circ h_*.$$

Proof We start by introducing the point-completed defect data $\mathbb{D}^\bullet$. It will differ from $\mathbb{D}$ only in the label set and adjacency map for 0–strata. Choose a subset

$$(2\text{-}69) \qquad\qquad \mathcal{S} \subset \mathrm{Sphere}_{n-1}^{\mathrm{def}}(\partial\mathbb{D})$$

of representatives of the classes in $[\mathrm{Sphere}_{n-1}^{\mathrm{def}}(\partial\mathbb{D})]$, ie for each $S \in [\mathrm{Sphere}_{n-1}^{\mathrm{def}}(\partial\mathbb{D})]$ there exists a unique $\Sigma \in \mathcal{S}$ such that $S = [\Sigma]$. An element of $D_0^\bullet$ is a pair consisting of a defect sphere $\Sigma \in \mathcal{S}$ and a state $\psi \in \mathcal{Y}_\Sigma$, while $f_0^\bullet$ simply forgets the state:

$$(2\text{-}70) \qquad\qquad D_0^\bullet := \{(\Sigma, \psi) \mid \Sigma \in \mathcal{S},\ \psi \in \mathcal{Y}_\Sigma\},$$

and

$$(2\text{-}71) \qquad f_0^\bullet \colon D_0^\bullet \times \{+\} \to [\mathrm{Sphere}_{n-1}^{\mathrm{def}}(\partial\mathbb{D})], \quad (\Sigma, \psi) \mapsto [\Sigma],$$

while $f_0^\bullet$ is determined on $D_0^\bullet \times \{-\}$ by (2-22). Finally, $\mathbb{D}^\bullet$ is given by

$$(2\text{-}72) \qquad\qquad \mathbb{D}^\bullet := (D_n, \dots, D_1, D_0^\bullet;\ f_{n-1}, \dots, f_1, f_0^\bullet).$$

By Lemma 2.15, there is a canonical map of defect data $h \colon \mathbb{D} \to \mathbb{D}^\bullet$ (recall Definition 2.6) which is given by the identity on D_i for $i \in \{1, \dots, n\}$ and by

$$(2\text{-}73) \qquad\qquad h_0 \colon D_0 \to D_0^\bullet, \quad x \mapsto Y_\Sigma(x),$$

where $\Sigma \in \mathcal{S}$ is such that $f_0(x, +) = [\Sigma]$, ie $x \in D_0(\Sigma)$, and Y_Σ is as defined in (2-67).

We can now describe the D_0–complete defect TQFT

$$(2\text{-}74) \qquad\qquad \mathcal{Z}^\bullet \colon \operatorname{Bord}_n^{\operatorname{def}}(\mathbb{D}^\bullet) \to \operatorname{Vect}_{\Bbbk}$$

obtained from $\mathcal{Z}$ by extending from $\mathbb{D}$ to $\mathbb{D}^\bullet$. Note that the objects of $\operatorname{Bord}_n^{\operatorname{def}}(\mathbb{D}^\bullet)$ are the same as those of $\operatorname{Bord}_n^{\operatorname{def}}(\mathbb{D})$, and so on objects X we set

$$(2\text{-}75) \qquad\qquad \mathcal{Z}^\bullet(X) := \mathcal{Z}(X).$$

Let

$$(2\text{-}76) \qquad\qquad M \colon X \to Y$$

be a bordism in $\operatorname{Bord}_n^{\operatorname{def}}(\mathbb{D}^\bullet)$. Let $p \in M_0$ be a 0–stratum of M and let $(\Sigma_p, \psi_p) \in D_0^\bullet$ be its label. Write $\bar{B}_p$ for the closed cone $C(\Sigma_p)$ with central point labelled (Σ_p, ψ_p). Fix a an isomorphism-onto-its-image $f_p \colon \bar{B}_p \to M$ of decorated stratified manifolds, such that $f_p(0) = p$. Such a local neighbourhood exists by the definition of defect bordisms (embed a slightly larger open ball $B_{p,1+\varepsilon}$ as above and restrict to $\bar{B}_p$). Repeat this procedure for all $p \in M_0$. By restricting the maps f_p to balls of smaller radii if necessary, we may assume that all images are disjoint.

We can now define a new bordism

$$(2\text{-}77) \qquad\qquad M((f_p)_{p\in M_0})\colon X \sqcup \bigsqcup_{p\in M_0} \Sigma_p \to Y$$

as follows. As a manifold, $M((f_p)_{p\in M_0}) = M \setminus \bigsqcup_{p\in M_0} f_p(B_p)$. The new boundary component arising from cutting out the open ball around p is parametrised by f_p, restricted to $\Sigma_p = \partial \bar{B}_p$. We make the ansatz

$$(2\text{-}78) \qquad \mathcal{Z}^\bullet(M)\colon \mathcal{Z}(X) \to \mathcal{Z}(Y), \quad u \mapsto \mathcal{Z}\big(M((f_p)_{p\in M_0})\big)\bigg(u \otimes \bigotimes_{p\in M_0} \psi_p\bigg).$$

The proof of the proposition is complete once we show the following:

Claim $\mathcal{Z}^\bullet(M)$ *in* (2-78) *is independent of the choice of the* f_p.

It is enough to consider the case that in (2-77), M_0 consists of a single point, the general case follows from gluing. Let thus $M_0 = \{p\}$ and let $f_p, g_p \colon \bar{B}_p \to M$ be two choices of local neighbourhood.

Choose open subsets $U, V \subset \bar{B}_p$ such that $\varphi := g_p^{-1} \circ f_p|_U \colon U \to V$ is an isomorphism. Pick $h \in \operatorname{Emb}(\bar{B}_p)$ such that $\operatorname{im}(h) \subset U$. Then $h' := \varphi \circ h$ is equally an element of

$\mathrm{Emb}(\bar{B}_p)$. With the thus-constructed maps we have the identities of bordisms

$$(2\text{-}79) \qquad M(f_p) \circ (1_X \sqcup H_h) = M(f_p \circ h) = M(g_p \circ h') = M(g_p) \circ (1_X \sqcup H_{h'}).$$

Now let $\psi \in \mathcal{Y}_{\Sigma_p}$. Then

$$(2\text{-}80) \qquad \begin{aligned} \mathcal{Z}(M(f_p))(u \otimes \psi) &= \mathcal{Z}(M(f_p) \circ (1_X \sqcup H_h))(u \otimes \psi) \\ &= \mathcal{Z}(M(g_p) \circ (1_X \sqcup H_{h'}))(u \otimes \psi) \\ &= \mathcal{Z}(M(g_p))(u \otimes \psi), \end{aligned}$$

where in the first and last step we used that $\psi \in \mathcal{Y}_{\Sigma_p}$, so that it is left invariant by $\mathcal{Z}(H_h)$ and $\mathcal{Z}(H_{h'})$. This proves the claim. $\qquad\square$

Remark 2.18   (i)   To justify the name "completion", we note that if $\mathcal{Z}$ is already D_0–complete, then $\mathcal{Z}^\bullet$ is isomorphic to $\mathcal{Z}$ in $\mathrm{TQFT}_n^{\mathrm{def}}$, and so in particular $\mathcal{Z}^\bullet \sim \mathcal{Z}$.

To see this, recall that, by construction, $D_0^\bullet(\Sigma) = \mathcal{Y}_\Sigma$ and that (2-73) states that $h_0|_{D_0(\Sigma)} = Y_\Sigma$. If $\mathcal{Z}$ is D_0–complete, by definition the latter is an isomorphism, so that $h \colon \mathbb{D} \to \mathbb{D}^\bullet$ is an isomorphism.

   (ii)   The construction of $\mathbb{D}^\bullet$ and $\mathcal{Z}^\bullet$ does not depend on the zero component D_0 of $\mathbb{D}$ at all. Furthermore, by Proposition 2.17 a defect TQFT for $\mathbb{D}$ factors through one for $\mathbb{D}^\bullet$, that is, every defect TQFT factors through a D_0–complete one. In this sense the label set D_0 is superfluous in the description of defect TQFTs.

Let $\mathcal{Z} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_\Bbbk$ be a D_0–complete defect TQFT. We will now describe how point defects on strata of dimension $\geqslant 1$ carry an algebra structure. This algebra structure describes the fusion of point defects and will be instrumental in the next section.

Fix an $i \in \{1, \dots, n\}$ and a defect label $x \in D_i$. Recall the open and closed x–defect balls B_x and $\bar{B}_x$ from (2-29) and (2-30), and that by definition of the defect bordisms, every point in an i–stratum labelled by x has a neighbourhood isomorphic to B_x. As before we abbreviate $\Sigma_x = \partial \bar{B}_x$ and

$$(2\text{-}81) \qquad\qquad\qquad A_x := D_0(\Sigma_x).$$

Recall that for D_0–complete theories, as we assume here, A_x is a $\Bbbk$–vector space.

For $u, v \in A_x$ let

$$(2\text{-}82) \qquad\qquad\qquad M_x(u, v) \colon \varnothing \to \Sigma_x$$

be the bordism in $\text{Bord}_n^{\text{def}}(\mathbb{D})$ obtained from $\bar{B}_x$ by adding two point defects p and q on the i–stratum of $\bar{B}_x$, that is, on the central fibre $(\{0\} \times [-1, 1]^i) \cap \bar{B}_x$. The 0–strata of $M_x(u, v)$ are then p and q, and we label p by u and q by v.

Suppose $\bar{B}_x$ contains a j–stratum $(\bar{B}_x)_j^{\alpha}$ with $j \neq i$. Note that necessarily $j > i$. Let $y \in D_j$ be the defect label of $(\bar{B}_x)_j^{\alpha}$. For $w \in A_y$ let

$$(2\text{-}83) \qquad N_{y,x}^{\alpha}(w) \colon \varnothing \to \Sigma_x$$

be the bordism in $\text{Bord}_n^{\text{def}}(\mathbb{D})$ obtained from $\bar{B}_x$ by adding one point defect on $(\bar{B}_x)_j^{\alpha}$ which is labelled w.

Proposition 2.19 *Let $x \in D_i$ for $i \geqslant 1$.*

(i) *A_x with multiplication*

$$(2\text{-}84) \qquad m_x \colon A_x \otimes A_x \to A_x, \quad (u, v) \mapsto Y_{\Sigma_x}^{-1}\big(\mathcal{Z}(M_x(u, v))\big),$$

and unit $1_{A_x} := Y_{\Sigma_x}^{-1}(\mathcal{Z}(\bar{B}_x))$ is a unital associative algebra. A_x is commutative for $i > 1$.

(ii) *The map*

$$(2\text{-}85) \qquad b_{y,x}^{\alpha} \colon A_y \to A_x, \quad w \mapsto Y_{\Sigma_x}^{-1}\big(\mathcal{Z}(N_{x,y}^{\alpha}(w))\big),$$

is an algebra homomorphism.

We call A_x the *algebra of point insertions on x*. For $i = n-1$ the above map $A_y \to A_x$ can be thought of as a variant of the familiar bulk-boundary map in open–closed TQFTs, to which it reduces if the defect label $x \in D_{n-1}$ is in fact a "boundary condition".

The proof of Proposition 2.19 relies on the next lemma, which allows us to move point defects on connected components without changing the value of the functor. We need to prepare a bit of notation to state it.

Let $M \colon X \to Y$ be a bordism in $\text{Bord}_n^{\text{def}}(\mathbb{D})$ and let $p_1, \ldots, p_m \in M$ be distinct points such that $p_k \in M_{i_k}^{\alpha_k}$ for some i_k–stratum $M_{i_k}^{\alpha_k}$ of dimension $i_k > 0$ with defect label $x_k \in D_{i_k}$. Let $u_1, \ldots, u_m \in D_0$ be point defect labels such that $u_k \in A_{x_k}$. Write

$$(2\text{-}86) \qquad M(p_1, \ldots, p_m; u_1, \ldots, u_m) \colon X \to Y$$

for the bordism obtained from M by declaring $p_1, \ldots, p_m$ to be new 0–strata such that p_k is labelled by $u_k \in D_0$. We have the following lemma, which holds also for non-D_0–complete defect TQFTs:

Lemma 2.20 Let $p_1, \ldots, p_m \in M$ and $q_1, \ldots, q_m \in M$ be two choices of m distinct points such that p_k and q_k lie in the same i_k–stratum $M_{i_k}^{\alpha_k}$. For a 1–dimensional stratum we demand in addition that all q_k on that stratum occur in the same order along that stratum as the p_k. Then

$$(2\text{-}87) \qquad \mathcal{Z}(M(p_1, \ldots, p_m; u_1, \ldots, u_m)) = \mathcal{Z}(M(q_1, \ldots, q_m; u_1, \ldots, u_m)).$$

Proof It is enough to show that we can replace p_1 by q_1, ie that

$$(2\text{-}88) \quad \mathcal{Z}(M(p_1, p_2, \ldots, p_m; u_1, \ldots, u_m)) = \mathcal{Z}(M(q_1, p_2, \ldots, p_m; u_1, \ldots, u_m)).$$

Pick a smooth path $\gamma\colon [0, 1] \to M$ such that $\gamma(0) = p_1$, $\gamma(1) = q_1$ and γ lies entirely in the stratum $M_{i_1}^{\alpha_1}$ and does not intersect any of the other points $p_2, \ldots, p_m$. For each point $\gamma(t)$ of the path there is an isomorphism-onto-its-image map $g_t\colon \bar{B}_{x_1} \to M$ such that $g_t(0) = \gamma(t)$ and $\mathrm{im}(g_t)$ does not contain any of $p_2, \ldots, p_m$. Pick a finite collection $0 = t_0, t_1, \ldots, t_N = 1$ such that the images $\mathrm{im}(g_{t_j})$ cover γ.

It is now enough to show that p_1 can be moved to a point r in the intersection of $\mathrm{im}(g_{t_0})$ and $\mathrm{im}(g_{t_1})$. Repeating this procedures allows one to move p_1 to q_1.

Both p_1 and r lie in the image of g_{t_0}. Let $\tilde{p}_1$ and $\tilde{r}$ be their preimages in $\bar{B}_{x_1}$. There is an isomophism ϕ of the decorated stratified manifold $\bar{B}_{x_1}$ which is the identity in some neighbourhood of the boundary of $\bar{B}_{x_1}$ and which maps $\tilde{p}_1$ and $\tilde{r}$. Using ϕ we obtain an isomorphism of decorated stratified manifolds $M \to M$ which is the identity outside $\mathrm{im}(g_{t_0})$ and which equals $g_{t_0} \circ \phi \circ g_{t_0}^{-1}$ on the image. This isomorphism maps p_1 to r, showing that $\mathcal{Z}(M(p_1, p_2, \ldots, p_m; u_1, \ldots, u_m)) = \mathcal{Z}(M(r, p_2, \ldots, p_m; u_1, \ldots, u_m))$. $\square$

A different way of stating the above lemma is that $\mathcal{Z}(M(p_1, \ldots, p_m; u_1, \ldots, u_m))$ depends on $p_1, \ldots, p_m$ only up to homotopy in the configuration space of m ordered distinct points, where during homotopies all points must remain in their respective strata.

Proof of Proposition 2.19 (i) Commutativity for $i > 1$ is immediate from Lemma 2.20. For unitality, pick a chart $g\colon \bar{B}_x \to B_x$ around p such that $g(0) = p$ and $q \notin \mathrm{im}(g)$. Let W be the bordism obtained from $M_x(u, v)$ by cutting out $g(B_x)$ and parametrising the boundary by $g|_{\partial \bar{B}_x}$. Recall that $\partial \bar{B}_x = \Sigma_x$. By definition of Y_{Σ_x} (see (2-67)) we have

$$(2\text{-}89) \qquad \mathcal{Z}(M_x(u, v)) = \mathcal{Z}(W)(Y_{\Sigma_x}(u)) \in \mathcal{Z}(\Sigma_x).$$

Denote by $M_x(v)$ the bordism obtained from $\bar{B}_x$ by adding a v–labelled 0–stratum at the point q. For $u = 1_{A_x}$ one obtains

$$(2\text{-}90) \qquad \mathcal{Z}(M_x(1_{A_x}, v)) = \mathcal{Z}(W)(Y_{\Sigma_x}(1_{A_x})) = \mathcal{Z}(W) \circ \mathcal{Z}(\bar{B}_x)$$
$$= \mathcal{Z}(M_x(v)) = Y_{\Sigma_x}(v),$$

where in the last equality we used once more Lemma 2.20.

The verification of associativity works along the same lines. One shows that the compositions $m_x(m_x(u, v), w)$ and $m_x(u, m_x(v, w))$ are equal to $\mathcal{Z}$ evaluated on the bordism given by $\bar{B}_x$ with three additional 0–strata p, q and r labelled u, v and w, respectively. We omit the details.

(ii)   We need to show $b^\alpha_{y,x}(m_y(u, v)) = m_x(b^\alpha_{y,x}(u), b^\alpha_{y,x}(v))$. Writing out both sides as a single bordism gives $\bar{B}_x$ with additional 0–strata p and q inserted on M^α_j and labelled u and v. Applying Lemma 2.20 gives the result; we again skip the details. $\square$

Considering the algebras of point defects provides another justification for our notion of equivalence of defect TQFTs from Definition 2.12.

Lemma 2.21   Let $\mathcal{Z}: \mathrm{Bord}^{\mathrm{def}}_n(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ and $\mathcal{Z}': \mathrm{Bord}^{\mathrm{def}}_n(\mathbb{D}') \to \mathrm{Vect}_{\Bbbk}$ be D_0–complete defect TQFTs. Suppose $\mathcal{Z} \sim \mathcal{Z}'$ via $(h, \varphi): \mathcal{Z} \rightleftarrows \mathcal{Z}' :(g, \psi)$. Then for every $x \in D_i$ the map $h_0|_{A_x}: A_x \to A_{h_i(x)}$ is an isomorphism of algebras.

Proof   Recall the definitions (2-30) and (2-31). Evaluating the naturality square of $\varphi: \mathcal{Z} \to \mathcal{Z}' \circ h_*$ for the bordism $\bar{B}_x: \varnothing \to \Sigma_x$ shows that the diagram

$$(2\text{-}91)$$
$$
\begin{array}{ccc}
D_0(\Sigma_x) & \xrightarrow{\ h_0\ } & D'_0(\Sigma_{h_i(x)}) \\
{\scriptstyle \mathcal{Z}(\bar{B}_x)}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{Z}'(\bar{B}_{h_i(x)})} \\
\mathcal{Z}(\Sigma_x) & \xrightarrow[\ \varphi_{\Sigma_x}\ ]{} & \mathcal{Z}'(\Sigma_{h_i(x)})
\end{array}
$$

commutes. Since $\mathcal{Z}$ is D_0–complete, the image of $\mathcal{Z}(\bar{B}_x)$ equals the subspace $\mathcal{Y}_{\Sigma_x}$ of invariant states. By Lemma 2.15, φ_{Σ_x} therefore induces a map $\varphi_{\mathcal{Y}}: \mathcal{Y}_{\Sigma_x} \to \mathcal{Y}_{\Sigma_{h_i(x)}}$ such that the diagram

$$(2\text{-}92)$$
$$
\begin{array}{ccc}
D_0(\Sigma_x) & \xrightarrow{\ h_0\ } & D'_0(\Sigma_{h_i(x)}) \\
{\scriptstyle Y_{\Sigma_x}}\Big\downarrow & & \Big\downarrow{\scriptstyle Y_{\Sigma_{h_i(x)}}} \\
\mathcal{Y}_{\Sigma_x} & \xrightarrow[\ \varphi_{\mathcal{Y}}\ ]{} & \mathcal{Y}_{\Sigma_{h_i(x)}}
\end{array}
$$

commutes. Since the bottom path of the square consists of three (linear) isomorphisms, also h_0 restricted to $D_0(\Sigma_x) = A_x$ is a linear isomorphism. To show that $h_0|_{A_x}$ is an algebra isomorphism, we compute, for $u, v \in A_x$,

$$(2\text{-}93) \qquad m'_{h_i(x)}(h_0(u), h_0(v)) = Y^{-1}_{\Sigma_{h_i(x)}} \circ \mathcal{Z}'\big(M_{h_i(x)}(h_0(u), h_0(v))\big)$$
$$= (\varphi_y \circ Y_{\Sigma_x} \circ h_0^{-1})^{-1} \circ \varphi_y \circ \mathcal{Z}(M_x(u, v))$$
$$= h_0 \circ m_x(u, v),$$

where in the second step we used that φ is natural. □

2.5 Euler-completing defect TQFTs

In the Euler defect TQFTs discussed in Example 2.14, each stratum contributed a weight calculated from its Euler character. Tensoring an arbitrary defect TQFT $\mathcal{Z}$ with an Euler theory allows one to attach such weights to defect strata for $\mathcal{Z}$ as well. One may ask if one can enlarge the set of defect labels of $\mathcal{Z}$ in such a way that assigning different weights just amounts to choosing a different defect label. This is indeed possible by "internalising" the construction of Example 2.14. As an additional bonus, this internal version allows for weights that are not just scalars, but arbitrary invertible point defects.

Given a D_0–complete defect TQFT $\mathcal{Z}: \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ (recall Definition 2.16) we will define Euler-completed defect data $\mathbb{D}^{\odot}$ together with an injection $\iota^{\odot}: \mathbb{D} \to \mathbb{D}^{\odot}$, as well as the Euler-completed TQFT

$$(2\text{-}94) \qquad\qquad \mathcal{Z}^{\odot}: \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\odot}) \to \mathrm{Vect}_{\Bbbk}.$$

The Euler completion has the following properties, which justify its name:

(i) $\mathcal{Z}$ factors through $\mathcal{Z}^{\odot}$ as $\mathcal{Z} = \mathcal{Z}^{\odot} \circ \iota_*^{\odot}$.

(ii) $\mathcal{Z}^{\odot}$ is equivalent to $(\mathcal{Z}^{\odot})^{\odot}$ in the sense of Definition 2.12.

(iii) $\mathcal{Z}^{\odot} \otimes \mathcal{Z}_{\Psi}^{\mathrm{eu}}$ is equivalent to $\mathcal{Z}^{\odot}$.

Therefore, let $\mathcal{Z}: \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ be a D_0–complete defect TQFT. Recall from Proposition 2.19 that the set of point defect labels on an i–stratum decorated with $x \in D_i$ naturally acquires the structure of an algebra, denoted by A_x in (2-81). We write $A_x^{\times}$ for the set of invertible elements in this algebra.

We would now like to say that the new sets $D_i^{\odot}$ of defect labels for i–strata consist of pairs (x, ϕ), where $x \in D_i$ and $\phi \in A_x^{\times}$ describes the modified weight given to

x–labelled i–strata. However, this is not quite enough, as one has to keep track of the thus-extended defect labels in the defect sphere $f_i(x)$. The correct definition below makes use of the notation

$$(2\text{-}95) \qquad\qquad \mathrm{Strat}(M) = \bigcup_{i=0}^{n} \bigcup_{\alpha_i} \{M_i^{\alpha_i}\}$$

to denote the set of strata $M_i^{\alpha_i}$ of a bordism M.

Definition 2.22 Let $\mathbb{D} \in \mathcal{D}^n$. The *Euler-completed defect data* $\mathbb{D}^{\odot} \in \mathcal{D}^n$ is given as follows:

- The label sets $D_i^{\odot}$ for $i \in \{1, \dots, n\}$ consist of triples

$$(2\text{-}96) \qquad\qquad\qquad (x, \phi, \Psi),$$

  where $x \in D_i$, $\phi \in A_x^{\times}$ and $\Psi = (\psi_S)_{S \in \mathrm{Strat}(f_i(x))}$. Here, $\psi_S \in A_y^{\times}$, where y is the label of the stratum S of the defect sphere $f_i(x)$. The set $D_0^{\odot}$ consists of pairs (x, Ψ) with $x \in D_0$ and Ψ as above.

- The value $f_i^{\odot}(x, \phi, \Psi)$ of the adjacency map $f_i^{\odot}$ is given by the defect sphere $f_i(x)$, except that a j–stratum $S \in \mathrm{Strat}(f_i(x))$ with label $y \in D_j$ is decorated by $(y, \psi_S, \widetilde{\Psi})$, where $\widetilde{\Psi}$ is determined from Ψ by the defect labels and weights adjacent to S. For $i = 0$, we set $f_0^{\odot}(x, \Psi) = f_0(x)$ with decorations from Ψ.

It is clear that $\mathbb{D}^{\odot}$ is again a set of defect data. We can realise $\mathbb{D}$ as a retract of $\mathbb{D}^{\odot}$ via the injection $\iota^{\odot}$ and surjection $\pi^{\odot}$ of defect data: for $i > 0$ we have

$$(2\text{-}97) \quad \iota_i^{\odot} \colon D_i \to D_i^{\odot}, \quad x \mapsto (x, 1_{A_x}, \Psi_{\mathbf{1}}), \qquad \pi_i^{\odot} \colon D_i^{\odot} \to D_i, \quad (x, \phi, \Psi) \mapsto x,$$

where $\Psi_{\mathbf{1}}$ assigns to each stratum of the defect sphere $f_i(x)$ the unit 1_{A_y} in the corresponding algebra A_y; for $i = 0$ we set $\iota_0^{\odot}(x) = (x, \Psi_{\mathbf{1}})$ and $\pi_0^{\odot}(x, \Psi) = x$.

We now describe the (nonfunctorial) map "insertion of point weights"

$$(2\text{-}98) \qquad\qquad W \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\odot}) \to \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$$

on objects and morphisms of $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\odot})$. To indicate the decoration with weights, we use the notation $\Sigma^{\odot}$ and $M^{\odot}$ for objects and morphisms of $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\odot})$, respectively. We write $\mathrm{Strat}_{>0}(M)$ for the set of all strata of M of dimension > 0.

- On objects, $W = \pi_*^{\odot}$, that is, $W(\Sigma^{\odot})$ agrees with $\Sigma^{\odot}$ as stratified manifold, but labels (x, Ψ) and (x, ϕ, Ψ) are replaced by just x.

- On morphisms, W maps the class $[M^{\odot}]$ to the class $[M((p_S), (\phi_S^{\chi_{\text{sym}}(S)}))]$, where the latter is defined as follows. Let M represent the $\mathbb{D}$–decorated bordism $\pi_*^{\odot}([M^{\odot}])$. For each $S \in \text{Strat}_{>0}(M)$, p_S is a choice of one point on S, ϕ_S is the corresponding component of the defect label (x_S, ϕ_S, Ψ_S) that S carries in $M^{\odot}$, and the product $\phi_S^{\chi_{\text{sym}}(S)}$ is computed in A_{x_S}. The defect bordism $M((p_S), (\phi_S^{\chi_{\text{sym}}(S)}))$ is defined as in (2-86): each p_S becomes an additional 0–stratum labelled by $\phi_S^{\chi_{\text{sym}}(S)}$ (with the symmetric Euler characteristic $\chi_{\text{sym}}(S)$ introduced in (2-57)). By Lemma 2.20 the class $[M((p_S), (\phi_S^{\chi_{\text{sym}}(S)}))]$ is independent of the choice of the p_S.

Even though W itself is not a functor (composition of bordisms may result in more than one additional 0–stratum on a given j–stratum), we have:

Lemma 2.23 $\mathcal{Z} \circ W \colon \text{Bord}_n^{\text{def}}(\mathbb{D}^{\odot}) \to \text{Vect}_{\mathbb{k}}$ *is a defect TQFT.*

Proof   It is easy to see that W is strictly monoidal and symmetric (these conditions can be formulated without W being compatible with composition), so it is enough to verify that $\mathcal{Z} \circ W$ is functorial.

Since all strata S in the cylinder $X = \Sigma^{\odot} \times [0, 1]$ over an object $\Sigma^{\odot}$ have $\chi_{\text{sym}}(S) = 0$, the additional 0–strata in $W(X)$ are labelled by $\phi_S^0 = 1$. Thus, $\mathcal{Z}(W(X)) = \text{id}$, as required.

Given two composable bordisms $M^{\odot}$ and $N^{\odot}$ we need to show

$$(2\text{-}99) \qquad \mathcal{Z}(W(M^{\odot})) \circ \mathcal{Z}(W(N^{\odot})) = \mathcal{Z}(W(M^{\odot} \circ N^{\odot})).$$

This will follow directly from additivity of χ_{sym} under gluing. Indeed, suppose two strata $S \in \text{Strat}_{>0}(M^{\odot})$ and $T \in \text{Strat}_{>0}(N^{\odot})$ with common label (x, ϕ, Ψ) get glued together, resulting in the stratum U of $M^{\odot} \circ N^{\odot}$. Then $\chi_{\text{sym}}(S) + \chi_{\text{sym}}(T) = \chi_{\text{sym}}(U)$.

Let p (labelled $\phi^{\chi_{\text{sym}}(S)}$) and q (labelled $\phi^{\chi_{\text{sym}}(T)}$) be the additional 0–strata in S and T, respectively, that are added by W. By Lemma 2.20 and by the definition of the product of A_x in (2-84), under $\mathcal{Z}$ the two point defects can be replaced by a single point defect anywhere on U, which is labelled by the product $\phi^{\chi_{\text{sym}}(S)} \phi^{\chi_{\text{sym}}(T)} = \phi^{\chi_{\text{sym}}(U)}$. But the latter is the 0–stratum added on U in $W(M^{\odot} \circ N^{\odot})$. $\qquad\square$

The above lemma allows us to define:

Definition 2.24 Let $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ be a D_0–complete defect TQFT. The defect TQFT

$$(2\text{-}100) \qquad\qquad \mathcal{Z}^{\odot} := \mathcal{Z} \circ W\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\odot}) \to \mathrm{Vect}_{\Bbbk}$$

is called the *Euler completion of* $\mathcal{Z}$.

It is immediate from the definition that

$$(2\text{-}101) \qquad\qquad \mathcal{Z}^{\odot} \circ \iota_*^{\odot} = \mathcal{Z}.$$

The Euler completion is also compatible with our notion of equivalence of defect TQFTs:

Lemma 2.25 *If* $\mathcal{Z}$ *and* $\mathcal{Z}'$ *are equivalent* D_0–complete defect TQFTs, $\mathcal{Z} \sim \mathcal{Z}'$, *then also* $\mathcal{Z}^{\odot} \sim \mathcal{Z}'^{\odot}$.

Proof Let $(h, \varphi)\colon \mathcal{Z} \rightleftarrows \mathcal{Z}' \colon (g, \psi)$ be maps witnessing the equivalence between $\mathcal{Z}$ and $\mathcal{Z}'$. The maps of defect data h and g extend to maps of Euler-completed defect data as follows. We define $h^{\odot}\colon \mathbb{D}^{\odot} \to \mathbb{D}'^{\odot}$ by mapping $(x, \phi, \Psi) \in D_i^{\odot}$ to $(h(x), h(\phi), h(\Psi))$, where h on A_x is as defined in Lemma 2.21 and we do not display indices on h for convenience. Since it is shown there that $h\colon A_x \to A_{h(x)}$ is an algebra isomorphism, it follows that $h(\phi) \in A_{h(x)}^{\times}$. Proceeding componentwise defines also $h(\Psi)$. As the symmetric Euler characteristics remain the same under $h_*^{\odot}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\odot}) \to \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}'^{\odot})$, it follows that φ induces a natural transformation $\varphi^{\odot}\colon \mathcal{Z}^{\odot} \Rightarrow \mathcal{Z}'^{\odot} \circ h_*^{\odot}$. The same argument applied to (g, ψ) concludes the proof. $\quad\square$

In the next two lemmas we investigate the structure of the point defects of $\mathcal{Z}^{\odot}$, working towards the equivalence $(\mathcal{Z}^{\odot})^{\odot} \sim \mathcal{Z}^{\odot}$.

Lemma 2.26 $\mathcal{Z}^{\odot}$ *is* D_0–complete.

Proof We proceed in two steps.

Step 1 We start with showing that

$$(2\text{-}102) \qquad \mathcal{Y}_{\Sigma^{\odot}} = \{\psi \in \mathcal{Z}^{\odot}(\Sigma^{\odot}) \mid \mathcal{Z}^{\odot}(H_f^{\odot})(\psi) = \psi \text{ for all } f \in \mathrm{Emb}(\bar{B})\}$$

is equal to $\mathcal{Y}_{\Sigma}$, as in Section 2.4 for $\Sigma^{\odot} \in \mathrm{Sphere}_{n-1}^{\mathrm{def}}(\partial \mathbb{D}^{\odot})$.

By definition, $\mathcal{Z}^{\circlearrowright}(\Sigma^{\circlearrowright}) = \mathcal{Z}(\Sigma)$ and we are left with showing that $\chi_{\mathrm{sym}}(S) = 0$ for all $(j+1)$–strata S of $H_f^{\circlearrowright}$ for all j. Indeed, then $\mathcal{Z}^{\circlearrowright}(H_f^{\circlearrowright}) = \mathcal{Z}(H_f)$ and thus $\mathcal{Y}_{\Sigma^{\circlearrowright}} = \mathcal{Y}_{\Sigma}$.

Fix a j–stratum T and of Σ and $f \in \mathrm{Emb}(\bar{B})$. There is a unique j–stratum T' of Σ such that f restricted to the cone $C(T)$ is a map $\overline{C(T)} \to C(T')$. By construction we have $\bar{B} = H_f \circ f(\bar{B})$ as stratified bordisms. The symmetric Euler characteristics are related by $\chi_{\mathrm{sym}}(C(T')) = \chi_{\mathrm{sym}}(S) + \chi_{\mathrm{sym}}(C(T))$, where $S := \overline{C(T')} \setminus f(C(T))$. Since $C(T)$ is contractible we have $\chi(C(T)) = 0$ and thus $\chi_{\mathrm{sym}}(CT) = 2\chi(C(T)) - \chi(\partial C(T)) = -\chi(T)$ and analogously $\chi_{\mathrm{sym}}(C(T')) = -\chi(T')$. Thus, we conclude that $\chi_{\mathrm{sym}}(S) = \chi(T) - \chi(T')$ and we are done with the first step once we established the following result:

Claim *The Euler characteristics of T and T' are equal.*

To establish this, we consider the smooth embedding

$$(2\text{-}103) \qquad\qquad f_T \colon T \times \mathbb{R} \to T' \times \mathbb{R},$$

which is obtained from the above map $\overline{C(T)} \to C(T')$ by removing the cone points and then composing with a map which inflates the interval $(0, 1)$ to the real line $\mathbb{R}$ (with ∞ corresponding to the cone point and $-\infty$ corresponding to the boundary of $\bar{B}$). We will show that f_T is a homotopy equivalence, thus proving the claim.

The map f_T induces a map $\pi_k(f_T) \colon \pi_k(T \times \mathbb{R}) \to \pi_k(T' \times \mathbb{R})$ between homotopy groups for all $k \in \mathbb{Z}_+$. Then by the Whitehead theorem, f_T is a homotopy equivalence if $\pi_k(f_T)$ is an isomorphism for all k.

Since we can find an $\varepsilon > 0$ such that a ball B_ε of radius ε is contained in $f(B)$, there exists an $r \in \mathbb{R}$ such that $T' \times (r, \infty) \subset f(T \times \mathbb{R})$. Choosing some $z > r$ guarantees that around $T' \times \{z\}$ also a collar lies in $f(T \times \mathbb{R})$. Pick a basepoint $f(p)$ on $T' \times \{z\}$.

To show surjectivity of $\pi_k(f_T)$, let $b \colon S^k \to T' \times \mathbb{R}$ be a based map. It is homotopic to a map $b_1 \colon S^k \to T' \times \{z\}$ by $b_t(x) = \bigl(b_{T'}(x), b_{\mathbb{R}}(x) + t(z - g_{\mathbb{R}}(x))\bigr)$ for $t \in [0, 1]$, where we write $b(x) = (b_{T'}(x), b_{\mathbb{R}}(x)) \in T' \times \mathbb{R}$. Clearly, $[b_1]$ is in the image of $\pi_k(f_T)$.

To show injectivity, consider now a continuous map $g \colon S^k \to T \times \mathbb{R}$ such that there is a homotopy in $T' \times \mathbb{R}$ from $f_T \circ g$ to the constant map. Pick such a homotopy h. Since h will in general leave the image $f(T \times \mathbb{R})$, we will "shift it above z", where z

is as in the previous paragraph. By continuity of f, there exists an $R \in \mathbb{R}$ such that $f(T \times (R, \infty)) \subset T' \times (z, \infty)$. First we apply the homotopy

$$(2\text{-}104) \qquad g_t(x) = \big(g_T(x), g_{\mathbb{R}}(x) + t(R - g_{\mathbb{R}}(x))\big)$$

from g to a map g_1. Then $f_T(g_1(x)) \in T' \times (z, \infty)$ for all $x \in S^k$, and $f_T \circ g_1$ is homotopic to $f_T \circ g$, but it does in general not preserve the basepoint. However, since T' is connected, $f_T \circ g_1$ is homotopic in $T' \times (z, \infty)$ to a based map g' that is also in the image of f_T. Composing the homotopies we obtain a homotopy $h' \colon S^k \times [0, 1] \to T' \times \mathbb{R}$ from g' to the constant map to $f(p)$. By compactness of S^k and by using the collar of $T' \times \{z\}$, which is still in $f(T \times \mathbb{R})$, there exists an interval $[a, b] \subset [0, 1]$ such that $h'(x, t) \in T' \times (r, \infty)$ for $t \in [0, 1] \setminus [a, b]$. Consider a smooth function $\rho \colon [0, 1] \to [0, 1]$ with $\rho(0) = 0 = \rho(1)$ and $\rho|_{[a,b]} = 1$, and set $\widetilde{h}(x, t) = \big(h'_{T'}(x, t), h'_{\mathbb{R}}(x, t) + \rho(t)(z - h'_{\mathbb{R}}(x, t))\big)$. By construction, $\widetilde{h}$ is a homotopy in $f(T \times \mathbb{R})$ to the constant map, and injectivity of f_T follows.

We conclude that $\mathcal{Z}^{\odot}(H_f^{\odot}) = \mathcal{Z}(H_f)$, and thus $\mathcal{Y}_{\Sigma \odot} = \mathcal{Y}_{\Sigma}$.

Step 2 Next we construct the inverse to the map $Y_{\Sigma \odot} \colon D_0^{\odot}(\Sigma^{\odot}) \to \mathcal{Y}_{\Sigma \odot}$. Let $(x, \Psi) \in D_0^{\odot}$ and consider the ball $\overline{B} = \overline{C(\Sigma^{\odot})}$ whose cone point is decorated with (x, Ψ). By definition, $Y_{\Sigma \odot}(x, \Psi) = \mathcal{Z}^{\odot}(\overline{B})$. After application of W, a stratum $T \in \mathrm{Strat}_{>0}(\overline{B})$ contains a point defect with label $\psi_{\partial T}^{\chi_{\mathrm{sym}}(T)}$, where $\partial T = T \cap \Sigma^{\odot}$ is the boundary of the stratum T. We define a defect bordism X_{Σ} as follows: first take the cylinder $\Sigma \times I$, then insert on each stratum $S \times I$ a defect point with decoration $\psi_S^{-\chi_{\mathrm{sym}}(T)}$, with T the stratum in $\overline{B}$ bounded by S. This yields $X_{\Sigma} \in \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$. Applying $\mathcal{Z}(X_{\Sigma})$ to an invariant vector in $\mathcal{Y}_{\Sigma \odot}$ yields again an invariant vector. The linear maps $\mathcal{Z}^{\odot}(\overline{B})$ and $\mathcal{Z}(X_{\Sigma})$ can be composed and, by construction, the diagram

$$(2\text{-}105)$$
$$
\begin{array}{ccccc}
D_0^{\odot}(\Sigma^{\odot}) & \xrightarrow{\;Y_{\Sigma \odot}\;} & \mathcal{Y}_{\Sigma \odot} & \xrightarrow{\;\mathcal{Z}(X_{\Sigma})\;} & \mathcal{Y}_{\Sigma \odot} \\
\mathrm{id} \downarrow & & & & \downarrow \mathrm{id} \\
D_0(\Sigma) & & \xrightarrow{\hspace{3cm}} & & \mathcal{Y}_{\Sigma} \\
& & Y_{\Sigma} & &
\end{array}
$$

commutes. By assumption, Y_{Σ} is an isomorphism and $\mathcal{Z}(X_{\Sigma})$ is an isomorphism by construction, and so we conclude that $Y_{\Sigma \odot}$ is an isomorphism as well. $\square$

We now turn to the algebra of point defects. For $(x, \phi, \Psi) \in D_i^{\odot}$ we have so far two point-set isomorphisms between $A_{(x, \phi, \Psi)}$ and A_x. However, neither the isomorphism

$Y_\Sigma^{-1} \circ Y_{\Sigma^\odot}$ nor the map $\pi^\odot \colon D_0^\odot(\Sigma^\odot) \to D_0(\Sigma)$ from (2-97) are algebra isomorphisms. Instead we have:

Lemma 2.27 For $i > 0$, let $(x, \phi, \Psi) \in D_i^\odot$ be an i–dimensional defect label. Consider the map

$$(2\text{-}106) \qquad \sigma_{(x,\phi,\Psi)} \colon A_{(x,\phi,\Psi)} \to A_x, \quad a \mapsto \phi^{E_i} \cdot \pi^\odot(a) \cdot \phi^{E_i},$$

where $E_i := \chi(S^{i-1}) - 1 = (-1)^{i-1}$ and the multiplication on the right-hand side takes place in A_x. The map $\sigma_{(x,\phi,\Psi)}$ is an algebra isomorphism.

Proof We first compute the change in the symmetric Euler characteristic when removing a point from a manifold. From covering an n–manifold X by $X = X \setminus \{p\} \cup B$ with B a ball around $p \in X$, one sees that the conventional Euler characteristic χ satisfies $\chi(X) = \chi(X \setminus \{p\}) + \chi(B) - \chi(B \setminus \{p\})$. Thus, using $\chi(B) = 1$ and $\chi(B \setminus \{p\}) = \chi(S^{n-1})$, we obtain $\chi(X \setminus \{p\}) = \chi(X) - 1 + \chi(S^{n-1})$. Hence, we have (recalling (2-57))

$$(2\text{-}107) \qquad \chi_{\text{sym}}(X \setminus \{p\}) = \chi_{\text{sym}}(X) + 2\chi(S^{n-1}) - 2 = \chi_{\text{sym}}(X) + 2E_n.$$

For simplicity we assume first that $i \geqslant 2$. The multiplication $m_{(x,\phi,\Psi)}$ of $a, b \in A_{(x,\phi,\Psi)}$ is defined by $m_{(x,\phi,\Psi)}(a, b) = Y_{\Sigma^\odot}^{-1}(\mathcal{Z}^\odot(\bar{B}(p, q; a, b)))$. Write $a = (a', \Psi')$ and $b = (b', \Psi')$, so that $\pi^\odot(a) = a'$, etc. We compute further, omitting the location of the 0–strata from the notation and denoting by M the i–stratum on which the point defects a and b are inserted,

$$(2\text{-}108) \qquad \mathcal{Z}^\odot(\bar{B}(a, b)) = \mathcal{Z}(\bar{B}(a', b', \phi^{\chi_{\text{sym}}(M) + 4E_i}, \dots))$$
$$= \mathcal{Z}(\bar{B}(a' \cdot b', \phi^{\chi_{\text{sym}}(M) + 2E_i}, \phi^{2E_i}, \dots)),$$

where $c = (a' \cdot \phi^{2E_i} \cdot b', \Psi') \in D_0^\odot(\Sigma^\odot)$, and where "$\dots$" indicates the point defect Euler weights inserted in $\bar{B}$ by W that are not located on the i–stratum M. Thus, $m_{(x,\phi,\Psi)}(a; b) = (a' \cdot \phi^{2E_i} \cdot b', \Psi')$. From this expression the claimed isomorphism follows straightforwardly. In the case $i = 1$ it follows along the same lines (but taking more care when inserting Euler weights disconnects a stratum) that the multiplication takes the same form $m_{(x,\phi,\Psi)}(a; b) = (a' \cdot \phi^{2E_i} \cdot b', \Psi')$. $\qquad \square$

Euler weights on line defects play a special role since inserting them disconnects the line defect. To obtain the equivalence $\mathcal{Z}^{\odot\odot} \sim \mathcal{Z}^\odot$ of defect TQFTs, they need to be treated separately. The next lemma shows that they do not add anything new to a given

defect TQFT, ie we show that it suffices to consider Euler weights just for defects of dimension $i \geqslant 2$. To this end we consider the truncation $\mathbb{D}^{\odot \geqslant 2}$ of $\mathbb{D}^{\odot}$ with defect labels $D_i^{\odot \geqslant 2} = D_i^{\odot}$ for $i \geqslant 2$, and with $D_1^{\odot \geqslant 2} = \{(x, \Psi) \mid x \in D_1, \Psi = (\psi_S)_{S \in \mathrm{Strat}(f_1(x))}\}$. The set $D_0^{\odot}$ consists of pairs (x, Ψ) with $x \in D_0$ and Ψ as before, where adjacent line defects have no Euler weights. $\mathbb{D}^{\odot \geqslant 2}$ is a set of defect data, and we have a map of defect data $k \colon \mathbb{D}^{\odot \geqslant 2} \to \mathbb{D}^{\odot}$ which is given by the identity on $D_i^{\odot \geqslant 2}$ for $i \geqslant 2$, and by choosing the Euler weight $1_x \in A_x$ for $i = 1$. We thus obtain a defect TQFT $\mathcal{Z}^{\odot \geqslant 2} = \mathcal{Z}^{\odot} \circ k_* \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\odot \geqslant 2}) \to \mathrm{Vect}_{\Bbbk}$.

Lemma 2.28 Let $\mathcal{Z} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ be a D_0–complete defect TQFT. Then the defect TQFTs $\mathcal{Z}^{\odot}$ and $\mathcal{Z}^{\odot \geqslant 2}$ are equivalent.

Proof We already have that, by definition, $\mathcal{Z}^{\odot \geqslant 2} = \mathcal{Z}^{\odot} \circ k_*$. Below we construct a map of defect data $l \colon \mathbb{D}^{\odot} \to \mathbb{D}^{\odot \geqslant 2}$ such that $\mathcal{Z}^{\odot} = \mathcal{Z}^{\odot \geqslant 2} \circ l_*$. This establishes the equivalence (with identity natural isomorphisms).

The maps l_i for $i \geqslant 2$ are again taken to be the identity. For $i = 1$ we define l_1 to map $(a, \phi, \Psi) \in \mathbb{D}_1^{\odot}$ to $(a, \Psi) \in D_1^{\odot \geqslant 2}$, ie it forgets the Euler weights on line defects. Those will be absorbed in the point defects D_0 by the map l_0:

Using the natural decomposition of $\mathrm{Strat}(M)$ according to the dimension of the strata, we can write a point defect as $(a, \Psi) = (a, ((\psi_{M_1^{\alpha_1}})_{\alpha_1}, (\psi_{M_2^{\alpha_2}})_{\alpha_2}, \dots)) \in \mathbb{D}_0^{\odot}$, where, in particular, $(\psi_{M_1^{\alpha_1}})_{\alpha_1}$ indicates the tuple of Euler-weights on the adjacent line defects. We define

(2-109) $$l_0((a, \Psi)) = \mathcal{Y}_{\Sigma}^{-1} \circ \mathcal{Z}\big(\bar{B}_a((\psi_{M_1^{\alpha_1}})_{\alpha_1})\big),$$

where $\bar{B}_a$ is the defect ball for a and in $\bar{B}_a((\psi_{M_1^{\alpha_1}})_{\alpha_1})$ we understand that for each α_1, the Euler weight $\psi_{M_1^{\alpha_1}}$ is inserted on the 1–stratum $M_1^{\alpha_1}$.

We claim that $\mathcal{Z}^{\odot}$ is equal to $\mathcal{Z}^{\odot \geqslant 2} \circ l_*$. To this end we consider $\mathcal{Z}^{\odot}(M) = \mathcal{Z}\big(M((p_S), (\phi_S^{\chi_{\mathrm{sym}}(S)}))\big)$ for a bordism $M \in \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\odot})$. For 1–stratum S in M the symmetric Euler characteristic can either be $\chi_{\mathrm{sym}}(S) = 0$ if S is a closed circle or if S has two boundary points on ∂M, or we can have $\chi_{\mathrm{sym}}(S) = 1$ if S has one endpoint on an inner 0–stratum of M and one endpoint in ∂M; or, finally, $\chi_{\mathrm{sym}}(S) = 2$ if both endpoints of S are an inner 0–stratum of M (which could be the same for both ends). In this last case we replace the single point defect with Euler weight ϕ_S^2 by two point defects on S, each labelled with ϕ_S. This produces a new bordism M' but does not affect the value of the TQFT: $\mathcal{Z}^{\odot}(M) = \mathcal{Z}(M')$. We can now choose small

balls around all inner 0–strata of M' which contain exactly one 0–stratum labelled with an Euler weight for each adjacent 1–stratum. Using functoriality of $\mathcal{Z}$ we can first evaluate these small balls and then the remaining bordism to obtain $\mathcal{Z}^\odot(M)$. But doing so is by definition of l_* the same as evaluating $\mathcal{Z}^{\odot_{\geqslant 2}}(l_*(M))$. This concludes the proof. $\qquad\square$

Proposition 2.29 *If* $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ *is a* D_0*–complete defect TQFT then* $\mathcal{Z}^{\odot\odot}$ *and* $\mathcal{Z}^\odot$ *are equivalent:* $\mathcal{Z}^{\odot\odot} \sim \mathcal{Z}^\odot$.

Proof We show that $(\mathcal{Z}^{\odot_{\geqslant 2}})^{\odot_{\geqslant 2}} \sim \mathcal{Z}^{\odot_{\geqslant 2}}$, the statement then follows by using Lemmas 2.28 and 2.25 in the series of equivalences

$$(2\text{-}110) \qquad \mathcal{Z}^{\odot\odot} \sim (\mathcal{Z}^{\odot_{\geqslant 2}})^\odot \sim (\mathcal{Z}^{\odot_{\geqslant 2}})^{\odot_{\geqslant 2}} \sim \mathcal{Z}^{\odot_{\geqslant 2}} \sim \mathcal{Z}^\odot.$$

From (2-101) we already have a map $h^{\odot\odot}\colon \mathbb{D}^\odot \to \mathbb{D}^{\odot\odot}$ such that $\mathcal{Z}^{\odot\odot} \circ h_*^{\odot\odot} = \mathcal{Z}^\odot$. By restriction we obtain a map

$$(2\text{-}111) \qquad\qquad h^{\odot_{\geqslant 2}\odot_{\geqslant 2}}\colon \mathbb{D}^{\odot_{\geqslant 2}} \to \mathbb{D}^{\odot_{\geqslant 2}\odot_{\geqslant 2}}$$

with $\mathcal{Z}^{\odot_{\geqslant 2}\odot_{\geqslant 2}} \circ h_*^{\odot_{\geqslant 2}\odot_{\geqslant 2}} = \mathcal{Z}^{\odot_{\geqslant 2}}$. Below we provide a degreewise surjective map of defect data $t\colon \mathbb{D}^{\odot_{\geqslant 2}\odot_{\geqslant 2}} \to \mathbb{D}^{\odot_{\geqslant 2}}$ such that $\mathcal{Z}^{\odot_{\geqslant 2}\odot_{\geqslant 2}}(M) = \mathcal{Z}^{\odot_{\geqslant 2}}(t_*(M))$ for all bordisms M in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\odot_{\geqslant 2}\odot_{\geqslant 2}})$. Since both defect TQFTs agree on objects by definition, this will imply $\mathcal{Z}^{\odot_{\geqslant 2}\odot_{\geqslant 2}} = \mathcal{Z}^{\odot_{\geqslant 2}} \circ t_*$, and hence prove the proposition.

The component maps of t for $i > 0$ are

$$(2\text{-}112) \quad t_i\colon D_i^{\odot_{\geqslant 2}\odot_{\geqslant 2}} \to D_i^{\odot_{\geqslant 2}}, \quad ((x,\phi,\Psi),\phi',\Psi') \mapsto (x, \phi \cdot \sigma_{(x,\phi,\Psi)}(\phi'), \Psi''),$$

where $\sigma_{(x,\phi,\Psi)}$ is the algebra isomorphism (2-106) and Ψ'' is obtained from Ψ' by applying the map (2-112) in each dimension. More precisely, starting from $i = n$, where Ψ' is not present in $D_n^{\odot_{\geqslant 2}\odot_{\geqslant 2}}$, the tuple Ψ'' is determined by applying t_i inductively, passing from i to $i-1$. Finally, for $i = 0$, $t_0\colon D_0^{\odot_{\geqslant 2}\odot_{\geqslant 2}} \to D_0^{\odot_{\geqslant 2}}$ maps $((x,\Psi),\Psi')$ to (x,Ψ'').

We have to show that for every bordism M in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\odot_{\geqslant 2}\odot_{\geqslant 2}})$, which we schematically write $(M,((x,\phi,\Psi),\phi',\Phi'))$ to indicate its decoration, we have

$$(2\text{-}113) \qquad \mathcal{Z}^{\odot_{\geqslant 2}\odot_{\geqslant 2}}\big(M,((x,\phi,\Psi),\phi',\Phi')\big) = \mathcal{Z}^{\odot_{\geqslant 2}}\big(M, t((x,\phi,\Psi),\phi',\Phi')\big).$$

To prove (2-113), we evaluate its left-hand side in two steps. Consider a stratum M_i with $i \geqslant 2$ that is decorated by $((x,\phi,\Psi),\phi',\Phi') \in \mathbb{D}_i^{\odot_{\geqslant 2}\odot_{\geqslant 2}}$. Below, by abuse of notation we write $\mathcal{Z}^{\odot_{\geqslant 2}\odot_{\geqslant 2}}(M_i)$ instead of $\mathcal{Z}^{\odot_{\geqslant 2}\odot_{\geqslant 2}}(M)$ to emphasise the changes

in the strata of M. By definition, $\mathcal{Z}^{\odot \geqslant 2 \odot \geqslant 2}(M_i) = \mathcal{Z}^{\odot \geqslant 2}(M_i((\phi')^{\chi_{\mathrm{sym}}(M_i)\odot}))$, where the notation $(\phi')^{\chi_{\mathrm{sym}}(M_i)\odot}$ indicates that the power is computed in the algebra $A_{(x,\phi,\Psi)}$. Computing further we obtain

$$
(2\text{-}114) \qquad \mathcal{Z}^{\odot \geqslant 2 \odot \geqslant 2}(M_i) = \mathcal{Z}\big(M_i((\phi')^{\chi_{\mathrm{sym}}(M_i)\odot}, \phi^{\chi_{\mathrm{sym}}(M_i)+2E_i})\big)
$$
$$
= \mathcal{Z}\big(M_i((\phi')^{\chi_{\mathrm{sym}}(M_i)\odot}, \phi^{\chi_{\mathrm{sym}}(M_i)}, \phi^{2E_i})\big).
$$

By Lemma 2.27 this last expression is equal to

$$
(2\text{-}115) \quad \mathcal{Z}\big(M_i\big(\sigma_{(x,\phi,\Psi)}((\phi')^{\chi_{\mathrm{sym}}(M_i)\odot}), \phi^{\chi_{\mathrm{sym}}(M_i)}\big)\big)
$$
$$
= \mathcal{Z}\big(M_i\big((\sigma_{(x,\phi,\Psi)}(\phi'))^{\chi_{\mathrm{sym}}(M_i)}, \phi^{\chi_{\mathrm{sym}}(M_i)}\big)\big)
$$
$$
= \mathcal{Z}\big(M_i\big((\sigma_{(x,\phi,\Psi)}(\phi') \cdot \phi)^{\chi_{\mathrm{sym}}(M_i)}\big)\big),
$$

where we used in the second step that $\sigma_{(x,\phi,\Psi)}$ is an algebra isomorphism. Now we are done, since the last expression is by definition $\mathcal{Z}^{\odot \geqslant 2}(M, t((x,\phi,\Psi),\phi',\Phi'))$ and this argument holds for all strata M_i. $\qquad\square$

We remark that the argument in the proof relies on commutativity of A_x and does not work directly for $i = 1$. This is the reason for the detour via $\mathcal{Z}^{\odot \geqslant 2}$.

Finally, let us come back to point (iii) mentioned in the introductory paragraph of the present subsection.

Lemma 2.30 $\mathcal{Z}^{\odot} \otimes \mathcal{Z}^{\mathrm{eu}}_{\Psi} \sim \mathcal{Z}^{\odot}$ *for any defect TQFT* $\mathcal{Z} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ *and all* $\Psi = (\psi_1, \ldots, \psi_n) \in (\Bbbk^{\times})^n$ *as in* (2-59).

Proof Recall that $\mathcal{Z}^{\mathrm{eu}}_{\Psi}$ has defect data $\mathbb{D}^{\mathrm{id}}$, but that it differs from the identity defect TQFT by its action on bordisms as given in (2-60). We will show that $\mathcal{Z}^{\odot} \otimes \mathcal{Z}^{\mathrm{eu}}_{\Psi}$ and $\mathcal{Z}^{\odot}$ are in fact isomorphic, not just equivalent.

First we give an isomorphism of defect data $f \colon \mathbb{D}^{\odot} \otimes \mathbb{D}^{\mathrm{id}} \to \mathbb{D}^{\odot}$ (which is not the standard unitor) such that $\mathcal{Z}^{\odot} \otimes \mathcal{Z}^{\mathrm{eu}}_{\Psi} = \mathcal{Z}^{\odot} \circ f_*$. We set f to be a composition of the unitor $\mathbb{D}^{\odot} \otimes \mathbb{D}^{\mathrm{id}} \to \mathbb{D}^{\odot}$ from Lemma 2.7 with an isomorphism $\widetilde{f} \colon \mathbb{D}^{\odot} \to \mathbb{D}^{\odot}$. The latter simply maps a defect label $(x, \phi, \Phi) \in D_j^{\odot}$ to $(x, \psi_j \phi, \Phi')$, using the linear structure of A_x, where in Φ' all weights are multiplied by the scalar ψ_k of the corresponding dimension. This modification absorbs the factor (2-60) into the evaluation of $\mathcal{Z}^{\odot} = \mathcal{Z} \circ W$ on $f_*(M)$ for any object or bordism M. We conclude that $\mathcal{Z}^{\odot} \otimes \mathcal{Z}^{\mathrm{eu}}_{\Psi} = (\mathcal{Z}^{\odot} \circ \widetilde{f}_*) \otimes \mathbf{1}_{\mathbb{D}^{\mathrm{id}}}$. $\qquad\square$

In Remark 3.14 below we will spell out details of Euler weights for 3–dimensional defect TQFT in relation to the orbifold construction, to which we now turn.

3 Orbifolds

In this section we construct orbifolds for defect TQFTs. We start with some preliminaries on oriented triangulations and Pachner moves in Section 3.1. Then we introduce the concept of an "orbifold datum" $\mathcal{A}$ for an n–dimensional defect TQFT $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ for arbitrary $n \in \mathbb{Z}_+$ in Section 3.2, and we explain how to obtain the associated orbifold theory $\mathcal{Z}_{\mathcal{A}}\colon \mathrm{Bord}_n \to \mathrm{Vect}_{\Bbbk}$. The case $n = 2$ had already been treated in the literature, and in Section 3.3 we briefly discuss how it fits into our broader picture. Our main example is the 3–dimensional case, which we study in detail in Section 3.4.

3.1 Oriented triangulations and Pachner moves

Orbifolds are constructed by decorating the Poincaré dual of oriented triangulations of bordisms. As preparation for that, below we recall basic facts about triangulations of n–dimensional manifolds, Pachner moves between them, and the oriented versions thereof which we will need. For illustration and later use, we spell out the cases $n = 2$ and $n = 3$ in some detail.

Let $n \in \mathbb{N}$. By an n–simplex K we mean the convex hull of $n + 1$ points $p_1, p_2, \ldots,$ $p_{n+1} \in \mathbb{R}^N$ for some integer $N \geqslant n$ such that $\{p_2 - p_1, p_3 - p_1, \ldots, p_{n+1} - p_1\}$ is linearly independent. Hence,

$$(3\text{-}1) \qquad K = \left\{ \sum_{i=1}^{n+1} t_i \, p_i \; \middle| \; t_i \in \mathbb{R}_{\geqslant 0} \text{ and } \sum_{i=1}^{n+1} t_i = 1 \right\}.$$

The *standard n–simplex* Δ^n is the special case where $\{p_1, \ldots, p_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$. By a *simplicial complex* we mean a finite collection C of simplices such that (1) all faces of all simplices in C are elements of C as well, and (2) if $\sigma, \sigma' \in C$, then $\sigma \cap \sigma'$ is either empty or a face of both σ and σ'.

A *triangulation* of a topological manifold M is a simplicial complex C together with a homeomorphism $|C| \to M$, where $|C|$ is the geometric realisation (or polyhedron) of C. A manifold together with a choice of triangulation is called a *triangulated manifold*.

A *smooth triangulation* of a smooth manifold M is a triangulation $f\colon |C| \to M$ where, for each simplex $S \subset C$, the restriction $f|_S$ is smooth and satisfies an extension property and a nondegeneracy condition on differentials; see [36, Definition 8.3] for details. To compare different triangulations we need that, given two smooth triangulations

$f\colon |C| \to M$ and $f'\colon |C'| \to M$, there exists a δ–approximation $\tilde{f}\colon |C'| \to M$ to f' such that $f^{-1} \circ \tilde{f}\colon |C'| \to |C|$ is piecewise linear [36, Corollary 10.13].

An *oriented n–simplex* is an n–simplex together with an equivalence class of total orders of its vertices. Two total orders are equivalent if they are related by an even permutation of vertices. Given an n–simplex K as in (3-1) together with a total order $(p_1, p_2, \ldots, p_{n+1})$ of its vertices p_i, one obtains an orientation of the n–dimensional tangent space (seen as a linear subspace of $\mathbb{R}^N$) at each point of K from the oriented basis $(p_2 - p_1, \, p_3 - p_1, \ldots, \, p_{n+1} - p_1)$.

By a *triangulation with total order* we mean a triangulation $|C| \to M$ of an n–manifold M with a total order of the vertices of C. In a triangulation with total order, every k–simplex with $k \in \{1, 2, \ldots, n\}$ is oriented by restricting the total order.

We will often work in the Poincaré dual picture for oriented n–manifolds. For $k \in \{0, 1, \ldots, n-1\}$, the orientation of a k–stratum in the dual of a triangulation is induced by that of the corresponding $(n{-}k)$–simplex by the rule that together (first the k–stratum, then the $(n{-}k)$–simplex) they produce the orientation of the underlying manifold M. We take the orientation of an n–stratum to be that of the underlying manifold M. As a consequence, a vertex in the Poincaré dual stratification is oriented positively if the orientation of the corresponding n–simplex agrees with the orientation of the manifold, and negatively otherwise.

It is convenient to describe the total order on the vertices by a "height function", that is, an injective function $h\colon C_0 \to \mathbb{R}$ from the set of vertices of the simplicial complex C to $\mathbb{R}$. Let us illustrate the orientation of a complex and its dual in the cases $n = 2$ and $n = 3$.

Example 3.1 (i) For $n = 2$, an n–simplex obtains a total order by assigning numbers $h(p_i)$ to the vertices p_1, p_2 and p_3. Then the edges are oriented towards the "higher" vertices, and the face is oriented such that the induced boundary orientation agrees with two of the edge orientations. For example,

(3-2)

is a 2–simplex with total order from a height function which takes values 1, 2 and 3, respectively.

If we take the standard orientation of $M = \mathbb{R}^2$, ie the ordered basis $\left(\binom{1}{0}, \binom{0}{1} \right)$, then according to our rule the oriented Poincaré dual of (3-2) is

(3-3)

where we also show the heights of the vertices corresponding to the 2–strata. The 0–stratum is oriented by $+$ since the vertex order in (3-2) induces the same orientation as that of M.

(ii)   For $n = 3$, an example of a 3–simplex (or tetrahedron) with total order is

(3-4)

where the orientation of the interior agrees with that induced from the standard orientation of $\mathbb{R}^3$, and the orientations of the edges and faces are induced as in the 2–dimensional case. Thus, for this orientation of $\mathbb{R}^3$ the Poincaré dual of (3-4) is

(3-5)

                            .
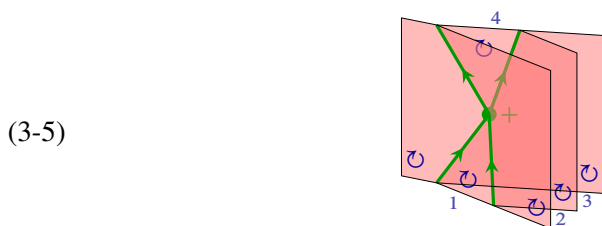
The orientation of the 2– and 1–strata in the dual are deduced with the right-hand rule:

(3-6)

                            .
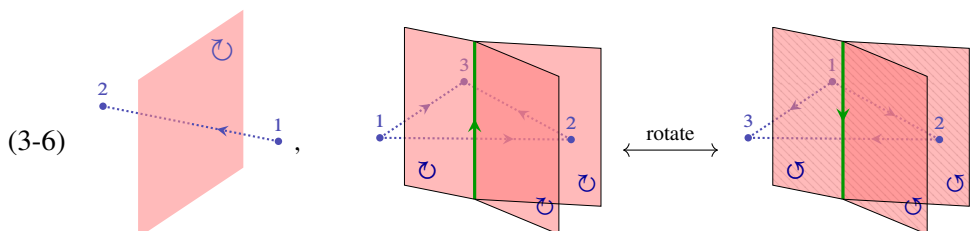
Here and below we indicate opposite orientations of 2–strata by a stripy pattern.

We note that the oriented 2– and 3–simplices with total order

(3-7)
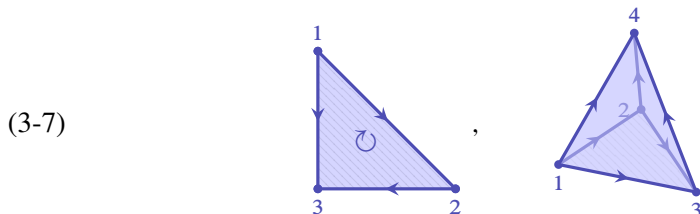
are not equivalent to (3-2) and (3-4) for $n = 2$ and $n = 3$, respectively. This is a special case of part (ii) of the following lemma, which will be important in determining the amount of defect data necessary as an input for the orbifold construction.

Lemma 3.2 Let K be an n–simplex in $\mathbb{R}^N$ as in (3-1), with $1 \leqslant n \leqslant N$.

(i)   If $n \leqslant N - 1$, any two total vertex orders of K are related by an orientation-preserving affine linear automorphism of $\mathbb{R}^N$ which maps K to K.

(ii) If $n = N$, two total orders are related by an automorphism as in (i) if and only if the two total orders induce the same orientation on K.

(iii) In both cases, the restriction of the automorphism to K is unique.

Proof   For part (ii), let $(p_1, p_2, \ldots, p_{n+1})$ be the ordered set of vertices of K with respect to the first order, and let $(p_{\sigma(1)}, \ldots, p_{\sigma(n+1)})$ for $\sigma \in S_{n+1}$ be the second order. There is a unique affine linear automorphism F of $\mathbb{R}^N$ such that $F(p_i) = p_{\sigma(i)}$ for $i \in \{1, \ldots, n+1\}$. Then F maps K to K by definition, and it is orientation-preserving if and only if σ is even.[7] This proves part (ii).

To get part (i), it is enough to add to the above argument one orientation-preserving affine automorphism of $\mathbb{R}^N$ which reverses the orientation of K. To do so, first extend $(p_1, p_2, \ldots, p_{n+1})$ to an affine basis of $\mathbb{R}^N$, that is, pick $p_{n+2}, \ldots, p_{N+1}$ such that $(p_2 - p_1, p_3 - p_1, \ldots, p_{N+1} - p_1)$ is a basis of $\mathbb{R}^N$. The affine linear automorphism which exchanges p_1 and p_2 and keeps $p_3, \ldots, p_{N+1}$ fixed maps K to K and is orientation-reversing on K and on $\mathbb{R}^N$. Compose this automorphism with an affine linear reflection along a hyperplane containing K. The resulting affine automorphism maps K to K, reverses the orientation of K, but is orientation-preserving on $\mathbb{R}^N$.

Part (iii) is clear.   □

---

[7]To see this, first assume that σ has a fixed point, say $\sigma(i_0) = i_0$. Write $F(x) = A(x - p_{i_0}) + p_{i_0}$ for a linear map A. Abbreviating $v_i := p_i - p_{i_0}$ we see that $A(v_i) = v_{\sigma(i)}$, showing that $\det(A) > 0$ if and only if σ is even. For $n \geqslant 2$, if σ has no fixed point, it can be written as the composition of two permutations, each of which has a fixed point. For $n = 1$ the statement is clear.

We now turn to Pachner moves. For an n–dimensional triangulated manifold M we consider a subcomplex $K \subset M$ that is isomorphic to a collection of n–simplices $F \subset \partial \Delta^{n+1}$ of the standard $(n+1)$–simplex; we write $\varphi \colon K \to F$ for the isomorphism. Then a *Pachner move* by definition is the replacement

$$(3\text{-}8) \qquad M \mapsto (M \setminus K) \cup_{\varphi|_{\partial K}} (\partial \Delta^{n+1} \setminus \overset{\circ}{F}).$$

Put differently, "it glues the other side of Δ^{n+1} into M instead of K". Since Δ^{n+1} has only finitely many (namely $n+2$) faces, there are only finitely many such moves.

Pachner's theorem [38] states that if two triangulated PL manifolds are PL isomorphic, then there exists a finite sequence of Pachner moves from one triangulation to the other. For fixed n and $k \in \{1, 2, \dots, n+1\}$, we will refer to the Pachner move which replaces k faces $F \subset \partial \Delta^{n+1}$ by the other $n+2-k$ faces in $\partial \Delta^{n+1}$ as a k–$(n+2-k)$ *Pachner move*.

Since any two smooth triangulations of a smooth n–manifold are PL isomorphic (up to an arbitrarily small δ–approximation), Pachner's result directly holds in our setting — if orientations are discarded. To relate triangulations with total order, we have to consider Pachner moves for all possible orders on the vertices of simplices. We will refer to Pachner moves between triangulations with total order as *oriented Pachner moves*.

Proposition 3.3 *For $n \in \mathbb{Z}_+$ and any two finite smooth triangulations with total order of a smooth n–manifold, there is a finite sequence of oriented Pachner moves which takes both to a common refinement with total order, up to an arbitrarily small δ–approximation.*

Proof Given Pachner's theorem, it is sufficient to show that the total order of a given triangulation can be changed to any other total order with finitely many oriented Pachner moves. For this it is enough to see how the height of any given vertex in the triangulation can be changed.

Let $f \colon |C| \to M$ be a triangulation with total order, and let $h \colon C_0 \to \mathbb{R}$ be its height function. For a vertex $v \in C_0$, we want to change the value $h(v) =: \gamma$ to γ' for any $\gamma' \in \mathbb{R} \setminus \{h(v) \mid v \in C_0\}$. Denote by $h' \colon C_0 \to \mathbb{R}$ the height function which only differs from h in v, where it takes the value γ'.

Let $S_v \subset M$ be the image under f of the star of v and let $p = f(v)$. Choose a diffeomorphism ϕ of M which is the identity on the complement of S_v and which maps p to $\tilde{p} \neq p$ (by construction of ϕ, we must have $\tilde{p} \in S_v$). Let the triangulation $\tilde{f} \colon |C| \to M$ be given by $\tilde{f} = \phi \circ f$. Note that $f(v) \neq \tilde{f}(v)$, and that in fact

$f(v) \notin \widetilde{f}(C_0)$. We can find [36, Corollary 10.13] a δ–approximation $f' \colon |C'| \to M$ to $\widetilde{f}$ such that $f'^{-1} \circ f \colon |C| \to |C'|$ is a PL isomorphism. For δ small enough we maintain the property $f(v) \notin f'(C_0')$.

Thanks to Pachner's theorem, there is a finite sequence of Pachner moves from the triangulation $f \colon |C| \to M$ to $f' \colon |C'| \to M$. We turn this into a sequence of oriented Pachner moves by applying the rule that whenever a new vertex is created at the position of one of the vertices in C_0, it must have the corresponding height from C (while the choice of heights at new vertices is arbitrary). Running the same argument (with the same choices for new heights, etc) but starting from h' instead of h, we arrive again at C', equipped with the same height function as before. Indeed, the only difference between h and h' was in the vertex v, which is no longer present in C'. $\square$

Example 3.4 We illustrate oriented Pachner moves and their Poincaré dual stratifications for $n = 2$ and $n = 3$.

(i) For $n = 2$, the $(n+1)$–simplex has four faces, and accordingly there are two types of Pachner moves: those replacing two triangles sharing a single edge by the "other" two faces of the tetrahedron Δ^3, and those replacing one triangle by the "other" three faces of Δ^3 meeting at a single vertex — as well as the inverse operations. These are the 2–2 and 1–3 *moves*, respectively, and locally they look as follows:

$$(3\text{-}9)$$
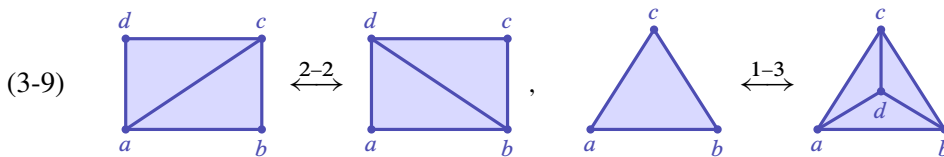
where $a, b, d, c \in \mathbb{R}$ are pairwise distinct. Hence, for fixed heights a, b and c, the unoriented 1–3 Pachner move yields four inequivalent oriented Pachner moves depending on the relative value of d. The Poincaré dual moves are

$$(3\text{-}10)$$
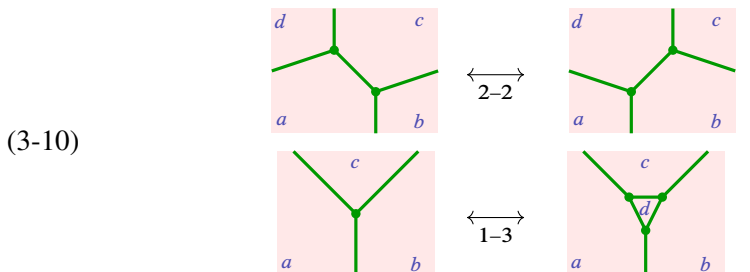
with the induced orientations as explained in Example 3.1(i).

(ii) For $n = 3$, there are again only two types of Pachner moves and their inverses. The 2–3 *move* is applied to two tetrahedra sharing precisely one face, and replaces them with the "other" tetrahedra on the boundary of Δ^4, ie with three tetrahedra that share a single edge:

(3-11)
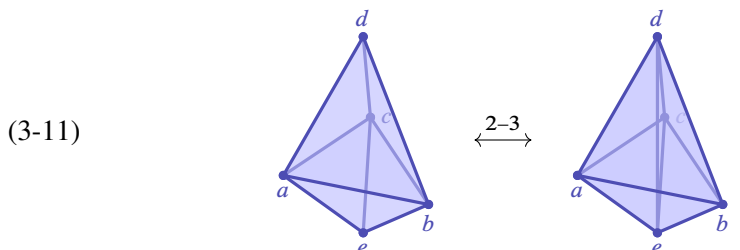
The remaining operation is the 1–4 *move*, which replaces one tetrahedron by four tetrahedra meeting at a single vertex:

(3-12)
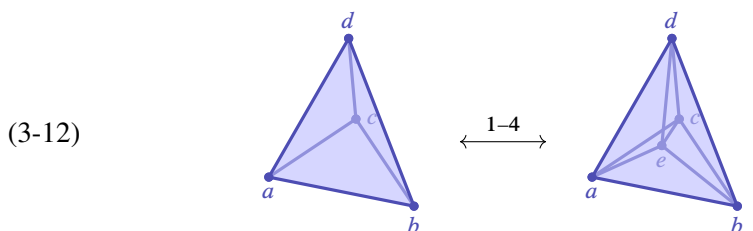
There are five inequivalent oriented 1–4 moves for fixed a, b, c and d, depending on the relative value of the height e. The Poincaré dual move to (3-11) is

(3-13)

,
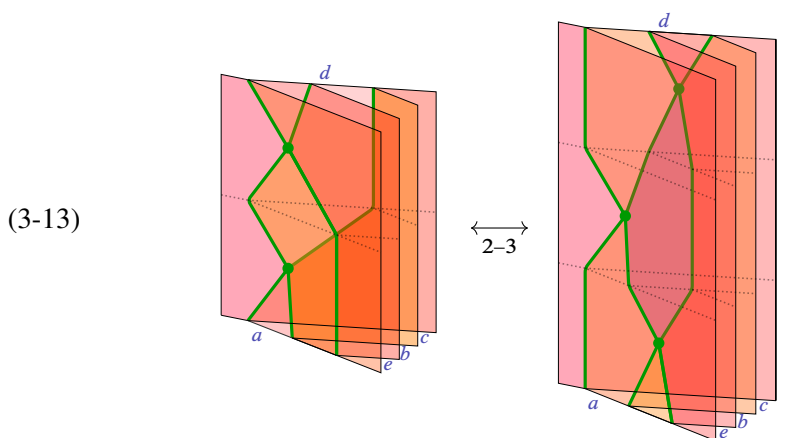
with the induced orientations as explained in Example 3.1(ii). We will not have need in this article to work with the move Poincaré dual to (3-12) directly.

3.2   Orbifolds of n–dimensional TQFTs

For any $n \in \mathbb{Z}_+$, we fix an n–dimensional defect TQFT

$$(3\text{-}14) \qquad\qquad \mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}.$$

We want to use $\mathcal{Z}$ to construct a new closed n–dimensional TQFT, the "orbifold theory"

$$(3\text{-}15) \qquad\qquad \mathcal{Z}_{\mathcal{A}}\colon \mathrm{Bord}_n \to \mathrm{Vect}_{\Bbbk},$$

which depends on a choice of "orbifold datum" $\mathcal{A}$. In this section we will first define what an orbifold datum $\mathcal{A}$ for a given defect TQFT $\mathcal{Z}$ is, and then construct the associated orbifold theory $\mathcal{Z}_{\mathcal{A}}$.[8]

Recall from Lemma 3.2 that for $k < n$ any two oriented k–simplices can be rotated into one another, while there are precisely two oriented n–simplices up to rotation. Hence, from the perspective of a *topological* QFT, there are only two oriented n–simplices and only one oriented k–simplex for every $k \in \{1, \ldots, n-1\}$. Thus, with $j := n-k$, the j–strata of the Poincaré dual of a triangulation in n dimensions can be decorated with the following data:

Definition 3.5   An *orbifold datum* $\mathcal{A}$ for a defect TQFT $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ is a choice of

- an element $\mathcal{A}_j \in D_j$ for all $j \in \{1, 2, \ldots, n\}$,
- two elements $\mathcal{A}_0^+, \mathcal{A}_0^- \in D_0$,

subject to the following constraints (using the notation of Section 2.2):

(i)   **Compatibility**   For $j \in \{1, 2, \ldots, n-1\}$, the representatives of the class $f_j(\mathcal{A}_j) \in [\mathrm{Sphere}_{n-j-1}^{\mathrm{def}}(\partial^{j+1}\mathbb{D})]$ are homeomorphic (as stratified topological manifolds) to the Poincaré dual (in $n-j-1$ dimensions) of the boundary of an $(n-j)$–simplex. The i–strata in $f_j(\mathcal{A}_j)$ are decorated by $\mathcal{A}_{i+j+1}$ for all $i \in \{0, \ldots, n-j-1\}$. Similarly, the classes $f_0(\mathcal{A}_0^+), f_0(\mathcal{A}_0^-) \in [\mathrm{Sphere}_{n-1}^{\mathrm{def}}(\partial\mathbb{D})]$ have representatives that are homeomorphic to the $(n-1)$–dimensional Poincaré dual of the boundary of the two inequivalent oriented n–simplices, respectively, and both $f_0(\mathcal{A}_0^+)$ and $f_0(\mathcal{A}_0^-)$ have their i–strata decorated by $\mathcal{A}_{i+1}$ for all $i \in \{0, \ldots, n-1\}$.

---

[8]It turns out that it is often useful to take $\mathcal{Z}$ in the orbifold construction described in this section to be the Euler completion of some other TQFT $\mathcal{Z}'$, that is, $\mathcal{Z} = (\mathcal{Z}')^{\circlearrowright}$; see Sections 2.5 and 3.4. Indeed, we will see in [16] that certain examples of orbifold data only become available after this completion.

(ii) **Invariance** Let $j \in \{1, \ldots, n+1\}$. For any oriented $j–(n+2-j)$ Pachner move, consider its Poincaré dual move between n–balls B and B', viewed as stratified bordisms in $\mathrm{Bord}_n^{\mathrm{strat}}$ with $\varnothing$ as their common source. B and B' also have a common target, and there is a unique way to make them into defect bordisms by decorating their k–strata with $\mathcal{A}_k$ for all $k \in \{0, 1, \ldots, n\}$. Then we demand the equality of vectors

(3-16) $$\mathcal{Z}(B) = \mathcal{Z}(B').$$

In Sections 3.3 and 3.4 we will spell out both conditions for $n = 2$ and $n = 3$ in detail. For now we note that condition (i) of Definition 3.5 ensures that, as a stratified manifold, the Poincaré dual of a triangulation with total vertex order has a unique decoration by the data in $\mathcal{A}$, and thus that condition (ii) can be consistently stated. In particular,

(3-17) $$(\{\mathcal{A}_n\}, \{\mathcal{A}_{n-1}\}, \ldots, \{\mathcal{A}_1\}, \{\mathcal{A}_0^+, \mathcal{A}_0^-\}; f_{n-1}, f_{n-2}, \ldots, f_0)$$

is a set of defect data in the sense of Definition 2.4. Condition (ii) itself implies that for any bordism M in Bord_n, if we decorate the Poincaré dual of any oriented triangulation t of M with the defect labels from the orbifold datum $\mathcal{A}$, then the evaluation on the resulting defect bordism in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ does not depend on the choice of t (after a limit construction to obtain source and target). This makes the orbifold theory $\mathcal{Z}_\mathcal{A}$ well defined, which is the content of Constructions 3.7–3.9 and Theorem/Definition 3.10.

Remark 3.6 We denote orbifold data by $\mathcal{A}$ as they are to be thought of as certain types of algebras: The element $\mathcal{A}_{n-1}$, which labels $(n-1)$–strata that are Poincaré dual to edges, appears as the underlying "space" of the algebra; its "multiplication" is provided by the element $\mathcal{A}_{n-2}$, which decorates $(n-2)$–strata that are Poincaré dual to a 2–simplex whose three edges (two "ingoing" and one "outgoing" for $n > 2$, and both options for $n = 2$) correspond to the $(n-1)$–strata associated with $\mathcal{A}_{n-1}$. This will be made precise for $n \in \{2, 3\}$ in the categorical formulation of Section 4. For example, for $n = 2$, $\mathcal{A}_1$ is literally an algebra (and coalgebra) with multiplication $\mathcal{A}_0^+$ (and comultiplication $\mathcal{A}_0^-$), while for $n = 3$, $\mathcal{A}_2$ is a monoidal category — ie an algebra in a higher category — with tensor product $\mathcal{A}_1$; see also Sections 3.3 and 3.4.

We begin by defining $\mathcal{Z}_\mathcal{A}(M)$ for a closed oriented n–manifold M, ie a morphism $\varnothing \to \varnothing$ in Bord_n. This is a special case of Construction 3.9 below, but it serves as a good warm-up and it highlights the core part of the full construction:

Construction 3.7 (evaluation of $\mathcal{Z}_{\mathcal{A}}$ on closed manifolds M) Let $\mathcal{A}$ be an orbifold datum for a defect TQFT $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$. The number $\mathcal{Z}_{\mathcal{A}}(M)$ is constructed as follows:

(i) Choose a triangulation t with total order of M, and denote the Poincaré dual stratification by M^t.[9]

(ii) Decorate M^t with the orbifold datum $\mathcal{A}$ to obtain a bordism $M^{t,\mathcal{A}}$ in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$. More precisely, decorate

- every j–stratum with $\mathcal{A}_j$ for all $j \in \{1, 2, \ldots, n\}$,
- every 0–stratum with either $\mathcal{A}_0^+$ or $\mathcal{A}_0^-$, as dictated by the orientation of the 0–stratum.

(iii) Apply $\mathcal{Z}$:

(3-18) $$\mathcal{Z}_{\mathcal{A}}(M) := \mathcal{Z}(M^{t,\mathcal{A}}).$$

By Theorem/Definition 3.10 below, $\mathcal{Z}_{\mathcal{A}}(M)$ is independent of the choice of triangulation with total order.

After the special bordisms $M\colon \varnothing \to \varnothing$, we will now define the functor $\mathcal{Z}_{\mathcal{A}}$ on all of Bord_n. We will obtain $\mathcal{Z}_{\mathcal{A}}$ as a limit construction, which is well established in the literature on state sum constructions (see eg [43; 6; 34; 32]). In fact in [16] we will show that, for $n = 3$, models of Turaev–Viro type are the special case of $\mathcal{Z}_{\mathcal{A}}$ where $\mathcal{Z}$ is taken to be "the trivial defect TQFT", and the orbifold datum $\mathcal{A}$ is extracted from a spherical fusion category.

Construction 3.8 (evaluation of $\mathcal{Z}_{\mathcal{A}}$ on objects) Let $\mathcal{A}$ be an orbifold datum for a defect TQFT $\mathcal{Z}$, and let $\Sigma \in \mathrm{Bord}_n$. We define $\mathcal{Z}_{\mathcal{A}}(\Sigma) \in \mathrm{Vect}_{\Bbbk}$ as follows:

(i) For every triangulation τ with total order of Σ, denote the Poincaré dual stratification by Σ^τ. Decorate Σ^τ with the orbifold datum $\mathcal{A}$. More precisely, decorate every j–stratum of Σ^τ with $\mathcal{A}_{j+1}$. This makes Σ into an object $\Sigma^{\tau,\mathcal{A}} \in \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ for every triangulation τ.[10]

---

[9] As for any Poincaré dual of a triangulation, the stratification M^t of M is only unique up to isotopy, but this ambiguity will be rendered inconsequential in point (iii) below.

[10] Here again the stratification Σ^τ of Σ is only unique up to isotopy. However, as opposed to Construction 3.7, evaluating $\mathcal{Z}$ on the object $\Sigma^{\tau,\mathcal{A}}$ now may depend on the choice of stratification in the isotopy class. But, since different choices lead to isomorphic spaces $\mathcal{Z}(\Sigma^{\tau,\mathcal{A}})$, the ambiguity disappears in the limit construction. We will use this to just speak of "the Poincaré dual" rather than of "a choice of Poincaré dual".

(ii) Let τ and τ' be triangulations as in (i). Consider the cylinder $C_\Sigma = \Sigma \times [0,1]$ viewed as a bordism $\Sigma \to \Sigma$ in Bord_n. Choose an oriented triangulation t of C_Σ extending the triangulations τ and τ' on the ingoing and outgoing boundaries, respectively. Decorate the Poincaré dual C_Σ^t with the orbifold datum $\mathcal{A}$ (analogously to Construction 3.7) to obtain a morphism

$$(3\text{-}19) \qquad C_\Sigma^{t,\mathcal{A}} \colon \Sigma^{\tau,\mathcal{A}} \to \Sigma^{\tau',\mathcal{A}}$$

in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$.

(iii) Note that $\mathcal{Z}(C_\Sigma^{t,\mathcal{A}})$ is independent of the choice of t. We define $\mathcal{Z}_\mathcal{A}(\Sigma)$ to be the limit of $\mathcal{Z}$ applied to (3-19) over all τ, ie $\mathcal{Z}_\mathcal{A}(\Sigma)$ is the universal cone

$$(3\text{-}20)$$

for all triangulations τ and τ'.

To compute $\mathcal{Z}_\mathcal{A}(\Sigma)$ explicitly, note that for $\tau = \tau'$ the linear map $\mathcal{Z}(C_\Sigma^{t,\mathcal{A}})$ is an idempotent. One may take $\mathcal{Z}_\mathcal{A}(\Sigma) = \mathrm{im}\mathcal{Z}(C_\Sigma^{\widetilde{t},\mathcal{A}})$ for a fixed choice of triangulation $\widetilde{\tau}$ and $C_\Sigma^{\widetilde{t},\mathcal{A}} \colon \Sigma^{\widetilde{\tau},\mathcal{A}} \to \Sigma^{\widetilde{\tau},\mathcal{A}}$. The diagonal arrows in (3-20) are then given by evaluating $\mathcal{Z}$ on $C_\Sigma^{t,\mathcal{A}}$ for appropriate t.

Construction 3.9 (action of $\mathcal{Z}_\mathcal{A}$ on morphisms) Let $\mathcal{A}$ be an orbifold datum for a defect TQFT $\mathcal{Z}$, and let $M \colon \Sigma_1 \to \Sigma_2$ be a morphism in Bord_n. We define $M_\cup$ to be M viewed as a bordism $\varnothing \to \Sigma := \Sigma_1^{\mathrm{rev}} \sqcup \Sigma_2$.

(i) For any fixed oriented triangulation τ' of Σ, choose an oriented triangulation t' of $M_\cup$ extending the triangulation τ' on the boundary.

(ii) Decorate the Poincaré dual stratification $M_\cup^{t'}$ with the orbifold datum $\mathcal{A}$ to produce a morphism $M_\cup^{t',\mathcal{A}} \colon \varnothing \to \Sigma^{\tau',\mathcal{A}}$ in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$.

(iii) Repeat steps (i) and (ii) for every triangulation τ'' of Σ to produce a morphism $M_\cup^{t'',\mathcal{A}} \colon \varnothing \to \Sigma^{\tau'',\mathcal{A}}$ in $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$.

(iv)   Note that the defining properties of $\mathcal{A}$ (ensuring triangulation invariance) imply that the diagrams

(3-21)

$$
\begin{array}{ccc}
 & \mathcal{Z}(\varnothing) & \\
\mathcal{Z}(M_{\cup}^{t',\mathcal{A}}) \swarrow & & \searrow \mathcal{Z}(M_{\cup}^{t'',\mathcal{A}}) \\
\mathcal{Z}(\Sigma^{\tau',\mathcal{A}}) & \xrightarrow{\quad \mathcal{Z}(C_{\Sigma}^{t,\mathcal{A}}) \quad} & \mathcal{Z}(\Sigma^{\tau'',\mathcal{A}})
\end{array}
$$

commute, where the cylinders $C_{\Sigma}^{t,\mathcal{A}}$ are as in Construction 3.8.

(v)   Since $\mathcal{Z}(\varnothing) = \Bbbk$ and because $\mathcal{Z}_{\mathcal{A}}(\Sigma)$ was defined to be the universal cone, we have that every face in the associated diagram

(3-22)

$$
\begin{array}{ccc}
 & \Bbbk & \\
\mathcal{Z}(M_{\cup}^{t',\mathcal{A}}) & \quad \exists! & \mathcal{Z}(M_{\cup}^{t'',\mathcal{A}}) \\
 & \mathcal{Z}_{\mathcal{A}}(\Sigma) & \\
\mathcal{Z}(\Sigma^{\tau',\mathcal{A}}) & \xrightarrow{\quad \mathcal{Z}(C_{\Sigma}^{t,\mathcal{A}}) \quad} & \mathcal{Z}(\Sigma^{\tau'',\mathcal{A}})
\end{array}
$$

commutes. Let us write $v_M \in \mathcal{Z}_{\mathcal{A}}(\Sigma)$ for the image of $1 \in \Bbbk$ under the above unique map $\Bbbk \to \mathcal{Z}_{\mathcal{A}}(\Sigma)$. Then we define the linear map

(3-23)
$$
\mathcal{Z}_{\mathcal{A}}(M)\colon \mathcal{Z}_{\mathcal{A}}(\Sigma_1) \to \mathcal{Z}_{\mathcal{A}}(\Sigma_2)
$$

to be the image of v_M under the canonical isomorphism $\mathrm{Hom}_{\Bbbk}(\Bbbk, \mathcal{Z}_{\mathcal{A}}(\Sigma)) = \mathrm{Hom}_{\Bbbk}(\Bbbk, \mathcal{Z}_{\mathcal{A}}(\Sigma_1^{\mathrm{rev}}) \otimes_{\Bbbk} \mathcal{Z}_{\mathcal{A}}(\Sigma_2)) \cong \mathrm{Hom}_{\Bbbk}(\mathcal{Z}_{\mathcal{A}}(\Sigma_1), \mathcal{Z}_{\mathcal{A}}(\Sigma_2))$.

To compute v_M (and hence $\mathcal{Z}_{\mathcal{A}}(M)$) explicitly, use the definition of $\mathcal{Z}_{\mathcal{A}}(\Sigma)$ in part (iii) of Construction 3.8 as the image of $C_{\Sigma}^{\tilde{t},\mathcal{A}}\colon \Sigma^{\tilde{\tau},\mathcal{A}} \to \Sigma^{\tilde{\tau},\mathcal{A}}$ under $\mathcal{Z}$. Let $\hat{t}$ be an extension of $\tilde{\tau}$ to $M_{\cup}$. The universal arrow in (3-22) is then given by $\mathcal{Z}(M_{\cup}^{\hat{t},\mathcal{A}})$. Thus, for this choice of $\mathcal{Z}_{\mathcal{A}}(\Sigma)$ we find

(3-24)
$$
v_M = \mathcal{Z}(M_{\cup}^{\hat{t},\mathcal{A}}).
$$

In summary, we have established the following:

Theorem/Definition 3.10 *Let* $\mathcal{Z}$: $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ *be a defect TQFT and let* $\mathcal{A}$ *be an orbifold datum for* $\mathcal{Z}$. *Then the output of Constructions 3.8 and 3.9 is a closed TQFT* $\mathcal{Z}_{\mathcal{A}}$: $\mathrm{Bord}_n \to \mathrm{Vect}_{\Bbbk}$, *which we call the* **$\mathcal{A}$–orbifold theory**.

In Sections 3.3 and 3.4 we will discuss the cases $n = 2$ and especially $n = 3$ in detail.

Remark 3.11 For any symmetric monoidal category $\mathcal{C}$, an *n–dimensional defect TQFT valued in* $\mathcal{C}$ is a symmetric monoidal functor $\mathcal{Z}$: $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathcal{C}$. Apart from the existence of limits, the above orbifold construction does not depend on special properties of the case $\mathcal{C} = \mathrm{Vect}_{\Bbbk}$.

3.3 Orbifolds of 2–dimensional TQFTs

Let us briefly consider the case of a 2–dimensional defect TQFT $\mathcal{Z}$: $\mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ (see also [21; 13] and Section 4.1 below). According to Definition 3.5, an orbifold datum $\mathcal{A}$ for $\mathcal{Z}$ is a list of elements $\mathcal{A}_j \in D_j$ for $j \in \{1, 2\}$ as well as $\mathcal{A}_0^+, \mathcal{A}_0^- \in D_0$ such that, in particular,

(3-25)

are local patches of bordisms in $\mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D})$. For any bordism M in Bord_2 together with a choice of triangulation t with total order, we can decorate the Poincaré dual with $\mathcal{A}$ to obtain a bordism $M^{t,\mathcal{A}}$ in $\mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D})$. Each inner point of $M^{t,\mathcal{A}}$ now has a neighbourhood isomorphic to one of the patches shown in (3-25). The constraints on $\mathcal{A}$ imply that evaluation of $\mathcal{Z}$ on any $\mathcal{A}$–decorated bordism is invariant under the Poincaré dual oriented Pachner moves (3-10).

A "special" type of solution to the constraints on the data in (3-25) to be an orbifold datum for $\mathcal{Z}$ has a well-studied purely algebraic description. Indeed, there is a natural monoidal category $\mathcal{B}_{\mathcal{Z}}(a, a)$ associated to every element $a \in D_2$ (see [19]), and we

may rewrite (3-25) as

$$*:= \mathcal{A}_2, \quad A := \mathcal{A}_1 \in \mathcal{B}_{\mathcal{Z}}(*, *),$$

(3-26)

$$\mu := \mathcal{Z}\left(\begin{array}{c} \mathcal{A}_0^+ \end{array} \right) : A \otimes A \to A,$$

$$\Delta := \mathcal{Z}\left(\begin{array}{c} \mathcal{A}_0^- \end{array} \right) : A \to A \otimes A.$$

It was shown in [13, Proposition 3.4] that a sufficient condition for (3-25) to form an orbifold datum is that (3-26) together with

(3-27) $$\varepsilon := \mathcal{Z}\left(\begin{array}{c} \mathcal{A}_0^- \end{array} \right) : \mathbb{1} \to A, \quad \eta := \mathcal{Z}\left(\begin{array}{c} \mathcal{A}_0 \end{array} \right) : A \to \mathbb{1}$$

form a Δ–separable symmetric Frobenius algebra in $\mathcal{B}_{\mathcal{Z}}(*, *)$ (whose definition we will recall in Section 4.1). The condition of Δ–separability means that $\mu \circ \Delta = \text{id}$; in terms of line defects, this amounts to leaving out a bubble (see the first identity below in (4-2)). While not itself Poincaré dual to a triangulation (instead the dual cell complex has triangles glued to each other along two edges), this "bubble-move" implies the 3–1 Pachner move.

Quite generally, if invariance under the $(n + 1)$–1 Pachner move — the only move that changes the number of simplices — in Definition 3.5 is replaced by invariance under appropriate "bubble moves", we refer to such an orbifold datum as "special". Hence, we define a 2–*dimensional special orbifold datum* for $\mathcal{Z}$ to be a Δ–separable symmetric Frobenius algebra in $\mathcal{B}_{\mathcal{Z}}$. (Note that Δ–separable Frobenius algebras are closely related to special Frobenius algebras [24, Definition 2.3], which is another reason for the term "special orbifold datum".)

3.4 Orbifolds of 3–dimensional TQFTs

Being able to treat 3–dimensional orbifolds was our main motivation for developing the general formalism in Section 3.2. Important earlier work on the significance of defects in 3–dimensional TQFT includes studies of examples in Rozansky–Witten

theory [30; 29], Dijkgraaf–Witten theory [22], Chern–Simons theory [31] and a general analysis of defects in Reshitikhin–Turaev TQFTs [26]. There is a close connection between 3–dimensional TQFTs and $(2+1)$–dimensional topological phases of matter. In the latter context, defects, group symmetries and orbifolding have been investigated eg in [33; 4; 25; 3; 18]. A general 3–categorical algebraic framework to accommodate the structure of defects in 3–dimensional TQFTs is developed in [5; 11], and in Section 4.2 we will place the present approach to orbifolds in this framework.

Fix a 3–dimensional defect TQFT $\mathcal{Z}\colon \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$. According to Definition 3.5, an orbifold datum $\mathcal{A}$ for $\mathcal{Z}$ is a list of elements $\mathcal{A}_j \in D_j$ for $j \in \{1, 2, 3\}$ as well as $\mathcal{A}_0^+, \mathcal{A}_0^- \in D_0$ such that (with all $\mathcal{A}_2$–decorated 2–strata oriented by the blackboard framing)

(3-28)
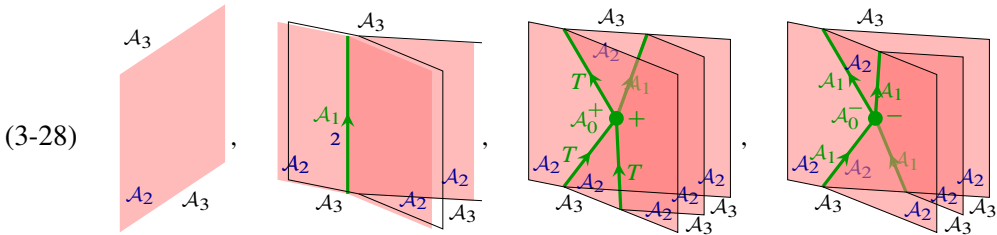
are local patches of bordisms in $\mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D})$. For any bordism M in Bord_3 together with a choice of triangulation t, we can decorate the Poincaré dual with $\mathcal{A}$ to obtain a bordism $M^{t,\mathcal{A}}$ in $\mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D})$. By the constraints on $\mathcal{A}$, $\mathcal{Z}$ is invariant under the Poincaré duals of the oriented Pachner moves (3-11) and (3-12). In particular, $\mathcal{Z}(M^{t,\mathcal{A}})$ is independent of the choice of triangulation t.

Up to rotations there are precisely 20 inequivalent oriented 2–3 moves and ten inequivalent 1–4 moves:

(3-29)
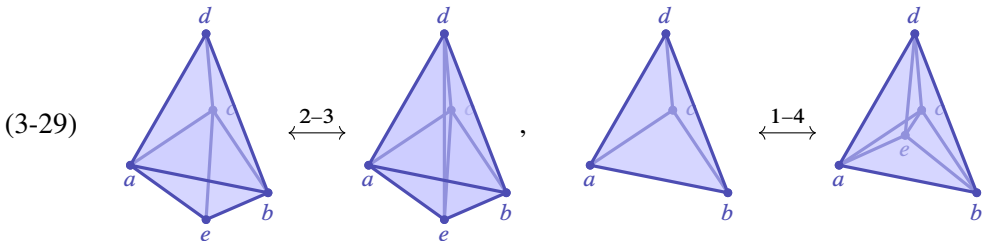
Indeed, for each of the ten inequivalent ways to assign heights d and e to the top and bottom vertices in the 2–3 move, there are up to rotations two ways to assign the remaining heights a, b and c ("clockwise" and "counterclockwise"). Similarly, for each of the two inequivalent oriented tetrahedra on the left-hand side of the 1–4 move

(recall Lemma 3.2(ii)), there are five possibilities to assign a height e to the new vertex on the right-hand side.

3.4.1 Special orbifold data   In this section we will describe a class of special orbifold data for which we demand only ten constraints, which fall into three classes. We will show that these ten constraints imply the above 30 constraints. To reflect the fact that we are dealing with special orbifold data, we will use special notation: we shall from now on write

$$* \text{ for } \mathcal{A}_3, \quad \mathcal{A} \text{ (as in "algebra") for } \mathcal{A}_2, \quad T \text{ (as in "tensor") for } \mathcal{A}_1,$$

$$\alpha \in \mathcal{Z}(S^2_{\mathcal{A},T}) \text{ and } \bar{\alpha} \in \mathcal{Z}((S^2_{\mathcal{A},T})^{\text{rev}}) \text{ (as in "associator") instead of } \mathcal{A}_0^+ \text{ and } rb_0^-,$$

where $S^2_{\mathcal{A},T}, (S^2_{\mathcal{A},T})^{\text{rev}} \in \text{Bord}_3^{\text{def}}(\mathbb{D})$ are

$$(3\text{-}30) \qquad S^2_{\mathcal{A},T} := \qquad\qquad , \quad (S^2_{\mathcal{A},T})^{\text{rev}} = \qquad\qquad .$$

We will now discuss and motivate the ten constraints imposed on the special orbifold datum $\mathcal{A} \equiv (*, \mathcal{A}, T, \alpha, \bar{\alpha})$. Then in Definition 3.13 we shall present the concise characterisation.

The first constraint on $\mathcal{A}$ is a single 2–3 move, namely the one involving only vertices of type α (and none of type $\bar{\alpha}$):

$$(3\text{-}31) \qquad \text{under } \mathcal{Z}: \qquad\qquad = \qquad\qquad .$$

Put differently, the two stratified 3–balls containing the two sides of (3-31) (viewed as bordisms from the empty set to their boundary spheres) evaluate identically under $\mathcal{Z}$. In examples, this amounts to a pentagon condition on α.

The next type of constraint on the data $(*, \mathcal{A}, T, \alpha, \overline{\alpha})$ demands that "if two tetrahedra in the cell decomposition Poincaré dual to the stratification by defects share two faces, the tetrahedra may be replaced by just the other two faces". We note immediately that in a triangulation any two tetrahedra can share at most one face, so applying this constraint leaves the realm of triangulations to one of more general stratifications. In Section 3.4.2 we will show how transiently leaving the domain of triangulations in this way will guarantee invariance under all 20 oriented 2–3 moves.

We want to make the above constraint precise. Let $a, b, c, d \in \mathbb{R}$ be pairwise distinct numbers, and consider the two tetrahedra

(3-32)
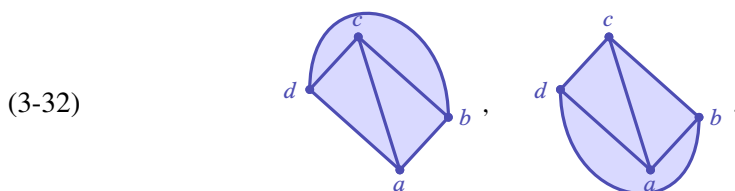
In these pictures we took a tetrahedron as in (3-4) and moved all of its vertices into the same 2–dimensional plane; in doing so we turned one of its edges into an arc to keep the tetrahedron from becoming degenerate. We may either glue the two tetrahedra along the two faces (abc) and (acd), or along (abd) and (bcd). (The second gluing is easier to visualise if one moves the corresponding faces into the same plane, thereby turning the opposite straight edge into an arc.) In either case we consider the move which replaces the two tetrahedra with the two nonshared faces (note that these moves decrease the number of 2– and 3–strata):

(3-33)
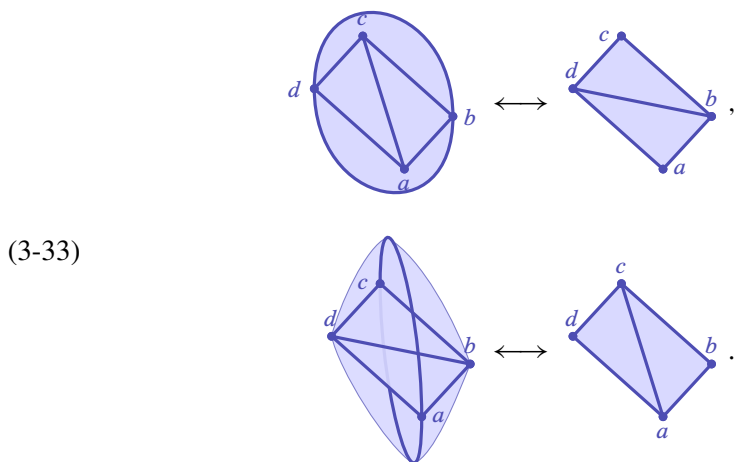
Note that these two figures are just two different ways of visualising the same move, but rotated by $90°$ relative to each other.

Lemma 3.12 *For $\{a, b, c, d\} = \{1, 2, 3, 4\}$ a move between oriented stratifications of the above type is up to rotation one of the following six* (the figures are rotated so that the highest vertex is the rightmost one)*:*

(3-34)

(3-35)

(3-36)
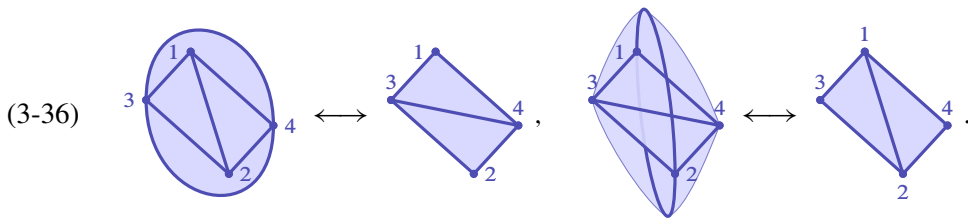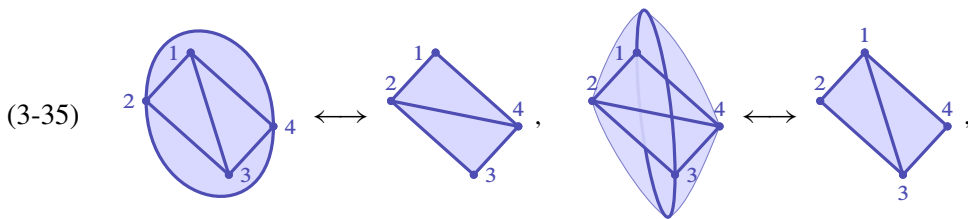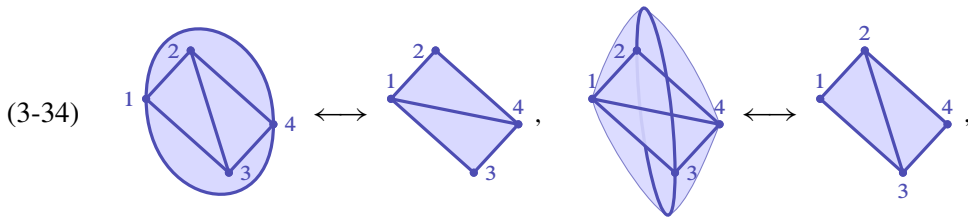
Proof Up to rotation, there are two choices where to place the highest number 4 on the vertices of the two tetrahedra:

(3-37)
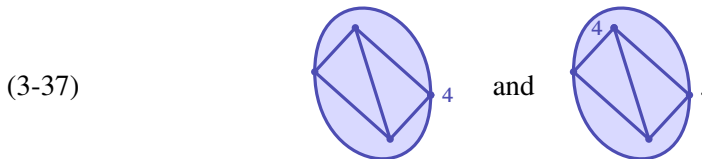
This amounts to the first and second column in the above list of moves.

Consider the left choice in (3-37). Up to rotation leaving the vertex with label 4 fixed, there are two choices on which vertex to place the label 1. One of these choices leaves
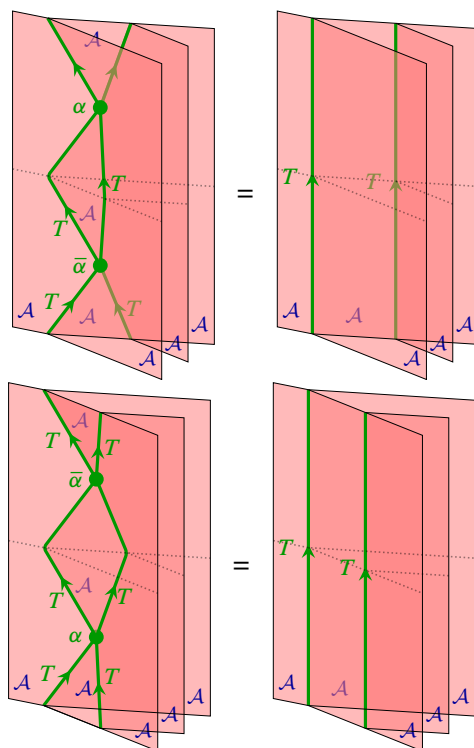
Figure 1

two inequivalent options how to distribute the remaining labels 2 and 3, the other leaves only one choice. This leaves us with the moves in the left column of (3-34)–(3-36).

The right column follows analogously from the right stratification in (3-37). □

We now translate the moves of Lemma 3.12 into constraints on the data $(*, \mathcal{A}, T, \alpha, \overline{\alpha})$. For this we pass to the stratifications Poincaré dual to the stratifications in (3-34)–(3-36), and demand that evaluation by $\mathcal{Z}$ is invariant under these moves. Hence, the constraints corresponding to these moves are that under $\mathcal{Z}$ we have the equalities of Figures 1, 2 and 3, and the stripy patterns in Figure 2 again indicate 2–strata oriented opposite to the paper plane.

Finally, we have to ensure invariance under a type of bubble move: Our last constraint on $(*, \mathcal{A}, T, \alpha, \overline{\alpha})$, is that, under the functor $\mathcal{Z}$:

(3-38)

where in the first three pictures two $\mathcal{A}$–labelled hemispheres are glued together along a T–line. In the first picture, all 2–strata have the same orientation as the paper plane, in the second picture the rear hemisphere has opposite orientation, while in the third picture it is the front hemisphere.

Now we collect all of the above in a single notion. As in Example 2.5 we shall work with the *source* and *target maps* $s, t\colon D_2 \to D_3$, defined via $f_2(x) = (s(x), t(x))$, and the *folding map* $f := f_1$, where f_j are the adjacency maps of the defect data $\mathbb{D}$.

Definition 3.13 Given a 3–dimensional defect TQFT $\mathcal{Z}\colon \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$, a *special orbifold datum* $\mathcal{A} \equiv (*, \mathcal{A}, T, \alpha, \bar{\alpha})$ for $\mathcal{Z}$ is a choice of

- $* \in D_3$,

Figure 2

Figure 3

- $\mathcal{A} \in D_2$ with $s(\mathcal{A}) = t(\mathcal{A}) = *$,

- $T \in D_1$ with $f(T) = (\mathcal{A}, +) \times (\mathcal{A}, +) \times (\mathcal{A}, -)$,

- $\alpha \in \mathcal{Z}(S^2_{\mathcal{A},T})$ and $\bar{\alpha} \in \mathcal{Z}(\bar{S}^2_{\mathcal{A},T})$ as in (3-30)

such that the constraints (3-31), Figures 1–3 and (3-38) are satisfied.

Remark 3.14 Here we spell out what a special orbifold datum for the Euler completion $\mathcal{Z}^{\odot} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}^{\odot}) \to \mathrm{Vect}_{\Bbbk}$ of a TQFT $\mathcal{Z} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ means in terms of $\mathcal{Z}$ and $\mathbb{D}$ directly. Recall Definition 2.22 for the Euler-completed defect data $\mathbb{D}^{\odot}$, and Definition 2.24 for $\mathcal{Z}^{\odot}$.

(i) By definition, to specify a special orbifold datum $\mathcal{A}^{\odot}$ for $\mathcal{Z}^{\odot}$ we first have to provide one element in each of $D_3^{\odot}$, $D_2^{\odot}$ and $D_1^{\odot}$ as well as two elements $\alpha, \bar{\alpha} \in D_0^{\odot}$. Recall that for $i > 0$, elements in $D_i^{\odot}$ are tuples of the form (x, ϕ, Ψ), where $x \in D_i$, ϕ is an invertible element in the algebra of point insertions on x, ie $\phi \in A_x^{\times}$ (see (2-81) and Proposition 2.19), and Ψ is a tuple $(\psi_S)_{S \in \mathrm{Strat}(f_i(x))}$ with $\psi_S \in A_y^{\times}$ and y the

label of the stratum S of $f_i(x)$. In the top dimension $i = 3$, the tuple Ψ is empty, so the 3–dimensional label of a special orbifold datum $\mathcal{A}^{\odot}$ for $\mathcal{Z}^{\odot}$ is of the form

$$(3\text{-}39) \qquad (*, \phi) \in D_3^{\odot}: \quad * \in D_3, \quad \phi \in \mathcal{Z}\left(\vcenter{\hbox{}} \right)^{\times}.$$

The 2–dimensional label for $\mathcal{A}^{\odot}$ hence must be of the form

$$(3\text{-}40) \quad (\mathcal{A}, \psi, (\phi, \phi)) \in D_2^{\odot}: \quad \mathcal{A} \in D_2, \quad s(\mathcal{A}) = t(\mathcal{A}) = *, \quad \psi \in \mathcal{Z}\left(\vcenter{\hbox{}} \right)^{\times}$$

as the Euler weights on both sides of an $(\mathcal{A}, \psi)$–labelled 2–stratum must be ϕ.

Thanks to Lemma 2.28 ($\mathcal{Z}^{\odot \geqslant 2} \sim \mathcal{Z}^{\odot}$), point insertions on line defects need not to be considered. Hence, a 1–dimensional label for our $\mathcal{A}^{\odot}$ can always be taken to be (T, Ψ), where T is as in Definition 3.13, and the tuple $\Psi = (\phi^{\times 3}, \psi^{\times 3})$ keeps track of the Euler weights inserted on the three 3–strata and the three 2–strata adjacent to the T–labelled 1–stratum:

$$(3\text{-}41) \quad (T, (\phi^{\times 3}, \psi^{\times 3})) \in D_1^{\odot}: \quad T \in D_1, \quad f(T) = (\mathcal{A}, +) \times (\mathcal{A}, +) \times (\mathcal{A}, -).$$

Finally, the 0–dimensional labels in the special orbifold datum $\mathcal{A}^{\odot}$ for $\mathcal{Z}^{\odot}$ features elements

$$(3\text{-}42) \qquad \alpha \in \mathcal{Z}\left(\vcenter{\hbox{}} \right)^{\times}, \quad \bar{\alpha} \in \mathcal{Z}\left(\vcenter{\hbox{}} \right)^{\times}$$

such that we have

$$(3\text{-}43) \qquad (\alpha, (\phi^{\times 4}, \psi^{\times 6})), (\bar{\alpha}, (\phi^{\times 4}, \psi^{\times 6})) \in D_0^{\odot}.$$

(ii)   The constraints on a special orbifold datum of the form (3-39)–(3-41) and (3-43) are that, under $\mathcal{Z}$, we have the equalities of Figures 4–8, where in Figure 8 there is an insertion of ψ^2 on each of the two hemispheres ending on a T–line, and the enclosed 3–strata feature one ϕ–insertion. Note that all of the above are identities between elements in vector spaces $\mathcal{Z}(B)$ for suitable decorated stratified 3–balls B.

(iii)   A datum $(*, \mathcal{A}, T, \alpha, \bar{\alpha})$ for $\mathcal{Z}$ that satisfies the constraints of Definition 3.13 only "up to normalisations" can sometimes be adjusted to form a special orbifold datum for the completion $\mathcal{Z}^{\odot}$. This amounts to finding invertible elements ϕ and ψ
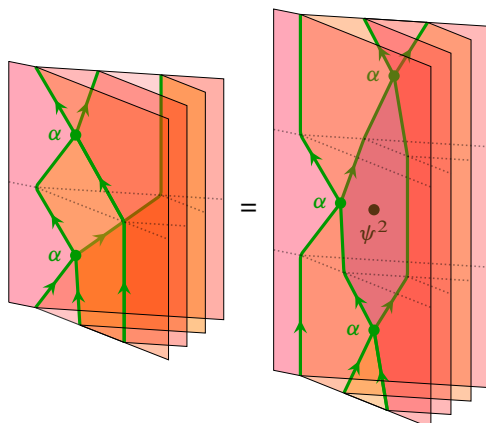
Figure 4

as in (3-39)–(3-40) which satisfy the constraints of Figures 4–8. If α is invertible, such field insertions ϕ and ψ are "unique for practical purposes" in the sense that their contribution to the orbifold theory $(\mathcal{Z}^{\circlearrowright})_{\mathcal{A}}$ can be determined from the data $(*, \mathcal{A}, T, \alpha)$ alone, as follows.

By pre- and postcomposing with α^{-1} in Figure 4 we find that $\mathcal{Z}$ evaluated on

(3-44)

is given by $\mathcal{Z}$ evaluated on a 3–ball containing only the known data $(*, \mathcal{A}, T, \alpha, \alpha^{-1})$.
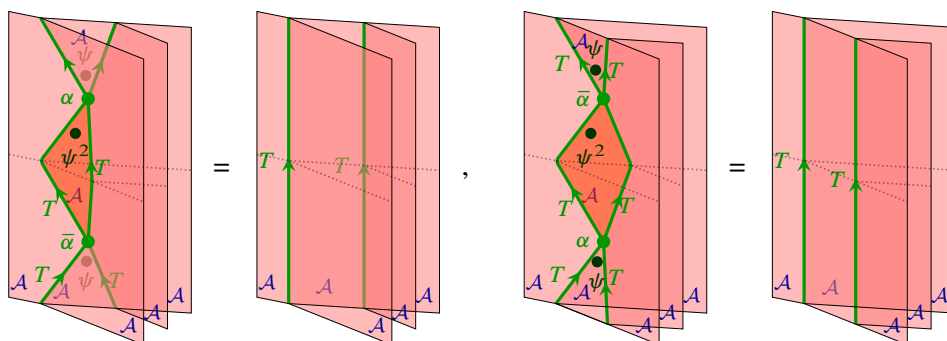
Figure 5
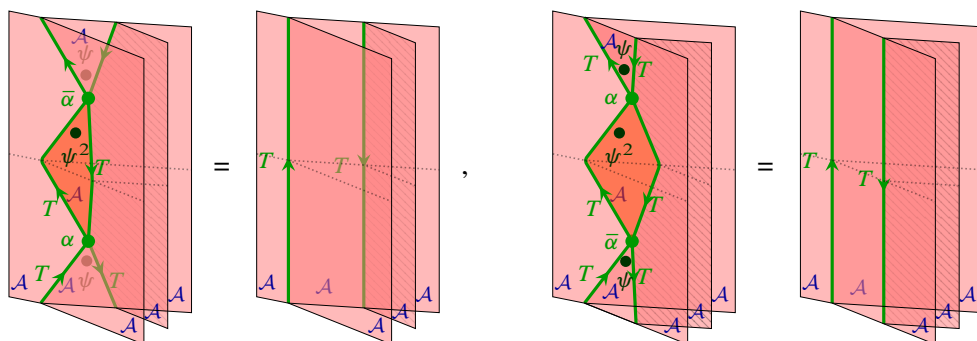
Figure 6

Furthermore, in the construction of $(\mathcal{Z}^{\circ})_{\mathcal{A}}$, the only local neighbourhood ψ will ever appear in is that of (3-44). Hence, knowing the action of $\mathcal{Z}$ on the 3–ball containing it is sufficient.

The insertion ϕ compensates for the T–rimmed bubble in Figure 8. In the examples we are considering in [16], this condition amounts to the computation of a quantum dimension and determines ϕ uniquely.
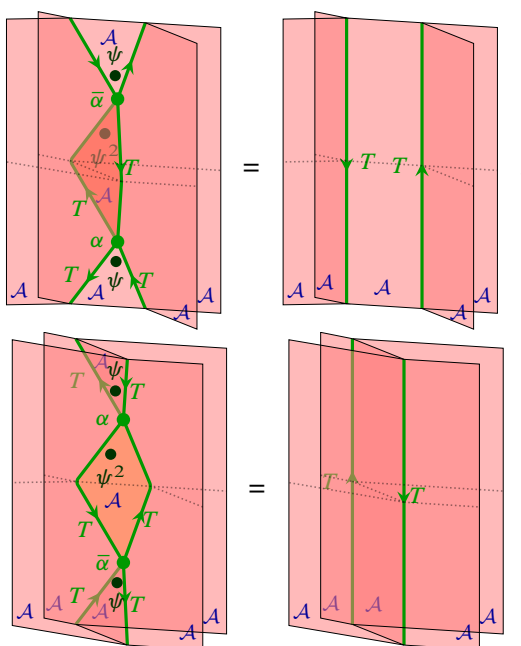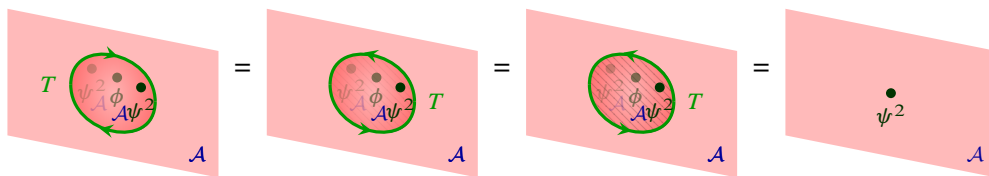
Figure 7

Figure 8

3.4.2 Special orbifold data are orbifold data Let $\mathcal{A} \equiv (*, \mathcal{A}, T, \alpha, \bar{\alpha})$ be a special orbifold datum for a defect TQFT $\mathcal{Z}\colon \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$. In a series of lemmas we will now show that $\mathcal{A}$ really is an orbifold datum in the sense of Definition 3.5. Hence, the orbifold theory $\mathcal{Z}_{\mathcal{A}}$ of Theorem/Definition 3.10 is well defined. We start with checking the remaining 19 oriented 2–3 moves:

Lemma 3.15 *The invariance condition* (ii) *in Definition 3.5 holds for all oriented 2–3 Pachner moves.*

Proof We need to show (3-16), ie that the Poincaré dual of the oriented 2–3 Pachner moves holds inside 3–balls after evaluating with $\mathcal{Z}$.

One of the defining conditions for special orbifold data $\mathcal{A}$, namely the constraint (3-31), is precisely (3-16) under the oriented 2–3 move

$$(3\text{-}45)$$
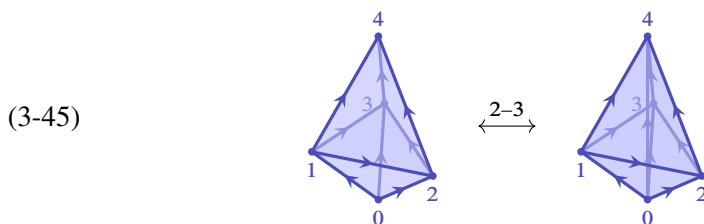
which only involves tetrahedra of type α. To show invariance under the remaining 19 oriented 2–3 moves we use the conditions of Figures 1–3 repeatedly.

For example, we may glue another tetrahedron to two faces on both sides of (3-45) by adding a new edge between the vertices labelled 4 and 0. The left-hand side then isotopically deforms into three tetrahedra joined along the edge $(1\,2)$:

$$(3\text{-}46)$$
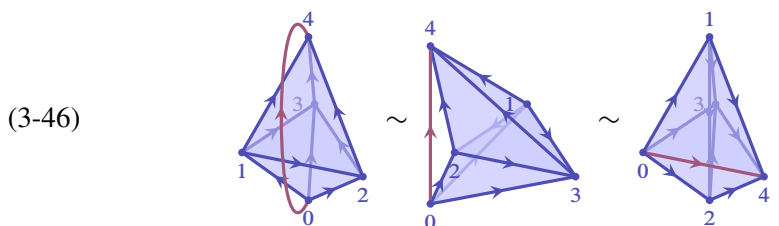
On the other hand, after gluing the new tetrahedron onto the right-hand side of (3-45), we can first use condition (3-34) and then rearrange isotopically:

(3-47)
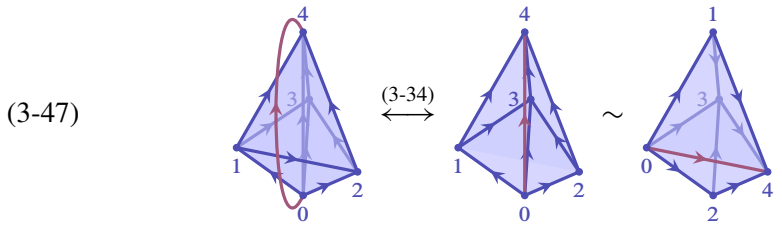
Combined, we have established invariance under the 2–3 move

(3-48)
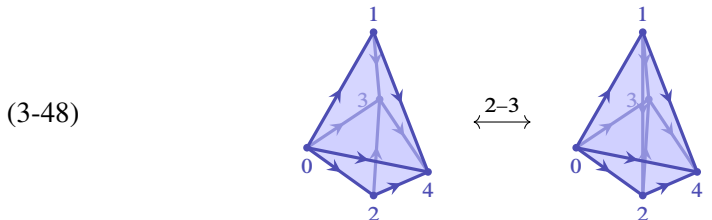
Repeating this argument with all three possible new edges on all incrementally established 2–3 moves produces all 20 inequivalent oriented 2–3 moves. More precisely, if we use the shorthand notation $\binom{a}{b}_\circlearrowleft$ and $\binom{a}{b}_\circlearrowright$ for the moves

(3-49)
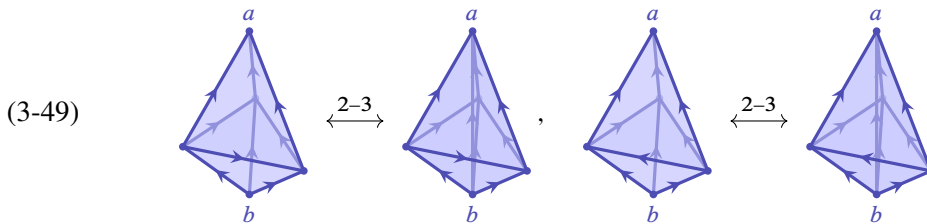
respectively, then we may abbreviate the above derivation of (3-48) from (3-45) as $\binom{4}{0}_\circlearrowleft \to \binom{1}{2}_\circlearrowright$. By applying the moves of Lemma 3.12 to the other two shared edges in (3-45), we obtain $\binom{2}{3}_\circlearrowright$ and $\binom{1}{3}_\circlearrowleft$ from $\binom{4}{0}_\circlearrowleft$. We summarise this as

$$(3\text{-}50) \qquad \binom{4}{0}_\circlearrowleft \to \binom{1}{2}_\circlearrowright, \binom{2}{3}_\circlearrowright, \binom{1}{3}_\circlearrowleft.$$

Similarly, one finds

$$
\begin{aligned}
(3\text{-}51) \qquad & \binom{1}{2}_\circlearrowright \to \binom{3}{4}_\circlearrowleft, \binom{3}{0}_\circlearrowleft, \quad \binom{2}{3}_\circlearrowright \to \binom{1}{0}_\circlearrowright, \binom{1}{4}_\circlearrowleft, \quad \binom{1}{3}_\circlearrowleft \to \binom{2}{0}_\circlearrowleft, \binom{2}{4}_\circlearrowright, \\
& \binom{1}{0}_\circlearrowright \to \binom{2}{4}_\circlearrowleft, \binom{3}{4}_\circlearrowright, \quad \binom{1}{4}_\circlearrowleft \to \binom{3}{0}_\circlearrowleft, \\
& \binom{2}{0}_\circlearrowleft \to \binom{1}{4}_\circlearrowright, \quad \binom{2}{4}_\circlearrowright \to \binom{1}{0}_\circlearrowleft, \\
& \binom{2}{4}_\circlearrowleft \to \binom{1}{3}_\circlearrowright, \quad \binom{3}{4}_\circlearrowright \to \binom{1}{2}_\circlearrowleft \to \binom{4}{0}_\circlearrowright, \quad \binom{1}{4}_\circlearrowright \to \binom{2}{3}_\circlearrowleft, \binom{2}{0}_\circlearrowright,
\end{aligned}
$$

establishing all 20 oriented 2–3 moves.   □

To verify invariance under all oriented 1–4 moves, we use an auxiliary lemma which we learned from [2]. It features the *bubble moves*, which are operations on oriented stratified 3–dimensional manifolds which locally act by replacing the two stratifications (2-1) and (2-2) for all possible orientations,

(3-52)
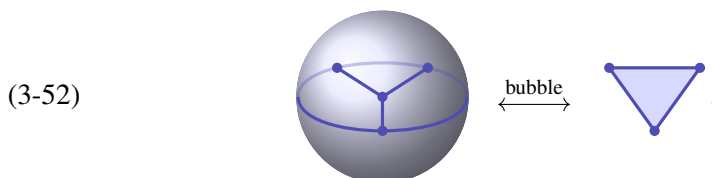
These are Poincaré dual to

(3-53)
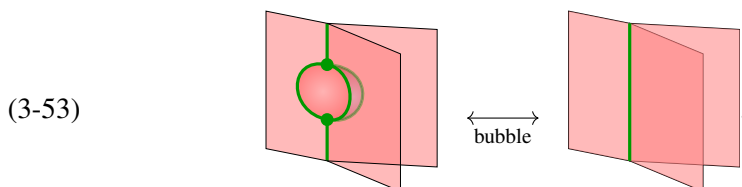
which we consider for all possible orientations that are consistent with $\mathcal{A}$–decorations.

Lemma 3.16 *A 1–4 move is a concatenation of a bubble move and a 2–3 move.*

Proof Given a tetrahedron, we pick one of its faces and apply the bubble move (3-52) to it. The result are three tetrahedra, one pair of which shares three faces, and another pair shares a single face. To the latter pair we apply the 2–3 move (3-11), producing a total of four tetrahedra meeting at a single vertex:
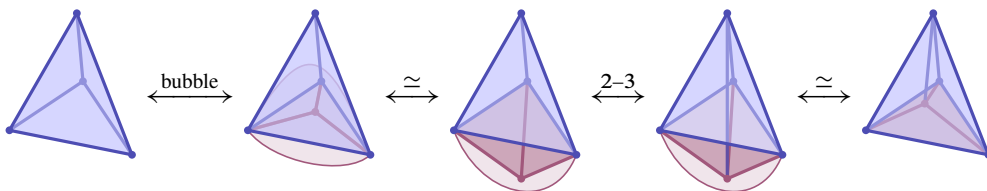
Thus, the lemma is proved. □

As a corollary, the above lemma shows that all oriented 1–4 moves are implied by the oriented bubble and 2–3 moves.

Lemma 3.17 *The invariance condition* (ii) *in Definition 3.5 holds for all oriented 1–4 Pachner moves.*

Proof Thanks to Lemmas 3.15 and 3.16 it suffices to show invariance under the bubble moves (3-52). For example, let us consider

(3-54)

where here and below to avoid clutter, we do not show the labels $\mathcal{A}$, T, α and $\bar{\alpha}$. Using isotopy invariance of $\mathcal{Z}$ as well as the defining properties of orbifold data, we find that, under $\mathcal{Z}$,

(3-55)
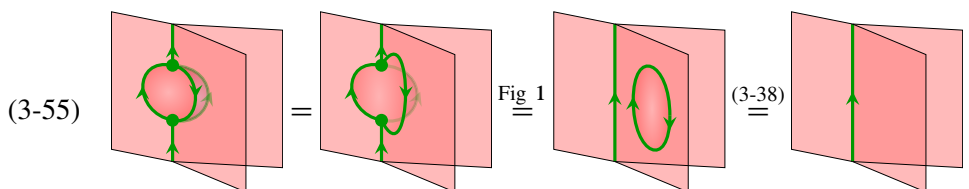
This establishes invariance under one of the oriented bubble moves and the proof of the other ones works analogously. $\square$

Thanks to Lemmas 3.15 and 3.17, we can now conclude:

Proposition 3.18 *A special orbifold datum for a TQFT* $\mathcal{Z}\colon \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\mathbb{k}}$ *in the sense of Definition 3.13 is an orbifold datum in the sense of Definition 3.5.*

In particular, by Theorem/Definition 3.10, a special orbifold datum $\mathcal{A}$ defines a closed TQFT $\mathcal{Z}_{\mathcal{A}}\colon \mathrm{Bord}_3 \to \mathrm{Vect}_{\mathbb{k}}$.

Remark 3.19 In [16] we will construct several examples of special orbifold data $\mathcal{A}$ and the associated orbifold theories. These include the following:

(i) $\mathcal{A}$ is extracted from the data of a spherical fusion category (which in turn is precisely the input data for Turaev–Viro models [43; 6]). This is a 3–dimensional analogue of viewing 2–dimensional state sum models [1; 27] as orbifolds via Δ–separable symmetric $\mathbb{k}$–algebras [19; 13]. More generally, one can think of special orbifold data for a 3–dimensional defect TQFT $\mathcal{Z}$ as "spherical fusion categories internal to a Gray category with duals" (see Section 4.2).

   (ii) $\mathcal{A}$ is extracted from a certain type of special symmetric Frobenius algebra internal to a modular tensor category, using an extension of the Reshetikhin–Turaev construction.

  (iii) $\mathcal{A}$ is extracted from a "surface defect with invertible bubble". This is an analogue of the Barr–Beck-type construction with invertible quantum dimensions of [13], which in fact generalises to arbitrary dimension n.

4 Higher categorical formulation

It is expected that the sets of defect labels in an n–dimensional defect TQFT arrange themselves into a "fairly strict n–category with duals" for any $n \in \mathbb{Z}_+$. This has been made precise for $n = 2$ and $n = 3$:

- It was shown in [19] that one can naturally extract a strictly pivotal 2–category from a 2–dimensional defect TQFT, and every bicategory with ambidextrous duals for 1–morphisms is biequivalent to a strictly pivotal 2–category, as follows from [7; 37]. Hence, for $n = 2$ "fairly strict" means strict, which in this case is the same as "as strict as achievable by biequivalence".

- For $n = 3$, it was shown in [11] that one naturally obtains a Gray category with duals from every 3–dimensional defect TQFT. This is generically the strictest form of 3–categorical structure (where the only nonidentity coherence 3–morphism that is allowed is in the interchange law for 2–morphisms) with ambidextrous duals for all 1– and 2–morphisms, and every tricategory with ambidextrous duals is triequivalent to a Gray category with duals; see [28; 5; 41]. In this sense, for $n = 3$ "fairly strict" means "as strict as achievable by triequivalence".

For $n > 3$ we may expect a similar state of affairs. However, to extract a specific notion of n–category from a given defect TQFT is no simple combinatorial task: Any flavour of n–category comes with a prescribed shape that is used to define a notion of source and target for all k–morphisms for $k \leqslant n$. An element $x \in D_k$ should correspond to an $(n-k)$–morphism, but it does so only after picking a certain decomposition of a sphere in $f_k(x)$ into the prescribed shape. The dualities in the category should then allow one to relate the morphisms defined by different decompositions. Instead of formalising the involved combinatorics, the notion of *disc-like n–category* was invented in [35], where now a $(k-1)$–sphere serves as the combined source and target for a k–morphism; see Remark 4.5 for a more detailed discussion.

In Sections 4.1 and 4.2 below we will reformulate the orbifold constructions of Sections 3.3 and 3.4 in 2– and 3–categorical language, respectively. In fact, we formalise the notion of special orbifold data internal to arbitrary pivotal bicategories and Gray categories with duals (which need not be associatedwith defect TQFTs).

4.1   Special orbifold data in pivotal bicategories

Recall from [19] that to every defect TQFT $\mathcal{Z}\colon \mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\mathbb{k}}$ one can naturally associate a 2–category $\mathcal{B}_{\mathcal{Z}}$. The objects of $\mathcal{B}_{\mathcal{Z}}$ are elements of D_2, and are to be thought of as D_2–decorated planes; 1–morphisms are lists of elements in $D_1 \times \{\pm\}$ (the signs encode the orientation of 1–strata), which we picture as parallel D_1–decorated lines. The 2–morphisms are $\mathbb{k}$–vector spaces which $\mathcal{Z}$ assigns to certain decorated circles. In fact, $\mathcal{B}_{\mathcal{Z}}$ has identical left and right adjoints for all 1–morphisms (corresponding to orientation reversal of 1–strata) and is in fact a pivotal 2–category. This construction is reviewed in detail in [10], which also discusses examples of pivotal bicategories from algebraic and symplectic geometry, differential graded algebras and categorified quantum groups.

Now let $(*, A, \mu, \eta, \Delta, \varepsilon)$ be a special orbifold datum for $\mathcal{Z}$ as in Section 3.3. In terms of $\mathcal{B}_{\mathcal{Z}}$, this means that A is a 1–endomorphism of the object $* \in \mathcal{B}_{\mathcal{Z}}$, and we have 2–morphisms $\mu\colon A \otimes A \to A$, $\eta\colon 1_* \to A$, $\Delta\colon A \to A \otimes A$ and $\varepsilon\colon A \to 1_*$. The constraints on these data, written in standard graphical calculus (with diagrams read from bottom to top), are as follows:

(4-1)

(4-2)

In other words, a special orbifold datum A for $\mathcal{Z}$ — the data needed to define the orbifold TQFT $\mathcal{Z}_A\colon \mathrm{Bord}_2 \to \mathrm{Vect}_{\mathbb{k}}$ — is a Δ–separable symmetric Frobenius algebra in $\mathcal{B}_{\mathcal{Z}}$.

Remark 4.1   The above realisation, originally due to [21] in the context of conformal field theory, is the starting point of the "orbifold completion" construction of [13]: The defect TQFT $\mathcal{Z}$ restricted to Bord_2, viewed as a nonfull subcategory of $\mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D})$ where all objects and morphisms are exclusively decorated by $* \in D_2$, is a closed TQFT.

One may ask whether there is a natural defect TQFT $\mathcal{Z}^{\mathrm{orb}}$ which analogously restricts to the closed orbifold TQFT $\mathcal{Z}_A$. Indeed, as explained in [13], the orbifold construction naturally lifts to produce the "complete" orbifold defect TQFT $\mathcal{Z}^{\mathrm{orb}}\colon \mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}^{\mathrm{orb}}) \to \mathrm{Vect}_{\Bbbk}$, where D_2^{orb} and D_1^{orb} are given by Δ–separable symmetric Frobenius algebras in $\mathcal{B}_{\mathcal{Z}}$ and their bimodules, respectively. Algebraically, for every pivotal bicategory $\mathcal{P}$ with idempotent complete morphism categories this construction motivates the definition of the orbifold completion $\mathcal{P}_{\mathrm{orb}}$ as the bicategory of Δ–separable symmetric Frobenius algebras, bimodules and bimodule maps internal to $\mathcal{P}$ [13, Section 5.1]. Then one has $\mathcal{B}_{(\mathcal{Z}^{\mathrm{orb}})} \cong (\mathcal{B}_{\mathcal{Z}})_{\mathrm{orb}}$ and $(\mathcal{P}_{\mathrm{orb}})_{\mathrm{orb}} \cong \mathcal{P}_{\mathrm{orb}}$.

## 4.2 Special orbifold data in Gray categories with duals

In analogy to the 2–dimensional case of Section 4.1, to a defect TQFT $\mathcal{Z}\colon \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ one can naturally associate a 3–categorical structure $\mathcal{T}_{\mathcal{Z}}$, as explained in [11]. Indeed, it was shown there that $\mathcal{T}_{\mathcal{Z}}$ has the structure of a Gray category with duals. We refer to [11] for the detailed construction of $\mathcal{T}_{\mathcal{Z}}$. Here we only recall some of the basic structure, so that we can formulate the notion of special orbifold data for $\mathcal{Z}$ internal to $\mathcal{T}_{\mathcal{Z}}$.

Roughly, the objects of $\mathcal{T}_{\mathcal{Z}}$ are elements of D_3, which we imagine as D_3–decorated patches of $\mathbb{R}^3$, say 3–cubes. In this picture, 1–morphisms are stacks of D_2–decorated planes, with the spaces in between them decorated compatibly with the maps s and t. Similarly, 2–morphisms can be represented as D_1–decorated lines with D_2–decorated planes ending on them as allowed by the adjacency map f_1. Hence, we may depict objects u, 1–morphisms $\alpha\colon u \to v$ and 2–morphisms $X\colon \beta \to \gamma$ in $\mathcal{T}_{\mathcal{Z}}$ as decorated cubes such as
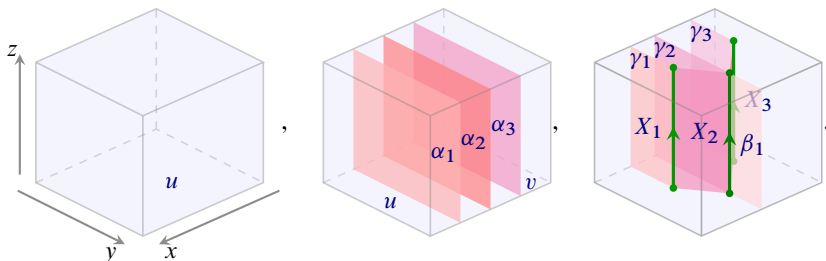

The functor $\mathcal{Z}$ is only used to construct the Hom sets for every two parallel 2–morphisms X and Y, namely as the vector space which $\mathcal{Z}$ assigns to the sphere around the potential meeting point of the lines X and Y, decorated accordingly.

Composition of 1–morphisms in $\mathcal{T}_{\mathcal{Z}}$, denoted by $\square$, corresponds to stacking D_2–decorated planes such as
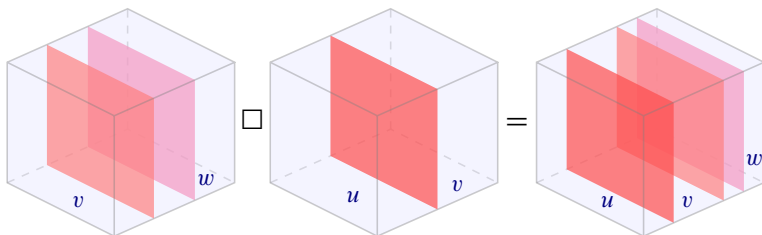

while horizontal composition of 2–morphisms, denoted by $\otimes$, is concatenation in the negative y–direction:
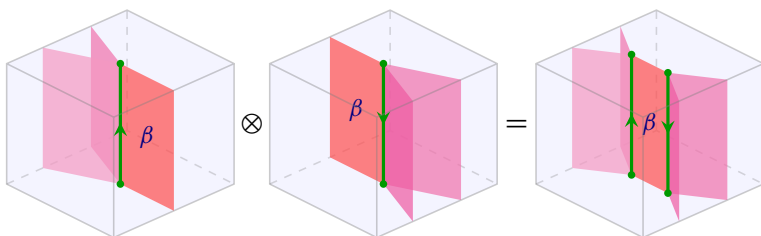

Vertical composition of 3–morphisms (as well as $\square$– and $\otimes$–composition of 3–morphisms) corresponds to cutting and pasting appropriately decorated 3–balls.

Orientation reversal of planes (1–morphisms) and lines (2–morphisms) gives rise to two distinct notions of duals in $\mathcal{T}_{\mathcal{Z}}$, #–duals and †–duals, respectively, which are compatible with one another in a natural way. In the graphical calculus, for 1–morphisms and 2–morphisms we have
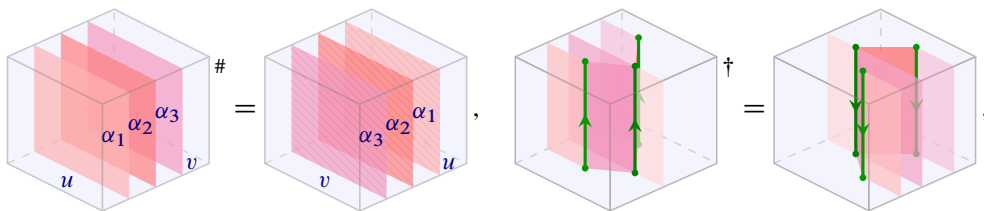

respectively, and the associated units of adjunction are the 2– and 3–morphisms

$$\mathrm{coev}_\alpha = \quad , \quad \mathrm{coev}_X = \mathcal{Z}\left(\right).$$
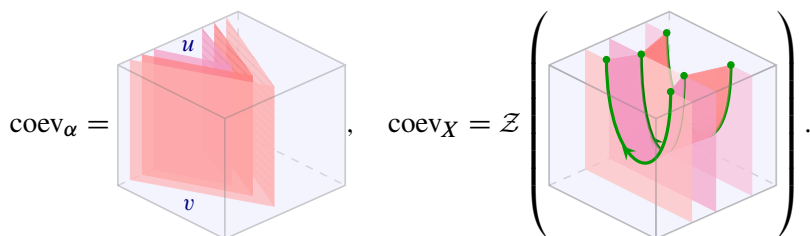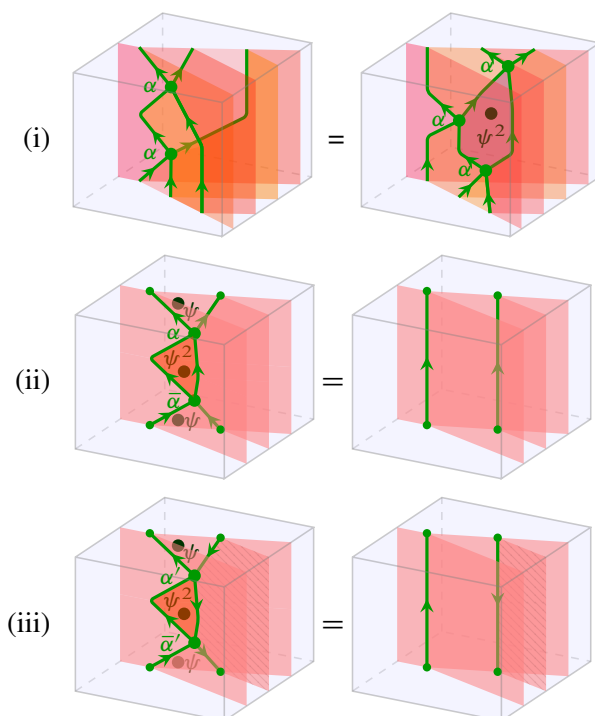
It is natural to express the constraints on special orbifold data for $\mathcal{Z}$ in terms of $\mathcal{T}_{\mathcal{Z}}$, or for any Gray category with duals:
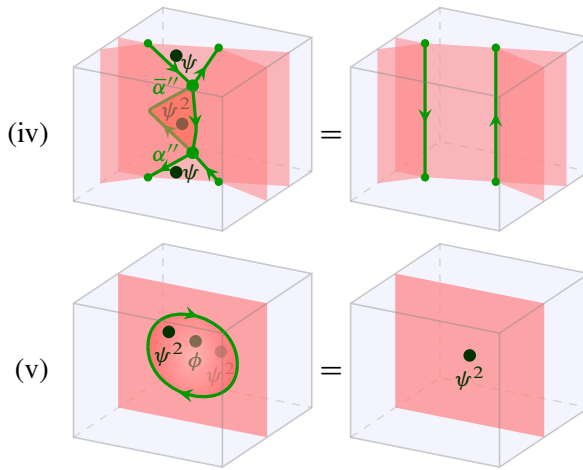
Definition 4.2 Let $\mathcal{G}$ be a Gray category with duals. A set of *special orbifold data in* $\mathcal{G}$ is

- an object $* \in \mathcal{G}$,

- a 1–morphism $\mathcal{A} \in \mathcal{G}(*, *)$,

- a 2–morphism $T \colon \mathcal{A} \square \mathcal{A} \to \mathcal{A}$,

- two 3–isomorphisms $\alpha \colon T \otimes (1_{\mathcal{A}} \square T) \rightleftarrows T \otimes (T \square 1_{\mathcal{A}}) \colon \bar{\alpha}$,
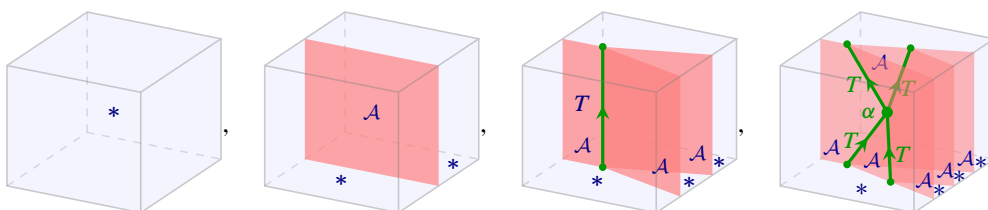
such that there are 3–isomorphisms $\phi \in \mathrm{Aut}(1_{1_*})$ and $\psi \in \mathrm{Aut}(1_{\mathcal{A}})$, and:

together with the opposite versions of (ii)–(iv) as in Figures 5–7, and with the two other orientations of the hemispheres in (v) as in Figure 8. The vertices α', $\bar{\alpha}'$, α'' and $\bar{\alpha}''$ are determined from α and $\bar{\alpha}$ using the duals in $\mathcal{G}$ (drawing only the T–lines around 0–strata for clarity):

$$(4\text{-}3) \quad \alpha' := \begin{array}{c}\includegraphics\end{array}, \quad \bar{\alpha}' := \begin{array}{c}\includegraphics\end{array}, \quad \alpha'' := \begin{array}{c}\includegraphics\end{array}, \quad \bar{\alpha}'' := \begin{array}{c}\includegraphics\end{array}.$$

In detail, α' is the 3–morphism (suppressing identities in the right column) in Figure 9, where σ and τ are the tensorator and triangulator of $\mathcal{G}$, respectively, using the conventions of [11, Definitions 3.4 and 3.8]. There are similar expressions for $\bar{\alpha}'$, α'' and $\bar{\alpha}''$.

Note that in the special case $\psi = 1_{1_\mathcal{A}}$ and $\phi = \delta \cdot 1_{1_{1_*}}$ for some $\delta \in \Bbbk$, the last condition (v) says that the quantum dimension of T is invertible.

By construction, in three dimensions a special orbifold datum $\mathcal{A}$ for a defect TQFT $\mathcal{Z}$ may be identified with a special orbifold datum $\mathcal{A}$ in $\mathcal{T}_\mathcal{Z}$. As we will explain in [16], the latter can be thought of as a spherical fusion category internal to the Gray category with duals $\mathcal{T}_\mathcal{Z}$, just as special orbifold data in two dimensions are Δ–separable symmetric Frobenius algebras.

Remark 4.3 Parallelling the constructions for the 2–dimensional case summarised in Remark 4.1, we expect there to be a natural defect TQFT $\mathcal{Z}^{\mathrm{orb}}$ which for $\mathcal{A}$–decorated bordisms restricts to the closed orbifold theory $\mathcal{Z}_\mathcal{A}$: the surface defect labels for $\mathcal{Z}^{\mathrm{orb}}$ are $\mathcal{A}'$–$\mathcal{A}$–bimodule categories internal to $\mathcal{T}_\mathcal{Z}$, while line defect labels are compatible
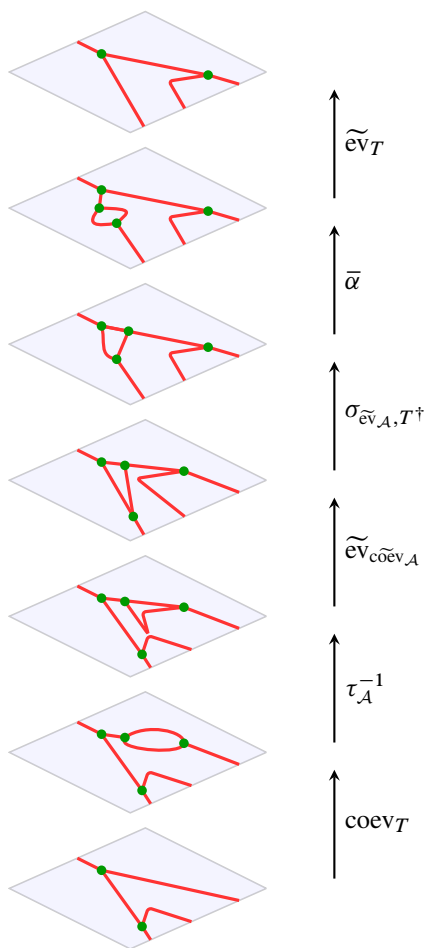
Figure 9

bimodule functors. Furthermore, we expect there to be an associated notion of orbifold completion $\mathcal{G}_{\mathrm{orb}}$ for any Gray category with duals $\mathcal{G}$ which is idempotent complete in an appropriate sense and such that $(\mathcal{T}_{\mathcal{Z}})_{\mathrm{orb}} \cong \mathcal{T}_{\mathcal{Z}^{\mathrm{orb}}}$ for $\mathcal{G} = \mathcal{T}_{\mathcal{Z}}$.

Next we address the relation between the Euler completion (see Definition 2.24) of a defect TQFT $\mathcal{Z} \colon \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ and its Gray category $\mathcal{T}_{\mathcal{Z}}$. We will show that different point insertions on surface defects for $\mathcal{Z}$ correspond to different pivotal structures on $\mathcal{T}_{\mathcal{Z}}$.

To see this, we fix a collection

(4-4) $$\psi_\alpha \in \mathrm{Aut}_{\mathcal{T}_{\mathcal{Z}}}(1_\alpha) \quad \text{for all } \alpha \in D_2$$

for the remainder of this section. Demanding $\psi_{\alpha \square \beta} = \psi_\alpha \square \psi_\beta$, we in fact have one 3–isomorphism for every 2–morphism α in $\mathcal{T}_{\mathcal{Z}}$. Restricting $\mathbb{D}^{\odot}$ to point insertions only of type ψ_α gives us a set of defect data $\mathbb{D}^\psi$ with $D_2^\psi = \{(\alpha, \psi_\alpha) \mid \alpha \in D_2\}$ and $D_j^\psi = D_j$ for $j \neq 2$, and we write

$$(4\text{-}5) \qquad\qquad \mathcal{Z}^\psi \colon \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}^\psi) \to \mathrm{Vect}_{\Bbbk}$$

for the restriction of $\mathcal{Z}^{\odot}$.

What is the relation between $\mathcal{T}_{\mathcal{Z}^\psi}$ and $\mathcal{T}_{\mathcal{Z}}$? The answer involves the following natural notion: given any Gray category with duals $\mathcal{G}$ together with a collection $c = \{c_\alpha \in \mathrm{Aut}_\mathcal{G}(1_\alpha) \mid \alpha \text{ is a 1–morphism in } \mathcal{G}\}$ satisfying $c_{\alpha \square \beta} = c_\alpha \square c_\beta$, the c–twist $\mathcal{G}^c$ is the Gray category with duals whose underlying tricategory and #–duals are those of $\mathcal{G}$, while the pivotal structures of $\mathcal{G}^c$ are those with the adjunction 3–morphisms

$$
\begin{aligned}
(4\text{-}6) \qquad
\mathrm{ev}_X^{\mathcal{G}^c} &= c_\alpha^{-1} \circ \mathrm{ev}_X^{\mathcal{G}} \circ (1_{X^\dagger} \otimes c_\beta \otimes 1_X) \colon X^\dagger \otimes X \to 1_\alpha, \\
\mathrm{coev}_X^{\mathcal{G}^c} &= (1_X \otimes c_\alpha \otimes 1_{X^\dagger}) \circ \mathrm{coev}_X^{\mathcal{G}} \circ c_\beta^{-1} \colon 1_\beta \to X \otimes X^\dagger, \\
\widetilde{\mathrm{ev}}_X^{\mathcal{G}^c} &= c_\beta^{-1} \circ \widetilde{\mathrm{ev}}_X^{\mathcal{G}} \circ (1_X \otimes c_\alpha \otimes 1_{X^\dagger}) \colon X \otimes X^\dagger \to 1_\beta, \\
\widetilde{\mathrm{coev}}_X^{\mathcal{G}^c} &= (1_{X^\dagger} \otimes c_\beta \otimes 1_X) \circ \widetilde{\mathrm{coev}}_X^{\mathcal{G}} \circ c_\alpha^{-1} \colon 1_\alpha \to X^\dagger \otimes X
\end{aligned}
$$

for every 2–morphism $X \colon \alpha \to \beta$ in $\mathcal{G}$. Put differently, $\mathcal{G}^c$ is the same as $\mathcal{G}$, except that the pivotal structures are "twisted" by the maps c_α.

Proposition 4.4 *For a defect TQFT $\mathcal{Z} \colon \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ and a collection of invertible 3–morphisms ψ_α as in* (4-4), *there is an equivalence*

$$(4\text{-}7) \qquad\qquad \mathcal{T}_{\mathcal{Z}^\psi} \cong (\mathcal{T}_{\mathcal{Z}})^\psi$$

of $\Bbbk$–linear Gray categories with duals.

Proof We will first express vertical composition in $\mathcal{T}_{\mathcal{Z}^\psi}$ in terms of $\mathcal{T}_{\mathcal{Z}}$. Then we will find a triequivalence $\Gamma \colon \mathcal{T}_{\mathcal{Z}^\psi} \to (\mathcal{T}_{\mathcal{Z}})^\psi$ which maps the adjunction 3–morphisms in $\mathcal{T}_{\mathcal{Z}^\psi}$ to those in $(\mathcal{T}_{\mathcal{Z}})^\psi$.

Let $f \colon X \to Y$ and $g \colon Y \to Z$ be 3–morphisms in $\mathcal{T}_{\mathcal{Z}^\psi}$. We denote the common source of X, Y and Z by (α, ψ), and the common target by (β, ψ). By definition (see [11, Section 3.3]), the vertical composition $g \circ_\psi f$ in $\mathcal{T}_{\mathcal{Z}^\psi}$ is $\mathcal{Z}^\psi(B_{g,f}) \in \mathrm{Hom}_{\mathcal{T}_{\mathcal{Z}^\psi}}(X, Z)$, where $B_{g,f}$ is the following defect bordism: $B_{g,f}$ is the solid 3–ball with two smaller 3–balls B_f and B_g removed whose boundaries are ingoing for $B_{g,f}$ and decorated such that $f \in \mathcal{Z}^\psi(\partial B_f)$ and $g \in \mathcal{Z}^\psi(\partial B_g)$, while the remaining boundary

component of $B_{g,f}$ is outgoing and its stratification is locally a cylinder over the boundary stratifications. Schematically,

(4-8)
$$B_{g,f} = \begin{array}{c}\end{array}$$

where the line-of-sight on the 3–ball is orthogonal to the two red 2–strata.

To evaluate $\mathcal{Z}^\psi(B_{g,f})$ we note that the two 2–strata in (4-8) decorated with (α, ψ) and (β, ψ) have two incoming and one outgoing boundary components. Thus, according to our convention in Example 2.14 they have symmetric Euler characteristic -1, and it follows that

(4-9)
$$g \circ_\psi f = (\psi_\beta^{-1} \otimes 1_Z \otimes \psi_\alpha^{-1}) \circ g \circ f,$$

where $g \circ f$ is the composition in $\mathcal{T}_\mathcal{Z}$. Similarly, we find that the identity 3–morphisms $1_X^{\mathcal{T}_{\mathcal{Z}^\psi}}$ in $\mathcal{T}_{\mathcal{Z}^\psi}$ are

(4-10)
$$1_X^{\mathcal{T}_{\mathcal{Z}^\psi}} = \mathcal{Z}^\psi\left(\begin{array}{c}\end{array} \right) = \psi_\beta \otimes 1_X \otimes \psi_\alpha$$

as the (α, ψ)– and (β, ψ)–decorated 2–strata now have symmetric Euler characteristic $+1$.

It follows that we obtain an equivalence Γ of Gray categories which is the identity on objects, 1– and 2–morphisms by setting $\Gamma(f) = (\psi_\beta^{-1} \otimes 1_Y \otimes \psi_\alpha^{-1}) \circ f$ for every 3–morphism $f \colon X \to Y$. The equivalence automatically respects the #–duals, and to verify that Γ also maps $\mathrm{ev}_X^{\mathcal{T}_{\mathcal{Z}^\psi}}$ to $\mathrm{ev}_X^{(\mathcal{T}_\mathcal{Z})^\psi}$, we compute

(4-11) $\mathrm{ev}_X^{\mathcal{T}_{\mathcal{Z}^\psi}} = \mathcal{Z}^\psi\left(\begin{array}{c}\end{array} \right) = \psi_\alpha \circ \mathrm{ev}_X^{\mathcal{T}_\mathcal{Z}} \circ (1_{X^\dagger} \otimes \psi_\beta \otimes 1_X)$

$$\xmapsto{\ \Gamma\ } \psi_\alpha^{-1} \circ \mathrm{ev}_X^{\mathcal{T}_\mathcal{Z}} \circ (1_{X^\dagger} \otimes \psi_\beta \otimes 1_X) = \mathrm{ev}_X^{(\mathcal{T}_\mathcal{Z})^\psi},$$

where the last step is due to the definition in (4-6). The argument for the other three types of adjunction maps is analogous. $\square$

Remark 4.5 Here we give a sketch of a construction of a disc-like n–category $\mathcal{D}_{\mathcal{Z}}$ from a defect TQFT $\mathcal{Z} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$.

Recall that the main datum for a disc-like n–category is a map that assigns to a k–ball B the set of "k–dimensional field configurations $C_k(B)$" for every $k \in \{0, \ldots, n\}$. The elements of $C_k(B)$ are thought of as the k–morphisms of the disc-like n–category. This assignment has to be functorial with respect to diffeomorphisms of balls, giving a functor

$$(4\text{-}12) \qquad\qquad\qquad C_k \colon \mathrm{Ball}_k \to \mathrm{Set}$$

from smooth k–balls and their diffeomorphisms to sets and bijections. For our defect TQFT $\mathcal{Z} \colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ we define C_k on a k–manifold X, for $k \in \{0, \ldots, n-1\}$, as

$$(4\text{-}13) \quad C_k(X) = \big\{ X' \mid X' \text{ a decorated stratified } k\text{–manifold}$$
$$\text{with underlying manifold } X \text{ such that}$$
$$[X'] \colon \varnothing \to \partial X' \text{ is a morphism in } \mathrm{Bord}_k^{\mathrm{def}}(\partial^{n-k+1}\mathbb{D}) \big\}.$$

Restricting to balls, this gives the functor C_k as in (4-12).

The combined source and target map for k–morphisms is specified by a natural transformation $\partial \colon C_k \Rightarrow C_{k-1}$ with components

$$(4\text{-}14) \qquad\qquad\qquad \partial_X \colon C_k(X) \to C_{k-1}(\partial X).$$

In the case of $\mathcal{D}_{\mathcal{Z}}$ we define $\partial_X(X')$ to be the decorated stratified manifold $\partial X'$ for every $X' \in C_k(X)$.

Disc-like n–categories have many compositions of morphisms modelled by the gluing of balls. If a k–ball $B = B_1 \circ_Y B_2$ is obtained as the gluing of two k–balls B_1 and B_2 along a (collar over a) $(k-1)$–ball Y with boundary $(k-2)$–sphere $S = \partial Y$, it is required that there exists a map

$$(4\text{-}15) \qquad\qquad \mathrm{gl}_Y \colon C(B_1)_S \times_{C(Y)} C(B_2)_S \to C(B)_S,$$

where the index S denotes elements that are "splittable along S". This notion is intrinsically defined, but in our case it has a clear candidate: in $\mathcal{D}_{\mathcal{Z}}$ the spaces $C(B_i)_S$ are those k–morphisms that intersect S transversely, and the map gl_Y is just given by gluing the defects in B_1 and B_2 along Y. Thus, gluing is strictly associative.

The axioms related to the units of disc-like n–categories pose a technical issue: To model all compositions with various units, Morrison and Walker [35] work in the PL

setting. In the case of $\mathcal{D}_{\mathcal{Z}}$ the k–dimensional units should be cylinders over lower-dimensional defect balls with certain singularities on the boundary. In this remark, however, we do not attempt to translate the unit axioms of [35] into our setting.

Up to now we have described the k–morphisms for $k < n$ without mentioning the functor $\mathcal{Z}$. Indeed, if we took isotopy classes of $(n-1)$–balls we should obtain a disc-like $(n-1)$–category that is "free over the defect data $\partial\mathbb{D}$" (compare [11, Section 3]). Instead we use $\mathcal{Z}$ to define n–morphisms on an n–ball B as the coproduct over the state spaces associated to all $\mathbb{D}$–decorations of the boundary ∂B,

$$(4\text{-}16) \qquad C_n(B) = \coprod_{[S]\in C_{n-1}(\partial B)} \mathcal{Z}(S),$$

where the index runs over all representatives of diffeomorphism classes of stratified decorated $(n-1)$–spheres.

The composition (4-15) of two n–balls B_1 and B_2 that are glued along an $(n-1)$–ball Y to produce an n–ball $B = B_1 \circ_Y B_2$ is given as follows. For each chosen decorated stratifications of B_1 and B_2 that are glued to a stratification of B, we need to give a map $\mathrm{gl}'_Y \colon \mathcal{Z}(\partial B_1) \times \mathcal{Z}(\partial B_2) \to \mathcal{Z}(\partial B)$. From the coproduct of the maps gl'_Y we then obtain the gluing map gl_Y. To this end we cut out one slightly smaller ball from the interior of both B_1 and B_2 inside B. This produces a two-holed ball on which we can evaluate $\mathcal{Z}$, giving the map gl'_Y. By diffeomorphism-invariance of $\mathcal{Z}$ we expect this to satisfy all axioms of a disc-like n–category.

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