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A finite $\mathbb{Q}$–bad space

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We prove that, for a free noncyclic group F, the second homology group $H_2(\widehat{F}_{\mathbb{Q}},\mathbb{Q})$ is an uncountable $\mathbb{Q}$–vector space, where $\widehat{F}_{\mathbb{Q}}$ denotes the $\mathbb{Q}$–completion of F. This solves a problem of A K Bousfield for the case of rational coefficients. As a direct consequence of this result, it follows that a wedge of two or more circles is $\mathbb{Q}$–bad in the sense of Bousfield–Kan. The same methods as used in the proof of the above result serve to show that $H_2(\widehat{F}_{\mathbb{Z}},\mathbb{Z})$ is not a divisible group, where $\widehat{F}_{\mathbb{Z}}$ is the integral pronilpotent completion of F.

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1 Introduction

In the foundational work [4], A K Bousfield and D M Kan introduced the concept of R–completion of a space for a commutative ring R. For a space X, there is an R–completion functor $X \mapsto R_\infty X$ such that a map between two spaces $f\colon X \to Y$ induces an isomorphism of reduced homology $\widetilde{H}_*(X,R) \cong \widetilde{H}_*(Y,R)$ if and only if it induces a homotopy equivalence $R_\infty X \simeq R_\infty Y$. Thus, R–completion can be viewed as an approximation of the R–homology localization of a space, defined by Bousfield [1]. For certain classes of spaces, such as nilpotent spaces, R–completion and R–homology localization coincide.

The R–completion functor for spaces is closely related to the R–completion functor for groups. For a group G, denote by $\{\gamma_i(G)\}_{i\ge 1}$ the lower central series of G. We will consider the pronilpotent completion $\widehat{G}_{\mathbb{Z}}$ of G as well as the $\mathbb{Q}$–completion $\widehat{G}_{\mathbb{Q}}$, defined as

$$\widehat{G}_{\mathbb{Z}} = \varprojlim G/\gamma_i(G) \quad \text{and} \quad \widehat{G}_{\mathbb{Q}} = \varprojlim G/\gamma_i(G)\otimes\mathbb{Q}.$$

Here $G/\gamma_i(G)\otimes\mathbb{Q}$ is the Maltsev $\mathbb{Q}$–localization of the nilpotent group $G/\gamma_i(G)$. One can find the definition of $\mathbb{Z}/p$–completion $\widehat{G}_{\mathbb{Z}/p}$ in [4; 2]. In this paper we do not use $\mathbb{Z}/p$–completion and work only over $\mathbb{Z}$ or $\mathbb{Q}$. It is shown in [4, Chapter 4] that the R–completion of a connected space X can be constructed explicitly as $\overline{W}\,\widehat{(GX)}_R$,

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where G is the Kan loop simplicial group, $\widehat{(GX)}_R$ is the R–completion of GX and $\overline{W}$ is the classifying space functor.

A space X is called R–good if the map $X \to R_\infty X$ induces an isomorphism of reduced homology $\widetilde{H}_*(X, R) \cong \widetilde{H}_*(R_\infty X, R)$, and called R–bad otherwise. In other words, for R–good spaces, R–homology localization and R–completion coincide.

There are a lot of examples of R–good and R–bad spaces. The key example of [4] is the projective plane $\mathbb{R}P^2$, which is $\mathbb{Z}$–bad. This fact implies that some finite wedge of circles is also $\mathbb{Z}$–bad. Bousfield [2] showed that a wedge of two circles is $\mathbb{Z}$–bad. In [3], Bousfield proved that, for any prime p, a wedge of circles is $\mathbb{Z}/p$–bad, thus providing the first example of a finite $\mathbb{Z}/p$–bad space. For R a subring of the rationals or $\mathbb{Z}/n$, where $n \geq 2$, and a free group F, there is a weak equivalence [4, Proposition 5.3]

$$R_\infty K(F, 1) \simeq K(\widehat{F}_R, 1).$$

Therefore, the question of R–goodness of a wedge of circles is reduced to the question of nontriviality of the higher R–homology of the R–completion of a free group. The same question naturally appears in the theory of HR–localizations of groups. In [2, Problem 4.11], Bousfield posed the following problem:

Problem (Bousfield) Does $H_2(\widehat{F}_R, R)$ vanish when F is a finitely generated free group and $R = \mathbb{Q}$ or $R = \mathbb{Z}/n$?

In the recent paper [7], we show for $R = \mathbb{Z}/n$ that $H_2(\widehat{F}_R, R)$ is an uncountable group, solving the above problem for the case $R = \mathbb{Z}/n$. The key step in [7] substantially uses the theory of profinite groups. Hence the method given in [7] cannot be directly transferred to the case $R = \mathbb{Q}$.

We answer Bousfield's problem over $\mathbb{Q}$. Our main results are the following theorems.

Theorem 1 *For F any finitely generated noncyclic free group, $H_2(\widehat{F}_\mathbb{Q}, \mathbb{Q})$ is uncountable.*

We also prove that the image of the map $H_2(\widehat{F}_\mathbb{Z}, \mathbb{Z}) \to H_2(\widehat{F}_\mathbb{Q}, \mathbb{Q})$ is uncountable.

Theorem 2 *For F any finitely generated noncyclic free group and p any prime, $H_2(\widehat{F}_\mathbb{Z}, \mathbb{Z}/p)$ is uncountable. In particular, $H_2(\widehat{F}_\mathbb{Z}, \mathbb{Z})$ is not divisible.*

Theorem 2 answers a problem we posted in [6]. As mentioned above, $\mathbb{Q}_\infty K(F, 1) = K(\widehat{F}_\mathbb{Q}, 1)$. Therefore, Theorem 1 implies the following:

Corollary *A wedge of ≥ 2 circles is $\mathbb{Q}$–bad.*

As far as the authors know, this is the first known example of a finite $\mathbb{Q}$–bad space.

The proof is organized as follows. In Section 2 we discuss technical results about power series. The main result of Section 2, Proposition 2.1, states that the kernel of the natural map between a rational power series ring and the coinvariants of the diagonal action of the rationals on the exterior square $\mathbb{Q}[\![x]\!] \to \Lambda^2(\mathbb{Q}[\![x]\!])_{\mathbb{Q}}$, given by $f \mapsto f \wedge 1$, is countable. (In the proof of the proposition we use the fact that the group algebra $\mathbb{Q}[\mathbb{Q}]$ is countable. In the similar statement for the $\mathbb{Z}/p$–completion we should consider the mod-p group algebra of the group of p–adic integers $\mathbb{Z}/p[\mathbb{Z}_p]$, which is uncountable. So this method fails for $\mathbb{Z}/p$–completions.) In Section 3, we consider the integral lamplighter group,

$$\mathcal{LG} = \langle a, b \mid [a, a^{b^i}] = 1,\ i \in \mathbb{Z} \rangle,$$

which is isomorphic to the wreath product of two infinite cyclic groups, as well as its p–analog $\mathbb{Z}/p \wr C$, where C denotes an infinite cyclic group. The group $\mathcal{LG}$ is metabelian; therefore, its completions $\widehat{\mathcal{LG}}_{\mathbb{Z}}$ and $\widehat{\mathcal{LG}}_{\mathbb{Q}}$ can be easily described (see (3-1) and (3-2)), and the homology group $H_2(\widehat{\mathcal{LG}}_{\mathbb{Q}}, \mathbb{Q})$ is isomorphic to the natural coinvariant quotient of the exterior square $\Lambda^2(\mathbb{Q}[\![x]\!])$. The key step in the proof of the main results occurs in Section 4, in Proposition 4.1. Let $F = F(a, b)$ be a free group of rank two with generators a and b. We construct (see Proposition 4.1) an uncountable collection of elements $r_q, s_q \in \widehat{F}_{\mathbb{Z}}$ such that $[r_q, a][s_q, b] = 1$ in $\widehat{F}_{\mathbb{Z}}$. One can consider the group homology $H_2(\widehat{F}_{\mathbb{Z}}, \mathbb{Z})$ as a kernel of the commutator map $\widehat{F}_{\mathbb{Z}} \wedge \widehat{F}_{\mathbb{Z}} \to \widehat{F}_{\mathbb{Z}}$ given by $a \wedge b \mapsto [a, b]$, where $\widehat{F}_{\mathbb{Z}} \wedge \widehat{F}_{\mathbb{Z}}$ is the nonabelian exterior square of $\widehat{F}_{\mathbb{Z}}$; see Brown and Loday [5]. Therefore, the pairs of elements $r_q, s_q \in \widehat{F}_{\mathbb{Z}}$ (through their association with $(r_q \wedge a)(s_q \wedge b) \in \widehat{F}_{\mathbb{Z}} \wedge \widehat{F}_{\mathbb{Z}}$) define certain elements of $H_2(\widehat{F}_{\mathbb{Z}}, \mathbb{Z})$. Next we consider the following natural maps between homology groups of different completions, which are induced by the standard projection $F \to \mathcal{LG}$:

$$\begin{tikzcd}
H_2(\widehat{F}_{\mathbb{Q}}, \mathbb{Q}) \arrow[d] & H_2(\widehat{F}_{\mathbb{Z}}, \mathbb{Z}) \arrow[l] \arrow[r] \arrow[dl] \arrow[dr] & H_2(\widehat{F}_{\mathbb{Z}}, \mathbb{Z}/p) \arrow[d] \\
H_2(\widehat{\mathcal{LG}}_{\mathbb{Q}}, \mathbb{Q}) & & H_2(\widehat{\mathcal{LG}}_{\mathbb{Z}}, \mathbb{Z}/p)
\end{tikzcd}$$

We show, in Section 5, that the sets of images of the elements $(r_q \wedge a)(s_q \wedge b)$ in $H_2(\widehat{\mathcal{LG}}_{\mathbb{Q}}, \mathbb{Q})$ and $H_2(\widehat{\mathcal{LG}}_{\mathbb{Z}}, \mathbb{Z}/p)$ are uncountable. Theorems 1 and 2 follow.

2 Technical results about power series

We denote by C an infinite cyclic group written multiplicatively as $C = \langle t \rangle$. For a commutative ring R we denote by $R[\![x]\!]$ the ring of formal power series over R and by $R[C]$ the group algebra of C. Consider the multiplicative homomorphism

$$\tau\colon C \to R[\![x]\!], \quad \tau(t) = 1 + x.$$

The induced ring homomorphism is denoted by the same letter:

$$\tau\colon R[C] \to R[\![x]\!].$$

Lemma 2.1 *Let I be the augmentation ideal of $R[C]$ and set $R[C]^\wedge = \varprojlim R[C]/I^i$. Then $\tau(I^n) \subseteq x^n \cdot R[\![x]\!]$ and τ induces isomorphisms*

$$R[C]/I^n \cong R[x]/x^n \quad \text{and} \quad R[C]^\wedge \cong R[\![x]\!].$$

Proof If we set $x = t - 1$, we obtain $R[C] = R[x, (1+x)^{-1}]$ and $I = x \cdot R[C]$. Observe that the image of the element $1 + x$ in $R[x]/x^n$ is invertible. Since localization at the element $1+x$ is an exact functor, the short exact sequence $x^n \cdot R[x] \rightarrowtail R[x] \twoheadrightarrow R[x]/x^n$ gives the short exact sequence $(x^n \cdot R[x])_{1+x} \rightarrowtail R[C] \twoheadrightarrow R[x]/x^n$. It follows that $R[C]/x^n \cong R[x]/x^n$. The assertion follows. □

Denote by σ the antipode of the group ring $R[C]$:

$$\sigma\colon R[C] \to R[C], \quad \sigma\left(\sum a_i t^i\right) = \sum a_i t^{-i}.$$

Obviously $\sigma(I^n) = I^n$, and hence it induces a continuous involution

$$\hat{\sigma}\colon R[C]^\wedge \to R[C]^\wedge.$$

Composing this involution with the isomorphism $R[C]^\wedge \cong R[\![x]\!]$ we obtain a continuous involution

$$\tilde{\sigma}\colon R[\![x]\!] \to R[\![x]\!]$$

such that

$$\tilde{\sigma}(x) = -x + x^2 - x^3 + x^4 - \cdots.$$

Consider the case $R = \mathbb{Q}$. Note that the set $1 + x \cdot \mathbb{Q}[\![x]\!]$ is a group and there is a unique way to define the r–power map $f \mapsto f^r$ for $r \in \mathbb{Q}$ that extends the usual power map $f \mapsto f^n$ so that $f^{r_1 r_2} = (f^{r_1})^{r_2}$ (see Lemma 4.4 of [6]). This map is defined by the formula

$$f^r = \sum_{n=0}^{\infty} \binom{r}{n} (f-1)^n,$$

where $\binom{r}{n} = r(r-1)\cdots(r-n+1)/n!$. Denote by $C \otimes \mathbb{Q}$ the group $\mathbb{Q}$ written multiplicatively as powers of t: $C \otimes \mathbb{Q} = \{t^r \mid r \in \mathbb{Q}\}$. Consider the multiplicative homomorphism

$$\tau_{\mathbb{Q}} \colon C \otimes \mathbb{Q} \to \mathbb{Q}[\![x]\!] \tag{2-1}$$

that extends $\tau \colon C \to \mathbb{Q}[\![x]\!]$:

$$\tau_{\mathbb{Q}}(t^r) = (1+x)^r.$$

The induced ring homomorphism is denoted by the same letter:

$$\tau_{\mathbb{Q}} \colon \mathbb{Q}[C \otimes \mathbb{Q}] \to \mathbb{Q}[\![x]\!].$$

This homomorphism allows us to consider $\mathbb{Q}[\![x]\!]$ as a $\mathbb{Q}[C \otimes \mathbb{Q}]$–module. We claim that the homomorphism $\tau_{\mathbb{Q}} \colon \mathbb{Q}[C \otimes \mathbb{Q}] \to \mathbb{Q}[\![x]\!]$ respects the involutions:

$$\tau_{\mathbb{Q}} \circ \sigma_{C \otimes \mathbb{Q}} = \widetilde{\sigma} \circ \tau_{\mathbb{Q}}, \tag{2-2}$$

where $\sigma_{C \otimes \mathbb{Q}}$ is the antipode on $\mathbb{Q}[C \otimes \mathbb{Q}]$. Indeed, we have that $(1+x)^{-1} = \widetilde{\sigma}(1+x) = \widetilde{\sigma}((1+x)^{1/n})^n$ and then $\widetilde{\sigma}((1+x)^{1/n}) = (1+x)^{-1/n}$, which implies $\widetilde{\sigma}((1+x)^r) = (1+x)^{-r}$ for any $r \in \mathbb{Q}$, and hence $\tau_{\mathbb{Q}}(\sigma_{C \otimes \mathbb{Q}}(t^r)) = \widetilde{\sigma}(\tau_{\mathbb{Q}}(t^r))$ for any $r \in \mathbb{Q}$.

Proposition 2.1 (1) *Denote by $\Lambda^2(\mathbb{Q}[\![x]\!])$ the exterior square of $\mathbb{Q}[\![x]\!]$ considered as a $(C \otimes \mathbb{Q})$–module with the diagonal action. Consider the space of $(C \otimes \mathbb{Q})$–coinvariants $(\Lambda^2(\mathbb{Q}[\![x]\!]))_{C \otimes \mathbb{Q}}$. Then the kernel of the homomorphism*

$$\theta_{\mathbb{Q}} \colon \mathbb{Q}[\![x]\!] \to (\Lambda^2(\mathbb{Q}[\![x]\!]))_{C \otimes \mathbb{Q}}, \quad \theta_{\mathbb{Q}}(f) = f \wedge 1,$$

is countable.

(2) *Let p be a prime. Denote by $\Lambda^2(\mathbb{Z}/p[\![x]\!])$ the exterior square of $\mathbb{Z}/p[\![x]\!]$ considered as a C–module with the diagonal action. Consider the space of C–coinvariants $(\Lambda^2(\mathbb{Z}/p[\![x]\!]))_C$. Then the kernel of the homomorphism*

$$\theta_{\mathbb{Z}/p} \colon \mathbb{Z}/p[\![x]\!] \to (\Lambda^2(\mathbb{Z}/p[\![x]\!]))_C, \quad \theta_{\mathbb{Z}/p}(f) = f \wedge 1,$$

is countable.

Proof (1) Consider the linear map

$$\alpha\colon \Lambda^2(\mathbb{Q}[\![x]\!]) \to \mathbb{Q}[\![x]\!]^{\otimes 2}, \quad \alpha(f \wedge g) = f \otimes g - g \otimes f.$$

Note that this is a homomorphism of $\mathbb{Q}[C \otimes \mathbb{Q}]$–modules, where the action of $C \otimes \mathbb{Q}$ is defined diagonally in both cases. Hence, it induces a linear map:

$$\alpha_{C\otimes Q}\colon (\Lambda^2(\mathbb{Q}[\![x]\!]))_{C\otimes\mathbb{Q}} \to (\mathbb{Q}[\![x]\!]^{\otimes 2})_{C\otimes\mathbb{Q}}.$$

Next, we consider the homomorphism

$$\beta\colon (\mathbb{Q}[\![x]\!]^{\otimes 2})_{C\otimes\mathbb{Q}} \to \mathbb{Q}[\![x]\!] \otimes_{\mathbb{Q}[C\otimes\mathbb{Q}]} \mathbb{Q}[\![x]\!], \quad \beta(f \otimes g) = f \otimes \tilde{\sigma}(g),$$

which is well defined because $\tau_{\mathbb{Q}}$ respects the involutions (2-2): $ft^r \otimes \tilde{\sigma}(gt^r) = ft^r \otimes \tilde{\sigma}(g)t^{-r} = f \otimes \tilde{\sigma}(g)$. Denote by K the subfield of the field of Laurent power series $\mathbb{Q}((x))$ generated by the image of $\tau_{\mathbb{Q}}$. Then there is a map

$$\gamma\colon \mathbb{Q}[\![x]\!] \otimes_{\mathbb{Q}[C\otimes\mathbb{Q}]} \mathbb{Q}[\![x]\!] \to \mathbb{Q}((x)) \otimes_K \mathbb{Q}((x)).$$

The composition

$$\gamma \circ \beta \circ \alpha_{C\otimes\mathbb{Q}} \circ \theta_{\mathbb{Q}}\colon \mathbb{Q}[\![x]\!] \to \mathbb{Q}((x)) \otimes_K \mathbb{Q}((x))$$

sends f to $f\otimes 1 - 1 \otimes \tilde{\sigma}(f)$. Note that for any vector spaces V and U over any field and any elements $v_1, v_2 \in V$ and $u_1, u_2 \in U$, if v_1 and v_2 are linearly independent, $u_1 \neq 0$ and $u_2 \neq 0$, then $v_1 \otimes u_1$ and $v_2 \otimes u_2$ are linearly independent in $V \otimes U$. It follows that for any $f \in \mathbb{Q}[\![x]\!] \setminus K$ we have that $f \otimes 1 - 1 \otimes \tilde{\sigma}(f) \neq 0$ in $\mathbb{Q}((x)) \otimes_K \mathbb{Q}((x))$. Therefore $\mathrm{Ker}(\theta_{\mathbb{Q}}) \subseteq K$. Since the fraction field of the countable algebra $\mathbb{Q}[C \otimes \mathbb{Q}]$ is countable, K is countable. The assertion follows.

(2) The proof is the same. □

3 Completions of lamplighter groups $\mathcal{LG}$ and $\mathcal{LG}(p)$

Recall the definition of the tensor square for a nonabelian group [5]. For a group G, the tensor square $G \otimes G$ is the group generated by the symbols $g \otimes h$, for $g, h \in G$, satisfying the defining relations

$$fg \otimes h = (g^{f^{-1}} \otimes h^{f^{-1}})(f \otimes h) \quad \text{and} \quad f \otimes gh = (f \otimes g)(f^{g^{-1}} \otimes h^{g^{-1}})$$

for all $f, g, h \in G$. The exterior square $G \wedge G$ is defined as

$$G \wedge G := G \otimes G / \langle g \otimes g,\ g \in G \rangle.$$

The images of the elements $g \otimes h$ in $G \wedge G$ will be denoted by $g \wedge h$. If $G = E/R$ for a free group E, there is a natural isomorphism $G \wedge G \cong [E, E]/[R, E]$.

For any group G, there is a natural short exact sequence

$$0 \to H_2(G, \mathbb{Z}) \to G \wedge G \xrightarrow{[-,-]} [G, G] \to 1$$

(see [5, (2.8)] and [8]). Let $g_1, \dots, g_n, h_1, \dots, h_n \in G$ be elements such that

$$[g_1, h_1] \cdots [g_n, h_n] = 1.$$

Then the element $(g_1 \wedge h_1) \cdots (g_n \wedge h_n)$ defines an element in $H_2(G, \mathbb{Z})$:

$$(g_1 \wedge h_1) \cdots (g_n \wedge h_n) \in H_2(G, \mathbb{Z}).$$

If R is a commutative ring, then the image of $(g_1 \wedge h_1) \cdots (g_n \wedge h_n)$ in $H_2(G, R)$ is denoted by

$$((g_1 \wedge h_1) \cdots (g_n \wedge h_n)) \otimes R \in H_2(G, R).$$

We will consider two versions of the lamplighter group. The integral lamplighter group

$$\mathcal{LG} = \mathbb{Z} \wr C = \langle a, b \mid [a, a^{b^i}] = 1,\ i \in \mathbb{Z} \rangle$$

and the p–lamplighter group for a prime p

$$\mathcal{LG}(p) = \mathbb{Z}/p \wr C = \langle a, b \mid [a, a^{b^i}] = a^p = 1,\ i \in \mathbb{Z} \rangle.$$

Observe that $\mathcal{LG} = \mathbb{Z}[C] \rtimes C$ and $\mathcal{LG}(p) = \mathbb{Z}/p[C] \rtimes C$. Using Lemma 2.1 and [6, Proposition 4.7], we obtain

$$\widehat{\mathcal{LG}}_{\mathbb{Z}} = \mathbb{Z}[\![x]\!] \rtimes C, \tag{3-1}$$

$$\widehat{\mathcal{LG}}_{\mathbb{Q}} = \mathbb{Q}[\![x]\!] \rtimes (C \otimes \mathbb{Q}) \quad \text{and} \quad \widehat{\mathcal{LG}(p)}_{\mathbb{Z}} = \mathbb{Z}/p[\![x]\!] \rtimes C, \tag{3-2}$$

where C acts on $\mathbb{Z}[\![x]\!]$ and $\mathbb{Z}/p[\![x]\!]$ via τ and $C \otimes \mathbb{Q}$ acts on $\mathbb{Q}[\![x]\!]$ via $\tau_{\mathbb{Q}}$.

Proposition 3.1 *There are isomorphisms*

$$(\Lambda^2(\mathbb{Q}[\![x]\!]))_{C \otimes \mathbb{Q}} \cong H_2(\widehat{\mathcal{LG}}_{\mathbb{Q}}, \mathbb{Q}) \quad \text{and} \quad (\Lambda^2(\mathbb{Z}/p[\![x]\!]))_C \cong H_2(\widehat{\mathcal{LG}(p)}_{\mathbb{Z}}, \mathbb{Z}/p),$$

in both cases given by

$$f \wedge f' \mapsto ((f, 1) \wedge (f', 1)) \otimes R,$$

where $R = \mathbb{Q}$ and $R = \mathbb{Z}/p$ respectively.

Proof Consider the short exact sequence $\mathbb{Q}[\![x]\!] \rightarrowtail \widehat{\mathcal{LG}}_{\mathbb{Q}} \twoheadrightarrow (C \otimes \mathbb{Q})$ and the associated spectral sequence E. Since $\mathbb{Q} = \varinjlim (1/n!)\mathbb{Z}$ and homology commutes with direct limits, we have $H_n(C \otimes \mathbb{Q}, -) = 0$ for $n \geq 2$. It follows that $E^2_{i,j} = 0$ for $i \geq 2$ and hence there is a short exact sequence

$$0 \to E^2_{0,2} \to H_2(\widehat{\mathcal{LG}}_{\mathbb{Q}}, \mathbb{Q}) \to E^2_{1,1} \to 0.$$

Observe that the action of C on $\mathbb{Q}[\![x]\!]$ has no invariants. Then

$$E^2_{1,1} = H_1(C \otimes \mathbb{Q}, \mathbb{Q}[\![x]\!]) = \varinjlim H_1\Big(C \otimes \frac{1}{n!}\mathbb{Z}, \mathbb{Q}[\![x]\!]\Big) = \varinjlim \mathbb{Q}[\![x]\!]^{C \otimes (1/n!)\mathbb{Z}} = 0.$$

It follows that the map

$$H_2(\mathbb{Q}[\![x]\!], \mathbb{Q})_{C \otimes \mathbb{Q}} = E^2_{0,2} \to H_2(\widehat{\mathcal{LG}}_{\mathbb{Q}}, \mathbb{Q}) \tag{3-3}$$

is an isomorphism. The map is induced by the map $\mathbb{Q}[\![x]\!] \rightarrowtail \widehat{\mathcal{LG}}_{\mathbb{Q}}$ that sends $f \in \mathbb{Q}[\![x]\!]$ to $(f, 1) \in \widehat{\mathcal{LG}}_{\mathbb{Q}}$. Then the isomorphism (3-3) sends $f \wedge f'$ to $((f, 1) \wedge (f', 1)) \otimes \mathbb{Q}$. Using the isomorphism $\Lambda^2(\mathbb{Q}[\![x]\!]) \cong H_2(\mathbb{Q}[\![x]\!], \mathbb{Q})$ we obtain the assertion.

The second isomorphism can be proved similarly. □

4 Completion of a free group

For elements of groups or Lie rings, we will use the left-normalized notation

$$[a_1, \dots, a_n] := [[a_1, \dots, a_{n-1}], a_n]$$

and the following notation for Engel commutators:

$$[a,_0 b] := a \qquad \text{and} \qquad [a,_{i+1} b] = [[a,_i b], b] \quad \text{for } i \geq 0.$$

For all elements a and b of a Lie ring, the Jacobi identity implies that

$$[a, b, a, b] + [b, [a, b], a] + [[a, b], [a, b]] = 0.$$

It follows that

$$[a, b, b, a] = [a, b, a, b]. \tag{4-1}$$

The following lemma is a generalization of this identity.

Lemma 4.1 *Let L be a Lie ring, $a, b \in L$ and $n \geq 1$. Then*

$$[[a,_{2n} b], a] = \Bigg[\sum_{i=0}^{n-1} (-1)^i [[a,_{2n-1-i} b], [a,_i b]], b\Bigg]. \tag{4-2}$$

Proof The Jacobi identity implies that

$$[[a,_{2n-i}b],[a,_i b]]+[[a,_{2n-1-i},b],[a,_{i+1}b]]=[[a,_{2n-1-i}b],[a,_i b],b] \tag{4-3}$$

for $0\le i\le n-1$. Taking the alternating sum of these identities and using the fact that $[[a,_n b],[a,_n b]]=0$, we obtain the assertion. □

Corollary 4.1 *Let $F=F(a,b)$ be a free group with generators a, b. For any $n\ge 1$,*

$$[[a,_{2n}b],a]\equiv\left[\prod_{i=0}^{n-1}[[a,_{2n-1-i}b],[a,_i b]]^{(-1)^i},b\right]\mod \gamma_{2n+3}(F).$$

We denote by F the free group on two variables $F=F(a,b)$ and denote by $\varphi\colon F\to\mathcal{LG}$ the obvious epimorphism to the integral lamplighter group. It induces a homomorphism between pronilpotent completions

$$\widehat{\varphi}\colon \widehat{F}_{\mathbb{Z}}\to\widehat{\mathcal{LG}}_{\mathbb{Z}}.$$

Note that

$$\varphi([u,v])=1\quad\text{for } u,v\in\langle a\rangle^F,$$

where $\langle a\rangle^F$ is the normal subgroup of F generated by a.

Proposition 4.1 *For any sequence of integers $q=(q_1,q_2,\dots)$, there exists a pair of elements $r_q,s_q\in\gamma_3(\widehat{F}_{\mathbb{Z}})$ such that*

(1) $[r_q,a][s_q,b]=1$;

(2) $\widehat{\varphi}(s_q)=1$;

(3) $\widehat{\varphi}(r_q)=\prod_{i=3}^{\infty}[a,_{i-1}b]^{n_i}$, *where $n_{2i+1}=q_i$ for $i\ge 1$ and n_{2i} are some integers* (we control only odd terms of the product).

Proof We claim there are sequences of elements $r_q^{(3)},r_q^{(4)},\dots\in F$ and $s_q^{(3)},s_q^{(4)},\dots\in F$ such that

(0) $r_q^{(k)},s_q^{(k)}\in\gamma_k(F)$;

(1) $\left[\prod_{i=3}^{k}r_q^{(i)},a\right]\left[\prod_{i=3}^{k}s_q^{(i)},b\right]\in\gamma_{k+2}(F)$;

(2) $\varphi(s_q^{(k)})=1$;

(3) $\varphi\Bigl(\prod_{i=3}^{k}r_q^{(i)}\Bigr)\equiv\prod_{i=3}^{k}[a,_{i-1}b]^{n_i}\mod\gamma_{k+1}(\mathcal{LG})$, where $n_{2i+1}=q_i$ for $2i+1\le k$.

Then we take $r_q=\prod_{i=3}^{\infty}r_q^{(i)}$ and $s_q=\prod_{i=3}^{\infty}s_q^{(i)}$ and the assertion follows. Thus it is sufficient to construct such elements $r_q^{(k)}$ and $s_q^{(k)}$ inductively.

In order to prove the base case we set

$$r_q^{(3)} := [a,b,b]^{q_1} \quad \text{and} \quad s_q^{(3)} := [a,b,a]^{-q_1}.$$

Corollary 4.1, with $n=1$, implies that

$$[r_q^{(3)}, a][s_q^{(3)}, b] \in \gamma_5(F).$$

Clearly $s_q^{(3)}, r_q^{(3)} \in \gamma_3(F)$, $\varphi(s_q^{(3)}) = 1$ and $\varphi(r_q^{(3)}) = [a,_2 b]^{q_1}$.

In order to prove the inductive step, assume that we already constructed

$$r_q^{(3)}, \dots, r_q^{(k)}, s_q^{(3)}, \dots, s_q^{(k)},$$

with properties (0)–(3). Construct $r_q^{(k+1)}$ and $s_q^{(k+1)}$. Note that any element of $\gamma_{k+2}(F)/\gamma_{k+3}(F)$ can be presented as $[A,a][B,b]\cdot\gamma_{k+3}(F)$, where $A, B \in \gamma_{k+1}(F)$. Then

$$\Bigl[\prod_{i=3}^{k} r_q^{(i)}, a\Bigr]\Bigl[\prod_{i=3}^{k} s_q^{(i)}, b\Bigr] \equiv [A,a][B,b] \mod \gamma_{k+3}(F). \tag{4-4}$$

Using that the images of $[A^{-1}, a]$ and $[B^{-1}, b]$ are in the center of $F/\gamma_{k+3}(F)$, that $\prod_{i=3}^{k} r_q^{(i)}, \prod_{i=3}^{k} s_q^{(i)} \in \gamma_3(F)$ and the identity $[xy,z] = [x,z]^y \cdot [y,z]$ we obtain

$$\Bigl[\prod_{i=3}^{k} r_q^{(i)} A^{-1}, a\Bigr]\cdot\Bigl[\prod_{i=3}^{k} s_q^{(i)} B^{-1}, b\Bigr] \in \gamma_{k+3}(F). \tag{4-5}$$

Next we prove that

$$\varphi(B) = 1.$$

Since $B \in \gamma_{k+1}(F)$ we have

$$B \equiv [a,_k b]^e c \mod \gamma_{k+2}(F),$$

where $e \in \mathbb{Z}$ and c is a product of powers of other basic commutators of weight $k+1$. All these other basic commutators contain a at least twice. It follows that $\varphi(c) = 1$. Since $A \in \gamma_3(F) \subseteq \langle a \rangle^F$, we have $\varphi([A,a]) = 1$. Moreover,

$$\varphi\Bigl(\Bigl[\prod_{i=3}^{k} r_q^{(i)}, a\Bigr]\Bigl[\prod_{i=3}^{k} s_q^{(i)}, b\Bigr]\Bigr) = 1.$$

Then

$$[a,_{k+1} b]^e \in \gamma_{k+3}(\mathcal{LG}).$$

This implies that $e = 0$ and hence $\varphi(B) = 1$.

If k is odd, we do need to care about (3) and we just take

$$r_q^{(k+1)} = A^{-1} \quad \text{and} \quad s_q^{(k+1)} = B^{-1}.$$

Indeed, it is easy to check that properties (0)–(2) are satisfied and property (3) automatically follows.

Suppose that k is even, say $k = 2k'$. Consider the image of the element $\prod_{i=3}^{k} r_q^{(i)} \cdot A^{-1}$ in the quotient $\mathcal{LG}/\gamma_{k+2}(\mathcal{LG})$. By the induction hypothesis,

$$\varphi\left(\prod_{i=3}^{k} r_q^{(i)}\right) \equiv \prod_{i=3}^{k} [a,_{i-1} b]^{n_i} \cdot c' \mod \gamma_{k+2}(\mathcal{LG}),$$

where $c' \in \gamma_{k+1}(\mathcal{LG})$. Since the quotient $\gamma_{k+1}(\mathcal{LG})/\gamma_{k+2}(\mathcal{LG})$ is cyclic with generator $[a,_k b] \cdot \gamma_{k+2}(\mathcal{LG})$,

$$c' \equiv [a,_k b]^y \mod \gamma_{k+2}(\mathcal{LG})$$

for some $y \in \mathbb{Z}$. For $n \geq 1$, denote

$$z_n := \prod_{i=0}^{n-1} [[a,_{2n-1-i} b], [a,_i b]]^{(-1)^i}.$$

Corollary 4.1 implies that

$$[[a,_k b], a][z_{k'}^{-1}, b] \in \gamma_{k+3}(F).$$

We set

$$r_q^{(k+1)} := A^{-1}[a,_k b]^{q_{k'}-e} \quad \text{and} \quad s_q^{(k+1)} := B^{-1} z_{k'}^{-(q_{k'}-e)}.$$

Now

$$\left[\prod_{i=3}^{k+1} r_q^{(i)}, a\right]\left[\prod_{i=3}^{k+1} s_q^{(i)}, b\right] \in \gamma_{k+3}(F)$$

and

$$\varphi\left(\prod_{i=3}^{k+1} r_q^{(i)}\right) \equiv \prod_{i=3}^{k+1} [a,_{i-1} b]^{n_i}.$$

Properties (0) and (2) are obvious. □

5 Proof of Theorems 1 and 2

Let F be a free group of rank ≥ 2 and p be a prime. We will show that the image of the homomorphism $H_2(\widehat{F}_{\mathbb{Z}}, \mathbb{Z}) \to H_2(\widehat{F}_{\mathbb{Q}}, \mathbb{Q})$ is uncountable. The proof that the image of the map $H_2(\widehat{F}_{\mathbb{Z}}, \mathbb{Z}) \to H_2(\widehat{F}_{\mathbb{Z}}, \mathbb{Z}/p)$ is uncountable is similar.

Since the free group with two generators is a retract of a free group of higher rank, it is enough to prove this only for $F = F(a,b)$. The map

$$H_2(\widehat{F}_{\mathbb{Z}}, \mathbb{Z}) \to H_2(\widehat{\mathcal{L}\mathcal{G}}_{\mathbb{Q}}, \mathbb{Q}) \tag{5-1}$$

factors through $H_2(\widehat{F}_{\mathbb{Q}}, \mathbb{Q})$. Then it is enough to prove that the image of the map (5-1) is uncountable.

For $q \in \{0,1\}^{\mathbb{N}}$ we denote by r_q and s_q some fixed elements of $\widehat{F}_{\mathbb{Z}}$ satisfying properties (1)–(3) of Proposition 4.1. Then

$$\widehat{\varphi}(r_q) = \prod_{i=3}^{\infty} [a,_{i-1} b]^{n_i(q)},$$

where $n(q)_{2i+1} = q_i$,

$$[r_q, a][s_q, b] = 1 \quad \text{and} \quad \widehat{\varphi}(s_q) = 1.$$

Set

$$f_q = \sum_{i=3}^{\infty} n_i(q) x^{i-1} \in \mathbb{Z}[\![x]\!].$$

If we consider $\widehat{\mathcal{L}\mathcal{G}}_{\mathbb{Z}}$ as the semidirect product $\mathbb{Z}[\![x]\!] \rtimes C$, we obtain that $[a,_{i-1} b] = (x^{i-1}, 1)$ and hence

$$\widehat{\varphi}(r_q) = (f_q, 1).$$

If we denote by $\widehat{\varphi}_{\mathbb{Q}}$ the composition of $\widehat{\varphi}$ with the map $\widehat{\mathcal{L}\mathcal{G}}_{\mathbb{Z}} \to \widehat{\mathcal{L}\mathcal{G}}_{\mathbb{Q}}$, we obtain

$$\widehat{\varphi}_{\mathbb{Q}}(r_q) = (f_q^{\mathbb{Q}}, 1),$$

where $f_q^{\mathbb{Q}}$ is the image of f_q in $\mathbb{Q}[\![x]\!]$. Consider the map

$$\Theta_{\mathbb{Q}} \colon \mathbb{Q}[\![x]\!] \to H_2(\widehat{\mathcal{L}\mathcal{G}}_{\mathbb{Q}}, \mathbb{Q}), \quad \text{given by } f \mapsto ((f,1) \wedge 1) \otimes \mathbb{Q}.$$

Observe that this map is the composition of the map from Proposition 2.1 and the isomorphism from Proposition 3.1. Therefore the kernel of $\Theta_{\mathbb{Q}}$ is countable. Set

$$A := \{ f_q^{\mathbb{Q}} \mid q \in \{0,1\}^{\mathbb{N}} \} \subseteq \mathbb{Q}[\![x]\!].$$

Using that $f_q^{\mathbb{Q}} = \sum_{i=3}^{\infty} n_i(q) x^{i-1}$, where $n_{2i+1}(q) = q_i$, we obtain that A is uncountable. Using that the kernel of $\Theta_{\mathbb{Q}}$ is countable, we obtain that its image

$$\Theta_{\mathbb{Q}}(A) = \big\{ ((f_q^{\mathbb{Q}}, 1) \wedge 1) \otimes \mathbb{Q} \mid q \in \{0,1\}^{\mathbb{N}} \big\} \subseteq H_2(\widehat{\mathcal{L}\mathcal{G}}_{\mathbb{Q}}, \mathbb{Q})$$

is uncountable. Finally, observe that any element $((f_q^{\mathbb{Q}}, 1) \wedge 1) \otimes \mathbb{Q}$ of $\Theta_{\mathbb{Q}}(A)$ has a preimage in $H_2(\widehat{F}_{\mathbb{Z}}, \mathbb{Z})$ given by $(r_q \wedge a)(s_q \wedge b)$, and then $\Theta_{\mathbb{Q}}(A)$ lies in the image of $H_2(\widehat{F}_{\mathbb{Z}}, \mathbb{Z}) \to H_2(\widehat{\mathcal{L}\mathcal{G}}_{\mathbb{Q}}, \mathbb{Q})$. This implies that the groups $H_2(\widehat{F}_{\mathbb{Q}}, \mathbb{Q})$ and $H_2(\widehat{F}_{\mathbb{Z}}, \mathbb{Z}/p) \cong H_2(\widehat{F}_{\mathbb{Z}}, \mathbb{Z}) \otimes \mathbb{Z}/p$ are uncountable and Theorems 1 and 2 follow.

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