Geometry & Topology 23 (2019) 1339–1413

Sasaki–Einstein metrics and K–stability

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We show that a polarized affine variety with an isolated singularity admits a Ricci flat Kähler cone metric if and only if it is K–stable. This generalizes the Chen–Donaldson–Sun solution of the Yau–Tian–Donaldson conjecture to Kähler cones, or equivalently, Sasakian manifolds. As an application we show that the five-sphere admits infinitely many families of Sasaki–Einstein metrics.

32Q20, 53C25; 32Q26

1 Introduction

The existence of Kähler–Einstein metrics is a fundamental problem in Kähler geometry. If M is a compact complex manifold with $c_1(M) = 0$ or $c_1(M) < 0$, then the work of Yau [80] shows that M admits Kähler–Einstein metrics with zero or negative Ricci curvature. The case when $c_1(M) > 0$ is more subtle, and the Yau–Tian–Donaldson conjecture [38; 78; 82], proved by Chen–Donaldson–Sun [21; 22; 23], relates the existence of a Kähler–Einstein metric on M to the K–stability of M, which is a certain algebro-geometric condition. Our goal in the present paper is to generalize this result to the setting of Kähler cones, giving a criterion for the existence of a Ricci flat Kähler cone metric, or equivalently, a Sasaki–Einstein metric on the link. The question of existence of such metrics has received increasing attention in the physics community through their connection to the AdS/CFT correspondence (see Klebanov–Witten [55] and Maldacena [63]), and we anticipate further developments along these lines (see eg Collins–Xie–Yau [29]).

Quite generally, if X is an affine variety with an isolated singular point, one can ask whether X admits a Ricci flat Kähler cone metric. We will address this question under the extra assumption that we fix the vector field on X that gives the homothetic scaling on the cone. More precisely, suppose that $X \subset \mathbb{C}^N$ is an affine variety, with an isolated singular point at the origin, invariant under the action of a torus $\mathbb{T} \subset U(N)$, which for simplicity we assume to be diagonal. We call $\xi \in \mathfrak{t}$ a *polarization* of X if it acts with

Published: 28 May 2019 DOI: 10.2140/gt.2019.23.1339

positive weights on the coordinate functions, ie if the corresponding holomorphic vector field ξ satisfies $L_\xi(z_i) = i a_i z_i$ with $a_i > 0$. We then seek a Kähler Ricci flat metric ω on X such that $L_{-J\xi}\omega = 2\omega$. We say that such a metric ω is a Ricci flat Kähler cone metric on the pair (X, ξ). Such a Ricci flat Kähler cone metric can only exist on X if the pair (X, ξ) is a normalized Fano cone singularity, in the terminology of Definition 2.1.

Our main result is the following.

Theorem 1.1 *Let (X, ξ) be a normalized Fano cone singularity. Then (X, ξ) admits a Ricci flat Kähler cone metric if and only if it is K–stable.*

We will give a precise definition of K–stability in this setting below. For now let us say that if (X, ξ) does not admit a Ricci flat Kähler cone metric, then there exists an embedding $X \hookrightarrow \mathbb{C}^{N'}$, a corresponding embedding of the torus $\mathbb{T} \subset U(N')$, and a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to \mathrm{GL}(N')^{\mathbb{T}}$ generated by a vector field w with the following properties:

(1) The limit $Y = \lim_{t\to 0} \lambda(t)\cdot X$ is normal.

(2) The Futaki invariant satisfies $\mathrm{Fut}(Y, \xi, w) \leqslant 0$, and $Y \ncong X$ if equality holds.

While in principle $\mathbb{T}$ just needs to be a torus of automorphisms of X for which $\xi \in \mathfrak{t}$, in practice it is useful to choose a maximal such torus. In fact as in Datar–Székelyhidi [31] we can also obtain an equivariant version of the theorem for the action of any compact group on X, but to simplify the exposition we will mostly focus on the case of a torus.

Usually there are infinitely many such degenerations that one needs to check in order to determine whether a pair (X, ξ) is K–stable, and so there does not seem to be an effective way to test K–stability. This become possible, however, in certain situations with large symmetry group, where there are only a finite number of possible normal limits Y under equivariant degenerations of X. Just as in [31], a simple example is when X is toric, in which case if we work equivariantly with respect to the maximal torus $\mathbb{T}$, then we necessarily have $Y \cong X$, and we only need to test the Futaki invariants $\mathrm{Fut}(X, \xi, \eta)$ for $\eta \in \mathfrak{t}$. If these vanish for all $\eta \in \mathfrak{t}$, then (X, ξ) admits a Ricci flat Kähler cone metric, recovering the result of Futaki–Ono–Wang [44].

A more general situation is when X has a complexity-one action of a torus $\mathbb{T}$, ie when $\dim \mathbb{T} = \dim X - 1$. In this case, using the methods in Ilten–Süß [52] we can still effectively test K–stability, by checking a finite number of degenerations. In Section 8 we will apply these techniques to several explicit families of hypersurface singularities.

For example, we study

$$Z_{\mathrm{BP}}(p,q)=\{x^2+y^2+z^p+w^q=0\}\subset\mathbb{C}^4,$$

where $p,q>1$. We show the following.

Theorem 1.2 *For a suitable choice of ξ the pair $(Z_{\mathrm{BP}}(p,q),\xi)$ admits a Ricci flat Kähler cone metric if and only if $2p>q$ and $2q>p$. As a consequence S^5 admits infinitely many families of Sasaki–Einstein metrics.*

The necessary conditions $2p>q$ and $2q>p$ follow from the Lichnerowicz obstruction of Gauntlett–Martelli–Sparks–Yau [46], while the existence result was only known previously for $(p,q)=(2,2)$ and $(2,3)$, where the latter was shown by Li–Sun [61].

To date, many Sasaki–Einstein manifolds have been found by employing estimates for the α–invariant; see Demailly–Kollár [35] and Tian [77]. For example, the affine varieties $Z_{\mathrm{BP}}(p,q)$ are a special case of the Brieskorn–Pham singularities, which have been thoroughly studied in the literature. Boyer–Galicki–Kollár [15] used estimates for the α–invariant of Brieskorn–Pham singularities to produce 68 distinct Sasaki–Einstein metrics on S^5, as well as SE metrics on all 28 oriented diffeomorphism types of S^7, and the standard and Kervaire spheres S^{4m+1}. Note that infinitely many Einstein (not Sasakian) metrics on spheres in dimensions 5 to 9 were constructed previously by Böhm [11].

Estimates for the α–invariant were also used by Boyer–Galicki [12; 13], Boyer–Nakamaye [17], Johnson–Kollár [54], Ghigi–Kollár [47], Kollár [58; 56] and others to produce many infinite families of Sasaki–Einstein metrics in dimensions 5 and 7, and higher. For example, $\#^k(S^2\times S^3)$ is known to admit infinite families of Sasaki–Einstein metrics for any $k\geqslant 1$. We refer the reader to Boyer–Galicki [14] for a thorough discussion of these results. We note that Kollár [56; 57; 59] has classified the possible topologies of Sasaki–Einstein manifolds. For example it is known that for affine varieties of complex dimension 3 with a 2–torus action, the only possible topologies of the links are S^5 and $\#^k(S^2\times S^3)$ for any $k\geqslant 1$ (see [14, Proposition 10.2.27]). Our techniques also produce new infinite families of distinct Sasaki–Einstein metrics on $\#^k(S^2\times S^3)$ for all $k\geqslant 1$, and hence cover all possible topologies that can occur with a 2–torus action.

We expect that many more examples can be found along the same lines. A particularly interesting problem is to find Sasaki–Einstein metrics with irregular Reeb vector fields. Remarkably, the first examples of irregular Sasaki–Einstein metrics were discovered by Gauntlett–Martelli–Sparks–Waldram [45] by explicitly writing down the metric in coordinates. We expect K–stability to be particularly useful for finding irregular Sasaki–Einstein manifolds in real dimension 5, since if the cone X has $\dim_{\mathbb{C}} X = 3$, and ξ is an irregular Reeb field, then X admits a complexity-one action of a 2–torus. In particular, using the methods of Ilten–Süß [52] we can effectively test whether (X, ξ) admits a Ricci flat Kähler cone metric.

The overall strategy of our proof is the same as that of Chen–Donaldson–Sun [21; 22; 23], as adapted in Datar–Székelyhidi [31] and Székelyhidi [74] to the smooth continuity method. We will set up this continuity method in Section 2, where we also give the precise definition of K–stability based on our previous work [28], extending the definition of Ross–Thomas [67] from the quasiregular case. The main technical results are contained in Sections 3 and 4. In Section 3 we discuss weak solutions of the equations along the continuity method, which is analogous to the theory of singular Kähler–Einstein metrics, as was studied by Eyssidieux–Guedj–Zeriahi [43]. Much of this discussion, such as the convexity of the Ding functional due to Berndtsson [9], extends to the case of cones without substantial difficulties. In Section 4 we generalize the partial C^0–estimate along the smooth continuity method from [74] to the setting of cones. The main new technical difficulty is that in the method of [74] the strict positivity of the Ricci curvature was a crucial ingredient, while in our setting the Ricci curvature on a cone is never strictly positive. Instead we need to exploit the transverse Kähler structure, which does have strictly positive Ricci curvature. The proof of one direction of Theorem 1.1 is given in Section 5, primarily along the lines of the argument in [31]. In Section 6 we collect some more algebraic results, with the goal of establishing the equality between the differential geometric and the algebraic definitions of the Futaki invariant. In Section 7 we prove the other implication in Theorem 1.1 along the lines of the work of Berman [6]. In Section 8 we give some example calculations of K–stability, including the proof of Theorem 1.2 and we finish with some further discussion and questions in Section 9.

2 Basic definitions

In this section we fix some basic definitions, and set up the continuity method that we would like to use to find Ricci flat Kähler cone metrics. The continuity method is equivalent to the usual continuity path for finding Kähler–Einstein metrics, but it involves a scaling to ensure that we have metrics of nonnegative Ricci curvature on our cones.

Definition 2.1 A *polarized affine variety* of dimension n is a triple $(X, \mathbb{T}, \xi)$, where X is a normal affine variety, $\dim_{\mathbb{C}} X = n$, $\mathbb{T}$ is a torus of automorphisms of X, and $\xi \in \mathfrak{t}$ acts on the ring of functions of X with positive weights in the following sense. We have a decomposition

$$R(X) = \bigoplus_{\chi \in \mathfrak{t}^*} R_\chi(X)$$

under the torus action into weight spaces, and we require that $\chi(\xi) > 0$ for all nonzero χ for which R_χ is nontrivial. Often we simply speak of a pair (X, ξ), where ξ is a vector field on X generating a compact torus of automorphisms, and then $\mathbb{T}$ is understood to be this torus. We call ξ a Reeb field or polarization on X. We denote by $\mathcal{C}_R \subset \mathfrak{t}$ the cone of Reeb fields.

We say that the pair (X, ξ) is a *Fano cone singularity* if X is $\mathbb{Q}$–Gorenstein and there is a trivializing section Ω of mK_X for some $m > 0$ such that $L_\xi \Omega = i\lambda\Omega$ for some $\lambda > 0$. The last condition is equivalent to X having log-terminal singularities (see Section 6). The Fano cone singularity (X, ξ) is *normalized* if $\lambda = nm$.

The basic example is obtained by taking a Fano manifold M, and letting X be the total space of mK_M, with the zero section blown down, for some m such that $-mK_M$ is very ample. In other words X is the cone over M under a projective embedding by $-mK_M$.

In [28] we defined a notion of K–semistability for a pair (X, ξ), in terms of test configurations for X that commute with a torus $\mathbb{T}$ whose Lie algebra contains ξ. Here we give a very similar definition, which is adapted to our work here, but is closer in spirit to the definition of K–stability by Tian [78], which only allows test configurations with normal central fibers. In addition, in view of possible future applications we work equivariantly for a compact group acting on X in analogy with [31].

Suppose that (X, ξ) is a normalized Fano cone singularity of dimension n, with X only having an isolated singularity, and G is a compact group of automorphisms of X such that ξ is in the center of its Lie algebra. In applications we will take G to be a maximal torus of automorphisms, containing the torus generated by ξ.

A G–equivariant *special degeneration* (or test configuration) of X consists of an embedding $X \to \mathbb{C}^N$ such that G acts linearly through an embedding $G \subset U(N)$, together with a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to \mathrm{GL}(N)^G$ commuting with G, such that $\lambda(S^1) \subset U(N)$ and $Y = \lim_{t\to 0} \lambda(t) \cdot X$ is normal. In this case (Y, ξ) is also a normalized Fano cone singularity, together with a $\mathbb{C}^*$–action given by λ commuting with ξ. Let us write $\mathbb{T}$ for the torus generated by λ and ξ. By a slight abuse of notation we will denote by $\lambda \in \mathfrak{t}$ the generator of the corresponding S^1–action, and note that for small $s \in \mathbb{R}$ the pairs $(Y, \xi + s\lambda)$ are also Fano cone singularities (which may not be normalized). We showed in [28] that the index character

$$F(\xi, t) = \sum_{\chi \in \mathfrak{t}^*} e^{-t\chi(\xi)} \dim R_\chi(Y)$$

can be extended meromorphically to a neighborhood of the origin, and we can define functions $a_i(\xi)$ by

$$F(\xi, t) = \frac{a_0(\xi)(n-1)!}{t^n} + \frac{a_1(\xi)(n-2)!}{t^{n-1}} + O(t^{2-n}).$$

As a matter of notation we will write

$$D_\lambda a_i(\xi) = \frac{d}{ds}\bigg|_{s=0} a_i(\xi + s\lambda). \tag{2-1}$$

Definition 2.2 The Futaki invariant of a special degeneration as above is defined to be

$$\mathrm{Fut}(X, \xi, \lambda) = \frac{a_0(\xi)}{n-1} D_\lambda\left(\frac{a_1}{a_0}\right)(\xi) + \frac{a_1(\xi) D_\lambda a_0(\xi)}{n(n-1)a_0(\xi)},$$

where each a_i is computed on the variety Y.

A normalized Fano cone singularity (X, ξ) is called G–equivariantly K–stable if for all special degenerations as above, we have $\mathrm{Fut}(X, \xi, \lambda) \geqslant 0$, and equality holds only if (Y, ξ) is isomorphic to (X, ξ).

Remark 2.3 In the case that (X, ξ) is the cone over a projective Fano manifold M, with ξ induced by the natural $\mathbb{C}^*$–action on the fibers of K_M, our definition of K–stability is equivalent to the usual definition of K–stability due to Tian [78] and refined by Donaldson [38]. This can be seen by showing that a test configuration as defined above gives rise to a filtration of the homogeneous coordinate ring $\bigoplus_m H^0(M, -mK_M)$. The central fiber Y is then the associated graded of this filtration, with finitely generated coordinate ring. It is well known (see [76; 79]) that this data is equivalent to the data of a test configuration for the projective variety M, and vice versa.

Our main result, Theorem 1.1, then says that X admits a Ricci flat Kähler cone metric with homotheties given by $-J\xi$ if and only if (X, ξ) is equivariantly K–stable.

Let us digress briefly on the Futaki invariant in Definition 2.2. As remarked above, given a special test configuration generated by a $\mathbb{C}^*$–action λ, the central fiber Y is again a $\mathbb{Q}$–Gorenstein variety with log-terminal singularities, and hence $(Y, \xi + s\lambda)$ is a Fano cone singularity, which is not necessarily normalized. As we will see in Proposition 6.4, the normalized Reeb vector fields form a linear subspace Σ_Y of the Reeb cone of Y defined by the *linear* equation

$$\frac{a_1(w)}{a_0(w)} = \frac{n(n-1)}{2}.$$

Let $N := \nabla(a_1/a_0)$ denote the normal vector to the normalized hyperplane Σ_Y.

First consider the case when $Y \cong X$, and that λ is generated by $w \in \mathrm{Lie}(\mathbb{T})$. Assume that w is tangent to Σ_X, then the Futaki invariant is just $\frac{1}{2} D_w a_0(\xi)$. Since we can also consider $-w$, this implies that if (X, ξ) is K–stable, then ξ must be an extremal value of a_0, on Σ_X. By [65, Equation (1.10)], a_0 is a convex function on Σ_X which has a unique minimum. Since a_0 can be interpreted as the volume of the link, this is called "volume minimization", and was discovered in fundamental work of Martelli–Sparks–Yau [65].

Now suppose we have a nontrivial test configuration, so that $Y \not\cong X$, and λ is generated by w and suppose that ξ is the Reeb vector field minimizing the volume. We can compute the Futaki invariant by the formula

$$\tfrac{1}{2} D_{w'} a_0(\xi) = \mathrm{Fut}(X, \xi, \lambda) \quad \text{if } w' = w - 2\frac{N \cdot w}{n(n-1)}\xi,$$

where now w' is normalized, but it may not generate a test configuration if ξ is irrational. This observation extends the interpretation of stability as volume minimization [65; 64; 28] from trivial test configurations to all test configurations, and will be useful in Section 6. Note that when $Y \not\cong X$ we cannot replace w with $-w$, since this will change the central fiber of the test configuration. These observations have applications in conformal field theory, where the AdS/CFT correspondence provides an interpretation of K–stability as a maximization problem for the central charge of the dual conformal field theory [29].

We next set up the relevant continuity method for polarized affine manifolds. Suppose that (X, ξ) is normalized Fano. Fixing any metric α on (X, ξ), our continuity method is to find metrics ω_t on (X, ξ) satisfying

$$\mathrm{Ric}(\omega_t) = 2n[t\omega_t^{\tau} + (1-t)\alpha^{\tau}] - 2n\omega_t^{\tau}, \tag{2-2}$$

where ω_t^τ and α^τ denote the transverse metrics induced by ω_t and α. In terms of the transverse metrics induced on the Reeb foliation of the link $L = \{r_t = 1\}$ the method of continuity is

$$\mathrm{Ric}^\tau(\omega_t^\tau) = 2n[t\omega_t^\tau + (1-t)\alpha^\tau].$$

In particular, (2-2) is the natural lift to the cone of the continuity method for Kähler–Einstein metrics (see for example [4]). We will call (2-2) the twisted equation, with twisting form α^τ.

Proposition 2.4 *Let $I = \{t \in [0,1] :$ (2-2) has a solution$\}$. Then I is nonempty, and relatively open in $[0,1]$.*

The nonemptiness follows from the transverse version of Yau's theorem [80], due to El Kacimi-Alaoui [42], while the openness is also analogous to the Kähler case as in Aubin [4].

As in the compact Kähler case, we must study the Gromov–Hausdorff limit of a sequence (X, ω_{t_i}) as $t_i \to T$. For this it is convenient to do a scaling of the Reeb fields to ensure that we have metrics with Ricci curvature bounded below. Let us denote the radial function of ω_t by r_t, and define $\tilde{r}_t = r_t^t$; in the Sasakian literature, this is often referred to as a D–homothetic transformation. It is straightforward to verify that $\tilde{\omega}_t = \sqrt{-1}\partial\bar{\partial}\tilde{r}_t$ satisfies

$$\mathrm{Ric}(\tilde{\omega}_t) = 2n\frac{1-t}{t}\alpha^\tau;$$

ie the Ricci curvature is nonnegative.

3 Weak solutions, twisted Futaki invariants and the Ding functional

The key result that we will ultimately need is that in the context of the continuity method defined in the previous section, as $t \to \sup I$, we can extract a limit that is a normalized Fano cone singularity (Y, ξ) together with a transverse positive current β^τ, and a weak solution ω_T on (Y, ξ) of the equation

$$\mathrm{Ric}(\omega_T) = 2n(1-T)[\beta^\tau - \omega_T^\tau].$$

In this section we will give a precise definition of such weak solutions, and describe how in analogy with the compact Kähler case, the existence of such a metric implies the reductivity of a certain automorphism group, and the vanishing of a twisted Futaki invariant.

We first define the weak solutions of the twisted equation. We assume that (Y, ξ) is a normalized Fano cone singularity, so we have a nonvanishing global holomorphic section Ω of mK_Y for some $m \geqslant 1$, with $L_\xi(\Omega) = imn\Omega$. This gives rise to the volume form

$$dV = i^{n^2}(\Omega \wedge \overline{\Omega})^{1/m},$$

which satisfies $L_\xi \, dV = 2n \, dV$. This volume form is uniquely defined up to a constant multiple, and we will call it the canonical volume form on a normalized Fano cone singularity.

Suppose that we have an embedding $Y \to \mathbb{C}^N$ such that the Reeb field (or rather the torus it generates) acts diagonally, and Y is not contained in a linear subspace. Then ξ defines a Reeb field on $\mathbb{C}^N$ and so we can then fix a smooth reference radial function $\hat{r}$ on $\mathbb{C}^N$ which is compatible with this Reeb field [28]. In the presence of the action of a torus $\mathbb{T}$, we can take our embedding to be $\mathbb{T}$–equivariant as well, where $\mathbb{T}$ acts diagonally.

The space of transverse psh potentials is the space of basic functions φ (ie $L_\xi \varphi = L_{J\xi}\varphi = 0$) such that $r_\varphi = \hat{r}e^\varphi$ is psh. Recall that a psh function on a normal variety can always be viewed locally as the restriction of a psh function from an ambient space, after embedding [33, Theorem 1.10]. In particular, r_φ is always the restriction of a psh function defined in a neighborhood of the origin. For smooth such φ we write

$$\omega_\varphi = \tfrac{1}{2}\sqrt{-1}\partial\bar{\partial} r_\varphi^2,$$

and we suppose that we have a twisting form β^τ given as

$$\beta^\tau = \sqrt{-1}\partial\bar{\partial}\log r_\psi,$$

where ψ is also a transverse psh potential. If Y and β^τ were smooth, then the twisted equation

$$\mathrm{Ric}(\omega_\varphi) = 2n(1-t)[\beta^\tau - \omega_t^\tau] \tag{3-1}$$

could be written on the level of volume forms as

$$\omega_\varphi^n = Ce^{2n(1-t)[\varphi-\psi]}\, dV, \tag{3-2}$$

and we can use this latter formulation to define solutions of the twisted equation in a weak sense (see [43] for the analogous definition of weak Kähler–Einstein metrics). More precisely, given ψ and t, a weak solution of (3-1) is a continuous transverse psh potential φ such that (3-2) holds as an equality of measures. In particular this implies that $e^{-2n(1-t)\psi}\,dV$ must be integrable in a neighborhood of the cone vertex.

Definition 3.1 Let (Y,ξ) be a normalized Fano cone singularity together with a reference radial function $\hat{r}$ as above. Suppose that ψ is a transverse psh potential such that $e^{-2n(1-t)\psi}\,dV$ is integrable in a neighborhood of the vertex for some $t\in[0,1]$. We say that $(Y,\xi,(1-t)\psi)$ admits a (weak) solution of the twisted equation if (Y,ξ) has a continuous transverse psh potential φ satisfying

$$\omega_\varphi^n = Ce^{2n(1-t)[\varphi-\psi]}\,dV$$

in the sense of measures; that is, for any Borel set $B\subset Y$ we have

$$\int_{B\cap Y_{\mathrm{reg}}}\omega_\varphi^n = C\int_{B\cap Y_{\mathrm{reg}}} e^{2n(1-t)[\varphi-\psi]}\,dV.$$

Often we will write $(Y,\xi,(1-t)\beta^\tau)$ when the twisting form β^τ is more natural than its potential ψ.

As in [31, Remark 4] (see also [7, Proposition 3.8]), it is enough to check that outside of a closed set Σ with vanishing $(2n-2)$–dimensional Hausdorff measure, ω_φ^n defines a singular metric e^{-f} on K_Y with $f\in L^1_{\mathrm{loc}}$, and in addition, on $Y\setminus\Sigma$,

$$\sqrt{-1}\partial\bar{\partial} f = 2n(1-t)[\beta^\tau-\omega_\varphi^\tau].$$

The main properties of weak solutions of the twisted equation are the reductivity of the automorphism group, and the vanishing of the twisted Futaki invariant, analogous to Propositions 7 and 8 in [31]. Let us first define the relevant automorphism group, or rather its Lie algebra.

Definition 3.2 Suppose we have a triple (Y,ξ,β^τ) as above. We define $\mathfrak{g}_{Y,\xi}$ to be the space of holomorphic vector fields on Y (more precisely on the regular part), commuting with ξ. We then define

$$\mathfrak{g}_{Y,\xi,\beta^\tau} = \{w\in\mathfrak{g}_{Y,\xi} : \iota_w\beta^\tau = 0\}.$$

As in [31], note that $\mathfrak{g}_{Y,\xi,\beta^\tau}$ is in general smaller than the space of holomorphic vector fields w on Y commuting with ξ, and preserving β^τ in the sense that $L_w\beta^\tau=0$. For instance if β^τ is a transverse Kähler form, then $\mathfrak{g}_{Y,\xi,\beta^\tau}$ is spanned by Ξ, where $\operatorname{Im}\Xi=\xi$.

The following is analogous to [31, Proposition 7], the special case of which, without the twisting form, has been shown in Donaldson–Sun [41]. The proof is based on a uniqueness theorem due to Berndtsson [10], generalizing the classical Bando–Mabuchi theorem [5], further extended by Berman–Boucksom–Eyssidieux–Guedj–Zeriahi [7], Berman–Witt Nyström [8], Chen–Donaldson–Sun [23] and Yi [84].

Proposition 3.3 *Suppose that $(Y, \xi, (1-t)\beta^{\tau})$ admits a solution ω_t of the twisted equation. Then $\mathfrak{g}_{Y,\xi,\beta^\tau}$ is reductive. Moreover if G is a compact group of biholomorphisms of Y, commuting with ξ and fixing ω_t, then the centralizer $(\mathfrak{g}_{Y,\xi,\beta^\tau})^G$ is also reductive.*

Next we define the twisted Futaki invariant. Suppose that w is a vector field commuting with ξ that preserves the radial function $\hat{r}$ above. In addition suppose that w is the real part of a holomorphic vector field in $\mathfrak{g}_{Y,\xi,\beta^\tau}$. The transverse Hamiltonian θ_w is defined by letting $\theta_w \hat{r}^2$ be a Hamiltonian for w, ie by requiring that it satisfy

$$\theta_w \hat{r}^2 = \iota_{Jw} d\left(\tfrac{1}{2}\hat{r}^2\right). \tag{3-3}$$

Note that with this convention the transverse Hamiltonian of the Reeb field ξ is $\theta_\xi = -1$. We then define the twisted Futaki invariant to be

$$\mathrm{Fut}_{Y,\xi,(1-t)\beta^\tau}(w) = \frac{t}{V}\int_Y \theta_w e^{-\frac{1}{2}\hat{r}^2}\omega^n - t\,\frac{\int_Y \theta_w e^{-2n(1-t)\psi} e^{-\frac{1}{2}\hat{r}^2}\, dV}{\int_Y e^{-2n(1-t)\psi} e^{-\frac{1}{2}\hat{r}^2}\, dV}.$$

Proposition 6.8 will show that for $t = 1$, this definition agrees with the algebraic definition given in Definition 2.2. Below we will also give a different formula for the twisted Futaki invariant when $t \neq 1$.

We have the following, analogous to [31, Proposition 8].

Proposition 3.4 *If (Y, ξ) admits a weak solution of the twisted equation above, then $\mathrm{Fut}_{Y,\xi,(1-t)\beta^\tau}(w) = 0$ for all vector fields w as above.*

The proofs of Propositions 3.3 and 3.4 both follow from convexity properties of the twisted Ding functional along weak geodesic segments, based essentially on work of Berndtsson [10]. The arguments follow those in [10] (see also [31, Section 6]) closely, together with the discussion in Donaldson–Sun [41] on extending these to the setting of cones. See also Guan–Zhang [49] for geodesics of Sasakian metrics.

Definition 3.5 Suppose that (Y,ξ) is a normalized Fano cone singularity, and ψ is a transverse psh potential as above (relative to a reference radial function), such that $e^{-2n(1-t)\psi}\,dV$ is locally integrable. The twisted Ding functional $\mathcal{D}_{(1-t)\psi}$ is defined for continuous transverse psh potentials φ by

$$\mathcal{D}_{(1-t)\psi}(\varphi) = -tE(\varphi) - \frac{1}{2n}\log\int_Y e^{2n(1-t)[\varphi-\psi]}e^{-\frac{1}{2}r_\varphi^2}\,dV.$$

Here E is defined by its variation:

$$\delta E(\varphi) = \frac{1}{V(\xi)}\int_Y \dot\varphi e^{-\frac{1}{2}r_\varphi^2}\omega_\varphi^n.$$

The properties of the twisted Ding functional in this setting follow calculations analogous to those in the compact Kähler case that was studied in [31] (see Ding–Tian [37], Berndtsson [10] and Chen–Donaldson–Sun [23] for earlier work), using some additional identities that are valid on cones. An alternative approach is to work with pluripotential techniques on the link, as developed recently by van Coevering [26], but we prefer to work directly on the cone. As a sample of the calculations involved we have the following simple result.

Lemma 3.6 *$E(\varphi)$ is well defined.*

Proof Let us consider first the restriction of E to smooth transverse psh potentials φ. We show that the 1–form defined by δE is closed. Note that the variation of $\frac{1}{2}r_\varphi^2$ is $\delta\big(\frac{1}{2}r_\varphi^2\big) = (\delta\varphi)r_\varphi^2$. The differential of δE is the map

$$(\psi_1,\psi_2)\mapsto -\int_Y \psi_1\psi_2 r_\varphi^2 e^{-\frac{1}{2}r_\varphi^2}\omega_\varphi^n + n\int_Y \psi_1 e^{-\frac{1}{2}r_\varphi^2}\sqrt{-1}\partial\bar\partial(\psi_2 r_\varphi^2)\wedge\omega_\varphi^{n-1},$$

and we need to show that this is symmetric in ψ_1 and ψ_2. We have

$$\begin{aligned}\int_Y \psi_1\Delta(\psi_2 r_\varphi^2)e^{-\frac{1}{2}r_\varphi^2}\omega_\varphi^n &= \int_Y [\psi_1\Delta\psi_2 r_\varphi^2 + \psi_1\nabla\psi_2\cdot\nabla r_\varphi^2 + \psi_1\psi_2\Delta r_\varphi^2]e^{-\frac{1}{2}r_\varphi^2}\omega_\varphi^n\\ &= \int_Y \big[-\tfrac{1}{2}\nabla\psi_1\cdot\nabla\psi_2 r_\varphi^2 + 2n\psi_1\psi_2\big]e^{-\frac{1}{2}r_\varphi^2}\omega_\varphi^n,\end{aligned}$$

where the integration by parts is justified since $r_\varphi^2 = O(\hat r^2)$ and $\nabla r_\varphi^2 = O(\hat r)$, and we used

$$\nabla\psi_2\cdot\nabla r_\varphi = 0 \quad\text{and}\quad \Delta\tfrac{1}{2}r_\varphi^2 = n.$$

We obtain that the differential of δE is

$$(\psi_1,\psi_2)\mapsto n\int_Y \nabla\psi_1\cdot\nabla\psi_2\, e^{-\frac{1}{2}r_\varphi^2}\omega_\varphi^n,$$

where we used the formula

$$\int_Y f r_\varphi^2 e^{-\frac{1}{2}r_\varphi^2}\,dV = 2n\int_Y f e^{-\frac{1}{2}r_\varphi^2}\,dV, \tag{3-4}$$

for any basic function f.

This shows that E is well defined on the space of smooth transverse psh potentials. We can then extend E to the space of continuous transverse psh potentials by continuity, since the formula for the variation of E implies that E is uniformly continuous for the L^∞–norm on potentials. □

The critical points of the twisted Ding functional are given by solutions of the twisted equation. To see this, note that the variation of $\mathcal{D}_{(1-t)\psi}(\varphi)$ is given by

$$\begin{aligned}\delta\mathcal{D}_{(1-t)\psi}(\varphi) &= \frac{-t}{V}\int_Y \dot\varphi e^{-\frac{1}{2}r_\varphi^2}\omega_\varphi^n - \frac{1}{2n}\frac{\int_Y [2n(1-t)\dot\varphi - \dot\varphi r_\varphi^2] e^{2n(1-t)[\varphi-\psi]}e^{-\frac{1}{2}r_\varphi^2}\,dV}{\int_Y e^{2n(1-t)[\varphi-\psi]}e^{-\frac{1}{2}r_\varphi^2}\,dV}\\ &= \frac{-t}{V}\int_Y \dot\varphi e^{-\frac{1}{2}r_\varphi^2}\omega_\varphi^n + t\,\frac{\int_Y \dot\varphi e^{2n(1-t)[\varphi-\psi]}e^{-\frac{1}{2}r_\varphi^2}\,dV}{\int_Y e^{2n(1-t)[\varphi-\psi]}e^{-\frac{1}{2}r_\varphi^2}\,dV},\end{aligned}$$

where we used (3-4) again. It follows that critical points satisfy

$$\omega_\varphi^n = Ce^{2n(1-t)[\varphi-\psi]}\,dV,$$

which is what we wanted.

In addition from this calculation of the variation we see that the variation of $\mathcal{D}_{(1-t)\psi}$ along a suitable one-parameter family of biholomorphisms recovers the twisted Futaki invariant. Suppose that w is a vector field as above, and let $f_s\colon Y\to Y$ denote the one-parameter group of biholomorphisms generated by $-Jw$. We claim that the twisted Futaki invariant is given by the variation of the twisted Ding functional along f_s. Writing φ_s for the induced family of potentials, we have

$$\tfrac{1}{2}\hat r^2 e^{2\varphi_s} = f_s^*\big(\tfrac{1}{2}\hat r^2 e^{2\varphi}\big),$$

and so

$$\dot\varphi_s r_\varphi^2 = -\iota_{Jw}\, d\big(\tfrac{1}{2}r_\varphi^2\big).$$

We obtain that $\dot\varphi = -\theta_w$ in terms of the transverse Hamiltonian of w as in (3-3). It follows that

$$\frac{d}{ds}\Big|_{s=0}\mathcal{D}_{(1-t)\psi}(\varphi_s) = \frac{t}{V}\int_Y \theta_w e^{-\frac{1}{2}r_\varphi^2}\omega_\varphi^n - t\,\frac{\int_Y \theta_w e^{2n(1-t)[\varphi-\psi]}e^{-\frac{1}{2}r_\varphi^2}\,dV}{\int_Y e^{2n(1-t)[\varphi-\psi]}e^{-\frac{1}{2}r_\varphi^2}\,dV},$$

which when $\varphi = 0$ is just the twisted Futaki invariant as we have defined it above. We can rewrite this in a different form, as in the proof of Proposition 8 in [31].

Proposition 3.7 *The twisted Futaki invariant is given by*

$$\mathrm{Fut}_{Y,\xi,(1-t)\psi}(w) = \mathrm{Fut}_{Y,\xi}(w) - n(n-1)\frac{1-t}{V}\int_Y \theta_w e^{-\frac{1}{2}\hat{r}^2}\sqrt{-1}\partial\bar{\partial}\psi \wedge \omega^{n-1},$$

where

$$\mathrm{Fut}_{Y,\xi}(w) = \frac{1}{V}\int_Y \theta_w e^{-\frac{1}{2}\hat{r}^2}\omega^n - \frac{\int_Y \theta_w e^{-\frac{1}{2}\hat{r}^2}\, dV}{\int_Y e^{-\frac{1}{2}\hat{r}^2}\, dV} \tag{3-5}$$

is the "untwisted" Futaki invariant. Note that here, as before, we are assuming that $\iota_{Jw}\sqrt{-1}\partial\bar{\partial}\psi = 0$, since w is the real part of a holomorphic vector field in $\mathfrak{g}_{Y,\xi,\beta^\tau}$.

Note that in addition when w is normalized, ie $L_w\Omega = 0$, we have

$$\mathrm{Fut}_{Y,\xi}(w) = \frac{1}{V}\int_Y \theta_w e^{-\frac{1}{2}\hat{r}^2}\omega^n, \tag{3-6}$$

since in this case we have

$$\int_Y \theta_w e^{-\frac{1}{2}\hat{r}^2}\, dV = 0,$$

as can be seen by considering the variation of the integral $\int_Y e^{-\frac{1}{2}\hat{r}^2}\, dV$ along the flow generated by Jw.

Proof of Proposition 3.7 We define

$$\begin{aligned} I(\varphi) &= \frac{1}{V}\int_Y \log \frac{\left(\int_Y e^{-\frac{1}{2}r_\varphi^2}\, dV\right)^{-1} e^{-\frac{1}{2}r_\varphi^2}\, dV}{\left(\int_Y e^{2n(1-t)[\varphi-\psi]}e^{-\frac{1}{2}r_\varphi^2}\, dV\right)^{-1} e^{2n(1-t)[\varphi-\psi]}e^{-\frac{1}{2}r_\varphi^2}\, dV} e^{-\frac{1}{2}r_\varphi^2}\omega_\varphi^n \\ &= \log \frac{\int_Y e^{2n(1-t)[\varphi-\psi]}e^{-\frac{1}{2}r_\varphi^2}\, dV}{\int_Y e^{-\frac{1}{2}r_\varphi^2}\, dV} - 2n\frac{1-t}{V}\int_Y [\varphi-\psi]e^{-\frac{1}{2}r_\varphi^2}\omega_\varphi^n. \end{aligned}$$

Differentiating along the one-parameter group generated by Jw we must get zero. To see this, we use that all the terms in the integral are invariant under biholomorphisms up to constant factors, and these constants cancel. For instance the fact that $\iota_{Jw}\beta^\tau = 0$ implies that $L_{Jw}\psi$ is a constant. The result of the differentiation is

$$\begin{aligned} 2n\frac{\int_Y \dot{\varphi}e^{-\frac{1}{2}r_\varphi^2}\, dV}{\int_Y e^{-\frac{1}{2}r_\varphi^2}\, dV} - 2nt\frac{\int_Y \dot{\varphi}e^{2n(1-t)[\varphi-\psi]}e^{-\frac{1}{2}r_\varphi^2}\, dV}{\int_Y e^{2n(1-t)[\varphi-\psi]}e^{-\frac{1}{2}r_\varphi^2}\, dV} - 2n\frac{1-t}{V}\int_Y \dot{\varphi}e^{-\frac{1}{2}r_\varphi^2}\omega_\varphi^n \\ - 2n\frac{1-t}{V}\int_Y (\varphi-\psi)[-\dot{\varphi}r_\varphi^2 + \Delta(\dot{\varphi}r_\varphi^2)]e^{-\frac{1}{2}r_\varphi^2}\omega_\varphi^n = 0. \end{aligned}$$

Similar calculations to before, using also that $r_\varphi^2\Delta(\varphi-\psi)$ is a basic function, show

$$\int_Y (\varphi-\psi)[-\dot\varphi r_\varphi^2+\Delta(\dot\varphi r_\varphi^2)]e^{-\frac12 r_\varphi^2}\omega_\varphi^n = 2n(n-1)\int_Y e^{-\frac12 r_\varphi^2}\dot\varphi\sqrt{-1}\partial\bar\partial(\varphi-\psi)\wedge\omega_\varphi^{n-1}.$$

We obtain

$$\begin{aligned}\frac{\int_Y \dot\varphi e^{-\frac12 r_\varphi^2}\,dV}{\int_Y e^{-\frac12 r_\varphi^2}\,dV} &- t\frac{\int_Y \dot\varphi e^{2n(1-t)[\varphi-\psi]}e^{-\frac12 r_\varphi^2}\,dV}{\int_Y e^{2n(1-t)[\varphi-\psi]}e^{-\frac12 r_\varphi^2}\,dV}\\ &= \frac{1-t}{V}\int_Y \dot\varphi e^{-\frac12 r_\varphi^2}\omega_\varphi^n + n(n-1)\frac{1-t}{V}\int_Y \dot\varphi e^{-\frac12 r_\varphi^2}\sqrt{-1}\partial\bar\partial(\varphi-\psi)\wedge\omega_\varphi^{n-1}.\end{aligned}$$

From this, using that at the reference metric $\varphi=0$ we have $\dot\varphi=\theta_w$, we obtain the required formula. □

We can write the twisting term above in a more intrinsic way as

$$-\int_Y \theta_w e^{-\frac12\hat r^2}\sqrt{-1}\partial\bar\partial\psi\wedge\omega^{n-1} = -\int_Y \theta_w e^{-\frac12\hat r^2}(\beta^\tau-\omega^\tau)\wedge\omega^{n-1},$$

recalling that $\beta^\tau=\sqrt{-1}\partial\bar\partial\log r_\psi$ and $\omega^\tau=\sqrt{-1}\partial\bar\partial\log\hat r$.

As in [31] we use the twisted Futaki invariant to define a notion of (equivariant) twisted stability following also Dervan [36]. We will also choose a twisting form of a special form for which we will calculate an alternative formula for the twisted Futaki invariant.

Suppose that (X,ξ) is a normalized Fano cone singularity, smooth away from the vertex, and write $R(X)$ for the coordinate ring as before. We also denote by $R_{<D}(X)$ the direct sum of weight spaces for the action of $\mathbb{T}$ with weights $\chi\neq 0$ such that $\chi(\xi)<D$. Suppose that D is large enough that the functions in $R_{<D}(X)$ give an embedding $X\hookrightarrow\mathbb{C}^N$. It will be convenient to separate the spaces corresponding to different characters as

$$\mathbb{C}^N=\mathbb{C}^{N_1}\times\cdots\times\mathbb{C}^{N_m},$$

and so ξ acts diagonally on the coordinate functions with weights $a_1,\dots,a_m>0$.

A test configuration for X commuting with $\mathbb{T}$ is given by a one-parameter subgroup $\lambda\colon\mathbb{C}^*\to\mathrm{GL}(N)^{\mathbb{T}}$, and we assume that $\lambda(S^1)\subset U(N)^{\mathbb{T}}$, generated by a vector field w. Note that $\mathrm{GL}(N)^{\mathbb{T}}$ is simply the product of $\mathrm{GL}(N_i)$ for $i=1,\dots,m$. By a further unitary change of basis we will assume that λ is diagonal. We define

$$Y=\lim_{t\to 0}\lambda(t)\cdot X,$$

and we suppose that Y is normal, and $\mathbb{Q}$–Gorenstein. We define the limiting current

$$\beta^\tau = \lim_{t\to 0} \lambda(t)\cdot\alpha^\tau,$$

whose existence follows from the proof of Proposition 3.8; see (3-9). The twisted Futaki invariant of the corresponding test configuration is then defined by

$$\mathrm{Fut}_{X,\xi,(1-t)\alpha^\tau}(w) = \mathrm{Fut}_{Y,\xi,(1-t)\beta^\tau}(w).$$

Note here that w is not tangent to X, but it is tangent to Y, and it is the real part of a holomorphic vector field in $\mathfrak{g}_{Y,\xi,\beta^\tau}$.

As in [31], a crucial role is played by an alternative formula for this twisted Futaki invariant. We assume that ξ is quasiregular. There is then a constant M such that $M/a_i \in \mathbb{Z}$. We define a reference radial function analogous to the Fubini–Study metric for projective varieties, with radial function $\hat{r}$ given by

$$\hat{r}^2 = \Bigg[\sum_{i=1}^m \bigg(\sum_{j=1}^{N_i} |z_j^{(i)}|^2\bigg)^{M/a_i}\Bigg]^{1/M}, \tag{3-7}$$

where the $\{z_j^{(i)}\}$ form an orthonormal basis for $\mathbb{C}^{N_i}$. We take the background metric to be $\omega = \frac{1}{2}\sqrt{-1}\partial\bar\partial\hat{r}^2$, and the transverse form $\alpha^\tau = \sqrt{-1}\partial\bar\partial\log\hat{r}$; ie $\psi = 0$ relative to this radial function. To see that ω is indeed a metric, note first that for any cone metric $\sqrt{-1}\partial\bar\partial r^2$ and $\gamma > 0$, the forms $\sqrt{-1}\partial\bar\partial r^{2\gamma}$ also define cone metrics, with Reeb fields obtained by scaling. In this way,

$$\sum_{i=1}^m \bigg(\sum_{j=1}^{N_i} |z_j^{(i)}|^2\bigg)^{M/a_i}$$

defines a product metric on $\mathbb{C}^N$, and it follows that ω is also a cone metric. In addition, $\hat{r}$ is preserved by the action of $U(N)^{\mathbb{T}}$. With this setup we have the following.

Proposition 3.8

$$\mathrm{Fut}_{X,\xi,(1-t)\alpha^\tau}(w) = \mathrm{Fut}_{X,\xi}(w) + c(n)\frac{1-t}{V}\int_Y (\max_Y \theta_w - \theta_w)e^{-\frac{1}{2}\hat{r}^2}\omega^n,$$

where $\omega = \sqrt{-1}\partial\bar\partial\frac{1}{2}\hat{r}^2$, and $c(n)$ is a dimensional constant.

Proof The proof follows a similar argument to that in [31, Proposition 11], expressing α^τ as an average of currents of integration along hypersurfaces in X. One new difficulty is that the limit Y may be contained in a coordinate hyperplane.

We will use hypersurfaces defined by functions of the form

$$f_\mu = \sum_{j=1}^{B} \mu_j u_j^{M/b_j},$$

where the u_j are monomials in the z_i (including each z_i as well), the b_j are corresponding weights, and we think of μ as $\mu \in \mathbb{P} = \mathbb{P}^{B-1}$. It may happen that X is contained in some of these hypersurfaces, but there is a linear subspace $E \subset \mathbb{P}$ such that for $\mu \in \mathbb{P} \setminus E$, the function f_μ does not vanish on X. Let us write $V_\mu = X \cap f_\mu^{-1}(0)$, which may have multiplicity. By Shiffman–Zelditch [70, Lemma 3.1], whose proof is entirely local, we have that on $\mathbb{C}^N$,

$$2\pi \int_{\mathbb{P}} [f_\mu^{-1}(0)]\, d\mu = \sqrt{-1}\partial\bar{\partial} \log\biggl(\sum_{j=1}^{B} |u_j|^{2M/b_j} \biggr),$$

in the sense of distributions, for the standard probability measure $d\mu$ on the projective space $\mathbb{P}$. Multiplying out the M/a_i power in (3-7) we see that for suitable choices of the u_j restricted to X we will have

$$\alpha^\tau = \frac{\pi}{M} \int_{\mathbb{P}\setminus E} [V_\mu]\, d\mu = \frac{1}{2M} \sqrt{-1}\partial\bar{\partial} \log\biggl(\sum_{j=1}^{B} |u_j|^{2M/b_j} \biggr). \tag{3-8}$$

We can then compute β^τ on Y by taking the limits of the currents $[V_\mu]$ under the $\mathbb{C}^*$–action λ. We will do this by computing the limit of the underlying schemes. Suppose that $\lambda(t)$ acts on the u_j diagonally with entries t^{w_j}. Then we have

$$\lambda(t) \cdot f_\mu = \sum_{j=1}^{B} \mu_j t^{w_j M/b_j} u_j^{M/b_j}.$$

If I denotes the homogeneous ideal defining X, then V_μ is defined by the ideal $I + (f_\mu)$. The weights of the $\mathbb{C}^*$–action λ define a partial order on the monomials, and the limit $\lambda(t) \cdot V_\mu$ has ideal defined by the lowest-weight parts of elements of $I + (f_\mu)$, ie the initial ideal $\mathrm{in}_\lambda(I + (f_\mu))$. In the same way the limiting variety Y is defined by the ideal $\mathrm{in}_\lambda(I)$.

Suppose that we find a function $g_\mu \in \mathrm{in}_\lambda(I + (f_\mu))$ such that $g_\mu \notin \mathrm{in}_\lambda(I)$, and in terms of the grading of the polynomial ring given by ξ the function g_μ has the same degree as the f_μ (ie degree M). In this case, since the coordinate ring of Y is an integral domain (we have assumed that Y is reduced and irreducible), we will see that the Hilbert functions of the ideals $I + (f_\mu)$ and $\mathrm{in}_\lambda(I) + (g_\mu)$ coincide, and so $\lim_{t\to 0} V_\mu$ is the hypersurface in Y cut out by g_μ. The key information that we need is the weight Λ of the λ–action on g_μ. We will determine this for generic μ and in fact we claim that generically $\Lambda = -M \max_Y \theta_w$.

For simplicity let us assume that the (relative) weights w_i/b_i are ordered so that $w_1/b_1 \leqslant w_2/b_2 \leqslant \cdots \leqslant w_B/b_B$. Let c be the smallest index such that u_c does not vanish on Y. Suppose that μ is chosen to be in general position in the sense that we cannot write $f_\mu = h + f'_\mu$, where $h \in I$, and f'_μ has strictly larger weights than u_c. If we write $u_{c'}, u_{c'+1}, \dots, u_B$ for the monomials with strictly larger weight (we have $c' \geqslant c+1$, but this inequality may be strict), then the condition is that $f_\mu \notin I + (u_{c'}, \dots, u_B)$, or in other words that f_μ does not vanish on the intersection of X with the subspace $H = \{u_{c'}, \dots, u_B = 0\}$. If this intersection were just the origin, then we would have $u_c \in I + (u_{c'}, \dots, u_B)$, so that $u_c \in \mathrm{in}_\lambda(I)$. But then u_c vanishes on Y contrary to our assumption. This means that the intersection $X \cap H$ is nontrivial, and so the space of μ for which f_μ vanishes on $X \cap H$ is contained in a hyperplane in $\mathbb{P}$. We can then enlarge the subspace E above by this hyperplane, and focus on $\mu \in \mathbb{P} \setminus E$.

By assumption $u_1, \dots, u_{c-1} \in \mathrm{in}_\lambda(I)$, and so I must contain u_i^{M/b_i} modulo higher-weight terms, for $i = 1, \dots, c-1$. It follows that I contains

$$\sum_{i=1}^{c-1} \mu_i u_i^{M/b_i}$$

modulo higher-weight terms, but it cannot contain

$$\sum_{i=1}^{c'-1} \mu_i u_i^{M/b_i},$$

modulo higher-weight terms, by our genericity assumption. Then $\mathrm{in}_\lambda(I + (f_\mu)) = \mathrm{in}_\lambda(I + (f'_\mu))$, where

$$f'_\mu = \sum_{i=c}^{B} \mu'_i u_i^{M/b_i},$$

for some new coefficients μ'_i with at least one of $\mu'_c, \mu'_{c+1}, \dots, \mu'_{c'-1}$ nonzero. We can then let g_μ be the part of f'_μ which has the same weight as u_c^{M/b_c} under λ, ie $\Lambda = M w_c/b_c$.

Since the transverse Hamiltonian is

$$\theta_w = \frac{\sum_{i=1}^{B} -(w_i/b_i)|u_i|^{2M/b_i}}{\sum_{i=1}^{B} |u_i|^{2M/b_i}},$$

and $u_1, \dots, u_{c-1}$ vanish on Y, but u_c does not, we have $Mw_c/b_c = -M \max_Y \theta_w$, and so the weight of g_μ is $\Lambda = -M \max_Y \theta_w$ as we claimed.

Let us write $Y_\mu = Y \cap g_\mu^{-1}(0)$. By the discussion above we then have

$$\beta^\tau = \frac{\pi}{M} \int_{\mathbb{P}\setminus E} [Y_\mu]\, d\mu, \tag{3-9}$$

where recall that now E is a suitable union of two hyperplanes. It follows that

$$\int_Y \theta_w e^{-\frac{1}{2}\hat{r}^2} \beta^\tau \wedge \omega^{n-1} = \frac{\pi}{M} \int_{\mathbb{P}\setminus E} \int_{Y_\mu} \theta_w e^{-\frac{1}{2}\hat{r}^2} \omega^{n-1}. \tag{3-10}$$

From Propositions 6.6 and 6.7 we have

$$\int_{Y_\mu} \theta_w e^{-\frac{1}{2}\hat{r}^2} \omega^{n-1} = (2\pi)^{n-1} [(n-2)!]^2 D_w a_0(Y_\mu, \xi), \tag{3-11}$$

where, as in (2-1), the notation D_w refers to varying the Reeb field ξ in the direction of w. We have that ξ acts on g_μ with weight M, while w acts with weight $-M \max_Y \theta_w$. The index characters of Y and Y_μ are related by

$$F_{Y_\mu}(t, \xi + sw) = F_Y(t, \xi + sw)[1 - e^{-tM(1 - s \max_Y \theta_w)}],$$

and so, differentiating with respect to s, at $s = 0$, we obtain

$$D_w F_{Y_\mu}(t, \xi) = D_w F_Y(t, \xi)[1 - e^{-tM}] - F_Y(t, \xi) t M \max_Y \theta_w.$$

Comparing the leading terms in the Laurent expansion, we have

$$(n-2)!\, D_w a_0(Y_\mu, \xi) = M(n-1)!\, D_w a_0(Y, \xi) - M \max_Y \theta_w (n-1)! a_0(Y, \xi).$$

It then follows from (3-10), (3-11), and Proposition 6.7 that

$$\int_Y \theta_w e^{-\frac{1}{2}\hat{r}^2} \beta^\tau \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{1}{2(n-1)} \int_Y [n\theta_w - \max_Y \theta_w] \frac{\omega^n}{n!}.$$

At the same time we have

$$\omega^\tau \wedge \omega^{n-1} = \frac{1}{\hat{r}^2} \frac{n-1}{n} \omega^n,$$

and so

$$\int_Y \theta_w e^{-\frac{1}{2}\hat{r}^2} \omega^\tau \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{1}{2} \int_Y \theta_w e^{-\frac{1}{2}\hat{r}^2} \frac{\omega^n}{n!}.$$

Combining these formulas we obtain the required result. □

Remark 3.9 We remark that $\max_Y \theta_w$ and the integral of θ_w on Y depend only on the induced action on Y and can be computed from the weights of the action. Indeed, in the argument above we have seen that $-\max_Y \theta_w$ is the minimal relative weight w_j/b_j of a monomial u_j that does not vanish on Y, under the action λ relative to the action of the Reeb field. This is the same as the minimal relative weight w_i/a_i of a coordinate function z_i which does not vanish on Y, and since the z_i generate the ring of algebraic functions of Y which vanish at the vertex, this is simply the minimum relative weight of any such function on Y. More precisely, if f is any algebraic function on Y vanishing at the vertex which is in a weight space of the torus spanned by λ and ξ, then the relative weight is the quotient of the weights of these two $\mathbb{C}^*$–actions, and $-\max_Y \theta_w$ is the minimum of this quotient over all such f. At the same time the integral of θ_w on Y can be interpreted as the variation of the a_0 coefficient in the Hilbert series of Y, by Propositions 6.6 and 6.7.

From the above proof we also see the following.

Proposition 3.10 *In the above setup, given X and λ, there is a union of two hyperplanes $E \subset \mathbb{P}$ such that if $\mu \in \mathbb{P} \setminus E$ and $V_\mu = X \cap f_\mu^{-1}(0)$, then*

$$\mathrm{Fut}_{X,\xi,(1-t)\alpha^\tau}(w) = \mathrm{Fut}_{X,\xi,(\pi/M)[V_\mu]}(w).$$

In other words, when we want to compute the twisted Futaki invariant, we can replace α^τ by a current of integration along a suitable hypersurface on X, as long as this hypersurface, as a point in a projective space $\mathbb{P}$, is not contained in a certain union of two hyperplanes. It is important to emphasize that these two hyperplanes can depend on the choice of X and λ.

4 The partial C^0–estimate

Our goal in this section is to prove the partial C^0–estimate, Theorem 4.7, for cone metrics satisfying a Ricci curvature equation. A special case of this will be the partial C^0–estimate along the continuity method. The partial C^0–estimate was introduced by Tian [77] in his study of Kähler–Einstein metrics on complex surfaces, and he conjectured a general version for compact Kähler manifolds with a positive lower bound on the Ricci curvature. For Kähler–Einstein metrics in arbitrary dimension this estimate was obtained by Donaldson–Sun [40], using the Cheeger–Colding convergence theory [19] under Ricci curvature bounds, together with the Hörmander technique [51] for constructing holomorphic functions. Many more general results followed this development (see [22; 23; 66; 74; 24; 53]).

Our method will be fairly close to that in [74] for the smooth continuity method. The main difficulty is that along our continuity method the Ricci curvature of the cone metrics is only nonnegative, while in the approach of [74] it is important to treat the Ricci form as a metric. On the other hand it is not clear how to extend the Cheeger–Colding theory to the transverse Kähler structure, which does have strictly positive Ricci curvature. We therefore use the convergence theory on the level of the cones, but at certain crucial steps we invoke the positivity of the Ricci curvature of the transverse metric.

We suppose that (X,ξ) is a normalized Fano cone singularity, and α is a smooth Kähler cone metric on (X,ξ). From the discussion in Section 2 we know that we have a family of metrics ω_t on $(X,t^{-1}\xi)$, satisfying

$$\mathrm{Ric}(\omega_t) = 2n\frac{1-t}{t}\alpha^\tau,$$

for $t\in[t_0,T)$, with $T\leqslant 1$ and $t_0>0$.

Since we will also have to obtain uniform estimates while varying the Reeb field, we suppose more generally that we have a family of Reeb fields ξ_t, and metrics α_t and ω_t on (X,ξ_t) satisfying

$$\mathrm{Ric}(\omega_t) = 2nc_t\alpha_t^\tau, \tag{4-1}$$

where $0\leqslant c_t\leqslant c_0<\infty$, and the pairs (ξ_t,α_t) move in a bounded family in the following sense.

Definition 4.1 We say that data (ξ_t,α_t) consisting of Reeb vector fields and a compatible cone metrics on X are in a C^2 bounded family if the ξ_t are in a compact subset of $\mathcal{C}_R$, and the metrics α_t are locally uniformly equivalent to a fixed reference cone metric, and locally bounded in C^2 when measured with respect to this reference metric.

Along our continuity method we will have uniform constants κ and C_L such that:

- (X,ω_t) are uniformly noncollapsed. That is, $\mathrm{Vol}(B_1(0,\omega_t))>\kappa>0$, where $0\in X$ denotes the cone point.
- $\mathrm{Ric}(\omega_t)\geqslant 0$ on X, and the corresponding Sasakian metric g_t on the link L satisfies $\mathrm{Ric}(g_t)=(2n-2)g_t+(2n-2)c_t\alpha_t^\tau$.
- $\mathrm{diam}(L,g_t)<C_L$ for some controlled constant C_L.

The first point follows from the lower bound for t_0. The second point is the formula relating the Ricci curvature on the cone to the Ricci curvature on L, while the last point follows from Myers's theorem. Let r_t denote the radial function of ω_t. Then the Bishop–Gromov comparison theorem implies the metrics ω_t are uniformly noncollapsed on the annuli $\left\{\frac{1}{2} < r_t < 2\right\}$ and so results of Croke [30] and Yau (see eg [81, page 9]) imply:

Lemma 4.2 *For metrics ω_t satisfying (4-1) there is a uniform Sobolev inequality on the set $\mathrm{Ann} := \left\{\frac{1}{2} < r_t < 2\right\}$. That is, there exists a constant $C(\kappa, C_L)$ independent of t such that for any $W^{1,2}$ function f on Ann we have*

$$C^{-1}\left(\int_{\mathrm{Ann}} |f|^{2n/(n-1)}\omega_t^n\right)^{\frac{n-1}{2n}} \leqslant \left(\int_{\mathrm{Ann}} |\nabla f|^2_{\omega_t}\omega_t^n\right)^{\frac{1}{2}} + \left(\int_{\mathrm{Ann}} |f|^2\omega_t^n\right)^{\frac{1}{2}}.$$

Our eventual estimates will depend only on the dimension, the noncollapsing condition, and a bound on the geometry of α, which roughly speaking says that we have good control of the transverse metric α^τ on sufficiently small balls (see Definition 4.5 for a precise statement). As motivation consider the following analogous property of a compact Kähler manifold, which is easily proven by covering the manifold with sufficiently small coordinate balls.

Lemma 4.3 *Let (M, ω) be a compact Kähler manifold, and let g be the associated Kähler metric. Then for any constant $K > 0$ sufficiently large the following holds: if $B \subset M$ is any g–ball of radius smaller than K^{-1}, then there exist holomorphic coordinates $\{z_1, \dots, z_n\}$ defined on B such that*

$$\tfrac{1}{2}\delta_{i\bar{j}} < g_{i\bar{j}} < 2\delta_{i\bar{j}} \quad \text{and} \quad \|g_{i\bar{j}}\|_{C^2(g_{\mathrm{Euc}})} < K.$$

Furthermore, K can be chosen to be uniform over C^2 bounded families of metrics.

We need a generalization of this to the transverse Kähler structure defined by α^τ. Let us denote by Q the quotient bundle $Q = TX/\mathbb{C}\xi$, where by $\mathbb{C}\xi \subset TX$ we denote the complex subbundle spanned by ξ. Note that Q has a natural integrable complex structure, and α^τ defines a Kähler form on it (see Boyer–Galicki [14]).

Definition 4.4 Let $B \subset \mathbb{C}^{n-1}$ be a ball. We say that an immersion $F\colon B \to X$ is a ξ–transverse immersion if for any point $p \in B$, the image $dF_p(T_pB)$ is transverse to $\mathbb{C}\xi$. In particular, we get an induced vector space isomorphism

$$dF_p\colon T_pB \to Q_{F(p)}.$$

We say F is a transverse holomorphic immersion if dF_p is complex linear with respect to the standard complex structure on B, and the transverse complex structure on Q.

For any transverse holomorphic immersion F, we obtain a Kähler metric $F^*\alpha^\tau$ on B, compatible with the standard structure on B. Note that we have a $\mathbb{C}$–action on X induced by ξ, and for any smooth $h\colon B \to \mathbb{C}$, the maps F and $h \cdot F$ induce the same Kähler structure on B. In particular we can assume that $F\colon B \to L$ maps into the unit link $L \subset X$.

Definition 4.5 We say that α (or equivalently α^τ) has geometry bounded by K if the following holds. Let $F\colon B \to L$ be a transversal holomorphic immersion as above, where $B \subset \mathbb{C}^{n-1}$ denotes a ball. We write $g = F^*\alpha^\tau$ for the induced Kähler metric on B. If (B, g) has diameter at most K^{-1}, then there are holomorphic functions $z_1, \dots, z_{n-1}$ on B (not necessarily giving an embedding of B into $\mathbb{C}^{n-1}$, but an immersion) such that

$$\tfrac{1}{2}\delta_{ij} < g_{i\bar{j}} < 2\delta_{ij} \quad \text{and} \quad \|g_{i\bar{j}}\|_{C^2(B, g_{\mathrm{Euc}})} < K.$$

Here $g_{i\bar{j}} = g(\partial/\partial z^i, \partial/\partial \bar{z}^j)$, and $g_{\mathrm{Euc}} = \delta_{ij}$ is the pullback of the Euclidean metric by the map $(z_1, \dots, z_{n-1})\colon B \to \mathbb{C}^{n-1}$.

We now prove the analog of Lemma 4.3.

Proposition 4.6 *For $K > 0$ sufficiently large, depending on α, the geometry of α is bounded by K. Moreover, K can be chosen uniformly over bounded families.*

Proof We consider the case of a fixed metric α. We first cover L by a finite number of adapted charts V_i. This means that on such a V we have coordinates

$$(x, z_1, \dots, z_{n-1})\colon V \to \mathbb{R} \times \mathbb{C}^{n-1},$$

in which the Reeb field ξ is given by $\partial/\partial x$ and the Sasakian metric α agrees with the Euclidean metric at the origin. If we denote by α_0 the metric on the slice $\{x = 0\}$, then $\alpha_0 = \alpha^\tau$ is a Kähler metric which in the coordinates z_i agrees with the Euclidean metric at the origin. By using a cover by smaller charts if necessary, we can assume that in these coordinates α_0 satisfies $\frac{1}{2}\delta_{ij} < \alpha_0 < 2\delta_{ij}$ and $\|\alpha_0\|_{C^2} < K$, for some K (independent of the chart). Increasing K if necessary, we can ensure that every α–ball of radius K^{-1} in L is contained in one of our adapted charts V_i.

Fixing again one of our charts V, suppose that we have a transverse holomorphic immersion $f\colon B \to V$, transverse to $\partial/\partial x$. Then the induced Kähler structure on B is simply the pullback $(\pi \circ f)^*\alpha_0$, where π is the projection in V onto the $\{x = 0\}$ slice. The holomorphic functions $z_i \circ \pi \circ f$ then satisfy our requirements.

To prove the proposition it would suffice to show that if $F\colon B \to L$ is any transversal holomorphic immersion such that $(B, F^*\alpha^\tau)$ has diameter smaller than K^{-1}, then it is contained in one of the V_i. This is clearly impossible, since given such an immersion one could easily stretch the immersion by the Reeb flow to obtain a new immersion which is not contained in a ball of radius K^{-1}. Instead, given such an immersion F, we will construct a new, "equivalent" immersion $f\colon B \to L$ whose image lies in one of our adapted charts V. Since the Reeb vector field is real holomorphic, we can flow our transverse holomorphic immersion F to a new transverse holomorphic immersion along the Reeb field. Writing $\Phi\colon L \times \mathbb{R} \to L$ for the Reeb action, we are looking for a smooth function $a\colon B \to \mathbb{R}$ such that the image of

$$f\colon B \to L, \quad p \mapsto \Phi(F(p), a(p)),$$

lies in one of our adapted charts. We can choose the function a so that f maps radial rays γ from the origin in B to curves $f(\gamma)$ in L that are orthogonal to ξ. The length of $f(\gamma)$ with respect to α is then equal to its transversal length — that is, its length in (B, g). By assumption the diameter of (B, g) is at most K^{-1}, and so the image $f(B)$ must be contained in an α–ball of radius K^{-1}, and so it is contained in one of our adapted charts. This completes the proof of bounded geometry of a fixed metric. Moreover it is clear from the above argument that we can choose a uniform K for metrics in a bounded family. $\square$

We now state the main result that we will prove in this section. Recall that we write $R_\chi(X)$ for the part of the ring of functions of X on which the torus $\mathbb{T}$ acts by the character χ. For any $D > 0$ let us write

$$R_{<D}(X) = \bigoplus_{0<\chi(\xi)<D} R_\chi(X).$$

Suppose that we have a sequence of solutions ω_k on (X, ξ_k) of

$$\mathrm{Ric}(\omega_k) = 2n c_k \alpha_k^\tau$$

for $c_k \in [0, c_0]$ for some $c_0 > 0$. Choosing L^2–orthonormal bases of $R_{<D}$ with respect to ω_k we obtain a sequence of maps $F_k\colon X \to \mathbb{C}^N$.

Theorem 4.7 *There exists a constant D, depending on the dimension, the noncollapsing constant and the bound K on the geometry of α, such that each F_k is an embedding, and up to choosing a subsequence we have $F_k(X) \to Y$ in the sense of currents,* *where Y is a normal, $\mathbb{Q}$–Gorenstein variety with a Reeb field $\xi = \lim \xi_k$. In addition $(F_k)_*(\alpha_k^\tau) \to \beta^\tau$ for a positive transverse current on Y, and $(Y, T^{-1}\xi, (1-T)\beta^\tau)$ admits a weak solution of the twisted equation, where $t_k \to T$.*

We will spend the rest of this section proving this result, based on work of Donaldson–Sun [40], Chen–Donaldson–Sun [22; 23] as well as the second author [74]. A key ingredient in the work of Chen–Donaldson–Sun is to make use of the Hörmander technique for producing holomorphic sections of positive line bundles. In our setting we will use the Hörmander technique to produce holomorphic functions on our affine varieties. The following estimate holds on noncompact manifolds (see Demailly [32, Theorem 4.1] or [9, Theorem 6.2]).

Theorem 4.8 *Let L be a holomorphic line bundle endowed with a metric $e^{-\varphi}$ over a complex manifold X which has some complete Kähler metric. Assume the metric $e^{-\varphi}$ has strictly positive curvature, and that*

$$\sqrt{-1}\partial\bar{\partial}\varphi \geqslant c\omega,$$

where ω is some Kähler form on X (not necessarily complete) and $c > 0$.

Let f be a $\bar{\partial}$–closed (n,q)–form (where $q > 0$) with values in L. Then there is an $(n, q-1)$–form u with values in L such that $\bar{\partial}u = f$, and

$$\|u\|^2_{L^2(X,e^{-\varphi},\omega)} \leqslant \frac{1}{cq}\|f\|^2_{L^2(X,e^{-\varphi},\omega)},$$

provided the right-hand side is finite.

Using a resolution of singularities (see Saper [68, Example 9.4]) we know that $X \setminus \{0\}$ admits a complete Kähler metric since $0 \in X$ is an isolated singular point. Hence Theorem 4.8 applies in our setting.

We are going to apply the Hörmander theorem to $L = \mathcal{O}_X \otimes K_X^{-1} \simeq K_X^{-1}$, where we recall that $\mathcal{O}_X$ is endowed with the metric $e^{-\frac{1}{2}r^2}$ and we use the corresponding volume form ω^n as a metric on K_X^{-1}. The reason we make this choice for L is that we have isomorphisms

$$\Lambda^{n,1}(K_X^{-1}) \cong \Lambda^{0,1} \otimes K_X \otimes K_X^{-1} \cong \Lambda^{0,1} \quad \text{and} \quad K_X^{-1} \otimes K_X \cong \mathcal{O}_X.$$

In particular, Theorem 4.8 implies that we can solve the $\bar{\partial}$ equation on $\mathcal{O}_X$. Following the ideas of Donaldson–Sun [40], we can then use the Hörmander technique to transplant holomorphic functions from tangent cones of Gromov–Hausdorff limits to our noncompact cone manifold (X,ω). In order to do this, we need to ensure that there exist iterated tangent cones which are "good".

Definition 4.9 Suppose (Z,d_Z) is a Gromov–Hausdorff limit of (X,ω_t) as $t\to T\leqslant 1$, and suppose $C(Y)$ is an (iterated) tangent cone at $p\in Z$. We say that the tangent cone is *good* if:

(1) The regular set $Y_{\mathrm{reg}}\subset Y$ is open in Y and smooth.

(2) The distance function on $C(Y_{\mathrm{reg}})$ is induced by a Ricci flat cone metric, and on $C(Y_{\mathrm{reg}})$ the scaled-up metrics along our sequence converge in L^p_{loc} for all p to this Ricci flat metric.

(3) For all $\delta>0$ there is a Lipschitz function g defined on Y which is identically 1 on a neighborhood of $Y_{\mathrm{sing}}=Y\setminus Y_{\mathrm{reg}}$, with support contained in the δ neighborhood of Y_{sing} and with $\|\nabla g\|_{L^2}\leqslant\delta$, where the L^2–norm is with respect to the Sasaki–Einstein metric on Y_{reg}.

Suppose that ω_t are Kähler cone metrics on X, solving (4-1) where (ξ_t,α_t) move in a bounded family. Suppose that, along a subsequence, (X,ω_t) converge in the Gromov–Hausdorff sense to (Z,d_Z). If we can show that each tangent cone of (Z,d_Z) is good, then the techniques of [40], together with the above remarks, will imply that there is a number ε_0 depending only on the dimension, the noncollapsing constant and a bound for the geometry of α_t with the following effect: Let r_t be the radial function for ω_t. For any point $p\in L=\{r_t=1\}$ there is a holomorphic function $f\in\mathcal{O}_X$ with $\|f\|_{L^2(e^{-\frac{1}{2}r_t^2})}=1$, and $|f(p)|>\varepsilon_0$.

At this point we will need to pass from arbitrary holomorphic functions to those with polynomial growth. This is done in Section 4.4, essentially by truncating the Taylor series of f at a sufficiently high (but controlled) order. Putting all of these results together with techniques from Donaldson–Sun [41] will imply Theorem 4.7. With this discussion, we state our first goal:

Proposition 4.10 *Suppose ω_t are solutions of (4-1) with data (α_t,ξ_t) which move in a bounded family. If (Z,d_Z) is any Gromov–Hausdorff limit of a sequence (X,ω_{t_i}), then Z has good tangent cones.*

4.1 Gromov–Hausdorff convergence

We will now specialize to sequences $(X,\omega_i,\alpha_i,\xi_i)$, where

$$(\xi_i,\alpha_i)\to(\xi,\alpha)$$

in the C^2 topology for a fixed background metric, and

$$\mathrm{Ric}(\omega_i) = 2nc_i\alpha_i^\tau \tag{4-2}$$

with $c_i \to c$.

Suppose we have a sequence of metrics solving (4-2), with $c_i \to c$. Since the links (L, g_i) have bounded diameter, positive Ricci curvature, and are uniformly noncollapsed, we can take a Gromov–Hausdorff limit

$$(L, g_i) \xrightarrow{d_{\mathrm{GH}}} (Z, d).$$

At the same time we will have convergence of the cones

$$(X, \omega_i) \xrightarrow{d_{\mathrm{GH}}} (C(Z), \hat{d}),$$

in the pointed Gromov–Hausdorff topology, where we can identify Z with the unit link in $C(Z)$.

To understand iterated tangent cones in the space $C(Z)$, for any $p \in Z$ we need to study very small balls centered around $p \in C(Z)$, scaled to unit size. This in turn means that we need to study small balls centered at points on the unit link in $(C(L), \omega_i)$, scaled to unit size. Such a ball B has the following structure: B is the unit ball with respect to a Kähler metric ω, satisfying the equation

$$\mathrm{Ric}(\omega) = c\alpha^\tau, \tag{4-3}$$

and a uniform noncollapsing condition $\mathrm{Vol}(B, \omega) > K^{-1} > 0$. There is a holomorphic vector field Ξ on B, whose imaginary part is the Reeb field ξ scaled down, satisfying

$$1 - \delta < |\Xi|_\omega < 1 + \delta, \quad \mathcal{L}_\Xi \omega = \lambda\omega$$

for some $\lambda \leqslant \delta \leqslant \frac{1}{2}$. If the ball that we scaled up is sufficiently small, then δ can be taken to be arbitrarily small. Finally, α^τ is a closed, nonnegative $(1,1)$–form, vanishing along Ξ, and defining a transverse Kähler metric with bounded geometry on $TB/\mathbb{C}\xi$ in the sense of Definition 4.5.

There are two different cases to study, depending on whether the c_i are bounded away from 0 or $c_i \to 0$.

4.2 The case c_i are bounded away from zero

Fix i, and suppress the index. Scaling α^τ by a bounded factor we can rewrite (4-3) as

$$\mathrm{Ric}(\omega) = \alpha^\tau.$$

The next proposition, which is based on [74, Proposition 8], shows that when (B,ω) is close to the Euclidean ball in the Gromov–Hausdorff sense, then on a smaller ball the Ricci curvature is bounded. The quantity $I(B)$, as defined in [22; 74], is

$$I(B)=\inf_{B(x,r)\subset B}\mathrm{VR}(x,r),$$

where $\mathrm{VR}(x,r)$ is the ratio of the volumes of the ball $B(x,r)$ and the Euclidean ball rB^{2n}.

Proposition 4.11 *There is a $\delta=\delta(K)>0$, depending on the bound K for the geometry of α, such that if $1-I(B)<\delta$, then $\mathrm{Ric}(\omega)<5\omega$ on $\frac{1}{2}B$.*

Proof The difference with [74, Proposition 8] is that α^τ is not strictly positive, but it is strictly positive on slices transverse to Ξ, and it is invariant under the flow of Ξ, since this flow simply scales ω.

As in [74, Proposition 8], if $\mathrm{Ric}(\omega)$ is not bounded by 5ω on $\frac{1}{2}B$, then we can find a small ball inside B, which when scaled to unit size $(\widetilde{B},\widetilde{\omega})$ satisfies $\alpha^\tau\leqslant\widetilde{\omega}$, and in addition there is a unit vector v at the origin (with respect to $\widetilde{\omega}$) such that

$$\alpha^\tau(v,\bar{v})\geqslant 1.$$

The equation for ω implies that on $\widetilde{B}$ the metric $\widetilde{\omega}$ has bounded Ricci curvature, and so if δ is sufficiently small, then Anderson's result [3] implies that we have holomorphic coordinates $z^1,\dots,z^n$ on the ball $\theta\widetilde{B}$, with respect to which $\widetilde{\omega}$ is close to the Euclidean metric in $C^{1,\alpha}$. In these coordinates the holomorphic vector field Ξ will satisfy $\frac{1}{4}<|\Xi|_{\mathrm{Euc}}<4$, and so by rotating the coordinates and shrinking θ we can assume that on $\theta\widetilde{B}$ the vector field Ξ is very close to ∂_{z^1}. In particular $\alpha(\partial_{z^1},\bar{w})$ is very small for any unit vector w.

It follows that the slice $U=\{z^1=0\}\cap\theta\widetilde{B}$ is transverse to Ξ, and so (U,α^τ) is a Kähler manifold with bounded geometry. The inequality $\alpha^\tau\leqslant\widetilde{\omega}$ implies that the diameter of (U,α^τ) is at most θ, so shrinking θ further if necessary, we have holomorphic functions $w^2,\dots,w^n$ on U, defining local coordinates near each point, in which the components of α^τ are controlled in C^2.

The vector v may not be tangent to the slice U, but we may simply discard its ∂_{z^1}–component, while still having $\alpha^\tau(v,\bar{v})>\frac{1}{2}$. Rotating the $z^2,\dots,z^n$ coordinates, we can then assume that

$$\alpha^\tau(\partial_{z^2},\partial_{\bar{z}^2})>\tfrac{1}{4}.$$

We now have that the components of α^τ in the z^i coordinates have bounded derivatives along the slice U, but in addition α^τ is also constant along the flow of Ξ, which is very close to ∂_{z^0}. It follows that just as in [74, Proposition 8] we can obtain a spherical sector in which the Ricci curvature of $\widetilde{\omega}$ is strictly positive. Applying the Bishop–Gromov volume comparison we get a contradiction to $1 - I(B) < \delta$ if δ is sufficiently small. □

Corollary 4.12 *If we have solutions of (X, g_i) of (4-2) with $c_i \geqslant c > 0$, and if $(B(p_i, 1), g_i) \subset X$ converge in the Gromov–Hausdorff sense to the Euclidean ball, then the convergence is $C^{1,\alpha}$ on compact sets. In particular, if $(B(p_i, 1), g_i) \to Z$, then the regular set in Z is open, and then convergence on the regular set is locally $C^{1,\alpha}$.*

Proof We combine Colding's volume convergence [27] with Proposition 4.11 to get a uniform Ricci bound, and then apply Anderson's result in [3] to get $C^{1,\alpha}$ convergence to a Euclidean ball. □

Now assume we have a sequence of balls as above such that $(B(p_i, 1), g_i) \to Z$, with $p_i \to p$, and a tangent cone at $p \in Z$ is of the form $\mathbb{C}_\gamma \times \mathbb{C}^{n-1}$. As in [74, Proposition 11] we have $\gamma \in (\gamma_1, \gamma_2)$ for some $0 < \gamma_1 < \gamma_2 < 1$. The results of [22] apply, and in particular, arguing as in [22, Section 2.5], after scaling up the ω_i, that is, letting

$$\widetilde{\omega}_i = k\omega_i, \qquad \widetilde{\Xi}_i = \frac{1}{\sqrt{k}}\Xi_i,$$

we can view $\widetilde{\omega}_i$ as a metric on the unit Euclidean ball B^{2n} whose coordinates are $(u, v_1, \dots, v_{n-1})$, in which $\widetilde{\omega}_i$ is close to the model cone metric

$$\eta_\gamma = \sqrt{-1}\,\frac{du \wedge d\bar{u}}{|u|^{2-2\gamma}} + \sum_{i=1}^{n-1} dv_i \wedge d\bar{v}_i.$$

More precisely, if we scale by a large integer k, and take i large depending on k, we have, for some fixed constant C:

- $\widetilde{\omega}_i = \sqrt{-1}\partial\bar{\partial}\varphi_i$, with $0 \leqslant \varphi_i \leqslant C$.
- $\omega_{\mathrm{Euc}} < C\widetilde{\omega}_i$.
- Given $\delta > 0$ and a compact set $K \subset B^{2n} \setminus \{u = 0\}$ we can suppose (by taking i large once k is taken sufficiently large) that $|\widetilde{\omega}_i - \eta_\gamma|_{C^{1,\alpha}(K, g_{\mathrm{Euc}})} < \delta$.

Lemma 4.13 *In the above setting, for every $\varepsilon \in \left(0, \frac{1}{2}\right)$ we have $|\widetilde{\Xi}_i|_{g_{\mathrm{Euc}}} > \frac{1}{2}$ on the set $\{|u| = \varepsilon\}$ provided i and the scaling factor k are sufficiently large.*

Proof We fix $\varepsilon > 0$, and suppose that the conclusion is false. Then we have a sequence of metrics $\widetilde{\omega}_i$ and holomorphic vector fields $\widetilde{\Xi}_i$ on B^{2n} (with scaling factors $k \to \infty$) converging in $C^{1,\alpha}$ to the standard cone metric η_γ locally away from $\{u=0\}$, and the $\widetilde{\Xi}_i$ satisfy

$$|\widetilde{\Xi}_i|_{g_{\mathrm{Euc}}} \leqslant C, \quad 1-\frac{1}{k} < |\widetilde{\Xi}_i|_{\widetilde{\omega}_i} < 1+\frac{1}{k}, \quad L_{\widetilde{\Xi}_i}\widetilde{\omega}_i = \lambda\widetilde{\omega}_i,$$

with $\lambda < 1/k$. Since the $\widetilde{\Xi}_i$ are holomorphic and bounded we obtain uniform $C^{3,\alpha}$–estimates on $\frac{1}{2}B^{2n}$. We can choose a subsequence so that $\widetilde{\Xi}_i$ converges to a holomorphic vector field $\widetilde{\Xi}$ in $C^3(g_{\mathrm{Euc}})$ on $\frac{1}{2}B$, and $\widetilde{\omega}_i \to \eta_\gamma$ on $\left\{|u| \geqslant \frac{1}{2}\varepsilon\right\} \cap \frac{1}{2}B$. Furthermore, we have

$$L_{\widetilde{\Xi}}\eta_\gamma = 0, \quad |\widetilde{\Xi}|_{\eta_\gamma} = 1.$$

By direct computation one verifies that the only holomorphic vector fields on $\frac{1}{2}B$ which are Killing for η_γ on $\left\{|u| \geqslant \frac{1}{2}\varepsilon\right\} \cap \frac{1}{2}B$ and of unit length are the translations in the v_i directions. In particular $|\widetilde{\Xi}|_{\mathrm{Euc}} = |\widetilde{\Xi}|_{\eta_\gamma}$. The result then follows from the convergence $\widetilde{\omega}_i \to \eta_\gamma$. □

Proposition 4.14 *There is a constant $c_0 > 0$ such that if $(B(p,1),\omega)$ is sufficiently close to the unit ball in the cone $\mathbb{C}_\gamma \times \mathbb{C}^{n-1}$ with $\gamma \in (\gamma_1,\gamma_2)$, then*

$$\int_{B(p,1)} \alpha^\tau \wedge \omega^{n-1} > c_0.$$

Proof We argue by contradiction and assume there is no such c_0. Then we will have a sequence $B(p_i,1) \to B(\underline{0},1)$, where $\underline{0}$ is the vertex in the cone $\mathbb{C}_\gamma \times \mathbb{C}^{n-1}$, such that

$$\lim_{i\to\infty} \int_{B(p_i,1)} \alpha_i^\tau \wedge \omega_i^{n-1} = 0. \tag{4-4}$$

As discussed above, we can then find a small $r_0 > 0$ such that for sufficiently large i, the scaled-up metric $r_0^{-2}\omega_i$ can be thought of as a metric on a set containing the Euclidean unit ball B^{2n} such that $\omega_{\mathrm{Euc}} < C r_0^{-2}\omega_i$ on B^{2n}. By the previous lemma we can assume (choosing r_0 smaller if necessary) that the rescaled Reeb vector field Ξ_i is a perturbation of the vector field $\partial/\partial z^0$ for one of the holomorphic coordinates z_0 on B^{2n}, and so the sets $U_c = \{z_0 = c\}$ are transverse to Ξ_i for $|c| < \frac{1}{2}$, say.

Suppose that

$$\int_{B^{2n}} \alpha_i^\tau \wedge \omega_{\mathrm{Euc}}^{n-1} < \varepsilon_1,$$

for some $\varepsilon_1 > 0$. Then we must have

$$\int_{U_c} \alpha_i^\tau \wedge \omega_{\mathrm{Euc}}^{n-2} < C_1\varepsilon_1$$

for at least one c with $|c| < \frac{1}{2}$, and a uniform constant C_1. The form α_i^τ defines a Kähler metric on U_c with bounded curvature, so the ε–regularity theorem of Schoen–Uhlenbeck [69] (see also [74, Proposition 7]) implies that if $C_1\varepsilon_1 < \varepsilon_0$, then we must have $\alpha_i^\tau < C'\omega_{\mathrm{Euc}} < C'Cr_0^{-2}\omega_i$ on U_c (evaluated on vectors tangent to this slice). Since Ξ_i is a small perturbation of $\partial/\partial z^0$, and α_i^τ vanishes along Ξ_i, we obtain a bound for α^τ on $\frac{1}{2}B^{2n}$. This implies a uniform bound for $\mathrm{Ric}(\omega_i)$ on this ball, independent of i. The result of Cheeger–Colding–Tian [20] implies that no conical singularity can then form, which is a contradiction.

As a result, we must have

$$\int_{B^{2n}} \alpha_i^\tau \wedge \omega_{\mathrm{Euc}}^{n-1} \geqslant \varepsilon_1$$

for sufficiently large i, where $\varepsilon_1 = C_1^{-1}\varepsilon_0$. This in turn implies

$$\int_{B^{2n}} \alpha_i^\tau \wedge (r_0^{-2}\omega_i)^{n-1} \geqslant C^{-n-1}\varepsilon_1,$$

contradicting (4-4). □

We also have the following, whose proof is the same as that of [74, Proposition 13].

Proposition 4.15 *There is a constant $A > 0$ such that if $(B(p,1),\omega)$ is sufficiently close to either the Euclidean unit ball or the unit ball in the cone $\mathbb{C}_\gamma \times \mathbb{C}^{n-1}$ with $\gamma \in (\gamma_1,\gamma_2)$, then*

$$\int_{B\left(p,\frac{1}{2}\right)} \alpha^\tau \wedge \omega^{n-1} < A.$$

We can now show that the iterated tangent cones are good, similarly to Chen–Donaldson–Sun [22], or [74]. The argument in the published version of [74] was incomplete, but it is corrected in the latest version on the arXiv, and that argument can be used verbatim in our setting, using the estimates on the "densities" given by Propositions 4.14 and 4.15.

4.3 The case $c_i \to 0$

In this case we study noncollapsed balls $B = B(p,1)$ with metrics ω satisfying

$$\mathrm{Ric}(\omega) = c_i\alpha^\tau,$$

where $c_i \to 0$. As above, the additional structure is the (rescaled) Reeb field Ξ, which is a holomorphic vector field satisfying $\frac{1}{2} < |\Xi|_\omega < 2$ and $L_\Xi \omega = \lambda\omega$, for some $\lambda \leqslant 2$. Note that as we scale up the metric, we must scale down the Reeb field, and so scale down λ. The form α^τ defines a transverse Kähler metric on $TB/\mathbb{C}\xi$, with bounded geometry as before. Once again, the difficulty when compared to [74] is that α^τ is not strictly positive, and so the ε–regularity theorem for harmonic maps cannot be applied. Our strategy, as above, is to find transverse slices.

We first need a slight refinement of [23, Proposition 1], which will allow us to control the Reeb field Ξ. First we recall some definitions. For a subset A in a $2n$–dimensional length space P, and for $\eta < 1$, let $m(\eta, A)$ be the infimum of those M for which A can be covered by Mr^{2-2n} balls of radius r for all $\eta \leqslant r < 1$.

For $x \in B$ and $r, \delta > 0$ a holomorphic map $\Gamma\colon B(x,r) \to \mathbb{C}^n$ is called an (r,δ)–chart centered at x if

- $\Gamma(x) = 0$,
- Γ is a homeomorphism onto its image,
- for all $x', x'' \in B(x,r)$ we have $|d(x',x'') - d(\Gamma(x'),\Gamma(x''))| \leqslant \delta$,
- for some fixed $p > 2n$, we have $\|\Gamma_*(\omega) - \omega_{\mathrm{Euc}}\|_{L^p} \leqslant \delta$.

With these definitions, we need the following slight modification of [23, Proposition 1].

Proposition 4.16 *Given M and c there are $\rho(M)$, $\eta(M,c)$, $\delta(M,c) > 0$ with the following effect. Suppose that $1 - I(B) < \delta$ and $W \subset B$ is a subset with $m(\eta, W) < M$ such that for any $x \in B \setminus W$ there is a $(c\eta,\delta)$–chart centered at x. There is a constant C, depending only on the dimension, such that:*

1. *There is a holomorphic map $F\colon B(p,\rho) \to \mathbb{C}^n$ which is a homeomorphism to its image, $|\nabla F| < C$, and the image of F lies between $0.9\rho B^{2n}$ and $1.1\rho B^{2n}$.*
2. *There is a local Kähler potential φ for ω on $B(p,\rho)$ with $|\varphi|\rho^{-2} < C$.*
3. *The slices $\{z_n = l\} \cap C^{-1}\rho B^{2n}$ are transverse to the Reeb field for $|l| < C^{-1}\rho$.*

The way this proposition is used is that under the assumptions we can use F to think of ω as a metric on the Euclidean ball $0.9\rho B^{2n}$, and because of the gradient bound for F we have $\omega_{\mathrm{Euc}} < C_1\omega$. The new statement is (3), which will essentially follow if we can show that $|\Xi|_{\omega_{\mathrm{Euc}}}$ is not too small near the origin. In fact we will show that given any $\kappa > 0$, the η and δ in the proposition can be chosen so that there exists a point $q \in \kappa\rho B^{2n}$ at which $F_*(\omega) < 2\omega_{\mathrm{Euc}}$.

In order to prove this claim, we recall briefly the proof of [23, Proposition 1]. For this, let $\Omega(W,s)\subset B$ denote the set of points in B at a distance greater than s from W and from the boundary of B. One part of the proof of the proposition is to produce an embedding $\varphi\colon \Omega(W,r)\to B^{2n}$, for which $\|\varphi_*(\omega)-\omega_{\mathrm{Euc}}\|_{L^p}\leqslant\widetilde{\theta}$, $|\varphi_*(J)-J_{\mathrm{Euc}}|\leqslant\widetilde{\theta}$ and $d(x,\varphi(x))\leqslant\widetilde{\theta}$, where $r,\widetilde{\theta}>0$ can be chosen a priori, and d denotes the distance function realizing the Gromov–Hausdorff distance from B to B^{2n}. In addition ω is the curvature of a metric on the trivial bundle on B, and φ can be lifted to a bundle map from the trivial bundle on B to the trivial bundle on B^{2n}, which almost identifies the corresponding connections (see [23, Proposition 4]).

The holomorphic function F is now obtained by taking the holomorphic functions $1,z_1,\dots,z_n$ on B^{2n}, and applying suitable cutoff functions to obtain approximately holomorphic functions (sections of the trivial bundle) $\sigma,\sigma_1,\dots,\sigma_n$ over B, vanishing on $\Omega(W,r)$, and near ∂B. Using the Hörmander L^2–estimate these can be projected to holomorphic sections $s,s_1,\dots,s_n$ (globally on our cone X), and s_i/s give the components of F. The result of [23, Proposition 1] is then obtained by choosing the parameters in the cutoff functions in a suitable way.

In order to obtain (3), we just note that given $\kappa>0$ we simply need to choose r much smaller than $\kappa\rho$, so that we can find some $q\in\kappa\rho B$, which is contained in a definite ball B_q disjoint from $\Omega(W,2r)$, say. If $\widetilde{\theta}$ above is sufficiently small, then on this ball B_q the geometry of ω will be almost identical in an L^p–sense to the Euclidean geometry. In particular the holomorphic function F will be very close to the identity map on $\frac{1}{2}B_q$. There will then exist a point $q'\in\frac{1}{2}B_q$ at which $F_*(\omega)<2\omega_{\mathrm{Euc}}$. Note that Ξ gives a holomorphic vector field on $0.9\rho B^{2n}$, and it has bounded length with respect to the Euclidean metric. In particular on $0.8\rho B^{2n}$ the derivatives of the components of Ξ are bounded. On the other hand we know that we can choose a point q' very close to the origin, where $|\Xi|_{\omega_{\mathrm{Euc}}}>\frac{1}{4}$. Rotating coordinates, we can assume that the $\partial/\partial z_n$ component of Ξ is nonzero inside $C_2^{-1}\rho B^{2n}$. This implies our claim (3).

Given this result, the rest of the argument is quite similar to that in [74]. We give the required modifications of the proofs. The following is analogous to [74, Proposition 16].

Proposition 4.17 *Given M, suppose that the ball B satisfies the hypotheses of Proposition 4.16 for some $c>0$. There are $A,\kappa>0$, depending on M, such that if*

$$\int_B \alpha^\tau\wedge\omega^{n-1}<\kappa,$$

then $\alpha^\tau<A\kappa\omega$ on $\frac{1}{3}C_2(K)^{-1}\rho B$, with $\rho=\rho(M)$ and $C_2(K)$ from Proposition 4.16.

Proof The assumption implies that

$$\int_{0.9\rho B^{2n}} \alpha^\tau \wedge \omega_{\mathrm{Euc}}^{n-1} < C_1 \kappa,$$

for some C_1. We argue similarly to the proof of Proposition 4.14. There is a slice $U_l = \{z_n = l\} \cap C_2(K)^{-1}\rho B^{2n}$, with $|l| < \frac{1}{2}K^{-1}\rho B^{2n}$ on which we have

$$\int_{U_l} \alpha^\tau \wedge \omega_{\mathrm{Euc}}^{n-1} < C_3 \kappa.$$

The ε–regularity implies that if κ is sufficiently small, then on a slightly smaller set we have $\alpha^\tau < C_4 \omega_{\mathrm{Euc}} < C_4' \omega$. The flow of the Reeb field preserves α^τ, and acts on ω by a controlled scaling factor, so we obtain the required estimate on a ball of a definite size, $\frac{1}{3}C_2(K)^{-1}\rho B^{2n}$. □

For any ball $B(q,r) \subset B$ we define

$$V(q,r) = r^{2-2n} \int_{B(q,r)} \alpha^\tau \wedge \omega^{n-1}.$$

We have the following.

Proposition 4.18 *There are $\delta, \varepsilon > 0$ depending on K satisfying the following: if $1 - I(B) < \delta$ and*

$$\sup_{B(q,r) \subset B} V(q,r) < \varepsilon,$$

then $\alpha^\tau \leqslant 4\omega$ on $\frac{1}{2}B$.

Proof The proof follows the argument of [74, Proposition 17], together with the idea we used in the proof of Proposition 4.11 to make use of the vector field Ξ. □

There is one other time when the strict positivity of α is used in [74], namely in the proof of [74, Proposition 19] where the ε–regularity is used again. In the present setting the same argument can be used, just like in the proof of Proposition 4.17. The remainder of the proof of Proposition 4.10 is identical to the argument in [74].

4.4 Polynomial growth holomorphic functions

In this section we assume that we have a Kähler cone $(C(L), \omega)$, which has nonnegative Ricci curvature, and is noncollapsed (ie we have a lower bound on the volume of L).

In addition we assume that there is a constant $\varepsilon_0 > 0$ with the following property: for every $x \in L$ there is a holomorphic function f on $C(L)$ such that $f(0) = 0$, $\|f\|_{L^2} = 1$, and $|f(x)|^2 > \varepsilon_0$. Here the L^2–norm is with respect to the weight $e^{-\frac{1}{2}r^2}$ as before. This extra property holds for any family of solutions of (4-1) with data moving in a bounded family by using the results of the previous section on having good tangent cones. Our goal is to show that up to replacing ε_0 by a smaller constant, we can take f to have polynomial growth with controlled degree. More precisely we have the following.

Proposition 4.19 *There are constants $D, \varepsilon_1 > 0$, depending on ε_0 and the lower bound on the volume of L, such that the following holds: if $x \in L$, then there is a holomorphic function f on $C(Y)$ with $f(0) = 0$, $\|f\|_{L^2} = 1$, $|f(x)|^2 > \varepsilon_1$, and in addition $|f| = O(r^D)$.*

Proof First let us write $\mathcal{H}$ for the space of L^2 holomorphic functions on $C(L)$, which we can decompose into weight spaces under the torus action

$$\mathcal{H} = \bigoplus_{\chi \in \mathfrak{t}^*} \mathcal{H}_\chi,$$

where infinite convergent sums are allowed. In addition we have the Reeb field $\xi \in \mathfrak{t}$.

As we mentioned above, we already have an f with the required properties, except for the growth condition. To restate our goal, we are trying to construct an f' that also satisfies the growth condition, which is equivalent to

$$f \in \bigoplus_{\substack{\chi \neq 0 \\ \langle \chi, \xi \rangle < D}} \mathcal{H}_\chi.$$

Lemma 4.21 shows that we have a constant C such that

$$\#\big(\mathcal{C} \cap \{\chi \in \mathfrak{t}^* : \langle \chi, \xi \rangle \in (w-1, w]\}\big) < C5^w$$

for all $w \geqslant 1$. Note that this estimate is far from optimal, but it is enough for our purposes here.

Suppose that we have f such that $f(0) = 0$, $|f(x)|^2 > \varepsilon_0$ and $\|f\|_{L^2} = 1$. We write

$$f = \sum_{w=1}^{\infty} \sum_{i=1}^{N_w} f_{w,i},$$

where each $f_{w,i}$ is in a weight space $\mathcal{H}_\chi$ with $\langle \chi, \xi \rangle \in (w-1, w]$. We have $N_w < C5^w$.

From Lemma 4.20 we have that on L

$$|f_{w,i}|^2 \frac{w!}{2^w} < C\|f_{w,i}\|^2_{L^2},$$

and so for any $D > 0$ we have

$$\begin{aligned}\left|\sum_{w=D}^{\infty}\sum_{i=1}^{N_w} f_{w,i}\right|^2 &\leqslant \left(\sum_{w=D}^{\infty}\sum_{i=1}^{N_w}\frac{2^w}{w!}\right)\left(\sum_{w=D}^{\infty}\sum_{i=1}^{N_w}|f_{w,i}|^2\frac{w!}{2^w}\right)\\ &\leqslant C\|f\|^2_{L^2}\sum_{w=D}^{\infty}\frac{10^w}{w!}.\end{aligned}$$

Since the series on the right converges, and $|f(x)|^2 > \varepsilon_0$, we can choose D large enough that at x we have

$$\left|\sum_{w=1}^{D}\sum_{i=1}^{N_w} f_{w,i}\right|^2 > \tfrac{1}{2}\varepsilon_0.$$

We can then let

$$f' = \sum_{w=1}^{D}\sum_{i=1}^{N_w} f_{w,i},$$

and $f'\|f'\|^{-1}_{L^2}$ will satisfy the required properties. □

Lemma 4.20 *Suppose that $f \in \mathcal{H}_\chi$, and let $w = \lceil\langle\chi,\xi\rangle\rceil$. Then we have*

$$|f|^2 < \frac{2^w C\|f\|^2_{L^2}}{w!} \quad \text{on } L.$$

Proof Assume $\|f\|_{L^2} = 1$. We have (with dimensional factors c_n)

$$\begin{aligned}1 = \|f\|^2_{L^2} &\geqslant c_n\|f\|^2_{L^2(L)}\int_1^{\infty} r^{2w+2n-3}e^{-\frac{1}{2}r^2}\,dr\\ &\geqslant c_n' 2^w(w+n-2)!\,\|f\|^2_{L^2(L)}.\end{aligned}$$

The L^2–norm on the link gives a C^0–estimate on the half-ball around the cone vertex, and then using the assumed growth rate on f we obtain a C^0–estimate on the link L:

$$\sup_L |f|^2 < \frac{2^w C}{(w+n-2)!}.$$ □

We have also used the following simple estimate on the dimension of the space of holomorphic functions of polynomial growth.

Lemma 4.21 *For an integer $w \geqslant 1$, let us write $\mathcal{H}_w$ for the space of holomorphic functions satisfying the growth condition $|f| = O(r^w)$ as $r \to \infty$. We then have*

$$\dim \mathcal{H}_w < C5^w.$$

Proof This follows a standard argument using the previous lemma. Let $\{f_1, \dots, f_N\}$ be an L^2–orthonormal basis for $\mathcal{H}_w$, and define the function B by

$$B(x) = \sum_{i=1}^{N} |f_i(x)|^2 e^{-\frac{1}{2}r^2}.$$

The previous lemma implies that

$$B \leqslant \frac{2^w C}{w!} \max\{1, r^{2w}\},$$

and so

$$\int_{C(L)} B\,\omega^n \leqslant C5^w.$$

On the other hand, by definition the integral of B is the dimension of $\mathcal{H}_w$. □

4.5 Synthesis

Let us recall where we stand. Suppose we have a sequence of cone Kähler metrics ω_{t_k} on $(X, t_k^{-1}\xi_k)$, with radial functions r_k, satisfying the equations

$$\mathrm{Ric}(\omega_{t_k}) = \frac{1-t_k}{t_k}\alpha_k^\tau,$$

and $t_k \to T$, $\xi_k \to \xi$ and $\alpha_k \to \alpha$. In addition the sequence (X, ω_{t_k}) converges in the Gromov–Hausdorff sense to (Z, d_Z). Our work so far leads to the following "partial C^0–estimate".

Proposition 4.22 *There exists a constant $D > 0$ depending only on the noncollapsing constant, the dimension and a bound for the geometry of (ξ_k, α_k) such that if $\{f_1^{(k)}, \dots, f_N^{(k)}\}$ denotes an L^2–orthonormal basis of $R_{<D}(X)$, then the map $F_k\colon X \to \mathbb{C}^N$ whose components are given by the $f_i^{(k)}$ gives an embedding of X. Furthermore, there is a uniform constant C such that*

$$C^{-1} < |F_k| < C \quad \text{and} \quad C^{-1} F_k^* \omega_{\mathrm{Euc}} < \omega_{t_k}$$

on the set $\{\frac{1}{2} < r_k < 2\}$.

Proof In Sections 4.2 and 4.3 we have shown that the iterated tangent cones in suitable Gromov–Hausdorff limits are good, and so the arguments of Donaldson–Sun [40] show that the assumptions of Proposition 4.19 apply: for each $x \in L$, we can find a holomorphic function f on X with polynomial growth of bounded degree, unit L^2–norm on L, and $|f(x)|^2 > \varepsilon_1$ for a fixed number ε_1 (independent of k). Increasing D if necessary, we can assume that F_k gives an embedding (note that X is fixed, only the metric is changing). It then follows directly that $C^{-1} < |F_k|$ on L, while the bound on the growth rate implies that the same estimate (with different C) also holds on the annulus $\frac{1}{2} < r < 2$. A uniform bound $|F_k| < C$ and derivative bound $|\nabla F_k|_{\omega_{t_k}} < C$ on the annulus follows by using Moser iteration, and this implies the estimate $C^{-1} F_k^* \omega_{\mathrm{Euc}} < \omega_{t_k}$. □

Arguing as in Donaldson–Sun [41, Section 2] we can deduce the convergence of the affine varieties

$$F_k(X) \to Y$$

where Y is a normal, $\mathbb{Q}$–Gorenstein affine variety with Reeb vector field ξ, and furthermore Y is homeomorphic to Z. In order to finish the proof of Theorem 4.7 it suffices to prove the last statement regarding convergence to a weak solution of the twisted equation, after possibly passing to a further subsequence.

Consider the pushed-forward forms $(F_k)_* \alpha^\tau$. We would like to take a weak limit of these forms. As a first step we prove the volume is bounded below. Suppose that $\alpha = \frac{1}{2}\sqrt{-1}\partial\bar\partial r^2$ is a cone metric, with transverse form $\alpha^\tau = \sqrt{-1}\partial\bar\partial \log r$. Recall [71] we have the form $\eta := \sqrt{-1}(\bar\partial - \partial)\log r$. In terms of η we have $\alpha^\tau = \frac{1}{2}\,d\eta$, and $\alpha = \frac{1}{2}\,d(r^2\eta)$. Then we compute

$$\alpha^\tau \wedge \alpha^{n-1} = 2^{1-n}(n-1) r^{2n-3}\,dr \wedge \eta \wedge (d\eta)^{n-1}.$$

Writing everything on the cone we get

$$\begin{aligned}\int_{\{r\leqslant 1\}} \alpha^\tau \wedge \alpha^{n-1} &= 2^{1-n}(n-1)\int_0^1 r^{2n-3}\,dr \int_{r=1} \eta\wedge(d\eta)^{n-1}\\ &= c_n \operatorname{Vol}(L,\xi) > 0,\end{aligned}$$

where $\operatorname{Vol}(L,\xi)$ is the volume of the link, which is a topological invariant depending only on the Reeb field. Furthermore, if ω is another metric compatible with ξ, with radial function $\tilde r$, then we can write $r = e^{\psi}\tilde r$ for a function ψ which is independent of r, and $L_\xi \psi = 0$. Then it is easy to check that

$$\int_{\{\tilde{r}\leqslant 1\}} \alpha^\tau \wedge \omega^{n-1} = c_n \operatorname{Vol}(L,\xi).$$

Now suppose we have sequence of maps $F_k\colon X \to \mathbb{C}^N$ as above and consider the closed positive currents

$$(F_k)_* \alpha^\tau \wedge [F_k(X)].$$

Let $A := \{p \in \mathbb{C}^N : M^{-1} < |p| < M\}$ for some constant $M > 0$. Let $A' := F_k^{-1}(A \cap F_k(X))$. Suppose that ν is a smooth positive $(n-1,n-1)$–form with compact support in A. Then we have

$$\int_{\mathbb{C}^N} (F_k)_* \alpha^\tau \wedge [F_k(X)] \wedge \nu = \int_{A'} \alpha^\tau \wedge F_k^* \nu.$$

We can find a constant C_1 such that on A we have $\nu \leqslant C_1 \omega_{\mathrm{Euc}}^{n-1}$ as $(n-1,n-1)$–forms. Furthermore, by the properties of F_k we have

$$F_k^* \omega_{\mathrm{Euc}} < C \omega_k$$

on A', where ω_k is our metric, and the partial C^0–estimate implies that

$$\left\{\tfrac{1}{2} < r_{t_k} < 2\right\} \subset A' \subset \{r < C'\}$$

for some constant C', provided M is sufficiently large. The above discussion implies a uniform upper bound

$$\int_{A'} \alpha^\tau \wedge F_k^* \nu < C,$$

with the constant C depending only on the form ν. It follows that $(F_k)_* \alpha^\tau \wedge [F_k(X)]$ converges weakly to a closed positive current β^τ on $Y = \lim_k F_k(X)$.

By arguing as in [31] we can show that Y admits a weak solution ω_T of the equation

$$\operatorname{Ric}(\omega_T) = \frac{1-T}{T} \beta^\tau.$$

Note that the proof of this result in [31] is essentially local, working in neighborhoods of points p in the limit Y where the complex structure of Y is smooth, and where in terms of the metric structure we have a tangent cone of the form $\mathbb{C}^n$ or $\mathbb{C}^{n-1} \times \mathbb{C}_\gamma$. On this set we have good local coordinates, and so we can study the limiting equation, while at the same time the complement of this set has Hausdorff codimension greater than 2, and therefore it can be ignored in terms of writing down weak solutions of the twisted equation, as in [31, Remark 4]. This completes the proof of Theorem 4.7.

5 Proof of the main result

In this section we prove one direction of Theorem 1.1. Recall that we have a normalized Fano cone singularity (X,ξ) together with the action of a torus $\mathbb{T}$, whose Lie algebra contains ξ. We assume that (X,ξ) is $\mathbb{T}$–equivariantly K–stable, and our goal is to show that (X,ξ) admits a Ricci flat Kähler cone metric. The proof naturally splits into two cases depending on whether (X,ξ) is quasiregular, or irregular. We will first focus on the former, and then we will deal with irregular ξ by approximating it with a sequence of quasiregular Reeb fields.

5.1 The quasiregular case

Suppose that (X,ξ) is quasiregular. We first fix a $\mathbb{T}$–invariant transverse Kähler metric α^τ, using an embedding $X\to\mathbb{C}^N$ by a collection of (nonconstant) holomorphic functions just as in (3-7). We can then solve the continuity method (2-2) up to some time $T\leqslant 1$.

According to Theorem 4.7 there is a number D, depending on the bound on the geometry of the twisting form α^τ and on the pair (X,ξ) through the noncollapsing condition, such that using orthonormal bases of holomorphic functions with growth rates less than D we obtain embeddings $F_k\colon X\to\mathbb{C}^N$ such that $F_k(X)\to Y$ with a normal limit space Y. In addition we have convergence of twisting forms $(F_k)_*\alpha^\tau\to\beta^\tau$ and the limit Y admits a weak solution of the twisted equation

$$\mathrm{Ric}(\omega_T)=\frac{1-T}{T}\beta^\tau.$$

In order to apply the results from Section 3 we need to make sure that α^τ can be written as an integral over currents of integration (with respect to a positive measure) over suitable hypersurfaces. The problem with α^τ defined as in (3-7) is that it does not use all the functions in $R_{<D}(X)$ and so in the integral expression (3-8) we are not using a positive measure on $\mathbb{P}$. To fix this, we will modify α^τ slightly by adding small terms corresponding to the remaining functions in $R_{<D}(X)$, and so that the new transverse metric still has the same bound on its geometry. In particular the above discussion still holds with the same constant D. Let us write $\alpha^\tau=\sqrt{-1}\partial\bar\partial\log\hat{R}$, and we will call the perturbed radial function $\hat{r}$.

Consider the decomposition

$$R_{<D}(X)=R_1\oplus\cdots\oplus R_m\oplus R_{m+1}\oplus\cdots\oplus R_k$$

into weight spaces of $\mathbb{T}$, where α^τ is defined as in (3-7) using bases of R_i for $i \leqslant m$. Suppose that ξ acts on the functions in R_i with weight a_i as before, and recall that we chose M so that $M/a_i \in \mathbb{Z}$ for $i \leqslant m$. We also choose K so that $K/a_i \in \mathbb{Z}$ for all i, and then for $\delta > 0$ we define a new radial function $\hat{r}$ by

$$\begin{aligned}\hat{r}^{2K} &= \Bigg[\sum_{i=1}^{m}\Bigg(\sum_{j=1}^{N_i}|z_j^{(i)}|^2\Bigg)^{M/a_i}\Bigg]^{K/M} + \delta\sum_{i=m+1}^{k}\Bigg(\sum_{j=1}^{N_i}|z_j^{(i)}|^2\Bigg)^{K/a_i}\\ &= \hat{R}^K + \delta\sum_{i=m+1}^{k}\Bigg(\sum_{j=1}^{N_i}|z_j^{(i)}|^2\Bigg)^{K/a_i}.\end{aligned}$$

Here the $z_j^{(i)}$ form a basis for R_i. As $\delta \to 0$, we recover the original radial function $\hat{R}$, and so for sufficiently small δ the transverse metric $\sqrt{-1}\partial\bar{\partial}\log\hat{r}$ has the same bounded geometry as $\sqrt{-1}\partial\bar{\partial}\log\hat{R}$, but at the same time the methods of Section 3, in particular Proposition 3.8, can be applied.

At this point we are in essentially the same setup as in [31, Section 3.1], and can follow the argument there closely. We have a sequence $\rho_k \in \mathrm{GL}(N)^{\mathbb{T}}$ such that $F_k = \rho_k \circ F_1$. For simplicity of notation let us write $F_1(X) = X$ and $(F_1)_*(\alpha^\tau) = \alpha^\tau$. Then in the notation of Section 3, on X we have

$$\alpha^\tau = \frac{\pi}{M}\int_{\mathbb{P}\setminus F}[V_\mu]\,d\mu, \tag{5-1}$$

where F is a hyperplane in the projective space $\mathbb{P}$ and each V_μ is a hypersurface in X. Similarly to [31, Lemma 14] we can choose a subsequence of the ρ_k such that $\rho_k(V_\mu)$ converges for all μ. Let us define

$$\rho_\infty(V_\mu) = \lim_{k\to\infty}\rho_k(V_\mu).$$

Generalizing Definition 3.2, for any $(1,1)$–current on Y we let $\mathfrak{g}_{Y,\xi,\chi}$ denote the holomorphic vector fields on Y commuting with ξ such that $\iota_\xi\chi = 0$. Then as in [31, Lemma 15] we have:

Lemma 5.1 *We can find $\mu_1, \dots, \mu_d$, for some d, such that*

$$\mathfrak{g}_{Y,\xi,\beta^\tau} = \bigcap_{i=1}^{d}\mathfrak{g}_{Y,\xi,[\rho_\infty(V_{\mu_i})]}.$$

The proof is the same as in [31], except instead of the Fubini–Study volume form we use $e^{-\frac{1}{2}\hat{r}^2}\omega^n$, where $\omega = \frac{1}{2}\sqrt{-1}\partial\bar{\partial}\hat{r}^2$ as usual.

By Lemma 5.1 and Proposition 3.3 we can choose hypersurfaces $V_1', \dots, V_d'$ in $\mathbb{P} \setminus F$ such that the automorphism group of the $(d+1)$–tuple $(Y, \rho_\infty(V_1'), \dots, \rho_\infty(V_d'))$ is reductive, by Proposition 3.3. Since Y may be contained in a hyperplane, the stabilizer of this $(d+1)$–tuple in the multigraded Hilbert scheme, under the action of $\mathrm{GL}(N)^{\mathbb{T}}$, may contain extra factors of $\mathrm{GL}(k_i)$ for suitable k_i, but this product is still reductive. At this point, we can use the Luna slice theorem (see [62] and also [39; 23]) to find a $\mathbb{C}^*$–subgroup $\lambda(t) \subset \mathrm{GL}(N)^{\mathbb{T}}$ with $\lambda(S^1) \subset U(N)^{\mathbb{T}}$, and an element $g \in \mathrm{GL}(N)^{\mathbb{T}}$ such that

$$(Y, \rho_\infty(V_1'), \dots, \rho_\infty(V_d')) = \lim_{t\to 0} \lambda(t) g \cdot (X, V_1', \dots, V_d').$$

More generally, enlarging F by a set of measure zero, if $V_1, \dots, V_K$ are hypersurfaces in $\mathbb{P} \setminus F$, then the stabilizer of the $(1+d+K)$–tuple

$$(Y, \rho_\infty(V_i'), \rho_\infty(V_j))_{i=1,\dots,d,\, j=1,\dots,K}$$

is unchanged, so the Luna slice theorem provides a corresponding $\mathbb{C}^*$–subgroup $\lambda(t)$ and element g. These will satisfy

$$Y = \lim_{t\to 0} \lambda(t) g \cdot X \quad \text{and} \quad \rho_\infty(V_i) = \lim_{t\to 0} \lambda(t) g \cdot V_i.$$

Note that we do not necessarily have $\beta^\tau = \lim_{t\to 0} \lambda(t) g \cdot \alpha^\tau$, and that the one-parameter subgroup $\lambda(t)$ and g may depend on the choice of $V_1, \dots, V_K$. Note also that the vector field w on Y induced by the $\mathbb{C}^*$–subgroup λ will stabilize $\rho_\infty(V_i')$ for $1 \leqslant i \leqslant d$, and hence by Lemma 5.1 we have $w \in \mathfrak{g}_{Y,\xi,\beta^\tau}$.

It is important to choose $V_1, \dots, V_K$ above correctly, and we will discuss how this is done shortly, but for the time being, let us assume we have a $\mathbb{C}^*$–subgroup λ generated by a vector field w commuting with ξ. Let θ_w denote the transverse Hamiltonian with respect to the radial function $\hat{r}$. We will assume that θ_w is normalized so that

$$\int_Y \theta_w e^{-\frac{1}{2}\hat{r}^2} \omega^n = 0,$$

and write $\|w\| = \sup_Y |\theta_w|$. Note that any two choices of norm are equivalent on the finite-dimensional space of holomorphic vector fields on Y commuting with ξ. In addition we cannot have $\theta_w = 0$ on Y, unless λ already acts trivially on X. To see this note that λ induces a filtration on the coordinate ring of X, and the coordinate ring of Y is the associated graded ring, with the action λ on Y being induced by the corresponding grading. If this grading is trivial, then the original filtration must have been trivial.

As in [31], the idea is to use Proposition 3.10 to estimate the twisted Futaki invariant of the test configuration $\lambda(t)$ for $g \cdot X$. In order to apply Proposition 3.10 we must ensure that our hypersurfaces avoid the set E, which is a union two hyperplanes in $\mathbb{P}$. On the other hand, if no $N+1$ of our hypersurfaces $V_1, \dots, V_K$ are on a hyperplane in $\mathbb{P}$, then at most $2N$ of them can be contained in E. So once we choose K very large compared to N, then we get a good approximation to the twisted Futaki invariant by replacing $g \cdot \alpha^\tau$ by the average of $g \cdot [V_i]$.

We now describe how to choose the $V_1, \dots, V_K$. For simplicity of notation we will suppose that g is the identity. From (5-1) we have that on Y

$$\beta^\tau = \frac{\pi}{M} \int_{\mathbb{P} \setminus F} [\rho_\infty(V_\mu)]\, d\mu.$$

In addition, since $(Y, \xi, (1-T)\beta^\tau)$ admits a weak solution of the twisted equation, from Proposition 3.4 we have

$$\mathrm{Fut}_{Y,\xi,(1-T)\beta^\tau}(w) = 0$$

for all $w \in \mathfrak{g}_{Y,\xi,\beta^\tau}$. By the formula for the twisted Futaki invariant in Proposition 3.7 this means that

$$\mathrm{Fut}_{Y,\xi}(w) - c_n \frac{1-T}{V} \int_Y \theta_w e^{-\frac{1}{2}\hat{r}^2} \beta^\tau \wedge \omega^{n-1} = 0,$$

and so if we define the function $h \colon \mathbb{P} \setminus F \to \mathbb{R}$ to be

$$h_w(\mu) = \frac{\pi}{M} \int_{\rho_\infty(V_\mu)} \theta_w e^{-\frac{1}{2}\hat{r}^2} \omega^{n-1},$$

then we have

$$\mathrm{Fut}_{Y,\xi}(w) = c_n \frac{1-T}{V} \int_{\mathbb{P} \setminus F} h_w(\mu)\, d\mu.$$

Since the possible w form a finite-dimensional space, for any $\varepsilon > 0$ we can choose a large K_0 with the following effect: for any $K \geqslant K_0$ we can find hypersurfaces $V_1, \dots, V_K$, with no $N+1$ on a hyperplane in $\mathbb{P}$, such that

$$\mathrm{Fut}_{Y,\xi}(w) \leqslant \varepsilon \|w\| + c_n \frac{1-T}{V} \cdot \frac{1}{K} \sum_{i=1}^{K} \frac{\pi}{M} \int_{\rho_\infty(V_i)} \theta_w e^{-\frac{1}{2}\hat{r}^2} \omega^{n-1}. \tag{5-2}$$

To see this more precisely, we choose a basis $w^1, \dots, w^k$ for $\mathfrak{g}_{Y,\xi,\beta^\tau}$, and then using Lusin's theorem applied to each h_{w^i} we can find a compact set $B \subset \mathbb{P} \setminus F$ with arbitrarily small complement, on which each h_{w^i} is continuous. We can then approximate the integrals on B using Riemann sums over discrete finite sets.

Assume $T<1$. There is a constant $\delta>0$ such that

$$\int_Y (\max_Y \theta_w - \theta_w) e^{-\frac{1}{2}\hat{r}^2}\omega^n > \delta\|w\|,$$

since the left-hand side is a norm on the finite-dimensional vector space $\mathfrak{g}_{Y,\xi,\beta^\tau}$. Choose $0<\varepsilon<4^{-1}c(n)(1-T)\delta$, where $c(n)$ is the constant from Proposition 3.10. Take K sufficiently large as above and such that $K>2N\varepsilon^{-1}$. Using the hypersurfaces $V_1,\dots,V_K$ we now find λ as discussed before (and for simplicity assume g is the identity). It follows that $\rho_\infty(V_i)=\lim_{t\to 0}\lambda(t)\cdot V_i$.

We now use Proposition 3.10, which implies that in the sum in (5-2), for all but $2N$ of the integrals we have

$$\frac{\pi}{M}\int_{\rho_\infty(V_i)} \theta_w e^{-\frac{1}{2}\hat{r}^2}\omega^{n-1} = -c(n)\int_Y \max_Y \theta_w e^{-\frac{1}{2}\hat{r}^2}\omega^n.$$

From our choice of K we have

$$\begin{aligned} \mathrm{Fut}_{Y,\xi}(w) &\leqslant 2\varepsilon\|w\| - c(n)\frac{1-T}{V}\int_Y \max_Y \theta_w e^{-\frac{1}{2}\hat{r}^2}\omega^n \\ &\leqslant -c(n)\tfrac{1}{2}(1-T)\max_Y \theta_w. \end{aligned} \tag{5-3}$$

This is a contradiction if (X,ξ) is K–stable.

If $T=1$, then (Y,ξ) admits a weak Ricci flat metric, and then as in [31] or Donaldson–Sun [41, Section 3.3], we obtain a test configuration for (X,ξ) with vanishing Futaki invariant.

5.2 The irregular case

Suppose now that (X,ξ) is irregular, and let $\xi_k\to\xi$ be a sequence of normalized quasiregular Reeb fields approximating ξ. Let $\alpha_k^\tau\to\alpha^\tau$ be a sequence of compatible transverse Kähler metrics, obtained by restricting suitable reference forms under an embedding $X\to\mathbb{C}^M$. Note that we may not be able to choose the α_k^τ to be exactly of the form considered in the previous section. On the other hand it is easy to show in a way identical to the argument in [73] that if $(X,\xi,(1-t)\alpha^\tau)$ admits a solution of the twisted equation, then so does $(X,\xi,(1-t)\tilde{\alpha}^\tau)$ for any $\tilde{\alpha}^\tau$ in the same transverse cohomology class as α^τ.

For each k we obtain a $T_k<1$ such that we can solve

$$\mathrm{Ric}(\omega_t^{(k)}) = 2n(1-t)[\alpha_k^\tau - \omega_t^{(k),\tau}]$$

on (X, ξ_k) for $t \in [0, T_k)$. Note that if we choose ξ to be the Reeb field with minimal volume, then necessarily $T_k < 1$, since each (X, ξ_k) will be strictly unstable if $\xi_k \neq \xi$. Indeed by [65] the Reeb field with minimal volume is unique and a Sasaki–Einstein metric can exist only for that Reeb field.

Proposition 5.2 *If (X, ξ) is K–stable, then $T_k \to 1$ as $k \to \infty$.*

Proof Let us suppose that $\limsup T_k < 1$. From (5-3), for each k we obtain a vector field w_k, inducing a test configuration for X with central fiber Y_k, such that

$$\mathrm{Fut}_{Y_k, \xi_k}(w_k) \leqslant -\delta \max_{Y_k} \theta_{w_k}, \tag{5-4}$$

for some $\delta > 0$. Applying Theorem 4.7 to a diagonal sequence, we can assume that $Y_k \to Y$, and moreover that Y is normal.

Our goal is to show that for sufficiently large k we have

$$\mathrm{Fut}_{Y_k, \xi}(w_k) < 0,$$

since this will contradict the K–stability of (X, ξ). Recall that we have normalized each θ_{w_k} to have zero integral on Y_k. It is worth pointing out that the background metrics on $\mathbb{C}^N$ are also varying with k, and so the transverse Hamiltonians are all computed with different metrics, but these background metrics also converge as $k \to \infty$ to a metric for the limiting Reeb field ξ. In addition, as discussed in Remark 3.9, the normalization of w_k and $\max_{Y_k} \theta_{w_k}$ only depends on the induced action on Y_k, and not for instance on the background metric used to compute the Hamiltonians.

By (3-5) the Futaki invariant is

$$\mathrm{Fut}_{Y_k, \xi_k}(w_k) = -\frac{\int_{Y_k} \theta_{w_k} e^{-\frac{1}{2}\hat{r}_k^2} \, dV_k}{\int_{Y_k} e^{-\frac{1}{2}\hat{r}_k^2} \, dV_k},$$

where dV_k is the canonical volume form on Y_k. Note that these can be scaled so that they converge to the canonical volume form on Y. In addition if we scale each w_k so that $\mathrm{osc}_{Y_k} \theta_{w_k} = 1$, then we can extract a limit θ_w on Y, corresponding to a vector field w. Note that w may not be defined on all of $\mathbb{C}^N$ if Y is contained in a hyperplane, since in this case our scaling might make some of the weights of w_k become unbounded. However we must still have $\mathrm{osc}_Y \theta_w = 1$.

From our normalization this implies a positive lower bound for $\max_Y \theta_w$ and hence a uniform positive lower bound for $\max_{Y_k} \theta_{w_k}$. From (5-4) we obtain

$$\mathrm{Fut}_{Y_k,\xi_k}(w_k) \leqslant -\delta'.$$

Now the required result follows, since both $\mathrm{Fut}_{Y_k,\xi_k}(w_k)$ and $\mathrm{Fut}_{Y_k,\xi}(w_k)$ converge to $\mathrm{Fut}_{Y,\xi}(w)$, which we have just seen must be negative. □

We can now choose metrics $\omega_{t_k}^{(k)}$ on (X,ξ_k), satisfying

$$\mathrm{Ric}(\omega_{t_k}^{(k)}) = 2n(1-t_k)[\alpha_k^\tau - \omega_{t_k}^{(k),\tau}],$$

with $t_k = T_k - k^{-1}$. We can apply the same arguments as above to the sequence $(X,\omega_{t_k}^{(k)})$, to obtain a normal limit space Y, which admits a weak Ricci flat metric, since $t_k \to 1$. As before, we can then realize Y as the central fiber of a test configuration for X, contradicting the assumption that (X,ξ) is K–stable, unless $Y \cong X$. But if $Y \cong X$, then we have obtained the desired Ricci flat Kähler cone metric on X.

6 The algebraic Futaki invariant

In this section we collect some results of a more algebraic nature. First we will consider the normalization (or gauge fixing) condition for a Fano cone singularity $(X,\mathbb{T},\xi)$, and show that normalized Reeb fields ξ form a linear subspace of the Lie algebra of $\mathbb{T}$. We then discuss the relation between the algebro-geometric definition of the Futaki invariant in Definition 2.2 and the differential geometric definition in (3-5). The analogous result in the smooth projective case was shown by Donaldson [38, Proposition 2.2.2], using the equivariant Riemann–Roch formula. Here our approach is more algebraic in order to avoid having to resolve possible singularities.

6.1 The gauge fixing condition

We begin with a lemma alluded to in Section 2.

Lemma 6.1 *Suppose $(X,\mathbb{T},\xi)$ is $\mathbb{Q}$–Gorenstein with an isolated singularity at the origin. Suppose $\Omega \in \Gamma(X, mK_X)$ is a nonvanishing section with $L_\xi\Omega = i\lambda\Omega$ for some $\lambda \in \mathbb{R}$. Then X has log-terminal singularities at 0 if and only if $\lambda > 0$.*

Proof Define a volume form $dV = i^{n^2}(\Omega \wedge \overline{\Omega})^{1/m}$. By [43, Lemma 6.4] it suffices to determine conditions for dV to have finite volume in a neighborhood of 0. Let r be a radial function for ξ. Using the flow by $-J\xi$, we can write

$$\int_{\{2^{-k}\leq r<2^{-(k-1)}\}} dV = e^{-\lambda k \log 2} \int_{\{\frac{1}{2}\leq r<1\}} dV$$

and so we get

$$\int_{\{0<r<1\}} dV = \sum_{k=0}^{\infty} e^{-\lambda k \log 2} \cdot \int_{\{\frac{1}{2}\leq r<1\}} dV,$$

which proves the lemma. □

Note that the proof implies slightly more. Namely, that if $(X, \mathbb{T}, \xi)$ is a normal, $\mathbb{Q}$–Gorenstein, with Ω as above, then X is log terminal if and only if $\lambda > 0$, and X is log terminal at all points away from the origin.

Suppose we have an n–dimensional polarized affine variety $(X, \mathbb{T}, \xi)$ which is a Fano cone singularity, so X is normal $\mathbb{Q}$–Gorenstein with log-terminal singularities. In addition suppose that $m \in \mathbb{N}$ is minimal such that mK_X is trivial. Let R be the coordinate ring of X, which is an integral domain since X is a variety. Furthermore, since X has log-terminal singularities, it is known that R is Cohen–Macaulay. When X is Gorenstein with an isolated singularity at the cone point, Martelli–Sparks–Yau [65] showed the existence of a $\mathbb{T}$–equivariant trivialization of K_X. Their argument applies verbatim to the singular, $\mathbb{Q}$–Gorenstein case. We include the short proof for the reader's convenience.

Lemma 6.2 *There exists a unique, up to scale, nonvanishing section $\Omega \in \Gamma(X, mK_X)$, and a unique linear function $\ell\colon \mathcal{C}_R \to \mathbb{R}_{>0}$ with the property that, for any Reeb field $\xi \in \mathcal{C}_R$, we have*

$$L_\xi \Omega = \ell(\xi)\Omega.$$

In particular, for any $c > 0$ the set $\{\xi \in \mathcal{C}_R : \ell(\xi) = c\}$ defines an affine hyperplane in $\mathfrak{t}$ intersecting $\mathcal{C}_R$ in a set of codimension 1.

Proof Fix a nonvanishing section $\Omega \in \Gamma(X, mK_X)$, and $\xi \in \mathcal{C}_R$. Since mK_X is trivial we have

$$L_\xi \Omega = k(z)\Omega$$

for some holomorphic function $k(z)$ on X. We decompose $k(z)$ into its graded pieces according to the grading defined by ξ:

$$k(z) = \sum_{\alpha \in \mathfrak{t}^*} k_\alpha(z).$$

This decomposition converges in L^2_{loc} and therefore also locally uniformly. Indeed, we can restrict k to the link of X, and consider its weight decomposition in L^2 for the torus action.

Since X has log-terminal singularities, we can assume that $k_0 \in \mathbb{R}_{>0}$. We project to the degree-zero part to conclude. Explicitly, define

$$f(z) = \sum_{\alpha \in \mathfrak{t}^* \setminus \{0\}} \frac{1}{\alpha(\xi)} k_\alpha(z),$$

and then $\widetilde{\Omega} := e^{-f(z)}\Omega$ satisfies our requirements. The last two claims are clear from the linearity of the projection and the positivity of k_0. The uniqueness follows from the fact that ξ induces a positive grading, and hence the only homogeneous, holomorphic, nonvanishing holomorphic functions on X are constant. □

The above discussion makes it possible to introduce the gauge fixing condition, as in Martelli–Sparks–Yau [65].

Definition 6.3 We say $\xi \in \mathcal{C}_R$ satisfies the gauge fixing condition (or is normalized) if

$$L_\xi \Omega = inm\Omega,$$

where Ω is as in Lemma 6.2.

An important point for us is that the gauge fixing condition can in fact be read off from the index character. This is implicit in the work of Martelli–Sparks–Yau [65] when X is Gorenstein with an isolated log-terminal singularity. We now extend this to the case of general $\mathbb{Q}$–Gorenstein affine varieties with log-terminal singularities. To this end, fix a trivializing section $\Omega \in \Gamma(X, mK_X)$ and suppose that $L_\xi \Omega = i\lambda\Omega$ for some $\lambda > 0$. Then we have an isomorphism of graded R–modules

$$R(-\lambda) \simeq \Gamma(X, mK_X) \tag{6-1}$$

by the map $f \mapsto f\Omega$. The index character of the ring R, denoted by $F_R(\xi, t)$, expands as a Laurent series

$$F_R(\xi, t) = \frac{a_0(\xi)(n-1)!}{t^n} + \frac{a_1(\xi)(n-2)!}{t^{n-1}} + O(t^{2-n}).$$

We have the following.

Proposition 6.4 *In the above setting, the coefficients $a_0(\xi)$ and $a_1(\xi)$ satisfy*

$$\frac{a_1(\xi)}{a_0(\xi)} = \frac{\lambda(n-1)}{2m}.$$

In particular, the set $\{\xi \in \mathcal{C}_R : 2a_1(\xi) = n(n-1)a_0(\xi)\}$ is an affine subset of $\mathcal{C}_R$ with codimension 1 which agrees with the normalized Reeb fields.

Proof The proof is essentially a computation in commutative algebra. Recall that a $\mathbb{Q}$–Gorenstein ring R with log-terminal singularities is Cohen–Macaulay. The main tool in the proof is a duality equation due to Stanley [72], Corollary 4.4.6 of [18], which says that if R is a Cohen–Macaulay, positively graded $\mathbb{C}$–algebra of dimension n with canonical module Ω_R, then the Hilbert series satisfies

$$H_{\Omega_R}(s) = (-1)^n H_R(s^{-1})$$

as rational functions, or in terms of the index character we have

$$F_{\Omega_R}(t) = (-1)^n F_R(-t). \tag{6-2}$$

Roughly speaking this is a form of Serre duality. Let us first explain the proof in the easier case that R is Gorenstein, so that $m = 1$. Then the isomorphism in (6-1) becomes

$$R(-\lambda) \simeq \Omega_R,$$

and so, in particular, $F_{\Omega_R}(t) = e^{-\lambda t} F_R(t)$. Combining this with (6-2) gives

$$F_R(t) = (-1)^n e^{\lambda t} F_R(-t).$$

In terms of the index character this implies

$$\frac{a_0(n-1)!}{t^n} + \frac{a_1(n-2)!}{t^{n-1}} = (1+\lambda t)\left(\frac{a_0(n-1)!}{t^n} - \frac{a_1(n-2)!}{t^{n-1}}\right) + O(t^{2-n}).$$

Comparing coefficients we get that

$$2a_1(n-2)! = \lambda a_0(n-1)!,$$

which proves the proposition in the Gorenstein case.

We now consider the case when X is $\mathbb{Q}$–Gorenstein. Since the coordinate ring R is a Cohen–Macaulay integral domain, [18, Proposition 3.3.18] says that there is a homogeneous ideal $I \subset R$ such that $\Omega_R \simeq I$ as R–modules. In the current case, X is normal and affine, so the canonical sheaf is given by $K_X = i_* K_U$, where $i\colon U = X_{\mathrm{reg}} \hookrightarrow X$ and K_U is the canonical sheaf of U. Then

$$\Omega_R = \Gamma(X, K_X)$$

as R–modules. Thanks to the fact that X is $\mathbb{Q}$–Gorenstein we have

$$I^{(m)} \simeq R\cdot\sigma \simeq R(-\lambda),$$

where σ is the trivializing section of $\Gamma(X, mK_X)$, and $I^{(m)}$ denotes the m^{th} symbolic power of I. Since I is a reflexive R–module of rank one, the symbolic power may be defined as

$$I^{(m)} = (\overbrace{I \otimes_R \cdots \otimes_R I}^{m \text{ times}})^{**} = (I^{\otimes m})^{**},$$

where if M is an R–module, then $M^* = \mathrm{Hom}_R(M, R)$ denotes the dual. Geometrically, we have an exact sequence of sheaves on X,

$$0 \to \mathcal{K} \to K_X^{\otimes m} \to (K_X^{\otimes m})^{**} \to \mathcal{Q} \to 0.$$

Since K_X is reflexive and locally free on X_{reg}, the sheaves $\mathcal{K}$ and $\mathcal{Q}$ are supported on a subvariety of codimension at least 2. By Serre's criterion for affineness, we have $H^1(X, \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$. In particular, by taking global sections, using [50, Proposition 5.2] we get an exact sequence of graded R–modules,

$$0 \to K := \Gamma(X, \mathcal{K}) \to I^{\otimes m} \to I^{(m)} \to Q := \Gamma(X, \mathcal{Q}) \to 0.$$

On the level of Hilbert series this implies that

$$H_{I^{(m)}}(s) = H_{I^{\otimes m}}(s) + H_Q(s) - H_K(s).$$

Since $\mathcal{Q}$ and $\mathcal{K}$ are supported in codimension 2, their associated index characters satisfy

$$F_Q(t) = O(t^{2-n}) \quad \text{and} \quad F_K(t) = O(t^{2-n}).$$

We now consider the index character (or Hilbert series) of $I^{\otimes m}$. We need the following lemma.

Lemma 6.5 *Suppose that M and N are graded R–modules, with M free in codimension 1. Then*

$$F_{M\otimes_R N}(t) = \frac{F_M(t) F_N(t)}{F_R(t)} + O(t^{2-n}).$$

Proof Let us take a free resolution of N, ie a complex

$$0 \to E_k \to E_{k-1} \to \cdots \to E_0 \to 0$$

whose only cohomology is $H^0 = N$. Tensoring with M we obtain a complex

$$0 \to M \otimes_R E_k \to \cdots \to M \otimes_R E_0 \to 0,$$

whose cohomology is $H^i = \mathrm{Tor}_i^R(M,N)$. It is easy to check that the alternating sum of index characters of a complex is the same as the alternating sum of the index characters of its cohomology; ie we have

$$\sum_{i=0}^{k} (-1)^i F_{M\otimes_R E_i}(t) = \sum_{i=0}^{k} (-1)^i F_{\mathrm{Tor}_i^R(M,N)}(t).$$

Since M is free in codimension 1, we have that $\mathrm{Tor}_i^R(M,N)$ is supported in codimension 2 for $i>0$; ie its index character is of order t^{2-n}. It follows that

$$\begin{aligned} F_{M\otimes_R N}(t) &= \sum_{i=0}^{k} (-1)^i \frac{F_M(t)F_{E_i}(t)}{F_R(t)} + O(t^{2-n}) \\ &= \frac{F_M(t)}{F_R(t)} \sum_{i=0}^{k} (-1)^i F_{E_i}(t) + O(t^{2-n}) \\ &= \frac{F_M(t)F_N(t)}{F_R(t)} + O(t^{2-n}). \end{aligned}$$ □

From this lemma a simple induction gives

$$F_{I^{\otimes m}}(t) = \frac{F_{\Omega_R}(t)^m}{F_R(t)^{m-1}} + O(t^{2-n}).$$

As remarked earlier, the $\mathbb{Q}$–Gorenstein assumption implies that $I^{(m)} \simeq R(-\lambda)$, so

(6-3) $$e^{-t\lambda} F_R(t) = \frac{F_{\Omega_R}(t)^m}{F_R(t)^{m-1}} + O(t^{2-n}).$$

Expanding this equation to order t^{1-n} we obtain, as required,

$$\frac{a_1}{a_0} = \frac{\lambda(n-1)}{2m}.$$ □

6.2 The Futaki invariant

Suppose that (X,ξ) is a normalized Fano cone singularity and we have a test configuration λ for X with central fiber Y. Recall that λ gives a $\mathbb{C}^*$–action on Y generated by a vector field w. We have given two versions of the Futaki invariant of this test configuration. One was purely algebraic in terms of the weights of the action λ on the coordinate ring of Y, in Definition 2.2. As discussed below that definition, for small s we can consider the Reeb field $\xi + sw$ on Y, and we can normalize the test configuration (ie modify the vector field w by adding a multiple of ξ) in such a way that

$$\frac{a_1(Y, \xi + sw)}{a_0(Y, \xi + sw)} = \frac{n(n-1)}{2}.$$

Under this normalization the Futaki invariant is given by

$$\mathrm{Fut}(Y, \xi, w) = \tfrac{1}{2} D_w a_0(Y, \xi). \tag{6-4}$$

At the same time, from Proposition 6.4 we see that this normalization is equivalent to requiring $L_w\, dV = 0$, where dV is the canonical volume form on Y. The differential geometric definition of the Futaki invariant in that case is given in (3-6) by

$$\mathrm{Fut}_{Y,\xi}(w) = \frac{1}{V} \int_Y \theta_w e^{-\frac{1}{2} r^2} \omega^n, \tag{6-5}$$

where $\omega = \frac{1}{2}\sqrt{-1}\partial\bar{\partial} r^2$ is a suitable reference metric with Reeb field ξ, the function θ_w is the transverse Hamiltonian of w and V is the volume of (Y, ξ). To relate this to the algebraic definition we have the following two results.

Proposition 6.6 *Suppose $X \subset \mathbb{C}^N$ is a polarized affine variety with Reeb field ξ which has weights w_i on the coordinates z_i; ie ξ is the imaginary part of*

$$2 \sum_{i=1}^N w_i z_i \frac{\partial}{\partial z_i}.$$

Let

$$\hat{r}^2 = \sum_{i=1}^N |z_i|^{2/w_i},$$

and $\omega = \frac{1}{2}\sqrt{-1}\partial\bar{\partial}\hat{r}^2$. We have

$$a_0 (n-1)! = \frac{1}{(2\pi)^n} \int_X e^{-\frac{1}{2}\hat{r}^2} \frac{\omega^n}{n!}, \tag{6-6}$$

where a_0 is defined by the index character

$$F(\xi, t) = \frac{a_0(n-1)!}{t^n} + O(t^{-n+1}).$$

Proof We can choose a generic $\mathbb{C}^*$–action $\lambda(t)$ commuting with ξ, and degenerate X to $Y = \lim_{t \to 0} \lambda(t) \cdot X$. This will not affect the integral in (6-6), since the integral over $\lambda(t)\cdot X$ is the same as the integral over X using a different metric. The volume, however, is a function of just the Reeb field. At the same time the index character is also unchanged in passing to the limit since Y is a flat limit.

For a generic $\mathbb{C}^*$–action the top-dimensional part of Y is a union of n–dimensional coordinate subspaces with multiplicity. The leading term in the index character will be the sum of the corresponding terms for these subspaces, with multiplicity, and the limiting integral on Y is also given by a corresponding sum. As such, we only need to check the formula on $\mathbb{C}^n$ for a given Reeb field. But both the index character and the integral is multiplicative when taking products of varieties, so it is enough to do the calculation for $\mathbb{C}$, with a Reeb field

$$\xi = \mathrm{Im}\left(wz\frac{\partial}{\partial z}\right),$$

with corresponding radial function $\hat{r}^2 = |z|^{2/w}$. The metric is then

$$\omega = \frac{i}{2}\cdot\frac{1}{w^2}|z|^{2/w-2}\,dz\wedge d\bar{z},$$

so a calculation gives

$$\int_{\mathbb{C}} e^{-\frac{1}{2}\hat{r}^2}\omega = \frac{2\pi}{w}.$$

At the same time the index character is

$$\sum_{k=0}^{\infty} e^{-tkw} = \frac{1}{1-e^{-tw}} = \frac{1}{tw} + O(1),$$

from which the result follows. □

We also have the following formula for the variation of the volume as we vary the Reeb field.

Proposition 6.7 *Suppose that we consider a variation $\delta\xi = w$ of the Reeb field. The corresponding variation in the volume*

$$V(\xi) = \int_X e^{-\frac{1}{2}r^2}\frac{\omega^n}{n!}$$

is given by

$$\delta V(\xi) = n\int_X \theta_w e^{-\frac{1}{2}r^2}\frac{\omega^n}{n!}.$$

Proof This was shown by Martelli–Sparks–Yau [65]; see also Donaldson–Sun [41]. In comparing these formulas recall that by our convention $\theta_\xi = -1$. The result and its proof are valid even if X is not normal, interpreting the integral as just a sum of integrals on the n–dimensional components of X, with multiplicity. □

Using these results we can now compare the definitions (6-4) and (6-5) to see that the two Futaki invariants agree up to a dimensional constant, obtaining the following.

Proposition 6.8 *We have $\mathrm{Fut}_{X,\xi}(w) = c(n)\,\mathrm{Fut}(X,\xi,\lambda)$ in terms of Definition 2.2, where λ is the $\mathbb{C}^*$–action generated by w, and $c(n) > 0$ is a dimensional constant.*

7 K–stability of affine varieties with Ricci flat Kähler cone metrics

The main theorem of this section is:

Theorem 7.1 *Suppose that (X,ξ) admits a Ricci flat Kähler cone metric. Then (X,ξ) is K–stable.*

The idea of the proof follows work of Berman [6], and goes as follows. First, we will show that the Ding functional is convex along (sub)geodesics. Geodesics in the space of Sasakian metrics have been studied by Guan–Zhang [49], but we will need a slightly different formulation than the one given there. Since the Ricci flat Kähler cone metric is a critical point of the Ding functional, the strict convexity along geodesics means that along a (sub)geodesic φ_s emanating from a Sasaki–Einstein potential φ_{SE} we have

$$\frac{d}{ds}\mathcal{D}(\varphi_s) \geqslant 0$$

and the limit slope $\lim_{s\to\infty}(d/ds)\mathcal{D}(\varphi_s)$ exists in $[0,\infty]$. Furthermore, a result of Berndtsson [10] says that the limit must be strictly positive unless the geodesic was generated by a real holomorphic vector field (see [41] for the generalization of Berndtsson's result to our setting). Next, we show that any special degeneration gives rise to a (sub)geodesic, and we show that the limit slope is precisely the Futaki invariant.

Let (X,ξ) be a polarized cone, which we assume is $\mathbb{Q}$–Gorenstein and log terminal. Recall that we have defined the Ding functional

$$\mathcal{D}(\varphi) = -E(\varphi) - \frac{1}{2n}\log\int_X e^{-\frac{1}{2}r_\varphi^2}\,dV,$$

where $dV = (\Omega\wedge\overline{\Omega})^{1/m}$ with Ω a $\mathbb{T}$–equivariant trivialization of mK_X, $r\colon X\to\mathbb{R}_+$ is a radial function compatible with ξ, and $r_\varphi = e^{\varphi}r$, where φ is basic and independent of r.

As before, the function E is defined by its variation

$$\delta E(\varphi) = \frac{1}{V(\xi)} \int_X \dot{\varphi} e^{-\frac{1}{2} r_\varphi^2} \omega_\varphi^n.$$

Our goal is to compute the second variation of $E(\varphi_s)$. For our computation, we will assume that φ_s is a *smooth* variation. Dropping the $V(\xi)$ term for convenience an easy computation shows that

$$-\frac{d}{ds} E(\varphi) = \frac{1}{2n} \int_X \dot{\varphi} r_\varphi^2 e^{-\frac{1}{2} r_\varphi^2} \omega_\varphi^n = \frac{1}{2n} \int_X \tfrac{1}{2} \dot{r_\varphi^2} e^{-\frac{1}{2} r_\varphi^2} \omega_\varphi^n.$$

Let us suppress the dependence on φ to ease notation. Then we have

$$\text{(7-1)} \quad -2^{n+2} n \frac{d^2}{ds^2} E = \int_X \big[\ddot{r^2} - \tfrac{1}{2} (\dot{r^2})^2\big] e^{-\frac{1}{2} r^2} (\sqrt{-1}\partial\bar{\partial} r^2)^n + n \int_X \dot{r^2} e^{-\frac{1}{2} r^2} \sqrt{-1}\partial\bar{\partial} \dot{r^2} \wedge (\sqrt{-1}\partial\bar{\partial} r^2)^{n-1}.$$

Let us manipulate the last term. Integrating by parts gives

$$\begin{aligned} n \int_X \dot{r^2} e^{-\frac{1}{2} r^2} \sqrt{-1}\partial\bar{\partial} \dot{r^2} \wedge (\sqrt{-1}\partial\bar{\partial} r^2)^{n-1} &= -n \int_X e^{-\frac{1}{2} r^2} \sqrt{-1}\partial \dot{r^2} \wedge \bar{\partial} \dot{r^2} \wedge (\sqrt{-1}\partial\bar{\partial} r^2)^{n-1} \\ &\quad + \frac{n}{2} \int_X \dot{r^2} e^{-\frac{1}{2} r^2} \sqrt{-1}\partial r^2 \wedge \bar{\partial} \dot{r^2} \wedge (\sqrt{-1}\partial\bar{\partial} r^2)^{n-1}. \end{aligned}$$

We focus now on the second term of this expression. Integration by parts gives

$$\begin{aligned} &\frac{n}{2} \int_X \dot{r^2} e^{-\frac{1}{2} r^2} \sqrt{-1}\partial r^2 \wedge \bar{\partial} \dot{r^2} \wedge (\sqrt{-1}\partial\bar{\partial} r^2)^{n-1} \\ &\quad = \frac{n}{2} \int_X \dot{r^2} e^{-\frac{1}{2} r^2} \sqrt{-1}\, \bar{\partial} \dot{r^2} \wedge \partial r^2 \wedge (\sqrt{-1}\partial\bar{\partial} r^2)^{n-1} \\ &\qquad - \frac{n}{4} \int_X (\dot{r^2})^2 e^{-\frac{1}{2} r^2} \sqrt{-1}\, \bar{\partial} r^2 \wedge \partial r^2 \wedge (\sqrt{-1}\partial\bar{\partial} r^2)^{n-1} \\ &\qquad + \frac{n}{2} \int_X (\dot{r^2})^2 e^{-\frac{1}{2} r^2} \sqrt{-1}\, \bar{\partial}\partial r^2 \wedge (\sqrt{-1}\partial\bar{\partial} r^2)^{n-1} \\ &\quad = -\frac{n}{2} \int_X \dot{r^2} e^{-\frac{1}{2} r^2} \sqrt{-1}\partial r^2 \wedge \bar{\partial} \dot{r^2} \wedge (\sqrt{-1}\partial\bar{\partial} r^2)^{n-1} \\ &\qquad + \frac{n}{4} \int_X (\dot{r^2})^2 e^{-\frac{1}{2} r^2} \sqrt{-1}\partial r^2 \wedge \bar{\partial} r^2 \wedge (\sqrt{-1}\partial\bar{\partial} r^2)^{n-1} \\ &\qquad - \frac{n}{2} \int_X (\dot{r^2})^2 e^{-\frac{1}{2} r^2} (\sqrt{-1}\partial\bar{\partial} r^2)^n. \end{aligned}$$

Thus, we get

$$\begin{aligned}&\frac{n}{2}\int_X \dot{r^2}e^{-\frac{1}{2}r^2}\sqrt{-1}\partial r^2\wedge\bar\partial\dot{r^2}\wedge(\sqrt{-1}\partial\bar\partial r^2)^{n-1}\\ &=\frac{n}{8}\int_X(\dot{r^2})^2e^{-\frac{1}{2}r^2}\sqrt{-1}\partial r^2\wedge\bar\partial r^2\wedge(\sqrt{-1}\partial\bar\partial r^2)^{n-1}-\frac{n}{4}\int_X(\dot{r^2})^2e^{-\frac{1}{2}r^2}(\sqrt{-1}\partial\bar\partial r^2)^n.\end{aligned}$$

Now, since $\omega^\tau=\sqrt{-1}\partial\bar\partial\log r^2$ satisfies $(\omega^\tau)^n=0$, a direct computation shows that

$$\partial r^2\wedge\bar\partial r^2\wedge(\sqrt{-1}\partial\bar\partial r^2)^{n-1}=\frac{r^2}{n}(\sqrt{-1}\partial\bar\partial r^2)^n.$$

From this observation, an easy computation shows that

$$\begin{aligned}\frac{n}{8}\int_X(\dot{r^2})^2e^{-\frac{1}{2}r^2}\sqrt{-1}\partial r^2\wedge\bar\partial r^2\wedge(\sqrt{-1}\partial\bar\partial r^2)^{n-1}&=\frac{1}{8}\int_X(\dot{r^2})^2e^{-\frac{1}{2}r^2}r^2(\sqrt{-1}\partial\bar\partial r^2)^n\\ &=\frac{n+2}{4}\int_X(\dot{r^2})^2e^{-\frac{1}{2}r^2}(\sqrt{-1}\partial\bar\partial r^2)^n,\end{aligned}$$

and hence

$$\frac{n}{2}\int_X \dot{r^2}e^{-\frac{1}{2}r^2}\sqrt{-1}\partial r^2\wedge\bar\partial\dot{r^2}\wedge(\sqrt{-1}\partial\bar\partial r^2)^{n-1}=\frac{1}{2}\int_X(\dot{r^2})^2e^{-\frac{1}{2}r^2}(\sqrt{-1}\partial\bar\partial r^2)^n.$$

Plugging this back into the (7-1) we get

$$-2^{n+2}n\frac{d^2}{ds^2}E=\int_X e^{-\frac{1}{2}r^2}[\ddot{r^2}\sqrt{-1}\partial\bar\partial r^2-n\sqrt{-1}\partial\dot{r^2}\wedge\bar\partial\dot{r^2}]\wedge(\sqrt{-1}\partial\bar\partial r^2)^{n-1}.$$

A geodesic is a path $\{\varphi_s\}$ along which the function E is affine. In particular, we can write the geodesic equation as

$$[\ddot{r^2}\sqrt{-1}\partial\bar\partial r^2-n\sqrt{-1}\partial\dot{r^2}\wedge\bar\partial\dot{r^2}]\wedge(\sqrt{-1}\partial\bar\partial r^2)^{n-1}=0.$$

We will also make use of subgeodesics, along which the functional E is concave; that is, paths φ_s satisfying

$$[\ddot{r^2}\sqrt{-1}\partial\bar\partial r^2-n\sqrt{-1}\partial\dot{r^2}\wedge\bar\partial\dot{r^2}]\wedge(\sqrt{-1}\partial\bar\partial r^2)^{n-1}\geqslant 0.$$

Another standard computation shows that if we introduce a holomorphic coordinate τ with $|\tau|=e^{-s}$, then the (sub)geodesic equation can be written as

$$(\sqrt{-1}D\bar{D}r^2)^{n+1}=0,$$

where now D and $\bar{D}$ denote the ∂ and $\bar\partial$ operators in the variables τ and z jointly. In particular, a subgeodesic is nothing but a family of radial functions $r(z,\tau)\colon X\to\mathbb{R}_+$, all compatible with ξ, and such that $\sqrt{-1}D\bar{D}r^2 \geqslant 0$. Furthermore, the above description of the geodesic equation makes it clear that we can produce (weak) geodesics using the standard techniques of envelopes and subsolutions, but we will not need this here.

In order to obtain the convexity of the Ding functional, we also need the convexity of the second term in its definition. Berndtsson's theorem [10] gives this and even more; we refer the reader to [41] for an extension of Berndtsson's theorem to our setting.

Proposition 7.2 *Let $r(x,\tau)\colon X\times\Delta^*\to\mathbb{R}_{>0}$ be a path of radial functions compatible with ξ, and S^1–invariant. Suppose that $\sqrt{-1}D\bar{D}r \geqslant 0$, and that $\mathcal{D}(r_s)$ is affine ($s=-\log|\tau|$). Then there exists a holomorphic vector field Ξ on X, commuting with ξ and $r\partial_r$, such that $r_s = F_s^* r(x,0)$, where*

$$F_s = \exp(s\,\mathrm{Re}(\Xi)).$$

We now explain how a special degeneration gives rise to a subgeodesic. Let $\mathbb{T}\subset \mathrm{Aut}(X)$ be a torus containing ξ. Recall that a special degeneration consists of an embedding $X\to\mathbb{C}^N$, which we may assume is not contained in a linear subspace, such that $\mathbb{T}\subset\mathrm{Aut}(X)$ acts linearly and diagonally through an embedding $\mathbb{T}\subset U(N)$, together with a one-parameter subgroup $\lambda\colon \mathbb{C}^*\to \mathrm{GL}(N)^{\mathbb{T}}$ commuting with $\mathbb{T}$ such that $\lambda(S^1)\subset U(N)$ and $Y=\lim_{t\to 0}\lambda(t)\cdot X$ is normal. We may package this as an affine scheme $\mathcal{Y}\subset\mathbb{C}^N\times\mathbb{C}$, together with a $\mathbb{C}^*$–equivariant projection

$$\pi\colon \mathcal{Y}\to\mathbb{C},$$

where the $\mathbb{C}^*$–action is by λ; we will usually restrict our attention to $\pi^{-1}(\Delta)$, where Δ is the closed unit disk. Abusing notation, we will also denote this by $\mathcal{Y}$. By definition $\xi\in\mathfrak{u}(N)$ induces a Reeb vector field on $\mathbb{C}^N$, and hence we may find $r_0\colon\mathbb{C}^N\to\mathbb{R}_+$, a $U(N)$–invariant radial function compatible with ξ. Let $p_1\colon\mathbb{C}^N\times\mathbb{C}\to\mathbb{C}^N$, and consider $p_1^*r_0\colon\mathbb{C}^N\times\mathbb{C}\to\mathbb{R}_+$. The $\mathbb{C}^*$–action allows us to identify $\mathcal{Y}^*:=\pi^{-1}(\Delta^*)$ with $X\times\Delta^*$, and hence $p_1^*r_0$ induces a radial function $r(\tau)$ compatible with ξ on each fiber of $X\times\Delta^*$. By the $U(N)$–invariance of r_0, the function $r(\tau)$ is S^1–invariant, and since the map $X\times\Delta^*\to\mathcal{Y}^*$ is holomorphic we have

$$\sqrt{-1}D\bar{D}r(\tau)\geqslant 0.$$

We can therefore write

$$r(\tau) = r_0 e^{\psi(|\tau|)}.$$

Let φ be any potential of a radial function on X. By taking A and C large and ε small we define

$$\Phi(z,\tau)=\Phi(z,|\tau|):=\begin{cases}\widetilde{\max}_\varepsilon\{\varphi(z)+A\log|\tau|,\psi(z,|\tau|)-C\} & \text{if } |\tau|\geqslant\frac{1}{2},\\ \psi(z,|\tau|)-C & \text{if } |\tau|\leqslant\frac{1}{2},\end{cases}$$

where $\widetilde{\max}_\varepsilon$ is the regularized maximum [34, Section 5.E]. A few words are in order about how to choose A and C. First, we choose $C\gg 1$ large enough that $\psi(z,\tau)-C<\varphi(z)-100$ on $X\times\{1\}$. Next we choose A large enough that

$$\varphi(z)+A\log|\tau|\leqslant\psi(z,|\tau|)-C-100$$

for all $\frac{1}{2}<|\tau|<\frac{3}{4}$, and finally choose $0<\varepsilon\ll 1$. Clearly A and C exist since φ and $\psi(\tau)$ are smooth on X, and uniformly bounded for τ in any compact subset of Δ^*. By our choices and the properties of the regularized maximum we obtain that for every $\tau\in\Delta^*$, $r_\Phi:=re^{\Phi(\tau)}$ defines a smooth radial function on X compatible with ξ, and furthermore, $re^{\Phi(\tau)}$ is plurisubharmonic on $X\times\Delta^*$. In particular, $\Phi(\tau)$ defines a subgeodesic emanating from φ.

We now compute the limit slope of the Ding functional along $\Phi(\tau)$. Note that $\Phi(\tau)$ for $|\tau|<\frac{1}{2}$ depends only on r_0 and the test configuration, and not on the initial data. Since we are computing the limit

$$\lim_{s\to\infty}\frac{d}{ds}\mathcal{D}(\Phi(e^{-s})),$$

we can assume that in fact $\Phi(\tau)=\psi(\tau)$. Before proceeding, we need a preliminary lemma.

Lemma 7.3 *The total space of the special degeneration $\mathcal{Y}$ is a polarized cone. In particular, it is $\mathbb{Q}$–Gorenstein, with log-terminal singularities, and admits a Reeb vector field.*

Proof That $\mathcal{Y}$ is $\mathbb{Q}$–Gorenstein and log terminal follow from the fact that $\mathcal{Y}$ is flat over $\mathbb{C}$ and every fiber is normal, $\mathbb{Q}$–Gorenstein with log-terminal singularities. We only need to show that $\mathcal{Y}$ has a Reeb vector field. Let η be the generator of the $\mathbb{C}^*$–action defining the test configuration. By definition η acts on the coordinate t on $\mathbb{C}$ with weight one, and commutes with ξ. Hence for s sufficiently small, $\xi+s\eta$ is a Reeb vector field for $\mathcal{Y}$. □

Let $\mathbb{T}'$ be the torus in $\mathrm{Aut}(\mathcal{Y})$ containing $\mathbb{T}$, and η the generator of the $\mathbb{C}^*$–action defining the special degeneration. It follows from Lemma 6.2 that we can choose $\widehat{\Omega}$

a $\mathbb{T}'$–equivariant trivializing section of $mK_{\mathcal{Y}}$ so that $\iota_{(\partial/\partial\tau)^{\otimes m}}\widehat{\Omega}$ is a $\mathbb{T}$–equivariant trivialization Ω of mK_{X_t} for all $t\in\mathbb{C}$, where $X_t=\pi^{-1}(t)$. By the uniqueness part of Lemma 6.2 we must have

$$(\lambda^{-1}(\tau))^*\,\Omega=c(\tau)\iota_{(\partial/\partial\tau)^{\otimes m}}\widehat{\Omega},$$

where $c(\tau)$ is a nonvanishing holomorphic function, constant on the fibers. In particular, on the level of volume forms we have

$$(\lambda^{-1}(\tau))^*\,dV=|\hat{c}(\tau)|^{2/m}\,dV_{X_\tau}:=|c(\tau)|^{2/m}(\iota_{(\partial/\partial\tau)^{\otimes m}}\widehat{\Omega}\wedge\overline{\iota_{(\partial/\partial\tau)^{\otimes m}}\widehat{\Omega}}).$$

We now compute the limit slope of the Ding functional. As explained above, it suffices to compute the limit slope of $\mathcal{D}(r_s)$, where $s=-\log|\tau|$, and $r_s=\lambda(e^{-s})^*p_1^*r_0$. Recall that

$$\mathcal{D}(r_s)=-E(r_s)-\frac{1}{2n}\log\int_X e^{-\frac{1}{2}r_s^2}\,dV.$$

Let us first focus on the E term. By definition we have

$$-\frac{d}{ds}E(r_s)=-\frac{1}{V(\xi)}\int_X\frac{d}{ds}\log\biggl(\frac{r_s}{r_0}\biggr)e^{-\frac{1}{2}r_s^2}\,\omega_s^n.$$

We now use the biholomorphism $\lambda(e^{-s})$ to push the integral forward to $X_{e^{-s}}$. Note

$$\lambda(e^{-s})_*\frac{d}{ds}\log\frac{\lambda(e^{-s})^*r_0}{r_0}=-\theta_\lambda,$$

where θ_λ is the Hamiltonian function, with respect to r_0, of the real part of the holomorphic vector field generating the action of λ on $\mathbb{C}^N$. Thus we have

$$-\frac{d}{ds}E(r_s)=\frac{1}{V(\xi)}\int_{X_{e^{-s}}}\theta_\lambda e^{-\frac{1}{2}r^2}\,\omega^n.$$

Since $\mathcal{Y}$ is flat over $\mathbb{C}$, the current of integration $[X_{e^{-s}}]$ converges to $[X_0]$ weakly and we obtain

$$\lim_{s\to\infty}-\frac{d}{ds}E(r_s)=\frac{1}{V(\xi)}\int_{X_0}\theta_\lambda e^{-\frac{1}{2}r^2}\,\omega^n,$$

which is justified by the weak convergence since $\theta_\lambda e^{-\frac{1}{2}r^2}\omega^n$ is a smooth (n,n)–form defined on the ambient space $\mathbb{C}^N$.

We now compute the contribution of the second term. Specifically, we are computing

$$-\frac{d}{ds}\frac{1}{2n}\log\int_X e^{-\frac{1}{2}r_s^2}\,dV=\frac{1}{2n}\frac{\int_X r_s^2\frac{d}{ds}\log(\frac{r_s}{r})e^{-\frac{1}{2}r_s^2}\,dV}{\int_X e^{-\frac{1}{2}r_s^2}\,dV}.$$

Pulling this back to X_τ and using that $(\lambda^{-1}(\tau))^* \, dV = |c(\tau)|^{2/m} \, dV_{X_\tau}$ we get

$$-\frac{d}{ds}\frac{1}{2n}\log\int_X e^{-\frac{1}{2}r_s^2}\,dV = -\frac{1}{2n}\frac{\int_{X_\tau} r^2\theta_\lambda e^{-\frac{1}{2}r^2}\,dV_\tau}{\int_{X_\tau} e^{-\frac{1}{2}r^2}\,dV_\tau}.$$

Since $\mathcal{L}_\xi \, dV_\tau = 2n\sqrt{-1}\, dV_\tau$ and θ_λ is basic we obtain

$$\int_{X_\tau} r^2\theta_\lambda e^{-\frac{1}{2}r^2}\,dV_\tau = 2n\int_{X_\tau}\theta_\lambda e^{-\frac{1}{2}r^2}\,dV_\tau.$$

Putting everything together we have

$$-\frac{d}{ds}\frac{1}{2n}\log\int_X e^{-\frac{1}{2}r_s^2}\,dV = -\frac{\int_{X_\tau}\theta_\lambda e^{-\frac{1}{2}r^2}\,dV_\tau}{\int_{X_\tau} e^{-\frac{1}{2}r^2}\,dV_\tau}.$$

Now, by definition $\widehat{\Omega}$ is a nonvanishing, holomorphic section of $mK_{\mathcal{Y}}$, which is in particular smooth on $\mathcal{Y}_{\mathrm{reg}}$. Thanks to the fact that $X_0 = \pi^{-1}(0)$ is reduced and normal, Hartogs's theorem implies that

$$\iota_{(\partial/\partial\tau)^{\otimes m}}\widehat{\Omega}|_{\tau=0} = c_0\Omega_0,$$

where Ω_0 is the unique (up to scale) $\mathbb{T}'$–equivariant trivialization of mK_{X_0}, and c_0 is a nonzero constant. In particular, it follows that

$$dV_\tau \to |c_0|^{2/m}(\Omega_0\wedge\overline{\Omega}_0)^{1/m}$$

smoothly on $X_{0,\mathrm{reg}}$. Furthermore, by the log-terminal assumption, $X_{\tau,\mathrm{sing}}$ and $X_{0,\mathrm{sing}}$ have zero volume with respect to dV_τ and dV_0 respectively. Finally, since θ_λ and r are smooth functions on $\mathbb{C}^N$, flatness implies

$$\lim_{\tau\to 0}\frac{\int_{X_\tau}\theta_\lambda e^{-\frac{1}{2}r^2}\,dV_\tau}{\int_{X_\tau} e^{-\frac{1}{2}r^2}\,dV_\tau} = \frac{\int_{X_0}\theta_\lambda e^{-\frac{1}{2}r^2}\,dV_0}{\int_{X_0} e^{-\frac{1}{2}r^2}\,dV_0}.$$

Putting everything together we get

$$\lim_{s\to\infty}\frac{d}{ds}\mathcal{D}(r_s) = \int_{X_0}\theta_\lambda e^{-\frac{1}{2}r^2}\omega^n - \frac{\int_{X_0}\theta_\lambda e^{-\frac{1}{2}r^2}\,dV}{\int_{X_0} e^{-\frac{1}{2}r^2}\,dV}.$$

By Proposition 6.8 this is (up to a positive constant $c(n)$), the algebraic Futaki invariant of the test configuration $(\mathcal{Y},\lambda)$. From the convexity of the Ding functional we conclude

$$\mathrm{Fut}(\mathcal{Y},\lambda,\xi)\geqslant 0.$$

If $\mathrm{Fut}(\mathcal{Y}, \lambda, \xi) = 0$, then we must have that $(d/ds)\mathcal{D}(r_s) = 0$ identically, and then Berndtsson's theorem implies that $r_s = F_s^* r_0$ on X, where

$$F_s = \exp(sV)$$

and V is the real part holomorphic vector field on X commuting with ξ. Consider the map

$$\rho := \lambda(\tau) \circ F_\tau^{-1} \colon X \times \Delta^* \to \mathcal{Y}^*,$$

and let $\rho_\tau = \rho(\cdot, \tau)$ for $\tau \in \Delta^*$. By definition we have $\rho_\tau^* r_0 = r_0$, and $(\rho_\tau)_* \xi = \xi$. Since r_0 is the potential for a Kähler cone metric on $\mathbb{C}^N$ compatible with ξ, this implies that for any compact set $K \subset X$, the image $\rho\big(K \times \big(\frac{1}{2}\overline{\Delta} \setminus \{0\}\big)\big)$ is compact in $\mathcal{Y}$. By Riemann's extension theorem ρ extends to a map

$$\rho \colon X \times \Delta \to \mathcal{Y}$$

which is an isomorphism away from $\tau = 0$. The same argument applied to ρ^{-1} shows that $\rho \colon X \times \Delta \to \mathcal{Y}$ is an isomorphism, and so $\mathcal{Y}$ is a trivial test configuration. This completes the proof of Theorem 7.1.

8 Examples and applications

In this section we check K–stability for a family of hypersurfaces of dimension 3 admitting a 2–torus action. Rational hypersurface singularities in $\mathbb{C}^4$ admitting a $\mathbb{C}^*$–action were classified by Yau–Yu [83]. In the terminology of the Yau–Yu classification, we will study the links of types I–III admitting a $\mathbb{T}^2$–action. For this section we will describe a holomorphic vector field in terms of its S^1–action by specifying the weights; in particular, a vector (a_1, a_2, a_3, a_4) should be understood to act on $\mathbb{C}^2_{(z_1,z_2,z_3,z_4)}$ by acting on z_i with weight a_i. With this notation, the main theorem of this section is:

Theorem 8.1 *The following affine varieties admit conical Ricci flat Kähler metrics with respect to the given Reeb field:*

(I) *The Brieskorn–Pham singularity*

$$Z_{\mathrm{BP}}(p,q) := \{uv + z^p + w^q = 0\} \subset \mathbb{C}^4_{(u,v,z,w)},$$
$$\xi = \frac{3}{2(p+q)}(pq, pq, 2q, 2p),$$

if $2p > q$ and $2q > p$. Topologically, the link is $\#^m(S^2 \times S^3)$, where $m = \gcd(p,q) - 1$ and $\#^0(S^2 \times S^3) = S^5$.

(II) *The Yau–Yu singularities of type* II

$$Z_{\mathrm{II}}(p,q) := \{uv + z^p + zw^q = 0\} \subset \mathbb{C}^4_{(u,v,z,w)},$$
$$\xi = \frac{3}{2(q+p-1)}(qp, qp, 2q, 2(p-1)),$$

provided $3(p-1) > (q+p-1)$ *and* $2qp+1 > p^2+q$. *Topologically, the link is* $\#^m(S^2\times S^3)$, *where* $m = \gcd(p-1,q)$.

(III) *The Yau–Yu singularities of type* III

$$Z_{\mathrm{III}}(p,q) := \{uv + z^p w + zw^q = 0\} \subset \mathbb{C}^4_{(u,v,z,w)},$$
$$\xi = \frac{3}{2(p+q-2)}(pq-1, pq-1, 2(q-1), 2(p-1)),$$

provided $3(p-1)^2(q-1) > (p+q-2)(pq-2p+1)$ *and* $3(q-1)^2(p-1) > (p+q-2)(pq-2q+1)$. *Topologically, the link is* $\#^m(S^2\times S^3)$, *where* $m = \gcd(p-1,q-1)+1$.

In particular, there exists

- *one infinite family of inequivalent Sasaki–Einstein metrics on* S^5,
- *two distinct infinite families of inequivalent Sasaki–Einstein metrics on* $S^2\times S^3$,
- *three distinct families of inequivalent Sasaki–Einstein metrics on* $\#^m(S^2\times S^3)$ *for any* $m \geqslant 2$.

The proof of this theorem will occupy the remainder of this section. We use the theory of polyhedral divisors. The original paper in this area is the work of Altmann–Hausen [1], and there is a nice survey by Altmann–Ilten–Petersen–Süß [2]. For applications of this theory to K–stability of Fano manifolds, see Ilten–Süß [52].

8.1 The Brieskorn–Pham singularities

We begin with the Brieskorn–Pham (BP) singularities, denoted by $Z_{\mathrm{BP}}(p,q)$ above, which appear as type I singularities in the Yau–Yu classification. In order to avoid the trivial cases, we will assume that $\max\{p,q\} > 2$. Let $\gcd(p,q) = m$, and choose relatively prime integers $a,b \in \mathbb{Z}$ such that

$$a\frac{q}{m} - b\frac{p}{m} = 1.$$

The affine variety admits a 2–torus action, which is generated by the $\mathbb{C}^*$–actions with weights $(1,-1,0,0)$, and $(0, pq/m, q/m, p/m)$ on (u,v,z,w), respectively. Let $\mathfrak{t}$ denote the Lie algebra of the compact torus, equipped with a basis $\{e_1,e_2\}$

corresponding to the above actions. The Reeb cone inside of $\mathfrak{t}$ is given by

$$\mathcal{C}_{\mathcal{R}} = \{(x, y) := xe_1 + ye_2 \mid x > 0 \text{ and } xpq - ym > 0\}.$$

A simple symmetry argument shows that the Reeb field minimizing the volume of the link is a multiple of $(2, pq)$, which corresponds to the $\mathbb{C}^*$–action $(pq, pq, 2q, 2p)$. Let $F\colon \mathbb{Z}^2 \to \mathbb{Z}^4$ be the inclusion of the algebra $\mathfrak{t}$ into the Lie algebra of the diagonal torus acting on $\mathbb{C}^4$, equipped with the standard basis. With these choices, F is represented by the matrix

$$F = \begin{bmatrix} 0 & 1 \\ pq/m & -1 \\ q/m & 0 \\ p/m & 0 \end{bmatrix}.$$

Let $P\colon \mathbb{Z}^4 \to \mathbb{Z}^2$ be the orthogonal projection to the cokernel of F. We have

$$P = \begin{bmatrix} 0 & 0 & -p/m & q/m \\ 1 & 1 & -p & 0 \end{bmatrix}.$$

So we have an exact sequence

$$0 \to \mathbb{Z}^2 \xrightarrow{F} \mathbb{Z}^4 \xrightarrow{P} \mathbb{Z}^2 \to 0.$$

We choose a splitting $s\colon \mathbb{Z}^4 \to \mathbb{Z}^2$ such that $s \circ F = 1$,

$$s = \begin{bmatrix} 0 & 0 & a & -b \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

In the language of [1], the tail cone is given by

$$\sigma := s(F(\mathbb{Q}^2) \cap \mathbb{Q}^4_{\geqslant 0}) = \operatorname{cone}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ pq/m \end{pmatrix}\right].$$

It is straightforward to check that $\sigma = \overline{\mathcal{C}_{\mathcal{R}}}$ is the closure of the Reeb cone. The 2–torus action on $\mathbb{C}^4$ induces a fibration of $\mathbb{C}^4$ over a surface whose fibers are noncompact toric surfaces. Using the techniques of [1] we can describe this fibration in terms of combinatorial data called a p–divisor. Roughly speaking, a p–divisor is a finite formal sum of divisors on the base of the fibration, with coefficients being noncompact polytopes corresponding to the noncompact toric varieties comprising the fibers. We refer the reader to [1; 2] for a thorough discussion and introduction to these techniques. We first compute the base of the fibration, which is given by the fan Σ_Y with maximal cones

$$\operatorname{cone}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -m \end{pmatrix}\right], \quad \operatorname{cone}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -m \end{pmatrix}\right], \quad \operatorname{cone}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right],$$

which is the fan corresponding to the projective space $\mathbb{P}^2_{[1:1:m]}$. The rays of this fan are the columns of P. The p–divisor is then a formal finite sum

$$\sum \Delta_\rho \otimes D_\rho,$$

where D_ρ is a divisor on $\mathbb{P}^2_{[1:1:m]}$ and Δ_ρ is a convex polytope with tail cone σ. For the case at hand we have

$$\Delta_{(1,0)} = \left(-\frac{bm}{q}, 0\right) + \sigma, \quad \Delta_{(-1,-m)} = \left(\frac{am}{p}, 0\right) + \sigma, \quad \Delta_{(0,1)} = \{0\} \times [0,1] + \sigma,$$

where $\Delta_\rho = s(P^{-1}(\rho) \cap \mathbb{Q}^4_{\geqslant 0})$. This restricts to define a p–divisor on the curve $C := \{X^m + Y^m + Z = 0\} \subset \mathbb{P}^2_{[1:1:m]}$, which is precisely the base of the induced fibration of Z_{BP}. This curve intersects the divisor $Z = 0$ at m points, so the polytope $\Delta_{(0,1)}$ will appear m times. Let $\sigma^\vee$ denote the dual of the tail cone as a subset of $\mathfrak{t}^\vee$, the dual of the Lie algebra. In this case $\sigma^\vee$ is described explicitly by

$$\sigma^\vee = \mathrm{cone}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} pq/m \\ -1 \end{pmatrix}\right].$$

For each p–divisor Δ_ρ we get a function $\Psi\colon \sigma^\vee \to \mathbb{R}$ defined by

$$\Psi_\rho(w) = \min_{u \in \Delta_\rho} \langle w, u \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{t}$ and $\mathfrak{t}^\vee$. In our case we get

$$\Psi_{(1,0)}(s,t) = -\frac{bm}{q} s, \quad \Psi_{(-1,-m)}(s,t) = \frac{am}{p} s, \quad \Psi_{(0,1)}(s,t) = \min\{t, 0\},$$

where again $\Psi_{(0,1)}$ is repeated m times. If am/p is an integer then we obtain an equivalent p–divisor by replacing $\Psi_{(-1,-m)}$ by zero, and replacing $\Psi_{(1,0)}$ by $(am/p - bm/q)s$. This new p–divisor only has two distinct polytopes, and so from the description of $\mathbb{T}$–equivariant test configurations with normal central fiber due to Ilten–Süß [52] we see that there are at most two nontrivial test configurations. The same applies when bm/q is an integer. When neither is an integer, then both $\Psi_{(1,0)}$ and $\Psi_{(0,1)}$ have nonintegral slope, and so again from [52, Proposition 4.2] we obtain at most two nontrivial test configurations. These can be obtained from the methods of [52], but here we can simply guess the test configurations. They are

$$\mathcal{X}_1 = \{uv + (t \cdot z)^p + w^q = 0\} \quad \text{and} \quad \mathcal{X}_2 = \{uv + z^p + (t \cdot w)^q = 0\}.$$

We will compute the Futaki invariant for $\mathcal{X}_1$, the other case being identical. The special fiber of $\mathcal{X}_1$ is

$$Z_0 := \operatorname{Spec} \frac{\mathbb{C}[u,v,z,w]}{uv+w^q},$$

which is polarized by the Reeb vector field $(pq, pq, 2q, 2p)$. Since $Z_{\mathrm{BP}}(p,q)$ is a hypersurface, it is straightforward to check (see eg [65]) that $(Z_{\mathrm{BP}}(p,q),\xi)$ is normalized Fano if

$$\xi = \frac{3}{2(p+q)}(pq, pq, 2q, 2p).$$

On the central fiber there is a new $\mathbb{C}^*$–action corresponding to the one-parameter subgroup induced by $\eta = (0,0,1,0)$. The index character for the Reeb field $\xi - s\eta$ can be computed directly since the special fiber is a hypersurface (and in particular, a complete intersection) — see [28, Proposition 4.11] — though one can equally use Macaulay2 [48] to compute the multigraded Hilbert series. If $\lambda = 3/(2(p+q))$, then

$$\begin{aligned} F(\xi - s\eta, t) &= \frac{1-e^{-2pq\lambda t}}{(1-e^{-pq\lambda t})^2(1-e^{-(2q\lambda - s)t})(1-e^{-2p\lambda t})} \\ &= \frac{1}{\lambda^2 p^2 q(2q\lambda - s)t^3} + \frac{2p\lambda + 2q\lambda - s}{2\lambda^2 p^2 q(2q\lambda - s)t^2} + O(t^{-1}). \end{aligned}$$

From this we read off

$$a_0(\xi - s\eta) = \frac{1}{2\lambda^2 p^2 q(2q\lambda - s)}, \qquad \left(\frac{a_1}{a_0}\right)(\xi - s\eta) = \lambda(2p+2q) - s$$

and so

$$D_\eta a_0(\xi) = -\frac{1}{2\lambda^2 p^2 q(2q\lambda)^2}, \qquad D_\eta\left(\frac{a_1}{a_0}\right)(\xi) = 1.$$

By definition, the Futaki invariant is

$$\begin{aligned} \operatorname{Fut}(\mathcal{X}_1,\xi) &= \tfrac{1}{2}a_0(\xi)D_\eta\left(\frac{a_1}{a_0}\right)(\xi) + \tfrac{1}{2}D_\eta a_0(\xi) \\ &= \tfrac{1}{2}a_0(\xi)\left(1 - \frac{1}{2q\lambda}\right) \\ &= \tfrac{1}{2}a_0(\xi)\left(\frac{2q-p}{3q}\right), \end{aligned}$$

which is positive if and only if $2q > p$. Similarly for the other test configuration the condition is $2p > q$, and so we obtain that $(Z_{\mathrm{BP}}(p,q),\xi)$ is K–stable, and hence admits a conical Ricci flat Kähler metric if and only if

$$2p > q \quad \text{and} \quad 2q > p,$$

which is precisely the Lichnerowicz obstruction discovered by Gauntlett–Martelli–Sparks–Yau [46]. In dimension 5 there is standard machinery for computing the topology of links of isolated hypersurface singularities; see [14] for a complete description. In particular, it is straightforward to compute that the link of $Z(p,q)$ is topologically $\#^{\gcd(p,q)-1}(S^2\times S^3)$. In particular, whenever $\gcd(p,q)=1$ with $2p>q$ and $2q>p$, we obtain a Sasaki–Einstein metric on S^5. Furthermore, as a function of p and q, the (unnormalized) volume of $(Z_{\mathrm{BP}}(p,q),\xi)$ is given by

$$a_0(\xi)=\frac{2(p+q)^3}{27p^2q^2},$$

and hence infinitely many of these metrics are inequivalent. For example, fix a positive integer m and let $p>2$. Then the affine varieties $Z_{\mathrm{BP}}(pm,(p-1)m)$ are K–stable, and the link is topologically $\#^m(S^2\times S^3)$. Furthermore, the volume is given by

$$\mathrm{Vol}(Z_{\mathrm{BP}}(pm,(p-1)m),\xi)=\frac{2(2p-1)^3}{27mp^2(p-1)^2},$$

which is a strictly decreasing function as $p\to\infty$. By taking a sequence of primes going to infinity we obtain the existence of infinite families of inequivalent, nontoric Sasaki–Einstein metrics on $\#^m(S^2\times S^3)$ for any $m\geqslant 0$ (where $m=0$ means S^5). Furthermore, we note that $Z_{\mathrm{BP}}(2,3)$ is also K–stable, confirming the result of Li–Sun [61] that the A_2 singularity admits a Ricci flat cone metric.

8.2 The Yau–Yu singularities of type II

One can apply similar techniques to treat the Yau–Yu links of types II and III. We mention these applications briefly. Consider the family of hypersurface singularities described by $Z_{\mathrm{II}}(p,q)$, which admits a 2–torus generated by the $\mathbb{C}^*$–actions with weights $(0,pq,q,p-1)$ and $(1,-1,0,0)$. An easy symmetry argument shows that the normalized Reeb field minimizing the volume is given by

$$\xi=\frac{3}{2(q+p-1)}(qp,qp,2q,2(p-1)),\qquad \mathrm{Vol}(Z_{\mathrm{II}}(p,q),\xi)=\frac{2(p+q-1)^3}{27pq^2(p-1)}.$$

By similar techniques used for the Brieskorn–Pham links one can show that there are only two $\mathbb{T}$–equivariant test configurations. The first of these test configurations is

$$\mathcal{X}_1:=\{uv+z^p+z(t\cdot w)^q=0\},$$

which is induced by the $\mathbb{C}^*$–action with weights $(0,0,0,1)$. A straightforward computation using the index character yields

$$\mathrm{Fut}(\mathcal{X}_1,\xi)>0 \iff \frac{3(p-1)}{(q+p-1)}>1.$$

The second test configuration is more interesting and given by

$$\mathcal{X}_2 := \{uv + t^{pq}w^p + wz^q = 0\},$$

which is induced by the $\mathbb{C}^*$–action with weights $(0,0,q,-1)$. Computing the Futaki invariant yields

$$\mathrm{Fut}(\mathcal{X}_2,\xi)>0 \iff 2qp+1>p^2+q.$$

Note, in particular, that this obstruction is *strictly stronger* than the Lichnerowicz. For example, the affine variety $Z_{\mathrm{II}}(6,3)$ is not obstructed by the Lichnerowicz bound, but is destabilized by the test configuration $\mathcal{X}_2$. Topologically, the link of $Z_{\mathrm{II}}(p,q)$ is $\#^{\gcd(p-1,q)}(S^2\times S^3)$. If $p\geqslant 2$ is a prime number then one can easily check that

$$Z_{\mathrm{II}}(m(p-1)+1,mp)$$

is K–stable, and hence generates a Sasaki–Einstein metric on $\#^m(S^2\times S^3)$ with volume

$$\mathrm{Vol}(Z_{\mathrm{II}}(m(p-1)+1,mp),\xi)=\frac{2(2p-1)^3}{27p^2(p-1)(m(p-1)+1)}.$$

Taking a sequence of primes going to ∞ yields a second infinite sequence of distinct Sasaki–Einstein metrics on $\#^m(S^2\times S^3)$ for any $m\geqslant 1$.

8.3 The Yau–Yu links of type III

Finally, a similar analysis works for the Yau–Yu links of type III, given by

$$Z_{\mathrm{III}}(p,q)=\{uv+z^pw+zw^q=0\}\subset\mathbb{C}^4_{(u,v,z,w)},$$

each of which has a 2–torus action generated by the $\mathbb{C}^*$–actions whose weights are $(0,(pq-1),(q-1),(p-1))$ and $(1,-1,0,0)$. The critical Reeb field is then

$$\xi=\frac{3}{2(p+q-2)}(pq-1,pq-1,2(q-1),2(p-1)).$$

There are two nontrivial $\mathbb{T}$–equivariant test configurations generated by the $\mathbb{C}^*$–actions with weights $(0,0,-1,p)$ and $(0,0,q,-1)$. Computing the Futaki invariants as above we find that the link of $Z_{\mathrm{III}}(p,q)$ admits a Sasaki–Einstein metric if and only if

$$3(p-1)^2(q-1) > (p+q-2)(pq-2p+1),$$
$$3(q-1)^2(p-1) > (p+q-2)(pq-2q+1).$$

If we let $m = \gcd(p-1,q-1)+1$, then using [14, Chapter 9] one can check that the link of $Z_{\mathrm{III}}(p,q)$ is topologically $\#^m(S^2\times S^3)$. As before, we obtain a third infinite family of distinct Sasaki–Einstein metrics on $\#^m(S^2\times S^3)$ for any $m\in\mathbb{N}$ with $m\geqslant 2$.

9 Further discussion

The results contained in this paper motivate the following picture, which is the Sasakian analog of the general picture described in [75]. Fix a polarized affine variety $(X,\mathbb{T},\xi)$ of dimension n, where $\xi\in\mathfrak{t}$ is normalized, and has minimal volume. We try to find a Ricci flat Kähler cone metric compatible with ξ by deforming along the method of continuity. If (X,ξ) is K–stable, then we succeed. If not, then the method of continuity breaks at some time $T_1\leqslant 1$, and we get a test configuration with central fiber a normal, polarized affine variety $(Y_1,\mathbb{T}_1,\xi)$, where $\mathbb{T}_1$ is the torus generated by $\mathbb{T}$ and the vector field w_1 giving the test configuration. In particular $\dim\mathbb{T}_1=\dim\mathbb{T}+1$, but it is possible that $Y_1\cong X$ if the torus $\mathbb{T}$ that we started with was not maximal.

The test configuration is destabilizing, and so by the discussion in Section 2, if w_1 is normalized, then we have

$$D_{w_1}a_0(Y_1,\xi)\leqslant 0,$$

with strict inequality if $Y_1\cong X$. We can now repeat the volume minimization for $(Y_1,\mathbb{T}_1,\xi)$ to obtain a new Reeb field ξ_1. We expect that it will be possible to restart the method of continuity with the data $(Y_1,\mathbb{T}_1,\xi_1)$. Assuming that the results here carry over to the case of nonisolated singularities, we can repeat the above process to get

$$X\to Y_1\to\cdots\to Y_k := Y,$$

where the final (Y,ξ_k) is K–stable, since after finitely many steps we must reach a toric variety. Once the variety is toric, then it is automatically K–stable, after volume minimization, since there are no nontrivial toric test configurations with normal central fiber. Note that it was previously shown by Futaki–Ono–Wang [44] that toric Fano cone singularities with an isolated singularity admit Ricci flat cone metrics, and we expect the same to hold when there are singularities away from the cone point. It then follows that given any $(X, \mathbb{T}, \xi)$, it should be possible to deform X to a K–stable affine variety $(Y, \mathbb{T}', \xi')$ by at most $n-1$ test configurations.

It is natural to wonder whether this process can be made canonical and it seems reasonable to expect that the K–stable variety (Y, ξ_k) is canonically associated to (X, ξ). In view of the discussion in Donaldson–Sun [41, Section 3.3] and the example of Hein–Naber mentioned there, we expect that the Ricci flat cone metric on (Y, ξ_k) is the metric tangent cone at the vertex of any Ricci flat Kähler metric on a neighborhood of the vertex on X.

One can also ask for a more algebraic description of each Y_i in the sequence above, and for this at each step it would be necessary to distinguish one particular destabilizing test configuration. Motivated by conformal field theory (see [29]) the natural way to choose between any two destabilizing test configurations with central fibers Y_1 and Y_2 is to compare their volumes, after volume minimization. That is, we repeat the volume minimization on Y_i and get new polarized affine varieties (Y_1, ξ_1) and (Y_2, ξ_2). We choose Y_1 over Y_2 if

$$\mathrm{Vol}(Y_1, \xi_1) > \mathrm{Vol}(Y_2, \xi_2),$$

and vice versa, where the volume can be computed algebraically from the index character. When equality occurs, it may be that either $Y_1 \cong Y_2$ or there is a test configuration taking (Y_1, ξ_1) to (Y_2, ξ_2) or vice versa. These statements are confirmed to some extent by example calculations, but so far there is still little evidence for them.

In a less speculative vein there are many interesting questions regarding the existence of Sasaki–Einstein metrics on various manifolds. From Cho–Futaki–Ono [25] we know that $\#^m(S^2 \times S^3)$ admits infinitely many irregular Sasaki–Einstein metrics for $m \geqslant 1$, and we have shown that S^5 admits infinitely many quasiregular Sasaki–Einstein metrics. It is natural to ask therefore:

Question 9.1 Does there exist an irregular Sasaki–Einstein metric on S^5? More generally can we classify all Sasaki–Einstein metrics on S^5 with an isometric 2–torus action?

The combinatorial description of $\mathbb{T}$–varieties should help with this classification, as long as one develops a method for reading off the topology of the link from the p–divisor (see the work [60] in this direction). A more thorough study should also lead to higher-dimensional existence results, in particular on odd-dimensional spheres. We expect the following.

Conjecture 9.2 *There are infinitely many families of Sasaki–Einstein metrics on S^{2n+1} for all n.*

The same question can be asked for exotic spheres which bound parallelizable manifolds, and the existence of Sasaki–Einstein metrics on these was conjectured by Boyer–Galicki–Kollár [15]. This conjecture was verified up to dimension 15 by Boyer–Galicki–Kollár–Thomas [16].

Acknowledgements

The authors would like to thank Charles Boyer and S-T Yau for their interest in this work, and for helpful comments. In addition we would like to thank Hendrik Süß for helpful discussions on $\mathbb{T}$–varieties. The authors are also thankful to the referee for some helpful suggestions.

Collins is supported in part by NSF grant DMS-1506652. Székelyhidi is supported in part by NSF grants DMS-1306298 and DMS-1350696.

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Proposed: Simon Donaldson | Received: 25 September 2017
Seconded: John Lott, Tobias H Colding | Revised: 11 July 2018