

Sharp entropy bounds for self-shrinkers in mean curvature flow

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Let $M \subset \mathbb{R}^{m+1}$ be a smooth, closed, codimension-one self-shrinker (for mean curvature flow) with nontrivial k^{th} homology. We show that the entropy of M is greater than or equal to the entropy of a round k -sphere, and that if equality holds, then M is a round k -sphere in \mathbb{R}^{k+1} .

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1 Introduction

A properly embedded hypersurface $M \subset \mathbb{R}^{m+1}$ is called a self-shrinker if $M_t := \sqrt{-t}M$ for $t \in (-\infty, 0)$ is an evolution by mean curvature, ie if

$$(\partial_t x)^\perp = \vec{H}(x)$$

holds for every $t \in (-\infty, 0)$ and $x \in M_t$. Equivalently, M is a self-shrinker if it satisfies

$$(1) \quad \vec{H} + \frac{1}{2}x^\perp = 0.$$

The study of self-shrinkers is central in the analysis of singularity formation of the mean curvature flow. Indeed, every limit of rescalings of a mean curvature flow around a fixed point in spacetime is modeled on a possibly singular self-shrinker; see Huisken [10], White [19] and Ilmanen [13]. It is straightforward to check that a hyperplane through the origin is a self-shrinker, as is $\mathbb{S}^k(\sqrt{2k})$, the k -sphere of radius $\sqrt{2k}$ in \mathbb{R}^{k+1} . Crossing with a plane through the origin leaves (1) unchanged, so the cylinder $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{m-k}$ in \mathbb{R}^{m+1} is also a self-shrinker. We regard spheres as a special cases of cylinders: $\mathbb{S}^k = \mathbb{S}^k \times \mathbb{R}^0$. Although many other self-shrinkers have been constructed — see Angenent [1], Kapouleas, Kleene and Møller [15] and Ketover [16] — Huisken [14] conjectured that for mean curvature flows from generic initial hypersurfaces, all singularities are cylindrical. When the initial hypersurface is mean-convex, all singularities are indeed cylindrical; see Huisken and Sinestrari [10; 11] and White [21].

In a recent fundamental paper [5], Colding and Minicozzi made an important step towards establishing Huisken’s genericity conjecture. In that paper, they defined the *Gaussian area* of a hypersurface M in \mathbb{R}^{m+1} to be

$$(2) \quad F[M] = \frac{1}{(4\pi)^{m/2}} \int_M e^{-|x|^2/4} d\mathcal{H}^m,$$

and they defined its *entropy* to be the supremum of the Gaussian area of all translates and rescalings of M ,

$$(3) \quad \mathcal{E}[M] = \sup_{x_0 \in \mathbb{R}^{m+1}, \lambda > 0} F[\lambda(x - x_0)].$$

Clearly, Gaussian area is invariant under rotations, and entropy is invariant under all rigid motions and rescalings. The normalization constant $1/(4\pi)^{m/2}$ in the definition of F is chosen so that linear hyperplanes have Gaussian area 1. It follows that $F[M] = F[M \times \mathbb{R}]$ for every M , and thus that $\mathcal{E}[M] = \mathcal{E}[M \times \mathbb{R}]$.

Entropy is related to mean curvature flow through Huisken’s monotonicity formula [10], which implies that entropy is nonincreasing under the flow. Moreover, given a mean curvature flow with initial surface N , if M is a self-shrinker that arises (as discussed above) by blowing up around a singular point of the flow, then

$$(4) \quad F[M] = \mathcal{E}[M] \leq \mathcal{E}[N].$$

(The first equality holds for every self-shrinker, as was shown in [5]). The main result of [5] states that every self-shrinker M other than the spheres and cylinders can be perturbed to a hypersurface with lower entropy. Thus, by (4), if we flow from the perturbed hypersurface, then M cannot appear as a singularity model.

Stone [17] calculated the F -areas of shrinking spheres (and thus also of shrinking cylinders). By (4), those F -areas are the entropies of round spheres. According to those calculations,

$$(5) \quad 2 > \mathcal{E}[\mathbb{S}^1] > \mathcal{E}[\mathbb{S}^2] > \dots \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathcal{E}[\mathbb{S}^n] = \sqrt{2}.$$

In this paper, we prove:

Theorem 1 *Suppose that $M \subset \mathbb{R}^{m+1}$ is a codimension-one, smooth, closed self-shrinker with nontrivial k^{th} homology. Then the entropy of M is greater than or equal to the entropy of a round k -sphere. If equality holds, then M is a round k -sphere in \mathbb{R}^{k+1} .*

The special case $k = m$ is the main result of Colding, Ilmanen, Minicozzi and White [4]. The special cases $(k, m) = (1, 2)$ and $(k, m) = (2, 3)$ of [Theorem 1](#) follow from recent work of Jacob Bernstein and Lu Wang. Indeed, they prove in [2] that any smooth closed hypersurface in \mathbb{R}^3 with entropy less than $\mathcal{E}(\mathbb{S}^1)$ is isotopic to \mathbb{S}^2 , and in [3] that any smooth closed hypersurface in \mathbb{R}^4 with entropy less than $\mathcal{E}(\mathbb{S}^2)$ is diffeomorphic to \mathbb{S}^3 .

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2 Proof of the sharp entropy bounds

Theorem 2 *Suppose that $M \subset \mathbb{R}^{m+1}$ is a codimension-one, smooth, closed self-shrinker, and that one of the components of $\mathbb{R}^{m+1} \setminus M$ has nontrivial k^{th} homotopy. Then the entropy of M is greater than or equal to the entropy of a round k -sphere. If equality holds, then M is a round k -sphere in \mathbb{R}^{k+1} .*

Before proving [Theorem 2](#), we show that it implies our main theorem.

Proof of [Theorem 1](#) By Mayer–Vietoris, one of the components of the complement has nontrivial k^{th} homology. By the Hurewicz theorem, that component has nontrivial j^{th} homotopy for some $j \leq k$. By [Theorem 2](#),

$$\mathcal{E}(M) \geq \mathcal{E}(\mathbb{S}^j),$$

with equality if and only if M is a round j -sphere in \mathbb{R}^{j+1} . The result follows immediately since $\mathcal{E}(\mathbb{S}^j) > \mathcal{E}(\mathbb{S}^k)$ for $j < k$ by (5). \square

Proof of [Theorem 2](#) Consider the vectorfield

$$X: \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1}, \quad X(x) = \frac{1}{2}x.$$

We say that a region K of \mathbb{R}^{m+1} with smooth boundary is strictly X -mean-convex if $H + X^\perp$ is nonzero and points into K at each point of ∂K , where H is the mean curvature vector of ∂K . Let Ω be a component of $\mathbb{R}^{m+1} \setminus M$ such that $\mathbb{R}^{m+1} \setminus \Omega$ has nontrivial k^{th} homotopy. We may suppose that M is not a round sphere (otherwise the

result is trivially true). By [4, Lemma 1.2], we can deform M by pushing it slightly into Ω to get a surface $M' \subset \Omega$ such that

$$(6) \quad \mathcal{E}(M') < \mathcal{E}(M)$$

and such that K' is strictly X -mean-convex, where K' is the closure of the component of $\mathbb{R}^{m+1} \setminus M'$ that is contained in Ω . Now, as M' is a smooth hypersurface, we can let it evolve for short time by X -mean curvature flow, ie with normal velocity $H + X^\perp$. Since $H + X^\perp$ points into K' , the surface immediately moves into the interior of K' . In fact, as explained in Section 3 (see Definition 3 and Theorem 4), we can extend the flow to all $t \geq 0$ (in particular, past singularities) by letting

$$t \in [0, \infty) \mapsto M'(t)$$

be the weak X -mean curvature flow with $M'(0) = M'$. For the particular vectorfield $X(x) = \frac{1}{2}x$ we are using, X -mean curvature flow is also called *renormalized mean curvature flow*: it differs from the ordinary mean curvature flow by a spacetime change-of-coordinates. To be precise, given our weak X -mean curvature flow $M'(\cdot)$, the flow

$$(7) \quad \tilde{M}: t \in [-1, 0) \mapsto \tilde{M}(t) = \sqrt{-t}M'(-\log|t|).$$

is a weak mean curvature flow in \mathbb{R}^{m+1} with initial surface $\tilde{M}(-1) = M'$. This is because (7) transforms smooth X -mean curvature flows to smooth mean curvature flows, and hence weak X -mean curvature flows to weak mean curvature flows, since the weak flows are defined by avoidance with smooth flows. Note also that Huisken’s monotonicity formula implies a modified monotonicity for $\tilde{M}(\cdot)$, and that existence of tangent flows to $M'(\cdot)$ implies existence of tangent flows to $\tilde{M}(\cdot)$. Indeed, the tangent flows to $M'(\cdot)$ at a specified spacetime point are the same as the tangent flows to $\tilde{M}(\cdot)$ at the corresponding spacetime point.

Just as in the mean-convex setting, we can think of $t \mapsto M'(t)$ as a flow of measures (see Theorem 4). Since entropy decreases under the flow $t \mapsto \tilde{M}(t)$, we see that it also decreases under the renormalized flow $t \mapsto M'(t)$. Consequently, if Θ is the Gauss density at a spacetime point of the flow $t \mapsto M'(t)$, then

$$(8) \quad \Theta \leq \mathcal{E}(M'(0)) = \mathcal{E}(M').$$

Now let

$$t \mapsto K'(t) \quad \text{for } t \in [0, \infty)$$

be the weak X -mean curvature flow with $K'(0) = K'$ (see Definition 3). By Theorem 4,

$$\partial K'(t) = M'(t).$$

Since $\mathbb{R}^{m+1} \setminus K'$ has nontrivial k^{th} homotopy, there is a continuous map $F: \partial \mathbb{B}^{k+1} \rightarrow \mathbb{R}^{m+1} \setminus K'$ such that F is homotopically nontrivial in $\mathbb{R}^{m+1} \setminus K'$. Extend F to a continuous map

$$F: \overline{\mathbb{B}^{k+1}} \rightarrow \mathbb{R}^{m+1}.$$

By Theorem 7, $F(\mathbb{B}^{k+1}) \cap K'(T) = \emptyset$ for T sufficiently large. Since $F|_{\partial \mathbb{B}^{k+1}}$ is homotopically nontrivial in $\mathbb{R}^{m+1} \setminus K'(0)$ and homotopically trivial in $\mathbb{R}^{m+1} \setminus K'(T)$, the flow must be singular at one or more intermediate times. In fact, the X -mean-convexity implies more (see Theorem 6): there is a $t \in (0, T)$ and an $x \in M(t)$ such that the tangent flow at (x, t) is a shrinking $\mathbb{S}^j \times \mathbb{R}^{m-j}$ for some $j \leq k$. Consequently, the Gauss density Θ at that point is

$$\Theta = \mathcal{E}(\mathbb{S}^j \times \mathbb{R}^{m-j}) = \mathcal{E}(\mathbb{S}^j).$$

Hence, by (5), (6) and (8),

$$\mathcal{E}(\mathbb{S}^k) \leq \mathcal{E}(\mathbb{S}^j) = \Theta \leq \mathcal{E}(M') < \mathcal{E}(M). \quad \square$$

3 Motion by mean curvature plus an ambient vectorfield

In this section we define weak X -mean curvature flow of closed sets, and we state precisely the properties of the flow that were used in the proof of Theorem 1.

The following definition is an adaptation of the ones in [18; 12]:

Definition 3 Suppose that K is a closed subset of \mathbb{R}^{m+1} and that X is a smooth vectorfield on \mathbb{R}^{m+1} . Let \mathcal{K} be the largest closed subset of $\mathbb{R}^{m+1} \times [0, \infty)$ such that

- (a) $K(0) = K$, and
- (b) if $t \in [a, b] \subset [0, \infty) \mapsto \Delta(t)$ is a X -mean curvature flow of smooth, compact hypersurfaces with $\Delta(a)$ disjoint from $K(a)$, then $\Delta(t)$ is disjoint from $K(t)$ for all $t \in [a, b]$,

where

$$K(t) := \{x \in \mathbb{R}^{m+1} : (x, t) \in \mathcal{K}\}.$$

We say that

$$t \in [0, \infty) \mapsto K(t)$$

is the *weak X -mean curvature flow* (or simply the *weak X -flow*) starting from K .

The largest set \mathcal{K} exists because the closure of the union of all sets \mathcal{K} having properties (a) and (b) also has those properties.

For a self-contained treatment of weak X -flows, see [8]. (In that paper any closed subset of \mathcal{K} satisfying (a) and (b) is called a *weak X -flow* starting from K , and the largest one is called *the biggest X -flow* starting from K . In this paper, the only weak X -flow starting from K that we need is the biggest one, and we write “the weak X -flow” rather than “the biggest X -flow”.)

The following theorem lists the main properties of weak X -flow of X -mean-convex regions:

Theorem 4 *Suppose that K is a closed region in \mathbb{R}^{m+1} with smooth, compact boundary. Suppose that X is a smooth vector field on \mathbb{R}^{m+1} such that*

$$(*) \quad \sup_x \frac{|X(x)|}{|x| + 1} < \infty.$$

and such that at each point of ∂K , the vector $\vec{H} + X^\perp$ is nonzero and points into K . Let $t \mapsto K(t)$ and $t \mapsto M(t)$ be the weak- X -flows starting from K and from ∂K . Then:

- (a) $K(t_2) \subset \text{Int}(K(t_1))$ whenever $0 \leq t_1 < t_2 < \infty$.
- (b) $M(t) = \partial K(t)$ for each $t < \infty$.
- (c) $M(t)$ is compact for each $t < \infty$.
- (d) $t \in [0, \infty) \mapsto \mathcal{H}^m \llcorner M(t)$ defines a unit-regular integral X -Brakke flow.
- (e) The flow $t \mapsto M(t)$ is smooth away from a closed set of parabolic Hausdorff dimension $m - 1$ in spacetime.
- (f) The singular points of the flow $t \mapsto M(t)$ are of convex type.

A spacetime singular point (x, t) is said to be of *convex type* provided the following holds: if $x_i \in M(t_i)$ are regular points with $(x_i, t_i) \rightarrow (x, t)$, then the mean curvature h_i of $M(t_i)$ at x_i tends to infinity, and $h_i(M(t_i) - x_i)$ converges smoothly (after passing to a subsequence) to a convex hypersurface M' of $\text{Tan}(N, x_i)$. (Here we regard N as isometrically embedded in some Euclidean space.)

For the definition of “ X -Brakke flow”, see [8, Section 12] or [7].

The hypothesis (*) guarantees that compactness is preserved, ie that $\bigcup_{t \in [0, T]} M(t)$ is compact for finite T . See [8, Theorem 23]. More generally, in smooth Riemannian

manifolds and without the hypothesis (*), the conclusions of the theorem hold as long as $\bigcup_{t \in [0, T]} M(t)$ is compact.

In the case of Euclidean space with no vectorfield (ie $X = 0$), [Theorem 4](#) was proved in [[20](#); [21](#); [23](#)]. That work was extended to compact K in Riemannian manifolds (still with $X \equiv 0$) by Haslhofer and Hershkovits [[6](#)]. The proof of [Theorem 4](#) is a modification of the proofs in those papers. See [[8](#)] for proofs of assertions (a), (b) and (c), and [[7](#)] for proofs of assertions (d), (e) and (f).

Remark 5 Although the proof of [Theorem 1](#) only used the vector field $X = \frac{1}{2}x$, in order to prove [Theorem 4](#) for this particular vector field when K is unbounded (which is key to identifying interior topology in [Theorem 1](#)), one is forced to consider more general vector fields. Thus, from the point of view of this current paper, it is (indirectly) essential that the analysis in [[8](#); [7](#)] holds for arbitrary vector fields X satisfying (*), and not just for $X = \frac{1}{2}x$.

Theorem 6 Suppose in [Theorem 4](#) that

$$F: \partial \mathbb{B}^{k+1} \rightarrow \mathbb{R}^{m+1} \setminus K$$

is homotopically nontrivial in $\mathbb{R}^{m+1} \setminus K$ and homotopically trivial in $\mathbb{R}^{m+1} \setminus K(T)$. Then there is a $t \in (0, T)$ and a singular point $x \in M(t)$ such that the tangent flow at (x, t) is a shrinking $\mathbb{S}^j \times \mathbb{R}^{m-j}$ for some j with $1 \leq j \leq k$.

[Theorem 6](#) is a special case of [[22](#), Theorem 4.4]. See [[9](#)] for a simpler, Morse-theoretic proof of [Theorem 6](#).

Theorem 7 (clearing-out theorem) Suppose in [Theorem 4](#) that $X(x) = \frac{1}{2}x$. Then $\text{dist}(0, K(t)) \rightarrow \infty$ as $t \rightarrow \infty$.

It is possible that $K(t)$ vanishes in finite time. [Theorem 7](#) includes that case: if $K(t) = \emptyset$, then $\text{dist}(0, K(t)) = \infty$.

Proof If $M_1(\cdot)$ and $M_2(\cdot)$ are weak mean curvature flows in Euclidean space with $M_2(0)$ compact, then

$$\text{dist}(M_1(t), M_2(t)) := \min_{x \in M_1(t), y \in M_2(t)} |x - y|$$

is a nondecreasing function of t . For mean curvature flows of smooth hypersurfaces, this is the standard avoidance principle. The proof of [[12](#), Lemma 4E] gives the general

result. It follows immediately from the transformation formula (7) that if $M_1(\cdot)$ and $M_2(\cdot)$ are renormalized weak mean curvature flows with $M_2(0)$ compact, then

$$t \mapsto e^{-t/2} \operatorname{dist}(M_1(t), M_2(t))$$

is nondecreasing.

Fix a $\tau > 0$. Then $t \mapsto K(\tau + t)$ and $t \mapsto M(t)$ are renormalized mean curvature flows, so

$$e^{-t/2} \operatorname{dist}(K(\tau + t), M(t)) \quad \text{is nondecreasing in } t.$$

Since $K(\tau + t) \subset K(t) \subset K(0)$ and since $M(\cdot) = \partial K(\cdot)$, we have

$$\operatorname{dist}(K(\tau + t), M(0)) \geq \operatorname{dist}(K(\tau + t), M(t)) \geq e^{t/2} \operatorname{dist}(K(\tau), M(0)),$$

which tends to ∞ as $t \rightarrow \infty$. (Note that $\operatorname{dist}(K(\tau), M(0)) > 0$ since $K(\tau)$ lies in the interior of $K(0)$ and since $M(0) = \partial K(0)$.) \square

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