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Central limit theorem for spectral partial Bergman kernels

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Partial Bergman kernels $\Pi_{k,E}$ are kernels of orthogonal projections onto subspaces $\mathcal{S}_k \subset H^0(M, L^k)$ of holomorphic sections of the k^{th} power of an ample line bundle over a Kähler manifold (M, ω). The subspaces of this article are spectral subspaces $\{\hat{H}_k \le E\}$ of the Toeplitz quantization $\hat{H}_k$ of a smooth Hamiltonian $H\colon M \to \mathbb{R}$. It is shown that the relative partial density of states satisfies $\Pi_{k,E}(z)/\Pi_k(z) \to \mathbf{1}_{\mathcal{A}}$ where $\mathcal{A} = \{H < E\}$. Moreover it is shown that this partial density of states exhibits "Erf" asymptotics along the interface $\partial\mathcal{A}$; that is, the density profile asymptotically has a Gaussian error function shape interpolating between the values 1 and 0 of $\mathbf{1}_{\mathcal{A}}$. Such "Erf" asymptotics are a universal edge effect. The different types of scaling asymptotics are reminiscent of the law of large numbers and the central limit theorem.

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This article is part of a series [28] devoted to partial Bergman kernels on polarized Kähler manifolds $(L, h) \to (M^m, \omega, J)$, ie Kähler manifolds of (complex) dimension m equipped with a Hermitian holomorphic line bundle whose curvature form F_∇ for the Chern connection ∇ satisfies $\omega = iF_\nabla$. Partial Bergman kernels

$$\Pi_{k,\mathcal{S}_k}\colon L^2(M, L^k) \to \mathcal{S}_k \subset H^0(M, L^k) \tag{1}$$

are Schwarz kernels for orthogonal projections onto proper subspaces $\mathcal{S}_k$ of the holomorphic sections of L^k. For certain sequences $\mathcal{S}_k$ of subspaces, the partial density of states $\Pi_{k,\mathcal{S}_k}(z)$ has an asymptotic expansion as $k \to \infty$ which roughly gives the probability density that a quantum state from $\mathcal{S}_k$ is at the point z. More concretely, in terms of an orthonormal basis $\{s_i\}_{i=1}^{N_k}$ of $\mathcal{S}_k$, the partial Bergman densities are defined by

$$\Pi_{k,\mathcal{S}_k}(z) = \sum_{i=1}^{N_k} \|s_i(z)\|^2_{h^k}. \tag{2}$$

When $\mathcal{S}_k = H^0(M, L^k)$, $\Pi_{k,\mathcal{S}_k} = \Pi_k\colon L^2(M, L^k) \to H^0(M, L^k)$ is the orthogonal (Szegő or Bergman) projection. We also call the ratio $\Pi_{k,\mathcal{S}_k}(z)/\Pi_k(z)$ the partial density of states.

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Corresponding to $\mathcal{S}_k$ there is an allowed region $\mathcal{A}$ where the relative partial density of states $\Pi_{k,\mathcal{S}_k}(z)/\Pi_k(z)$ is 1, indicating that the states in $\mathcal{S}_k$ "fill up" $\mathcal{A}$, and a forbidden region $\mathcal{F}$ where the relative density of states is $O(k^{-\infty})$, indicating that the states in $\mathcal{S}_k$ are almost zero in $\mathcal{F}$. On the boundary $\mathcal{C} := \partial\mathcal{A}$ between the two regions there is a shell of thickness $O(k^{-\frac{1}{2}})$ in which the density of states decays from 1 to 0. One of the principal results of this article is that the $\sqrt{k}$–scaled relative partial density of states is asymptotically Gaussian along this interface, in a way reminiscent of the central limit theorem. This was proved by Ross and Singer [21] for certain Hamiltonian holomorphic S^1 actions, then in greater generality by Zelditch and Zhou [28]. The results of this article show it is a universal property of partial Bergman kernels defined by C^∞ Hamiltonians.

Before stating our results, we explain how we define the subspaces $\mathcal{S}_k$. In [28] and in this article, they are defined as spectral subspaces for the quantization of a smooth function $H\colon M \to \mathbb{R}$. By the standard (Kostant) method of geometric quantization, one can quantize H as the self-adjoint zeroth-order Toeplitz operator

(3) $$H_k := \Pi_k\Big(\frac{i}{k}\nabla_{\xi_H} + H\Big)\Pi_k\colon H^0(M, L^k) \to H^0(M, L^k)$$

acting on the space $H^0(M, L^k)$ of holomorphic sections. Here, ξ_H is the Hamiltonian vector field of H, ∇_{ξ_H} is the Chern covariant derivative on sections, and H acts by multiplication. Let E be a regular value of H. We denote the partial Bergman kernels for the corresponding spectral subspaces by

(4) $$\Pi_{k,E}\colon H^0(M, L^k) \to \mathcal{H}_{k,E},$$

where

(5) $$\mathcal{S}_k := \mathcal{H}_{k,E} := \bigoplus_{\mu_{k,j} < E} V_{\mu_{k,j}},$$

with $\mu_{k,j}$ the eigenvalues of H_k and

(6) $$V_{\mu_{k,j}} := \{s \in H^0(M, L^k) : H_k s = \mu_{k,j} s\}.$$

We denote by $\Pi_{k,j}\colon H^0(M, L^k) \to V_{\mu_{k,j}}$ the orthogonal projection to $V_{\mu_{k,j}}$. The associated allowed region $\mathcal{A}$, which is the classical counterpart to (5), the forbidden region $\mathcal{F}$ and the interface $\mathcal{C}$ are

(7) $$\mathcal{A} := \{z : H(z) < E\}, \quad \mathcal{F} = \{z : H(z) > E\}, \quad \mathcal{C} = \{z : H(z) = E\}.$$

For each $z \in \mathcal{C}$, let ν_z be the unit normal vector to $\mathcal{C}$ pointing towards $\mathcal{A}$. And let $\gamma_z(t)$ be the geodesic curve with respect to the Riemannian metric $g(X,Y) = \omega(X, JY)$ defined by the Kähler form ω such that $\gamma_z(0) = z$ and $\dot{\gamma}_z(0) = \nu_z$. For small enough $\delta > 0$, the map

$$\Phi\colon \mathcal{C} \times (-\delta,\delta) \to M, \quad (z,t) \mapsto \gamma_z(t), \tag{8}$$

is a diffeomorphism onto its image.

Main Theorem *Let $(L,h) \to (M,\omega,J)$ be a polarized Kähler manifold, and let $H\colon M \to \mathbb{R}$ be a smooth function and E a regular value of H. Let $\mathcal{S}_k \subset H^0(X, L^k)$ be defined as in* (5)*. Then we have the following asymptotics on partial Bergman densities $\Pi_{k,\mathcal{S}_k}(z)$:*

$$\left(\frac{\Pi_{k,\mathcal{S}_k}}{\Pi_k}\right)(z) = \begin{cases} 1 \bmod O(k^{-\infty}) & \text{if } z \in \mathcal{A}, \\ 0 \bmod O(k^{-\infty}) & \text{if } z \in \mathcal{F}. \end{cases}$$

For small enough $\delta > 0$, let $\Phi\colon \mathcal{C} \times (-\delta,\delta) \to M$ be given by (8)*. Then for any $z \in \mathcal{C}$ and $t \in \mathbb{R}$, we have*

$$\left(\frac{\Pi_{k,\mathcal{S}_k}}{\Pi_k}\right)(\Phi(z, t/\sqrt{k})) = \mathrm{Erf}(2\sqrt{\pi}t) + O(k^{-\frac{1}{2}}), \tag{9}$$

where $\mathrm{Erf}(x) = \int_{-\infty}^{x} e^{-\frac{1}{2}s^2} \frac{ds}{\sqrt{2\pi}}$ is the cumulative distribution function of the Gaussian, ie $\mathbb{P}_{X \sim N(0,1)}(X < x)$.[1]

Remark 0.1 The analogous result for critical levels is proved in [30]. We could also choose an interval (E_1, E_2) with E_i regular values of H,[2] and define $\mathcal{S}_k$ as the span of eigensections with eigenvalue within (E_1, E_2). However the interval case can be deduced from the half-ray case $(-\infty, E)$ by taking the difference of the corresponding partial Bergman kernels for $(-\infty, E_2)$ and $(-\infty, E_1)$, hence we only consider allowed regions of the type in (7).

Example 0.2 As a quick illustration, holomorphic sections of the trivial line bundle over $\mathbb{C}$ are holomorphic functions on $\mathbb{C}$. We equip the bundle with the Hermitian metric where 1 has the norm-square $e^{-|z|^2}$. The k^{th} power has metric $e^{-k|z|^2}$. Fix $\epsilon > 0$ and define the subspaces $\mathcal{S}_k = \bigoplus_{j \le \epsilon k} z^j$ of sections vanishing to order at most ϵk at 0, or

[1] The usual Gaussian error function $\mathrm{erf}(x) = (2\pi)^{-\frac{1}{2}} \int_{-x}^{x} e^{-\frac{1}{2}s^2} ds$ is related to Erf by $\mathrm{Erf}(x) = \frac{1}{2}\big(1 + \mathrm{erf}(\frac{1}{\sqrt{2}}x)\big)$.

[2] It does not matter whether the endpoints are included in the interval, since contributions from the eigenspaces $V_{k,\mu}$ with $\mu = E_i$ are of order lower than k^m.

sections with eigenvalues $\mu < \epsilon$ for the operator $H_k = (1/ik)\partial_\theta$ quantizing $H = |z|^2$. The full and partial Bergman densities are

$$\Pi_k(z) = \frac{k}{2\pi}, \quad \Pi_{k,\epsilon}(z) = \frac{k}{2\pi} \sum_{j \le \epsilon k} \frac{k^j}{j!} |z^j|^2 e^{-k|z|^2}.$$

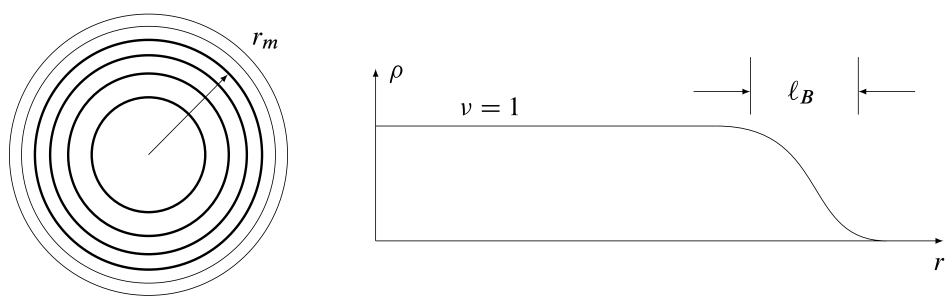
As $k \to \infty$, we have

$$\lim_{k\to\infty} k^{-1} \Pi_{k,\epsilon}(z) = \begin{cases} 1 & \text{if } |z|^2 < \epsilon, \\ 0 & \text{if } |z|^2 > \epsilon. \end{cases}$$

For the boundary behavior, consider a sequence z_k such that $|z_k|^2 = \epsilon(1 + k^{-\frac{1}{2}} u)$:

$$\lim_{k\to\infty} k^{-1} \Pi_{k,\epsilon}(z_k) = \mathrm{Erf}(u).$$

This example is often used to illustrate the notion of "filling domains" in the integer quantum Hall (IQH) effect. Filling domains will be discussed further in Section 0.8. In IQH, one considers a free electron gas confined in the plane $\mathbb{R}^2 \simeq \mathbb{C}$, with a uniform magnetic field in the perpendicular direction. A one-particle electron state is said to be in the lowest Landau level (LLL) if it has the form $\Psi(z) = e^{-\frac{1}{2}|z|^2} f(z)$, where $f(z)$ is holomorphic as in Example 0.2. The following image of the density profile is copied from Wen [25]. The picture on the right illustrates how the states $((\sqrt{k}\, z)^j / \sqrt{j!}) e^{-\frac{1}{2}k|z|^2}$ with $j \le \epsilon k$ fill the disc of radius $\sqrt{\epsilon}$, so that the density profile drops from 1 to 0:



The example is S^1 symmetric and therefore the simpler results of [28] apply. For more general domains $D \subset \mathbb{C}$, it is not obvious how to fill D with LLL states. Douglas and Klevtsov [8] generalized the IQH to line bundles over curved surfaces or higher-dimensional Kähler manifolds, and D can be a domain in any Kähler manifold. The Main Theorem answers the question when $D = \{H \le E\}$ for some H. Other approaches are discussed in Section 0.9. For a physics discussion of Erf asymptotics and their (as yet unknown) generalization to the fractional QH effect, see Can, Forrester, Téllez and Wiegmann [3; 26].

0.1 Three families of measures at different scales

The rationale for viewing the Erf asymptotics of scaled partial Bergman kernels along the interface $\mathcal{C}$ is explained by considering three different scalings of the spectral problem:

$$d\mu_k^z(x) = \sum_j \Pi_{k,j}(z)\delta_{\mu_{k,j}}(x), \tag{10a}$$

$$d\mu_k^{z,\frac{1}{2}}(x) = \sum_j \Pi_{k,j}(z)\delta_{\sqrt{k}(\mu_{k,j}-H(z))}(x), \tag{10b}$$

$$d\mu_k^{z,1,\tau}(x) = \sum_j \Pi_{k,j}(z)\delta_{k(\mu_{k,j}-H(z))+\sqrt{k}\tau}(x), \tag{10c}$$

where as usual, δ_y is the Dirac point mass at $y \in \mathbb{R}$. We use $\mu(x) = \int_{-\infty}^x d\mu(y)$ to denote the cumulative distribution function.

We view these scalings as analogous to three scalings of the convolution powers μ^{*k} of a probability measure μ supported on $[-1, 1]$ (say). The third scaling (10c) corresponds to μ^{*k}, which is supported on $[-k, k]$. The first scaling (10a) corresponds to the law of large numbers, which rescales μ^{*k} back to $[-1, 1]$. The second scaling (10b) corresponds to the central limit theorem (CLT), which rescales the measure to $[-\sqrt{k}, \sqrt{k}]$. For more on the CLT connection we refer to Section 0.7.

Our main results give asymptotic formulae for integrals of test functions and characteristic functions against these measures. To obtain the remainder estimate (9), we need to apply semiclassical Tauberian theorems to $\mu_k^{z,\frac{1}{2}}$ and that forces us to find asymptotics for $\mu_k^{z,1,\tau}$.

0.2 Unrescaled bulk results on $d\mu_k^z$

The first result is that the behavior of the partial density of states in the allowed region $\{z : H(z) < E\}$ is essentially the same as for the full density of states, while it is rapidly decaying outside this region.

We begin with a simple and general result about partial Bergman kernels for smooth metrics and Hamiltonians.

Theorem 1 *Let ω be a C^∞ metric on M and let $H \in C^\infty(M)$. Fix a regular value E of H and let $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{C}$ be given by (7). Then for any $f \in C^\infty(\mathbb{R})$, we have*

$$\Pi_k(z)^{-1} \int_{-\infty}^{E} f(\lambda)\, d\mu_k^z(\lambda) \to \begin{cases} f(H(z)) & \text{if } z \in \mathcal{A}, \\ 0 & \text{if } z \in \mathcal{F}. \end{cases} \tag{11}$$

In particular, the density of states of the partial Bergman kernel is given by the asymptotic formula

$$\Pi_k(z)^{-1}\Pi_{k,E}(z) \sim \begin{cases} 1 \mod O(k^{-\infty}) & \text{if } z \in \mathcal{A}, \\ 0 \mod O(k^{-\infty}) & \text{if } z \in \mathcal{F}, \end{cases} \tag{12}$$

where the asymptotics are uniform on compact sets of $\mathcal{A}$ or $\mathcal{F}$.

In effect, the leading-order asymptotics says that the normalized measure satisfies $\Pi_k(z)^{-1}\, d\mu_k^z \to \delta_{H(z)}$. This is a kind of law of large numbers for the sequence $d\mu_k^z$. The theorem does not specify the behavior of $\mu_k^z(-\infty, E)$ when $H(z) = E$. The next result pertains to the edge behavior.

0.3 $\sqrt{k}$–scaling results on $d\mu_k^{z,\frac{1}{2}}$

The most interesting behavior occurs in $k^{-\frac{1}{2}}$–tubes around the interface $\mathcal{C}$ between the allowed region $\mathcal{A}$ and the forbidden region $\mathcal{F}$. For any $T > 0$, the tube of "radius" $Tk^{-\frac{1}{2}}$ around $\mathcal{C} = \{H = E\}$ is the flowout of $\mathcal{C}$ under the gradient flow of H,

$$F^t := \exp(t\nabla H)\colon M \to M,$$

for $|t| < Tk^{-\frac{1}{2}}$. Thus it suffices to study the partial density of states $\Pi_{k,E}(z_k)$ at points $z_k = F^{\beta/\sqrt{k}}(z_0)$ with $z_0 \in H^{-1}(E)$. The interface result for any smooth Hamiltonian is the same as if the Hamiltonian flow generates a holomorphic S^1 action, and thus our result shows that it is a universal scaling asymptotic around $\mathcal{C}$.

Theorem 2 *Let ω be a C^∞ metric on M and let $H \in C^\infty(M)$. Fix a regular value E of H and let $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{C}$ be given by (7). Let $F^t\colon M \to M$ denote the gradient flow of H by time t. We have the following results:*

(1) *For any point $z \in \mathcal{C}$, any $\beta \in \mathbb{R}$, and any smooth function $f \in C^\infty(\mathbb{R})$, there exists a complete asymptotic expansion,*

$$\sum_j f(\sqrt{k}(\mu_{k,j} - E))\Pi_{k,j}(F^{\beta/\sqrt{k}}(z)) \simeq \left(\frac{k}{2\pi}\right)^m (I_0 + k^{-\frac{1}{2}} I_1 + \cdots), \tag{13}$$

in descending powers of $k^{\frac{1}{2}}$*, with the leading coefficient as*

$$I_0(f,z,\beta)=\int_{-\infty}^{\infty} f(x)\exp\biggl(-\biggl(\frac{x}{|\nabla H|(z)|}-\beta|\nabla H(z)|\biggr)^2\biggr)\frac{dx}{\sqrt{\pi}|\nabla H(z)|}.$$

(2) *For any point* $z\in\mathcal{C}$*, and any* $\alpha\in\mathbb{R}$*, the cumulative distribution function* $\mu_k^{z,\frac{1}{2}}(\alpha)=\int_{-\infty}^{\alpha}d\mu_k^{z,\frac{1}{2}}$ *is given by*

$$\mu_k^{z,\frac{1}{2}}(\alpha)=\sum_{\mu_{k,j}<E+\alpha/\sqrt{k}}\Pi_{k,j}(z)=\Bigl(\frac{k}{2\pi}\Bigr)^m\operatorname{Erf}\biggl(\frac{\sqrt{2}\alpha}{|\nabla H(z)|}\biggr)+O(k^{m-\frac{1}{2}}). \tag{14}$$

(3) *For any point* $z\in\mathcal{C}$*, and any* $\beta\in\mathbb{R}$*, the Bergman kernel density near the interface is given by*

$$\begin{aligned}&\Pi_{k,E}(F^{\beta/\sqrt{k}}(z))\\&\quad=\sum_{\mu_{j,k}<E}\Pi_{k,j}(F^{\beta/\sqrt{k}}(z))=\Bigl(\frac{k}{2\pi}\Bigr)^m\operatorname{Erf}(-\sqrt{2}\beta|\nabla H(z)|)+O(k^{m-\frac{1}{2}}).\end{aligned} \tag{15}$$

Remark 0.3 The leading power $(k/2\pi)^m$ is the same as in Theorem 1, despite the fact that we sum over a packet of eigenvalues of width (and cardinality) $k^{-\frac{1}{2}}$ times the width (and cardinality) in Theorem 1. This is because the summands $\Pi_{k,j}(z)$ already localize the sum to $\mu_{k,j}$ satisfying $|\mu_{k,j}-H(z)|<Ck^{-\frac{1}{2}}$.

0.4 Energy level localization and $d\mu_k^{z,1,\alpha}$

To obtain the remainder estimate for the $\sqrt{k}$–rescaled measure $d\mu_k^{z,\frac{1}{2}}$ in (14) and (15), we apply the Tauberian theorem. Roughly speaking, one approximates $d\mu_k^{z,\frac{1}{2}}$ by convoluting the measure with a smooth function W_h of width h, and the difference of the two is proportional to h. The smoothed measure $d\mu_k^{z,\frac{1}{2}}*W_h$ has a density function, the value of which can be estimated by an integral of the propagator $U_k(t,z,z)$ for $|t|\sim k^{-1}/(hk^{-\frac{1}{2}})$. Thus if we choose $h=k^{-\frac{1}{2}}$, and W_h to have Fourier transform supported in $(-\epsilon,+\epsilon)/h$, we only need to evaluate $U_k(t,z,z)$ for $|t|<\epsilon$, where ϵ can be taken to be arbitrarily small.

Theorem 3 *Let* E *be a regular value of* H *and* $z\in H^{-1}(E)$*. If* ϵ *is small enough that the Hamiltonian flow trajectory starting at* z *does not loop back to* z *for time* $|t|<2\pi\epsilon$*, then for any Schwarz function* $f\in\mathcal{S}(\mathbb{R})$ *with* $\hat{f}$ *supported in* $(-\epsilon,\epsilon)$ *and* $\hat{f}(0)=\int f(x)\,dx=1$*, and for any* $\alpha\in\mathbb{R}$*, we have*

$$\int_{\mathbb{R}}f(x)\,d\mu_k^{z,1,\alpha}(x)=\Bigl(\frac{k}{2\pi}\Bigr)^{m-\frac{1}{2}}e^{-\alpha^2/\|\xi_H(z)\|^2}\frac{\sqrt{2}}{2\pi\|\xi_H(z)\|}(1+O(k^{-\frac{1}{2}})).$$

It is an interesting and well-known problem, in other settings, to find the asymptotics when $\mathrm{supp}(\hat{f})$ is a general interval. When $z \in H^{-1}(E)$ they depend on whether or not z is a periodic point for $\exp t\xi_H$. In this article we only need to consider the singularity at $t = 0$ or, equivalently, test functions for which the support of $\hat{f}$ is sufficiently close to 0. In [31], we use the long-time dynamics of the Hamiltonian flow to give a complete asymptotic expansion for (23) when $f \in C_c^\infty(\mathbb{R})$ and a two-term asymptotics with remainder when $f = \mathbf{1}_{[E_1,E_2]}$.

0.5 Sketch of the proofs

The theorems are proved by first smoothing the sharp interval cutoff $\mathbf{1}_{[E_{\min},E]}$ to a smooth cutoff f and obtaining asymptotics, and then applying a Tauberian argument. The jump discontinuity of $\mathbf{1}_{[E_{\min},E]}$ produces the universal error function transition between the allowed and forbidden regions. This error function arises in classical approximation arguments involving Bernstein polynomials (see [28] for background and references). This is a standard method in proving sharp pointwise Weyl asymptotics by combining smoothed asymptotics with Tauberian theorems.

Given a function $f \in \mathcal{S}(\mathbb{R})$ (Schwartz space) one defines

$$f(H_k) = \int_{\mathbb{R}} \hat{f}(\tau) e^{i\tau H_k} \frac{d\tau}{2\pi} \tag{16}$$

to be the operator on $H^0(M, L^k)$ with the same eigensections as H_k and with eigenvalues $f(\mu_{k,j})$. Thus, if $s_{k,j}$ is an eigensection of H_k, then

$$f(H_k) s_{k,j} = f(\mu_{k,j}) s_{k,j}. \tag{17}$$

Let $E_{\min}$ and $E_{\max}$ be such that $H(M) = [E_{\min}, E_{\max}]$. Given a regular value of E of H, the subspace $\mathcal{S}_k$ in (5) is defined as the range of $f(H_k)$ where $f = \mathbf{1}_{[E_{\min},E]}$ and the partial density of states is given by the metric contraction of the kernel,

$$\Pi_{k,E}(z) = f(H_k)(z) = \sum_{\{j : \mu_{k,j} \le E\}} \Pi_{k,j}(z). \tag{18}$$

For a smooth test function f, $\Pi_{k,f}(z)$ is the metric contraction of the Schwartz kernel of $f(H_k)$ at $z = w$, given by

$$\Pi_{k,f}(z) = \sum_j f(\mu_{k,j}) \Pi_{k,j}(z). \tag{19}$$

Note that $\Pi_k e^{itH_k} \Pi_k$ is the exponential of a bounded Toeplitz pseudodifferential operator H_k and itself is a Toeplitz pseudodifferential operator. To obtain a dynamical operator, ie one which quantizes a Hamiltonian flow, one needs to exponentiate the first-order Toeplitz operator kH_k. In that case,

$$f(k(H_k - E)) = \int_{\mathbb{R}} \hat{f}(\tau) e^{ik\tau H_k} \frac{d\tau}{2\pi} = \int_{\mathbb{R}} \hat{f}(\tau) U_k(\tau) \frac{d\tau}{2\pi}, \tag{20}$$

where

$$U_k(\tau) = \exp(i\tau k H_k) \tag{21}$$

is the unitary group on $H^0(M, L^k)$ generated by kH_k.

In Section 4 it is shown that $U_k(t)$ is a semiclassical Toeplitz Fourier integral operator of a type defined in [27]. To construct a semiclassical parametrix for $U_k(t)$ it is convenient to lift the Hamiltonian flow g^t of H to a contact flow $\hat{g}^t$ on the unit circle bundle $X_h = \{\zeta \in L^* : h(\zeta) = 1\}$ associated to the Hermitian metric h on L^*. That is, we lift sections s of L^k to equivariant functions $\hat{s}\colon X_h \to \mathbb{C}$ transforming by $e^{ik\theta}$ under the natural S^1 action on L^*. Holomorphic sections lift to CR–holomorphic functions and the space $H^0(M, L^k)$ lifts to the space $\mathcal{H}_k(X_h)$ of equivariant CR functions. The orthogonal projection Π_k onto $H^0(M, L^k)$ lifts to the orthogonal projection $\hat{\Pi}_{h^k}$ onto $\mathcal{H}_k(X_h)$. In Proposition 4.5 it is shown that the lift $\hat{U}_k(t)$ to $\mathcal{H}_k(X_h)$ has the form $\hat{\Pi}_{h^k}(\hat{g}^t)^* \sigma_{k,t} \hat{\Pi}_{h^k}$, where $(\hat{g}^t)^*$ is the pullback of functions on X_h by $\hat{g}^t$ and where $\sigma_{k,t}$ is a semiclassical symbol.

The main tool in the proof of Theorem 2 is to use the Boutet de Monvel–Sjöstrand parametrix to study the integrals $\int_{\mathbb{R}} \hat{f}(t) U_k(t/\sqrt{k}, z, z)\, dt$. Since the relevant time interval is "infinitesimal" (of the order $k^{-\frac{1}{2}}$) the result can be proved by linearizing the kernel $U_k(t/\sqrt{k}, z, z)$. The smoothed interface asymptotics thus amount to the asymptotics of the dilated sums,

$$\begin{aligned} \sum_j f(\sqrt{k}(\mu_{k,j} - E)) \Pi_{k,j}(F^{\beta/\sqrt{k}}(z_0)) \\ = \int_{\mathbb{R}} \hat{f}(t) e^{-iE\sqrt{k}t} \hat{\Pi}_{h^k}(\hat{g}^t)^* \sigma_{k,t} \hat{\Pi}_{h^k}(F^{\beta/\sqrt{k}}(z)) \frac{dt}{2\pi}, \end{aligned} \tag{22}$$

where $z \in \partial\mathcal{A} = H^{-1}(E)$ and where $\hat{f} \in L^1(\mathbb{R})$, so that the integral on the right side converges. We employ the Boutet de Monvel–Sjöstrand parametrix to give an explicit formula for the right side of (22) modulo small remainders.

At this point, we are essentially dealing with $(1/\sqrt{k})$–scaling asymptotics of Szegő kernels, as studied first by Bleher, Shiffman and Zelditch [2], then in more detail by Lu, Ma, Marinescu, Shiffman and Zelditch [12; 13; 23] and for related dynamical purposes by Paoletti [17; 18]. The scaling asymptotics of the Bergman kernel infinitesimally off the diagonal at z_0 are expressed in terms of the osculating Bargmann–Fock–Heisenberg kernel $\Pi_{\mathrm{BF}}^{T_{z_0}M}$ of the (complex) tangent space at z_0. This tangent space is equipped with a complex structure J_z and a Hermitian metric H_z and therefore with a Bargmann–Fock space $H^2(T_{z_0}M, \gamma_{J_{z_0},H_{z_0}})$ of entire J_{z_0}–holomorphic functions on $T_{z_0}M$ which are in L^2 with respect to the Gaussian weight determined by J_z and H_z. This is the linear model for semiclassical Toeplitz calculations. Since we reduce the general calculation of scaling asymptotics to the linear ones, we present the calculations in the Bargmann–Fock model in detail first (see Sections 1.5, 3.2 and 5.1). We emphasize that the linear model is not only an example, but constitutes a fundamental part of the proofs.

To prove Theorem 3, we study the integrals

$$\mu_k^{z,1,\tau}(f) := \sum_j f(k(\mu_{k,j} - H(z)) + \sqrt{k}\tau)\Pi_{k,j}(z) = \int_{\mathbb{R}} \hat{f}(t) e^{-iH(z)kt + i\sqrt{k}\tau t}\, \hat{\Pi}_{h^k}(\hat{g}^t)^* \sigma_{k,t} \hat{\Pi}_{h^k}(z)\, \frac{dt}{2\pi}. \tag{23}$$

For the purposes of this article, we only need the infinitesimal-time behavior of the Hamilton flow. In [31], we use the long-time asymptotics of the Hamilton flow to obtain a two-term Weyl law.

After obtaining scaling asymptotics for smoothed partial densities of states, we employ Tauberian theorems with different scalings to obtain asymptotics with remainders for the sharp partial densities of states or the measures (10). Background on Tauberian theorems is given in Appendix B.

0.6 Comparison to the S^1 case

For purposes of comparison, we review the results of Ross and Singer in [21] and of [28] in the case where the Hamiltonian generates an S^1 action by holomorphic maps.

In [21] and [28], it is assumed that (M, L, h) is invariant under a Hamiltonian holomorphic S^1 action. The action naturally quantizes or linearizes on the spaces $H^0(M, L^k)$, and $\mathcal{S}_k$ is defined in terms of the weights (eigenvalues) of the S^1 action on the spaces $H^0(M, L^k)$. The eigenvalues of the generator H_k of the quantized S^1 action are "lattice points" j/k, where $j \in \mathbb{Z}$. The partial Bergman kernel $\Pi_{k,j}$ onto a single weight space $V_k(j) \subset H^0(M, L^k)$ already has asymptotics as $k \to \infty$ and $j/k \to E$, and the asymptotics of the partial Bergman kernels corresponding to intervals of weights is obtained by integrating these "equivariant" Bergman kernels.

The "equivariant Bergman kernels" $\Pi_{k,j}\colon H^0(M, L^k) \to V_k(j)$ resemble transverse Gaussian beams along the energy level $H^{-1}(j/k)$, and have complete asymptotic expansions that can be summed over j to give asymptotics of the density of states $\Pi_{h^k,[E_1,E_2]}(z,z)$ of partial Bergman kernels as $k \to \infty$. In the allowed region $H^{-1}([E_1,E_2])$, we have $k^{-m}\Pi_{h^k,[E_1,E_2]}(z,z) \simeq k^{-m}\Pi_k(z,z) \simeq 1$, while in the complementary forbidden region the asymptotics are rapidly decaying, and exponentially decaying when the metric ω is real analytic. The equivariant kernels additionally possess Gaussian scaling asymptotics along the energy surface, and by summing in j one obtains incomplete Gaussian asymptotics of the partial Bergman kernels along the boundary $H^{-1}\{E_1, E_2\}$ as in [21].

In the nonperiodic case of this article, the Hamilton flow

$$g^t := \exp t\xi_H\colon M \to M,$$

generated by $H\colon M \to \mathbb{R}$ with respect to ω, is not holomorphic. Here and below, we use the notation $\exp tX$ for the flow of a real vector field X. The gradient flow is denoted by

$$F^t := \exp t\nabla H\colon M \to M. \tag{24}$$

This change from a holomorphic Hamiltonian S^1 action to a Hamiltonian $\mathbb{R}$ action brings many new features into the asymptotics of partial Bergman kernels. First, the gradient flow of ∇H no longer commutes with the Hamilton flow of ξ_H, so that one does not have a global $\mathbb{C}^*$ action to work with. As mentioned above, the eigenvalues generally have multiplicity one and the eigenspace projections

$$\Pi_{k,\mu_{k,j}}\colon H^0(M, L^k) \to V_{\mu_{k,j}} \tag{25}$$

do not have individual asymptotics. One only obtains asymptotics if one sums over a "packet" of k^{m-1} eigenvalues in a "smooth" way. The asymptotics of this counting function are given by the reasonably well-understood Weyl law for semiclassical Toeplitz operators. When H is real analytic, exponential decay of density of states for $z \in \mathcal{F}$ still occurs but the rate has a different shape from that in the holomorphic case. But as shown in this article, the interface asymptotics of [21] and [28] are universal, and continue to hold even in the C^∞ case.

Remark 0.4 The corresponding result for holomorphic Hamiltonian S^1 actions [28, Theorem 4] states the following. Let ω be a C^∞ $\mathbb{T}$–invariant Kähler metric, and let H generate the holomorphic $\mathbb{T}$ action. Fix $E \in H(M)$, and let $z = e^{\beta/\sqrt{k}} \cdot z_0$ for some $z_0 \in H^{-1}(E) \in \mathbb{R}$, where $e^{\beta/\sqrt{k}}$ denotes the imaginary-time part of the $\mathbb{C}^*$ action generated by H. The gradient flow (24) of this article is the same as the imaginary-time Hamiltonian flow $e^{\beta/\sqrt{k}}$ in the S^1 case. Moreover, in the S^1 case, $H = \frac{1}{2}\partial_\rho \varphi$ and $\pi \partial_\rho^2 \varphi = |\nabla H|^2$. The asymptotics also hold if $f = \mathbf{1}_{[E_1,E_2]}$ is the characteristic function of an interval, and then, for $f(x) = 1_{[0,\infty]}(x)$,

$$k^{-m} \sum_{j>kE} \Pi_{k,j}(e^{\beta/\sqrt{k}} \cdot z_0) = \int_{-\infty}^{\beta\sqrt{\partial_\rho^2 \varphi(z_0)}} e^{-\frac{1}{2}t^2} \frac{dt}{\sqrt{2\pi}} + O(k^{-\frac{1}{2}}).$$

Hence this agrees with the formula of Theorem 2.

0.7 Relation to the CLT

In Section 0.1 we mentioned that the three scalings of $d\mu_k^z$ are analogous to the three scalings of convolution powers μ^{*k} of probability measures μ on $\mathbb{R}$. The simplest example of the classical CLT is that of the Bernoulli measures $\mu_p = (1-p)\delta_0 + p\delta_1$ and their convolution powers on the unit interval $[0,1]$. The k^{th} convolution power $\mu_p^{*k} = 2^{-k} \sum_{n=0}^k p^k (1-p)^{n-k} \binom{k}{n} \delta_n$ has its support in $[0,k]$. In the law of large numbers one rescales the measure back to $[0,1]$ as $2^{-k} \sum_{n=0}^k p^k (1-p)^{n-k} \binom{k}{n} \delta_{n/k}$, which tends weakly to δ_p. This is thus analogous to our k–scaling in Section 0.1. In the CLT one recenters the measure at 0 and then dilates it by $\sqrt{k}$ so that it spreads out to $[-\sqrt{k}, \sqrt{k}]$, and then it tends to the Gaussian of mean 0 and variance 1. The parameter $p \in [0,1]$ of μ_p is analogous to the parameter $z \in M$ in the Kähler setting. In a sequel to this article [29] the authors expand on this analogy by proving a CLT for toric Kähler manifolds, in which the μ_k^z are refined so that they live on the polytope of a toric Kähler manifold. In the special case of $\mathbb{CP}^1$ with the Fubini–Study metric, the measures are precisely the Bernoulli measures μ_p with $p \in [0,1]$ being the image of z under the moment map. Moreover, the CLT is in fact the classical CLT in this special case, ie μ_k^z is the k^{th} convolution power of μ_1^z. For almost any other metric, the CLT of [29] and of this article does not involve a sequence of convolution powers, nor has any relation to the classical CLT. But these special cases show that the analogy to the CLT is quite apt.

0.8 Filling domains in the quantum Hall effect

As mentioned above, this article is partly motivated by the somewhat vague question that arises in the quantum Hall effect (QHE): *How do you fill a domain D with quantum* *states in the lowest Landau level?* The LLL is the term in QHE for holomorphic sections ψ of line bundles. The curvature form ω of a Hermitian line bundle models a magnetic field, and the LLL represents single-particle states of electrons in a strong magnetic field. The problem is to construct states ψ_j in the LLL so that $(1/N)\sum_{j=1}^{N}|\psi_j(z)|^2 \simeq \mathbf{1}_D$ is approximately equal to the characteristic function $\mathbf{1}_D$ of D.

In this article we answer the question by representing $D=\{H \le E\}$ for some smooth Hamiltonian H. We then use the quantization $\widehat{H}_k$ of H to construct the states that fill D, by proving that the eigensections of $\widehat{H}_k$ (3) from the spectral subspace $\mathcal{H}_{k,E}$ (5) asymptotically fill D. The Main Theorem and Theorem 1 show that, as $k \to \infty$,

$$k^{-m}\Pi_{h^k,[E_1,E_2]}(z,z) \to \mathbf{1}_{H^{-1}[E_1,E_2]}$$

and thus it "fills the domain $H^{-1}[E_1,E_2]$ with lowest Landau levels" without spilling outside the domain.

Another approach is to use $\Pi_k \mathbf{1}_D \Pi_k$, a Toeplitz operator with discontinuous symbol $\mathbf{1}_D$. To leading order it agrees with $\Pi_{k,E}$ when $D=\{H \le E\}$, but it is not a projection operator. Its eigensections with eigenvalues close to 1 should be the states which fill up D. It would be interesting to compare this operator to $\Pi_{k,E}$ more precisely.

0.9 Subspaces of sections vanishing to high order on a divisor

Another way to generalize Example 0.2 is to think of the exterior of the disc as sections vanishing to high order $\ge \epsilon k$ on the divisor $\{0\}$. In general, one may introduce a smooth integral divisor D in X, and let the subspace $\mathcal{S}_k$ be holomorphic sections in L^k that vanish at least to order $\lfloor \epsilon k \rfloor$ along D, where ϵ is a small enough positive number. Articles by Berman, Pokorny, Ross and Singer [1; 19; 21] study this problem. The allowed region is given by

$$\mathcal{A}=\{\varphi_{\mathrm{eq},\epsilon,D}(z)=0\},$$

where

$$\varphi_{\mathrm{eq},\epsilon,D}(z)=\sup\{\widetilde{\varphi}(z) : \omega + i\partial\bar{\partial}\widetilde{\varphi} \ge 0,\ \widetilde{\varphi} \le 0,\ \nu(\widetilde{\varphi})_w \ge \epsilon \text{ for all } w \in D\}$$

and $\nu(\widetilde{\varphi})_w$ is the Lelong number at w (see [1, Section 4]). As yet, there is no spectral interpretation of this subspace except in the S^1 equivariant case, where D is a component of the fixed-point set of the S^1 action. Erf asymptotics along the boundary of the allowed region have only been proved in that case [21; 28].

1 Background

The background to this article is largely the same as in [28], and we refer there for many details. Here we give a lightning review to set up the notation. First we introduce the cocircle bundle $X \subset L^*$ for a positive Hermitian line bundle (L, h) such that holomorphic sections of L^k for different k can all be represented in the same space $\mathcal{H}(X) = \bigoplus_k \mathcal{H}_k(X)$ of CR–holomorphic functions on X. Then we define the Szegő projection kernel and state the Boutet de Monvel–Sjöstrand parametrix. In the end, we give the Bergman kernel for the Bargmann–Fock model on $\mathbb{C}^n$.

1.1 Positive line bundle (L, h) and the dual unit circle bundle X

Let (M, ω, J, g) be a Kähler manifold, where ω is a J–invariant symplectic two-form and g is a Riemannian metric determined by ω and J as $g(-,-) = \omega(-, J-)$. Let (L, h) be a holomorphic Hermitian line bundle on M. Let $e_L \in \Gamma(U, L)$ be a local frame over an open subset $U \subset M$, and the Kähler potential $\varphi(z)$ over U is defined by $e^{-\varphi(z)} = h(e_L, e_L)$. The local expressions for the Chern connection ∇ and Chern curvature F_∇ are $\nabla = \partial + \bar\partial - \partial\varphi \wedge (-)$ and $F_\nabla = \partial\bar\partial\varphi$. We say (M, ω) is polarized by (L, h) if the Kähler form ω is

$$\omega = 2\pi c_1(L) = iF_\nabla = i\partial\bar\partial\varphi = -\tfrac{1}{2} d d^c \varphi,$$

where $d^c = i(\partial - \bar\partial)$ such that $\langle d^c f, -\rangle = \langle df, J(-)\rangle$.

Let (L^*, h^*) be the dual bundle to L with the induced Hermitian metric h^*, and $e_L^* \in \Gamma(U, L^*)$ the dual frame to e_L with $\|e_L^*\| = 1/\|e_L\| = e^{\frac{1}{2}\varphi(z)}$. The unit open disk bundle is $D = D(L^*, h^*) = \{p \in L^* : \|p\| < 1\}$ and the unit circle bundle is $X = \partial D$. Let $\pi\colon X \to M$ be the projection. Then there is a canonical circle action r_θ on X. Let ρ be a smooth function defined in a neighborhood of X such that $\rho > 0$ in D, $\rho|_X = 0$ and $d\rho|_X \neq 0$. In this paper, we fix a choice of ρ as

$$\rho(x) = -\log \|x\|_h^2 = -2\log|\lambda| - \log\|e_L^*\|^2 = -2\log|\lambda| - \varphi(z),$$

where $x = \lambda e_L^*(z) \in L^*|_U$, $z \in U$ and $\lambda \in \mathbb{C}^*$. Then X can be equipped with a contact one-form α:

$$\alpha = -\mathrm{Re}(i\bar\partial\rho)|_X = d\theta + \pi^* \mathrm{Re}(i\bar\partial\varphi(z)) = d\theta - \tfrac{1}{2} d^c\varphi(z) \quad \text{and} \quad d\alpha = \pi^*\omega, \tag{26}$$

where (z, θ) is a local coordinate on X, given by

$$(z, \theta) \mapsto e^{i\theta} \cdot e_L^*(z)/\|e_L^*(z)\| = e^{i\theta - \frac{1}{2}\varphi(z)} e_L^*(z). \tag{27}$$

The Reeb vector field on X is given by $R = \partial_\theta$.

Naming convention For points in the base space M, we use names such as $z, w, \dots$; for points in the circle bundle X, we use $\hat{z}$, $\hat{w}, \dots$ to denote that $\pi(\hat{z}) = z$, $\pi(\hat{w}) = w, \dots$ under the projection $\pi\colon X \to M$. In general, objects upstairs in the circle bundle X with a corresponding object in M are equipped with a hat.

1.2 Holomorphic sections in L^k and CR–holomorphic functions on X

Since (L, h) is a positive Hermitian line bundle, X is a strictly pseudoconvex CR manifold. The *CR structure* on X is defined as follows: The kernel of α defines a horizontal hyperplane bundle $HX \subset TX$ which is invariant under J since $\ker\alpha = \ker d\rho \cap \ker d^c\rho$. Thus there is a splitting of the complexification of HX as $HX_{\mathbb{C}} = HX \otimes_{\mathbb{R}} \mathbb{C} = HX^{1,0} \oplus HX^{0,1}$ compatible with the splitting $TM_{\mathbb{C}} = TM^{1,0} \oplus TM^{0,1}$. We define the almost-CR $\bar\partial_b$ operator by $\bar\partial_b = d|_{H^{0,1}}$. More concretely, if $z_1, \dots, z_m$ are complex local coordinates on M, and $\bar\partial$ on M is given by $\bar\partial = \sum_{j=1}^m d\bar{z}_j \otimes \partial_{\bar{z}_j}$, then the $\bar\partial_b$ operator on X is given by $\bar\partial_b = \sum_{j=1}^m \pi^* d\bar{z}_j \otimes \partial^h_{\bar{z}_j}$, where $\partial^h_{\bar{z}_j}$ is the horizontal lift of $\partial_{\bar{z}_j}$ from $TM^{0,1}$ to $HX^{0,1}$. Similarly one can define ∂_b. A function $f\colon X \to \mathbb{C}$ is CR–holomorphic if $\bar\partial_b f = 0$.

A smooth section s_k of L^k determines a smooth function $\hat{s}_k$ on X by

$$\hat{s}_k(x) := \langle x^{\otimes k}, s_k \rangle, \quad x \in X \subset L^*.$$

Furthermore, $\hat{s}_k$ is of degree k under the canonical S^1 action r_θ on X, that is, $\hat{s}_k(r_\theta x) = e^{ik\theta}\hat{s}_k(x)$. We denote the space of smooth sections of degree k by $C^\infty(X)_k$. If s_k is holomorphic, then $\hat{s}_k$ is CR–holomorphic.

We equip X and M with volume forms

$$d\mathrm{Vol}_X = \frac{\alpha}{2\pi} \wedge \frac{(d\alpha)^m}{m!} \quad \text{and} \quad d\mathrm{Vol}_M = \frac{\omega^m}{m!}$$

so that the pushforward measure of X equals that of M. Then, given two smooth section s_1 and s_2 of L^k, we may define the inner product

$$\langle s_1, s_2 \rangle := \int_M h^k(s_1(z), s_2(z))\, d\mathrm{Vol}_M(z).$$

Similarly, given two smooth functions f_1 and f_2 on X, we may define

$$\langle f_1, f_2 \rangle := \int_X f_1(x)\overline{f_2(x)}\, d\mathrm{Vol}_X(x).$$

Let $L^2(M, L^k)$ and $L^2_k(X)$ be the Hilbert spaces of L^2–integral sections. Then sending s_k to $\hat{s}_k$ is an isomorphism of Hilbert spaces: $L^2(M, L^k) \xrightarrow{\sim} L^2_k(X)$. Moreover, if $\mathcal{H}^2_k(X) \subset L^2_k(X)$ is the subspace of CR–holomorphic functions, then the isomorphism restricts to an isomorphism between the holomorphic sections in L^k and the CR–holomorphic functions of degree k, that is, $H^0(M, L^k) \xrightarrow{\sim} \mathcal{H}^2_k(X)$.

1.3 Szegő projection kernel on X

On the circle bundle X over M, we define the orthogonal projection from $L^2(X)$ to the CR–holomorphic subspace $\mathcal{H}^2(X) = \widehat{\bigoplus}_{k\geq 0}\mathcal{H}^2_k(X)$ and the degree-k subspace $\mathcal{H}^2_k(X)$:

$$\widehat{\Pi}\colon L^2(X) \to \mathcal{H}^2(X), \quad \widehat{\Pi}_k\colon L^2(X) \to \mathcal{H}^2_k(X), \quad \widehat{\Pi} = \sum_{k\geq 0} \widehat{\Pi}_k.$$

The Schwarz kernel $\widehat{\Pi}_k(x, y)$ of $\widehat{\Pi}_k$ is called the degree-k Szegő kernel, ie

$$(\widehat{\Pi}_k F)(x) = \int_X \widehat{\Pi}_k(x, y) F(y)\, d\mathrm{Vol}_X(y) \quad \text{for all } F \in L^2(X).$$

If we have an orthonormal basis $\{\hat{s}_{k,j}\}_j$ of $\mathcal{H}^2_k(X)$, then

$$\widehat{\Pi}_k(x, y) = \sum_j \hat{s}_{k,j}(x)\overline{\hat{s}_{k,j}(y)}.$$

Similarly, one can define the full Szegő kernel $\widehat{\Pi}(x, y)$. The degree-k kernel can be extracted as the Fourier coefficient of $\widehat{\Pi}(x, y)$:

$$\widehat{\Pi}_k(x, y) = \frac{1}{2\pi}\int_0^{2\pi} \widehat{\Pi}(r_\theta x, y) e^{-ik\theta}\, d\theta.$$

We denote the value of the kernels on the diagonal as $\widehat{\Pi}(x) = \widehat{\Pi}(x, x)$ and $\widehat{\Pi}_k(x) = \widehat{\Pi}_k(x, x)$. Then the Bergman density on M can be obtained from the Szegő kernel as $\Pi_k(z) = \widehat{\Pi}_k(\hat{z})$.

1.4 Boutet de Monvel–Sjöstrand parametrix for the Szegő kernel

Near the diagonal in $X \times X$, there exists a parametrix due to Boutet de Monvel and Sjöstrand [16] for the Szegő kernel of the form

$$\widehat{\Pi}(x, y) = \int_{\mathbb{R}+} e^{\sigma\hat{\psi}(x,y)} s(x, y, \sigma)\, d\sigma + \widehat{R}(x, y), \tag{28}$$

where $\hat{\psi}(x, y)$ is the almost CR–analytic extension of $\hat{\psi}(x, x) = -\rho(x) = \log \|x\|^2$.

In local coordinates, let $x = e^{i\theta_x} e_L^*(z)/\|e_L^*(z)\|$ and $y = e^{i\theta_y} e_L^*(w)/\|e_L^*(w)\|$. Then

$$\widehat{\psi}(x, y) = i(\theta_x - \theta_y) + \psi(z, w) - \tfrac{1}{2}\varphi(z) - \tfrac{1}{2}\varphi(w),$$

where $\psi(z, w)$ is the almost analytic extension of $\varphi(z)$.

1.5 Bargmann–Fock Model

Here we consider the trivial line bundle L over $\mathbb{C}^m$, both as a first example to illustrate the various definitions and the normalization convention and as a local model for a general Kähler manifold.

We fix a nonvanishing holomorphic section e_L of L, and choose the Hermitian metric h on L so that $\varphi(z) = -\log \|e_L\|_h^2(z) = |z|^2$. The Kähler form ω is then

$$\omega = i\partial\overline{\partial}\varphi(z) = i\sum_j dz_j \wedge d\overline{z}_j.$$

The unit circle bundle X in the dual line bundle $L^* \cong \mathbb{C}^m \times \mathbb{C}$ is given by

$$X = \{(z, \lambda) \in \mathbb{C}^m \times \mathbb{C} : \|\lambda e_L^*\| = 1\} = \{(z, \lambda) \in \mathbb{C}^m \times \mathbb{C} : |\lambda| = e^{-\frac{1}{2}|z|^2}\}.$$

We may then choose a trivialization of $X \cong \mathbb{C}^m \times S^1$ with coordinate (z, θ):

$$(z, \theta) \mapsto e^{i\theta} e_L^*(z)/\|e_L^*(z)\| = e^{i\theta - \frac{1}{2}|z|^2} e_L^*(z) \in L^*.$$

The contact form α on X is then

$$\alpha = d\theta - \frac{i}{2}\sum_j (\overline{z}_j dz_j - z_j d\overline{z}_j).$$

If $s(z)$ is a holomorphic function (section of L^k) on $\mathbb{C}^m$, then its CR–holomorphic lift to X is

$$\widehat{s}(z, \theta) = e^{k\left(i\theta - \frac{1}{2}|z|^2\right)} s(z).$$

Indeed, the horizontal lift of $\partial_{\overline{z}_j}$ is $\partial_{\overline{z}_j}^h = \partial_{\overline{z}_j} - \frac{1}{2} i z_j \partial_\theta$, and $\partial_{\overline{z}_j}^h \widehat{s}(z, \theta) = 0$.

The Szegő kernel $\widehat{\Pi}(\widehat{z}, \widehat{w})$ for $X = \mathbb{C}^m \times S^1$ is given by

$$\begin{aligned}\widehat{\Pi}(\widehat{z}, \widehat{w}) &= \sum_{k>0} \left(\frac{k}{2\pi}\right)^m e^{ik(\theta_z - \theta_w) + k\left(z\overline{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2\right)} \\ &= \sum_{k>0} \left(\frac{k}{2\pi}\right)^m e^{ik(\theta_z - \theta_w + \operatorname{Im}(z\overline{w})) - \frac{1}{2}k|z-w|^2},\end{aligned}$$

where $\widehat{z} = (z, \theta_z)$, $\widehat{w} = (w, \theta_w)$, and where the k^{th} summand is $\widehat{\Pi}_k(\widehat{z}, \widehat{w})$.

The Bergman kernel $\Pi_k(z,w)$ for $M=\mathbb{C}^m$ is given by

$$\Pi_k(z,w)=\Big(\frac{k}{2\pi}\Big)^m e^{kz\bar{w}}.$$

The Bergman density is the norm contraction of $\Pi_k(z,w)$ on the diagonal

$$\Pi_k(z)=\Pi_k(z,z)\|e_L^{\otimes k}\|^2=\Big(\frac{k}{2\pi}\Big)^m e^{kz\bar{z}}e^{-k|z|^2}=\Big(\frac{k}{2\pi}\Big)^m.$$

In general, for an m–dimensional complex vector space (V,J) with a constant Kähler form ω, we may define a "ground state"

$$\Omega_{\omega,J}(v)=e^{-\frac{1}{2}\omega(v,Jv)},$$

as a real-valued function on V. The Bargmann–Fock Hilbert space for (V,ω,J) is

$$\mathcal{H}_{\omega,J}=\{\psi\Omega_{\omega,J}:\psi \text{ is } J\text{–holomorphic and } \psi\Omega_{\omega,J}\in L^2(V,dL)\},$$

where L is the standard Lebesgue measure on V.

1.6 Osculating Bargmann–Fock model and near-diagonal scaling asymptotics

At each $z\in M$ there is an osculating Bargmann–Fock or Heisenberg model, defined as above for the data (T_zM,J_z,ω_z). We denote the model Heisenberg Szegő kernel on the tangent space by

$$\widehat{\Pi}^{T_zM}_{\omega_z,J_z}\colon L^2(T_zM\times S^1)\to\mathcal{H}(T_zM,J_z,\omega_z)=\mathcal{H}_J. \tag{29}$$

If we choose linear coordinates $(z_1,\dots,z_m)$ on T_zM such that $(T_zM,\omega_z,J_z)\cong(\mathbb{C}^m,\omega_{\mathrm{std}},J_{\mathrm{std}})$, and the obvious coordinate θ on $S^1=\mathbb{R}/2\pi\mathbb{Z}$, then we have

$$\begin{aligned}\widehat{\Pi}^{T_zM}_{\omega_z,J_z}(u,\theta_1,v,\theta_2)&=(2\pi)^{-m}e^{i(\theta_1-\theta_2)}e^{u\cdot\bar{v}-\frac{1}{2}(|u|^2+|v|^2)}\\&=e^{i(\theta_1-\theta_2)}e^{i\,\mathrm{Im}\,u\cdot\bar{v}-\frac{1}{2}|u-v|^2}.\end{aligned}\tag{30}$$

The following near-diagonal asymptotics of the Szegő kernel is the key analytical result on which our analysis of the scaling limit for correlations of zeros is based. The lifted Szegő kernel is shown by Shiffman and Zelditch [23] and by Lu and Shiffman [12, Theorems 2.2–2.3] to have the scaling asymptotics. Here we only state the first version since it will be enough for our purpose.

Theorem 1.1 [23] *Let $(L,h)\to(M,\omega)$ be a positive line bundle over an m–dimensional compact Kähler manifold with Kähler form $\omega=iF_\nabla$. Let e_L be a* *holomorphic local frame for* L *and* $z_1, \dots, z_m$ *be complex coordinates about a point* $z_0 \in M$ *such that the Kähler potential satisfies* $\varphi = -\log \|e_L(z)\|_h^2 = |z|^2 + O(|z|^3)$*. Then, for some* $k \in \mathbb{Z}^+$*,*

$$\Pi_k\left(\frac{u}{\sqrt{k}}, \frac{\theta_1}{k}; \frac{v}{\sqrt{k}}, \frac{\theta_2}{k}\right) = k^m \widehat{\Pi}^{T_z M}_{\omega_z, J_z}(u, \theta_1; v, \theta_2)\left(1 + \sum_{r=1}^{N} k^{-\frac{1}{2}r} b_r(u,v) + k^{-\frac{1}{2}(N+1)} E_{kN}\right),$$

where

- *each* $b_r(u,v)$ *is a polynomial* (in $u, v, \bar{u}, \bar{v}$) *of degree at most* $5r$*;*
- *for all* $a > 0$ *and* $j \geq 0$*, there exists a positive constant* C_{jka} *such that*

$$|D^j E_{kN}(u,v)| \leq C_{jNa} \quad \text{for } |u| + |v| < a.$$

2 Proof of Theorem 1

This is the simplest of the results because it does not involve the dynamics of the Hamiltonian flow. That is, it only involves the unitary group of "pseudodifferential" Toeplitz operators (16) and not the group $U_k(t)$ of Fourier integral Toeplitz operators.

We first prove a smoothed version of Theorem 1. We recall from (19) that a smoothly weighted Bergman density is defined by

$$(31)\quad \Pi_{k,f}(z) = \Pi_k f(H_k) \Pi_k(z) = \sum_{\mu_{k,j}} f(\mu_{k,j}) \Pi_{k,j}(z) = \sum_{\mu_{k,j}} f(\mu_{k,j}) \|s_{k,j}(z)\|^2_{h^k},$$

where $f \in \mathcal{S}(\mathbb{R})$ and the sum is over the eigenvalues $\mu_{k,j}$ of the operator H_k defined in (3). Note that since H_k commutes with Π_k and $\Pi_k \circ \Pi_k = \Pi_k$, we could have written $f(H_k)\Pi_k$ instead of $\Pi_k f(H_k) \Pi_k$. The sharp partial Bergman kernels $\Pi_{k,E}$ morally corresponds to the case $f = \mathbf{1}_{(E_{\min}, E)}$, where $E_{\min} = \min_{z \in M}\{H(z)\}$.

Proposition 2.1 *Let* ω *be a* C^∞ *metric on* M *and let* $H \in C^\infty(M)$*. Let* $f \in C_c^\infty(\mathbb{R})$*. Then the density of states of the smoothly weighted Bergman kernel is given by the asymptotic formulae*

$$\Pi_{k,f}(z) \simeq \Pi_k(z)(f(H(z)) + c_{1,f}(z)k^{-1} + c_{2,f}(z)k^{-2} + \cdots),$$

where the $c_{i,f}(z)$ *depend on* f *up to its* i^{th} *derivative at* $H(z)$*.*

We give two proofs for the above proposition, one using the Helffer–Sjöstrand formula and the fact $\Pi_k(\lambda - H_k)^{-1}\Pi_k$ is a Toeplitz operator, the other using the Fourier transformation of f and the fact that $\Pi_k e^{itH_k}\Pi_k$ is a Toeplitz operator.

2.1 Proof of Proposition 2.1 using the Helffer–Sjöstrand formula

Proof We use the Helffer–Sjöstrand formula [7]. Let $f \in C_c^\infty(\mathbb{R})$ and let $\widetilde{f}(\lambda) \in C_c^\infty(\mathbb{C})$ be an almost analytic extension of f to $\mathbb{C}$, satisfying $\overline{\partial}\widetilde{f} = 0$ to infinite order on $\mathbb{R}$. Then

$$f(H_k) = -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(\lambda)(\lambda - H_k)^{-1}\, dL(\lambda),$$

where dL is the Lebesgue measure on $\mathbb{C}$. We recall that the almost analytic extension is defined as follows: Let $\psi(x) = 1$ on $\operatorname{supp} f$ and let χ be a standard cutoff function. Then, define

$$\widetilde{f}(x+iy) = \frac{\psi(x)}{2\pi}\int_{\mathbb{R}} e^{i(x+iy)\xi}\chi(y\xi)\hat{f}(\xi)\, d\xi. \tag{32}$$

It is verified in [7, page 94] that this defines an almost analytic extension. This formula has previously been adapted to Toeplitz operators in [4].

If $\operatorname{Im}\lambda \neq 0$, then there exists a unique semiclassical symbol $\sum_{j=0}^\infty k^{-j} b_j(w;\lambda)$ with $b_0 = (\lambda - H(w))^{-1}$ and a Toeplitz operator $\hat{B}_k(\lambda)$ as an approximation for $\Pi_k(\lambda - H_k)^{-1}\Pi_k$ such that

$$\hat{B}_k(\lambda)\circ(\lambda - H_k) = \Pi_k + R_{B,k}(\lambda), \quad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

where $R_{B,k}(\lambda)$ is a residual Toeplitz operator (ie of order $k^{-\infty}$). Thus

$$\Pi_{k,f} := \Pi_k f(H_k)\Pi_k \approx -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(\lambda)\hat{B}_k(\lambda)\, dL(\lambda)$$

is a Toeplitz operator with complete symbol

$$\sigma_{k,f}(w) \sim -\sum_{j=0}^\infty \frac{k^{-j}}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(\lambda) b_j(w;\lambda)\, dL(\lambda),$$

and principal symbol

$$\sigma_f^{\mathrm{prin}}(w) = -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(\lambda)(\lambda - H(w))^{-1}\, dL(\lambda) = f(H(w)).$$

We can then express the density for $\Pi_{k,f}$ as

$$\Pi_{k,f}(z) = \int_X \hat{\Pi}_k(\hat{z},\hat{w})\sigma_{k,f}(w)\hat{\Pi}_k(\hat{w},\hat{z})\, d\mathrm{Vol}_X(w)$$

and use the Boutet de Monvel–Sjöstrand parametrix to compute the resulting expansion stated in Proposition 2.1.

Finally, we show how the coefficients in the expansion $c_{i,f}(w)$ depend on the j^{th} derivative of f at $H(w)$ for $j \le i$. For this it suffices to prove the same statements for the terms

$$\int_{\mathbb{C}} \bar\partial \tilde f(\lambda) b_j(w;\lambda)\, dL(\lambda)$$

for $j \ge 1$. As discussed in [7], the lower-order terms $b_j(w;\lambda)$ of $\widehat{B}_k$ have the form

$$b_j(w;\lambda) = P_j(w;\lambda)(\lambda - H(w))^{-j-1},$$

where P_j is polynomial in λ. Writing

$$(\lambda - H(w))^{-j-1} = \frac{1}{j!}\frac{\partial^j}{\partial\lambda^j}(\lambda - H(w))^{-1}$$

and integrating by parts gives the value as a sum of holomorphic derivatives $\bar\partial(\partial^\ell \tilde f(\lambda))$ evaluated at $\lambda = H(w)$ of degree $\ell \le j$. This shows $c_{i,f}(w)$ depends on f through $f^{(0)}(H(z)), \dots, f^{(i)}(H(z))$. □

We get immediately the following corollary:

Corollary 2.2 *If f vanishes in an open neighborhood of $H(z)$, then*

$$\Pi_{k,f}(z) = O(k^{-\infty}).$$

2.2 Proof of Proposition 2.1 using Fourier transformation

Since we use the Fourier inversion formula in other applications of the functional calculus, we digress to describe an alternative to the Helffer–Sjöstrand formula above.

We may construct $W_k(t) := \Pi_k e^{itH_k}\Pi_k$ as a Toeplitz operator of the form

$$\Pi_k u_k(w;t)\Pi_k, \quad u_k(w;t) = e^{itH(w)} + \sum_{j=1}^{\infty} k^{-j} u_j(w;t), \tag{33}$$

where $u_k(w;t)$ denotes multiplication by the function $u_k(w;t)$, where $w \in M$ and $t \in \mathbb{R}$. For instance, we can write the equation for $W_k(t)$ as the unique solution of the propagator initial value problem

$$\Big(\frac{1}{i}\frac{d}{dt} - H_k\Big) W_k(t) = 0, \quad \text{with } W_k(0) = \Pi_k. \tag{34}$$

There exists a semiclassical symbol

$$\sigma_{k,f}(w) = f(H(w)) + k^{-1} c_{f,1}(z) + \cdots$$

such that

$$\Pi_k f(H_k) \Pi_k = \Pi_k \sigma_{k,f}(w) \Pi_k. \tag{35}$$

Indeed,

$$\begin{aligned} \Pi_{k,f} &= \Pi_k f(H_k) \Pi_k \\ &= \int_{\mathbb{R}} \hat{f}(t) \Pi_k e^{it\hat{H}_k} \Pi_k(z,z) \frac{dt}{2\pi} = \Pi_k \left(\int_{\mathbb{R}} \hat{f}(t) u_k(w;t) \frac{dt}{2\pi} \right) \Pi_k. \end{aligned}$$

Hence, the symbol for $\Pi_{k,f}$ is then $\sigma_{k,f}(w) = \int_{\mathbb{R}} \hat{f}(t) u_k(w;t) \frac{dt}{2\pi}$, with the principal symbol $f(H)$.

2.3 Proof of Theorem 1

We first prove the result (12) in Theorem 1 about $\Pi_k(z)^{-1} \Pi_{k,E}(z)$. It suffices to prove the result in the case where z lies in the forbidden region $H(z) > E$. Indeed, since for $H(z) < E$, we may use $-H$ and $-E$ as the energy function and the threshold, write

$$\Pi_k(z)^{-1} \Pi_{k,H<E}(z) = 1 - \Pi_k(z)^{-1} \Pi_{k,-H<-E}(z),$$

and use $\Pi_{k,-H<-E}(z) = O(k^{-\infty})$ from the result in the forbidden region. Hence from now on, we assume $H(z) > E$.

Let ϵ be a positive number small enough that $H(z) > E + \epsilon$. It suffices to assume $f \in C_0^\infty(\mathbb{R})$, $f \geq 0$, with $f \equiv 1$ on the interval $(E_{\min}, E)$ and $f \equiv 0$ outside of $(E_{\min} - \epsilon, E + \epsilon)$, since the general result can be obtained by taking the difference of two such functions. Then, $f(x) > \mathbf{1}_{[E_{\min}, E]}(x)$, and

$$\Pi_{k,f}(z) = \sum_j f(\mu_{k,j}) \|s_{k,j}(z)\|^2 \geq \sum_{\mu_{k,j} < E} \|s_{k,j}(z)\|^2 = \Pi_{k,E}(z).$$

Since we have $f(H(w)) \equiv 0$ for w in an open neighborhood of z, by Corollary 2.2, $\Pi_{k,f}(z) = O(k^{-\infty})$. Since $\Pi_{k,E}(z) > 0$, we have

$$|\Pi_{k,E}(z)| = \Pi_{k,E}(z) < \Pi_{k,f}(z) = O(k^{-\infty}).$$

To prove the statement in (11), we split the sum in $\Pi_{k,f}(z)$ according to the eigenvalue:

$$\Pi_{k,f}(z) = \Pi_{k,f,<E}(z) + \Pi_{k,f,>E}(z),$$

where

$$\Pi_{k,f,<E}(z) := \sum_{\mu_{k,j}<E} f(\mu_{k,j})\|s_{k,j}(z)\|^2,$$
$$\Pi_{k,f,>E}(z) := \sum_{\mu_{k,j}>E} f(\mu_{k,j})\|s_{k,j}(z)\|^2.$$

For $z \in \mathcal{F}$ such that $H(z) > E$, we need to prove that

$$\lim_{k\to\infty} k^{-m}\Pi_{k,f,<E}(z) = 0.$$

We may define a nonnegative smooth function χ on $\mathbb{R}$ that is 1 for $x < E$, and 0 for $x > H(z) > E$. Then

$$0 < \Pi_{k,f,<E}(z) < \Pi_{k,f\chi}(z) = O(k^{-\infty}),$$

where we used Corollary 2.2 and that $f\chi(w)$ vanishes identically for w in a neighborhood of $H(z)$. Similarly, if $H(z) < E$, then

$$\lim_{k\to\infty} k^{-m}\Pi_{k,f,>E}(z) = 0,$$

and we get

$$\lim_{k\to\infty} k^{-m}\Pi_{k,f,<E}(z) = \lim_{k\to\infty} k^{-m}\Pi_{k,f}(z) = f(H(z)).$$

This finishes the proof of the statement of Theorem 1 regarding the leading-order asymptotic behavior of $\Pi_{k,f}(z)$.

3 Hamiltonian flow and its contact lifts to X_h

In order to prove the main results, we need to quantize the Hamiltonian flow of a Hamiltonian $H: M \to \mathbb{R}$. The definition is based on lifting the Hamiltonian flow to a contact flow on X_h. The purpose of this section is to explain this lift.

3.1 Lifting the Hamiltonian flow to a contact flow on X_h

Let H be a Hamiltonian function on (M,ω). Let ξ_H be the Hamiltonian vector field associated to H, that is,

$$dH(Y) = \omega(\xi_H, Y)$$

for all vector fields Y on M. Let g^t be the flow generated by ξ_H.

Recall (X,α) is a contact manifold, and $X \to M$ is a circle bundle, such that $d\alpha = \pi^*\omega$. We abuse notation and still use H for the lifted Hamiltonian function $\pi^* H$ on X. The horizontal lift ξ_H^h of ξ_H is given by the conditions

$$\alpha(\xi_H^h) = 0 \quad \text{and} \quad \pi_* \xi_H^h = \xi_H.$$

The contact lift $\hat{\xi}_H$ of ξ_H is then defined by

$$\hat{\xi}_H := \xi_H^h - HR. \tag{36}$$

We define $\hat{g}^t\colon X_h \to X_h$ to be the flow generated by $\hat{\xi}_H$:

$$\hat{g}^t = \exp t\hat{\xi}_H. \tag{37}$$

Lemma 3.1 *The flow $\hat{g}^t$ preserves the contact form α and commutes with the S^1 action of rotation in the fibers of $X \to M$.*

Proof Since $d\alpha = \omega$, we have

$$\begin{aligned}\mathcal{L}_{\hat{\xi}_H}\alpha &= \mathcal{L}_{\xi_H^h - HR}\alpha \\ &= (\iota_{\xi_H^h - HR} \circ d + d \circ \iota_{\xi_H^h - HR})\alpha = \iota_{\xi_H^h}\pi^*\omega + d(-H\alpha(R)) = H - H = 0.\end{aligned}$$

Since $\hat{\xi}_H$ preserves α, it follows that $\mathcal{L}_{\hat{\xi}_H}(R) = 0$, ie $[\hat{\xi}_H, \partial_\theta] = 0$, and the flow $\hat{g}^t$ commutes with the fiberwise rotation. □

Let (z,θ) be local coordinates for $X|_U \cong U \times S^1$ over an open neighborhood U, as in (27).

Lemma 3.2 *In the coordinates (z,θ), we can write ξ_H^h and $\hat{\xi}_H$ as*

$$\xi_H^h = \big(\xi_H, \tfrac{1}{2}\langle d^c\varphi, \xi_H\rangle\partial_\theta\big) \quad \text{and} \quad \hat{\xi}_H = \big(\xi_H, \big(\tfrac{1}{2}\langle d^c\varphi, \xi_H\rangle - H\big)\partial_\theta\big).$$

And the flow $\hat{g}^t$ has the form

$$\begin{aligned}\hat{g}^t(z,\theta) &= \left(g^t(z), \theta + \int_0^t \tfrac{1}{2}\langle d^c\varphi, \xi_H\rangle(g^s(z))\,ds - \int_0^t H(g^s(z))\,ds\right) \\ &= \left(g^t(z), \theta + \int_0^t \tfrac{1}{2}\langle d^c\varphi, \xi_H\rangle(g^s(z))\,ds - tH(z)\right).\end{aligned}$$

Proof Since $\alpha = d\theta - \frac{1}{2}\pi^* d^c\varphi$, and $\langle \alpha, \xi_H^h\rangle = 0$, we have then

$$\langle d\theta, \xi_H^h\rangle = \big\langle \tfrac{1}{2}\pi^* d^c\varphi, \xi_H^h\big\rangle = \big\langle \tfrac{1}{2} d^c\varphi, \xi_H\big\rangle.$$

Hence $\xi_H^h = \left(\xi_H, \frac{1}{2}\, d^c\varphi(\xi_H)\partial_\theta\right)$. The formula for $\widehat{\xi}_H$ follows from (36). The statement for $\widehat{g}^t$ follows since H is constant along the Hamiltonian flow. □

Since $\widehat{g}^t$ preserves α it preserves the horizontal distribution $H(X_h) = \ker\alpha$, ie

$$D\widehat{g}^t\colon H(X)_x \to H(X)_{\widehat{g}^t(x)}. \tag{38}$$

It also preserves the vertical (fiber) direction and therefore preserves the splitting $V \oplus H$ of TX. When g^t is nonholomorphic, $\widehat{g}^t$ is not CR–holomorphic, ie does not preserve the horizontal complex structure J on $H(X)$ or the splitting of $H(X)\otimes\mathbb{C}$ into its $\pm i$ eigenspaces.

3.2 Linear Hamiltonian function and Heisenberg group action

Let $\mathbb{C}^m$ be equipped with coordinates $z_j = x_j + iy_j$ with $j = 1,\dots,m$. We use x_i, y_i as coordinates on $\mathbb{R}^{2m}$. Let L be the trivial bundle over $\mathbb{C}^m$, and $\varphi(z) = |z|^2$.

A linear Hamiltonian function H on $\mathbb{C}^m$ has the form

$$H(x,y) = \mathrm{Re}(\alpha\cdot\bar{z}) = \tfrac{1}{2}(\alpha\bar{z} + \bar{\alpha}z) \tag{39}$$

for some $0 \neq \alpha \in \mathbb{C}^m$. Then

$$\begin{aligned}
\xi_H &= \sum_j \frac{1}{2i}(\alpha_j\partial_{z_j} - \bar{\alpha}_j\partial_{\bar{z}_j}),\\
\xi_H^h &= \sum_j \frac{1}{2i}\Big(\alpha_j\Big(\partial_{z_j} + \frac{i}{2}\bar{z}_j\partial_\theta\Big) - \bar{\alpha}_j\Big(\partial_{\bar{z}_j} - \frac{i}{2}z_j\partial_\theta\Big)\Big) = \xi_H + \tfrac{1}{2}H\partial_\theta,\\
\widehat{\xi}_H &= \xi_H^h - H\partial_\theta = \xi_H - \tfrac{1}{2}H\partial_\theta,
\end{aligned}$$

where we abuse notation and write $(\xi_H, 0)$ on $U\times S^1$ as ξ_H.

The lifted Hamiltonian flow $\widehat{g}^t(z) = \exp(t\widehat{\xi}_H)$ is then

$$\widehat{g}^t(z,\theta) = \Big(z + \frac{\alpha t}{2i}, \theta - \frac{t}{4}(\alpha\bar{z} + \bar{\alpha}z)\Big). \tag{40}$$

The linear Hamiltonian function generates translations on $\mathbb{C}^m\times S^1$, which is exactly the action of the reduced Heisenberg group, which we now review. The simply connected Heisenberg group [9; 24] of dimension $2m+1$ is $\mathbb{H}^m \cong \mathbb{R}^{2m}\times\mathbb{R} = \{(x,y,t) : x,y\in\mathbb{R}^n,\ t\in\mathbb{R}\}$, with a multiplication law

$$(x,y,t)(x',y',t') = (x+x', y+y', t+t' - (xy' - yx')). \tag{41}$$

The reduced Heisenberg group is $\mathbb{H}^m_{\mathrm{red}} = \mathbb{R}^{2m} \times (\mathbb{R}/2\pi\mathbb{Z})$. The center of $\mathbb{H}^m_{\mathrm{red}}$ is the circle subgroup $S^1 \cong \{(0,0,\theta)\}$. If we identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ by $z_i = x_i + \sqrt{-1}y_i$, and $\mathbb{R}/2\pi\mathbb{Z}$ with the argument of $\mathbb{C}$, then we may make the identification $\mathbb{H}^m_{\mathrm{red}} \cong \mathbb{C}^m \times S^1$. And the group action is given by

$$(z,\theta) \circ (z',\theta') = (z + z', \theta + \theta' + \operatorname{Im}(z\overline{z}')).$$

Furthermore, $\mathbb{H}^m_{\mathrm{red}}$ can be identified with the unit cocircle bundle X of the dual bundle L^* of the positive Hermitian line bundle over $\mathbb{C}^m$ with Kähler potential $\varphi(z) = |z|^2$.

Lemma 3.3 *The contact form $\alpha = d\theta + \frac{1}{2}i\sum_j (z_j\,d\overline{z}_j - \overline{z}_j\,dz_j)$ on $\mathbb{H}^m_{\mathrm{red}}$ is invariant under the left multiplication*

$$R_{(z_0,\theta_0)}\colon (z,\theta) \mapsto (z_0,\theta_0)\circ(z,\theta) = \left(z + z_0, \theta + \theta_0 + \frac{z_0\overline{z} - \overline{z}_0 z}{2i}\right).$$

Proof We have

$$\begin{aligned} R^*_{(z_0,\theta_0)}\alpha|_{(z,\theta)} &= d\left(\theta + \theta_0 + \frac{\overline{z}z_0 - \overline{z}_0 z}{2i}\right) + \frac{i}{2}\sum_j ((z_j + z_{0j})\,d\overline{z}_j - (\overline{z}_j + \overline{z}_{0j})\,dz_j) \\ &= \alpha|_{(z,\theta)}. \end{aligned}$$

□

Lemma 3.4 *The lifted contact flow $\hat{g}^t$ generated by $H = \frac{1}{2}(\alpha\overline{z} + \overline{\alpha}z)$ acts on $\mathbb{H}^m_{\mathrm{red}}$ by left group multiplication by $\left(\frac{1}{2}\alpha t/i, 0\right) \in \mathbb{H}^m_{\mathrm{red}}$.*

Proof We have

$$\left(\frac{\alpha t}{2i}, 0\right)\circ(z,\theta) = \left(z + \frac{\alpha t}{2i}, \theta + \operatorname{Im}\left(\frac{\alpha t}{2i}\overline{z}\right)\right) = \left(z + \frac{\alpha t}{2i}, \theta - \frac{1}{2}\operatorname{Re}(\alpha\overline{z})t\right) = \hat{g}^t(z,\theta).$$ □

The Hardy space of square-integrable CR–holomorphic functions $\mathcal{H}$ on X is preserved under the reduced Heisenberg group $\mathbb{H}^m_{\mathrm{red}}$, and decomposes into Fourier components according to the action by the central subgroup S^1:

$$\mathcal{H} = \bigoplus_{n\in\mathbb{Z}, n>0} \mathcal{H}_n,$$

$$\mathcal{H}_n = \left\{e^{in(\theta - \frac{1}{2}|z|^2)} f(z) : f(z) \text{ holomorphic and } \textstyle\int_{\mathbb{C}^m} |f|^2 e^{-n|z|^2}\, dL < \infty\right\},$$

where dL is the Lebesgue measure on $\mathbb{C}^m$ and the integrability condition forces $n > 0$.

In the case of the simply connected $(2m+1)$–dimensional Heisenberg group $\mathbb{H}^m$, the Szegö projector $S(x,y)$ is given by convolution with

$$K(x) = C_m \frac{\partial}{\partial t}\big(t+\tfrac{1}{2}i|\zeta|^2\big)^{-m} = C_m \int_0^\infty e^{r(it-\frac{1}{2}|z|^2)} r^m \, dr, \quad x=(z,t), \tag{42}$$

for some constant C_m depending on the dimension m. More precisely, if $x=(z,t)$ and $y=(w,s)$, then the projector is given by

$$(Sf)(x) = \int S(x,y) f(y)\,dy = \int K(y^{-1}x) f(y)\,dy$$

and

$$\begin{aligned} S(x,y) &= K(y^{-1}x) = K(z-w, t-s-\operatorname{Im}(w\bar{z})) \\ &= C_m \int_0^\infty e^{ir(t-s)} e^{r(z\bar{w}-\frac{1}{2}|z|^2-\frac{1}{2}|w|^2)} r^m \, dr. \end{aligned} \tag{43}$$

In the reduced Heisenberg group,

$$\begin{aligned} S_{\mathrm{red}}(x,y) &= \sum_{n\in\mathbb{Z}} K(y^{-1}x(0,2\pi n)) \\ &= \sum_{n\in\mathbb{Z}} C_m \int_0^\infty e^{ir(t-s)} e^{r(z\bar{w}-\frac{1}{2}|z|^2-\frac{1}{2}|w|^2)} r^m \, dr. \end{aligned} \tag{44}$$

Using the Poisson summation formula $\sum_{m\in\mathbb{Z}} e^{2\pi ikx} = \sum_{n\in\mathbb{Z}} \delta(x-n)$, we have

$$S_{\mathrm{red}}(x,y) = C_m \sum_{k\in\mathbb{Z}_{>0}} k^m e^{ik(t-s)} e^{k(z\bar{w}-\frac{1}{2}|z|^2-\frac{1}{2}|w|^2)}, \quad x=(z,t) \text{ and } y=(w,s).$$

If we denote the Fourier components relative to the S^1 action of $\mathbb{H}^m_{\mathrm{red}}$ by $\widehat{\Pi}^{\mathbb{H}}_k(x,y)$, then we have

$$\widehat{\Pi}^{\mathbb{H}}_k(x,y) = C_m k^m e^{ik(t-s)} e^{k(z\cdot\bar{w}-\frac{1}{2}|z|^2-\frac{1}{2}|w|^2)}, \tag{45}$$

and $S_{\mathrm{red}}(x,y) = \sum_{k>0} \widehat{\Pi}^{\mathbb{H}}_k(x,y)$.

4 Toeplitz quantization of contact transformation

Let (M,ω,L,h) be a polarized Kähler manifold, and $\pi\colon X\to M$ the unit circle bundle in the dual bundle (L^*,h^*). X is a contact manifold, and can be equipped with a contact one-form α, whose associated Reeb flow R is the rotation ∂_θ in the fiber direction of X. Any Hamiltonian vector field ξ_H on M generated by a smooth function $H\colon M\to R$ can be lifted to a contact Hamiltonian vector field $\hat{\xi}_H$ on X.

In this section, we review the relevant results about the quantization of this contact vector field $\hat{\xi}_H$, acting on holomorphic sections of $H^0(M, L^k)$, or equivalently on CR–holomorphic functions on X, that is, $\mathcal{H}(X) = \bigoplus_{k\geq 0} \mathcal{H}_k(X)$. We follow the exposition of [22; 15] closely.

An operator $T\colon C^\infty(X) \to C^\infty(X)$ is called a *Toeplitz operator of order* k, denoted as $T \in \mathcal{T}^k$, if it can be written as $T = \hat{\Pi} \circ Q \circ \hat{\Pi}$, where Q is a pseudodifferential operator on X. Its principal symbol $\sigma(T)$ is the restriction of the principal symbol of Q to the symplectic cone

$$\Sigma = \{(x, r\alpha(x)) : r > 0\} \cong X \times \mathbb{R}_+ \subset T^*X.$$

The symbol satisfies the properties

$$\begin{cases} \sigma(T_1T_2) = \sigma(T_1)\sigma(T_2); \\ \sigma([T_1, T_2]) = \{\sigma(T_1), \sigma(T_2)\}; \\ \text{if } T \in \mathcal{T}^k \text{ and } \sigma(T) = 0, \text{ then } T \in \mathcal{T}^{k-1}. \end{cases}$$

The choice of the pseudodifferential operator Q in $T = \hat{\Pi} \circ Q \circ \hat{\Pi}$ is not unique. However, there exists some particularly nice choices.

Lemma 4.1 [15, Proposition 2.13] *Let T be a Toeplitz operator on Σ of order p. Then there exists a pseudodifferential operator Q of order p on X such that $[Q, \hat{\Pi}] = 0$ and $T = \hat{\Pi} \circ Q \circ \hat{\Pi}$.*

Now we specialize to the setup here, following closely [21]. Consider an order-one self-adjoint Toeplitz operator

$$T = \hat{\Pi} \circ (H \cdot \boldsymbol{D}) \circ \hat{\Pi},$$

where $\boldsymbol{D} = (-i\partial_\theta)$ and ∂_θ is the fiberwise rotation vector field on X, and H is multiplication by $\pi^{-1}H$. We note that $\boldsymbol{D}$ decomposes $L^2(X)$ into eigenspaces with eigenvalues $k \in \mathbb{Z}$, that is, $L^2(X) = \bigoplus_{k\in\mathbb{Z}} L^2(X)_k$. The symbol of T is a function on $\Sigma \cong X \times \mathbb{R}_+$, given by

$$\sigma(T)(x, r) = (\sigma(H)\sigma(\boldsymbol{D})|_\Sigma)(x, r) = H(x)r \quad \text{for all } (x, r) \in \Sigma.$$

Definition 4.2 [21, Definition 5.1] Let $\hat{U}(t)$ denote the one-parameter subgroup of unitary operators on $L^2(X)$ given by

$$\hat{U}(t) = \hat{\Pi} e^{it\hat{\Pi}(\boldsymbol{D}H)\hat{\Pi}} \hat{\Pi}, \tag{46}$$

and let $\widehat{U}_k(t)$ (21) denote the Fourier component acting on $L^2(X)_k$:

$$\widehat{U}_k(t) = \widehat{\Pi}_k \, e^{it\widehat{\Pi}(kH)\widehat{\Pi}} \, \widehat{\Pi}_k. \tag{47}$$

We use $U_k(t)$ to denote the corresponding operator on $H^0(M, L^k)$.

Proposition 4.3 [21, Proposition 5.2] *$\widehat{U}(t)$ is a group of Toeplitz Fourier integral operators on $L^2(X)$, whose underlying canonical relation is the graph of the time t Hamiltonian flow of rH on the symplectic cone Σ of the contact manifold (X, α).*

We warn the reader that $\widehat{\Pi} e^{it\widehat{\Pi}(\boldsymbol{D}H)\widehat{\Pi}} \widehat{\Pi}$ in general is not equal to $\widehat{\Pi} e^{it\boldsymbol{D}H} \widehat{\Pi}$, since $\boldsymbol{D}H$ and $\widehat{\Pi}$ may not commute. However, thanks to Lemma 4.1, one can always find a Q replacing $\boldsymbol{D}H$ such that $[Q, \widehat{\Pi}] = 0$ and $\widehat{\Pi}(\boldsymbol{D}H)\widehat{\Pi} = \widehat{\Pi} Q \widehat{\Pi} = Q\widehat{\Pi} = \widehat{\Pi} Q$. Thus

$$\widehat{U}(t_1) \circ \widehat{U}(t_2) = \widehat{\Pi} e^{it_1\widehat{\Pi} Q \widehat{\Pi}} \widehat{\Pi} \widehat{\Pi} e^{it_2\widehat{\Pi} Q \widehat{\Pi}} \widehat{\Pi} = \widehat{\Pi} e^{i(t_1+t_2)Q} \widehat{\Pi} = \widehat{U}(t_1 + t_2),$$

and $\widehat{U}(t)$ indeed forms a group.

Remark 4.4 In the introduction, we defined a semiclassical (ie depending on k) Toeplitz operator H_k acting downstairs on $L^2(M, L^k)$, or equivalently an operator $\widehat{H}_k$ acting on $L^2(X)$ by

$$\widehat{H}_k := \widehat{\Pi}_k \circ \Big[\frac{i}{k}\xi_H^h + H\Big] \circ \widehat{\Pi}_k.$$

The collections $\{\widehat{H}_k\}$ assemble into a homogeneous degree-one Toeplitz operator $\widehat{H}$ acting on $L^2(X)$:

$$\widehat{H} = \bigoplus_{k \geq 0} k\widehat{H}_k = \widehat{\Pi} \circ [i\xi_H^h + (-i\partial_\theta)H] \circ \widehat{\Pi}\colon L^2(X) \to \mathcal{H}(X).$$

Compared with $T = \widehat{\Pi} \circ [(-i\partial_\theta)H] \circ \widehat{\Pi}$, we claim that they have the same principal symbol on Σ. Indeed,

$$\sigma(\widehat{H}) - \sigma(T) = \sigma(i\xi_H^h)|_\Sigma = \langle p, -\xi_H^h(x)\rangle|_{p=r\alpha(x)} = 0,$$

since $\langle \alpha, \xi_H^h\rangle = 0$.

When we say a one-parameter family of unitary operators $U(t)$ on $\mathcal{H}(X)$ *quantizes the Hamiltonian flow* $\exp(t\xi_H)$ on (M, ω), we mean the real points of the canonical relation of the complex Fourier integral operator $U(t)$ in $T^*X \times T^*X$ is the graph $\{(x, \Psi_t(x)) : x \in \Sigma\}$ of the Hamiltonian flow Ψ_t generated by rH on the symplectic cone (Σ, ω_Σ). The quantization is not unique. Indeed, if A is a pseudodifferential operator of degree zero on X, and $V = e^{iA}$ is unitary pseudodifferential operator, then $V^* U(t) V$ is another quantization with the same principal symbol.

Proposition 4.5 *There exists a semiclassical symbol $\sigma_k(t)$ such that the unitary group* (47) *has the form*

$$\hat{U}_k(t) = \hat{\Pi}_k (\hat{g}^{-t})^* \sigma_k(t) \hat{\Pi}_k \tag{48}$$

modulo smooth kernels of order $k^{-\infty}$.

Proof It follows from the theory of Toeplitz Fourier integral operators and Fourier integral operators with complex phase [15; 14] that (47) is a unitary group of semiclassical Fourier integral operators with complex phase associated to the graph of the lifted Hamiltonian flow of H to X. On the other hand, the operator on the right-hand side of (48) is manifestly such a Fourier integral operator. To prove that they are equal modulo smoothing operators of order $k^{-\infty}$ it suffices to construct $\sigma_k(t)$ so that they have the same complete symbol.

In [27] the principal symbol $\sigma_k^{\text{prin}}(t)$ of (47) was calculated, and by using this as the principal symbol of (48) one has that

$$\hat{U}_k(t) - \hat{\Pi}_k (\hat{g}^{-t})^* \sigma_k^{\text{prin}}(t) \hat{\Pi}_k \in I_{\text{sc}}^{-1}(\mathbb{R} \times X \times X, \mathcal{C} \circ \hat{\Gamma} \circ \mathcal{C}). \tag{49}$$

where $I^{-1}(\cdots)$ is the class of semiclassical Fourier integrals of order -1 with complex phase associated to the canonical relation $\mathcal{C} \circ \hat{\Gamma} \circ \mathcal{C}$; see [10, Chapter 25] and [22]. By the same method as in the homogeneous case, one may then construct the Toeplitz symbol $\sigma_k(t)$ to all orders so that the complete symbols of $\hat{U}_k(t)$ and (48) agree to all orders in k. □

It follows from the above proposition and the Boutet de Monvel–Sjöstrand parametrix construction that $\hat{U}_k(t,x,x)$ admits an oscillatory integral representation of the form

$$\begin{aligned} \hat{U}_k(t,x,x) &\simeq \Big(\frac{k}{2\pi}\Big)^{2m} \int_X \int_0^\infty \int_0^\infty \int_{S^1} \int_{S^1} \Big(e^{\sigma_1 \hat{\psi}(r_{\theta_1} x, \hat{g}^t y) + \sigma_2 \hat{\psi}(r_{\theta_2} y, x) - ik\theta_1 - ik\theta_2} \\ &\qquad \times A_k(t,y,\theta_i,\sigma_i)\Big)\, d\theta_1\, d\theta_2\, d\sigma_1\, d\sigma_2\, dy \\ &\simeq \Big(\frac{k}{2\pi}\Big)^{2m} \int_X e^{k\hat{\psi}(x,\hat{g}^t y) + k\hat{\psi}(y,x)} A_k(t,y)\, dy, \end{aligned} \tag{50}$$

where A_k is a semiclassical symbol, the integral for θ_1, θ_2 is to extract the Fourier component, and the integral for y comes from operator composition. The asymptotic symbol $\simeq$ means that the difference of the two sides is rapidly decaying in k.

5 Short-time propagator $\widehat{U}_k(t,x,x)$ and proof of Theorem 3

In this section, we give an explicit expression for the short time propagator $\widehat{U}_k(t)$, by applying the stationary phase method to the integral (50). As a warm-up, we illustrate the quantization of Hamiltonian flows in the model case of quantizations of linear Hamiltonian functions (constant flow) on Bargmann–Fock space. In effect, this is the construction of the Bargmann–Fock representation of the Heisenberg group in terms of Toeplitz quantization. Quantizations of linear Hamiltonians on $\mathbb{C}^m$ are "phase space translations", and belong to the representation theory of the Heisenberg group. Their relevance to our problem is that Hamilton flows on general Kähler manifolds (M, J, ω) can be approximated by the linear models at a tangent space (T_zM, J_z, ω_z). Quadratic Hamiltonians and the metaplectic representation are constructed by the Toeplitz method in [5].

5.1 Propagator with linear Hamiltonian function

Let $M=\mathbb{C}^m$, $\omega=i\sum_j dz_j\wedge d\bar{z}_j$ be the Bargmann–Fock model, and let $H=\mathrm{Re}(\alpha\cdot\bar{z})$ (see Sections 1.5 and 3.2).

Proposition 5.1 *The kernel for the propagator $\widehat{U}_k(t)=\widehat{\Pi}_k e^{ikt\widehat{H}_k}\widehat{\Pi}_k$ is given by*

$$\widehat{U}_k(t,\hat{z},\hat{w})=\Big(\frac{k}{2\pi}\Big)^m e^{ik\left(\theta_z+\frac{1}{2}\mathrm{Re}(\alpha\bar{z})t-\theta_w+\mathrm{Im}\left(\left(z-\frac{1}{2}\alpha t/i\right)\bar{w}\right)\right)-k\frac{1}{2}\left|z-\frac{1}{2}\alpha t/i-w\right|^2}. \tag{51}$$

In particular, if $\hat{z}=\hat{w}$, we have

$$\widehat{U}_k(t,\hat{z},\hat{z})=\Big(\frac{k}{2\pi}\Big)^m e^{ikH(z)t}e^{-kt^2\frac{1}{4}\|\xi_H(z)\|^2}. \tag{52}$$

Proof We may start from the integral expression

$$\widehat{U}_k(t;\hat{z},\hat{w})=\Big(\frac{k}{2\pi}\Big)^{2m}\int_{\hat{y}\in X} e^{k[\hat{\psi}(\hat{z},\hat{g}^t(\hat{y}))+\hat{\psi}(\hat{y},\hat{w})]}\,d\mathrm{Vol}_X(\hat{y}), \tag{53}$$

where the function $\hat{\psi}(\hat{x}_1,\hat{x}_2)$ is given by

$$\hat{\psi}(\hat{x}_1,\hat{x}_2)=i(\theta_1-\theta_2)+z\bar{w}-\tfrac{1}{2}|z|^2-\tfrac{1}{2}|w|^2,\quad \hat{x}_1=(z,\theta_1)\text{ and }\hat{x}_2=(w,\theta_2),$$

and $\hat{g}^t(\hat{y})$ is given by

$$\hat{g}^t(\hat{y})=\Big(y+\frac{\alpha t}{2i},\theta_y-\frac{1}{2}H(y)t\Big).$$

Then after a Gaussian integral of dy we get the desired answer. The diagonal case is straightforward, except to note that $\|\xi_H(z)\|^2=\omega\big(\xi_H(z),J\xi_H(z)\big)=\frac{1}{2}|\alpha|^2$. □

Remark 5.2 Since the Hamiltonian flow for a linear Hamiltonian is just translation on $\mathbb{C}^m$, which is compatible with the complex structure on $\mathbb{C}^m$, we have that $\widehat{H}_k$ commutes with $\widehat{\Pi}_k$, and hence $\widehat{U}_k(t) = \widehat{\Pi}_k(\hat{g}^{-t})^* \widehat{\Pi}_k = (\hat{g}^{-t})^* \widehat{\Pi}_k$. Thus

$$\widehat{U}_k(t; \hat{z}, \hat{w}) = \widehat{\Pi}_k(\hat{g}^{-t}\hat{z}, \hat{w}) = \left(\frac{k}{2\pi}\right)^m e^{k\hat{\psi}(\hat{g}^{-t}\hat{z}, \hat{w})},$$

which gives the desired result.

5.2 Short-time propagator $\widehat{U}_k(t, \hat{z}, \hat{z})$ for $t \sim 1/\sqrt{k}$

Let (M, ω) be a Kähler manifold polarized by an ample line bundle (L, h) such that $\omega = c_1(L)$. Let $X \to M$ be the dual circle bundle over M. We have the following result about the kernel $\widehat{U}(t)$ on the diagonal, generalizing the special case of the Bargmann–Fock model.

Proposition 5.3 *If $z_0 \in M$ such that $dH(z_0) \neq 0$, then for any $\tau \in \mathbb{R}$,*

$$\widehat{U}_k\left(\frac{\tau}{\sqrt{k}}, \hat{z}_0, \hat{z}_0\right) = \left(\frac{k}{2\pi}\right)^m e^{i\tau\sqrt{k}H(z_0)} e^{-\tau^2 \frac{1}{4}\|\xi_H(z_0)\|^2}(1 + O(|\tau|^3 k^{-\frac{1}{2}})),$$

where the constant in the error term is uniform as τ varies over compact subsets of $\mathbb{R}$.

Proof First, we pick Kähler normal coordinates $z_1, \dots, z_m$ centered at the point z_0. That is, for a small enough $\epsilon > 0$, we may find coordinates in $U = B(z_0, \epsilon)$ such that the coordinate of z_0 is $0 \in \mathbb{C}^m$, and

$$\omega(z) = i \sum_{j=1}^{m} dz_j \wedge d\bar{z}_j + O(|z|^2).$$

We may also choose a local reference frame e_L of the line bundle in a neighborhood of z_0 so that the induced Kähler potential φ takes the form

$$\varphi(z) = |z|^2 + O(|z|^3).$$

And the almost analytic continuation is

$$\psi(z, w) = z \cdot \bar{w} + O(|z|^3 + |w|^3).$$

The Hamiltonian assumes the form

$$H(z) = H(0) + H_1(z) + O(|z|^2),$$

where $H_1(z)$ is a real-valued linear function on $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Locally the circle bundle X over M can be trivialized with fiber coordinate θ.

Now we are ready to evaluate the integral in (50):

$$\widehat{U}_k\Big(\frac{\tau}{\sqrt{k}},\hat{z}_0,\hat{z}_0\Big)=\Big(\frac{k}{2\pi}\Big)^{2m}\int_{S^1}\int_M e^{k[\widehat{\psi}(0,\hat{g}^t\hat{w})+\widehat{\psi}(\hat{w},0)]}A_k\,dw\,\frac{d\theta_w}{2\pi}+O(k^{-\infty}).$$

If $|w|>\epsilon$ or $|g^t w|>\epsilon$, the integrand is fast decaying in k, due to the off-diagonal decay of $\Pi_k(z_1,z_2)$. Hence we may restrict the integral to $|w|<\epsilon$ and $|g^t w|<\epsilon$.

For the phase function in the integral (50), when we plug in

$$w=\frac{u}{\sqrt{k}},\quad t=\frac{\tau}{\sqrt{k}},$$

and write $\hat{w}=(w,\theta_w)$ and $\hat{g}^t\hat{w}=(w(t),\theta_w(t))$, if $|w|<\epsilon$ and $|g^t w|<\epsilon$, we have

$$\begin{aligned}
&\widehat{\psi}(0,\hat{g}^t\hat{w})+\widehat{\psi}(\hat{w},0)\\
&\quad=i(\theta_w(0)-\theta_w(t))-\tfrac12|w(0)|^2-\tfrac12|w(t)|^2+O(|w|^3+|w(t)|^3)\\
&\quad=k^{-\frac12}[iH(0)\tau]+k^{-1}\big[i\tfrac12 H_1(u)\tau-\tfrac12|u|^2-\tfrac12|u+\xi_{H_1}\tau|^2\big]+O(k^{-\frac32}(|u|^3+|\tau|^3)).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\widehat{U}_k(t,\hat{z}_0,\hat{z}_0)\\
&=\Big(\frac{k}{2\pi}\Big)^{2m}\int_{S^1}\int_W e^{k[\widehat{\psi}(0,\hat{g}^t\hat{w})+\widehat{\psi}(\hat{w},0)]}A_k\,dw\,\frac{d\theta_w}{2\pi}+O(k^{-\infty})\\
&=\Big(\frac{k}{2\pi}\Big)^{2m}\int_{S^1}\int_W e^{k\left[i(\theta_w(0)-\theta_w(t))-\frac12|w(0)|^2-\frac12|w(t)|^2+O(|w|^3+|w(t)|^3)\right]}A_k\,dw\,d\theta_w\\
&\qquad\qquad+O(k^{-\infty})\\
&=\Big(\frac{k}{2\pi}\Big)^{2m}\int_{S^1}\int_W e^{k\left[i(\theta_w(0)-\theta_w(t))-\frac12|w(0)|^2-\frac12|w(t)|^2\right]}\,dw\,d\theta_w\times(1+O(k^{-\frac12}))\\
&=\Big(\frac{k}{2\pi}\Big)^{2m}\int_{|u|<\epsilon k^{\frac12}} e^{k^{\frac12}[iH(0)\tau]+\left[i\frac12 H_1(u)\tau-\frac12|u|^2-\frac12|u+\xi_{H_1}\tau|^2\right]}k^{-m}\,d\mathrm{Vol}_{\mathbb{C}^m}(u)\\
&\qquad\qquad\times(1+O(|\tau|^3k^{-\frac12}))\\
&=\Big(\frac{k}{2\pi}\Big)^{2m}e^{k^{\frac12}[iH(0)\tau]}\int_{u\in\mathbb{C}^m} e^{\left[i\frac12 H_1(u)\tau-\frac12|u|^2-\frac12|u+\xi_{H_1}\tau|^2\right]}k^{-m}\,d\mathrm{Vol}_{\mathbb{C}^m}(u)\\
&\qquad\qquad\times(1+O(|\tau|^3k^{-\frac12}))\\
&=\Big(\frac{k}{2\pi}\Big)^{m}e^{i\tau\sqrt{k}H(z_0)}e^{-\tau^2\frac14\|\xi_H(z_0)\|^2}(1+O(|\tau|^3k^{-\frac12})),
\end{aligned}$$

where $W=B(z_0,\epsilon)\cap g^{-t}(B(z_0,\epsilon))$, and the calculation in the last step is the exactly the same as in the Bargmann–Fock model. □

5.3 Short-time propagator $U_k(t, z, z)$ for nonperiodic trajectories

For $z \in M$ such that $dH(z) \neq 0$, let $T(z) \in (0, \infty]$ be the period of the flow trajectory starting at z, and if the trajectory is not periodic we let $T(z) = \infty$.

Lemma 5.4 *The function $T(z)$ is uniformly bounded from below over compact subsets in the complement of the fixed-point set of ξ_H.*

Proof Let $K \subset M$ be a compact subset such that if $z \in K$ then $dH(z) \neq 0$. Then for small enough $\epsilon > 0$, the graph of the flow ξ_H,

$$\Phi\colon K \times (-\epsilon, \epsilon) \to M \times M, \quad (z, t) \mapsto (z, \exp(t\xi_H)(z)),$$

is injective by the implicit function theorem. Thus $T(z) > \epsilon$ for all $z \in K$. □

Lemma 5.5 *Fix any $T > 0$. For any $z, w \in M$ and $t \in \mathbb{R}$ with $|t| < T$, there exists constants $C, \beta > 0$ depending on (T, H, M, ω, J) such that for any lifts $\hat{z}, \hat{w} \in X$ of z, w, we have*

$$|\hat{U}_k(t, \hat{z}, \hat{w})| < Ck^m e^{-\beta\sqrt{k}d(z, g^t w)},$$

where d is the Riemannian metric of M. In addition, for any $n > 0$, we have

$$|\partial_t^n \hat{U}_k(t, \hat{z}, \hat{w})| < C_n k^{m+n} e^{-\beta\sqrt{k}d(z, g^t w)}$$

for some constants C_n.

Proof Modulo a smooth remainder term, we have

$$\hat{U}_k(t, \hat{z}, \hat{w}) = \int_{\hat{u}\in X} \widehat{\Pi}_k(\hat{z}, \hat{g}^t\hat{u})\widehat{\Pi}_k(\hat{u}, \hat{w})A_k(t, \hat{z}, \hat{u}, \hat{w})\, d\hat{u}.$$

From the off-diagonal decay estimate of the Szegő kernel (Theorem A.1), there exists C_1 and β_1 such that

$$|\widehat{\Pi}_k(\hat{u}, \hat{v})| < C_1 k^m e^{-\beta_1\sqrt{k}d(u,v)}.$$

Thus there exists a constant $C_2 > 0$ such that

$$\begin{aligned}|\hat{U}_k(t, \hat{z}, \hat{w})| &< C_2 \sup_{\hat{u}\in X} |\widehat{\Pi}_k(\hat{z}, \hat{g}^t\hat{u})| \cdot |\widehat{\Pi}_k(\hat{u}, \hat{w})| \\ &< C_1C_2k^m \sup_{u\in M} e^{-\beta_1\sqrt{k}(d(z, g^t u) + d(u, w))}.\end{aligned}$$

To finish the proof, it suffices to bound $d(z, g^t u) + d(u, w)$ from below by $d(z, g^t w)$. Since g^t is a diffeomorphism on M, there exists $1 > \epsilon > 0$ such that

$$d(g^t u, g^t v) > \epsilon d(u, v) \quad \text{for all } u, v \in M \text{ and } |t| < T,$$

and then

$$d(z,g^tu)+d(u,w)>\epsilon d(z,g^tu)+\epsilon d(g^tu,g^tw)\geq \epsilon d(z,g^tw).$$

To prove the claim for the t–derivative, we note that

$$\partial_t^n \hat{U}_k(t)=(ik\hat{H}_k)^n\hat{U}_k(t).$$

Since $\hat{H}_k$ is an operator with bounded spectrum on $\mathcal{H}_k(X)$, we get the desired estimate. □

5.4 Proof of Theorem 3

Proof By the Fourier inversion formula,

$$\begin{aligned}\int_{\mathbb{R}} f(x)\,d\mu_k^{z,1,\alpha}(x) &= \sum_j f(k(\mu_{k,j}-H(z))+\sqrt{k}\alpha)\hat{\Pi}_{k,j}(\hat{z},\hat{z})\\ &= \int e^{-ikt(H(z)-\alpha/\sqrt{k})}\hat{f}(t)\hat{U}_k(t,\hat{z},\hat{z})\,\frac{dt}{2\pi}.\end{aligned}$$

To complete the proof of the theorem, it suffices to prove:

Lemma 5.6 *Let K be a compact subset of M that does not contain any fixed points of ξ_H, and let $\epsilon>0$ be small enough that $T(z)>\epsilon$ for all $z\in K$. Then for any $z\in K$ and any nonnegative Schwarz function $f\in\mathcal{S}(\mathbb{R})$ such that $\int f(x)\,dx=1$, ie $\hat{f}(0)=1$, and $\hat{f}(t)$ is supported in $(-\epsilon,+\epsilon)$, and for any $\alpha\in\mathbb{R}$, we have*

$$\int_{\mathbb{R}}\hat{f}(t)U_k(t,z,z)e^{-itkH(z)+it\sqrt{k}\alpha}\,\frac{dt}{2\pi} = \left(\frac{k}{2\pi}\right)^{m-\frac12}e^{-\alpha^2/\|\xi_H(z)\|^2}\frac{\sqrt{2}}{2\pi\|\xi_H(z)\|}(1+O(k^{-\frac12})). \tag{54}$$

Fix any $0<\delta<\frac12$. We claim that the contribution to the integral is $O(k^{-\infty})$ if $\epsilon>|t|>k^{-\frac12+\delta}$. From Lemma 5.4,

$$U_k(t,z,z)=\hat{U}_k(t,\hat{z},\hat{z})\leq Ck^me^{-\sqrt{k}\beta d(z,g^tz)}\leq Ck^me^{-\beta' k^\delta}=O(k^{-\infty})$$

for all t such that $|t|\in[k^{-\frac12+\delta},\epsilon]$.

Now, we may approximate the integral on the left side of (54), and rescale the integration variable by $t=\tau/\sqrt{k}$:

$$\int\hat{f}(t)U_k(t,z,z)e^{-itkH(z)+it\sqrt{k}\alpha}\,\frac{dt}{2\pi} = k^{-\frac12}\int_{|\tau|<k^\delta}\hat{f}(\tau k^{-\frac12})U_k(\tau k^{-\frac12},z,z)e^{-i\tau\sqrt{k}H(z)+i\tau\alpha}\,\frac{d\tau}{2\pi}.$$

With the Taylor expansion $\widehat{f}(\tau k^{-\frac{1}{2}}) = \widehat{f}(0)(1+O(|\tau| k^{-\frac{1}{2}})) = 1+O(|\tau| k^{-\frac{1}{2}})$, using $\widehat{f}(0) = 1$, and using the expansion of $U_k(\tau k^{-\frac{1}{2}}, z, z)$ in Proposition 5.3, we have

$$\begin{aligned}\int \widehat{f}(t) U_k(t,z,z) e^{-itkH(z)+it\alpha\sqrt{k}} \frac{dt}{2\pi} \\ &= k^{-\frac{1}{2}} \int_{|\tau|<k^\delta} \Big(\frac{k}{2\pi}\Big)^m e^{-\tau^2 \frac{1}{4}\|\xi_H(z)\|^2 + i\tau\alpha} (1+O(|\tau|^3 k^{-\frac{1}{2}})) \frac{d\tau}{2\pi} \\ &= \Big(\frac{k}{2\pi}\Big)^{m-\frac{1}{2}} e^{-\alpha^2/\|\xi_H(z)\|^2} \frac{\sqrt{2}}{2\pi \|\xi_H(z)\|} (1+O(k^{-\frac{1}{2}})).\end{aligned}$$

This completes the proof of Lemma 5.6 and of Theorem 3 . □

6 Smooth density of states: proof of Theorem 2(1)

We start by proving (13). As above, let $\hat{g}^t\colon X \to X$ denote the lifted Hamiltonian flow and let F^t denote the gradient flow. For any $z \in M$, we choose a lift $\hat{z}$ in X, and since $\widehat{\Pi}_k(\hat{z},\hat{z})$ and $\widehat{U}_k(t,\hat{z},\hat{z})$ do not depend on the choice of the lift, we will use $\hat{z}$ and z interchangeably in these cases.

Proof of (13) Let $z_k = F^{\beta/\sqrt{k}} z$. By (20) we have

$$I := \sum_{j=1}^{d_k} f(\sqrt{k}(\mu_{k,j} - E)) \Pi_{k,\mu_{k,j}}(z_k, z_k) = \int_{\mathbb{R}} \widehat{f}(t) e^{-iE\sqrt{k}t} U_k(t/\sqrt{k}, z_k, z_k) \frac{dt}{2\pi}.$$

Using Proposition 5.3 to express U_k, we get

$$I = \Big(\frac{k}{2\pi}\Big)^m \int_{\mathbb{R}} \widehat{f}(t) e^{-iE\sqrt{k}t} k^m e^{it\sqrt{k}H(z_k)} e^{-\frac{1}{4}|t\xi_H(z_k)|^2} [1+O(|t|^3 k^{-\frac{1}{2}})] \frac{dt}{2\pi}.$$

In the exponent, using $E = H(z)$, we get

$$\begin{aligned}-iE\sqrt{k}t + it\sqrt{k}H(z_k) &= it\sqrt{k}(H(z_k) - H(z)) \\ &= it\sqrt{k}\Big[g\Big(\nabla H(z), \frac{\beta}{\sqrt{k}} \nabla H(z)\Big) + O\Big(\Big(\frac{\beta}{\sqrt{k}}\Big)^2\Big)\Big] \\ &= it\beta \|\nabla H(z)\|^2 + O(|t| k^{-\frac{1}{2}}).\end{aligned}$$

Furthermore, $-\frac{1}{4}|t\xi_H(z_k)|^2 = -\frac{1}{4}|t\xi_H(z)|^2 + O(k^{-\frac{1}{2}}|t|^2)$.

Hence

$$I=\Big(\frac{k}{2\pi}\Big)^m\int_{\mathbb{R}}\hat f(t)e^{it\beta\|\nabla H(z)\|^2-\frac14|t\nabla H(z)|^2}\,\frac{dt}{2\pi}\times(1+O(k^{-\frac12}))$$
$$=\Big(\frac{k}{2\pi}\Big)^m\int_{t\in\mathbb{R}}\int_{x\in\mathbb{R}}f(x)e^{-ixt}e^{it\beta\|\nabla H(z)\|^2-\frac14|t\nabla H(z)|^2}\,\frac{dx\,dt}{2\pi}\times(1+O(k^{-\frac12}))$$
$$=\Big(\frac{k}{2\pi}\Big)^m\int_{x\in\mathbb{R}}f(x)e^{-(x/\|\nabla H\|-\beta\|\nabla H(z)\|)^2}\,\frac{dx}{\sqrt{\pi}\|\nabla H(z)\|}\times(1+O(k^{-\frac12})).$$

To obtain the complete asymptotic expansion stated in Theorem 2(1), it is only necessary to Taylor expand $F^{\beta/\sqrt{k}}z$ and use the expansion of Proposition 5.3 for $\hat U_k(t/\sqrt{k})$ to higher order in $k^{-\frac12}$. □

7 Tauberian argument: proof of Theorem 2(2)–(3)

We denote by W a positive function in $\mathcal{S}(\mathbb{R})$ with $\int W(x)\,dx=1$ and $\operatorname{supp}\hat W\subset(-\epsilon,\epsilon)$, where ϵ is small enough that there is no closed orbit of the flow ξ_H passing through z with time less than ϵ. Let $W_h(x)=h^{-1}W(x/h)$ for $h>0$.

For clarity of notation, we let $F_k(x)=\Pi_k(z)^{-1}\mu_k^{z,\frac12}(x)$.

Proposition 7.1 *There exists a k_0 large enough that for any $k>k_0$, and for all $x\in\mathbb{R}$, we have*

$$F_k*W_{k^{-\frac12}}(x_0)=\int_{-\infty}^{x_0}e^{-x^2/|\xi_H|^2}\,\frac{dx}{\sqrt{\pi}|\xi_H|}+O(k^{-\frac12}).$$

Proof Denoting $F_k*W_{k^{-\frac12}}(x_0)$ by I_1, and letting $h=k^{-\frac12}$,

$$I_1=\int_{-\infty}^{x_0}\int_{y\in\mathbb{R}}\Pi_k(z)^{-1}\sum_{j=1}^{N_k}\|s_{k,j}(z)\|^2\delta_{\sqrt{k}(\mu_{k,j}-H(z))}(y)W_h(x-y)\,dy\,dx$$
$$=\int_{-\infty}^{x_0}\Pi_k(z)^{-1}\sum_{j=1}^{N_k}\|s_{k,j}(z)\|^2W_h\big(x-\sqrt{k}(\mu_{k,j}-H(z))\big)\,dx$$
$$=\int_{-\infty}^{x_0}\Pi_k(z)^{-1}\sum_{j=1}^{N_k}\|s_{k,j}(z)\|^2\int_{\mathbb{R}}\hat W_h(\tau)e^{i\tau[x-\sqrt{k}(\mu_{k,j}-H(z))]}\,\frac{d\tau\,dx}{2\pi}$$
$$=\Pi_k(z)^{-1}\int_{-\infty}^{x_0}\int_{\mathbb{R}}e^{i\tau x}\hat W\Big(\frac{\tau}{\sqrt{k}}\Big)e^{i\tau\sqrt{k}H(z)}U_k\Big(-\frac{\tau}{\sqrt{k}},z,z\Big)\,\frac{d\tau\,dx}{2\pi}.$$

Since $\operatorname{supp} \widehat{W} \subset (-\epsilon, \epsilon)$, the integral of t is limited to $|t/\sqrt{k}| < \epsilon$. We first show that one can further cut off the $d\tau$ integral to $|\tau| < k^{\delta}$ for any $0 < \delta \ll \frac{1}{2}$. Let $\chi(x) \in C_c^{\infty}(\mathbb{R})$ which is identically 1 in a neighborhood of $x = 0$. Then we claim

$$\Pi_k(z)^{-1} \int_{-\infty}^{x_0} \int_{\mathbb{R}} \Big(1-\chi\Big(\frac{\tau}{k^{\delta}}\Big)\Big) \widehat{W}\Big(\frac{\tau}{\sqrt{k}}\Big) e^{i\tau\sqrt{k}H(z)+i\tau x} U_k\Big(-\frac{\tau}{\sqrt{k}}, z, z\Big) \frac{d\tau\, dx}{2\pi} = O(k^{-\infty}). \tag{55}$$

Indeed, we may define

$$G_k(\tau) := \Big(1 - \chi\Big(\frac{\tau}{k^{\delta}}\Big)\Big) \widehat{W}\Big(\frac{\tau}{\sqrt{k}}\Big) e^{i\tau\sqrt{k}H(z)} U_k\Big(-\frac{\tau}{\sqrt{k}}, z, z\Big),$$

and let $\widehat{G}_k(x)$ be its Fourier transformation. Then (55) can be written as

$$\int_{-\infty}^{x_0} \int_{\mathbb{R}} e^{i\tau x} G_k(\tau) \frac{d\tau\, dx}{2\pi} = \int_{-\infty}^{x_0} \widehat{G}_k(x)\, dx.$$

We note that $G_k(\tau)$ is a smooth and compactly supported function in τ, hence its Fourier transform $\widehat{G}_k(x)$ is also a Schwarz function in x. Since when $k^{-\frac{1}{2}+\delta} < |\tau|/\sqrt{k} < \epsilon$, we have that $U_k(-\tau/\sqrt{k}, z, z)$ and all its derivatives in τ are bounded by $k^{\gamma} e^{-\beta k^{\delta}}$ for some $\beta, \gamma > 0$, all the Schwarz function seminorms of $G_k(\tau)$ and $\widehat{G}_k(x)$ are $O(k^{-\infty})$. Thus we have proved (55).

With the above claim, we can write

$$\begin{aligned} I_1 &= \Pi_k(z)^{-1} \int_{-\infty}^{x_0} \int_{\mathbb{R}} \chi(\tau/k^{\delta}) \widehat{W}\Big(\frac{\tau}{\sqrt{k}}\Big) e^{i\tau x + i\tau\sqrt{k}H(z)} U_k\Big(-\frac{\tau}{\sqrt{k}}, z, z\Big) \frac{d\tau\, dx}{2\pi} \\ &\qquad + O(k^{-\infty}) \\ &= \int_{-\infty}^{x_0} \int_{\mathbb{R}} \chi(\tau/k^{\delta}) \widehat{W}\Big(\frac{\tau}{\sqrt{k}}\Big) e^{i\tau x} e^{-\tau^2 \frac{1}{4}\|\xi_H(z)\|^2} (1 + R_k(\tau)) \frac{d\tau\, dx}{2\pi} + O(k^{-\infty}), \end{aligned}$$

where $e^{-\tau^2 \frac{1}{4}\|\xi_H(z)\|^2} R_k(\tau)$ is a Schwarz function in τ with all the Schwarz norms bounded by $k^{-\frac{1}{2}}$. With the Gaussian suppression factor $e^{-\tau^2 \frac{1}{4}\|\xi_H(z)\|^2}$, we may replace the factor $\chi(\tau/k^{\delta})$ by 1, while adding an error term of $O(k^{-\infty})$. We may also write

$$\widehat{W}\Big(\frac{\tau}{\sqrt{k}}\Big) = \widehat{W}(0) + O\Big(\frac{|\tau|}{\sqrt{k}}\Big) = 1 + O\Big(\frac{|\tau|}{\sqrt{k}}\Big),$$

and absorb the error term into $R_k(\tau)$ as well. Thus we have

$$\begin{aligned} I_1 &= \int_{-\infty}^{x_0} \int_{\mathbb{R}} e^{i\tau x} e^{-\tau^2 \frac{1}{4}\|\xi_H(z)\|^2} (1 + R_k(\tau)) \frac{d\tau\, dx}{2\pi} + O(k^{-\infty}) \\ &= \int_{-\infty}^{x_0} e^{-x^2/|\xi_H|^2} \frac{dx}{\sqrt{\pi}|\xi_H|} + O(k^{-\frac{1}{2}}). \end{aligned}$$

□

This shows the smoothed measure of $d\mu_k^{z,\frac{1}{2}}$ has the desired cumulative distribution function. To prove the sharp result, we need a Tauberian argument.

Lemma 7.2 *Let $\sigma_h = dF_h$ be a family of finite measures, where $F_h\colon \mathbb{R} \to \mathbb{R}$ is a family of nondecreasing functions with $h \in [0,1)$, such that:*

(1) $\operatorname{supp} \sigma_h(x) \subset [-h^{-1}, h^{-1}]$.

(2) $F_h(x) = \sigma_h[-\infty, x]$.

(3) *There exists a nonnegative integer n such that $F_h(x) \le h^{-n}$ uniformly in x as $h \to 0$.*

(4) $\frac{d}{dx} F_h * W_h(x) = O(h^{-n})$ *uniformly in x.*

Then

$$F_h(x) = F_h * W_h(x) + O(h^{-n+1}) \quad \text{as } h \to 0.$$

This lemma is almost the same as in [20, Theorem V-13, page 266], except that the latter Tauberian lemma assumes that $\operatorname{supp} d\mathbb{F}_h$ is a fixed interval $[\tau_0, \tau_1]$ whereas our $d\mu_k^{z,\frac{1}{2}}$ each have support in $C[-\sqrt{k}, \sqrt{k}]$. It turns out that the proof of [20, Theorem V-13, page 266] extends to this situation with no change in the proof. For the sake of completeness, we review the proof in Appendix B to ensure that the extension is correct.[3]

Proposition 7.3 *Let $h = k^{-\frac{1}{2}}$ and $n = 0$, and $F_h = F_{h_k} = (k/2\pi)^{-m} \mu_k^{z,\frac{1}{2}}(-\infty, x)$. Then we have*

(1) $\sup_x F_h(x) < C$ *for some positive constant C.*

(2) $\frac{d}{dx} F_h * W_h(x) = O(1)$ *uniformly in x.*

Proof (1) Since $F_h(x)$ is nondecreasing and $\lim_{x\to\infty} F_h(x) = \Pi_k(z)^{-1}\Pi_k(z) = 1$, $F_h(x)$ is uniformly bounded by 1.

(2) By an argument similar to that used in Proposition 7.1, we have

$$\frac{d}{dx} F_h * W_h(x) = e^{-x^2/|\xi_H|^2} \frac{1}{\sqrt{\pi}|\xi_H|(z)} + O(k^{-\frac{1}{2}}),$$

which is uniformly bounded in x. □

In particular, the condition in Lemma 7.2 is satisfied, and we have

$$F_h(x_0) = F_h * W_h(x_0) + O(k^{-\frac{1}{2}}) = \int_{-\infty}^{x_0} e^{-x^2/|\xi_H|^2} \frac{dx}{\sqrt{\pi}|\xi_H|} + O(k^{-\frac{1}{2}}).$$

[3] We thank D Robert for corroborating that the compact support condition is unnecessary.

7.1 Proof of Theorem 2(3)

In the statement of Theorem 2(2), the sequence of points $(z_k, E_k) = (z, H(z)+\alpha/\sqrt{k})$ approaches $(z, H(z))$ while keeping z_k fixed. The following proposition would relax the direction of approach.

Proposition 7.4 *Let (L, h, M, ω), H and E be as in Theorem 2. Pick any $z \in H^{-1}(E)$. If there is a sequence of $(z_k, E_k) \in M \times \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $|z_k - z| = O(k^{-\frac{1}{2}})$ and $|E_k - E| = O(k^{-\frac{1}{2}})$, and $|\sqrt{k}(E_k - H(z_k)) - \alpha| = O(k^{-\frac{1}{2}})$, then*

$$\Pi_{k,E_k}(z_k) = \left(\frac{k}{2\pi}\right)^m \operatorname{Erf}\left(\frac{\sqrt{2}\alpha}{|\nabla H|(z)}\right) + O(k^{m-\frac{1}{2}}). \tag{56}$$

Proof By Theorem 2(2), the constant in (14) is uniform for z for a compact neighborhood K (not containing the critical point of H) and $|\alpha| < T$. Thus, we have for k large enough, $z_k \in K$, and

$$\Pi_{k,H(z_k)+\alpha/\sqrt{k}}(z_k) = \left(\frac{k}{2\pi}\right)^m \operatorname{Erf}\left(\frac{\sqrt{2}\alpha}{|\nabla H|(z_k)}\right) + O(k^{m-\frac{1}{2}}).$$

Let $\alpha_k = \sqrt{k}(E_k - H(z_k))$. Then by hypothesis $|\alpha - \alpha_k| = O(k^{-\frac{1}{2}})$, and then

$$\Pi_{k,E_k}(z_k) = \Pi_{k,H(z_k)+\alpha_k/\sqrt{k}}(z_k) = \left(\frac{k}{2\pi}\right)^m \operatorname{Erf}\left(\frac{\sqrt{2}\alpha_k}{|\nabla H|(z_k)}\right) + O(k^{m-\frac{1}{2}}). \quad \square$$

Given this proposition, we may take the sequence $(z_k, E_k) = (F^{\beta/\sqrt{k}}(z), H(z))$, and $\alpha = -\beta|\nabla H(z)|^2$, then verify that

$$\begin{aligned}
&\sqrt{k}(E_k - H(z_k)) - \alpha \\
&\qquad = -\sqrt{k}\left(H(z) + \langle dH, \nabla H\rangle\left(\frac{\beta}{\sqrt{k}}\right) + O(k^{-1}) - H(z)\right) + \beta|\nabla H(z)|^2 \\
&\qquad = O(k^{-\frac{1}{2}}).
\end{aligned}$$

Thus

$$\Pi_{k,H(z)}(F^{\beta/\sqrt{k}}(z)) = \left(\frac{k}{2\pi}\right)^m \operatorname{Erf}(-\sqrt{2}\beta|\nabla H|(z)) + O(k^{m-\frac{1}{2}}),$$

which proves (15).

Appendix A Off-diagonal decay estimates

Theorem A.1 (see [6, Theorem 2; 11, Proposition 9]) *Let M be a compact Kähler manifold, and let $(L,h)\to M$ be a positive Hermitian line bundle. Then there exists a constant $\beta=\beta(M,L,h)>0$ such that*

$$|\widetilde{\Pi}_N(x,y)|_{\tilde{h}^N}\le CN^m e^{-\beta\sqrt{N}d(x,y)},$$

where $d(x,y)$ is the Riemannian distance with respect to the Kähler metric $\widetilde{\omega}$.

The theorem is stated for strictly pseudoconvex domains in $\mathbb{C}^n$ but applies with no essential change to unit codisc bundles of positive Hermitian line bundles.

Appendix B Tauberian theory

In this section, we review the semiclassical Tauberian theorem of Robert [20].

Let $\theta\in C_0^\infty(-1,1)$ satisfy $\theta(0)=1$, and $\theta(-x)=\theta(x)$. We may also assume $\widehat{\theta}\ge 0$ and $|\widehat{\theta}(x)|\ge r_0$ for $|x|\le\delta_0$. Let

$$W_h(x)=(2\pi h)^{-1}\widehat{\theta}\Big(-\frac{x}{h}\Big).$$

Theorem B.1 *Let $\sigma_h\colon\mathbb{R}\to\mathbb{R}$ be a family of nondecreasing functions with $h\in[0,1)$ satisfying:*

1. *$\sigma_h(x)=0$ for $x\le x_0$.*
2. *There exists $x_1>x_0$ such that σ_h is constant on $[x_1,\infty]$.*
3. *There exists a positive integer $n\ge 1$ such that $\sigma_h(x)\le h^{-n}$ uniformly in x as $h\to 0$.*
4. *$\frac{d}{dx}\sigma_h * W_h(x)=O(h^{-n})$ uniformly in x.*

Then

$$\sigma_h(x)=\sigma_h * W_h(x)+O(h^{-n+1})\quad\text{as } h\to 0.$$

Proof One has

$$\begin{aligned}\sigma_h(\tau)-\sigma_h * W_a(\tau)&=\int_{\mathbb{R}}(\sigma_h(\tau)-\sigma_h(\tau-\mu))W_a(\mu)\,d\mu\\&=\int_{\mathbb{R}}(\sigma_h(\tau)-\sigma_h(\tau-a\nu))\widehat{\theta}(-\nu)\,d\nu.\end{aligned}$$

Due to the fact that $\widehat{\theta}\in\mathcal{S}(\mathbb{R})$ and assumptions (1)–(2), this theorem reduces to the following estimate on the a–scale increments.

Lemma B.2 *There exists* $\Gamma > 0$ *such that*

$$|\sigma_h(\tau) - \sigma_h(\tau + h\nu)| \leq \Gamma(|\nu| + 1)\, h^{-n+1},$$

for all $\tau, \nu \in \mathbb{R}$.

The proof is broken up into three cases:

Case (i) ($|\nu| \leq \delta_0$, where $|\widehat{\theta}| \geq r_0$ on $|x| \leq \delta_0$) We have

$$|\sigma_h(\tau) - \sigma_h(\tau + h\nu)| = \int_{\tau - h|\nu|}^{\tau + h|\nu|} d\sigma_h(\mu) \leq h\, d\sigma_h * W_h(\tau) = \int_{\mathbb{R}} \widehat{\theta}\Big(\frac{\tau - \mu}{h}\Big)\, d\sigma_h(\mu).$$

The inequality holds because $|\mu - \tau| \leq h\delta_0$ on the interval of integration and $|\widehat{\theta}| \geq r_0$ on $|x| \leq \delta_0$. The statement of the lemma then follows from Theorem B.1(4).

Case (ii) ($|\nu| = j\delta_0$, where $j \in \mathbb{Z}$) One has

$$|\sigma_h(\tau) - \sigma_h(\tau + hj\delta_0)| = \sum_{k=1}^{j} |\sigma_h(\tau + hk\delta_0) - \sigma_h(\tau + h(k-1)\delta_0)|.$$

Applying Case (i) to each term gives

$$|\sigma_h(\tau) - \sigma_h(\tau + hj\delta_0)| \leq C|j|h^{-n+1}.$$

Case (iii) ($j\delta_0 < |\nu| < (j+1)\delta_0$, where $j \in \mathbb{Z}$) In this case,

$$|\sigma_h(\tau) - \sigma_h(\tau + h\nu)| \leq |\sigma_h(\tau) - \sigma_h(\tau + jh\delta_0)| + |\sigma_h(\tau + jh\delta_0) - \sigma_h(\tau)|.$$

By the previous two cases,

$$|\sigma_h(\tau) - \sigma_h(\tau + h\nu)| \leq Chh^{-n+1}(1 + |j|).$$

It follows that

$$|\sigma_h(\tau) - \sigma_h(\tau + h\nu)| \leq ah^{-n} C\delta_0^{-1}(\delta_0 + |\nu|). \qquad \square$$

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