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Fourier–Mukai and autoduality for compactified Jacobians, II

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To every reduced (projective) curve X with planar singularities one can associate, following E Esteves, many fine compactified Jacobians, depending on the choice of a polarization on X, which are birational (possibly nonisomorphic) Calabi–Yau projective varieties with locally complete intersection singularities. We define a Poincaré sheaf on the product of any two (possibly equal) fine compactified Jacobians of X and show that the integral transform with kernel the Poincaré sheaf is an equivalence of their derived categories, hence it defines a Fourier–Mukai transform. As a corollary of this result, we prove that there is a natural equivariant open embedding of the connected component of the scheme parametrizing rank-1 torsion-free sheaves on X into the connected component of the algebraic space parametrizing rank-1 torsion-free sheaves on a given fine compactified Jacobian of X.

The main result can be interpreted in two ways. First of all, when the two fine compactified Jacobians are equal, the above Fourier–Mukai transform provides a natural autoequivalence of the derived category of any fine compactified Jacobian of X, which generalizes the classical result of S Mukai for Jacobians of smooth curves and the more recent result of D Arinkin for compactified Jacobians of integral curves with planar singularities. This provides further evidence for the classical limit of the geometric Langlands conjecture (as formulated by R Donagi and T Pantev). Second, when the two fine compactified Jacobians are different (and indeed possibly nonisomorphic), the above Fourier–Mukai transform provides a natural equivalence of their derived categories, thus it implies that any two fine compactified Jacobians of X are derived equivalent. This is in line with Kawamata's conjecture that birational Calabi–Yau (smooth) varieties should be derived equivalent and it seems to suggest an extension of this conjecture to (mildly) singular Calabi–Yau varieties.

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1 Introduction

Let C be a smooth irreducible projective curve over an algebraically closed field k and let $J(C)$ be its Jacobian variety. Since $J(C)$ is an autodual abelian variety, ie it is canonically isomorphic to its dual abelian variety, there exists a Poincaré line bundle $\mathcal{P}$ on $J(C) \times J(C)$ which is universal as a family of algebraically trivial line bundles on $J(C)$. In the breakthrough work [37], S Mukai proved that the integral transform with kernel $\mathcal{P}$ is an auto-equivalence of the bounded derived category of coherent sheaves on $J(C)$, or in other words it defines what is, nowadays, called a Fourier–Mukai transform.[1]

Motivated by the classical limit of the geometric Langlands duality (see Donagi and Pantev [11] and the discussion below), D Arinkin [4; 5] extended the above Fourier–Mukai transform to the compactified Jacobians of integral projective curves with planar singularities.

The aim of this paper, which is heavily based on our previous manuscripts [33; 34], is to extend this autoequivalence to fine compactified Jacobians (as defined by E Esteves [12]) of *reduced projective curves with planar singularities.* The main novelty for reducible curves is that compactified Jacobians are not canonically defined but they depend on the choice of a polarization on the curve itself. Indeed we also prove that given any two fine compactified Jacobians (which are always birational but possibly nonisomorphic) of a reduced curve X with planar singularities, there is a Fourier–Mukai transform between their derived categories, hence all fine compactified Jacobians of X are derived equivalent.

1.1 Fine compactified Jacobians of singular curves

Before stating our main result, we need to briefly recall how Esteves' fine compactified Jacobians of *reduced* curves are defined in [12]; we refer the reader to Section 2.1 for

[1] More generally, for an arbitrary abelian variety A with dual abelian variety $A^\vee$, Mukai proved that the Fourier–Mukai transform associated to the Poincaré line bundle on $A \times A^\vee$ gives an equivalence between the bounded derived category of A and that of $A^\vee$.

more details. Fine compactified Jacobians of a reduced projective curve X parametrize torsion-free rank-1 sheaves on X that are semistable with respect to a general polarization on X. More precisely, a *polarization* on X is a tuple of rational numbers $\underline{q} = \{\underline{q}_{C_i}\}$, one for each irreducible component C_i of X, such that $|\underline{q}| := \sum_i \underline{q}_{C_i} \in \mathbb{Z}$. A torsion-free rank-1 sheaf I on X of Euler characteristic $\chi(I) := h^0(X, I) - h^1(X, I)$ equal to $|\underline{q}|$ is called *$\underline{q}$–semistable* (resp. *$\underline{q}$–stable*) if for every nontrivial subcurve $Y \subset X$, we have that

$$\chi(I_Y) \geq \sum_{C_i \subseteq Y} \underline{q}_{C_i} \quad \text{(resp. } >\text{)},$$

where I_Y is the biggest torsion-free quotient of the restriction $I|_Y$ of I to the subcurve Y. A polarization $\underline{q}$ is called *general* if there are no strictly $\underline{q}$–semistable sheaves, ie if every $\underline{q}$–semistable sheaf is also $\underline{q}$–stable; see Definition 2.4 for a numerical characterization of general polarizations. A fine compactified Jacobian of X is the fine moduli space $\overline{J}_X(\underline{q})$ of torsion-free rank-1 sheaves of degree $|\underline{q}|$ on X that are $\underline{q}$–semistable (or equivalently $\underline{q}$–stable) with respect to a general polarization $\underline{q}$ on X.

If the curve X has planar singularities, then we proved in [33, Theorem A] that any fine compactified Jacobian $\overline{J}_X(\underline{q})$ of X has the following remarkable properties (see Fact 2.7):

- $\overline{J}_X(\underline{q})$ is a connected reduced scheme with locally complete intersection singularities and trivial canonical sheaf, ie it is a Calabi–Yau singular variety in the weak sense.
- The smooth locus of $\overline{J}_X(\underline{q})$ coincides with the open subset $J_X(\underline{q}) \subseteq \overline{J}_X(\underline{q})$ parametrizing line bundles; in particular, $J_X(\underline{q})$ is dense in $\overline{J}_X(\underline{q})$ and $\overline{J}_X(\underline{q})$ is of pure dimension equal to the arithmetic genus $p_a(X)$ of X.
- $J_X(\underline{q})$ is the disjoint union of a number of copies of the generalized Jacobian $J(X)$ of X (which is the smooth irreducible algebraic group parametrizing line bundles on X of multidegree 0) and such a number is independent of the chosen polarization $\underline{q}$ and it is denoted by $c(X)$. In particular, all the fine compactified Jacobians of X have $c(X)$ irreducible components, all of dimension $p_a(X)$, and they are all birational among themselves.

Note also that we have found in [33] examples of reducible curves (indeed even nodal curves, whose fine compactified Jacobians are studied in detail in [35]) that admit nonisomorphic (and even nonhomeomorphic if $k = \mathbb{C}$) fine compactified Jacobians.

1.2 Main results

Let $\bar{J}_X(\underline{q})$ and $\bar{J}_X(\underline{q}')$ be two (possibly equal) fine compactified Jacobians of X such that $|\underline{q}| = |\underline{q}'| = 0$. Starting from the universal sheaves on $X \times \bar{J}_X(\underline{q})$ and on $X \times \bar{J}_X(\underline{q}')$, it is possible to define, using the formalism of the determinant of cohomology, a (canonical) Poincaré line bundle $\mathcal{P}$ on $\bar{J}_X(\underline{q}) \times J_X(\underline{q}) \cup J_X(\underline{q}') \times \bar{J}_X(\underline{q}')$; we refer the reader to Section 4.1 for details.

Consider the inclusion $j\colon \bar{J}_X(\underline{q}) \times J_X(\underline{q}') \cup J_X(\underline{q}) \times \bar{J}_X(\underline{q}') \hookrightarrow \bar{J}_X(\underline{q}) \times \bar{J}_X(\underline{q}')$ and define $\overline{\mathcal{P}} := j_*(\mathcal{P})$. In Theorem 4.6, we prove that $\overline{\mathcal{P}}$ is a maximal Cohen–Macaulay (coherent) sheaf on $\bar{J}_X(\underline{q}) \times \bar{J}_X(\underline{q}')$, flat with respect to the projections over the two factors, and whose restrictions over the fibers of each projection are again maximal Cohen–Macaulay sheaves.

The main result of this paper is the following.

Theorem A *Let X be a reduced connected projective curve with planar singularities over an algebraically closed field k of characteristic either zero or bigger than the arithmetic genus $p_a(X)$ of X. Let $\bar{J}_X(\underline{q})$ and $\bar{J}_X(\underline{q}')$ be two (possibly equal) fine compactified Jacobians of X with $|\underline{q}| = |\underline{q}'| = 1 - p_a(X)$, and let $D^b_{\mathrm{qcoh}}(\bar{J}_X(\underline{q}))$ and $D^b_{\mathrm{qcoh}}(\bar{J}_X(\underline{q}'))$ (resp. $D^b_{\mathrm{coh}}(\bar{J}_X(\underline{q}))$ and $D^b_{\mathrm{coh}}(\bar{J}_X(\underline{q}'))$) be their bounded derived categories of quasicoherent sheaves (resp. of coherent sheaves). The integral transform with kernel $\overline{\mathcal{P}}$ on $\bar{J}_X(\underline{q})\times, \bar{J}_X(\underline{q}')$*

$$\Phi^{\overline{\mathcal{P}}}\colon D^b_{\mathrm{qcoh}}(\bar{J}_X(\underline{q})) \to D^b_{\mathrm{qcoh}}(\bar{J}_X(\underline{q}')), \quad \mathcal{E}^\bullet \mapsto \boldsymbol{R}p_{2*}(p_1^*(\mathcal{E}^\bullet) \otimes^{\boldsymbol{L}} \overline{\mathcal{P}}),$$

is an equivalence of triangulated categories (ie it defines a Fourier–Mukai transform) *whose inverse is the integral transform $\Phi^{\overline{\mathcal{P}}^\vee[g]}$ with kernel*

$$\overline{\mathcal{P}}^\vee[g] := \mathcal{H}om(\overline{\mathcal{P}}, \mathcal{O}_{\bar{J}_X(\underline{q}) \times \bar{J}_X(\underline{q}')})[g].$$

Moreover, $\Phi^{\overline{\mathcal{P}}}$ restricts to an equivalence of categories between $D^b_{\mathrm{coh}}(\bar{J}_X(\underline{q}))$ and $D^b_{\mathrm{coh}}(\bar{J}_X(\underline{q}'))$.

Some comments on the hypothesis of Theorem A are in order.

First of all, the assumption that $|\underline{q}| = |\underline{q}'| = 1 - p_a(X)$, ie that we are dealing with fine compactified Jacobians parametrizing sheaves of Euler characteristic $1 - p_a(X)$ (or equivalently degree 0) on X, guarantees that the Poincaré sheaf $\mathcal{P}$ (and hence, a fortiori, its extension $\overline{\mathcal{P}}$) is canonically defined, independently of the universal sheaves on $X \times \bar{J}_X(\underline{q})$ and on $X \times \bar{J}_X(\underline{q}')$ which are used in its definition (4-1) (recall that such universal sheaves are only well-defined up to the pullback of a line bundle on $\bar{J}_X(\underline{q})$ or on $\bar{J}_X(\underline{q}')$, respectively); see Remark 4.2 for a discussion of this issue. However, if $|\underline{q}| \neq 1 - p_a(X)$ or $|\underline{q}'| \neq 1 - p_a(X)$, then one can fix once and for all a Poincaré sheaf $\mathcal{P}$ (together with its extension $\overline{\mathcal{P}}$) and all our arguments go through, giving also in this case a Fourier–Mukai transform, although not canonically defined.

Second, the assumption that either $\operatorname{char}(k) = 0$ or $\operatorname{char}(k) > p_a(X)$ is needed because of the following two facts: first of all, the results of M Haiman on the isospectral Hilbert scheme $\widetilde{\operatorname{Hilb}}^n(S)$ of a smooth surface S, originally proved under the assumption that $\operatorname{char}(k) = 0$, are known to hold also if $\operatorname{char}(k) > n$ (as pointed out by M Groechenig in [17, pages 18–19]); second, for any fine compactified Jacobian $\bar{J}_X(\underline{q})$ of X, the rational twisted Abel maps from $\operatorname{Hilb}^{p_a(X)}$ to $\bar{J}_X(\underline{q})$ are, locally on the codomain, smooth and surjective; see Fact 3.7 for the precise statement.

Theorem A can be interpreted in two ways depending on whether $\bar{J}_X(\underline{q}) = \bar{J}_X(\underline{q}')$ or $\bar{J}_X(\underline{q}) \neq \bar{J}_X(\underline{q}')$.

On one hand, when applied to the case $\bar{J}_X(\underline{q}) = \bar{J}_X(\underline{q}')$, Theorem A provides a Fourier–Mukai autoequivalence on any fine compactified Jacobian of a reduced curve with planar singularities, thus extending the classical result of Mukai [37] for Jacobians of smooth curves and the more recent result of Arinkin [5, Theorem C] for compactified Jacobians of integral curves with planar singularities. Note that in loc. cit. Arinkin states his result under the assumption that $\operatorname{char}(k) = 0$; however it was observed by Groechenig in [17, Theorem 4.8] that Arinkin's proof works verbatim also under the assumption that $\operatorname{char}(k) > 2p_a(X) - 1$. Our proof of Theorem A uses twisted Abel maps (see (3-9)) instead of the global Abel map used by Arinkin for integral curves; this explains why we are able to improve the hypothesis on the characteristic of the base field even for integral curves. As a consequence, it follows that all the results of [17, Section 4] are true under the weaker assumption that $\operatorname{char}(k) \geq n^2(h-1)+1$.

The above Fourier–Mukai autoequivalence provides further evidence for the classical limit of the (conjectural) geometric Langlands correspondence for the general linear group GL_r, as formulated by Donagi–Pantev in [11]. More precisely, in loc. cit. the authors conjectured that there should exist a Fourier–Mukai autoequivalence, induced by a suitable Poincaré sheaf, of the derived category of the moduli stack of Higgs bundles. Moreover, among other properties, such a Fourier–Mukai autoequivalence is expected to induce an autoequivalence of the derived category of the fibers of the Hitchin map. Since the fibers of the Hitchin map can be described in terms of compactified Jacobians of spectral curves (see Melo, Rapagnetta and Viviani [34, Appendix] for the precise description), it is natural to expect that such a Fourier–Mukai autoequivalence should exist on each fine compactified Jacobian of a spectral curve, which has always planar singularities. Our Theorem A shows that this is indeed the case for reduced spectral curves (ie over the so called regular locus of the Hitchin map), extending the result of Arinkin for integral spectral curves, ie over the so-called elliptic locus of the Hitchin map.

On the other hand, in the general case when $\bar{J}_X(\underline{q})$ is different from $\bar{J}_X(\underline{q}')$ (and possibly nonisomorphic to it; see the examples in [33]), Theorem A implies that $\bar{J}_X(\underline{q})$ and $\bar{J}_X(\underline{q}')$ (which are birational Calabi–Yau singular projective varieties by what we have said above) are derived equivalent via a canonical Fourier–Mukai transform. This result seems to suggest an extension to singular varieties of the conjecture of Kawamata [30], which predicts that birational Calabi–Yau smooth projective varieties should be derived equivalent.

We point out that a topological counterpart of the above result is obtained by L Migliorini, V Shende and the third author in [36]: any two fine compactified Jacobians of X (under the same assumptions on X) have the same perverse Leray filtration on their cohomology. This result again seems to suggest an extension to (mildly) singular varieties of the result of Batyrev [7] which says that birational Calabi–Yau smooth projective varieties have the same Betti numbers.

As a corollary of Theorem A, we can generalize the autoduality result of Arinkin [5, Theorem B] for compactified Jacobians of integral curves, which extends the previous result of Esteves and Kleimann [14] for integral curves with nodes and cusps. In order to state our autoduality result, we need first to introduce some notation.

For a projective k–scheme Z, denote by $\mathrm{Spl}(Z)$ the (possibly nonseparated) algebraic space, locally of finite type over k, parametrizing simple sheaves on Z; see [3, Theorem 7.4]. Denote by $\mathrm{Pic}^{=}(Z) \subseteq \mathrm{Spl}(Z)$ the open subset parametrizing simple, torsion-free sheaves having rank 1 on each irreducible component of Z, and by $\mathrm{Pic}^{-}(Z) \subseteq \mathrm{Pic}^{=}(Z)$ the open subset parametrizing simple, Cohen–Macaulay sheaves having rank 1 on each irreducible component of Z; see [3, Proposition 5.13]. If Z does not have embedded components (or, equivalently, if the structure sheaf $\mathcal{O}_Z$ is torsion-free) then $\mathrm{Pic}^{=}(Z)$ contains the Picard group scheme $\mathrm{Pic}(Z)$ of Z as an open subset; under this hypothesis, we will denote by $\overline{\mathrm{Pic}}^{\mathrm{o}}(Z)$ the connected component of $\mathrm{Pic}^{=}(Z)$ that contains $\mathcal{O}_Z \in \mathrm{Pic}(Z) \subseteq \mathrm{Pic}^{=}(Z)$. Clearly, $\overline{\mathrm{Pic}}^{\mathrm{o}}(Z)$ contains as an open subset the connected component $\mathrm{Pic}^{\mathrm{o}}(Z)$ of $\mathrm{Pic}(Z)$ that contains $\mathcal{O}_Z \in \mathrm{Pic}(Z)$.

If X is a projective reduced curve with locally planar singularities, then $\mathrm{Pic}^{-}(X) = \mathrm{Pic}^{=}(X)$ (since on a curve torsion-free sheaves are also Cohen–Macaulay) is known to be a scheme, which is denoted by $\overline{\mathbb{J}}_X$ in Section 2.1, and $\overline{\mathrm{Pic}}^{\mathrm{o}}(X)$ is contained in the subscheme $\overline{\mathbb{J}}_X^{1-g} \subset \overline{\mathbb{J}}_X$ parametrizing torsion-free rank-1 sheaves on X of Euler characteristic $1 - p_a(X)$ (or equivalently degree 0); see Section 2.1. Note that every fine compactified Jacobian $\overline{J}_X(\underline{q})$ of X such that $|\underline{q}| = 1 - p_a(X)$ is an open and proper subscheme of $\overline{\mathbb{J}}_X^{1-p_a(X)}$ (see Section 2.1) and that the Poincaré sheaf considered above is actually a restriction of a Cohen–Macaulay Poincaré sheaf $\overline{\mathcal{P}}$ on $\overline{\mathbb{J}}_X^0 \times \overline{\mathbb{J}}_X^0$ (see Section 4).

In our previous paper [34, Theorem C] we proved that there is an isomorphism of algebraic groups

$$\beta_{\underline{q}}\colon J(X) = \mathrm{Pic}^{\mathrm{o}}(X) \to \mathrm{Pic}^{\mathrm{o}}(\overline{J}_X(\underline{q})), \quad L \mapsto \mathcal{P}_L := \mathcal{P}|_{\overline{J}_X(\underline{q}) \times \{L\}}.$$

In this paper, we prove the following theorem, which can be seen as a natural generalization of the above autoduality result.

Theorem B *Let X be a reduced connected projective curve with planar singularities over an algebraically closed field k of characteristic either zero or bigger than the arithmetic genus $p_a(X)$ of X. Let $\overline{J}_X(\underline{q})$ be a fine compactified Jacobian of X with $|\underline{q}| = 1 - p_a(X)$. Then the morphism*

$$\rho_{\underline{q}}\colon \overline{\mathrm{Pic}}^{\mathrm{o}}(X) \to \overline{\mathrm{Pic}}^{\mathrm{o}}(\overline{J}_X(\underline{q})), \quad I \mapsto \overline{\mathcal{P}}_I := \overline{\mathcal{P}}|_{\overline{J}_X(\underline{q}) \times \{I\}}, \tag{1-1}$$

is an open embedding, which is equivariant with respect to the isomorphism of algebraic groups $\beta_{\underline{q}}\colon J(X) = \mathrm{Pic}^{\mathrm{o}}(X) \xrightarrow{\cong} \mathrm{Pic}^{\mathrm{o}}(\overline{J}_X(\underline{q}))$, where $\mathrm{Pic}^{\mathrm{o}}(X)$ (resp. $\mathrm{Pic}^{\mathrm{o}}(\overline{J}_X(\underline{q}))$) acts on $\overline{\mathrm{Pic}}^{\mathrm{o}}(X)$ (resp. on $\overline{\mathrm{Pic}}^{\mathrm{o}}(\overline{J}_X(\underline{q}))$) by tensor product. Moreover:

- (i) *The image of $\rho_{\underline{q}}$ is contained in $\mathrm{Pic}^{-}(\overline{J}_X(\underline{q})) \cap \overline{\mathrm{Pic}}^{\mathrm{o}}(\overline{J}_X(\underline{q}))$.*
- (ii) *The morphism $\rho_{\underline{q}}$ induces a morphism of algebraic groups*
$$\rho_{\underline{q}}\colon \overline{\mathrm{Pic}}^{\mathrm{o}}(X) \cap \mathrm{Pic}(X) \to \overline{\mathrm{Pic}}^{\mathrm{o}}(\overline{J}_X(\underline{q})) \cap \mathrm{Pic}(\overline{J}_X(\underline{q})).$$
- (iii) *If every singular point of X that lies on at least two different irreducible components of X is a separating node (eg if X is an irreducible curve or a nodal curve of compact type) then $\rho_{\underline{q}}$ is an isomorphism between integral projective varieties.*

Theorem B(iii) is a slight generalization of the result for irreducible curves proved by Arinkin in [5, Theorem B]. It would be interesting to know if $\rho_{\underline{q}}$ is an isomorphism for any reduced curve X with locally planar singularities.

1.3 Sketch of the proof of Theorem A

Let us now give a brief outline of the proof of Theorem A.

Using the well-known description of the kernel of a composition of two integral transforms, Theorem A is equivalent to the following equality in $D^b_{\mathrm{coh}}(\bar{J}_X(\underline{q}) \times \bar{J}_X(\underline{q}))$:

$$\Psi[g] := Rp_{13*}(p_{12}^*((\bar{\mathcal{P}})^\vee) \otimes^{\boldsymbol{L}} p_{23}^*(\bar{\mathcal{P}}))[g] \cong \mathcal{O}_\Delta, \tag{1-2}$$

where p_{ij} denotes the projection of $\bar{J}_X(\underline{q}) \times \bar{J}_X(\underline{q}') \times \bar{J}_X(\underline{q})$ onto the i^{th} and j^{th} factors and $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal $\Delta \subset \bar{J}_X(\underline{q}) \times \bar{J}_X(\underline{q})$.

In order to prove (1-2), the key idea, which we learned from Arinkin in [4; 5], is to prove a similar formula for the effective semiuniversal deformation[2] family $\pi\colon \mathcal{X} \to \operatorname{Spec} R_X$ of the curve X. The fine compactified Jacobians $\bar{J}_X(\underline{q})$ and $\bar{J}_X(\underline{q}')$ deform over $\operatorname{Spec} R_X$ to the universal fine compactified Jacobians $\bar{J}_{\mathcal{X}}(\underline{q})$ and $\bar{J}_{\mathcal{X}}(\underline{q}')$, respectively; see Section 2.2. Moreover, the Poincaré sheaf $\bar{\mathcal{P}}$ on $\bar{J}_X(\underline{q}) \times \bar{J}_X(\underline{q}')$ deforms to a universal Poincaré sheaf $\bar{\mathcal{P}}^{\mathrm{un}}$ on the fiber product $\bar{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q}')$. Equation (1-2) will follow, by restricting to the central fiber, from the following universal version of it, which we prove in Theorem 6.2:

$$\begin{aligned}\Psi^{\mathrm{un}}[g] &:= Rp_{13*}(p_{12}^*((\bar{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes^{\boldsymbol{L}} p_{23}^*(\bar{\mathcal{P}}^{\mathrm{un}}))[g] \\ &\cong \mathcal{O}_{\Delta^{\mathrm{un}}} \in D^b_{\mathrm{coh}}(\bar{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q})),\end{aligned} \tag{1-3}$$

where p_{ij} denotes the projection of $\bar{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q}') \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q})$ onto the i^{th} and j^{th} factors and $\mathcal{O}_{\Delta^{\mathrm{un}}}$ is the structure sheaf of the universal diagonal $\Delta^{\mathrm{un}} \subset \bar{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q})$.

A key intermediate step in proving (1-3) consists of showing:

(∗) $\Psi^{\mathrm{un}}[g]$ *is a Cohen–Macaulay sheaf such that* $\operatorname{supp} \Psi^{\mathrm{un}}[g] = \Delta^{\mathrm{un}}$.

[2] There Arinkin considered the stack of all (integral) curves with planar singularities. Here (and in our previous related papers [33; 34]), we need to work with the semiuniversal deformation space of X in order to be able to define universal fine compactified Jacobians with respect to any general polarization on the central fiber; see Section 2.2.

The two main ingredients in proving (∗) are the equigeneric stratification of Spec R_X (see Fact 2.9) and a lower bound for the codimension of the support of the restriction of $\Psi^{\mathrm{un}}[g]$ on the fibers of $\bar{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q}) \to \operatorname{Spec} R_X$ (see Proposition 6.3).

Outline of the paper

The paper is organized as follows.

In Section 2.1 we collect several facts about fine compactified Jacobians of reduced curves, with special emphasis on the case of curves with planar singularities. In Section 2.2 we recall some facts about deformation theory that will be crucial in the proof of Theorem A: the equigeneric stratification of the semiuniversal deformation space of a curve with planar singularities (Fact 2.9) and the universal fine compactified Jacobians (Fact 2.10).

Section 3 is devoted to Hilbert schemes of points on smooth surfaces and on curves with planar singularities. More precisely, in Section 3.1 we recall some classical facts about the Hilbert scheme of points on a smooth surface and on the Hilbert–Chow morphism together with the recent results of Haiman on the isospectral Hilbert scheme. In Section 3.2, we recall some facts about the Hilbert scheme of a curve X with planar singularities and on the local Abel map from the Hilbert scheme of X to any fine compactified Jacobian of X.

In Section 4, we define the Poincaré sheaf $\bar{\mathcal{P}}$ and we prove that it is a maximal Cohen–Macaulay sheaf, flat over each factor; see Theorem 4.6. The proof of Theorem 4.6 is based on the work of Arinkin [5], which uses in a crucial way the properties of Haiman's isospectral Hilbert scheme of a surface.

In Section 5, we establish several properties of the Poincaré sheaf $\bar{\mathcal{P}}$, while Section 6 contains the proofs of Theorem A and Theorem B.

Notation

The following notation will be used throughout the paper:

- Unless otherwise stated, k will denote an algebraically closed field (of arbitrary characteristic). All *schemes* are k–schemes, and all morphisms are implicitly assumed to respect the k–structure.
- A *curve* is a *reduced* projective scheme over k of pure dimension 1.

Given a curve X, we denote by X_{sm} the smooth locus of X, by X_{sing} its singular locus and by $\nu\colon X^\nu \to X$ the normalization morphism. We denote by γ_X, or simply by γ where there is no danger of confusion, the number of irreducible components of X.

We denote by $p_a(X)$ the *arithmetic genus* of X, ie

$$p_a(X) := 1 - \chi(\mathcal{O}_X) = 1 - h^0(X, \mathcal{O}_X) + h^1(X, \mathcal{O}_X).$$

We denote by $g^\nu(X)$ the *geometric genus* of X, ie the sum of the genera of the connected components of the normalization X^ν, and by $p_a^\nu(X)$ the arithmetic genus of the normalization X^ν of X. Note that $p_a^\nu(X) = g^\nu(X) + 1 - \gamma_X$.

- A *subcurve* Z of a curve X is a closed k–scheme $Z \subseteq X$ that is reduced and of pure dimension 1. We say that a subcurve $Z \subseteq X$ is nontrivial if $Z \neq \varnothing, X$.

Given two subcurves Z and W of X without common irreducible components, we denote by $Z \cap W$ the 0–dimensional subscheme of X that is obtained as the scheme-theoretic intersection of Z and W and we denote by $|Z \cap W|$ its length.

Given a subcurve $Z \subseteq X$, we denote by $Z^c := \overline{X \setminus Z}$ the *complementary subcurve* of Z and we set $\delta_Z = \delta_{Z^c} := |Z \cap Z^c|$.

- A curve X is called *Gorenstein* if its dualizing sheaf ω_X is a line bundle.

- A curve X has *locally complete intersection* (*lci*) *singularities at* $p \in X$ if the completion $\widehat{\mathcal{O}}_{X,p}$ of the local ring of X at p can be written as

$$\widehat{\mathcal{O}}_{X,p} = k[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_{r-1})$$

for some $r \geq 2$ and some $f_i \in k[\![x_1, \dots, x_r]\!]$. A curve X has locally complete intersection (lci) singularities if X is lci at every $p \in X$. Clearly, a curve with lci singularities is Gorenstein.

- A curve X has *planar singularities at* $p \in X$ if the completion $\widehat{\mathcal{O}}_{X,p}$ of the local ring of X at p has embedded dimension two, or equivalently if it can be written as

$$\widehat{\mathcal{O}}_{X,p} = k[\![x, y]\!]/(f)$$

for a reduced series $f = f(x, y) \in k[\![x, y]\!]$. A curve X has planar singularities if X has planar singularities at every $p \in X$. Clearly, a curve with planar singularities has lci singularities, hence it is Gorenstein.

- Given a curve X, the *generalized Jacobian* of X, denoted by $J(X)$ or by $\mathrm{Pic}^{\underline{0}}(X)$, is the algebraic group whose k–valued points are the group of line bundles on X

of multidegree $\underline{0}$ (ie having degree 0 on each irreducible component of X) together with the multiplication given by the tensor product. The generalized Jacobian of X is a connected commutative smooth algebraic group of dimension equal to $h^1(X, \mathcal{O}_X)$ and it coincides with the connected component of the Picard scheme $\mathrm{Pic}(X)$ of X containing the identity.

- Given a scheme Y, we will denote by $D(Y)$ the *derived category* of complexes of $\mathcal{O}_Y$–modules with quasicoherent cohomology sheaves and by $D^b(Y) \subset D(Y)$ the *bounded derived category* consisting of complexes with only finitely many nonzero cohomology sheaves. We denote by $D_{\mathrm{coh}}(Y) \subset D(Y)$ (resp. $D^b_{\mathrm{coh}}(Y) \subset D^b(Y)$) the full category consisting of complexes with coherent cohomology and by $D_{\mathrm{qcoh}}(Y) \subset D(Y)$ (resp. $D^b_{\mathrm{qcoh}}(Y) \subset D^b(Y)$) the full category consisting of complexes with quasicoherent cohomology.

- Given a scheme Y and a closed point $y \in Y$, we will denote by $\boldsymbol{k}(y)$ the *skyscraper sheaf* supported at y.

2 Fine compactified Jacobians and their universal deformations

The aim of this section is to summarize some properties of fine (universal) compactified Jacobians of connected reduced curves with planar singularities which were proved in [33; 34]. Throughout this section, we fix a connected reduced curve X over an algebraically closed field k.

2.1 Fine compactified Jacobians

We begin by reviewing the definition and the main properties of fine compactified Jacobians of reduced curves with planar singularities, referring to [33, Section 2] for complete proofs.

Fine compactified Jacobians of a curve X will parametrize sheaves on X of a certain type, which we now define.

Definition 2.1 A coherent sheaf I on a connected reduced curve X is said to be

(i) *rank-*1 if I has generic rank 1 at every irreducible component of X,

(ii) *torsion-free* if $\mathrm{Supp}(I) = X$ and every nonzero subsheaf $J \subseteq I$ is such that $\dim \mathrm{Supp}(J) = 1$,

(iii) *simple* if $\mathrm{End}_k(I) = k$.

Note that any line bundle on X is a simple rank-1 torsion-free sheaf.

If the curve X is Gorenstein, then rank-1 torsion-free sheaves on X correspond to linear equivalence classes of generalized divisors in the sense of Hartshorne; see [25, Proposition 2.8]. This allows us to describe these sheaves in terms of (usual) effective divisors as follows.

Lemma 2.2 *Let X be a (reduced) Gorenstein curve and let I be a rank-1 torsion-free sheaf on X. Then there exist two disjoint effective divisors E_1 and E_2 on X, with E_2 being a Cartier divisor supported on the smooth locus of X, such that*

$$I = I_{E_1} \otimes I_{E_2}^{-1},$$

where I_{E_i} denotes the ideal sheaf of E_i for $i = 1, 2$.

Proof It follows from [25, Proposition 2.11] that we can write $I = I_{E_1} \otimes I_{E_2}^{-1}$ for two effective divisors E_1 and E_2 on X such that E_2 is Cartier and linearly equivalent to an arbitrary high power of a fixed ample line bundle. Thus, up to by replacing E_2 with a divisor linearly equivalent to it, we can assume that the support of E_2 is disjoint from the singular locus of X and from the support of E_1. □

Rank-1 torsion-free simple sheaves on X can be parametrized by a scheme. More precisely, there exists a k–scheme $\overline{\mathbb{J}}_X$, locally of finite type and universally closed over k, which represents the Zariski (or, equivalently, étale or fppf) sheafification of the functor

$$\overline{\mathbb{J}}_X^*\colon \{\text{schemes}/k\} \to \{\text{sets}\}$$

which associates to a k–scheme T the set of isomorphism classes of T–flat, coherent sheaves on $X \times_k T$ whose fibers over T are simple rank-1 torsion-free sheaves. The fact that $\overline{\mathbb{J}}_X$ represents the Zariski sheafification of the functor $\overline{\mathbb{J}}_X^*$ amounts to the existence of a coherent sheaf $\mathcal{I}$ on $X \times \overline{\mathbb{J}}_X$, flat over $\overline{\mathbb{J}}_X$, such that for every $\mathcal{F} \in \overline{\mathbb{J}}_X^*(T)$ there exists a unique map $\alpha_{\mathcal{F}}\colon T \to \overline{\mathbb{J}}_X$ with the property that $\mathcal{F} = (\mathrm{id}_X \times \alpha_{\mathcal{F}})^*(\mathcal{I}) \otimes \pi_2^*(N)$ for some $N \in \mathrm{Pic}(T)$, where $\pi_2\colon X \times T \to T$ is the projection onto the second factor. The sheaf $\mathcal{I}$ is uniquely determined up to tensor product with the pullback of an invertible sheaf on $\overline{\mathbb{J}}_X$ and it is called a *universal sheaf*. Moreover, there exists a k–smooth open subset $\mathrm{Pic}(X) = \mathbb{J}_X \subseteq \overline{\mathbb{J}}_X$, whose k–points parametrize line bundles on X. The restriction of a universal sheaf $\mathcal{I}$ to $X \times \mathbb{J}_X$ is a line bundle that enjoys a similar universal property with respect to families of line bundles on X. A proof of the above results can be found in [33, Fact 2.2], where they are deduced from results of Murre–Oort, Altmann–Kleiman [3; 2] and Esteves [12].

Since the Euler characteristic $\chi(I) := h^0(X,I) - h^1(X,I)$ of a sheaf I on X is constant under deformations, we get a decomposition

$$\overline{\mathbb{J}}_X = \coprod_{\chi\in\mathbb{Z}} \overline{\mathbb{J}}_X^{\chi}, \quad \mathbb{J}_X = \coprod_{\chi\in\mathbb{Z}} \mathbb{J}_X^{\chi} = \coprod_{\chi\in\mathbb{Z}} \operatorname{Pic}^{\chi+p_a(X)-1}(X), \tag{2-1}$$

where $\overline{\mathbb{J}}_X^{\chi}$ (resp. $\mathbb{J}_X^{\chi}$) denotes the open and closed subscheme of $\overline{\mathbb{J}}_X$ (resp. $\mathbb{J}_X$) parametrizing simple rank-1 torsion-free sheaves I (resp. line bundles L) such that $\chi(I) = \chi$ (resp. $\chi(L) = \chi$, or equivalently $\deg(L) = \chi + p_a(X) - 1$). We will sometimes refer to the degree of a rank-1 torsion-free sheaf I, which is defined by $\deg I := \chi(I) + p_a(X) - 1$.

If X has planar singularities, then $\overline{\mathbb{J}}_X$ has the following properties.

Fact 2.3 *Let X be a connected reduced curve with planar singularities. Then*:

(i) *$\overline{\mathbb{J}}_X$ is a reduced scheme with locally complete intersection singularities.*

(ii) *$\mathbb{J}_X$ is the smooth locus of $\overline{\mathbb{J}}_X$. In particular, $\mathbb{J}_X$ is dense in $\overline{\mathbb{J}}_X$.*

Proof See [33, Theorem 2.3]. □

For any integer $\chi \in \mathbb{Z}$, the scheme $\overline{\mathbb{J}}_X^{\chi}$ is neither of finite type nor separated over k if X is not irreducible. However, it can be covered by open subsets that are proper (and even projective) over k: the fine compactified Jacobians of X. The fine compactified Jacobians depend on the choice of a general polarization, whose definition is as follows (using the notation of [33]).

Definition 2.4 Let X be a connected reduced curve.

(1) A *polarization* on a connected curve X is a tuple of rational numbers $\underline{q} = \{\underline{q}_{C_i}\}$, one for each irreducible component C_i of X, such that $|\underline{q}| := \sum_i \underline{q}_{C_i} \in \mathbb{Z}$. We call $|\underline{q}|$ the total degree of $\underline{q}$. Given any subcurve $Y \subseteq X$, we set $\underline{q}_Y := \sum_j \underline{q}_{C_j}$, where the sum runs over all the irreducible components C_j of Y.

(2) A polarization $\underline{q}$ is called *integral* at a subcurve $Y \subseteq X$ if $\underline{q}_Z \in \mathbb{Z}$ for any connected component Z of Y and of Y^c. A polarization is called *general* if it is not integral at any nontrivial subcurve $Y \subset X$.

The choice of a polarization on X allows us to define the concepts of stability and semistability.

Definition 2.5 Let $\underline{q}$ be a polarization on X and let I be a torsion-free rank-1 sheaf on X of Euler characteristic $\chi(I) = |\underline{q}|$ (not necessarily simple). We say that I is *(semi)stable* with respect to $\underline{q}$, or simply $\underline{q}$–(semi)stable, if for every nontrivial subcurve $Y \subset X$, we have that

$$\chi(I_Y) \geq \underline{q}_Y \quad (\text{resp.} >), \tag{2-2}$$

where I_Y is the quotient of the restriction $I|_Y$ modulo its biggest zero-dimensional subsheaf (or, in other words, I_Y is the biggest torsion-free quotient of $I|_Y$).

Given a polarization $\underline{q}$ on X, we denote by $\bar{J}_X^{\mathrm{ss}}(\underline{q})$ (resp. $\bar{J}_X^{\mathrm{s}}(\underline{q})$) the subscheme of $\bar{\mathbb{J}}_X$ parametrizing simple rank-1 torsion-free sheaves I on X which are $\underline{q}$–semistable (resp. $\underline{q}$–stable). By [12, Proposition 34], the inclusions

$$\bar{J}_X^{\mathrm{s}}(\underline{q}) \subseteq \bar{J}_X^{\mathrm{ss}}(\underline{q}) \subset \bar{\mathbb{J}}_X$$

are open.

Fact 2.6 (Esteves) *Let X be a connected reduced curve.*

(i) *If $\underline{q}$ is general then $\bar{J}_X^{\mathrm{ss}}(\underline{q}) = \bar{J}_X^{\mathrm{s}}(\underline{q})$ is a projective scheme over k (not necessarily reduced).*

(ii) $\bar{\mathbb{J}}_X = \bigcup_{\underline{q} \text{ general}} \bar{J}_X^{s}(\underline{q})$.

Proof See [33, Fact 2.19]. □

If $\underline{q}$ is general, we set $\bar{J}_X(\underline{q}) := \bar{J}_X^{\mathrm{ss}}(\underline{q}) = \bar{J}_X^{\mathrm{s}}(\underline{q})$ and we call it the *fine compactified Jacobian* with respect to the polarization $\underline{q}$. We denote by $J_X(\underline{q})$ the open subset of $\bar{J}_X(\underline{q})$ parametrizing line bundles on X. Note that $J_X(\underline{q})$ is isomorphic to the disjoint union of a certain number of copies of the generalized Jacobian $J(X) = \mathrm{Pic}^{\underline{0}}(X)$ of X.

If X has planar singularities, then any fine compactified Jacobian of X enjoys the following properties.

Fact 2.7 *Let X be a connected reduced curve with planar singularities and $\underline{q}$ a general polarization on X. Then:*

(i) *$\bar{J}_X(\underline{q})$ is a connected reduced scheme with locally complete intersection singularities and trivial dualizing sheaf.*

(ii) *The smooth locus of* $\overline{J}_X(\underline{q})$ *coincides with the open subset* $J_X(\underline{q}) \subseteq \overline{J}_X(\underline{q})$ *parametrizing line bundles; in particular,* $J_X(\underline{q})$ *is dense in* $\overline{J}_X(\underline{q})$ *and of pure dimension equal to* $p_a(X)$*.*

(iii) $J_X(\underline{q})$ *is the disjoint union of a number of copies of* $J(X)$*, and such a number is independent of the chosen polarization* $\underline{q}$ *and it is denoted by* $c(X)$*. In particular, all the fine compactified Jacobians of* X *have* $c(X)$ *irreducible components, all of dimension equal to the arithmetic genus* $p_a(X)$ *of* X*, and they are all birational among them.*

Proof See [33, Theorem A]. □

In [33, Section 5.1], we prove a formula for the number $c(X)$ (which is called the complexity of X) in terms of the combinatorics of the curve X. The above properties rely heavily on the fact that the curve X has planar singularities and indeed we expect that many of the above properties fail to hold without this assumption; see the discussion in [33, Remark 2.7].

2.2 Universal fine compactified Jacobians

The aim of this subsection is to introduce and describe universal fine compactified Jacobians following the presentation given in [33, Sections 4 and 5; 34, Section 3].

Consider the effective semiuniversal deformation of a reduced curve X (in the sense of [40]),

(2-3)
$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow \pi \\ o := [\mathfrak{m}_X] & \hookrightarrow & \operatorname{Spec} R_X \end{array}$$

where R_X is a Noetherian complete local k–algebra with maximal ideal $\mathfrak{m}_X$ and residue field k. Note that if X has locally complete intersection singularities (eg if X has planar singularities), then $\operatorname{Spec} R_X$ is formally smooth or, equivalently, R_X is a power series ring over k; see eg [33, Fact 4.1] and the references therein. For any (schematic) point $s \in \operatorname{Spec} R_X$, we will denote by $\mathcal{X}_s := \pi^{-1}(s)$ the fiber of π over s and by $\mathcal{X}_{\overline{s}} := \mathcal{X}_s \otimes_{k(s)} \overline{k(s)}$ an associated geometric fiber. For later use, we recall the following.

Lemma 2.8 *Let* U *be the open subset of* $\operatorname{Spec} R_X$ *consisting of all the points* $s \in \operatorname{Spec} R_X$ *such that the fiber* $\mathcal{X}_{\overline{s}}$ *of the universal family* $\pi\colon \mathcal{X} \to \operatorname{Spec} R_X$ *is smooth or* *has a unique singular point which is a node. If X has locally planar singularities, then the codimension of the complement of U inside* $\operatorname{Spec} R_X$ *is at least two.*

Proof See [33, Lemma 4.3]. □

The scheme $\operatorname{Spec} R_X$ admits two stratifications into closed subsets according to either the arithmetic genus or the geometric genus of the normalization of the geometric fibers of the family π. More precisely, using the notation introduced in (1.2), consider the two functions

(2-4)
$$p_a^\nu\colon \operatorname{Spec} R_X \to \mathbb{N}, \quad s \mapsto p_a^\nu(\mathcal{X}_{\bar{s}}) := p_a(\mathcal{X}_{\bar{s}}^\nu),$$
$$g^\nu\colon \operatorname{Spec} R_X \to \mathbb{N}, \quad s \mapsto g^\nu(\mathcal{X}_{\bar{s}}) = g^\nu(\mathcal{X}_{\bar{s}}^\nu).$$

Since the number of connected components of $\mathcal{X}_{\bar{s}}^\nu$ is the number $\gamma(\mathcal{X}_{\bar{s}})$ of irreducible components of $\mathcal{X}_{\bar{s}}$, we have the relation

(2-5)
$$p_a^\nu(\mathcal{X}_{\bar{s}}) = g^\nu(\mathcal{X}_{\bar{s}}) - \gamma(\mathcal{X}_{\bar{s}}) + 1 \le g^\nu(\mathcal{X}_{\bar{s}}).$$

The functions p_a^ν and g^ν are lower semicontinuous; see [34, Lemma 3.2]. Moreover, using (2-5) and the fact that the arithmetic genus p_a stays constant in the family π because of flatness, we get that

$$p_a(X^\nu) = p_a^\nu(X) \le p_a^\nu(\mathcal{X}_{\bar{s}}) \le g^\nu(\mathcal{X}_{\bar{s}}) \le p_a(\mathcal{X}_{\bar{s}}) = p_a(X).$$

Therefore for any $p_a(X^\nu) \le l \le p_a(X)$ we have two closed subsets of $\operatorname{Spec} R_X$,

(2-6)
$$(\operatorname{Spec} R_X)^{g^\nu \le l} := \{s \in \operatorname{Spec} R_X : g^\nu(\mathcal{X}_{\bar{s}}) \le l\}$$
$$\subseteq (\operatorname{Spec} R_X)^{p_a^\nu \le l} := \{s \in \operatorname{Spec} R_X : p_a^\nu(\mathcal{X}_{\bar{s}}) \le l\}.$$

If X has planar singularities, then the stratification (called *equigeneric stratification*) by the latter closed subsets has the following remarkable properties.

Fact 2.9 *Assume that X is a reduced curve with planar singularities. Then, for any $p_a(X^\nu) \le l \le p_a(X)$, we have:*

(i) *The codimension of the closed subset* $(\operatorname{Spec} R_X)^{p_a^\nu \le l} \subset \operatorname{Spec} R_X$ *is at least $p_a(X) - l$. Hence, the same is true for the closed subset* $(\operatorname{Spec} R_X)^{g^\nu \le l}$.

(ii) *For each generic point s of* $(\operatorname{Spec} R_X)^{p_a^\nu \le l}$, *$\mathcal{X}_{\bar{s}}$ is a nodal curve.*

Proof See [34, Theorem 3.3]. □

The schemes $\mathbb{J}_X \subseteq \overline{\mathbb{J}}_X$ of Section 2.1 can be deformed over $\operatorname{Spec} R_X$. More precisely, there a scheme $\overline{\mathbb{J}}_{\mathcal{X}}$ endowed with a morphism $u\colon \overline{\mathbb{J}}_{\mathcal{X}} \to \operatorname{Spec} R_X$, which is locally of finite type and universally closed, and which represents the Zariski (or, equivalently, étale or fppf) sheafification of the functor

$$\overline{\mathbb{J}}^*_{\mathcal{X}}\colon \{\operatorname{Spec} R_X\text{–schemes}\} \to \{\text{sets}\}$$

which sends a scheme $T \to \operatorname{Spec} R_X$ to the set of isomorphism classes of T–flat, coherent sheaves on $\mathcal{X}_T := T \times_{\operatorname{Spec} R_X} \mathcal{X}$ whose fibers over T are simple rank-1 torsion-free sheaves. The fact that $\overline{\mathbb{J}}_{\mathcal{X}}$ represents the Zariski sheafification of the functor $\overline{\mathbb{J}}^*_{\mathcal{X}}$ amounts to the existence of a coherent sheaf $\hat{I}$ on $\mathcal{X} \times_{\operatorname{Spec} R_X} \overline{\mathbb{J}}_{\mathcal{X}}$, flat over $\overline{\mathbb{J}}_{\mathcal{X}}$, such that for every $\mathcal{F} \in \overline{\mathbb{J}}^*_{\mathcal{X}}(T)$ there exists a unique $\operatorname{Spec} R_X$–map $\alpha_{\mathcal{F}}\colon T \to \overline{\mathbb{J}}_{\mathcal{X}}$ with the property that $\mathcal{F} = (\mathrm{id}_{\mathcal{X}} \times \alpha_{\mathcal{F}})^*(\hat{I}) \otimes \pi_2^*(N)$ for some $N \in \operatorname{Pic}(T)$, where $\pi_2\colon \mathcal{X} \times_{\operatorname{Spec} R_X} T \to T$ is the projection onto the second factor. The sheaf $\hat{I}$ is uniquely determined up to tensor product with the pullback of an invertible sheaf on $\overline{\mathbb{J}}_{\mathcal{X}}$ and it is called a *universal sheaf* on $\overline{\mathbb{J}}_{\mathcal{X}}$. Moreover, there exists an open subscheme $\mathbb{J}_{\mathcal{X}} \subseteq \overline{\mathbb{J}}_{\mathcal{X}}$ which is smooth over $\operatorname{Spec} R_X$ and parametrizes families of line bundles on the family $\pi\colon \mathcal{X} \to \operatorname{Spec} R_X$. Furthermore, the geometric fiber of $\overline{\mathbb{J}}_{\mathcal{X}}$ (resp. of $\mathbb{J}_{\mathcal{X}}$) over $s \in \operatorname{Spec} R_X$ is isomorphic to $\overline{\mathbb{J}}_{\mathcal{X}_{\bar{s}}}$ (resp. $\mathbb{J}_{\mathcal{X}_{\bar{s}}}$), and the pullback of $\hat{I}$ to $\mathcal{X}_{\bar{s}} \times \overline{\mathbb{J}}_{\mathcal{X}_{\bar{s}}}$ is a universal sheaf for $\overline{\mathbb{J}}_{\mathcal{X}_{\bar{s}}}$. In particular, the fiber of $\overline{\mathbb{J}}_{\mathcal{X}}$ (resp. of $\mathbb{J}_{\mathcal{X}}$) over the closed point $o \in \operatorname{Spec} R_X$ is isomorphic to $\overline{\mathbb{J}}_X$ (resp. $\mathbb{J}_X$), and the restriction of $\hat{I}$ to $X \times \overline{\mathbb{J}}_X$ is equal to a universal sheaf as in Section 2.1. A proof of the above results can be found in [33, Fact 4.1], where they are deduced from results of Altmann–Kleiman [3; 2] and Esteves [12].

We now introduce universal fine compactified Jacobians, which are certain open subsets of $\overline{\mathbb{J}}_{\mathcal{X}}$ that are projective over $\operatorname{Spec} R_X$ and whose central fiber is a fine compactified Jacobian of X. The universal fine compactified Jacobian will depend on a general polarization $\underline{q}$ on X as in Definition 2.4. Indeed, the polarization $\underline{q}$ induces a polarization on each geometric fiber of the effective semiuniversal deformation family $\pi\colon \mathcal{X} \to \operatorname{Spec} R_X$, in the following way. For any (schematic) point $s \in \operatorname{Spec} R_X$, denote by $\psi_s\colon \mathcal{X}_{\bar{s}} \to \mathcal{X}_s$ the natural base change map. There is a natural map

$$\Sigma_s\colon \{\text{subcurves of } \mathcal{X}_{\bar{s}}\} \to \{\text{subcurves of } X\}, \quad \mathcal{X}_{\bar{s}} \supseteq Z \mapsto \overline{\psi_s(Z)} \cap X \subseteq X, \tag{2-7}$$

where $\overline{\psi_s(Z)}$ is the Zariski closure inside $\mathcal{X}$ of the subcurve $\psi_s(Z) \subseteq \mathcal{X}_s$, and the intersection $\overline{\psi_s(Z)} \cap X$ is endowed with the reduced scheme structure; see [33, Section 5] for more details. Using the above map Σ_s, we can define a polarization $\underline{q}^s$ on $\mathcal{X}_{\bar{s}}$

starting from a polarization $\underline{q}$ on X by the rule

$$q^s_Z := \underline{q}_{\Sigma_s(Z)} \quad \text{for any subcurve } Z \subseteq \mathcal{X}_{\overline{s}}. \tag{2-8}$$

It turns out that if $\underline{q}$ is a general polarization on X, then $\underline{q}^s$ is a general polarization on $\mathcal{X}_{\overline{s}}$ for any point $s \in \operatorname{Spec} R_X$; see [33, Lemma–Definition 5.3].

Given a general polarization $\underline{q}$ on X, it is proved in [33, Theorem 5.4] that there exists an open subscheme $\overline{J}_{\mathcal{X}}(\underline{q}) \subseteq \overline{\mathbb{J}}_{\mathcal{X}}$, called the *universal fine compactified Jacobian* of X with respect to the polarization $\underline{q}$, which is projective over $\operatorname{Spec} R_X$ and such that the geometric fiber of $u\colon \overline{J}_{\mathcal{X}}(\underline{q}) \to \operatorname{Spec} R_X$ over a point $s \in \operatorname{Spec} R_X$ is isomorphic to $\overline{J}_{\mathcal{X}_{\overline{s}}}(\underline{q}^s)$. In particular, the fiber of $\overline{J}_{\mathcal{X}}(\underline{q}) \to \operatorname{Spec} R_X$ over the closed point o in $\operatorname{Spec} R_X$ is isomorphic to $\overline{J}_X(\underline{q})$. We let $J_{\mathcal{X}}(\underline{q})$ denote the open subset of $\overline{J}_{\mathcal{X}}(\underline{q})$ parametrizing line bundles, ie $J_{\mathcal{X}}(\underline{q}) = \overline{J}_{\mathcal{X}}(\underline{q}) \cap \mathbb{J}_{\mathcal{X}} \subseteq \overline{\mathbb{J}}_{\mathcal{X}}$.

If the curve X has planar singularities, then the universal fine compactified Jacobians of X have several nice properties, which we collect in the following statement.

Fact 2.10 *Assume that X is a reduced and connected curve with planar singularities, and let $\underline{q}$ be a general polarization on X. Then we have:*

(i) *The scheme $\overline{J}_{\mathcal{X}}(\underline{q})$ is smooth and irreducible.*

(ii) *The surjective map $u\colon \overline{J}_{\mathcal{X}}(\underline{q}) \to \operatorname{Spec} R_X$ is projective and flat of relative dimension $p_a(X)$.*

(iii) *The smooth locus of u is $J_{\mathcal{X}}(\underline{q})$.*

Proof See [33, Theorem 5.5]. □

3 Punctual Hilbert schemes

The aim of this section is to recall some properties of the punctual Hilbert schemes (ie Hilbert schemes of points) on curves with planar singularities and on smooth surfaces, which will be needed in Section 4.

For any projective scheme Z and $n \geq 1$, let Hilb^n_Z be the (punctual) Hilbert scheme parametrizing 0–dimensional subschemes $D \subset Z$ of length n, in other words such that $k[D] := \Gamma(D, \mathcal{O}_D)$ is a k–algebra of dimension n. The Hilbert scheme Hilb^n_Z is endowed with a universal divisor $\mathcal{D}$, giving rise to the diagram

(3-1)
$$\begin{array}{ccccc} & & \mathcal{D} & \hookrightarrow & \mathrm{Hilb}^n_Z \times Z \\ & \swarrow^{h} & & \searrow^{f} & \\ \mathrm{Hilb}^n_Z & & & & Z \end{array}$$

where the morphism h is finite and flat of degree n. The sheaf $\mathcal{A} := h_*\mathcal{O}_{\mathcal{D}}$ is a coherent sheaf of algebras on Hilb^n_Z which is locally free of rank n. The fiber of $\mathcal{A}$ over $D \in \mathrm{Hilb}^n_Z$ is canonically isomorphic to the k–algebra $k[D]$ of regular functions on D. We refer the reader to [15, Chapters 5 and 6] for a detailed account of the theory of Hilbert schemes.

The punctual Hilbert scheme Hilb^n_Z contains a remarkable open subset ${}^{\mathrm{c}}\mathrm{Hilb}^n_Z \subseteq \mathrm{Hilb}^n_Z$, called the *curvilinear Hilbert scheme* of Z, consisting of all the 0–dimensional subschemes $D \in \mathrm{Hilb}^n_Z$ such that Z can be embedded into a smooth curve, or equivalently such that
$$k[D] \cong \prod_i \frac{k[x]}{(x^{n_i})}.$$

In what follows, we will be concerned with the punctual Hilbert schemes of curves and surfaces. Observe that if a curve X is contained in a smooth surface S, then X has planar singularities. Indeed, the converse is also true due to the following result.

Fact 3.1 (Altman–Kleiman [31]) *If X is a connected projective reduced curve with planar singularities, then there exists a smooth projective integral surface S such that $X \subset S$.*

If $X \subset S$ is as above, then we get a closed embedding
$$\mathrm{Hilb}^n_X \subseteq \mathrm{Hilb}^n_S .$$

In the next two subsections, we will review some of the properties of Hilb^n_S and of Hilb^n_X that we will need later on.

3.1 Punctual Hilbert schemes of surfaces

Throughout this subsection, we fix a projective smooth integral surface S over an algebraically closed field k.

The following properties of Hilb^n_S are due to J Fogarty; see [15, Theorem 7.2.3] for a modern proof.

Fact 3.2 (Fogarty) *For any $n\in\mathbb{N}$ and any projective smooth connected surface S, the punctual Hilbert scheme Hilb^n_S is smooth and irreducible of dimension $2n$.*

Let $\mathrm{Sym}^n(S)$ be the n^{th} symmetric product of S, ie $\mathrm{Sym}^n(S)=S^n/\Sigma_n$, where Σ_n is the symmetric group on n letters acting on the n^{th} product S^n by permuting the factors. The n^{th} symmetric product $\mathrm{Sym}^n(S)$ parametrizes 0–cycles $\zeta=\sum_{p\in\mathrm{supp}\,\zeta}\zeta_p\cdot p$ on S of length n. There is a surjective morphism, called the *Hilbert–Chow morphism* (see [15, Section 7.1]), defined by

$$\text{(3-2)}\qquad \mathrm{HC}\colon \mathrm{Hilb}^n_S\to\mathrm{Sym}^n(S),\quad D\mapsto\sum_{p\in S} l(\mathcal{O}_{D,p})\cdot p,$$

where $l(\mathcal{O}_{D,p})$ is the length of the Artinian ring $\mathcal{O}_{D,p}$.

The fiber of the Hilbert–Chow morphism over a divisor $\sum_i n_ip_i\in\mathrm{Sym}^n S$ is isomorphic (see [28, page 820]) to

$$\text{(3-3)}\qquad \mathrm{HC}^{-1}\Big(\sum_i n_ip_i\Big)\cong\prod_i\mathrm{Hilb}^{n_i}(\widehat{\mathcal{O}}_{S,p_i}),$$

where, for any $m\geq 1$ and any $p\in S$, $\mathrm{Hilb}^m(\widehat{\mathcal{O}}_{S,p}):=\mathrm{Hilb}^m(k[\![x,y]\!])$ is the *local Hilbert scheme* parametrizing ideals $I\subset k[\![x,y]\!]$ of colength m, ie such that $k[\![x,y]\!]/I$ is a k–algebra of dimension m. Denote by ${}^{\mathrm{c}}\mathrm{Hilb}^m(k[\![x,y]\!])\subseteq\mathrm{Hilb}^m(k[\![x,y]\!])$ the open subset (called the *curvilinear local Hilbert scheme*) parametrizing ideals $I\subset k[\![x,y]\!]$ such that $k[\![x,y]\!]/I\cong k[z]/(z^m)$.

The following result was proved by J Briançon; see also [29] and [16].

Fact 3.3 (Briançon [8]) *For $m\geq 1$, the local Hilbert scheme $\mathrm{Hilb}^m(k[\![x,y]\!])$ is irreducible of dimension $m-1$. In particular, the curvilinear local Hilbert scheme ${}^{\mathrm{c}}\mathrm{Hilb}^m(k[\![x,y]\!])\subseteq\mathrm{Hilb}^m(k[\![x,y]\!])$ is an open dense subset.*

Note that Facts 3.2 and 3.3, together with equation (3-3), imply that HC is a resolution of singularities; see also [15, Theorem 7.3.4].

Consider now the reduced fiber product $\widetilde{\mathrm{Hilb}}^n_S:=(S^n\times_{\mathrm{Sym}^n(S)}\mathrm{Hilb}^n_S)_{\mathrm{red}}$, ie the reduced scheme associated to the fiber product of S^n and Hilb^n_S over $\mathrm{Sym}^n(S)$. The scheme

$\widetilde{\mathrm{Hilb}}^n_S$ was introduced by Haiman [22, Definition 3.2.4] under the name of *isospectral Hilbert scheme* of S. Consider the diagram

(3-4)
$$\begin{array}{ccc} \widetilde{\mathrm{Hilb}}^n_S & \xrightarrow{\psi} & \mathrm{Hilb}^n_S \\ \sigma\downarrow & & \downarrow \\ S^n & \longrightarrow & \mathrm{Sym}^n(S) \end{array}$$

Clearly, there is a natural action of Σ_n on $\widetilde{\mathrm{Hilb}}^n_S$ that makes σ a Σ_n–equivariant morphism and ψ a Σ_n–invariant morphism. Haiman proved the following properties of $\widetilde{\mathrm{Hilb}}^n_S$.

Fact 3.4 (Haiman [22]) *Assume that either* $\mathrm{char}(k)=0$ *or* $\mathrm{char}(k)>n$.[3] *Then*:

(i) *The isospectral Hilbert scheme* $\widetilde{\mathrm{Hilb}}^n_S$ *is normal, Gorenstein and integral of dimension* $2n$.

(ii) *The morphism* $\psi\colon \widetilde{\mathrm{Hilb}}^n_S \to \mathrm{Hilb}^n_S$ *is finite and flat of degree* $n!$.

The inverse image of the curvilinear Hilbert scheme ${}^{\mathrm{c}}\mathrm{Hilb}^n_S \subseteq \mathrm{Hilb}^n_S$ via the map ψ of (3-4) admits a modular description that we now recall. Denote by Flag^n_S the moduli space of flags

$$D_1 \subset \cdots \subset D_n,$$

where $D_i \in {}^{\mathrm{c}}\mathrm{Hilb}^n_S$ has length i for every $i=1,\dots,n$. There is a natural morphism

(3-5)
$${}^{\mathrm{c}}\psi\colon \mathrm{Flag}^n_S \to {}^{\mathrm{c}}\mathrm{Hilb}^n_S, \quad (D_1\subset\cdots\subset D_n)\mapsto D_n.$$

Fact 3.5 *Assume that either* $\mathrm{char}(k)=0$ *or* $\mathrm{char}(k)>n$. *Then there is a cartesian diagram*

$$\begin{array}{ccc} \mathrm{Flag}^n_S & \xrightarrow{{}^{\mathrm{c}}\psi} & {}^{\mathrm{c}}\mathrm{Hilb}^n_S \\ \downarrow & \square & \downarrow \\ \widetilde{\mathrm{Hilb}}^n_S & \xrightarrow[\psi]{} & \mathrm{Hilb}^n_S \end{array}$$

For a proof, see [5, Proposition 3.7]. Moreover, in loc. cit. it is also shown that the composition of the inclusion $\mathrm{Flag}^n_S \hookrightarrow \widetilde{\mathrm{Hilb}}^n_S$ given in Fact 3.5 with the map $\sigma\colon \widetilde{\mathrm{Hilb}}^n_S \to S^n$ of (3-4) is equal to the modular map

(3-6)
$${}^{\mathrm{c}}\sigma\colon \mathrm{Flag}^n_S \to S^n, \quad (D_1\subset\cdots\subset D_n)\mapsto (\mathrm{supp}\ker(\mathcal{O}_{D_i}\to\mathcal{O}_{D_{i-1}}))_i.$$

[3] Haiman stated his results in [22] under the assumption that $\mathrm{char}(k)=0$. However, his results are true also if $\mathrm{char}(k)>n$, as observed by Groechenig in [17, Remark 4.9].

3.2 Punctual Hilbert scheme of curves with planar singularities

Throughout this subsection, we fix a connected projective reduced curve X with planar singularities over an algebraically closed field k.

Note that if $D \in \mathrm{Hilb}_X^n$ then its ideal sheaf I_D is a torsion-free rank-1 sheaf on X (in the sense of Definition 2.1), which, however, is in general neither a line bundle (unless X is smooth) nor simple (unless X is irreducible). We refer to [12, Example 38] for an example of $D \in \mathrm{Hilb}_X^n$ with I_D not simple. Letting $X_{\mathrm{sm}} \subseteq X$ denote the smooth locus, we introduce the following subschemes of $\mathrm{Hilb}^d(X)$:

$$
\begin{aligned}
{}^{\mathrm{r}}\mathrm{Hilb}_X^n &:= \{D \in \mathrm{Hilb}_X^n : D \text{ is reduced and contained in } X_{\mathrm{sm}} \subseteq X\},\\
{}^{\mathrm{l}}\mathrm{Hilb}_X^n &:= \{D \in \mathrm{Hilb}_X^n : I_D \text{ is a line bundle}\},\\
{}^{\mathrm{s}}\mathrm{Hilb}_X^n &:= \{D \in \mathrm{Hilb}_X^n : I_D \text{ is simple}\}.
\end{aligned}
\tag{3-7}
$$

By [3, Proposition 5.15], the natural inclusions

$$
{}^{\mathrm{r}}\mathrm{Hilb}_X^n \subseteq {}^{\mathrm{l}}\mathrm{Hilb}_X^n \subseteq {}^{\mathrm{s}}\mathrm{Hilb}_X^n \subseteq \mathrm{Hilb}_X^n \tag{3-8}
$$

are open inclusions.

The punctual Hilbert scheme of a curve with planar singularities was studied by Altman–Iarrobino–Kleiman and by Briançon–Granger–Speder, who proved the following.

Fact 3.6 (Altman–Iarrobino–Kleiman [1], Briançon–Granger–Speder [9]) *Let X be a connected projective reduced curve with planar singularities. Then the Hilbert scheme Hilb_X^n satisfies the following properties*:

(i) Hilb_X^n *is a connected and reduced projective scheme of pure dimension n with locally complete intersection singularities.*

(ii) ${}^{\mathrm{r}}\mathrm{Hilb}_X^n$ *is dense in* Hilb_X^n.

(iii) ${}^{\mathrm{l}}\mathrm{Hilb}_X^n$ *is the smooth locus of* Hilb_X^n.

Proof Property (i) follows from [1, Corollary 7] (see also [9, Proposition 1.4]) and [9, Proposition 3.1]. Property (ii) follows from [9, Proposition 1.4]. Property (iii) follows from [9, Proposition 2.3]. □

Note that properties (ii) and (iii) of Hilb_X^n are inherited by its open subset ${}^{\mathrm{s}}\mathrm{Hilb}_X^n$. This also holds for the reducedness and the lci singularities part of (i).

The punctual Hilbert scheme of X and the moduli space $\overline{\mathbb{J}}_X$ are related via the Abel map, which is defined as follows. Given a line bundle M on X, we define the *(M–twisted) Abel map of degree d* by

$$A^d_M\colon {}^{\mathrm{s}}\mathrm{Hilb}^d_X \to \overline{\mathbb{J}}_X, \quad D \mapsto I_D \otimes M. \tag{3-9}$$

Note that the image of $A_M({}^{\mathrm{s}}\mathrm{Hilb}^d_X)$ is contained in $\overline{\mathbb{J}}_X^{1-p_a(X)-d+\deg M}$, since for any $D \in \mathrm{Hilb}^d_X$ we have that

$$\chi(I_D \otimes M) = \chi(I_D) + \deg M = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_D) + \deg M = 1 - p_a(X) - d + \deg M.$$

The following result shows that, locally on the codomain, the M–twisted Abel maps of degree $p_a(X)$ are smooth and surjective (for suitable choices of $M \in \mathrm{Pic}(X)$) for Gorenstein curves.

Fact 3.7 *Let X be a connected projective reduced Gorenstein curve of arithmetic genus $g = p_a(X)$. For any $\chi \in \mathbb{Z}$, there exists a cover of $\overline{\mathbb{J}}_X^\chi$ by k–finite-type open subsets $\{U_\beta\}$ such that, for each such U_β, there exists $M_\beta \in \mathrm{Pic}^{\chi+2g-1}(X)$ with the property that*

$${}^{\mathrm{s}}\mathrm{Hilb}^g_X \supseteq V_\beta := (A^g_{M_\beta})^{-1}(U_\beta) \xrightarrow{A^g_{M_\beta}} U_\beta$$

is smooth and surjective.

Proof See [33, Proposition 2.5]. □

Remark 3.8 (i) The integer $g = p_a(X)$ is the smallest integer for which Fact 3.7 is true for any X; see [33, Remark 2.6].

(ii) If the curve X is irreducible (and Gorenstein) of arithmetic genus g, then we can get a global result although using a bigger punctual Hilbert scheme, namely: for any integer $\chi \in \mathbb{Z}$ and for any line bundle M on X of degree $3g-2+\chi$, the M–twisted Abel map $A^{2g-1}_M\colon {}^{\mathrm{s}}\mathrm{Hilb}^{2g-1}(X) = \mathrm{Hilb}^{2g-1}(X) \to \overline{\mathbb{J}}_X^\chi$ is smooth and surjective; see [3, Theorem 8.6]. It is easy to see that $2g-1$ is the smallest integer for which the above property holds for any X of arithmetic genus g.

Consider now the curvilinear Hilbert scheme ${}^{\mathrm{c}}\mathrm{Hilb}^n_X \subset \mathrm{Hilb}^n_X$ of X. Observe that, since X is assumed to have planar singularities, $D \in \mathrm{Hilb}^n_X$ belongs to ${}^{\mathrm{c}}\mathrm{Hilb}^n_X$ if and only if $I_D \not\subseteq I_p^2$ for every $p \in X_{\mathrm{sing}}$, where I_p denotes the defining ideal of $p \in X$.

Furthermore, if we chose a projective smooth integral surface S such that $X \subset S$ (see Fact 3.1), then we have the equality

$$ {}^{\mathrm{c}}\mathrm{Hilb}^n_X = {}^{\mathrm{c}}\mathrm{Hilb}^n_S \cap \mathrm{Hilb}^n_X \subset \mathrm{Hilb}^n_S . \tag{3-10}$$

Lemma 3.9 *The complement of* ${}^{\mathrm{c}}\mathrm{Hilb}^n_X$ *inside* Hilb^n_X *has codimension at least two.*

Proof First of all, chose a projective smooth integral surface S such that $X \subset S$; this is possible by Fact 3.1. The Hilbert–Chow morphism of (3-2) induces the commutative diagram

$$\begin{array}{ccc} \mathrm{Hilb}^n_X & \hookrightarrow & \mathrm{Hilb}^n_S \\ {\scriptstyle \mathrm{HC}}\downarrow & & \downarrow{\scriptstyle \mathrm{HC}} \\ \mathrm{Sym}^n(X) & \hookrightarrow & \mathrm{Sym}^n(S) \end{array} \tag{3-11}$$

Note that if $D \in \mathrm{Hilb}^n_X$ is such that $\mathrm{HC}(D) = \sum_i n_i p_i \in \mathrm{Sym}^n(X)$, then D can be written as a disjoint union

$$D = \bigcup_{p_i \in \mathrm{HC}(D)} D|_{p_i},$$

where $D|_{p_i}$ is a 0–dimensional subscheme of X supported at p_i and of length n_i. We can look at $D|_{p_i}$ as an element of $\mathrm{Hilb}^{n_i}(\widehat{\mathcal{O}}_{S,p_i})$. Clearly $D \in {}^{\mathrm{c}}\mathrm{Hilb}^n_X$ if and only if $D|_{p_i} \in {}^{\mathrm{c}}\mathrm{Hilb}^{n_i}(\widehat{\mathcal{O}}_{S,p_i})$ for every $p_i \in \mathrm{HC}(D)$.

Consider now an irreducible component W of $\mathrm{Hilb}^n_X \setminus {}^{\mathrm{c}}\mathrm{Hilb}^n_X$ and endow it with the reduced scheme structure. The above discussion implies that there exists a singular point $p \in X_{\mathrm{sing}}$ and an integer $m \geq 2$ such that for the generic $D \in W$ we have that

$$mp \subseteq \mathrm{HC}(D) \quad \text{and} \quad D|_p \in \mathrm{Hilb}^m(\widehat{\mathcal{O}}_{S,p}) \setminus {}^{\mathrm{c}}\mathrm{Hilb}^m(\widehat{\mathcal{O}}_{S,p}).$$

Therefore there exists an open and dense subset $U \subseteq W$ that admits an embedding

$$U \hookrightarrow [\mathrm{Hilb}^m(\widehat{\mathcal{O}}_{S,p}) \setminus {}^{\mathrm{c}}\mathrm{Hilb}^m(\widehat{\mathcal{O}}_{S,p})] \times \mathrm{Hilb}^{n-m}_X, \quad D \mapsto \Big(D|_p, \bigcup_{p \neq q \in \mathrm{HC}(D)} D|_q\Big). \tag{3-12}$$

Facts 3.3 and 3.6(i) imply that the right side of (3-12) has dimension $m-2+(n-m) = n-2$; therefore W, and hence $\mathrm{Hilb}^n_X \setminus {}^{\mathrm{c}}\mathrm{Hilb}^n_X$, can have dimension at most $n-2$. This concludes the proof, since Hilb^n_X is pure of dimension n by Fact 3.6(i). □

If we intersect the open subsets of (3-8) with ${}^{\mathrm{c}}\mathrm{Hilb}_X^n \subset \mathrm{Hilb}_X^n$, we obtain the following chain of open inclusions:

(3-13) $$ {}^{\mathrm{r}}\mathrm{Hilb}_X^n \subseteq {}^{\mathrm{cl}}\mathrm{Hilb}_X^n := {}^{\mathrm{c}}\mathrm{Hilb}_X^n \cap {}^{\mathrm{l}}\mathrm{Hilb}_X^n \subseteq {}^{\mathrm{cs}}\mathrm{Hilb}_X^n := {}^{\mathrm{c}}\mathrm{Hilb}_X^n \cap {}^{\mathrm{s}}\mathrm{Hilb}_X^n \subseteq {}^{\mathrm{c}}\mathrm{Hilb}_X^n \subseteq \mathrm{Hilb}_X^n. $$

4 Definition of the Poincaré sheaf

The aim of this section is to introduce the Poincaré sheaf $\overline{\mathcal{P}}$ on $\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g}$, where X is a reduced connected projective curve of arithmetic genus $g := p_a(X)$. Let us start by describing the restriction of $\overline{\mathcal{P}}$ to the open subset

$$ (\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural := \overline{\mathbb{J}}_X^{1-g} \times \mathbb{J}_X^{1-g} \cup \mathbb{J}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g} \subseteq \overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g} $$

consisting of pairs of torsion-free rank-1 simple sheaves I on X of Euler characteristic $1-g$ (or equivalently degree 0) such that at least one of the two sheaves is a line bundle.

4.1 The Poincaré line bundle $\mathcal{P}$

Consider the open subset $X \times (\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural \subseteq X \times \overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g}$ and, for any $1 \le i < j \le 3$, denote by p_{ij} the projection onto the product of the i^{th} and j^{th} factors. Consider the trivial family of curves

$$ p_{23}\colon X \times (\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural \to (\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural. $$

For any coherent sheaf $\mathcal{F}$ on $X \times (\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural$, flat over $(\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural$, the complex $Rp_{23*}(\mathcal{F})$ is perfect of amplitude $[0,1]$, ie there is a Zariski open cover $(\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural = \bigcup_\alpha U_\alpha$ and, for each open subset U_α, a complex $\mathcal{G}_*^\alpha := \{\mathcal{G}_0^\alpha \to \mathcal{G}_1^\alpha\}$ of locally free sheaves of finite rank over U_α which is quasiisomorphic to $Rp_{23*}(\mathcal{F})|_{U_\alpha}$; see [12, Observation 43]. The line bundles $\det(\mathcal{G}_*^\alpha) := \det(\mathcal{G}_0^\alpha) \otimes \det(\mathcal{G}_1^\alpha)^{-1}$ on U_α glue together to give a (well-defined) line bundle on $(\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural$, which is denoted by $\mathcal{D}_{p_{23}}(\mathcal{F})$ and is called the *determinant of cohomology of* $\mathcal{F}$ *with respect to* p_{23}; see [12, Section 6.1] for details.

Choose now a universal sheaf $\mathcal{I}$ on $X \times \overline{\mathbb{J}}_X^{1-g}$ as in Section 2.1 and form the line bundle on $(\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural$, called the *Poincaré line bundle*,

(4-1) $$ \mathcal{P} := \mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I} \otimes p_{13}^*\mathcal{I})^{-1} \otimes \mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}) \otimes \mathcal{D}_{p_{23}}(p_{13}^*\mathcal{I}). $$

Remark 4.1 The above definition of $\mathcal{P}$ makes sense since $p_{12}^*\mathcal{I}$ and $p_{13}^*\mathcal{I}$ are coherent sheaves flat over $(\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g})^\natural$ (because $\mathcal{I}$ is coherent and flat over $\overline{\mathbb{J}}_X^{1-g}$), and $p_{12}^*\mathcal{I}\otimes p_{13}^*(\mathcal{I})$ is flat over $(\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g})^\natural$ since $p_{12}^*(\mathcal{I})$ is a line bundle over $X\times\mathbb{J}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$ and $p_{13}^*(\mathcal{I})$ is a line bundle over $X\times\overline{\mathbb{J}}_X^{1-g}\times\mathbb{J}_X^{1-g}$. However $p_{12}^*(\mathcal{I})\otimes p_{13}^*(\mathcal{I})$ is not flat over $\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$ (in general), hence definition (4-1) does not extend over $\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$.

Remark 4.2 The above definition of $\mathcal{P}$ is independent of the chosen universal sheaf $\mathcal{I}$ since we are working over $(\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g})^\natural\subseteq\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$. Indeed, one could define a Poincaré line bundle $\mathcal{P}$ on $\overline{\mathbb{J}}_X\times\mathbb{J}_X\cup\mathbb{J}_X\times\overline{\mathbb{J}}_X$ using the same formula (4-1). Then, considering another universal sheaf $\widetilde{\mathcal{I}}=\mathcal{I}\otimes\pi_2^*(N)$ for some $N\in\operatorname{Pic}(\overline{\mathbb{J}}_X)$ (see Section 2.1) and defining a new Poincaré line bundle $\widetilde{\mathcal{P}}$ with respect to $\widetilde{\mathcal{I}}$, one can check that

$$\widetilde{\mathcal{P}}|_{\overline{\mathbb{J}}_X^{\chi_1}\times\mathbb{J}_X^{\chi_2}}=\mathcal{P}|_{\overline{\mathbb{J}}_X^{\chi_1}\times\mathbb{J}_X^{\chi_2}}\otimes p_1^*(N|_{\overline{\mathbb{J}}_X^{\chi_1}})^{1-g-\chi_2}\otimes p_2^*(N|_{\mathbb{J}_X^{\chi_2}})^{1-g-\chi_1}\tag{4-2}$$

for every $\chi_1,\chi_2\in\mathbb{Z}$, where p_1 and p_2 denote the projection of $\overline{\mathbb{J}}_X\times\mathbb{J}_X$ onto $\overline{\mathbb{J}}_X$ and $\mathbb{J}_X$, respectively. (See [13, Proposition 2.2] for a similar computation.)

The following lemma will be used throughout.

Lemma 4.3 *Let S be a scheme and consider the trivial family $p_2\colon X\times S\to S$. Let I be a rank-1 torsion-free sheaf on X, and write $I=I_{E_1}\otimes I_{E_2}^{-1}$ as in Lemma 2.2. Let $\mathcal{F}$ be a coherent sheaf on $X\times S$, flat over S, and assume that $\mathcal{F}$ is locally free along $p_1^{-1}(E_i)$ for $i=1,2$. Then*

$$\mathcal{D}_{p_2}(\mathcal{F}\otimes p_1^*I)\otimes\mathcal{D}_{p_2}(\mathcal{F})^{-1}=\mathcal{D}_{p_2}(\mathcal{F}|_{p_1^{-1}(E_2)})\otimes\mathcal{D}_{p_2}(\mathcal{F}|_{p_1^{-1}(E_1)})^{-1}.\tag{4-3}$$

Proof Consider the exact sequences associated to the effective divisors $E_i\subseteq X$,

$$0\to I_{E_1}\to\mathcal{O}_X\to\mathcal{O}_{E_1}\to 0,\tag{4-4}$$

$$0\to I_{E_2}=\mathcal{O}(-E_2)\to\mathcal{O}_X\to\mathcal{O}_{E_2}\to 0.\tag{4-5}$$

Tensoring the sequence (4-5) with I and using the fact that $I=I_{E_1}\otimes I_{E_2}^{-1}$, we get the new sequence

$$0\to I_{E_2}\otimes I=I_{E_1}\to I\to I|_{E_2}=\mathcal{O}(E_2)|_{E_2}=\mathcal{O}_{E_2}\to 0,\tag{4-6}$$

which remains exact since $I_{E_2}^{-1}=\mathcal{O}(E_2)$ is a line bundle, and $(I_{E_1})|_{E_2}=\mathcal{O}_{E_2}$ because E_1 and E_2 have disjoint supports by construction. By pulling back the exact sequences

(4-4) and (4-6) via $p_1\colon X\times S\to X$ and tensoring them with $\mathcal{F}$, we get the following two sequences, which remain exact by the hypothesis on $\mathcal{F}$:

$$0\to\mathcal{F}\otimes p_1^*I_{E_1}\to\mathcal{F}\to\mathcal{F}|_{p_1^{-1}(E_1)}\to 0,$$
$$0\to\mathcal{F}\otimes p_1^*I_{E_1}\to\mathcal{F}\otimes p_1^*I\to\mathcal{F}|_{p_1^{-1}(E_2)}\to 0.$$

Using the additivity of the determinant of cohomology [12, Proposition 44(4)], we get

$$\mathcal{D}_{p_2}(\mathcal{F})=\mathcal{D}_{p_2}(\mathcal{F}\otimes p_1^*I_{E_1})\otimes\mathcal{D}_{p_2}(\mathcal{F}|_{p_1^{-1}(E_1)}),$$
$$\mathcal{D}_{p_2}(\mathcal{F}\otimes p_1^*I)=\mathcal{D}_{p_2}(\mathcal{F}\otimes p_1^*I_{E_1})\otimes\mathcal{D}_{p_2}(\mathcal{F}|_{p_1^{-1}(E_2)}).$$

By taking the difference of the above two equalities, we get the desired formula (4-3). □

Corollary 4.4 *Take the same assumptions as in Lemma 4.3. Moreover, let L be a line bundle on X. Then*

$$\mathcal{D}_{p_2}(\mathcal{F}\otimes p_1^*L\otimes p_1^*I)=\mathcal{D}_{p_2}(\mathcal{F}\otimes p_1^*L)\otimes\mathcal{D}_{p_2}(\mathcal{F}\otimes p_1^*I)\otimes\mathcal{D}_{p_2}(\mathcal{F})^{-1}.$$

Proof It follows from Lemma 4.3 together with the fact that, for $i=1,2$,

$$p_1^*L|_{p_1^{-1}(E_i)}=p_1^*(L|_{E_i})=p_1^*\mathcal{O}_{E_i}=\mathcal{O}_{p_1^{-1}(E_i)}.$$ □

4.2 Definition of the Poincaré sheaf $\overline{\mathcal{P}}$

In this subsection we construct a Poincaré sheaf $\overline{\mathcal{P}}$ on $\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$, which is an extension of $\mathcal{P}$. The definition of $\overline{\mathcal{P}}$ is as follows.

Definition 4.5 Let X be a reduced and connected curve. Denote by

$$j\colon(\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g})^{\natural}=\mathbb{J}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}\cup\overline{\mathbb{J}}_X^{1-g}\times\mathbb{J}_X^{1-g}\hookrightarrow\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$$

the natural inclusion. The $\mathcal{O}_{\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}}$–module

$$\overline{\mathcal{P}}:=j_*(\mathcal{P})$$

is called the *Poincaré sheaf.*

If X has planar singularities, then the Poincaré sheaf enjoys the following properties.

Theorem 4.6 *Let X be a reduced connected curve with planar singularities and of arithmetic genus $g:=p_a(X)$. Assume that either* $\mathrm{char}(k)=0$ *or* $\mathrm{char}(k)>g$.

(i) *$\overline{\mathcal{P}}$ is a maximal Cohen–Macaulay (coherent) sheaf on $\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$.*

(ii) *$\overline{\mathcal{P}}$ is flat with respect to the second projection $p_2\colon \overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}\to\overline{\mathbb{J}}_X^{1-g}$ and, for every $I\in\overline{\mathbb{J}}_X^{1-g}$, the restriction $\overline{\mathcal{P}}_I:=\overline{\mathcal{P}}|_{\overline{\mathbb{J}}_X^{1-g}\times\{I\}}$ is a maximal Cohen–Macaulay sheaf on $\overline{\mathbb{J}}_X^{1-g}$.*

Remark 4.7 Under the assumptions of Theorem 4.6, we observe that, since the complement of $(\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g})^\natural$ inside $\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$ has codimension greater than or equal to two by Fact 2.3(ii), the sheaf $\overline{\mathcal{P}}$ is the Cohen–Macaulay extension of $\mathcal{P}$; in other words, $\overline{\mathcal{P}}$ can be characterized as the unique Cohen–Macaulay sheaf on $\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$ whose restriction to $(\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g})^\natural$ is equal to $\mathcal{P}$; see [20, Theorem 5.10.5].

In proving Theorem 4.6, we will adapt the strategy used by Arinkin [5] to prove the same result for integral curves: we will first construct a sheaf $\mathcal{Q}^n$ on $\mathrm{Hilb}^n_X\times\overline{\mathbb{J}}_X^{1-g}$ for any $n\geq 1$ and then we will descend it to a sheaf $\overline{\mathcal{P}}$ on $\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$ using the Abel map (3-9).

4.2.1 The sheaf $\mathcal{Q}$ on $\left(\coprod_{n\in\mathbb{N}}\mathrm{Hilb}^n_X\right)\times\overline{\mathbb{J}}_X^{1-g}$

Choose an embedding $i\colon X\hookrightarrow S$ as in Fact 3.1, fix an integer $n\in\mathbb{N}$ and, using the notation of Section 3.1, consider the diagram

$$\begin{array}{ccccccccc}\mathrm{Hilb}^n_X\times\overline{\mathbb{J}}_X^{1-g} & \hookrightarrow & \mathrm{Hilb}^n_S\times\overline{\mathbb{J}}_X^{1-g} & \xleftarrow{\psi\times\mathrm{id}} & \widetilde{\mathrm{Hilb}}^n_S\times\overline{\mathbb{J}}_X^{1-g} & \xrightarrow{\sigma\times\mathrm{id}} & S^n\times\overline{\mathbb{J}}_X^{1-g} & \hookleftarrow_{i^n\times\mathrm{id}} & X^n\times\overline{\mathbb{J}}_X^{1-g}\\ & & \downarrow{\scriptstyle p_1} & & & & & & \\ & & \mathrm{Hilb}^n_S & & & & & & \end{array}\tag{4-7}$$

where the maps $\sigma\times\mathrm{id}$ and $i^n\times\mathrm{id}$ are clearly Σ_n–equivariant.

Choose a universal sheaf $\mathcal{I}$ on $X\times\overline{\mathbb{J}}_X^{1-g}$ as in Section 2.1 and define a sheaf $\mathcal{I}^n$ on $X^n\times\overline{\mathbb{J}}_X^{1-g}$ by

$$\mathcal{I}^n:=p_{1,n+1}^*(\mathcal{I})\otimes\cdots\otimes p_{n,n+1}^*(\mathcal{I}),\tag{4-8}$$

where $p_{i,n+1}\colon X^n\times\overline{\mathbb{J}}_X^{1-g}\to X\times\overline{\mathbb{J}}_X^{1-g}$ denotes the projection onto the i^{th} and $(n+1)^{\text{st}}$ factors. Observe that $\mathcal{I}^n$ is clearly Σ_n–equivariant.

Now define a coherent sheaf $\mathcal{Q}^n$ on $\mathrm{Hilb}^n_S\times\overline{\mathbb{J}}_X^{1-g}$ by the formula

$$\mathcal{Q}^n:=\left[(\psi\times\mathrm{id})_*(\sigma\times\mathrm{id})^*(i^n\times\mathrm{id})_*\mathcal{I}^n\right]^{\mathrm{sign}}\otimes p_1^*(\det\mathcal{A})^{-1},\tag{4-9}$$

where $\mathcal{A}$ is the locally free rank n sheaf on Hilb^n_S defined after the diagram (3-1) and the upper index sign stands for the space of anti-invariants with respect to the natural action of Σ_n.

As shown by Arinkin,[4] the sheaf $\mathcal{Q}^n$ enjoys the following properties.

Fact 4.8 (Arinkin [5, Proposition 4.1, Section 4.1]) *Assume that either* $\mathrm{char}(k)=0$ *or* $\mathrm{char}(k)>n$. *Let* X *be a connected projective reduced curve with planar singularities and choose an embedding* $i\colon X\hookrightarrow S$ *as in Fact 3.1. Then the sheaf* $\mathcal{Q}^n$ *defined by (4-9) satisfies the following properties*:

(i) $\mathcal{Q}^n$ *is supported schematically on* $\mathrm{Hilb}^n_X\times\overline{\mathbb{J}}_X^{1-g}$ *and it does not depend on the chosen embedding* $i\colon X\hookrightarrow S$.

(ii) $\mathcal{Q}^n$ *is a maximal Cohen–Macaulay sheaf on* $\mathrm{Hilb}^n_X\times\overline{\mathbb{J}}_X^{1-g}$.

(iii) $\mathcal{Q}^n$ *is flat over* $\overline{\mathbb{J}}_X^{1-g}$.

(iv) *For any* $I\in\overline{\mathbb{J}}_X^{1-g}$, *the restriction* $\mathcal{Q}^n|_{\mathrm{Hilb}^n_X\times\{I\}}$ *is a maximal Cohen–Macaulay sheaf on* Hilb^n_X.

Denote by $\mathcal{Q}$ the sheaf on $\left(\coprod_{n\in\mathbb{N}}\mathrm{Hilb}^n_X\right)\times\overline{\mathbb{J}}_X^{1-g}$ which is equal to $\mathcal{Q}^n$ on $\mathrm{Hilb}^n_X\times\overline{\mathbb{J}}_X^{1-g}$.

4.2.2 The sheaf $\mathcal{Q}'$ on $\left(\coprod_{n\in\mathbb{N}}\mathrm{Hilb}^n_X\right)\times\overline{\mathbb{J}}_X^{1-g}$ The restriction of the sheaf $\mathcal{Q}^n$ to ${}^{c}\mathrm{Hilb}^n_X\times\overline{\mathbb{J}}_X^{1-g}$ coincides with the restriction of another sheaf, $\mathcal{Q}'^n$ on $\mathrm{Hilb}^n_X\times\overline{\mathbb{J}}_X^{1-g}$, which we now introduce.

Consider the universal divisor $\mathcal{D}\subseteq\mathrm{Hilb}^n_X\times X$ of (3-1). Recall that $\mathcal{A}:=h_*\mathcal{O}_{\mathcal{D}}$ is a coherent sheaf of algebras on Hilb^n_X, which is locally free of rank n. Denote by $\mathcal{A}^\times$ the subsheaf of $\mathcal{A}$ of invertible elements. Clearly, $\mathcal{A}^\times$ is the sheaf of sections of a flat abelian group scheme over Hilb^n_X, whose fiber over $D\in\mathrm{Hilb}^n_X$ is canonically isomorphic to the group $k[D]^\times$ of invertible elements of the algebra $k[D]$ of regular functions on D. Clearly $\mathcal{A}^\times$ acts on $\mathcal{A}$ and therefore also on the line bundle $\det\mathcal{A}$; the action of $\mathcal{A}^\times$ on $\det\mathcal{A}$ is given by the norm character $\mathcal{N}\colon\mathcal{A}^\times\to\mathcal{O}^\times$.

Consider the pullback $p_1^{-1}(\mathcal{A}^\times)$ (resp. $p_1^{-1}(\mathcal{A})$) of $\mathcal{A}^\times$ (resp. $\mathcal{A}$) to $\mathrm{Hilb}^n_X\times\overline{\mathbb{J}}_X^{1-g}$. For any sheaf $\mathcal{F}$ of $p_1^{-1}(\mathcal{A})$–algebras on $\mathrm{Hilb}^n_X\times\overline{\mathbb{J}}_X^{1-g}$, we will denote by $\mathcal{F}_{\mathcal{N}}$ the maximal quotient of $\mathcal{F}$ on which $p_1^{-1}(\mathcal{A}^\times)$ acts via the norm character $\mathcal{N}$.

Define a coherent sheaf $\mathcal{Q}'^n$ on $\mathrm{Hilb}^n_X\times\overline{\mathbb{J}}_X^{1-g}$ by the formula

$$\begin{aligned}(4\text{-}10)\quad \mathcal{Q}'^n&:=\left[\textstyle\bigwedge^n(h\times\mathrm{id})_*(f\times\mathrm{id})^*\mathcal{I}\right]_{\mathcal{N}}\otimes p_1^*(\det\mathcal{A})^{-1}\\ &=\left[\textstyle\bigwedge^n(h\times\mathrm{id})_*(f\times\mathrm{id})^*\mathcal{I}\right]_{\mathcal{N}}\otimes\left[\textstyle\bigwedge^n(h\times\mathrm{id})_*\mathcal{O}_{\mathcal{D}\times\overline{\mathbb{J}}_X^{1-g}}\right],\end{aligned}$$

[4] Arinkin stated his results in [5] under the assumption that $\mathrm{char}(k)=0$. However, his results are true also if $\mathrm{char}(k)>n$, as observed by Groechenig in [17, Remark 4.9].

where $\mathcal{I}$ is a universal sheaf on $X \times \overline{\mathbb{J}}_X^{1-g}$ as in Section 2.1 and where the maps involved in the above formula are collected in the diagram

$$\begin{array}{ccccc} & & \mathcal{D} \times \overline{\mathbb{J}}_X^{1-g} & \hookrightarrow & \mathrm{Hilb}_X^n \times X \times \overline{\mathbb{J}}_X^{1-g} \\ & \swarrow^{h\times \mathrm{id}} & & \searrow^{f\times \mathrm{id}} & \\ \mathrm{Hilb}_X^n \xleftarrow{p_1} \mathrm{Hilb}_X^n \times \overline{\mathbb{J}}_X^{1-g} & & & & X \times \overline{\mathbb{J}}_X^{1-g} \end{array}$$

Remark 4.9 The restriction of $\mathcal{Q}'^n$ to the open subset

$$\mathrm{Hilb}_X^n \times \mathbb{J}_X^{1-g} \cup {}^{\mathrm{r}}\mathrm{Hilb}_X^n \times \overline{\mathbb{J}}_X^{1-g} \subseteq \mathrm{Hilb}_X^n \times \overline{\mathbb{J}}_X^{1-g}$$

is a line bundle and is equal to

$$\begin{aligned} (4\text{-}11)\quad & \mathcal{Q}'^n_{|\mathrm{Hilb}_X^n \times \mathbb{J}_X^{1-g} \cup {}^{\mathrm{r}}\mathrm{Hilb}_X^n \times \overline{\mathbb{J}}_X^{1-g}} \\ &= \det((h\times \mathrm{id})_*(f\times \mathrm{id})^*\mathcal{I}) \otimes \det((h\times \mathrm{id})_*\mathcal{O}_{\mathcal{D}\times \overline{\mathbb{J}}_X^{1-g}})^{-1}. \end{aligned}$$

Indeed, the universal sheaf $\mathcal{I}$ is a line bundle on the open subset

$$(f\times\mathrm{id})(h\times\mathrm{id})^{-1}(\mathrm{Hilb}_X^n \times \mathbb{J}_X^{1-g} \cup {}^{\mathrm{r}}\mathrm{Hilb}_X^n \times \overline{\mathbb{J}}_X^{1-g}) = X\times \mathbb{J}_X^{1-g} \cup X_{\mathrm{sm}} \times \overline{\mathbb{J}}_X^{1-g} \subseteq X\times \overline{\mathbb{J}}_X^{1-g}.$$

This implies that $\bigwedge^n (h\times\mathrm{id})_*(f\times\mathrm{id})^*\mathcal{I} = \det((h\times\mathrm{id})_*(f\times\mathrm{id})^*\mathcal{I})$ is a line bundle on $\mathrm{Hilb}_X^n \times \mathbb{J}_X^{1-g} \cup {}^{\mathrm{r}}\mathrm{Hilb}_X^n \times \overline{\mathbb{J}}_X^{1-g}$ on which $p_1^{-1}(\mathcal{A}^\times)$ acts via the norm character $\mathcal{N}$. The expression (4-11) now follows.

The relation between $\mathcal{Q}^n$ and $\mathcal{Q}'^n$ is clarified by the following result of Arinkin.[5]

Fact 4.10 (Arinkin [5, Proposition 4.4, Section 4.2]) *Assume that either* $\mathrm{char}(k) = 0$ *or* $\mathrm{char}(k) > n$*. Let* X *be a connected projective reduced curve with planar singularities. The sheaves* $\mathcal{Q}^n$ *and* $\mathcal{Q}'^n$ *coincide on the open subset* ${}^{\mathrm{c}}\mathrm{Hilb}_X^n \times \overline{\mathbb{J}}_X^{1-g} \subseteq \mathrm{Hilb}_X^n \times \overline{\mathbb{J}}_X^{1-g}$*.*

Let $\mathcal{Q}'$ be the sheaf on $\left(\coprod_{n\in\mathbb{N}} \mathrm{Hilb}_X^n\right) \times \overline{\mathbb{J}}_X^{1-g}$ which is equal to $\mathcal{Q}'^n$ on $\mathrm{Hilb}_X^n \times \overline{\mathbb{J}}_X^{1-g}$.

Remark 4.11 The sheaves $\mathcal{Q}^n$ and $\mathcal{Q}'^n$ depend on the choice of the universal sheaf $\mathcal{I}$ on $X \times \overline{\mathbb{J}}_X^{1-g}$. By taking another universal sheaf $\widetilde{\mathcal{I}} = \mathcal{I} \otimes \pi_2^*(N)$ for some N in

[5] The result [5, Proposition 4.4] is stated only for an integral curve X (with locally planar singularities). However, the proof of loc. cit. consists of choosing an embedding of X into a smooth and projective surface S and then using [5, Lemma 3.6], which is a statement about ${}^{\mathrm{c}}\mathrm{Hilb}_S^n$. Therefore, the same proof works for a reduced curve X with locally planar singularities using Fact 3.1.

$\operatorname{Pic}(\overline{\mathbb{J}}_X^{1-g})$ (see Section 2.1) and defining $\widetilde{\mathcal{Q}}^n$ and $\widetilde{\mathcal{Q}}'^n$ by replacing $\mathcal{I}$ with $\widetilde{\mathcal{I}}$ in formulas (4-9) and (4-10), we have that

$$\widetilde{\mathcal{Q}}^n = \mathcal{Q}^n \otimes \pi_2^*(N)^{\otimes n} \quad \text{and} \quad \widetilde{\mathcal{Q}}'^n = \mathcal{Q}'^n \otimes \pi_2^*(N)^{\otimes n}.$$

4.2.3 The relation between $\mathcal{Q}'$ and $\mathcal{P}$ We want now to compare the sheaf $\mathcal{Q}'$ to the Poincaré line bundle $\mathcal{P}$ on $(\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural = \mathbb{J}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g} \cup \overline{\mathbb{J}}_X^{1-g} \times \mathbb{J}_X^{1-g}$ (see Section 4.1) via the Abel map of (3-9).

Consider an open cover $\overline{\mathbb{J}}_X^{1-g} = \bigcup_\beta U_\beta$, as in Fact 3.7, such that for each U_β there exists $M_\beta \in \operatorname{Pic}^g(X)$ with the property that

$$V_\beta := (A^g_{M_\beta})^{-1}(U_\beta) \xrightarrow{A^g_{M_\beta}} U_\beta$$

is smooth and surjective. Fix one such U_β and consider the smooth and surjective map

$$\text{(4-12)} \qquad {}^{\mathrm{s}}\operatorname{Hilb}^g_X \times \overline{\mathbb{J}}_X^{1-g} \supseteq V_\beta \times \overline{\mathbb{J}}_X^{1-g} \xrightarrow{A^g_{M_\beta}\times \mathrm{id}} U_\beta \times \overline{\mathbb{J}}_X^{1-g} \subset \overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g}.$$

Define the open subset

$$\text{(4-13)} \quad W_\beta := [({}^{\mathrm{r}}\operatorname{Hilb}^g_X \cap V_\beta) \times \overline{\mathbb{J}}_X^{1-g}] \cup [V_\beta \times \mathbb{J}_X^{1-g}] \subseteq V_\beta \times \overline{\mathbb{J}}_X^{1-g} \subseteq {}^{\mathrm{s}}\operatorname{Hilb}^g_X \times \overline{\mathbb{J}}_X^{1-g},$$

and observe that $(A^g_{M_\beta} \times \mathrm{id})(W_\beta) \subseteq (\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural$.

Proposition 4.12 *With the same notation as above, assume that either* $\operatorname{char}(k) = 0$ *or that* $\operatorname{char}(k) > g$. *The restrictions of $\mathcal{Q}'$ and of $(A^g_{M_\beta} \times \mathrm{id})^*\mathcal{P}$ to W_β differ by the pullback of a line bundle from $\overline{\mathbb{J}}_X^{1-g}$.*

Proof Denote by π_{ij} and π_i the projections of $X \times \operatorname{Hilb}^g_X \times \overline{\mathbb{J}}_X^{1-g}$ (or of its open subsets $X \times W_\beta \subseteq X \times V_\beta \times \overline{\mathbb{J}}_X^{1-g}$) onto the factors corresponding to the subscripts, and consider the commutative diagram

$$\text{(4-14)} \quad \begin{array}{ccccccc}
 & \nearrow^{f\times\mathrm{id}} & X\times\overline{\mathbb{J}}_X^{1-g} & & & & \\
 & & \uparrow{\scriptstyle \pi_{13}} & & & & \\
\mathcal{D}\times\overline{\mathbb{J}}_X^{1-g} & \hookrightarrow & X\times\operatorname{Hilb}^g_X\times\overline{\mathbb{J}}_X^{1-g} & \hookleftarrow & X\times W_\beta & \xrightarrow{\mathrm{id}\times A^g_{M_\beta}\times\mathrm{id}} & X\times(\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g})^\natural \\
 & \searrow_{h\times\mathrm{id}} & \downarrow{\scriptstyle \pi_{23}} & & \downarrow{\scriptstyle \pi_{23}} & & \downarrow{\scriptstyle p_{23}} \\
 & & \operatorname{Hilb}^g_X\times\overline{\mathbb{J}}_X^{1-g} & \hookleftarrow & W_\beta & \xrightarrow{A^g_{M_\beta}\times\mathrm{id}} & (\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g})^\natural
\end{array}$$

From the definition of the Abel map (3-9), it follows that the pullback of the universal sheaf $\mathcal{I}$ via the map $\mathrm{id}\times A^g_{M_\beta}\colon X\times V_\beta\to X\times U_\beta\subseteq X\times\overline{\mathbb{J}}^{1-g}_X$ is equal to

$$(\mathrm{id}\times A^g_{M_\beta})^*\mathcal{I}=\mathcal{I}(\mathcal{D})|_{X\times V_\beta}\otimes p_1^*(M_\beta)\otimes p_2^*(N), \tag{4-15}$$

where $\mathcal{I}(\mathcal{D})$ is the ideal sheaf of the universal divisor $\mathcal{D}\subset X\times\mathrm{Hilb}^g_X$, p_1 and p_2 are the projection maps from $X\times U_\beta$ onto X and U_β, respectively, and N is some line bundle on V_β.

By the base change property of the determinant of cohomology [12, Proposition 44(1)] applied to the definition (4-1) of $\mathcal{P}$, and using (4-15), we get that

$$\begin{aligned}(A^g_{M_\beta}\times\mathrm{id})^*\mathcal{P}\\ &=\mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D})\otimes\pi_1^*M_\beta\otimes\pi_2^*N\otimes\pi_{13}^*\mathcal{I})^{-1}\\ &\qquad\otimes\mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D})\otimes\pi_1^*M_\beta\otimes\pi_2^*N)\otimes\mathcal{D}_{\pi_{23}}(\pi_{13}^*\mathcal{I}).\end{aligned} \tag{4-16}$$

By the projection property of the determinant of cohomology [12, Proposition 44(3)], and using that $\pi_{12}^*\mathcal{I}(\mathcal{D})\otimes\pi_1^*M_\beta$ and $\pi_{13}^*\mathcal{I}$ have relative Euler characteristic equal to $1-g$, we get

$$\begin{aligned}&\mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D})\otimes\pi_1^*M_\beta\otimes\pi_2^*N\otimes\pi_{13}^*\mathcal{I})\\ &\qquad=\mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D})\otimes\pi_1^*M_\beta\otimes\pi_{13}^*\mathcal{I})\otimes(\pi_2^*N)^{1-g},\\ &\mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D})\otimes\pi_1^*M_\beta\otimes\pi_2^*N)=\mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D})\otimes\pi_1^*M_\beta)\otimes(\pi_2^*N)^{1-g}.\end{aligned} \tag{4-17}$$

Substituting (4-17) into (4-16), we get

$$\begin{aligned}(A^g_{M_\beta}\times\mathrm{id})^*\mathcal{P}\\ &=\mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D})\otimes\pi_1^*M_\beta\otimes\pi_{13}^*\mathcal{I})^{-1}\otimes\mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D})\otimes\pi_1^*M_\beta)\\ &\qquad\otimes\mathcal{D}_{\pi_{23}}(\pi_{13}^*\mathcal{I}).\end{aligned} \tag{4-18}$$

Claim *The two line bundles*

$$\mathcal{M}:=\mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D})\otimes\pi_1^*M_\beta\otimes\pi_{13}^*\mathcal{I})^{-1}\otimes\mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D})\otimes\pi_1^*M_\beta)\otimes\mathcal{D}_{\pi_{23}}(\pi_{13}^*\mathcal{I}),$$
$$\mathcal{N}:=\mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D})\otimes\pi_{13}^*\mathcal{I})^{-1}\otimes\mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D}))\otimes\mathcal{D}_{\pi_{23}}(\pi_{13}^*\mathcal{I})$$

on $\mathrm{Hilb}^g_X\times\overline{\mathbb{J}}^{1-g}_X$ *differ by the pullback of a line bundle from* $\overline{\mathbb{J}}^{1-g}_X$*.*

Indeed, since Hilb^g_X is a connected and reduced projective scheme (by Fact 3.6) and $\overline{\mathbb{J}}^{1-g}_X$ is reduced and locally Noetherian (by Facts 2.3 and 2.6), the claim will follow from the seesaw principle [39, Section II.5, Corollary 6] if we show that

$$\mathcal{M}_{|\mathrm{Hilb}^g_X\times\{I\}}=\mathcal{N}_{|\mathrm{Hilb}^g_X\times\{I\}}\quad\text{for any } I\in\overline{\mathbb{J}}^{1-g}_X. \tag{4-19}$$

By the base change property of the determinant of cohomology, we get

$$\begin{aligned}\mathcal{M}_{|\mathrm{Hilb}^g_X\times\{I\}} &= \mathcal{D}_{\pi_2}(\mathcal{I}(\mathcal{D})\otimes\pi_1^*M_\beta\otimes\pi_1^*I)^{-1}\otimes\mathcal{D}_{\pi_2}(\mathcal{I}(\mathcal{D})\otimes\pi_1^*M_\beta),\\ \mathcal{N}_{|\mathrm{Hilb}^g_X\times\{I\}} &= \mathcal{D}_{\pi_2}(\mathcal{I}(\mathcal{D})\otimes\pi_1^*I)^{-1}\otimes\mathcal{D}_{\pi_2}(\mathcal{I}(\mathcal{D})),\end{aligned}\tag{4-20}$$

where π_i (for $i=1,2$) is the projection of $X\times\mathrm{Hilb}^g_X$ onto the i^{th} factor. Using these formulas, the equality (4-19) follows from Corollary 4.4.

Consider now the exact sequence associated to the universal divisor $\mathcal{D}\subset X\times\mathrm{Hilb}^g_X$,

$$0\to\mathcal{I}(\mathcal{D})\to\mathcal{O}_{X\times\mathrm{Hilb}^g_X}\to\mathcal{O}_{\mathcal{D}}\to 0.\tag{4-21}$$

By pulling back (4-21) via $\pi_{12}\colon X\times W_\beta\to X\times\mathrm{Hilb}^g_X$, tensoring the result with $\pi_{13}^*\mathcal{I}$ (it remains exact since, by the definition of W_β, $\pi_{13}^*\mathcal{I}$ is a line bundle on $\pi_{12}^{-1}(D)\cap(X\times W_\beta)\subseteq(X_{\mathrm{sm}}\times{}^{\mathrm{r}}\mathrm{Hilb}^g_X\times\overline{\mathbb{J}}^{1-g}_X)\cup(X\times\mathrm{Hilb}^g_X\times\bar{J}^0_X))$ and using the additivity property of the determinant of cohomology, we get

$$\begin{aligned}\mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D})\otimes\pi_{13}^*\mathcal{I})^{-1} &= \mathcal{D}_{\pi_{23}}(\pi_{13}^*\mathcal{I})^{-1}\otimes\mathcal{D}_{\pi_{23}}((\pi_{13}^*\mathcal{I})|_{\pi_{12}^{-1}(\mathcal{D})}),\\ \mathcal{D}_{\pi_{23}}(\pi_{12}^*\mathcal{I}(\mathcal{D})) &= \mathcal{D}_{\pi_{23}}(\mathcal{O}_{X\times W_\beta})\otimes\mathcal{D}_{\pi_{23}}(\mathcal{O}_{\pi_{12}^{-1}(\mathcal{D})})^{-1}.\end{aligned}\tag{4-22}$$

By the base change property of the determinant of cohomology, we get that

$$\mathcal{D}_{\pi_{23}}(\mathcal{O}_{X\times W_\beta})=\mathcal{O}_{W_\beta}.\tag{4-23}$$

By the definition of W_β, the restriction of the sheaf $\pi_{13}^*\mathcal{I}$ over the relative divisor $\pi_{12}^{-1}(\mathcal{D})\subset X\times W_\beta\to W_\beta$ is a line bundle. Therefore $\pi_{23*}(\pi_{13}^*\mathcal{I}|_{\pi_{12}^{-1}(\mathcal{D})})$ and $\pi_{23*}(\mathcal{O}_{\pi_{12}^{-1}(\mathcal{D})})$ are locally free sheaves of rank g over W_β. From the definition of the determinant of cohomology (see Section 4.1) and the commutative diagram (4-14), it follows that over W_β we have the equalities

$$\begin{aligned}\mathcal{D}_{\pi_{23}}(\pi_{13}^*\mathcal{I}|_{\pi_{12}^{-1}(\mathcal{D})}) &= \det[\pi_{23*}(\pi_{13}^*\mathcal{I}|_{\pi_{12}^{-1}(\mathcal{D})})] = \det[(h\times\mathrm{id})_*(f\times\mathrm{id})^*\mathcal{I}],\\ \mathcal{D}_{\pi_{23}}(\mathcal{O}_{\pi_{12}^{-1}(\mathcal{D})}) &= \det[\pi_{23*}(\mathcal{O}_{\pi_{12}^{-1}(\mathcal{D})})] = \det[(h\times\mathrm{id})_*\mathcal{O}_{\mathcal{D}\times\overline{\mathbb{J}}^{1-g}_X}].\end{aligned}\tag{4-24}$$

Observe that since W_β is contained in the open subset $\mathrm{Hilb}^g_X\times\mathbb{J}^{1-g}_X\cup{}^{\mathrm{r}}\mathrm{Hilb}^g_X\times\overline{\mathbb{J}}^{1-g}_X\subseteq\mathrm{Hilb}^g_X\times\overline{\mathbb{J}}^{1-g}_X$, the restriction of $\mathcal{Q}'$ to W_β is given by the expression (4-11). Therefore, using (4-22), (4-23) and (4-24), we get

$$\mathcal{Q}'|_{W_\beta}=\mathcal{Q}'^{g}|_{W_\beta}=\mathcal{N}|_{W_\beta}.\tag{4-25}$$

We now conclude the proof by combining (4-25), (4-18) and the above claim. □

Remark 4.13 We cannot hope to have equality in Proposition 4.12 since $\mathcal{Q}'$ is only well-defined up to the pullback of a line bundle from $\overline{\mathbb{J}}_X^{1-g}$ (Remark 4.11) while $\mathcal{P}$ is well-defined (Remark 4.2).

4.2.4 Descending $\mathcal{Q}$ to $\overline{\mathcal{P}}$ We are now ready to prove Theorem 4.6. In order to do so, we need the following result, which will allow us to descend the sheaf $\mathcal{Q}$ of Section 4.2.1 to our desired Poincaré sheaf $\overline{\mathcal{P}}$.

Lemma 4.14 *Let $f\colon Y \to Z$ be a faithfully flat morphism of finite type between locally Noetherian schemes. Let $\mathcal{F}$ be a (maximal) Cohen–Macaulay coherent sheaf on Y and let $\mathcal{G}$ be a Cohen–Macaulay coherent sheaf on an open subset $j\colon V \hookrightarrow Z$. Assume that there exists an open subset $i\colon U \hookrightarrow Y$ such that*

(i) *the complement of U inside Y has codimension at least two,*

(ii) $f(U) \subseteq V$,

(iii) $(f|_U)^*(\mathcal{G}|_{f(U)}) = \mathcal{F}|_U \otimes f^*(N)|_U$ *for some line bundle N on Z.*

Then there exists a unique coherent sheaf $\widetilde{\mathcal{G}}$ on Z such that

(a) *$\widetilde{\mathcal{G}}$ is a (maximal) Cohen–Macaulay sheaf,*

(b) $\widetilde{\mathcal{G}}|_V = \mathcal{G}$,

(c) $f^*(\widetilde{\mathcal{G}}) = \mathcal{F} \otimes f^*(N)$.

Proof First of all, observe that f is both quasicompact (hence an fpqc morphism) by Section 1.5 of [19], and of finite presentation (hence an fppf morphism) by Section 1.6. Moreover, by replacing $\mathcal{G}$ with $\mathcal{G} \otimes N^{-1}$, we can assume that $N = \mathcal{O}_Z$.

Let us first prove the uniqueness of $\widetilde{\mathcal{G}}$. From hypotheses (i) and (ii) and the fact that f is open with equidimensional fibers (being fppf; see [20, Theorem 2.4.6] and [21, Corollary 14.2.2]), we get that the complement of V inside Z has codimension at least two. Since $\widetilde{\mathcal{G}}$ is Cohen–Macaulay by (a) and $\widetilde{\mathcal{G}}|_V = \mathcal{G}$ by (b), we get that $\widetilde{\mathcal{G}} = j_*\mathcal{G}$ by [20, Theorem 5.10.5]; hence $\widetilde{\mathcal{G}}$ is unique.

Let us now prove the existence of $\widetilde{\mathcal{G}}$. Using fpqc descent for quasicoherent sheaves [15, Theorem 4.23], the existence of a quasicoherent sheaf $\widetilde{\mathcal{G}}$ satisfying (c) will follow if we find a descent data for $\mathcal{F}$ relative to the fpqc morphism $f\colon Y \to Z$, ie an isomorphism $\phi\colon p_1^*(\mathcal{F}) \xrightarrow{\cong} p_2^*(\mathcal{F})$ which satisfies the cocycle condition $p_{13}^*(\phi) = p_{12}^*(\phi) \circ p_{23}^*(\phi)$, where $p_i\colon Y \times_Z Y \to Y$ and $p_{ij}\colon Y \times_Z Y \times_Z Y \to Y \times_Z Y$

denote the projection onto the i^{th} and ij^{th} factors, respectively. Because of (iii), a descent data exists for $\mathcal{F}|_U$ relative to the fpqc morphism $f\colon U \to f(U)$, ie there exists an isomorphism $\psi\colon q_1^*(\mathcal{F}|_U) \xrightarrow{\cong} q_2^*(\mathcal{F}|_U)$ such that $q_{13}^*(\psi) = q_{12}^*(\psi) \circ q_{23}^*(\psi)$, where $q_i\colon U \times_{f(U)} U \to U$ and $q_{ij}\colon U \times_{f(U)} U \times_{f(U)} U \to U \times_{f(U)} U$ denote the projection onto the i^{th} and ij^{th} factors, respectively. Observe now that, since $\mathcal{F}$ is Cohen–Macaulay and the complement of U inside Y has codimension at least two by (i), it holds that $\mathcal{F} = i_*(\mathcal{F}|_U)$ by [20, Theorem 5.10.5]. By taking the pushforward of ψ with respect to the open embedding $i \times i\colon U \times_{f(U)} U \hookrightarrow Y \times_Z Y$, we obtain an isomorphism

$$p_1^*(\mathcal{F}) = p_1^*(i_*(\mathcal{F}|_U)) = (i \times i)_*(q_1^*(\mathcal{F}|_U)) \xrightarrow{\phi := (i\times i)_*(\psi)} (i \times i)_*(q_2^*(\mathcal{F}|_U)) = p_2^*(i_*(\mathcal{F}|_U)) = p_2^*(\mathcal{F}),$$

which clearly satisfies the cocycle condition since ψ does. Therefore, by fpqc descent, we obtain a quasicoherent sheaf $\widetilde{\mathcal{G}}$ on Z which satisfies (c). Observe now that $\widetilde{\mathcal{G}}$ is of finite type by faithful descent [20, Proposition 2.5.2], hence coherent because Y is locally Noetherian [18, Section 6.1]. Moreover, $\widetilde{\mathcal{G}}$ is (maximal) Cohen–Macaulay by faithful descent [20, Proposition 6.4.1], hence (a) is satisfied. Finally, since the descent data for $\mathcal{F}$ that were used above to construct $\widetilde{\mathcal{G}}$ are induced by the descent data for $\mathcal{F}|_U = f^*(\mathcal{G})|_U$, it follows that $\widetilde{\mathcal{G}}|_{f(U)} = \mathcal{G}|_{f(U)}$. Therefore, the two Cohen–Macaulay sheaves $\widetilde{\mathcal{G}}|_V$ and $\mathcal{G}$ on V have the same restriction to the open subset $f(U) \subseteq V$ whose complement has codimension at least two by the observation above; hence $\widetilde{\mathcal{G}}|_V = \mathcal{G}$ by [20, Theorem 5.10.5] and (b) is satisfied. □

Proof of Theorem 4.6 Consider an open cover $\overline{\mathbb{J}}_X^{1-g} = \bigcup_\beta U_\beta$ as in Fact 3.7, such that for each U_β there exists $M_\beta \in \mathrm{Pic}^g(X)$ with the property that

$$V_\beta := (A^g_{M_\beta})^{-1}(U_\beta) \xrightarrow{A^g_{M_\beta}} U_\beta$$

is smooth and surjective.

We want to apply the descent Lemma 4.14 to the smooth and surjective (hence faithfully flat of finite type) morphism $A^g_{M_\beta} \times \mathrm{id}\colon V_\beta \times \overline{\mathbb{J}}_X^{1-g} \to U_\beta \times \overline{\mathbb{J}}_X^{1-g}$ with respect to the sheaf $\mathcal{Q}$ on $V_\beta \times \overline{\mathbb{J}}_X^{1-g}$ (which is a maximal Cohen–Macaulay sheaf by Fact 4.8(ii)) and to the line bundle $\mathcal{P}$ defined on the open subset

$$(U_\beta \times \overline{\mathbb{J}}_X^{1-g}) \cap (\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural \subseteq U_\beta \times \overline{\mathbb{J}}_X^{1-g}.$$

Let us check that the hypotheses of Lemma 4.14 are satisfied if we choose the open subset

$$W'_\beta := W_\beta \cap {}^{\mathrm{c}}\mathrm{Hilb}^g_X \times \overline{\mathbb{J}}^{1-g}_X = [({}^{\mathrm{r}}\mathrm{Hilb}^g_X \cap V_\beta) \times \overline{\mathbb{J}}^{1-g}_X] \cup [({}^{\mathrm{c}}\mathrm{Hilb}^g_X \cap V_\beta) \times \mathbb{J}^{1-g}_X] \subseteq V_\beta \times \overline{\mathbb{J}}^{1-g}_X,$$

where W_β is as defined in (4-13). We have already observed in Section 4.2.3 that

$$(A^g_{M_\beta} \times \mathrm{id})(W'_\beta) \subseteq (A^g_{M_\beta} \times \mathrm{id})(W_\beta) \subseteq (\overline{\mathbb{J}}^{1-g}_X \times \overline{\mathbb{J}}^{1-g}_X)^\natural,$$

which gives hypothesis (ii) of the lemma. Hypothesis (iii) follows from Fact 4.10 and Proposition 4.12. In order to prove hypothesis (i), observe that the complement of W'_β inside $V_\beta \times \overline{\mathbb{J}}^{1-g}_X$ is given by the closed subset

$$[(V_\beta \cap (\mathrm{Hilb}^g_X \setminus {}^{\mathrm{c}}\mathrm{Hilb}^g_X)) \times \overline{\mathbb{J}}^{1-g}_X] \cup [(V_\beta \cap (\mathrm{Hilb}^g_X \setminus {}^{\mathrm{r}}\mathrm{Hilb}^g_X)) \times (\overline{\mathbb{J}}^{1-g}_X \setminus \mathbb{J}^{1-g}_X)].$$

This closed subset has codimension at least two, since $\mathrm{Hilb}^g_X \setminus {}^{\mathrm{c}}\mathrm{Hilb}^g_X$ has codimension at least two by Lemma 3.9, $\mathrm{Hilb}^g_X \setminus {}^{\mathrm{r}}\mathrm{Hilb}^g_X$ has codimension at least one by Fact 3.6(ii), and $\overline{\mathbb{J}}^{1-g}_X \setminus \mathbb{J}^{1-g}_X$ has codimension at least one by Fact 2.3(ii). Hypothesis (i) of Lemma 4.14 is therefore satisfied.

We can now apply Lemma 4.14 to obtain a unique maximal Cohen–Macaulay sheaf $\widetilde{\mathcal{P}}_\beta$ on $U_\beta \times \overline{\mathbb{J}}^{1-g}_X$ that agrees with $\mathcal{P}$ on the open subset $(U_\beta \times \overline{\mathbb{J}}^{1-g}_X) \cap (\overline{\mathbb{J}}^{1-g}_X \times \overline{\mathbb{J}}^{1-g}_X)^\natural$ and whose pullback via $A^g_{M_\beta} \times \mathrm{id}$ agrees with $\mathcal{Q}$, up to the pullback of a line bundle on $\overline{\mathbb{J}}^{1-g}_X$. Because of the uniqueness of $\widetilde{\mathcal{P}}_\beta$ and the fact that $\mathcal{P}$ is defined on the whole $(\overline{\mathbb{J}}^{1-g}_X \times \overline{\mathbb{J}}^{1-g}_X)^\natural$, the sheaves $\widetilde{\mathcal{P}}_\beta$ glue together to give a maximal Cohen–Macaulay sheaf $\widetilde{\mathcal{P}}$ on $\overline{\mathbb{J}}^{1-g}_X \times \overline{\mathbb{J}}^{1-g}_X$ that agrees with $\mathcal{P}$ on the open subset $(\overline{\mathbb{J}}^{1-g}_X \times \overline{\mathbb{J}}^{1-g}_X)^\natural \subseteq \overline{\mathbb{J}}^{1-g}_X \times \overline{\mathbb{J}}^{1-g}_X$. We have already observed in Remark 4.7 that this is enough to ensure that $\widetilde{\mathcal{P}} = \overline{\mathcal{P}} := j_*(\mathcal{P})$. Part (i) of the theorem now follows.

For part (ii), it is clearly enough to prove the desired properties for the sheaves $\widetilde{\mathcal{P}}_\beta$ on $U_\beta \times \overline{\mathbb{J}}^{1-g}_X$. By construction (see Proposition 4.12 and Lemma 4.14(c)), we have that

$$(A^g_{M_\beta} \times \mathrm{id})^*(\widetilde{\mathcal{P}}_\beta) = \mathcal{Q}|_{V_\beta \times \overline{\mathbb{J}}^{1-g}_X} \otimes \pi_2^* N$$

for some line bundle N on $\overline{\mathbb{J}}^{1-g}_X$. Now the flatness of $\widetilde{\mathcal{P}}_\beta$ with respect to the second projection follows from the analogous property of $\mathcal{Q}$; see Fact 4.8(iii). For any fixed $I \in \overline{\mathbb{J}}^{1-g}_X$, the fact that $(\widetilde{\mathcal{P}}_\beta)|_{U_\beta \times \{I\}}$ is maximal Cohen–Macaulay follows from the fact that $(\mathcal{Q} \otimes \pi_2^* N)|_{V_\beta \times \{I\}} = \mathcal{Q}|_{V_\beta \times \{I\}}$ is maximal Cohen–Macaulay (Fact 4.8(iv)) using faithful descent with respect to the smooth and surjective morphism $A^g_{M_\beta}\colon V_\beta \to U_\beta$; see [20, Proposition 6.4.1]. □

5 Properties of the Poincaré sheaf

Throughout this section, we assume that X is a connected reduced curve with planar singularities of arithmetic genus $g := p_a(X)$ and that either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > g$. The aim of this section is to prove several properties of the Poincaré sheaf $\overline{\mathcal{P}}$ on $\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g}$ constructed in Section 4.

First of all, $\overline{\mathcal{P}}$ is symmetric with respect to the two factors.

Proposition 5.1 *The Poincaré sheaf is equivariant under the permutation*

$$\sigma\colon \overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g} \to \overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g}$$

of the two factors.

In particular, for any $I \in \overline{\mathbb{J}}_X^{1-g}$, we have that

$$\overline{\mathcal{P}}_I := \overline{\mathcal{P}}|_{\overline{\mathbb{J}}_X^{1-g} \times \{I\}} = \overline{\mathcal{P}}|_{\{I\} \times \overline{\mathbb{J}}_X^{1-g}}.$$

Proof From the definition in equation (4-1) it is clear that the Poincaré line bundle $\mathcal{P}$ on $(\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^{\natural}$ is equivariant under the permutation of the two factors. The same result for $\overline{\mathcal{P}}$ follows now from Remark 4.7. □

We now study the behavior of $\overline{\mathcal{P}}$ under the Serre dualizing functor. Recall that, for any Cohen–Macaulay scheme Z with dualizing sheaf ω_Z, the Serre dualizing functor is defined as

$$\mathbb{D}_Z\colon D^b_{\mathrm{coh}}(Z) \to D^b_{\mathrm{coh}}(Z), \quad \mathcal{K}^\bullet \mapsto (\mathcal{K}^\bullet)^D := \mathcal{RH}om(\mathcal{K}^\bullet, \omega_Z).$$

We will need the following well-known facts.

Fact 5.2 *Let Z be a Cohen–Macaulay scheme with dualizing functor $\mathbb{D}_Z$. Then:*

(i) *$\mathbb{D}_Z$ is an involution; in other words $((\mathcal{K}^\bullet)^D)^D = \mathcal{K}^\bullet$ for any $\mathcal{K}^\bullet \in D^b_{\mathrm{coh}}(Z)$.*

(ii) *A coherent sheaf $\mathcal{F}$ on Z is maximal Cohen–Macaulay if and only if $\mathcal{F}^D$ is concentrated in degree zero, ie if $\mathcal{F}^D = \mathcal{H}om(\mathcal{F}, \omega_Z)$.*

Proof See [23, Chapter V, Proposition 2.1] for (i), and see [10, Corollary 3.5.11] for (ii). □

By Fact 2.3(i), $\overline{\mathbb{J}}_X^{1-g}$ is a Gorenstein (and in particular Cohen–Macaulay) scheme. Therefore, the same is true for $\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g}$.

Proposition 5.3 *The Serre dual complex* $\overline{\mathcal{P}}^D$ *and the dual sheaf*

$$\overline{\mathcal{P}}^\vee := \mathcal{H}om(\overline{\mathcal{P}}, \mathcal{O}_{\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}})$$

of the Poincaré sheaf $\overline{\mathcal{P}}$ *satisfy the following properties:*

(i) $\overline{\mathcal{P}}^D$ *is concentrated in degree* 0 *and*

$$\overline{\mathcal{P}}^D = \overline{\mathcal{P}}^\vee \otimes \omega_{\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}}. \tag{5-1}$$

(ii) $\overline{\mathcal{P}}^D$ *(and hence also* $\overline{\mathcal{P}}^\vee$*) is a maximal Cohen–Macaulay sheaf on* $\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$*.*

(iii) $\overline{\mathcal{P}}^\vee$ *and* $\overline{\mathcal{P}}^D$ *are equivariant with respect to the permutation* $\sigma\colon \overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}\to\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$ *of the two factors. In particular,*

$$(\overline{\mathcal{P}}^\vee)|_{\overline{\mathbb{J}}_X^{1-g}\times\{I\}} = (\overline{\mathcal{P}}^\vee)|_{\{I\}\times\overline{\mathbb{J}}_X^{1-g}} := (\overline{\mathcal{P}}^\vee)_I$$

for every $I\in\overline{\mathbb{J}}_X^{1-g}$*, and similarly for* $(\overline{\mathcal{P}}^D)_I$*.*

(iv) $\overline{\mathcal{P}}^D$ *(and hence also* $\overline{\mathcal{P}}^\vee$*) is flat with respect to the two projections*

$$p_i\colon \overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}\to\overline{\mathbb{J}}_X^{1-g} \quad \text{for } i=1,2,$$

and for every $I\in\overline{\mathbb{J}}_X^{1-g}$*, the restriction* $(\overline{\mathcal{P}}^D)_I = (\overline{\mathcal{P}}^\vee)_I\otimes\omega_{\overline{\mathbb{J}}_X^{1-g}}$ *is a maximal Cohen–Macaulay sheaf on* $\overline{\mathbb{J}}_X^{1-g}$*. Moreover,*

$$(\overline{\mathcal{P}}^\vee)_I = (\overline{\mathcal{P}}_I)^\vee \quad\text{and}\quad (\overline{\mathcal{P}}^D)_I = (\overline{\mathcal{P}}_I)^D.$$

Proof Property (i): the fact that $\overline{\mathcal{P}}^D$ is concentrated in degree 0 follows from the fact that $\overline{\mathcal{P}}$ is a maximal Cohen–Macaulay sheaf on $\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$ (by Theorem 4.6(i)) together with Fact 5.2(ii). Formula (5-1) follows from the previous fact together with the fact that the dualizing sheaf of $\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$ is a line bundle (because $\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$ is a Gorenstein scheme).

Property (ii) follows by combining Facts 5.2(i) and 5.2(ii).

Property (iii) follows from the corresponding statement for $\overline{\mathcal{P}}$; see Proposition 5.1.

Property (iv): combining Theorem 4.6(ii) with [5, Lemma 2.1(2)] (which can be applied since $\overline{\mathbb{J}}_X^{1-g}$ is Gorenstein), we deduce that $\overline{\mathcal{P}}^D$ (and hence also $\overline{\mathcal{P}}^\vee$ by property (i)) is flat with respect to the two projections p_1 and p_2, and that $(\overline{\mathcal{P}}^D)_I = (\overline{\mathcal{P}}_I)^D$ for every $I\in\overline{\mathbb{J}}_X^{1-g}$. Moreover, since $\overline{\mathcal{P}}_I$ is maximal Cohen–Macaulay by Theorem 4.6(ii), Fact 5.2 implies that also $(\overline{\mathcal{P}}^D)_I$ is maximal Cohen–Macaulay. We conclude using the fact that $(\overline{\mathcal{P}}^D)_I = (\overline{\mathcal{P}}^\vee)_I\otimes\omega_{\overline{\mathbb{J}}_X^{1-g}}$ by formula (5-1), and the analogous formula $(\overline{\mathcal{P}}_I)^D = (\overline{\mathcal{P}}_I)^\vee\otimes\omega_{\overline{\mathbb{J}}_X^{1-g}}$. □

Let us now study the behavior of the Poincaré sheaf $\overline{\mathcal{P}}$ under the natural multiplication map

$$\mu\colon \mathbb{J}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g} \to \overline{\mathbb{J}}_X^{1-g}, \quad (L, I) \mapsto L \otimes I. \tag{5-2}$$

Proposition 5.4 *Consider the diagram*

$$\begin{array}{ccccc} \overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g} & \xleftarrow{\pi_{23}} & \mathbb{J}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g} & \xrightarrow{\pi_{13}} & \mathbb{J}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g} \\ & & \Big\downarrow{\scriptstyle \mu\times \mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}}} & & \\ & & \overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g} & & \end{array}$$

The Poincaré sheaf satisfies the property

$$(\mu \times \mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}})^*(\overline{\mathcal{P}}) = \pi_{13}^*(\mathcal{P}) \otimes \pi_{23}^*(\overline{\mathcal{P}}).$$

In particular, for any $(L, I) \in \mathbb{J}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g}$, it holds that $\overline{\mathcal{P}}_{L\otimes I} = \mathcal{P}_L \otimes \overline{\mathcal{P}}_I$.

Proof We are going to apply Lemma 5.5, with $T = \mathbb{J}_X^{1-g}$, $Z = \overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g}$, $\mathcal{F} = \pi_{13}^*(\mathcal{P}) \otimes \pi_{23}^*(\overline{\mathcal{P}})$ and $\mathcal{G} = (\mu \times \mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}})^*(\overline{\mathcal{P}})$. Let us check that the hypotheses of the lemma are satisfied.

First of all, by Facts 2.3(i) and 2.6(ii), $\mathbb{J}_X^{1-g}$ and $\overline{\mathbb{J}}_X^{1-g}$ are reduced locally Noetherian schemes. Moreover, $\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g}$ can be covered by the countably many connected and proper open subsets $\bar{J}_X(\underline{q}) \times \bar{J}_X(\underline{q}')$, by Facts 2.6 and Fact 2.7(i).

For any $L \in \mathbb{J}_X^{1-g}$ and any $\bar{J}_X(\underline{q}) \times \bar{J}_X(\underline{q}') \subseteq \overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g}$, the sheaf

$$\pi_{13}^*(\mathcal{P}) \otimes \pi_{23}^*(\overline{\mathcal{P}})|_{\{L\}\times \bar{J}_X(\underline{q})\times \bar{J}_X(\underline{q}')}$$

is simple because $\mathcal{P}$ is a line bundle and, by Definition 4.5, $\overline{\mathcal{P}}_{|\bar{J}_X(\underline{q})\times \bar{J}_X(\underline{q}')}$ is the pushforward of the line bundle $\mathcal{P}$ from the open subset $J_X(\underline{q})\times \bar{J}_X(\underline{q}') \cup \bar{J}_X(\underline{q}) \times J_X(\underline{q}')$ of $\bar{J}_X(\underline{q}) \times \bar{J}_X(\underline{q}')$, whose complement has codimension greater than or equal to two. Therefore, hypothesis (ii) of Lemma 5.5 is satisfied.

In order to check that hypothesis (iii) of Lemma 5.5 is satisfied, we will check that for any $(I_1, I_2) \in \mathbb{J}_X^{1-g} \times \mathbb{J}_X^{1-g} \subseteq \overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g}$, it holds that

$$(\mu \times \mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}})^*(\overline{\mathcal{P}})|_{\mathbb{J}_X^{1-g}\times\{I_1\}\times\{I_2\}} = \pi_{13}^*(\mathcal{P}) \otimes \pi_{23}^*(\overline{\mathcal{P}})|_{\mathbb{J}_X^{1-g}\times\{I_1\}\times\{I_2\}}. \tag{5-3}$$

This will imply that hypothesis (iii) holds since $\mathbb{J}_X^{1-g}\times\mathbb{J}_X^{1-g}$ is dense in $\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$ by Fact 2.3(ii). After identifying $\mathbb{J}_X^{1-g}\times\{I_1\}\times\{I_2\}$ with $\mathbb{J}_X^{1-g}$, (5-3) is equivalent to

$$t_{I_1}^*(\mathcal{P}_{I_2})=\mathcal{P}_{I_2}, \tag{5-4}$$

where $t_{I_1}\colon \mathbb{J}_X^{1-g}\to\mathbb{J}_X^{1-g}$ is the translation map sending L into $L\otimes I_1$. Equality (5-4) follows now from [34, Lemma 5.4].

Finally, in order to check that hypothesis (i) of Lemma 5.5 is satisfied, we need to prove that for any $L\in\mathbb{J}_X^{1-g}$, we have that

$$(\mu\times\mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}})^*(\overline{\mathcal{P}})|_{\{L\}\times\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}}=\pi_{13}^*(\mathcal{P})\otimes\pi_{23}^*(\overline{\mathcal{P}})|_{\{L\}\times\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}}. \tag{5-5}$$

Identifying $\{L\}\times\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$ with $\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}$, (5-5) is equivalent to

$$(t_L\times\mathrm{id})^*(\overline{\mathcal{P}})=\pi_2^*(\mathcal{P}_L)\otimes\overline{\mathcal{P}}, \tag{5-6}$$

where $t_L\colon\overline{\mathbb{J}}_X^{1-g}\to\overline{\mathbb{J}}_X^{1-g}$ is the translation map which sends I to $I\otimes L$, and the map $\pi_2\colon\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g}\to\overline{\mathbb{J}}_X^{1-g}$ is projection onto the second factor. Since the sheaves appearing on the left- and right-hand sides of (5-6) are Cohen–Macaulay sheaves by Theorem 4.6(i), it is enough, by Remark 4.7, to show that we have the equality

$$(t_L\times\mathrm{id})^*(\mathcal{P})=\pi_2^*(\mathcal{P}_L)\otimes\mathcal{P} \tag{5-7}$$

of sheaves on $(\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g})^\natural$.

From the definition of $\mathcal{P}$ in Section 4.1 (keeping the same notation) and using the base change property of the determinant of cohomology [12, Proposition 44(1)], we get

$$(t_L\times\mathrm{id})^*\mathcal{P}=\mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}\otimes p_1^*L\otimes p_{13}^*\mathcal{I})^{-1}\otimes\mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}\otimes p_1^*L)\otimes\mathcal{D}_{p_{23}}(p_{13}^*\mathcal{I}), \tag{5-8}$$

$$\pi_2^*(\mathcal{P}_L)\otimes\overline{\mathcal{P}}=\mathcal{D}_{p_{23}}(p_1^*L\otimes p_{13}^*\mathcal{I})^{-1}\otimes\mathcal{D}_{p_{23}}(p_1^*L)\otimes\mathcal{D}_{p_{23}}(p_{13}^*\mathcal{I})\otimes\mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}\otimes p_{13}^*\mathcal{I})^{-1}\otimes\mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I})\otimes\mathcal{D}_{p_{23}}(p_{13}^*\mathcal{I}). \tag{5-9}$$

Since L is a line bundle of degree zero on X, we can find two reduced Cartier divisors $E_1=\sum_{j=1}^n q_j^1$ and $E_2=\sum_{j=1}^n q_j^2$ of the same degree on X, supported on the smooth locus of X, such that $L=I_{E_1}\otimes I_{E_2}^{-1}=\mathcal{O}_X(-E_1+E_2)$. Using (5-8) and (5-9), together with the easy fact that $\mathcal{D}_{p_{23}}(p_1^*L)=\mathcal{O}$, and applying Lemma 4.3 three times to the sheaves $p_{12}^*\mathcal{I}$, $p_{13}^*\mathcal{I}$ and $p_{12}^*\mathcal{I}\otimes p_{13}^*\mathcal{I}$, we get

$$\begin{aligned}(t_L\times\mathrm{id})^*(\mathcal{P})\otimes\pi_2^*(\mathcal{P}_L)^{-1}\otimes\mathcal{P}^{-1} &= \mathcal{D}_{p_{23}}((p_{12}^*\mathcal{I}\otimes p_{13}^*\mathcal{I})|_{p_1^{-1}(E_2)})^{-1}\otimes\mathcal{D}_{p_{23}}((p_{12}^*\mathcal{I}\otimes p_{13}^*\mathcal{I})|_{p_1^{-1}(E_1)}) \\ &\quad\otimes\mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}|_{p_1^{-1}(E_2)})\otimes\mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}|_{p_1^{-1}(E_1)})^{-1} \\ &\quad\otimes\mathcal{D}_{p_{23}}(p_{13}^*\mathcal{I}|_{p_1^{-1}(E_2)})\otimes\mathcal{D}_{p_{23}}(p_{13}^*\mathcal{I}|_{p_1^{-1}(E_1)})^{-1}.\end{aligned}\tag{5-10}$$

Observe now that for any coherent sheaf $\mathcal{F}$ on $X\times(\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g})^\natural$ whose restriction to $p_1^{-1}(E_i)=p_1^{-1}\big(\sum_j q_j^i\big)$ for $i=1,2$ is a line bundle, from the definition of the determinant of cohomology it follows that

$$\mathcal{D}_{p_{23}}(\mathcal{F}|_{p_1^{-1}(E_i)})=\bigotimes_j\mathcal{F}|_{\{q_j^i\}\times(\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g})^\natural}.$$

This implies that for any $i=1,2$ we have

$$\mathcal{D}_{p_{23}}((p_{12}^*\mathcal{I}\otimes p_{13}^*\mathcal{I})|_{p_1^{-1}(E_i)})=\mathcal{D}_{p_{23}}(p_{12}^*\mathcal{I}|_{p_1^{-1}(E_i)})\otimes\mathcal{D}_{p_{23}}(p_{13}^*\mathcal{I}|_{p_1^{-1}(E_i)}).\tag{5-11}$$

Substituting (5-11) into (5-10), we get that

$$(t_L\times\mathrm{id})^*(\mathcal{P})\otimes\pi_2^*(\mathcal{P}_L)^{-1}\otimes\mathcal{P}^{-1}=\mathcal{O}_{(\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g})^\natural},$$

which shows that (5-7) holds true.

Therefore, all the hypotheses of Lemma 5.5 are in our case satisfied, and the thesis of the lemma concludes our proof. □

The following lemma, which is a generalization of the classical seesaw principle [39, Section II.5, Corollary 6], was used in the proof of Proposition 5.4.

Lemma 5.5 (seesaw principle) *Let Z and T be two reduced locally Noetherian schemes such that Z admits an open cover $Z=\bigcup_{m\in\mathbb{N}}U_m$ with U_m proper and connected. Let $\mathcal{F}$ and $\mathcal{G}$ be two coherent sheaves on $Z\times T$, flat over T, such that*

(i) *$\mathcal{F}|_{Z\times\{t\}}\cong\mathcal{G}|_{Z\times\{t\}}$ for every $t\in T$,*

(ii) *$\mathcal{F}|_{U_m\times\{t\}}$ is simple for every $t\in T$ and $m\in\mathbb{N}$,*

(iii) *for every connected component W of Z, there exists $z_0\in W$ and isomorphism $\psi\colon\mathcal{F}|_{\{z_0\}\times T}\xrightarrow{\cong}\mathcal{G}|_{\{z_0\}\times T}$ of line bundles.*

Then $\mathcal{F}\cong\mathcal{G}$.

Proof Clearly, it is enough to prove the result for every connected component of Z; hence, we can assume that Z is connected. Moreover, up to reordering the open subsets U_m, we can assume that $z_0 \in U_0$ and that for every $m \in \mathbb{N}$ the open subset $V_m := \bigcup_{0 \le n \le m} U_m$ is connected. For every $m \in \mathbb{N}$, we set $\mathcal{F}_m := \mathcal{F}|_{V_m \times T}$ and $\mathcal{G}_m := \mathcal{G}|_{V_m \times T}$. The hypotheses (i) and (ii) on $\mathcal{F}$ and $\mathcal{G}$, together with the fact that $V_{m-1} \cap U_m \neq \varnothing$ (since V_m is connected), imply for every $m \in \mathbb{N}$ that

(a) $(\mathcal{F}_m)|_{V_m \times \{t\}} \cong (\mathcal{G}_m)|_{V_m \times \{t\}}$ for every $t \in T$, and

(b) $(\mathcal{F}_m)|_{V_m \times \{t\}}$ is simple for every $t \in T$.

Claim *For every $m \in \mathbb{N}$, there exists a unique isomorphism $\phi_m \colon \mathcal{F}_m \xrightarrow{\cong} \mathcal{G}_m$ whose restriction to $\{z_0\} \times T$ coincides with ψ.*

Indeed, because of (a) and (b), the sheaf $(p_2)_*(\mathcal{H}om(\mathcal{F}_m, \mathcal{G}_m))$ is a line bundle on T, where $p_2 \colon V_m \times T \to T$ denotes the projection onto the second factor. Moreover, since $z_0 \in U_0 \subseteq V_m$ and using (iii), we get that $(p_2)_*(\mathcal{H}om(\mathcal{F}_m, \mathcal{G}_m)) = \mathcal{O}_T$. The isomorphism ψ of (iii) defines a nonzero constant section of $(p_2)_*(\mathcal{H}om(\mathcal{F}_0, \mathcal{G}_0)) = \mathcal{O}_T$, which gives rise to an isomorphism $\phi_m \colon \mathcal{F}_m \xrightarrow{\cong} \mathcal{G}_m$ via the natural evaluation morphism $\mathcal{F}_m \otimes p_2^*((p_2)_*(\mathcal{H}om(\mathcal{F}_m, \mathcal{G}_m))) \to \mathcal{G}_m$. By construction, ϕ_m is the unique isomorphism whose restriction to $\{z_0\} \times T$ is the isomorphism ψ of (iii).

The claim implies that for any $m \ge 1$, we have that $(\phi_m)|_{V_{m-1} \times T} = \phi_{m-1}$; hence, the isomorphisms ϕ_m glue together producing an isomorphism $\phi \colon \mathcal{F} \xrightarrow{\cong} \mathcal{G}$. □

From the above lemma, we get the following corollary, which will be used later on.

Corollary 5.6 *Let Z and T be two reduced locally Noetherian schemes, and assume that Z admits an open cover $Z = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ with U_α proper and connected. Let $\mathcal{L}$ and $\mathcal{M}$ be two line bundles on $Z \times T$ such that*

(a) $\mathcal{L}|_{Z \times \{t\}} = \mathcal{M}|_{Z \times \{t\}}$ *for every $t \in T$, and*

(b) $\mathcal{L}|_{\{z\} \times T} = \mathcal{M}|_{\{z\} \times T}$ *for every $z \in Z$.*

Then $\mathcal{L} = \mathcal{M}$.

Combining Propositions 5.1 and 5.4 we immediately get the following corollary, which describes the behavior of $\overline{\mathcal{P}}$ under translations.

Corollary 5.7 *Given two line bundles $M_1, M_2 \in \operatorname{Pic}^0(X)$, consider the translation morphism*

$$t_{(M_1,M_2)}\colon \overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g} \to \overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g}, \quad (I_1, I_2) \mapsto (I_1 \otimes M_1, I_2 \otimes M_2). \tag{5-12}$$

Then we have

$$t^*_{(M_1,M_2)}\overline{\mathcal{P}} = \overline{\mathcal{P}} \otimes p_1^*(\overline{\mathcal{P}}_{M_2}) \otimes p_2^*(\overline{\mathcal{P}}_{M_1}). \tag{5-13}$$

Let us finally examine the behavior of the Poincaré sheaf $\overline{\mathcal{P}}$ under the duality involution

$$\nu\colon \overline{\mathbb{J}}_X^{1-g} \to \overline{\mathbb{J}}_X^{1-g}, \quad I \mapsto I^\vee := \mathcal{H}om(I, \mathcal{O}_X). \tag{5-14}$$

Proposition 5.8 *The Poincaré sheaf $\overline{\mathcal{P}}$ satisfies the following properties*:

(i) $(\nu \times \mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}})^*\overline{\mathcal{P}} = (\mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}} \times \nu)^*\overline{\mathcal{P}} = \overline{\mathcal{P}}^\vee$. *In particular*, $\overline{\mathcal{P}}_{I^\vee} = (\overline{\mathcal{P}}^\vee)_I$ *for any* $I \in \overline{\mathbb{J}}_X^{1-g}$.

(ii) $(\nu \times \nu)^*\overline{\mathcal{P}} = \overline{\mathcal{P}}$.

Proof First of all, notice that (ii) follows from (i) by observing that

$$(\nu \times \nu) = (\nu \times \mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}}) \circ (\mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}} \times \nu)$$

and using the fact that, since ν is an involution, if (i) holds for $\overline{\mathcal{P}}$ then it also holds for $\overline{\mathcal{P}}^\vee$.

Let us prove (i) for the morphism $(\nu \times \mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}})$; the case of $(\mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}} \times \nu)$ is dealt with in the same way. Using Remark 4.7, it is enough to prove the result for the Poincaré line bundle $\mathcal{P}$ on $(\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural$. Consider the diagram

$$\begin{array}{ccc} X \times (\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural & \xrightarrow{\mathrm{id}_X \times \nu \times \mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}}} & X \times (\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural \\ \downarrow{\scriptstyle p_{23}} & & \downarrow{\scriptstyle p_{23}} \\ (\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural & \xrightarrow[\nu \times \mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}}]{} & (\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g})^\natural \end{array}$$

By definition (4-1), we get

$$\mathcal{P}^\vee = \mathcal{D}(p_{12}^*\mathcal{I} \otimes p_{13}^*\mathcal{I}) \otimes \mathcal{D}(p_{12}^*\mathcal{I})^{-1} \otimes \mathcal{D}(p_{13}^*\mathcal{I})^{-1}, \tag{5-15}$$

where $\mathcal{D}$ denotes the determinant of cohomology with respect to the projection map p_{23}.

By the base change property of the determinant of cohomology [12, Proposition 44(1)], and using the equalities

$$
\text{(5-16)}\qquad
\begin{aligned}
(\mathrm{id}_X\times \nu \times \mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}})^*(p_{13}^*\mathcal{I}) &= p_{13}^*\mathcal{I},\\
(\mathrm{id}_X\times \nu \times \mathrm{id}_{\overline{\mathbb{J}}_X^{1-g}})^*(p_{12}^*\mathcal{I}) &= p_{12}^*\mathcal{I}^\vee,
\end{aligned}
$$

we get that

$$
\text{(5-17)}\qquad (\nu\times \mathrm{id}_{\mathbb{J}_X})^*\mathcal{P} = \mathcal{D}(p_{12}^*\mathcal{I}^\vee\otimes p_{13}^*\mathcal{I})^{-1}\otimes \mathcal{D}(p_{12}^*\mathcal{I}^\vee)\otimes \mathcal{D}(p_{13}^*\mathcal{I}).
$$

Claim 1 *For any $I\in \mathbb{J}_X^{1-g}$, we have that*

$$
\mathcal{P}^\vee|_{\{I\}\times \overline{\mathbb{J}}_X^{1-g}} = [(\nu\times \mathrm{id}_{\mathbb{J}_X})^*\mathcal{P}]|_{\{I\}\times \overline{\mathbb{J}}_X^{1-g}}.
$$

Indeed, using (5-15) and (5-17) together with the base change property of the determinant of cohomology, we get

$$
\text{(5-18)}\qquad
\begin{aligned}
\mathcal{P}^\vee|_{\{I\}\times \overline{\mathbb{J}}_X^{1-g}} &= \mathcal{D}(p_1^*I\otimes \mathcal{I})\otimes \mathcal{D}(p_1^*I)^{-1}\otimes \mathcal{D}(\mathcal{I})^{-1}\\
&= \mathcal{D}(p_1^*I\otimes \mathcal{I})\otimes \mathcal{D}(\mathcal{I})^{-1},\\
[(\nu\times \mathrm{id}_{\mathbb{J}_X})^*\mathcal{P}]|_{\{I\}\times \overline{\mathbb{J}}_X^{1-g}} &= \mathcal{D}(p_1^*I^\vee\otimes \mathcal{I})^{-1}\otimes \mathcal{D}(p_1^*I^\vee)\otimes \mathcal{D}(\mathcal{I})\\
&= \mathcal{D}(p_1^*I^\vee\otimes \mathcal{I})^{-1}\otimes \mathcal{D}(\mathcal{I}),
\end{aligned}
$$

where $\mathcal{D}$ denotes now the determinant of cohomology with respect to the projection map $p_2\colon X\times \overline{\mathbb{J}}_X^{1-g}\to \overline{\mathbb{J}}_X^{1-g}$ and p_1 is the projection onto the first factor. Write now $I=I_{E_1}\otimes I_{E_2}^{-1}$ as in Lemma 2.2. Since I is a line bundle by assumption, we have that E_1 is Cartier; moreover, arguing similarly to the proof of Lemma 2.2, we can choose E_1 to be supported on the smooth locus of X. Hence, the representation $I^\vee = I^{-1} = I_{E_2}\otimes I_{E_1}^{-1}$ satisfies the same assumptions as in Lemma 2.2. We can now apply Lemma 4.3 twice in order to conclude that

$$
\text{(5-19)}\qquad
\begin{aligned}
\mathcal{D}(p_1^*I\otimes \mathcal{I})\otimes \mathcal{D}(\mathcal{I})^{-1} &= \mathcal{D}(\mathcal{I}|_{p_1^{-1}(E_2)})\otimes \mathcal{D}(\mathcal{I}|_{p_1^{-1}(E_1)})^{-1},\\
\mathcal{D}(p_1^*I^\vee\otimes \mathcal{I})\otimes \mathcal{D}(\mathcal{I})^{-1} &= \mathcal{D}(\mathcal{I}|_{p_1^{-1}(E_1)})\otimes \mathcal{D}(\mathcal{I}|_{p_1^{-1}(E_2)})^{-1}.
\end{aligned}
$$

Claim 1 follows from (5-18) and (5-19).

Claim 2 *For any $I\in \mathbb{J}_X^{1-g}$, we have that*

$$
\mathcal{P}^\vee|_{\overline{\mathbb{J}}_X^{1-g}\times\{I\}} = [(\nu\times \mathrm{id}_{\mathbb{J}_X})^*\mathcal{P}]|_{\overline{\mathbb{J}}_X^{1-g}\times\{I\}}.
$$

Again, using (5-15) and (5-17) together with the base change property of the determinant of cohomology, we get

$$
\begin{aligned}
\mathcal{P}^\vee|_{\overline{\mathbb{J}}_X^{1-g}\times\{I\}} &= \mathcal{D}(\mathcal{I}\otimes p_1^* I)\otimes\mathcal{D}(\mathcal{I})^{-1}\otimes\mathcal{D}(p_1^*I)^{-1}\\
&= \mathcal{D}(\mathcal{I}\otimes p_1^* I)\otimes\mathcal{D}(\mathcal{I})^{-1},\\
[(\nu\times \mathrm{id}_{\mathbb{J}_X})^*\mathcal{P}]|_{\overline{\mathbb{J}}_X^{1-g}\times\{I\}} &= \mathcal{D}(\mathcal{I}^\vee\otimes p_1^* I)^{-1}\otimes\mathcal{D}(\mathcal{I}^\vee)\otimes\mathcal{D}(p_1^*I)\\
&= \mathcal{D}(\mathcal{I}^\vee\otimes p_1^* I)^{-1}\otimes\mathcal{D}(\mathcal{I}^\vee),
\end{aligned}
\tag{5-20}
$$

where $\mathcal{D}$ now denotes the determinant of cohomology with respect to the projection map $p_2\colon X\times\overline{\mathbb{J}}_X^{1-g}\to\overline{\mathbb{J}}_X^{1-g}$, and p_1 is the projection onto the first factor. Now write $I=I_{E_1}\otimes I_{E_2}^{-1}$ as in Lemma 2.2. As before, since I is a line bundle by assumption, we can assume that E_1 is Cartier and supported on the smooth locus of X. We can apply Lemma 4.3 to the sheaf $\mathcal{I}$ and to its dual $\mathcal{I}^\vee$ in order to get that

$$
\begin{aligned}
\mathcal{D}(\mathcal{I}\otimes p_1^*I)\otimes\mathcal{D}(\mathcal{I})^{-1} &= \mathcal{D}(\mathcal{I}|_{p_1^{-1}(E_2)})\otimes\mathcal{D}(\mathcal{I}|_{p_1^{-1}(E_1)})^{-1},\\
\mathcal{D}(\mathcal{I}^\vee\otimes p_1^*I)\otimes\mathcal{D}(\mathcal{I}^\vee)^{-1} &= \mathcal{D}(\mathcal{I}^\vee|_{p_1^{-1}(E_2)})\otimes\mathcal{D}(\mathcal{I}^\vee|_{p_1^{-1}(E_1)})^{-1}.
\end{aligned}
\tag{5-21}
$$

Now fix $i\in\{1,2\}$ and denote by π the projection $X\times\overline{\mathbb{J}}_X^{1-g}\supset p_1^{-1}(E_i)\to\overline{\mathbb{J}}_X^{1-g}$. Since $\mathcal{I}$ is locally free along the divisor $p_1^{-1}(E_i)$ (because E_i is supported on the smooth locus of X), we have that $R^1\pi_*(I_{|p_1^{-1}(E_i)})=0$ and that $\pi_*(I_{|p_1^{-1}(E_i)})$ is locally free; the same statements hold with $\mathcal{I}$ replaced by $\mathcal{I}^\vee$. Therefore, from the definition of the determinant of cohomology (see the discussion in Section 4.1), it follows that

$$
\mathcal{D}(\mathcal{I}|_{p_1^{-1}(E_i)})=\det\pi_*(\mathcal{I}|_{p_1^{-1}(E_i)}),\quad \mathcal{D}(\mathcal{I}^\vee|_{p_1^{-1}(E_i)})=\det\pi_*(\mathcal{I}^\vee|_{p_1^{-1}(E_i)}).
\tag{5-22}
$$

Applying the relative duality to the finite morphism π (see [23, Section III.6]) and using that the relative dualizing sheaf $\omega_\pi=p_1^*(\omega_{E_i})$ of π is trivial because E_i is a curvilinear 0–dimensional scheme (hence Gorenstein), we have that

$$
\pi_*(\mathcal{I}|_{p_1^{-1}(E_i)})^\vee=\pi_*\mathcal{RH}om(\mathcal{I}|_{p_1^{-1}(E_i)},\omega_\pi)=\pi_*(\mathcal{I}^\vee|_{p_1^{-1}(E_i)}).
\tag{5-23}
$$

Claim 2 now follows by combining (5-20)–(5-23).

Now, by applying Corollary 5.6 and using Claims 1 and 2, we get that $\mathcal{P}^\vee$ is isomorphic to $(\nu\times\mathrm{id}_{\mathbb{J}_X})^*\mathcal{P}$ on $\overline{\mathbb{J}}_X^{1-g}\times\mathbb{J}_X^{1-g}$ and on $\mathbb{J}_X\times\overline{\mathbb{J}}_X^{1-g}$. Finally, using the invariance of $\mathcal{P}$ under the permutation σ of the two factors, it can be checked that the above isomorphisms glue to an isomorphism between $\mathcal{P}^\vee$ and $(\nu\times\mathrm{id}_{\mathbb{J}_X})^*\mathcal{P}$ on the entire $(\overline{\mathbb{J}}_X^{1-g}\times\overline{\mathbb{J}}_X^{1-g})^\natural$. □

6 Proof of the main results

The aim of this section is to prove Theorem A and Theorem B from the introduction.

Fix two fine compactified Jacobians $\overline{J}_X(\underline{q}), \overline{J}_X(\underline{q}') \subseteq \overline{\mathbb{J}}_X^{1-g}$ associated to two general polarizations q and q' on a connected reduced k–curve X with planar singularities and arithmetic genus $g := p_a(X)$, as in Section 2.1. Assume throughout this section that either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > g$. With a slight abuse of notation, we will also denote by $\overline{\mathcal{P}}$ the restriction of the Poincaré sheaf (see Section 4.2) to $\overline{J}_X(\underline{q}) \times \overline{J}_X(\underline{q}')$.

The constructions of Section 4 can be repeated for the universal family

$$\overline{\mathbb{J}}_{\mathcal{X}}^{1-g} \times_{\operatorname{Spec} R_X} \overline{\mathbb{J}}_{\mathcal{X}}^{1-g} \to \operatorname{Spec} R_X,$$

and they provide a maximal Cohen–Macaulay sheaf $\overline{\mathcal{P}}^{\mathrm{un}}$ over $\overline{\mathbb{J}}_{\mathcal{X}}^{1-g} \times_{\operatorname{Spec} R_X} \overline{\mathbb{J}}_{\mathcal{X}}^{1-g}$, called the *universal Poincaré sheaf*, whose restriction to the fiber over the closed point o of $\operatorname{Spec} R_X$ coincides with the Poincaré sheaf $\overline{\mathcal{P}}$ on $\overline{\mathbb{J}}_X^{1-g} \times \overline{\mathbb{J}}_X^{1-g}$. With a slight abuse of notation, we will also denote by $\overline{\mathcal{P}}^{\mathrm{un}}$ the restriction of the universal Poincaré sheaf to $\overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}')$, where $u\colon \overline{J}_{\mathcal{X}}(\underline{q}) \to \operatorname{Spec} R_X$ (resp. $u'\colon \overline{J}_{\mathcal{X}}(\underline{q}') \to \operatorname{Spec} R_X$) is the universal fine compactified Jacobian with respect to the polarization $\underline{q}$ (resp. $\underline{q}'$) as in Section 2.2.

Consider now the complex

$$\Psi^{\mathrm{un}} := Rp_{13*}\big(p_{12}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})\big) \in D^b_{\mathrm{coh}}(\overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q})), \tag{6-1}$$

where p_{ij} denotes the projection of $\overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}') \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q})$ onto the i^{th} and j^{th} factors.

As $\overline{\mathcal{P}}^{\mathrm{un}}$ is flat over $\overline{J}_{\mathcal{X}}(\underline{q}')$, its pullback $p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})$ is flat over $\overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}')$ with respect to the (flat) morphism p_{12}. The tensor product $p_{12}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})$ is therefore quasi-isomorphic to the derived tensor product $p_{12}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes^{\boldsymbol{L}} p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})$.

Remark 6.1 The derived dual

$$\mathcal{RH}om\big(p_{12}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}}), \mathcal{O}_{\overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}') \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q})}\big)$$

is isomorphic in $D_{\mathrm{coh}}\big(\overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}') \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q})\big)$ to the sheaf

$$p_{12}^*(\overline{\mathcal{P}}^{\mathrm{un}}) \otimes p_{23}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee).$$

In particular, $p_{12}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})$ is a maximal Cohen–Macaulay sheaf.

To prove this isomorphism,[6] let $L^\bullet \to (\overline{\mathcal{P}}^{\mathrm{un}})^\vee$ be a locally free resolution. As p_{12} is flat, it induces a locally free resolution $p_{12}^*(L^\bullet) \to p_{12}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee)$. Since $p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})$ is flat over $\overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}')$ with respect to p_{12}, tensoring with $p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})$ still gives a resolution

$$p_{12}^*(L^\bullet) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}}) \to p_{12}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}}).$$

As p_{23} is flat and $\overline{\mathcal{P}}^{\mathrm{un}}$ is maximal Cohen–Macaulay, the pullback $p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})$ is maximal Cohen–Macaulay too. It follows that $p_{12}^*(L^\bullet) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})$ is a complex of Cohen–Macaulay sheaves. Therefore

$$\mathcal{RH}om\big(p_{12}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}}), \mathcal{O}_{\overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}') \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q})}\big)$$

is isomorphic to

$$(p_{12}^*(L^\bullet) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}}))^\vee = p_{12}^*(L^\bullet)^\vee \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})^\vee$$

in $D_{\mathrm{coh}}(\overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}') \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}))$.

Finally, as $(\overline{\mathcal{P}}^{\mathrm{un}})^\vee$ is maximal Cohen–Macaulay, the complex $(\overline{\mathcal{P}}^{\mathrm{un}}) \to (L^\bullet)^\vee$ is exact and, as p_{12} is flat and $p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})^\vee$ is flat with respect to p_{12}, the complex

$$p_{12}^*(\overline{\mathcal{P}}^{\mathrm{un}}) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})^\vee \to p_{12}^*(L^\bullet)^\vee \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})^\vee$$

is also exact. Hence $p_{12}^*(\overline{\mathcal{P}}^{\mathrm{un}}) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})^\vee$ is isomorphic to $p_{12}^*(L^\bullet)^\vee \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})^\vee$ in the derived category $D_{\mathrm{coh}}(\overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}') \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}))$.

Theorem A will descend easily from the following key result.

Theorem 6.2 *Take notation as above. Then there is a natural isomorphism*

(6-2) $$\vartheta^{\mathrm{un}} \colon \Psi^{\mathrm{un}}[g] \to \mathcal{O}_{\Delta^{\mathrm{un}}}$$

in $D^b_{\mathrm{coh}}(\overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}))$*, where* $\mathcal{O}_{\Delta^{\mathrm{un}}}$ *is the structure sheaf of the universal diagonal* $\Delta^{\mathrm{un}} \subset \overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q})$*.*

Before proving Theorem 6.2, we need to bound the dimension of the support of the complex

(6-3) $$\Psi := Rp_{13*}\big(p_{12}^*((\overline{\mathcal{P}})^\vee) \otimes p_{23}^*(\overline{\mathcal{P}})\big) \in D^b_{\mathrm{coh}}(\overline{J}_X(\underline{q}) \times \overline{J}_X(\underline{q})).$$

[6]The proof of the isomorphism would be elementary if $\overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}') \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q})$ were smooth.

Proposition 6.3 *Take the same assumptions as in Theorem 6.2. Then the complex* Ψ *of (6-3) satisfies*

$$\operatorname{codim}(\operatorname{supp}(\Psi)) \geq g^\nu(X),$$

with strict inequality if X *is irreducible and singular.*

Proof For any $I \in \overline{\mathbb{J}}_X^{1-g}$, set $\overline{\mathcal{P}}_I := \overline{\mathcal{P}}_{|\{I\}\times\overline{\mathbb{J}}_X^{1-g}}$ and $\mathcal{P}_I := \mathcal{P}_{|\{I\}\times\mathbb{J}_X^{1-g}}$. Observe that, since the tensor product $p_{12}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})$ is isomorphic to the derived tensor product $p_{12}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes^{\boldsymbol{L}} p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})$, for any $(I_1, I_2) \in \overline{J}_X(\underline{q}) \times \overline{J}_X(\underline{q})$ the derived restriction of $p_{12}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})$ to $p_{13}^{-1}((I_1, I_2))$ is isomorphic to $\overline{\mathcal{P}}_{I_1}^\vee \otimes^{\boldsymbol{L}} \overline{\mathcal{P}}_{I_2}$ in $D^b_{\mathrm{coh}}(\overline{J}_X(\underline{q}) \times \overline{J}_X(\underline{q}))$. Thus, using base change and Proposition 5.3(iv), we have

(6-4) $$\operatorname{supp}(\Psi) := \{(I_1, I_2) \in \overline{J}_X(\underline{q}) \times \overline{J}_X(\underline{q}) : \mathbb{H}^i(\overline{J}_X(\underline{q}'), \overline{\mathcal{P}}_{I_1}^\vee \otimes^{\boldsymbol{L}} \overline{\mathcal{P}}_{I_2}) \neq 0 \text{ for some } i \in \mathbb{Z}\}.$$

Claim 1 *If* $(I_1, I_2) \in \operatorname{supp}(\Psi)$, *then* $\mathcal{P}_{I_1} = \mathcal{P}_{I_2}$.

The proof of this claim follows the same lines of the proof of [5, Proposition 7.2] using Proposition 5.4 (see also [34, Proposition 6.1] and [4, Proposition 1]) and therefore is left to the reader.

Claim 2 *If* $(I_1, I_2) \in \operatorname{supp}(\Psi)$, *then* $(I_1)|_{X_{\mathrm{sm}}} = (I_2)|_{X_{\mathrm{sm}}}$.

Consider the L–twisted Abel map $A_L \colon X \to \overline{\mathbb{J}}_X$ for some $L \in \operatorname{Pic}(X)$; see [33, Section 6.1]. Since $A_L(X_{\mathrm{sm}}) \subseteq \mathbb{J}_X$ and $A_L^*(\mathcal{P}_I) = I|_{X_{\mathrm{sm}}}$ for any $I \in \overline{\mathbb{J}}_X$ (see [34, Proposition 5.6]), Claim 2 follows from Claim 1.

Consider now the map

$$(\mu \times \mathrm{id}) \colon J(X) \times \overline{J}_X(\underline{q}) \times \overline{J}_X(\underline{q}) \to \overline{J}_X(\underline{q}) \times \overline{J}_X(\underline{q}), \quad (L, I_1, I_2) \mapsto (L \otimes I_1, I_2),$$

and set $\Phi := (\mu \times \mathrm{id})^{-1}(\Psi)$. Since $\mu \times \mathrm{id}$ is a smooth and surjective map, hence with equidimensional fibers, it is enough to prove that

(6-5) $$\operatorname{codim}(\operatorname{supp}(\Phi)) \geq g^\nu(X),$$

with strict inequality if X is irreducible and singular. Consider now the projection $p \colon \Phi \to \overline{J}_X(\underline{q}) \times \overline{J}_X(\underline{q})$ on the last two factors. Using Claim 2, the fiber of p over the point $(I_1, I_2) \in \overline{J}_X(\underline{q}) \times \overline{J}_X(\underline{q})$ is contained in the locus of $J(X)$ consisting of those line bundles $L \in J(X)$ such that $L|_{X_{\mathrm{sm}}} = (I_1)|_{X_{\mathrm{sm}}}^{-1} \otimes (I_2)|_{X_{\mathrm{sm}}}$. Arguing as in [34, Corollary 6.3], it follows that

(6-6) $$\operatorname{codim}(p^{-1}(I_1, I_2)) \geq g^\nu(X) \quad \text{for any } (I_1, I_2) \in \overline{J}_X(\underline{q}) \times \overline{J}_X(\underline{q}).$$

Moreover, if X is irreducible and $(I_1, I_2) \in J_X(\underline{q}) \times J_X(\underline{q})$, then we can apply [4, Theorem B(i)], together with Propositions 5.8 and 5.4, in order to deduce that

$$\begin{aligned} p^{-1}(I_1, I_2) &= \{L \in J(X) : H^i(\bar{J}_X(\underline{q}'), \mathcal{P}_{(L \otimes I_1)^{-1} \otimes I_2}) \neq 0 \text{ for some } i \in \mathbb{Z}\} \\ &= \{I_1^{-1} \otimes I_2\}. \end{aligned}$$

Therefore, if X is irreducible and singular, it holds that

$$\text{(6-7)} \quad \operatorname{codim}(p^{-1}(I_1, I_2)) = p_a(X) > g^\nu(X) \quad \text{for any } (I_1, I_2) \in J_X(\underline{q}) \times J_X(\underline{q}).$$

Combining (6-6) and (6-7), we get condition (6-5). □

We can now give the proof of Theorem 6.2.

Proof of Theorem 6.2 We will first explain how the natural morphism ϑ^{un} is defined. Consider the Cartesian diagram

$$\text{(6-8)} \quad \begin{array}{ccc} \widetilde{\Delta}^{\mathrm{un}} & \xrightarrow{\ \tilde{\imath}\ } & \bar{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q}') \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q}) \\ \downarrow \tilde{p} & \square & \downarrow p_{13} \\ \Delta^{\mathrm{un}} & \xrightarrow{\ i\ } & \bar{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q}) \end{array}$$

and notice that the morphism $\tilde{p}$ can be identified with the projection

$$p_1 \colon \bar{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q}') \to \bar{J}_{\mathcal{X}}(\underline{q})$$

onto the first factor. By applying base change (see eg [6, Proposition A.85]) to the diagram (6-8), we get a morphism

$$\text{(6-9)} \quad \begin{aligned} Li^*\Psi^{\mathrm{un}} = Li^* Rp_{13*}\big(p_{12}^*((\bar{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes^{\boldsymbol{L}} p_{23}^*(\bar{\mathcal{P}}^{\mathrm{un}})\big) \\ \to R\tilde{p}_* L\tilde{\imath}^*\big(p_{12}^*((\bar{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes^{\boldsymbol{L}} p_{23}^*(\bar{\mathcal{P}}^{\mathrm{un}})\big). \end{aligned}$$

By a standard spectral sequence argument, $L\tilde{\imath}^*\big(p_{12}^*((\bar{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes^{\boldsymbol{L}} p_{23}^*(\bar{\mathcal{P}}^{\mathrm{un}})\big)$ is isomorphic to $(\bar{\mathcal{P}}^{\mathrm{un}})^\vee \otimes^{\boldsymbol{L}} \bar{\mathcal{P}}^{\mathrm{un}}$ in $D_{\mathrm{coh}}(\widetilde{\Delta}^{\mathrm{un}})$, hence it admits a natural morphism to its top degree cohomology (ie the usual tensor product $(\bar{\mathcal{P}}^{\mathrm{un}})^\vee \otimes \bar{\mathcal{P}}^{\mathrm{un}}$) and to the structure sheaf $\mathcal{O}_{\widetilde{\Delta}^{\mathrm{un}}}$. Composing with $R\tilde{p}_*$ finally gives a morphism

$$\text{(6-10)} \quad \begin{aligned} R\tilde{p}_* L\tilde{\imath}^*\big(p_{12}^*((\bar{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes^{\boldsymbol{L}} p_{23}^*(\bar{\mathcal{P}}^{\mathrm{un}})\big) \simeq R\tilde{p}_*((\bar{\mathcal{P}}^{\mathrm{un}})^\vee \otimes^{\boldsymbol{L}} \bar{\mathcal{P}}^{\mathrm{un}}) \\ \to R\tilde{p}_*((\bar{\mathcal{P}}^{\mathrm{un}})^\vee \otimes \bar{\mathcal{P}}^{\mathrm{un}}) \to R\tilde{p}_*(\mathcal{O}_{\widetilde{\Delta}^{\mathrm{un}}}). \end{aligned}$$

Since the complex of sheaves $R\widetilde{p}_*(\mathcal{O}_{\widetilde{\Delta}^{\mathrm{un}}})$ is concentrated in cohomological degrees from 0 to g, we get a morphism of complexes of sheaves

$$R\widetilde{p}_*(\mathcal{O}_{\widetilde{\Delta}^{\mathrm{un}}}) \to R^g\widetilde{p}_*(\mathcal{O}_{\widetilde{\Delta}^{\mathrm{un}}})[-g]. \tag{6-11}$$

Moreover, since the morphism $\widetilde{p}$ is proper of relative dimension g, with trivial relative dualizing sheaf and geometrically connected fibers (by Facts 2.10 and 2.7), then the relative duality applied to $\widetilde{p}$ (see [24, Corollary 11.2(g)]) gives

$$R^g\widetilde{p}_*(\mathcal{O}_{\widetilde{\Delta}^{\mathrm{un}}}) \cong \mathcal{O}_{\Delta^{\mathrm{un}}}. \tag{6-12}$$

By composing the morphisms (6-9)–(6-11) and using the isomorphism (6-12), we get a morphism

$$Li^*\Psi^{\mathrm{un}} \to \mathcal{O}_{\Delta^{\mathrm{un}}}[-g]. \tag{6-13}$$

Since $i_* = (Ri_*)$ is right adjoint to Li^*, the morphism (6-13), shifted by $[g]$, gives rise to the morphism ϑ^{un}.

In order to show that ϑ^{un} is an isomorphism, we divide the proof into several steps, which we present as a sequence of claims.

The first claim states that ϑ^{un} is an isomorphism on an interesting open subset. More precisely, let $(\operatorname{Spec} R_X)_{\mathrm{sm}}$ be the open subset of $\operatorname{Spec} R_X$ consisting of all the (schematic) points $s \in \operatorname{Spec} R_X$ such that the geometric fiber $\mathcal{X}_{\overline{s}}$ of the universal family $\pi\colon \mathcal{X} \to \operatorname{Spec} R_X$ is smooth.

Claim 1 *The morphism ϑ^{un} is an isomorphism over the open subset*

$$(u\times u)^{-1}(\operatorname{Spec} R_X)_{\mathrm{sm}} \subseteq \overline{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \overline{J}_{\mathcal{X}}(\underline{q}).$$

It is enough to prove that ϑ^{un} is an isomorphism when restricted to $(u\times u)^{-1}(s)$ for any $s \in (\operatorname{Spec} R_X)_{\mathrm{sm}}$. This is a classical result due to Mukai [37, Theorem 2.2]; it may also be seen as a particular case of [5, Proposition 7.1], which holds more generally for any irreducible curve.

Claim 2 $\Psi^{\mathrm{un}}[g]$ *is a Cohen–Macaulay sheaf of codimension g.*

Let us first prove that

$$\operatorname{codim}(\operatorname{supp}(\Psi^{\mathrm{un}})) = g. \tag{6-14}$$

First of all, Claim 1 gives that $\operatorname{codim}(\operatorname{supp}(\Psi^{\mathrm{un}})) \le g$. In order to prove the reverse inequality, we stratify the scheme $\operatorname{Spec} R_X$ into locally closed subsets according to the geometric genus of the geometric fibers of the universal family $\mathcal{X} \to \operatorname{Spec} R_X$:

$$(\operatorname{Spec} R_X)^{g^\nu = l} := \{s \in \operatorname{Spec} R_X : g^\nu(\mathcal{X}_{\bar{s}}) = l\}$$

for any $g^\nu(X) \le l \le p_a(X) = g$. Fact 2.9(i) gives that $\operatorname{codim}(\operatorname{Spec} R_X)^{g^\nu = l} \ge g - l$. On the other hand, on the fibers of $u \times u$ over $(\operatorname{Spec} R_X)^{g^\nu = l}$, the sheaf Ψ^{un} has support of codimension at least l by Proposition 6.3. Therefore, we get

$$\text{(6-15)} \quad \operatorname{codim}(\operatorname{supp}(\Psi^{\mathrm{un}}) \cap (u \times u)^{-1}((\operatorname{Spec} R_X)^{g^\nu = l})) \ge g \quad \text{for any } g^\nu(X) \le l \le g.$$

Since the locally closed subsets $(\operatorname{Spec} R_X)^{g^\nu = l}$ form a stratification of $\operatorname{Spec} R_X$, we deduce that $g \le \operatorname{codim}(\operatorname{supp}(\Psi^{\mathrm{un}}))$, which concludes the proof of (6-14).

Observe next that, since p_{23} has relative dimension g and $p_{12}^*((\overline{\mathcal{P}})^\vee) \otimes^{\boldsymbol{L}} p_{23}^*(\overline{\mathcal{P}})$ is concentrated in nonpositive degrees, we have that

$$\text{(6-16)} \qquad H^i(\Psi^{\mathrm{un}}[g]) = 0 \quad \text{for } i > 0.$$

Let $\mathbb{D}$ be the dualizing functor of $\bar{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q})$. Applying the relative duality (see [23, Chapter VII.3]) to the morphism p_{13}, which is projective and flat of relative dimension g by Fact 2.10(ii) and with trivial dualizing sheaf by Fact 2.7(i), we get that

$$\text{(6-17)} \quad \mathbb{D}(\Psi^{\mathrm{un}}[g]) = Rp_{13*}\big(\mathcal{RH}om\big(p_{12}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}}), \mathcal{O}_{\bar{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q}') \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q})}\big)\big).$$

By Remark 6.1, the second term of (6-17) equals

$$\text{(6-18)} \quad Rp_{13*}\big([p_{12}^*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})]^\vee\big) = Rp_{13*}\big(p_{12}^*(\overline{\mathcal{P}}^{\mathrm{un}}) \otimes p_{23}^*(\overline{\mathcal{P}}^{\mathrm{un}})^\vee\big) = \sigma^*(\Psi^{\mathrm{un}}),$$

where σ is the automorphism of $\bar{J}_{\mathcal{X}}(\underline{q}) \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q})$ that exchanges the two factors. From (6-16), (6-17) and (6-18) it follows that

$$\text{(6-19)} \qquad H^i(\mathbb{D}(\Psi^{\mathrm{un}}[g])) = 0 \quad \text{for } i > g.$$

Since $\Psi^{\mathrm{un}}[g]$ satisfies (6-14), (6-16) and (6-19), Lemma 7.6 of [5] gives that $\Psi^{\mathrm{un}}[g]$ is a Cohen–Macaulay sheaf of codimension g.[7]

Claim 3 *We have a set-theoretic equality* $\operatorname{supp}(\Psi^{\mathrm{un}}[g]) = \Delta^{\mathrm{un}}$.

Let Z be an irreducible component of $\operatorname{supp}(\Psi^{\mathrm{un}}[g])$. Since $\Psi^{\mathrm{un}}[g]$ is a Cohen–Macaulay sheaf of codimension g, then by [32, Theorem 6.5(iii) and Theorem 17.3(i)]

[7] An expanded proof that $\Psi^{\mathrm{un}}[g]$ is Cohen–Macaulay can be given by copying the proofs of Claims 3 and 4 in the proof of Theorem 8.1 in [33].

we get that

$$\operatorname{codim} Z = g. \tag{6-20}$$

Let η be the generic point of $(u \times u)(Z)$. Clearly, $(u \times u)(Z)$ is contained in $(\operatorname{Spec} R_X)^{p_a^\nu \le p_a^\nu(\mathcal{X}_{\bar\eta})}$, from which we deduce, using Fact 2.9(i), that

$$\operatorname{codim}(u \times u)(Z) \ge \operatorname{codim}(\operatorname{Spec} R_X)^{p_a^\nu \le p_a^\nu(\mathcal{X}_{\bar\eta})} = g - p_a^\nu(\mathcal{X}_{\bar\eta}). \tag{6-21}$$

Moreover, denoting by $\Psi_{\bar\eta}$ the analogue of Ψ for the curve $\mathcal{X}_{\bar\eta}$, Proposition 6.3 gives that

$$\operatorname{codim} \operatorname{supp}(\Psi_{\bar\eta}) \ge g^\nu(\mathcal{X}_{\bar\eta}). \tag{6-22}$$

Putting together (6-20)–(6-22), we compute that

$$g = \operatorname{codim} Z = \operatorname{codim}(u \times u)(Z) + \operatorname{codim} \operatorname{supp}(\Psi_{\bar\eta}) \ge g - p_a^\nu(\mathcal{X}_{\bar\eta}) + g^\nu(\mathcal{X}_{\bar\eta}) \ge g,$$

which implies that $p_a^\nu(\mathcal{X}_{\bar\eta}) = g^\nu(\mathcal{X}_{\bar\eta})$ (ie $\mathcal{X}_{\bar\eta}$ is irreducible) and that equality holds in (6-22). By Proposition 6.3, this can happen only if $\mathcal{X}_{\bar\eta}$ is smooth. Then Claim 1 implies that $Z = \Delta^{\mathrm{un}}$.

Claim 4 *We have a scheme-theoretic equality* $\operatorname{supp}(\Psi^{\mathrm{un}}[g]) = \Delta^{\mathrm{un}}$.

Since the subscheme Δ^{un} is reduced, Claim 3 gives the inclusion of subschemes $\Delta^{\mathrm{un}} \subseteq \operatorname{supp}(\Psi^{\mathrm{un}}[g])$. Moreover, Claim 1 says that this inclusion is generically an equality; in particular $\operatorname{supp}(\Psi^{\mathrm{un}}[g])$ is generically reduced. Moreover, since $\Psi^{\mathrm{un}}[g]$ is a Cohen–Macaulay sheaf by Claim 2, we get that $\operatorname{supp}(\Psi^{\mathrm{un}}[g])$ is reduced by [34, Lemma 8.2]. Therefore, we must have the equality of subschemes $\Psi^{\mathrm{un}}[g] = \Delta^{\mathrm{un}}$.

We can now finish the proof of Theorem 6.2. Combining Claims 1, 2 and 4, we get that the sheaf $\Psi^{\mathrm{un}}[g]$ is a Cohen–Macaulay sheaf supported (schematically) on Δ^{un}, hence

$$\vartheta^{\mathrm{un}} \colon \Psi^{\mathrm{un}}[g] \to \mathcal{O}_{\Delta^{\mathrm{un}}}$$

is a morphism of sheaves supported (schematically) on Δ^{un}, therefore it is an isomorphism if and only if $i^*\vartheta^{\mathrm{un}}$ is an isomorphism.

The morphism $i^*\vartheta^{\mathrm{un}}$ is, by definition, the morphism induced on degree-g cohomology groups (ie the top cohomology groups) by the composition

$$Li^*\Psi^{\mathrm{un}} \xrightarrow{a_1} R\tilde{p}_*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee \otimes^{\boldsymbol{L}} \overline{\mathcal{P}}^{\mathrm{un}}) \xrightarrow{a_2} R\tilde{p}_*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee \otimes \overline{\mathcal{P}}^{\mathrm{un}}) \xrightarrow{a_3} R\tilde{p}_*(\mathcal{O}_{\tilde{\Delta}^{\mathrm{un}}}),$$

where a_1 is the base change morphism of (6-9) and a_2 and a_3 are the morphisms appearing in (6-10). Using the isomorphism (6-12) and denoting by $H^g(a_i)$ the morphisms induced by a_i on the g^{th} cohomology sheaves, it remains to show that the $H^g(a_i)$ are isomorphisms.

The morphism $H^g(a_1)$ is an isomorphism because g is the relative dimension of p_{13}, hence a_1 is a top-degree base change map. The morphism $H^g(a_2)$ is an isomorphism by a spectral sequence argument, because the tensor product $(\overline{\mathcal{P}}^{\mathrm{un}})^\vee \otimes \overline{\mathcal{P}}^{\mathrm{un}}$ is the degree-0 and top cohomology sheaf of $(\overline{\mathcal{P}}^{\mathrm{un}})^\vee \otimes^{\boldsymbol{L}} \overline{\mathcal{P}}^{\mathrm{un}}$ and g is the relative dimension of the flat morphism $\widetilde{p}$.

Finally, $H^g(a_3)$ is an isomorphism as the kernel and cokernel of $(\overline{\mathcal{P}}^{\mathrm{un}})^\vee \otimes \overline{\mathcal{P}}^{\mathrm{un}} \to \mathcal{O}_{\widetilde{\Delta}^{\mathrm{un}}}$ are supported on the locus where $\overline{\mathcal{P}}^{\mathrm{un}}$ is not locally free, ie in the fiber product over Spec R_X of the singular loci of $\overline{J}_{\mathcal{X}}(\underline{q})$ and $\overline{J}_{\mathcal{X}}(\underline{q}')$. As this locus intersects each fiber of $\widetilde{p}$ in dimension at most $g-2$, the morphism $(\overline{\mathcal{P}}^{\mathrm{un}})^\vee \otimes \overline{\mathcal{P}}^{\mathrm{un}} \to \mathcal{O}_{\widetilde{\Delta}^{\mathrm{un}}}$ induces the desired isomorphism $R^g \widetilde{p}_*((\overline{\mathcal{P}}^{\mathrm{un}})^\vee \otimes \overline{\mathcal{P}}^{\mathrm{un}}) \simeq R^g \widetilde{p}_*(\mathcal{O}_{\widetilde{\Delta}^{\mathrm{un}}})$. □

By passing to the central fiber, Theorem 6.2 implies the following result, which is a generalization of the result of Mukai for Jacobians of smooth curves [37, Theorem 2.2], and of Arinkin for compactified Jacobians of irreducible curves [5, Proposition 7.1].

Corollary 6.4 *Let X be a connected and reduced curve with planar singularities and arithmetic genus $g := p_a(X)$ and let $\underline{q}$ and $\underline{q}'$ be two general polarizations on X, of total degree $1-g$. Assume that either* $\mathrm{char}(k)=0$ *or* $\mathrm{char}(k) > g$. *Then there is a natural isomorphism in $D^b_{\mathrm{coh}}(\overline{J}_X(\underline{q}) \times \overline{J}_X(\underline{q}))$,*

(6-23) $$\vartheta\colon \Psi[g] \to \mathcal{O}_\Delta,$$

where $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal $\Delta \subset \overline{J}_X(\underline{q}) \times \overline{J}_X(\underline{q})$.

Proof The natural morphism ϑ is defined similarly to the morphism ϑ^{un} in (6-2). By base changing the isomorphism ϑ^{un} to the central fiber, we obtain the diagram

(6-24) $$\begin{array}{ccc} \Psi^{\mathrm{un}}[g]|_{\overline{J}_X(\underline{q})\times\overline{J}_X(\underline{q})} & \xrightarrow[\cong]{\vartheta^{\mathrm{un}}} & (\mathcal{O}_{\Delta^{\mathrm{un}}})|_{\overline{J}_X(\underline{q})\times\overline{J}_X(\underline{q})} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle \cong} \\ \Psi[g] & \xrightarrow{\quad\vartheta\quad} & \mathcal{O}_\Delta \end{array}$$

in $D^b_{\mathrm{coh}}(\overline{J}_X(\underline{q}) \times \overline{J}_X(\underline{q}))$, where b is the base-change morphism; see for instance [27, Remark 3.33]. Note that the complex $\Psi[g]$ is supported in nonpositive degree (as follows easily from its definition) and, since the base change morphism is an isomorphism in top degree (see eg [24, Theorem 12.11]), we deduce that the morphism b induces an isomorphism on the 0^{th} cohomology sheaves,

$$\mathcal{H}^0(b)\colon \Psi^{\mathrm{un}}[g]|_{\bar{J}_X(\underline{q})\times\bar{J}_X(\underline{q})} = \mathcal{H}^0(\Psi^{\mathrm{un}}[g]|_{\bar{J}_X(\underline{q})\times\bar{J}_X(\underline{q})}) \xrightarrow{\cong} \mathcal{H}^0(\Psi[g]). \tag{6-25}$$

Moreover, since the sheaf $\Psi^{\mathrm{un}}[g]_{|\bar{J}_X(\underline{q})\times\bar{J}_X(\underline{q})}$ is supported on Δ, by the base change theorem (see eg [24, Theorem 12.11]) we deduce that $\Psi[g]$ has set-theoretic support on Δ, which has codimension g inside $\bar{J}_X(\underline{q})\times\bar{J}_X(\underline{q})$. Therefore, Proposition 2.26 of [38] gives that Ψ is supported in degree g, or in other words that

$$\Psi[g] \xrightarrow{\cong} \mathcal{H}^0(\Psi[g]). \tag{6-26}$$

From (6-25) and (6-26), it follows that the base change morphism b is an isomorphism; using the diagram (6-24), we deduce that ϑ is an isomorphism. □

With the help of the above corollary, we can now prove Theorem A from the introduction.

Proof of Theorem A Consider the integral functor

$$\Phi^{\overline{\mathcal{P}}^\vee[g]}\colon D^b_{\mathrm{qcoh}}(\bar{J}_X(\underline{q}'))\to D^b_{\mathrm{qcoh}}(\bar{J}_X(\underline{q})),\quad \mathcal{E}^\bullet\mapsto \boldsymbol{R}p_{1*}(p_2^*(\mathcal{E}^\bullet)\otimes^{\boldsymbol{L}}\overline{\mathcal{P}}^\vee[g]),$$

where $\overline{\mathcal{P}}^\vee$ is the dual sheaf of $\overline{\mathcal{P}}$ as in Proposition 5.3. The composition $\Phi^{\overline{\mathcal{P}}^\vee[g]}\circ\Phi^{\overline{\mathcal{P}}}$ is the integral functor

$$\Phi^{\overline{\mathcal{P}}^\vee[g]}\circ\Phi^{\overline{\mathcal{P}}} = \Phi^{\Psi[g]}\colon D^b_{\mathrm{qcoh}}(\bar{J}_X(\underline{q}))\to D^b_{\mathrm{qcoh}}(\bar{J}_X(\underline{q})),$$
$$\mathcal{E}^\bullet\mapsto \boldsymbol{R}p_{2*}(p_1^*(\mathcal{E}^\bullet)\otimes^{\boldsymbol{L}}\Psi[g]),$$

with kernel given by $\Psi[g]$, where Ψ is the complex of (6-3); see eg [26, Section 1.4]. Since $\Psi[g]=\mathcal{O}_\Delta$ by Corollary 6.4, we have that $\Phi^{\overline{\mathcal{P}}^\vee[g]}\circ\Phi^{\overline{\mathcal{P}}}=\mathrm{id}$. By exchanging the roles of $\bar{J}_X(\underline{q})$ and $\bar{J}_X(\underline{q}')$, we get similarly that $\Phi^{\overline{\mathcal{P}}}\circ\Phi^{\overline{\mathcal{P}}^\vee[g]}=\mathrm{id}$, which proves the first statement of Theorem A.

The second statement follows from the first one together with the fact that $\Phi^{\overline{\mathcal{P}}}$ sends $D^b_{\mathrm{coh}}(\bar{J}_X(\underline{q}'))$ into $D^b_{\mathrm{coh}}(\bar{J}_X(\underline{q}))$ (and similarly for $\phi^{\overline{\mathcal{P}}^\vee[g]}$) because $\overline{\mathcal{P}}$ is a coherent sheaf and $\bar{J}_X(\underline{q})$ and $\bar{J}_X(\underline{q}')$ are proper varieties. □

Finally, we can prove Theorem B from the introduction.

Proof of Theorem B First of all, let us check that the morphism $\rho_{\underline{q}}$ is well-defined. For any $I\in\bar{\mathbb{J}}_X^{1-g}$, $\overline{\mathcal{P}}_I:=\overline{\mathcal{P}}_{|\bar{J}_X(\underline{q})\times\{I\}}$ is a Cohen–Macaulay sheaf on $\bar{J}_X(\underline{q})$ (by Theorem 4.6(ii)) whose restriction to the dense open subset $J_X(\underline{q})\subseteq\bar{J}_X(\underline{q})$ is a line bundle (see Section 4.1), which implies that $\overline{\mathcal{P}}_I$ has rank 1 on each irreducible component of $\bar{J}_X(\underline{q})$. Moreover, by the definition of the integral transform $\Phi^{\overline{\mathcal{P}}}$, it follows that $\Phi^{\overline{\mathcal{P}}}(\boldsymbol{k}(I)) = \overline{\mathcal{P}}_I$, which, using the fully faithfulness of $\Phi^{\overline{\mathcal{P}}}$ (see Theorem A), gives that

$$\operatorname{Hom}(\overline{\mathcal{P}}_I, \overline{\mathcal{P}}_I) = \operatorname{Hom}(\boldsymbol{k}(I), \boldsymbol{k}(I)) = k,$$

or, in other words, that $\overline{\mathcal{P}}_I$ is simple. Therefore, we get a morphism

$$\tilde{\rho}_{\underline{q}} \colon \overline{\mathbb{J}}_X^{1-g} \to \operatorname{Pic}^{=}(\bar{J}_X(\underline{q})), \quad I \mapsto \overline{\mathcal{P}}_I, \tag{6-27}$$

whose image is contained in $\operatorname{Pic}^{-}(\bar{J}_X(\underline{q})) \subseteq \operatorname{Pic}^{=}(\bar{J}_X(\underline{q}))$. Since

$$\tilde{\rho}_{\underline{q}}(\mathcal{O}_X) = \beta_{\underline{q}}(\mathcal{O}_X) = \mathcal{O}_{\bar{J}_X(\underline{q})} \in \operatorname{Pic}^{\mathrm{o}}(\bar{J}_X(\underline{q})) \subset \operatorname{Pic}^{=}(\bar{J}_X(\underline{q})),$$

the morphism $\tilde{\rho}_{\underline{q}}$ induces the required morphism $\rho_{\underline{q}}$ by passing to the connected components containing the structure sheaves. This concludes the proof that $\rho_{\underline{q}}$ is well-defined and it also shows that (i) holds true.

The morphism $\tilde{\rho}_{\underline{q}}$ (and hence also $\rho_{\underline{q}}$) is equivariant with respect to the isomorphism $\beta_{\underline{q}}$ since for any $I \in \overline{\mathbb{J}}_X^{1-g}$ and $L \in \operatorname{Pic}^{\mathrm{o}}(X)$ it holds that $\overline{\mathcal{P}}_I \otimes \mathcal{P}_L = \overline{\mathcal{P}}_{I\otimes L}$ by Proposition 5.4.

Moreover, $\tilde{\rho}_{\underline{q}}$ induces a homomorphism of group schemes

$$(\tilde{\rho}_{\underline{q}})|_{\mathbb{J}_X^{1-g}} \colon \mathbb{J}_X^{1-g} = \operatorname{Pic}^0(X) \to \operatorname{Pic}(\bar{J}_X(\underline{q})), \quad I \mapsto \overline{\mathcal{P}}_I = \mathcal{P}_I,$$

since for any $I, I' \in \mathbb{J}_X^{1-g}$, we have that $\mathcal{P}_{I\otimes I'} = \mathcal{P}_I \otimes \mathcal{P}_{I'}$ by Proposition 5.4 and $\mathcal{P}_I^{-1} = \mathcal{P}_{I^{-1}}$ by Proposition 5.8. This shows that (ii) holds true.

Consider now the restriction of $\tilde{\rho}_{\underline{q}}$ to a fine compactified Jacobian $\bar{J}_X(\underline{q}') \subseteq \overline{\mathbb{J}}_X^{1-g}$,

$$\tilde{\rho}_{\underline{q}'/\underline{q}} \colon \bar{J}_X(\underline{q}') \to \operatorname{Pic}^{=}(\bar{J}_X(\underline{q})), \quad I \mapsto \overline{\mathcal{P}}_I. \tag{6-28}$$

Claim 1 *$\tilde{\rho}_{\underline{q}'/\underline{q}}$ is an open embedding.*

The proof of this claim is similar to [5, Proof of Theorem B]; let us sketch the argument for the benefit of the reader. Fix a polarization $\mathcal{O}(1)$ on $\bar{J}_X(\underline{q}')$ and, for any coherent sheaf S on $\bar{J}_X(\underline{q}')$, denote by $\theta(S) \in \mathbb{Q}[t]$ the Hilbert polynomial of S with respect to $\mathcal{O}(1)$. Consider the locus $\mathcal{L}$ inside $\operatorname{Pic}^{=}(\bar{J}_X(\underline{q}'))$ consisting of the sheaves $F \in \operatorname{Pic}^{=}(\bar{J}_X(\underline{q}'))$ such that, with the notation of the proof of Theorem A,

$$\theta(H^i(\Phi^{\overline{\mathcal{P}}^\vee[g]}(F))) = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases} \tag{6-29}$$

Using the upper semicontinuity of the Hilbert polynomial, it follows that $\mathcal{L}$ is an open subset of $\mathrm{Pic}^{=}(\bar{J}_X(\underline{q}'))$; see [5, Proof of Theorem B]. From (6-29), we deduce that $F \in \mathcal{L}$ if and only if $\Phi^{\overline{\mathcal{P}}^\vee[g]}(F) = \boldsymbol{k}(I)$ for some $I \in \bar{J}_X(\underline{q}')$, which, using Theorem A, is equivalent to the fact that $F = \Phi^{\overline{\mathcal{P}}}(\boldsymbol{k}(I))$. In other words, the image of $\widetilde{\rho}_{\underline{q}'/\underline{q}}$ is equal to the open subset $\mathcal{L} \subseteq \mathrm{Pic}^{=}(\bar{J}_X(\underline{q}))$. Moreover, $\widetilde{\rho}_{\underline{q}'/\underline{q}}$ is an isomorphism into its image $\mathcal{L} = \mathrm{Im}\widetilde{\rho}_{\underline{q}'/\underline{q}}$, whose inverse is given by the morphism

$$\sigma\colon \mathcal{L} \to \bar{J}_X(\underline{q}'), \quad F \mapsto \mathrm{supp}\,\Phi^{\overline{\mathcal{P}}^\vee[g]}(F).$$

Before proving part (iii), let us examine how the map $\rho_{\underline{q}}$ behaves with respect to the decomposition of X into its separating blocks. Consider the partial normalization $\widetilde{X} \to X$ at the separating nodes of X and denote by $Y_1, \dots, Y_r$ the images in X of the connected components of $\widetilde{X}$ (in [33, Section 6.2], the curves $\{Y_1, \dots, Y_r\}$ are called the separating blocks of X). Note that $Y_1, \dots, Y_r$ are connected (reduced and projective) curves with planar singularities. From [33, Proposition 6.6(i)], it follows that we have an isomorphism

$$\bar{\mathbb{J}}_X \xrightarrow{\cong} \bar{\mathbb{J}}_{Y_1} \times \cdots \times \bar{\mathbb{J}}_{Y_r}, \quad I \mapsto (I|_{Y_1}, \dots, I|_{Y_r}),$$

which implies that

$$\overline{\mathrm{Pic}}^{\mathrm{o}}(X) = \overline{\mathrm{Pic}}^{\mathrm{o}}(Y_1) \times \cdots \times \overline{\mathrm{Pic}}^{\mathrm{o}}(Y_r). \tag{6-30}$$

Using [33, Proposition 6.6], we get the existence of general polarizations $\underline{q}^i$ on Y_i such that

$$\bar{J}_X(\underline{q}) = \bar{J}_{Y_1}(\underline{q}^1) \times \cdots \times \bar{J}_{Y_r}(\underline{q}^r),$$

which implies that

$$\overline{\mathrm{Pic}}^{\mathrm{o}}(\bar{J}_X(\underline{q})) = \overline{\mathrm{Pic}}^{\mathrm{o}}(\bar{J}_{Y_1}(\underline{q}^1)) \times \cdots \times \overline{\mathrm{Pic}}^{\mathrm{o}}(\bar{J}_{Y_r}(\underline{q}^r)). \tag{6-31}$$

Moreover, arguing as in [34, Lemma 5.5], the fibers of the Poincaré sheaf $\overline{\mathcal{P}}$ over a sheaf $I = (I_1, \dots, I_r) \in \overline{\mathrm{Pic}}^{\mathrm{o}}(X) = \overline{\mathrm{Pic}}^{\mathrm{o}}(Y_1) \times \cdots \times \overline{\mathrm{Pic}}^{\mathrm{o}}(Y_r)$ are such that

$$\overline{\mathcal{P}}_I = \overline{\mathcal{P}}^1_{I_1} \boxtimes \cdots \boxtimes \overline{\mathcal{P}}^r_{I_r} := p_1^*(\overline{\mathcal{P}}^1_{I_1}) \times \cdots \times p_r^*(\overline{\mathcal{P}}^r_{I_r}), \tag{6-32}$$

where $p_i\colon \overline{\mathrm{Pic}}^{\mathrm{o}}(\bar{J}_X(\underline{q})) \to \overline{\mathrm{Pic}}^{\mathrm{o}}(\bar{J}_{Y_i}(\underline{q}^i))$ denotes the projection onto the i^{th} factor in the decomposition (6-31) and $\overline{\mathcal{P}}^i$ is the Poincaré sheaf on $\bar{\mathbb{J}}_{Y_i}^{1-p_a(Y_i)} \times \bar{\mathbb{J}}_{Y_i}^{1-p_a(Y_i)}$. Putting together the decompositions (6-30), (6-31) and (6-32), we get that the morphism $\rho_{\underline{q}}$

decomposes as

$$(6\text{-}33)\quad \rho_{\underline{q}} = \prod_{i=1}^{r} \rho_{\underline{q}^i} \colon \overline{\mathrm{Pic}}^{\mathrm{o}}(X) = \prod_{i=1}^{r} \overline{\mathrm{Pic}}^{\mathrm{o}}(Y_i) \to \overline{\mathrm{Pic}}^{\mathrm{o}}(\bar{J}_X(\underline{q})) = \prod_{i=1}^{r} \overline{\mathrm{Pic}}^{\mathrm{o}}(\bar{J}_{Y_i}(\underline{q}^i)),$$
$$I = (I_1, \dots, I_r) \mapsto \overline{\mathcal{P}}_I = \overline{\mathcal{P}}^1_{I_1} \boxtimes \cdots \boxtimes \overline{\mathcal{P}}^r_{I_r}.$$

We can now easily prove part (iii). Indeed, if the curve X is such that every singular point of X that lies on two different irreducible components is a separating node, then the separating blocks $\{Y_1, \dots, Y_r\}$ are integral curves with planar singularities. Therefore, [5, Theorem B] implies that each $\rho_{\underline{q}^i} \colon \overline{\mathrm{Pic}}^{\mathrm{o}}(Y_i) \to \overline{\mathrm{Pic}}^{\mathrm{o}}(\bar{J}_{Y_i}(\underline{q}^i)) = \overline{\mathrm{Pic}}^{\mathrm{o}}(\bar{\mathbb{J}}^0_{Y_i})$ is an isomorphism (for any $i = 1, \dots, r$). We conclude that $\rho_{\underline{q}}$ is an isomorphism by (6-33).

Finally, it remains to show that $\rho_{\underline{q}}$ is an open embedding. According to (6-33), we can (and will) assume that the curve X does not have separating nodes. Under this assumption, we will prove more generally that $\tilde{\rho}_{\underline{q}}$ is an open embedding. Since fine compactified Jacobians form an open cover of $\bar{\mathbb{J}}^{1-g}_X$ (see Fact 2.6(ii)), Claim 1 gives that $\tilde{\rho}_{\underline{q}}$ is a local isomorphism. Therefore it remains to show that $\tilde{\rho}_{\underline{q}}$ is injective on geometric points, or in other words it is enough to establish the following.

Claim 2 *Assume that X does not have separating nodes. If $I_1, I_2 \in \bar{\mathbb{J}}^{1-g}_X$ are such that $\overline{\mathcal{P}}_{I_1} = \overline{\mathcal{P}}_{I_2}$, then $I_1 = I_2$.*

In order to prove this claim, we are going to extend the morphism $\tilde{\rho}_{\underline{q}}$ over the effective semiuniversal deformation $\pi\colon \mathcal{X} \to \mathrm{Spec}\, R_X$ of X; see Section 2.2. Consider the universal fine compactified Jacobian $\bar{J}_{\mathcal{X}}(\underline{q}) \to \mathrm{Spec}\, R_X$ with respect to the polarization $\underline{q}$ (see Section 2.2) and form the algebraic space $w\colon \mathrm{Pic}^{=}(\bar{J}_{\mathcal{X}}(\underline{q})) \to \mathrm{Spec}\, R_X$ parametrizing coherent sheaves on $\bar{J}_{\mathcal{X}}(\underline{q})$, flat over $\mathrm{Spec}\, R_X$, which are relatively simple and torsion-free of rank 1; see [3, Theorem 7.4, Proposition 5.13(ii)]. Using the universal Poincaré sheaf $\overline{\mathcal{P}}^{\mathrm{un}}$ over $u \times u\colon \bar{\mathbb{J}}^{1-g}_{\mathcal{X}} \times_{\mathrm{Spec}\, R_X} \bar{\mathbb{J}}^{1-g}_{\mathcal{X}} \to \mathrm{Spec}\, R_X$ introduced at the beginning of Section 6, we can define a $\mathrm{Spec}\, R_X$–morphism

$$(6\text{-}34)\qquad \tilde{\rho}^{\mathrm{un}}_{\underline{q}}\colon \bar{\mathbb{J}}^{1-g}_{\mathcal{X}} \to \mathrm{Pic}^{=}(\bar{J}_{\mathcal{X}}(\underline{q})), \quad \mathcal{I} \mapsto \overline{\mathcal{P}}^{\mathrm{un}}_{\mathcal{I}} := \overline{\mathcal{P}}^{\mathrm{un}}|_{\bar{J}_{\mathcal{X}}(\underline{q}) \times \{\mathcal{I}\}},$$

whose restriction to the closed point o of $\mathrm{Spec}\, R_X$ is the morphism $\tilde{\rho}_{\underline{q}}$ introduced in (6-27).

Now, by Fact 2.6(ii), we can choose two general polarizations $\underline{q}^1$ and $\underline{q}^2$ on X such that $I_i \in \bar{J}_X(\underline{q}^i)$ for $i = 1, 2$. From a relative version of Claim 1, we deduce that the restrictions of $\tilde{\rho}^{\mathrm{un}}_{\underline{q}}$ to $\bar{J}_{\mathcal{X}}(\underline{q}^i)$ for $i = 1, 2$ are open embeddings. Consider the open subset

$$V := \tilde{\rho}^{\mathrm{un}}_{\underline{q}}(\bar{J}_{\mathcal{X}}(\underline{q}^1)) \cap \tilde{\rho}^{\mathrm{un}}_{\underline{q}}(\bar{J}_{\mathcal{X}}(\underline{q}^2)) \subseteq \mathrm{Pic}^{=}(\bar{J}_{\mathcal{X}}(\underline{q})),$$

which clearly contains the point $\mathcal{P}_{I_1} = \mathcal{P}_{I_2} \in \operatorname{Pic}^{=}(\bar{J}_X(\underline{q}))$. On the fiber product $\mathcal{X} \times_{\operatorname{Spec} R_X} V$ there are two coherent sheaves $\mathcal{J}^1$ and $\mathcal{J}^2$, flat over V and relatively simple, torsion-free of rank 1, that are obtained by pushforward via $\mathrm{id} \times \tilde{\rho}_{\underline{q}}^{\mathrm{un}}$ of the universal sheaves on $\mathcal{X} \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q}^1)$ and on $\mathcal{X} \times_{\operatorname{Spec} R_X} \bar{J}_{\mathcal{X}}(\underline{q}^2)$. If we denote by $\mathcal{J}_v^i$ for $i = 1, 2$ the restriction of $\mathcal{J}^i$ to the fiber of $\mathcal{X} \times_{\operatorname{Spec} R_X} V \to V$ over $v \in V$, then by construction we have that $\overline{\mathcal{P}}^{\mathrm{un}}_{\mathcal{J}_v^1} = \overline{\mathcal{P}}^{\mathrm{un}}_{\mathcal{J}_v^2}$ for any $v \in V$. Claim 2 will be proved if we show that $\mathcal{J}_v^1 = \mathcal{J}_v^2$ for any $v \in V$, or in other words that

$$\mathcal{J}^1 = \mathcal{J}^2 \otimes p_2^*(L) \quad \text{for some line bundle } L \text{ on } V. \tag{6-35}$$

Denote by V_0 the open subset of V consisting of all the points $v \in V$ whose image $w(v)$ in $\operatorname{Spec} R_X$ is such that the fiber $\mathcal{X}_{w(v)}$ of the universal curve $\pi\colon \mathcal{X} \to \operatorname{Spec} R_X$ over $w(v)$ is smooth or has a unique singular point which is a node. By Lemma 2.8, the complement of V_0 has codimension at least two inside V. Since X does not have separating nodes and separating nodes are preserved under specialization [34, Corollary 3.8], for every $v \in V$ the curve $\mathcal{X}_{w(v)}$ does not have separating nodes or, equivalently, it is an integral curve with at most one node. Therefore, [14, Theorem 4.1] (or [5, Theorem B]) implies that (6-35) is true over V_0, ie $\mathcal{J}^1|_{V_0} = \mathcal{J}^2|_{V_0} \otimes p_2^*(L_0)$ for some line bundle $L_0 \in V_0$. Since V is smooth (as $\bar{J}_{\mathcal{X}}(\underline{q}^i)$ is smooth by Fact 2.10(i)) and the complement of V_0 has codimension at least two inside V, we can extend the line bundle L_0 on V_0 to a line bundle L on V. With this choice of L, the two sheaves appearing on the left- and right-hand sides of (6-35) are Cohen–Macaulay sheaves on V that agree on the open subset V_0 whose complement has codimension at least two; therefore the two sheaves agree on V by [20, Theorem 5.10.5]. □

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References

[1] **A B Altman**, **A Iarrobino**, **S L Kleiman**, *Irreducibility of the compactified Jacobian*, from "Real and complex singularities, Oslo 1976" (P Holm, editor), Sijthoff & Noordhoff, Alphen aan den Rijn (1977) 1–12 MR

[2] **A B Altman**, **S L Kleiman**, *Compactifying the Picard scheme, II*, Amer. J. Math. 101 (1979) 10–41 MR

[3] **A B Altman**, **S L Kleiman**, *Compactifying the Picard scheme*, Adv. Math. 35 (1980) 50–112 MR

[4] **D Arinkin**, *Cohomology of line bundles on compactified Jacobians*, Math. Res. Lett. 18 (2011) 1215–1226 MR

[5] **D Arinkin**, *Autoduality of compactified Jacobians for curves with plane singularities*, J. Algebraic Geom. 22 (2013) 363–388 MR

[6] **C Bartocci**, **U Bruzzo**, **D Hernández Ruipérez**, *Fourier–Mukai and Nahm transforms in geometry and mathematical physics*, Progr. Math. 276, Birkhäuser, Boston, MA (2009) MR

[7] **V V Batyrev**, *Birational Calabi–Yau n–folds have equal Betti numbers*, from "New trends in algebraic geometry" (K Hulek, F Catanese, C Peters, M Reid, editors), London Math. Soc. Lecture Note Ser. 264, Cambridge Univ. Press (1999) 1–11 MR

[8] **J Briançon**, *Description de* $\mathrm{Hilb}^n C\{x, y\}$, Invent. Math. 41 (1977) 45–89 MR

[9] **J Briançon**, **M Granger**, **J-P Speder**, *Sur le schéma de Hilbert d'une courbe plane*, Ann. Sci. École Norm. Sup. 14 (1981) 1–25 MR

[10] **W Bruns**, **J Herzog**, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Math. 39, Cambridge Univ. Press (1993) MR

[11] **R Donagi**, **T Pantev**, *Langlands duality for Hitchin systems*, Invent. Math. 189 (2012) 653–735 MR

[12] **E Esteves**, *Compactifying the relative Jacobian over families of reduced curves*, Trans. Amer. Math. Soc. 353 (2001) 3045–3095 MR

[13] **E Esteves**, **M Gagné**, **S Kleiman**, *Autoduality of the compactified Jacobian*, J. London Math. Soc. 65 (2002) 591–610 MR

[14] **E Esteves**, **S Kleiman**, *The compactified Picard scheme of the compactified Jacobian*, Adv. Math. 198 (2005) 484–503 MR

[15] **B Fantechi**, **L Göttsche**, **L Illusie**, **S L Kleiman**, **N Nitsure**, **A Vistoli**, *Fundamental algebraic geometry: Grothendieck's FGA explained*, Mathematical Surveys and Monographs 123, Amer. Math. Soc., Providence, RI (2005) MR

[16] **M Granger**, *Géométrie des schémas de Hilbert ponctuels*, Mém. Soc. Math. France 8, Soc. Mat. de France, Paris (1983) MR

[17] **M Groechenig**, *Moduli of flat connections in positive characteristic*, Math. Res. Lett. 23 (2016) 989–1047 MR

[18] **A Grothendieck**, *Éléments de géométrie algébrique, II: Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. 8 (1961) 5–222 MR

[19] **A Grothendieck**, *Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, I*, Inst. Hautes Études Sci. Publ. Math. 20 (1964) 5–259 MR

[20] **A Grothendieck**, *Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, II*, Inst. Hautes Études Sci. Publ. Math. 24 (1965) 5–231 MR

[21] **A Grothendieck**, *Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, III*, Inst. Hautes Études Sci. Publ. Math. 28 (1966) 5–255 MR

[22] **M Haiman**, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. 14 (2001) 941–1006 MR

[23] **R Hartshorne**, *Residues and duality*, Lecture Notes in Math. 20, Springer (1966) MR

[24] **R Hartshorne**, *Algebraic geometry*, Graduate Texts in Math. 52, Springer (1977) MR

[25] **R Hartshorne**, *Generalized divisors on Gorenstein schemes*, K–Theory 8 (1994) 287–339 MR

[26] **D Hernández Ruipérez**, **A C López Martín**, **F Sancho de Salas**, *Fourier–Mukai transforms for Gorenstein schemes*, Adv. Math. 211 (2007) 594–620 MR

[27] **D Huybrechts**, *Fourier–Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, Oxford Univ. Press (2006) MR

[28] **A Iarrobino**, *Punctual Hilbert schemes*, Bull. Amer. Math. Soc. 78 (1972) 819–823 MR

[29] **A A Iarrobino**, *Punctual Hilbert schemes*, Mem. Amer. Math. Soc. 188, Amer. Math. Soc., Providence, RI (1977) MR

[30] **Y Kawamata**, *D–equivalence and K–equivalence*, J. Differential Geom. 61 (2002) 147–171 MR

[31] **S L Kleiman**, **A B Altman**, *Bertini theorems for hypersurface sections containing a subscheme*, Comm. Algebra 7 (1979) 775–790 MR

[32] **H Matsumura**, *Commutative ring theory*, 2nd edition, Cambridge Studies in Advanced Math. 8, Cambridge Univ. Press (1989) MR

[33] **M Melo**, **A Rapagnetta**, **F Viviani**, *Fine compactified Jacobians of reduced curves*, Trans. Amer. Math. Soc. 369 (2017) 5341–5402 MR

[34] **M Melo**, **A Rapagnetta**, **F Viviani**, *Fourier–Mukai and autoduality for compactified Jacobians, I*, J. Reine Angew. Math. 755 (2019) 1–65 MR

[35] **M Melo**, **F Viviani**, *Fine compactified Jacobians*, Math. Nachr. 285 (2012) 997–1031 MR

[36] **L Migliorini**, **V Shende**, **F Viviani**, *A support theorem for Hilbert schemes of planar curves, II*, preprint (2018) arXiv

[37] **S Mukai**, *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves*, Nagoya Math. J. 81 (1981) 153–175 MR

[38] **S Mukai**, *On the moduli space of bundles on $K3$ surfaces, I*, from "Vector bundles on algebraic varieties", Tata Inst. Fund. Res. Stud. Math. 11, Oxford Univ. Press (1987) 341–413 MR

[39] **D Mumford**, *Abelian varieties*, Tata Inst. Fund. Res. Stud. Math. 5, Oxford Univ. Press (1970) MR

[40] **E Sernesi**, *Deformations of algebraic schemes*, Grundl. Math. Wissen. 334, Springer (2006) MR

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