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The extended Bogomolny equations and generalized Nahm pole boundary condition

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We develop a Kobayashi–Hitchin-type correspondence between solutions of the extended Bogomolny equations on $\Sigma \times \mathbb{R}^+$ with Nahm pole singularity at $\Sigma \times \{0\}$ and the Hitchin component of the stable $\mathrm{SL}(2,\mathbb{R})$ Higgs bundle; this verifies a conjecture of Gaiotto and Witten. We also develop a partial Kobayashi–Hitchin correspondence for solutions with a knot singularity in this program, corresponding to the non-Hitchin components in the moduli space of stable $\mathrm{SL}(2,\mathbb{R})$ Higgs bundles. We also prove existence and uniqueness of solutions with knot singularities on $\mathbb{C} \times \mathbb{R}^+$.

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1 Introduction

An intriguing proposal by Witten [27] interprets the Jones polynomial and Khovanov homology of knots on a 3–manifold Y by counting solutions to certain gauge-theoretic equations; see Kapustin and Witten [18; 27] and Haydys [12] for much more on this. In this picture, the Jones polynomial for a knot $K \subset Y$ is realized by a count of solutions to the Kapustin–Witten equations on $Y \times \mathbb{R}^+$ satisfying a new type of singular boundary conditions. We refer to Gaiotto and Witten [11; 28; 29] for a more detailed explanation, along with Mazzeo and Witten [21; 22] and He [13] for the beginnings of the analytic theory for this program. In the absence of a knot, the problem is still of interest and may lead to 3–manifold invariants. When $K = \varnothing$, the singular boundary conditions are called the Nahm pole boundary conditions, while in the presence of a knot, they are called the generalized Nahm pole boundary conditions, or Nahm pole boundary conditions with knot singularities. For simplicity, we usually just refer to solutions with Nahm pole or with Nahm pole and knot singularities.

There are two main sets of technical difficulties in this program. The first arises from the singular boundary conditions, which turn the problem into one of nonstandard elliptic type. These are now understood; see [21; 22]. A more serious difficulty

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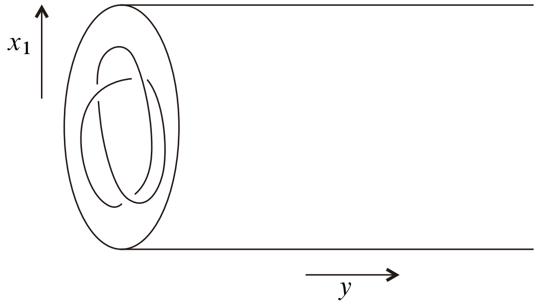

Figure 1: A knot placed at the boundary of $Y \times \mathbb{R}^+$.

involves whether it is possible to prove compactness of the space of solutions to the Kapustin–Witten (KW) equations. An important first step was accomplished by Taubes in [25; 24], but at present there is no understanding about how the Nahm pole boundary conditions interact with these compactness issues.

Gaiotto and Witten [11] proposed the study of a more tractable aspect of this problem. Suppose that we stretch the 3–manifold across a separating Riemann surface Σ in a Heegaard decomposition of Y which meets the knot transversely. In the limit, Y separates into two components $Y^{\pm}$ and zooming in on the transition region leads to a problem on $\Sigma\times\mathbb{R}\times\mathbb{R}^+$ which is independent of the $\mathbb{R}$ direction normal to the separating surface. We are thus led to study the dimensionally reduced problem, called the extended Bogomolny equations, on $\Sigma\times\mathbb{R}^+$ with the induced singular boundary condition.

A further motivation for studying the moduli space of solutions of the extended Bogomolny equations on $\Sigma\times\mathbb{R}^+$ is provided by the Atiyah–Floer conjecture [5]. In terms of a handlebody decomposition $Y^3 = Y^+ \cup_\Sigma Y^-$, the Atiyah–Floer conjecture states that the instanton Floer homology of Y can be recovered from Lagrangian Floer homology of two Lagrangians associated to the handlebodies in the moduli space $\mathcal{M}(\Sigma)$ of flat $\mathrm{SU}(2)$ connections of Σ. These Lagrangians consist of the flat connections which extend into Y^+ or Y^-. Another way to view $\mathcal{M}(\Sigma)$ is as the moduli space for the reduction of the antiselfdual equations to Σ. One then expects to use Lagrangian intersectional Floer theory to define invariants. We refer to Daemi and Fukaya [9] and Abouzaid and Manolescu [1] for recent progress on this.

In any case, we are presented with the problem of studying the dimensionally reduced Kapustin–Witten equations on $\Sigma\times\mathbb{R}^+$ with generalized Nahm pole boundary conditions. We describe these now; their derivation and further explicit computations appear in Section 2 below. Let P be a principal $\mathrm{SU}(2)$ bundle over Σ, pulled back to $\Sigma\times\mathbb{R}^+$,

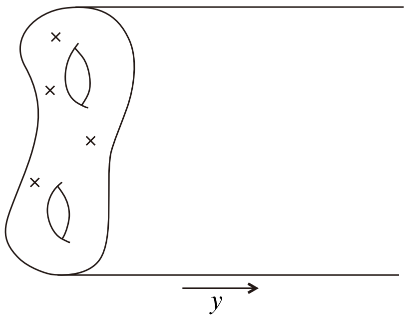

Figure 2: $\Sigma \times \mathbb{R}^+$; the "knots" correspond to points on $\Sigma \times \{0\}$.

and $\mathfrak{g}_P$ its adjoint bundle. The extended Bogomolny equations are the following set of equations for a connection A on P, and $\mathfrak{g}_P$–valued 1– and 0–forms ϕ and ϕ_1, respectively:

$$F_A - \phi \wedge \phi = \star d_A \phi_1, \quad d_A \phi = \star[\phi, \phi_1], \quad d_A^\star \phi = 0. \tag{1}$$

The knot corresponds in this setting to where the stretched knot crosses Σ, or in other words, to a set of marked points $\{p_1, \dots, p_N\}$ on Σ; see Figure 2.

In the following we denote the standard linear coordinate y on $\mathbb{R}^+$. Define $\mathcal{M}_{\mathrm{NP}}^{\mathrm{EBE}}$ and $\mathcal{M}_{\mathrm{KS}}^{\mathrm{EBE}}$ to be the moduli spaces of solutions to (1) which satisfy the Nahm pole and generalized Nahm pole boundary conditions at $y = 0$, and which converge to an $\mathrm{SL}(2, \mathbb{R})$ flat connection as $y \to \infty$. For the second of these spaces, we tacitly restrict to the subset of solutions which are compatible with an $\mathrm{SL}(2, \mathbb{R})$ structure, as explained more carefully in Section 3. The subscripts NP and KS here stand for "Nahm pole" and "knot singularity". We also write $\mathcal{M}$ for the moduli space of stable $\mathrm{SL}(2, \mathbb{R})$ Higgs bundles and recall that $\mathcal{M} = \mathcal{M}^{\mathrm{Hit}} \sqcup \mathcal{M}^{\mathrm{Hit}^c}$, where the first term on the right is the Fuchsian, or Hitchin, component and $\mathcal{M}^{\mathrm{Hit}^c}$ the union of the other components. It is well known that $\mathcal{M}^{\mathrm{Hit}}$ is identified with a finite cover of the Teichmüller space for Σ.

In the spirit of Donaldson [10] and Uhlenbeck and Yau [26], Gaiotto and Witten [11] define maps

$$I_{\mathrm{NP}}\colon \mathcal{M}_{\mathrm{NP}}^{\mathrm{EBE}} \to \mathcal{M}^{\mathrm{Hit}}, \quad I_{\mathrm{KS}}\colon \mathcal{M}_{\mathrm{KS}}^{\mathrm{EBE}} \to \mathcal{M}^{\mathrm{Hit}^c}, \tag{2}$$

which we recall in Section 3. They conjecture that I_{NP} is one-to-one. We prove this here and also describe the map I_{KS}. Our main result is:

Theorem 1.1 (i) *The map I_{NP} is a bijection. Explicitly, to every element in the Hitchin component $\mathcal{M}^{\mathrm{Hit}}$, there exists a solution to (1) satisfying the Nahm* *pole boundary condition. If two solutions to (1) satisfying these boundary conditions map to the same element in $\mathcal{M}^{\mathrm{Hit}}$ under I_{NP}, then they are* SU(2)*–gauge equivalent.*

(ii) *The map I_{KS} is two-to-one: For every element in the $\mathcal{M}^{\mathrm{Hit}^c}$, there exist two solutions to (1) which satisfy generalized Nahm pole boundary conditions with knot singularities and which are compatible with the* $\mathrm{SL}(2,\mathbb{R})$ *structure as $y \to \infty$. Any solution to (1) satisfying these boundary and compatibility conditions is equal, up to* SU(2)*–gauge equivalence, to one of these two solutions.*

We define in Section 3 what it means for solutions of (1) with knot singularities to be compatible with the $\mathrm{SL}(2,\mathbb{R})$ structure as $y \to \infty$. This condition allows (1) to be reduced to a scalar equation. There are almost surely solutions to (1) which do not satisfy this condition.

The expectation, explained in [27], is that the Jones polynomial should be recovered by counting solutions to the extended Bogomolny equations on $\mathbb{R}^3 \times \mathbb{R}^+$, with a knot singularity at some $K \subset \mathbb{R}^3$. Thus, as a dimensionally reduced version of this problem, we also consider these equations on $\mathbb{C} \times \mathbb{R}^+$:

Theorem 1.2 *Given any positive divisor $D = \sum n_i p_i$ on $\mathbb{C}$, there exists a solution to (1) which has knot singularities of order n_i at p_i. This solution is unique to the scalar equation.*

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2 Preliminaries

We begin by considering various ways in which the extended Bogomolny equations (1) may be interpreted.

2.1 S^1–invariant Kapustin–Witten equations

Let X be a smooth 4–manifold with boundary, P an SU(2) bundle over X and $\mathfrak{g}_P$ the adjoint bundle of P. If $\widehat{A}$ is a connection on P and $\widehat{\Phi}$ is a $\mathfrak{g}_P$–valued one-form, then the Kapustin–Witten equations for the pair $(\widehat{A}, \widehat{\Phi})$ are

$$F_{\widehat{A}} - \widehat{\Phi} \wedge \widehat{\Phi} + \star d_{\widehat{A}} \widehat{\Phi} = 0, \quad d_{\widehat{A}}^{\star} \widehat{\Phi} = 0. \tag{3}$$

Consider the special case where $X = S^1 \times Y$ is the product of a circle and a 3–manifold, and where $(\widehat{A}, \widehat{\Phi})$ is an S^1 invariant solution to (3). We then set

$$\widehat{A} = A + A_1 dx_1, \quad \widehat{\Phi} = \phi + \phi_1 dx_1, \tag{4}$$

where $A, \phi \in \Omega^1_Y(\mathfrak{g}_P)$ and $A_1, \phi_1 \in \Omega^0_Y(\mathfrak{g}_P)$ are independent of $x_1 \in S^1$. Then (3) becomes

$$\begin{aligned} F_A - \phi \wedge \phi - \star d_A \phi_1 - \star [A_1, \phi] &= 0, \\ \star d_A \phi + [\phi_1, \phi] + d_A A_1 &= 0, \\ d_A^{\star} \phi - [A_1, \phi_1] &= 0. \end{aligned} \tag{5}$$

Denoting the quantities on the left of these three qualities by $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$, respectively, we define the expressions

$$\begin{aligned} I_0 &= \int_Y |\mathcal{X}_1|^2 + |\mathcal{X}_2|^2 + |\mathcal{X}_3|^2, \\ I_1 &= \int_Y |F_A - \phi \wedge \phi - \star d_A \phi_1|^2 + |\star d_A \phi + [\phi_1, \phi]|^2 + |d_A^{\star} \phi|^2, \\ I_2 &= \int_Y |[A_1, \phi]|^2 + |d_A A_1|^2 + |[A_1, \phi_1]|^2, \end{aligned} \tag{6}$$

and also, if Y is a 3–manifold with boundary,

$$I_3 = -\int_{\partial Y} \mathrm{Tr}(d_A A_1 \wedge \phi_1) - \int_{\partial Y} \mathrm{Tr}([A_1, \phi_1] \wedge \star \phi).$$

After a straightforward calculation, assuming that all integrations are valid, we have

$$I_0 = I_1 + I_2 + I_3. \tag{7}$$

Since I_0, I_1 and I_2 are all nonnegative, we deduce the

Proposition 2.1 *If (A_1, ϕ_1) satisfies a boundary condition which guarantees that $I_3 = 0$, and if (A, ϕ) is irreducible, then $A_1 = 0$ and (5) reduces to the equations corresponding to $I_1 = 0$.*

The case of principal interest in this paper is when $Y = \Sigma \times \mathbb{R}^+_y$ and $(\widehat{A}, \widehat{\Phi})$ satisfy the Nahm pole boundary conditions at $y = 0$ and converge as $y \to \infty$ to a flat $\mathrm{SL}(2, \mathbb{C})$ connection. The conditions of this proposition are then satisfied. We recall the claim —

see [24, page 36] as well as [13, Corollary 4.7] — that for solutions satisfying these boundary conditions, the dy component of ϕ vanishes. Results from [21] show that as $y \searrow 0$, $A_1 \sim y^2$ and $\phi_1 \sim \frac{1}{y}$, hence $\star\phi = 0$ at $y = 0$. In addition, A_1 and ϕ_1 both converge to 0 as $y \to \infty$. These facts together imply that I_3 vanishes at both $y = 0$ and $y = \infty$, so Proposition 2.1 holds.

When the boundary condition for a knot appears, we refer to Definition 3.4 for an explicit definition. If there is a knot $p \in \Sigma \times \mathbb{R}^+$, let R be the distance to p; then the boundary condition requires that for some $\epsilon > 0$, we have $A_1 \sim R^{-1+\epsilon}$, $\phi_1 \sim R^{-1}$ and $\phi \sim R^{-1}$. Thus, both $d_A A_1 \wedge \phi_1$ and $[A_1, \phi_1] \wedge \star\phi$ are asymptotic to $R^{-3+\epsilon}$ as $R \to 0$. Using the spherical coordinate, we have found the boundary condition for the knot doesn't contribute to the integral I_3.

If an S^1–invariant solution satisfies the Nahm pole boundary condition at $y = 0$ and converges to a flat $\mathrm{SL}(2,\mathbb{C})$ connection as $y \to \infty$, then the pair (A, Φ) satisfies the so-called extended Bogomolny equations on $\Sigma \times \mathbb{R}^+$,

$$F_A - \phi \wedge \phi = \star d_A \phi_1, \quad d_A \phi = \star[\phi, \phi_1], \quad d_A^\star \phi = 0. \tag{8}$$

Here A is a connection, $\phi \in \Omega^1(\mathfrak{g}_P)$, $\phi_1 \in \Omega^0(\mathfrak{g}_P)$ and the dy component of ϕ vanishes.

These equations reduce, when $\phi_1 = 0$, to the Hitchin equations; when $\phi = 0$, to the Bogomolny equations; and when $A = 0$ and ϕ is independent of Σ, to the Nahm equations. Thus one expects that all known techniques for these special cases should be applicable to these hybrid equations as well.

2.2 Hermitian geometry

Choose a holomorphic coordinate $z = x_2 + i x_3$ on Σ and let y be the linear coordinate on $\mathbb{R}^+$. In these coordinates, define $d_A = \nabla_2\, dx_2 + \nabla_3\, dx_3 + \nabla_y\, dy$ and $\phi = \phi_2\, dx_2 + \phi_3\, dx_3 = \frac{1}{2}(\varphi_z\, dz + \varphi_{\bar z}^\dagger\, d\bar z)$, where $\varphi_z = \phi_2 - i\phi_3$; we also write $\varphi = \varphi_z dz$. Using these, we can rewrite (1) in the "three D's" formalism: with $\mathcal{A}_y = A_y - i\phi_1$, set

$$\begin{aligned} \mathcal{D}_1 &= \nabla_2 + i\nabla_3, \\ \mathcal{D}_2 &= \mathrm{ad}\,\varphi_z = [\varphi_z, \cdot], \\ \mathcal{D}_3 &= \nabla_y - i\phi_1 = \partial_y + \mathcal{A}_y = \partial_y + A_y - i\phi_1. \end{aligned} \tag{9}$$

The adjoints of these operators are

$$\mathcal{D}_1^\dagger = -\nabla_2 + i\nabla_3, \quad \mathcal{D}_2^\dagger = -[\phi_2 + i\phi_3, \cdot\,], \quad \mathcal{D}_3^\dagger = -\nabla_y - i\phi_1. \tag{10}$$

The extended Bogomolny equations can then be written in the alternative form

$$[\mathcal{D}_i, \mathcal{D}_j] = 0 \quad \text{for } i,j = 1,2,3 \qquad \text{and} \qquad \sum_{i=1}^{3} [\mathcal{D}_i, \mathcal{D}_i^\dagger] = 0. \tag{11}$$

We write out the last of these, which is the most intricate. Noting that

$$\begin{aligned} &[\mathcal{D}_1, \mathcal{D}_1^\dagger] = [\nabla_2 + i\nabla_3, -\nabla_2 + i\nabla_3] = 2iF_{23}, \\ &[\mathcal{D}_2, \mathcal{D}_2^\dagger] = -2i[\phi_2, \phi_3], \\ &[\mathcal{D}_3, \mathcal{D}_3^\dagger] = -2i\nabla_y\phi_1, \end{aligned} \tag{12}$$

we have

$$\frac{1}{2i}\sum_{k=1}^{3} [\mathcal{D}_k, \mathcal{D}_k^\dagger] = F_{23} - [\phi_2, \phi_3] - \nabla_y\phi_1 = 0.$$

As is standard for such equations — see [27] — the smaller system $[\mathcal{D}_i, \mathcal{D}_j] = 0$ is invariant under the complex ($\mathrm{SL}(2;\mathbb{C})$–valued) gauge group $\mathcal{G}_P^{\mathbb{C}}$, while the full system (11) is invariant under the unitary gauge group, $\mathcal{D}_i \to g^{-1}\mathcal{D}_i g$ for $g \in \mathcal{G}_P$, and the final equation is a real moment map condition. Following the spirit of Donaldson, Uhlenbeck and Yau [10; 26], we thus expect that Hermitian geometric data from the $\mathcal{G}_P^{\mathbb{C}}$–invariant equations play a role in solving the moment map equation.

Suppose that E is a rank 2 Hermitian bundle over $\Sigma \times \mathbb{R}^+$. As we now explain, for any function f and section s, $\mathcal{D}_1(fs) = \partial_{\bar{z}} fs + f\mathcal{D}_1 s$, which is a ∂–operator in the Newlander–Nirenberg sense; $\mathcal{D}_2$ is then a K_Σ–valued endomorphism of $\mathcal{E}$, while $\mathcal{D}_3$ specifies a parallel transport in the y direction. In terms of these, the equations $[\mathcal{D}_i, \mathcal{D}_j] = 0$ have a nice geometric meaning.

Denote by $E_y := E|_{\Sigma \times \{y\}}$ the restriction of E to each slice $\Sigma \times \{y\}$. Since $\mathcal{D}_1^2 = 0$ is always true for dimensional reasons, the Newlander–Nirenberg theorem gives that $\mathcal{D}_1$ induces a holomorphic structure on E_y for each y, ie in some gauge, we can write $\mathcal{D}_1 = \bar{\partial}$. A connection A is compatible with this holomorphic structure if $A^{0,1}$ equals $\bar{\partial}$.

Next, $[\mathcal{D}_1, \mathcal{D}_2] = 0$ says that the endomorphism φ is holomorphic with respect to this structure, so $(E, \mathcal{D}_1, \varphi)$ is a Higgs bundle over each slice. Finally, the equations $[\mathcal{D}_2, \mathcal{D}_3] = 0$ and $[\mathcal{D}_1, \mathcal{D}_3] = 0$ show that this family of Higgs bundles is parallel in y, ie there is a specified identification of these objects at different values of y.

Following [10], a data set for our problem consists of a rank two bundle E over $\Sigma\times\mathbb{R}^+$ and a triplet of operators $\Theta=(\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3)$ on $\mathcal{C}^\infty(E)$ satisfying

- $\mathcal{D}_1(fs)=\partial_{\overline{z}} fs+f\mathcal{D}_1 s$ and $\mathcal{D}_3(fs)=(\partial_y f)s+f\mathcal{D}_3 s$ for $f\in\mathcal{C}^\infty(\Sigma\times\mathbb{R}^+)$ and $s\in C^\infty(E)$;
- $\mathcal{D}_2=[\varphi,\cdot]$ for some $\varphi\in\Omega^{1,0}(\mathfrak{g}_P)$;
- $[\mathcal{D}_i,\mathcal{D}_j]=0$ for all i and j.

Given (E,Θ), a choice of Hermitian metric H on E determines Hermitian adjoints $\mathcal{D}_i'$ of the operators $\mathcal{D}_i$ by the requirements that for any smooth functions f and sections s,

- $\mathcal{D}_1'$ and $\mathcal{D}_3'$ are derivations, ie $\mathcal{D}_1'(fs)=(\partial_z f)s+f\mathcal{D}_1's$ and $\mathcal{D}_3'(fs)=(\partial_y f)s+f\mathcal{D}_3's$, while $\mathcal{D}_2(fs)=f\mathcal{D}_2(s)$;
- $\mathcal{D}_1$ and $\mathcal{D}_3$ satisfy $\partial_{\overline{z}}H(s,s')=H(\mathcal{D}_1 s,s')+H(s,\mathcal{D}_1's')$ and $\partial_y H(s,s')=H(\mathcal{D}_3 s,s')+H(s,\mathcal{D}_3's')$;
- $H(\mathcal{D}_2 s,s')+H(s,\mathcal{D}_2's')=0$.

The moment map equation in (11) can be regarded as an equation for the Hermitian metric H. Indeed, setting $\mathcal{D}_y=\frac{1}{2}(\mathcal{D}_3+\mathcal{D}_3')$, $\mathcal{D}_{\overline{z}}=\mathcal{D}_1$ and $\mathcal{D}_z=\mathcal{D}_1'$, we define a unitary connection $\mathcal{D}_A$, and an endomorphism-valued 1–form ϕ and 0–form ϕ_1 on (E,Θ,H) by

$$
\begin{aligned}
\mathcal{D}_A(s) &:= \mathcal{D}_1(s)\,d\overline{z}+\mathcal{D}_1'(s)\,dz+\mathcal{D}_y(s)\,dy,\\
[\phi,s] &:= [\mathcal{D}_2,s]\,dz+[\mathcal{D}_2',s]\,d\overline{z},\\
\phi_1 &:= \tfrac{1}{2}i(\mathcal{D}_3-\mathcal{D}_3').
\end{aligned}
\tag{13}
$$

We call (A,ϕ,ϕ_1) a unitary triplet. Note however that in an arbitrary trivialization of E, (A,ϕ,ϕ_1) may not consist of unitary matrices. We recall a standard result [4] which provides the link between connections in unitary and holomorphic frames. In the following, and later, we refer to parallel holomorphic gauges. These are, as the moniker suggests, holomorphic gauges for each E_y which are parallel with respect to $\mathcal{D}_3$.

Proposition 2.2 *With (E,Θ,H) as above, there is a unique triplet (A,ϕ,ϕ_y) compatible with the unitary structure and with the structure defined by Θ. In other words, in every unitary gauge, $A^\star=-A$, $\phi^\star=\phi$ and $\phi_1^\star=-\phi_1$, while in every parallel holomorphic gauge, $\mathcal{D}_1=\overline{\partial}_E$ and $\mathcal{D}_3=\partial_y$, ie $A^{(0,1)}=A_y-i\phi_1=0$.*

Proof With the convention $H(s,s')=\overline{s}^\top Hs'$, we compute first in a holomorphic parallel gauge, from the defining equations for the $\mathcal{D}_i'$, that $\overline{\partial}H=(\overline{A^{(1,0)}})^\top H$ and

$\partial_y H = H(-A_y - i\phi_1)$, so in this gauge, $A = A^{(1,0)} = H^{-1}\partial H$ and $A_y + i\phi_1 = -H^{-1}\partial_y H$.

Suppose next that we know H with respect to a holomorphic frame. If g is a complex gauge transformation such that $H = g^\dagger g$, then in the parallel holomorphic gauge,

$$A^{(1,0)} = H^{-1}\partial H = g^{-1}(g^\dagger)^{-1}(\partial_z g^\dagger)g + g^{-1}\partial_z g, \quad A^{(0,1)} = 0. \tag{14}$$

If $\widehat{A}$ is the connection form in unitary gauge, then

$$\widehat{A}_z = (g^\dagger)^{-1}\partial_z g^\dagger, \quad \widehat{A}_{\bar{z}} = -(\partial_{\bar{z}} g)g^{-1}, \tag{15}$$

and $\widehat{A}_{\bar{z}}^\dagger = -\widehat{A}_z$. Thus g transforms the holomorphic form to the unitary one.

Similarly, the same Higgs field in holomorphic and unitary gauge, φ and ϕ, are related by

$$\phi_z = g\varphi g^{-1}, \quad \phi_{\bar{z}} = (g^\dagger)^{-1}\overline{\varphi}^\top g^\dagger. \tag{16}$$

For the final component, suppose that $\mathcal{A}_y$ is given in holomorphic gauge. Then in unitary gauge,

$$A_y = \tfrac{1}{2}((\partial_y g)g^{-1} - (g^\dagger)^{-1}\partial_y g^\dagger), \quad \phi_1 = \tfrac{1}{2}i((g^\dagger)^{-1}\partial_y g^\dagger + \partial_y g^\dagger (g^\dagger)^{-1}). \tag{17}$$

This concludes the proof. □

We now record some computations in a local holomorphic coordinate chart. Writing $\mathcal{D}_1 = \partial_{\bar{z}} + \alpha$, $\mathcal{D}_1' = \partial_z + A^{(1,0)}$, $\mathcal{D}_3 = \partial_y + \mathcal{A}_y$ and $\mathcal{D}_3' = \partial_y + \mathcal{A}_y'$, we compute

$$\begin{aligned} A^{(1,0)} &= H^{-1}\partial_z H - H^{-1}(\overline{\alpha})^\top H, \\ A = A^{(1,0)} + \alpha &= H^{-1}\partial_z H - H^{-1}\overline{\alpha}^\top H + \alpha, \\ \varphi^\dagger &= H^{-1}\overline{\varphi}^\top H, \\ \mathcal{A}_y' &= H^{-1}\partial_y H - H^{-1}\overline{\mathcal{A}}_y^\top H. \end{aligned} \tag{18}$$

Thus, if $\alpha = \mathcal{A}_y = 0$, then the adjoint operators become

$$\begin{aligned} \mathcal{D}_1^\dagger &= -\mathcal{D}_1' = -(\partial_z + H^{-1}\partial_z H), \\ \mathcal{D}_2^\dagger &= -\mathcal{D}_2' = [\varphi^\dagger, \cdot], \\ \mathcal{D}_3^\dagger &= -\mathcal{D}_3' = -\partial_y - H^{-1}\partial_y H. \end{aligned} \tag{19}$$

Altogether, in a local holomorphic coordinate z for which the metric on Σ equals $g_0^2|dz|^2$, and in the holomorphic parallel gauge where $\mathcal{D}_1 = \overline{\partial}$ and $\mathcal{D}_3 = \partial_y$, then in local coordinates (z, y), the extended Bogomolny equations (11) become

$$-\overline{\partial}_{\overline{z}}(H^{-1}\partial_z H) - g_0^2\partial_y(H^{-1}\partial_y H) + [\varphi_z, \varphi_{\overline{z}}^{\star}] = 0. \tag{20}$$

Two sets of data (E, Θ) and $(E, \widetilde{\Theta})$ are called equivalent if there exists a complex gauge transform g such that $g^{-1}\widetilde{\mathcal{D}}_i g = \mathcal{D}_i$ for $i = 1, 2, 3$. A key fact is that (E, Θ) is completely determined by a Higgs bundle $(\mathcal{E}, \varphi)$ over the Riemann surface Σ.

Proposition 2.3 (i) *Suppose that (E, Θ) and $(E, \widetilde{\Theta})$ are two data sets. If the restrictions of Θ to E_y and $\widetilde{\Theta}$ to some possibly different $E_{y'}$ are complex gauge equivalent, then (E, Θ) and $(E, \widetilde{\Theta})$ are equivalent.*

(ii) *If (E, Θ, H) is a solution to the extended Bogomolny equations, and if g is a complex gauge transform, then (E, Θ^g), where*

$$\Theta^g = (g^{-1}D_1 g, g^{-1}D_2 g, g^{-1}D_3 g), \quad H^g = Hg^{\star_H} g$$

is also a solution.

Proof Since $\mathcal{D}_3$ and $\widetilde{\mathcal{D}}_3$ both define isomorphisms of the Higgs bundles, (i) follows immediately. Then, recalling that $D_i^{\dagger}$ is the conjugate of D_i with respect to H, one may check (ii) directly from the definition. □

3 Boundary conditions

In this section we introduce boundary conditions for the extended Bogomolny equations over $\Sigma \times \mathbb{R}^+$ at $y = 0$ and as $y \to +\infty$.

3.1 SL(2, ℝ) Higgs-bundles

We impose an asymptotic boundary condition as $y \to +\infty$ by requiring that solutions of (1) converge to flat $\mathrm{SL}(2, \mathbb{R})$ connections. To explain this more carefully, we recall some basic facts about the moduli space of stable $\mathrm{SL}(2, \mathbb{R})$ Higgs-bundles; see [15; 16].

Consider a Riemann surface Σ of genus $g > 1$. A Higgs bundle consists of a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a holomorphic vector bundle over a complex vector bundle E and $\varphi \in H^0(\mathrm{End}(\mathcal{E}) \otimes K)$ is a Higgs field. Let $(\mathcal{E}, \varphi)$ be a rank 2 Higgs bundle such that $\deg E = 0$. It is proved in [15] that once an $\mathrm{SL}(2, \mathbb{R})$ structure is fixed, there is an isomorphism $\mathcal{E} \cong L^{-1} \oplus L$, where L is a line bundle with $0 \le \deg L \le g-1$, in terms of which the Higgs field takes the form

$$\varphi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}, \tag{21}$$

where $\alpha \in H^0(L^{-2} \otimes K)$ and $\beta \in H^0(L^2 \otimes K)$. When $\deg L = g-1$ and $L = K^{1/2}$ for one of the 2^{2g} square roots of K, then we write this canonical form for the Higgs field in the familiar form

$$\varphi = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}. \tag{22}$$

Here 1 is the canonical identity element in

$$\mathrm{Hom}(L, L^{-1}) \otimes K = \mathrm{Hom}(K^{1/2}, K^{-1/2}) \otimes K = \mathcal{O}$$

and $q \in H^0(L^2 \otimes K) = H^0(K^2)$ is a holomorphic quadratic differential. This set of Higgs bundles constitutes the Hitchin component of the $\mathrm{SL}(2, \mathbb{R})$ moduli space.

The splittings with $|\deg L| < g-1$ constitute the non-Hitchin components. Write $k = \deg L$, so that $\deg(L^{-2} \otimes K) = \deg K - 2\deg L = 2g-2-2k$. Thus when $0 \le k < g-1$, the section α has $2g-2-2k$ zeros; these are of course invariant under complex gauge transform.

If $\phi_1 = 0$ in (1), or if $D_3 = 0$ in (11), we obtain the Hitchin equation

$$F_H + [\varphi, \varphi^\star] = 0, \quad \bar{\partial}_A \varphi = 0. \tag{23}$$

A rank 2 Higgs bundle $(\mathcal{E}, \varphi)$ with $\det(\mathcal{E}) = \mathcal{O}$ is stable if $\deg S < 0$ for every φ–invariant subbundle $S \subset E$. We say in general that $(\mathcal{E}, \varphi)$ is polystable if it is a direct sum of stable Higgs bundles. In the rank 2 case, a polystable Higgs bundle takes the form $\left(E = L^{-1} \oplus L,\ \varphi = \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)\right)$, but by assumption we shall exclude these.

The solvability of the Hitchin equation (23) was analyzed completely in [15]:

Theorem 3.1 [15] *Let $(\mathcal{E}, \varphi)$ be a Higgs bundle over Σ. There exists an irreducible solution H to the Hitchin equations if and only if the Higgs bundle is stable, and a reducible solution if and only if it is polystable.*

When $\deg L > 0$, the Higgs bundles $\left(L^{-1} \oplus L,\ \varphi = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right)\right)$ are all stable. If $\deg L = 0$, then $L \cong \mathcal{O}$ and E is holomorphically trivial. If $\varphi = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right)$, then the pair is stable if and only if neither α nor β are identically zero. If precisely one of α and β vanishes, the pair is neither stable nor polystable and the Hitchin equation has no solution. If $\alpha = \beta = 0$, then the Higgs bundle is polystable and there exists a reducible solution.

In this paper we restrict attention to irreducible solutions. The moduli space of stable $\mathrm{SL}(2,\mathbb{R})$ Higgs bundles can then be described as follows:

Theorem 3.2 [15] *The* $\mathrm{SL}(2,\mathbb{R})$ *Higgs bundle moduli space contains* $2g-1$ *components, classified by the degree* k *of the line bundle* L*, where* $|k| \le g-1$*. The component* $\mathcal{M}_k^{\mathrm{SL}(2,\mathbb{R})}$ *is a smooth manifold of dimension* $6g-6$ *diffeomorphic to a complex vector bundle of rank* $g-1+2k$ *over the* 2^{2g}*–fold cover of the symmetric product* $S^{2g-2-2k}\Sigma$*.*

Proof We sketch the proof. For the $\mathrm{SL}(2,\mathbb{R})$ Higgs bundle $\bigl(L^{-1}\oplus L, \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right)\bigr)$, the zeros of $\alpha \in H^0(L^{-2}\otimes K)$ give a divisor D where $\mathcal{O}(D) = L^{-2}\otimes K$, and hence an element of $S^{2g-2-2k}\Sigma$. Then $\beta \in H^0(\Sigma, \mathcal{O}(-D)K^2)$ determines a line bundle.

Note that since we are working with $\mathrm{SL}(2,\mathbb{R})$, given D we can only determine $L^2 = \mathcal{O}(-D)K$, but L itself can only be recovered up to the choice of a line bundle I with $I^2 = \mathcal{O}$. There are precisely 2^{2g} such choices. □

We recall finally a well-known result:

Proposition 3.3 *The harmonic metric* H *corresponding to a stable* $\mathrm{SL}(2,\mathbb{R})$ *Higgs bundle splits with respect to the decomposition* $E = L^{-1}\oplus L$ *as* $H = \left(\begin{smallmatrix} h & 0 \\ 0 & h^{-1} \end{smallmatrix}\right)$*.*

A proof appears in [8, Theorem 2.10].

3.2 The Nahm pole boundary condition and holomorphic data

We next recall the Nahm pole boundary condition and its associated Hermitian geometry, following [11].

The starting point is the model solution [27]. Consider a trivial rank 2 bundle E over $\mathbb{C}\times\mathbb{R}^+$. The model Nahm pole solution is

$$A_z = 0, \quad \phi_z = \frac{1}{y}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A}_y = -i\phi_1 = \frac{1}{2y}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{24}$$

Under the singular complex gauge transformation, these fields become $g = \left(\begin{smallmatrix} y^{-1/2} & 0 \\ 0 & y^{1/2} \end{smallmatrix}\right)$ to $\varphi = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $A_z = 0$ and $\mathcal{A}_y = 0$, ie the connection in the $\mathbb{R}^+$ direction transforms to ∂_y.

Now, $s = \binom{ay^{-1/2}}{by^{1/2}}$ is an $\mathcal{D}_3$ parallel section of E for any $a, b \in \mathbb{R}$, and indeed is a solution of the full extended Bogomolny equations. A generic solution of this form blows up as $y \to 0$, but there is a well-defined subbundle $L \subset E$, called the *vanishing line bundle*, defined as the space of solutions which tend to 0 as $y \to 0$. For this model solution and line bundle, $\operatorname{span}\{\varphi(L), L \otimes K\} = E \otimes K$ at all points.

We say that a solution (A, φ, ϕ_1) to (1) on a rank 2 Hermitian bundle E with determinant zero over Σ satisfies the Nahm pole boundary condition if, in terms of any local trivialization,

$$A_z \sim \mathcal{O}(y^{-1+\epsilon}), \quad \varphi = \frac{1}{y}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(y^{-1+\epsilon}), \quad \mathcal{A}_y = \frac{1}{2y}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \mathcal{O}(y^{-1+\epsilon}) \tag{25}$$

as $y \to 0$. As described in [21], it is of course necessary to consider fields which lie in some function space, eg a weighted Hölder space, and the error estimate $\mathcal{O}(y^{-1+\epsilon})$ is interpreted in terms of that norm. The regularity theory in that paper shows that a solution of the extended Bogomolny equations, or indeed of the full Kapustin–Witten system, is then much more regular after being put into gauge.

In exactly the same way as in the model case, this boundary condition defines a line bundle $L \subset E$, and since $\det E = \mathcal{O}$, we have $E/L \cong L^{-1}$. On the other hand, $\operatorname{span}\{\varphi(L), L \otimes K\} = E \otimes K$, so that pushing forward L via

$$L \xrightarrow{\varphi} E \otimes K \to (E/L) \otimes K \tag{26}$$

shows that $L \cong L^{-1} \otimes K$, ie $L \cong K^{1/2}$, and then $E/L \cong K^{-1/2}$. In other words,

$$0 \to K^{1/2} \to E \to K^{-1/2} \to 0. \tag{27}$$

In addition, write $i_1\colon \varphi(L) \to E \otimes K$ and $i_2\colon L \otimes K \to E \otimes K$, and define

$$i\colon \varphi(L) \oplus L \otimes K \to E \otimes K, \quad i = i_1 + i_2. \tag{28}$$

As $\operatorname{span}\{\varphi(L), L \otimes K\} = E \otimes K$, we obtain that i is surjective between two rank two bundles, thus an isomorphism. Tensoring by K^{-1}, we obtain $E \cong K^{-1/2} \oplus K^{1/2}$.

Under a complex gauge transform, we can then put the Higgs field into the form $\varphi = \begin{pmatrix} t & 1 \\ \beta' & -t \end{pmatrix}$. Setting $g = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$, we compute that $g^{-1}\varphi g = \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}$. This shows that an $\mathrm{SL}(2,\mathbb{R})$ Higgs bundle lies in the Hitchin component of the $\mathrm{SL}(2,\mathbb{R})$ Higgs bundle moduli space.

In summary, recalling that $\mathcal{M}_{\mathrm{NP}}^{\mathrm{EBE}}$ is the moduli space of solutions of the extended Bogomolny equations with limit in $\mathrm{SL}(2,\mathbb{R})$ and $\mathcal{M}^{\mathrm{Hit}}$ is the Hitchin component of the stable $\mathrm{SL}(2,\mathbb{R})$ Higgs bundle, we have now explained the map $I_{\mathrm{NP}}\colon \mathcal{M}_{\mathrm{NP}}^{\mathrm{EBE}} \to \mathcal{M}^{\mathrm{Hit}}$. Gaiotto and Witten [11] conjectured that this map is a bijection, and we show below that this is the case.

3.3 Knot singularity

We next define the model knot singularity introduced by Witten in [27], and the modified Nahm pole condition for knots. In the Riemann surface picture, knot singularities correspond to marked points, at which monopoles are wrapped.

Fix coordinates $z = x_2 + ix_3 \in \mathbb{C}$ and $y \in \mathbb{R}^+$ on $\mathbb{C}\times\mathbb{R}^+$. Then, with respect to the Higgs field $\varphi = \begin{pmatrix} 0 & z^n \\ 0 & 0 \end{pmatrix}$ and Hermitian metric $H = \begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix}$, equation (20) takes the form

$$-(\Delta + \partial_y^2)u + r^{2n}e^{2u} = 0, \tag{29}$$

where $\Delta = \partial_{x_2}^2 + \partial_{x_3}^2$ and $r = |z|$.

Assuming homogeneity in (z, y) and radial symmetry in z, Witten [27] obtained the model solution

$$U_n(r, y) = \log\left(\frac{2(n+1)}{(\sqrt{r^2+y^2}+y)^{n+1} - (\sqrt{r^2+y^2}-y)^{n+1}}\right). \tag{30}$$

To investigate this further, introduce spherical coordinates (R, ψ, θ),

$$R = \sqrt{r^2+y^2}, \quad z = re^{i\theta}, \quad \sin\psi = \frac{y}{R}, \quad \cos\psi = \frac{r}{R}.$$

Writing $a = \sqrt{r^2+y^2} + y$ and $b = \sqrt{r^2+y^2} - y$, then

$$\frac{a}{R} = 1 + \frac{y}{R} = 1 + \sin\psi, \quad \frac{b}{R} = 1 - \frac{y}{R} = 1 - \sin\psi,$$

and hence

$$U_n = -\log y - n\log R + \log\frac{n+1}{S_n(\psi)},$$

where

$$S_n(\psi) = \mathcal{S}_n(a,b) = \sum_{k=0}^{n} a^{n-k}b^k.$$

Note that $U_0 = -\log y$ when $n = 0$, which recovers the model Nahm pole solution. Moreover, U_n is compatible with the Nahm pole singularity in the sense that $U_n \sim -\log y$ as $y \to 0$ for $r \geq \epsilon > 0$.

Define $g_n = \begin{pmatrix} e^{u_n/2} & 0 \\ 0 & e^{-u_n/2} \end{pmatrix}$; then, in unitary gauge,

$$\begin{aligned} A_z &= g_n^{-1}\partial g_n, & \phi_z &= g_n \varphi g_n^{-1}, \\ A_{\bar{z}} &= -(\bar{\partial} g_n) g_n^{-1}, & \phi_1 &= \tfrac{1}{2} i (g_n^{-1}\partial_y g_n + \partial_y g_n g_n^{-1}), \end{aligned} \tag{31}$$

or, explicitly,

$$\begin{aligned} \phi_z &= \begin{pmatrix} 0 & z^n e^{U_n} \\ 0 & 0 \end{pmatrix} = \frac{2}{R} \frac{(n+1)\cos^n \psi}{(1+\sin\psi)^{n+1} - (1-\sin\psi)^{n+1}} e^{in\theta} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{R\sin\psi} \frac{(n+1)\cos^n\psi}{S_n(\psi)} e^{in\theta} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \phi_1 &= -U_n' \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix} = \frac{n+1}{R} \frac{(1+\sin\psi)^{n+1} + (1-\sin\psi)^{n+1}}{(1+\sin\psi)^{n+1} - (1-\sin\psi)^{n+1}} \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}, \\ A_y &= 0. \end{aligned} \tag{32}$$

Suppose that s is a section with $\mathcal{D}_3 s = 0$. Then $s = \begin{pmatrix} a e^{U_n/2} \\ b e^{-U_n/2} \end{pmatrix}$ is a solution for any $a, b \in \mathbb{R}$, where $e^{U_n} = (n+1)/(yR^n S_n(\psi))$. As in the Nahm pole case, we can still define a line subbundle L corresponding to parallel sections whose limits as $y \to 0$ vanish; generic parallel sections blow up. However, a new feature here is that $\operatorname{span}(L \otimes K, \varphi(L)) \neq E \otimes K$ precisely at the knot singularities, reflecting the zeros of φ.

For any $p \in \Sigma$ we can transport the model solution to $\Sigma \times \mathbb{R}^+$ using the local coordinates (z, y), giving an approximate solution (A^p, ϕ^p, ϕ_1^p) in a neighborhood of $(p, 0)$. It is convenient to define:

Definition 3.4 A solution (A, ϕ, ϕ_1) to the extended Bogomolny equations satisfies the general Nahm pole boundary condition with knot singularity of order n at $(p, 0) \in \Sigma \times \mathbb{R}^+$ if in a suitable gauge it satisfies

$$(A, \phi, \phi_1) = (A^p, \phi^p, \phi_1^p) + \mathcal{O}(R^{-1+\epsilon}(\sin\psi)^{-1+\epsilon}) \tag{33}$$

for some $\epsilon > 0$, where R and ψ are the spherical coordinates used above.

Corresponding to a solution with knot singularity is a set of holomorphic data. Suppose (A, ϕ, ϕ_1) is a solution with a knot singularity at the points $\{p_j\}$ with orders n_j for $j = 1, \dots, N$. We define the line subbundle L of E and obtain the exact sequence

$$0 \to L \to E \to L^{-1} \to 0. \tag{34}$$

Using the asymptotic boundary condition at $y \to +\infty$ and the Milnor–Wood inequality [23; 30], we have $|\deg L| \le g-1$.

The knot singularity and Higgs field induce a map

$$P\colon L \xrightarrow{\varphi} E \otimes K \to L^{-1} \otimes K. \tag{35}$$

Regarding P as an element of $H^0(L^{-2} \otimes K)$, we deduce that there are $2g-2-2\deg L$ marked points, counted with multiplicity.

The data we must specify then consists of the following:

(a) An $\mathrm{SL}(2;\mathbb{C})$ Higgs bundle with a line subbundle L.

(b) Marked points $\{p_j\}$ with orders n_j.

(c) Generic parallel sections of E over $\Sigma \setminus \{p_j\}$ blow up at the rate $y^{-1/2}$.

(d) The section $P \in H^0(L^{-2}K)$ in (35) has zeros precisely at p_j of order n_j.

Just as for the Nahm pole case, we impose an $\mathrm{SL}(2,\mathbb{R})$ structure on the Higgs bundle. The following assumption simplifies the Hermitian geometric data.

Definition 3.5 Suppose we have a solution to (1) which satisfies the general Nahm pole boundary conditions, and assume that the solution converges to an $\mathrm{SL}(2,\mathbb{R})$ Higgs bundle $\big(L^{-1} \oplus L,\ \varphi = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right)\big)$ as $y \to \infty$. We say that this solution is compatible with the $\mathrm{SL}(2,\mathbb{R})$ structure at $y = \infty$ if either L or L^{-1} is the vanishing line bundle.

Merely assuming that the Higgs bundle converges to an $\mathrm{SL}(2,\mathbb{R})$ Higgs bundle, as above, is not enough to imply that L is the vanishing line bundle.

Remark If the exact sequence (34) splits, the Higgs field may take the slightly more general form $\varphi = \left(\begin{smallmatrix} t & \alpha \\ \beta & -t \end{smallmatrix}\right)$. Such Higgs fields with $t \neq 0$ exist, but at present we do not know whether it is possible to solve the extended Bogomolny equations with knot singularity with this data. The vanishing of t will play a minor but important technical role below in Proposition 3.9, which we need in proving uniqueness theorems later.

The compatibility of the solution with the $\mathrm{SL}(2,\mathbb{R})$ structure is a technical condition that allows us to reduce the Bogomolny equation to a scalar equation. There is one special case where we do not need to assume this compatibility condition. Under the assumption of Definition 3.5, denote the vanishing line bundle by L'. We then obtain:

Proposition 3.6 *If $L' \neq L$ or L^{-1}, then $\deg L' \le -|\deg L|$,*

Proof The line subbundle L' induces the exact sequence

$$0 \to L' \to L^{-1} \oplus L \to L'^{-1} \to 0,$$

which defines the holomorphic map $\gamma_1\colon L \to L'^{-1}$ and $\gamma_2\colon L^{-1} \to L'^{-1}$. Since $L' \neq L$ or L^{-1}, we obtain that neither γ_1 nor γ_2 equals the identity. In other words, we obtain nonzero elements $\gamma_1 \in H^0(L^{-1} \otimes L'^{-1})$ and $\gamma_2 \in H^0(L \otimes L'^{-1})$. Since γ_1 and γ_2 do not have poles, we obtain $\deg(L^{-1} \otimes L'^{-1}) \geq 0$ and $\deg(L \otimes L'^{-1}) \geq 0$, which implies $\deg L' \leq -|\deg L|$. □

Denoting by $N := \sum n_j$ the sum of the orders of the marked points, we conclude:

Corollary 3.7 *If* $\deg L > 0$ *and* $N < 2g - 2 + 2\deg L$, *then* $L' = L$.

Proof Recall that $N = 2g - 2 - 2\deg L'$, and furthermore, if $N < 2g - 2 + 2\deg L$, then $\deg L' > -\deg L$. Proposition 3.6 then implies this result. □

3.4 Regularity

We have defined these boundary conditions both at $y = 0$ and at the knot singular points by requiring the fields (A, ϕ) to differ from the corresponding model solutions by an error term, the relative size of which is smaller than the model. In the existence theorems later in this paper this may be all we know about solutions at first. However, to be able to carry out many further arguments it is important to know that, in an appropriate gauge, solutions have much stronger regularity properties. Fortunately there is an appropriate regularity theory available which was developed in [21] in the Nahm pole case and [22] near the knot singularities. We note that in those papers solutions to the full four-dimensional KW system are treated, but those results specialize directly to the present setting, and in fact there are some minor but important strengthenings here which we point out inter alia.

Regularity theory relies on ellipticity, and to turn the extended Bogomolny equations into an elliptic system we must add an appropriate gauge condition. We recall the choice made in [21] for the KW system on a four-manifold and then specialize it in our dimensionally reduced setting. Fix a pair of fields $(\widehat{A}^0, \widehat{\phi}^0)$ on a four-manifold which are either solutions or approximate solutions of KW equations. Then nearby fields can be written in the form $(\widehat{A}, \widehat{\phi}) = (\widehat{A}^0, \widehat{\phi}^0) + (\alpha, \psi)$. The gauge-fixing equation is then

$$d^*_{\widehat{A}^0}\alpha + \star[\widehat{\phi}^0, \star\psi] = 0. \tag{36}$$

It is shown in [21] that adjoining (36) to the KW equations is elliptic.

Denote by $\mathcal{L}$ the linearization of this system at $(\hat{A}^0, \hat{\phi}^0)$. This is a Dirac-type operator with coefficients which blow up at $y=0$ and $R=0$ in a very special manner. In the absence of knots, $\mathcal{L}$ is (up to a multiplicative factor) a *uniformly degenerate* operator, while near a knot it lies in a slightly more general class of incomplete iterated edge operators. These are classes of degenerate differential operators for which tools of geometric microlocal analysis may be applied to construct parametrices, which in turn lead to strong mapping and regularity properties. We refer to [21; 22] for further discussion about all of this and simply state the consequences of this theory here.

Before doing this we first recall that for degenerate elliptic problems it is too restrictive to expect solutions to be smooth up to the boundary. Instead we consider polyhomogeneous regularity. Let X be a manifold with boundary, with coordinates (s,z) near a boundary point, with $s \geq 0$ and z a coordinate in the boundary. We say that a function u is polyhomogeneous at ∂X if

$$u(s,z) \sim \sum_{j=0}^{\infty} \sum_{\ell=0}^{N_j} a_{j\ell}(z) s^{\gamma_j} (\log s)^{\ell} \quad \text{with } a_{j\ell} \in \mathcal{C}^{\infty}(\partial X).$$

The exponents γ_j here is a sequence of (possibly complex) numbers with real parts tending to infinity; importantly, for each j, only finitely many factors with (positive integral) powers of $\log s$ can appear. The set of pairs (γ_j, ℓ) which appear in this expansion is called the index set for this expansion. Denoting this index set by $\mathcal{I}$, we say that u is $\mathcal{I}$–smooth, which emphasizes that this regularity is a very close relative of and satisfactory replacement for ordinary smoothness. Similarly, if X is a manifold with corners of codimension 2, with coordinates (s_1, s_2, z) near a point on the corner, then u is polyhomogeneous if

$$u(s_1, s_2, z) \sim \sum_{i,j=0}^{\infty} \sum_{p,q=0}^{N_{i,j}} a_{ijpq}(z) s_1^{\gamma_i} s_2^{\lambda_j} (\log s_1)^p (\log s_2)^q.$$

In other words, we require the expansion for u to be of product type near the corner. These are all classical expansions with the usual meaning and the corresponding expansions for any number of derivatives hold as well. The reason for introducing this more general notion is precisely because, at least in favorable situations, solutions of this have regularity but are not smooth in a classical sense. The important point is that this is a perfectly satisfactory replacement for smoothness up to the boundary and allows one to analyze and manipulate expressions using these "Taylor series"-type expansions.

We first consider the case where there are no knot singularities, but note that this result is a local one and can be applied away from knot singular points. Here the manifold with boundary is simply $\Sigma \times \mathbb{R}^+$ and we use coordinates (y, z).

Proposition 3.8 [21] *Let (A, φ, ϕ_1) be a solution to the extended Bogomolny equations near $y = 0$ which satisfies the Nahm pole boundary conditions and is in gauge relative to the model approximate solution. Then these fields are polyhomogeneous with*

$$A = \mathcal{O}(1), \quad \varphi = \frac{1}{y}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(y), \quad \phi_1 = \frac{1}{y}\begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix} + \mathcal{O}(y \log y).$$

This statement incorporates recent work in [14], which provides much more detail about the expansions than is present in [21].

To state the corresponding result in the presence of a knot singularity, we first define the manifold with corners X to be the blowup of $\Sigma \times \mathbb{R}^+$ around each of the knot singular points $(p_j, 0)$. In other words, we replace each $(p_j, 0)$ by the hemisphere $R = 0$ (parametrized by the spherical coordinate variables (ψ, θ)), points of which label directions of approach to that point. The discussion is local near each p_j, so we may as well fix coordinates (R, ψ, θ). The corner of X is defined by $R = \psi = 0$.

Proposition 3.9 [22] *Let (A, ϕ, ϕ_1) satisfy the extended Bogomolny equations near $(0, 0)$ as well as the gauge condition relative to the model knot solution U_n. Then these fields are polyhomogeneous with the same asymptotics as in the previous proposition when $y \to 0$ away from the knot, while*

$$A = A^n + \mathcal{O}(R^\epsilon \sin\psi), \quad \varphi = \varphi^n + \mathcal{O}(R^\epsilon \sin\psi), \quad \phi_1 = \phi_1^n + \mathcal{O}(R^\epsilon (\sin\psi) \log(\sin\psi))$$

near the knot. Here $(A^n, \varphi^n, \phi_1^n)$ is the model solution described in Section 3.3 associated to U_n.

Referring to the language of [22], these rates of decay, ie the first exponents in the expansions beyond the initial model terms, are indicial roots of type II and II′. The exponent 0 is a possible indicial root of type II', but does not appear in our setting because the $\mathrm{SL}(2, \mathbb{R})$ structure forces φ to have no diagonal terms — see the remark on page 2490 — and it is precisely in these diagonal terms where the exponent 0 might appear in the expansion.

3.5 The boundary condition for the Hermitian metric

Since we must deal with singularities of the gauge field, it is often simpler to work in holomorphic gauge but consider singular Hermitian metrics. We now describe a boundary condition for the Hermitian metric compatible with the unitary boundary condition defined above. We use the Riemannian metric $g = g_0^2 |dz|^2 + dy^2$ on $\Sigma \times \mathbb{R}^+$. The following result is a direct consequence of the previous computations in Sections 3.2 and 3.3.

Proposition 3.10 *Consider the Higgs bundle* $\bigl(E \cong L^{-1} \oplus L,\ \varphi = \bigl(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\bigr)\bigr)$. *Fix* $p \in \Sigma \times \{0\}$ *and an open set* U_p *containing* p. *Let* H *be a polyhomogeneous solution to the Hermitian extended Bogomolny equations (11).*

(i) *Suppose that* $\varphi|_{U_p} = \bigl(\begin{smallmatrix} 0 & 1 \\ \star & 0 \end{smallmatrix}\bigr)$ *in a local trivialization on* U_p. *If, for some* $\epsilon > 0$,

$$H \sim \begin{pmatrix} y^{-1}(g_0 + \mathcal{O}(y^\epsilon)) & 0 \\ 0 & y(g_0^{-1} + \mathcal{O}(y^\epsilon)) \end{pmatrix} \quad \text{as } y \to 0, \tag{37}$$

then the unitary solution with respect to H *satisfies the Nahm pole boundary condition near* p *and* $\bigl(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\bigr)$ *is the vanishing line bundle in this trivialization.*

(ii) *Suppose that* $\varphi|_{U_p} = \bigl(\begin{smallmatrix} 0 & z^n \\ \star & 0 \end{smallmatrix}\bigr)$ *in a local trivialization on* U_p *(where* $z = 0$ *is the point* p*). In spherical coordinates* (R, θ, ψ)*, suppose, for some* $\epsilon > 0$,

$$H = \begin{pmatrix} e^{U_n}(1 + \mathcal{O}(R^\epsilon)) & 0 \\ 0 & e^{-U_n}(1 + \mathcal{O}(R^\epsilon)) \end{pmatrix} \quad \text{as } R \to 0. \tag{38}$$

Then the unitary solution with respect to H *satisfies the Nahm pole condition with knot singularity at* p *and* $\bigl(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\bigr)$ *is the vanishing line bundle in this trivialization.*

Since we wish to work with holomorphic gauge fields and singular Hermitian metrics, we obtain some restrictions. Let P be an $\mathrm{SU}(2)$ bundle and (A, ϕ, ϕ_1) a solution to the extended Bogonomy equations (1) with Nahm pole boundary and knot singularities of order n_j at the points p_j for $j = 1, \dots, n$. For each j choose small balls B_j around p_j, and also let B_0 be a neighborhood of $\Sigma \setminus \{B_1, \dots, B_k\}$ which does not contain any of the p_j. Choosing a partition of unity χ_j subordinate to this cover, define the approximate solution $u = \sum_{j=0} \chi_j U_{n_j}$, where U_{n_j} is the model solution, and with $U_{n_0} = -\log y$.

Proposition 3.11 *There exists a Hermitian bundle* (E, H) *such that*

(i) $(H, A^{(0,1)}, \varphi, \mathcal{A}_y)$ *is a solution to the Hermitian extended Bogomolny equations*;

(ii) $(A^{(0,1)}, \varphi, \mathcal{A}_y)$ *is bounded as* $y \to 0$;

(iii) $H = \begin{pmatrix} e^u h_{11} & h_{12} \\ h_{21} & e^{-u} h_{22} \end{pmatrix}$, *where* u *is the approximate function above and the* h_{ij} *are bounded.*

Proof We have explained that (A, ϕ, ϕ_1) is polyhomogeneous, ie $(A, \phi, \phi_1) = (A^{p_j}, \phi^{p_j}, \phi_1^{p_j}) + (a, b, c)$ near p_j, where (a, b, c) are bounded. Near other points of $\Sigma \times \{0\}$, (A, ϕ, ϕ_1) is the sum of a Nahm pole and a bounded term. Since P is an SU(2) bundle over $\Sigma \times \mathbb{R}^+$, it is necessarily trivial, so consider the associated rank 2 Hermitian bundle (E, H_0), with $H_0 = \mathrm{Id}$ in some trivialization. Now write $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$, where the h_{ij} are bounded. Then $(H_0, A^{(0,1)}, \varphi, \mathcal{A}_y)$, where $\varphi = \phi_z$, $\mathcal{A}_y = A_y - i\phi_1$, is a solution to the Hermitian extended Bogomolny equations (11). Consider the complex gauge transform $g = \begin{pmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{pmatrix}$. Since u is compatible with the knot singularity, we obtain a new solution $(H', A^{(0,1)'}, \varphi', \mathcal{A}'_y)$, where $H' = H_0 g^\dagger g = \begin{pmatrix} e^u h_{11} & h_{12} \\ h_{21} & e^{-u} h_{22} \end{pmatrix}$, and $A^{(0,1)'}$, φ' and $\mathcal{A}'_y$ are all bounded. □

We conclude this section with a brief discussion about the regularity of a harmonic metric which satisfies the boundary conditions described here. Such metrics correspond precisely to the solutions $(A_z, A_y, \varphi, \phi_1)$ of the original extended Bogomolny equations, and for this reason one obvious route to obtain this regularity is to exhibit the direct formula from the set of A's and ϕ's to the metric H. Another reasonable approach is to simply look at the equation (20) defining H and prove the necessarily regularity directly from this equation. In fact, the methods used in [21; 22] are sufficiently robust that this adaptation is quite straightforward. In the interests of efficiency, we simply state the conclusion:

Lemma 3.12 *A harmonic metric* H *which satisfies the boundary conditions discussed above is necessarily polyhomogeneous.*

The terms which appear in the polyhomogeneous expansion of H may be determined by the obvious formal calculations once we know that the expansion actually exists.

4 Existence of solutions

We shall prove in this section an existence theorem for the extended Bogomolny equations on $\Sigma \times \mathbb{R}^+$, either without or with knot singularities at $y = 0$. The proofs employ the classical barrier method, which we review briefly.

4.1 Semilinear elliptic equations on noncompact manifolds

We consider on a Riemannian manifold (W, g) the elliptic equation

$$N(u) := -\Delta u + F(x,u) = 0, \qquad F \in \mathcal{C}^\infty(W \times \mathbb{R}). \tag{39}$$

A $\mathcal{C}^2$ function u^+ is called a supersolution for this problem if $N(u^+) \geq 0$, while u^- is called a subsolution if $N(u^-) \leq 0$. These are called *barriers* for the operator. It is often much simpler to construct such functions which are only continuous, and which satisfy the corresponding differential inequalities weakly (either in the distributional or viscosity sense). We refer to [6] for more details about the viscosity solutions.

Proposition 4.1 *Suppose that W is a possibly open manifold, and that there exist continuous barriers $u^\pm$ which satisfy $u^- \leq u^+$ everywhere on W. Then there exists a solution u to $N(u) = 0$ which satisfies $u^- \leq u \leq u^+$.*

Proof sketch We first assume that W is a compact manifold with boundary. Then $u^\pm$ are bounded functions and we may choose $\lambda > 0$ so that $\partial_u F(x,u) \leq \lambda$ for all numbers u lying in the interval $[u^-(x), u^+(x)]$ for every $x \in W$. The equation can then be written as

$$(\Delta - \lambda)u = \widetilde{F}(x,u) := F(x,u) - \lambda u.$$

We then define a sequence of functions u_j for $j = 0, 1, 2, \dots$ by setting $u_0 = u^-$ and successively solving $(\Delta - \lambda)u_{j+1} = \widetilde{F}(x, u_j)$, and with u_{j+1} equal to some fixed function ψ on ∂W which satisfies $u^-|_{\partial W} \leq \psi \leq u^+|_{\partial W}$. The monotonicity of $\widetilde{F}$ in u and the maximum principle can be used to prove inductively that $u^- = u_0 \leq u_1 \leq u_2 \leq \cdots \leq u^+$. When W is a manifold with boundary we require a version of the maximum principle which holds up to the boundary even for weak solutions; one version appears in [17, Theorem II.1].

It is then obvious that u_j converges pointwise to an L^∞ function u, and standard elliptic regularity implies that $u \in \mathcal{C}^\infty$ and that $N(u) = 0$.

Now suppose that W is an open manifold. Choose a sequence of compact smooth manifolds with boundary W_k with $W_1 \subset W_2 \subset \cdots$ which exhaust all of W. For each k, choose a function ψ_k on ∂W_k which lies between u^- and u^+ on this boundary, and then find a solution u_k to $N(u_k) = 0$ on W_k with $u_k = \psi_k$ on ∂W_k. The sequence u_k is uniformly bounded on any compact subset of W, so we may choose a sequence which converges (by elliptic regularity) in the $\mathcal{C}^\infty$ topology on any compact subset of W. The limit function is a solution of N and still satisfies $u^- \le u \le u^+$ on all of W. □

We conclude this general discussion by making a few comments about the construction of weak barriers. A very convenient principle is that sub- and supersolutions may be constructed locally in the following sense. Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ are two open sets in W and that w_j is a supersolution for N on $\mathcal{U}_j$ for $j = 1, 2$. Define the function w on $\mathcal{U}_1 \cup \mathcal{U}_2$ by setting $w = w_1$ on $\mathcal{U}_1 \setminus (\mathcal{U}_1 \cap \mathcal{U}_2)$, $w = w_2$ on $\mathcal{U}_2 \setminus (\mathcal{U}_1 \cap \mathcal{U}_2)$ and $w = \min\{w_1, w_2\}$ on $\mathcal{U}_1 \cap \mathcal{U}_2$. Then w is a supersolution for N on $\mathcal{U}_1 \cup \mathcal{U}_2$. Similarly, the maximum of two (or any finite number) of subsolutions is again a subsolution. In our work below, the individual w_j will typically be smooth, but the new barrier w produced in this way is only piecewise smooth, but is still a sub- or supersolution in the weak sense. We refer to [7, Appendix A] for a proof.

4.2 The scalar form of the extended Bogomolny equations

Following the discussion in Section 3, suppose that $E \cong L \oplus L^{-1}$ and

(40) $$\varphi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}.$$

When $\deg L = g - 1$, $L = K^{1/2}$ and $\alpha = 1$, we seek a solution of the extended Bogomolny equations which satisfies the Nahm pole boundary condition at $y = 0$, while if $\deg L < g - 1$, then the zeros of α determine points and multiplicities p_j and n_j on Σ at $y = 0$ and we search for a solution which satisfies the Nahm pole boundary condition with knot singularities at these points.

Fix a metric $g = g_0^2 |dz|^2 + dy^2$ on $\Sigma \times \mathbb{R}^+$ (where $z = x_2 + i x_3$ is a local holomorphic coordinate on Σ), and assume also that the solution metric splits as $H = \left(\begin{smallmatrix} h & 0 \\ 0 & h^{-1} \end{smallmatrix}\right)$, where h is a bundle metric on L^{-1}. We are then looking for a solution to

(41) $$-\Delta_g \log h + g_0^{-2}(h^2 \alpha\overline{\alpha} - h^{-2} \beta\overline{\beta}) = 0.$$

We simplify this slightly further. Choose a background metric h_0 on L^{-1} and recall that its curvature equals $-\Delta_{g_0} \log h_0$. Then, writing $h = h_0 e^u$ and calculating the norms of α and β in terms of g_0 and h_0, (41) becomes

(42) $$K_{h_0} - (\Delta_{g_0} + \partial_y^2) u + |\alpha|^2 e^{2u} - |\beta|^2 e^{-2u} = 0.$$

In the remainder of this paper, we denote by $N(u)$ the operator on the left in (42).

An explicit solution to this equation was noted by Mikhaylov in a special case (personal communication, 2017):

Example 4.2 Consider the Higgs bundle $\big(E \cong K^{1/2} \oplus K^{-1/2},\ \varphi = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\big)$. Let g_0 be the hyperbolic metric on Σ with curvature -2 and h_0 the naturally induced metric on $K^{-1/2}$, for which $K_{h_0} = -1$. Then, restricted to Σ–independent functions, (42) equals

$$-1 - \partial_y^2 u + e^{2u} = 0. \tag{43}$$

We seek a solution for which $u \sim -\log y$ as $y \to 0$ and $v \to 0$ as $y \to \infty$. The first integral of (43) is $u' = -\sqrt{e^{2u} - 2u - 1}$, and hence the unique solution is

$$\int_u^\infty \frac{ds}{\sqrt{e^{2s} - 2s - 1}} = y. \tag{44}$$

Note that u is monotone decreasing and strictly positive for all $y > 0$.

We now describe the precise asymptotics of this solution. If $u \to \infty$, then s is large; write the denominator as $e^s \sqrt{1 - (2s+1)e^{-2s}}$, whence

$$y = \int_u^\infty e^{-s}\big(1 + \tfrac{1}{2}(2s+1)e^{-2s} + \cdots\big)\, ds \sim e^{-u} + \cdots,$$

so $u \sim -\log y$. Similarly, if $u < \epsilon$ for some small ϵ, then $e^{2s} - 2s - 1 \sim 2s^2 + \cdots$ when $u < s < \epsilon$, so

$$u = \int_\epsilon^\infty \frac{ds}{\sqrt{e^{2s} - 2s - 1}} + \int_u^\epsilon \Big(\frac{1}{\sqrt{2}s} + \cdots\Big)\, ds = A - \frac{1}{\sqrt{2}} \log u + \cdots,$$

so $u = Ce^{-\sqrt{2}y} + \cdots$. Obviously, with only a little more effort, one may develop full asymptotics in both regimes.

4.3 Limiting solution at infinity

We first consider the simpler problem of finding a solution of the reduction of (42) reduced to Σ, ie of

$$K - \Delta u_\infty + |\alpha|^2 e^{2u_\infty} - |\beta|^2 e^{-2u_\infty} = 0, \tag{45}$$

where $K = K_{h_0}$ and $\Delta = \Delta_{g_0}$. Without loss of generality, we assume $\deg L \geq 0$ and note that since $\deg L^{-1} \leq 0$, $\int_\Sigma K \leq 0$ (and is strictly negative if the degree of L is positive). A solution to (45) is the obvious candidate for the limit as $y \to \infty$ of solutions on $\Sigma \times \mathbb{R}^+$.

Proposition 4.3 *If $\alpha \not\equiv 0$, which is equivalent to the stability of the pair (E, φ), there exists a solution $u_\infty \in \mathcal{C}^\infty(\Sigma)$ to (45).*

Proof Since this is an equation on Σ rather than $\Sigma \times \mathbb{R}^+$, this follows immediately from the existence of solutions to the Hitchin equations [15]. However, we give another proof, at least when $\deg L > 0$, using the barrier method. A proof in the same style when $\deg L = 0$ requires more work so we omit it.

Solve $\Delta w^- = K - \overline{K}$, where $\overline{K} < 0$ is the average of K, and set $u^- = w^- - A$ for some constant A. Then $K - \Delta u^- + |\alpha|^2 e^{2u^-} - |\beta|^2 e^{-2u^-} \le \overline{K} + |\alpha|^2 e^{w^- - A}$, which is negative when A is sufficiently large. Thus u^- is a subsolution.

To obtain a supersolution, first modify the background metric h_0 by multiplying it by a suitable positive factor so that its curvature K is positive near the zeros of α. Next solve $\Delta w^+ = |\alpha|^2 - B$, where B is the average of $|\alpha|^2$, and set $u^+ = w^+ + A$. Then

$$\begin{aligned} K - \Delta u^+ + |\alpha|^2 e^{2u^+} - |\beta|^2 e^{-2u^+} &= K + B + |\alpha|^2 (e^{2(w^+ + A)} - 1) - |\beta|^2 e^{-2(w^+ + A)} \\ &\ge K + B + 2|\alpha|^2 (w^+ + A) - |\beta|^2 e^{-2(w^+ + A)}. \end{aligned}$$

Away from the zeros of α this is certainly positive if we choose A sufficiently large. Near these zeros we obtain positivity, using that $K + B > 0$ there, and since the final term can be made arbitrarily small. Thus u^+ is a supersolution.

Noting that $u^- < u^+$ and applying Proposition 4.1, we obtain a solution of (45). □

Observe that since it is only the boundary condition, but not the equation, which depends on y, this limiting solution is actually a solution of (42) on any semi-infinite region $\Sigma \times [y_0, \infty)$ with $y_0 > 0$.

4.4 Approximate solutions and regularity near $y = 0$

As a complement to the result in the previous subsection, we now construct an approximate solution u_0 to (42) near $\{y = 0\}$. Unlike there, however, we do not find an exact solution, but rather show how to build an initial approximate solution and then incrementally correct it so that it solves (42) to all orders as $y \to 0$. In the next subsection we use u_0 and u_∞ together to construct global barriers.

We first begin with the simpler situation where there is only a Nahm pole singularity without knots.

Proposition 4.4 *Let $L = K^{1/2}$ and $\alpha \equiv 1$. Then there exists a function u_0 which is polyhomogeneous as $y \to 0$ and is such that $N(u_0) = f$ decays faster than y^ℓ for any $\ell \geq 0$.*

Proof We seek u_0 with a polyhomogeneous expansion of the form

$$-\log y + \sum_{j,\ell} a_{j\ell}(z) y^j (\log y)^\ell := -\log y + v,$$

where all the coefficients are smooth in z, and where the number of $\log y$ factors is finite for each j. Rewriting $N(-\log y + v)$ as

$$\left(-\partial_y^2 + \frac{2}{y^2}\right) v + \frac{1}{y^2}(e^{2v} - 2v - 1) - |\beta|^2 y^2 e^{-2v} - \Delta_{g_0} v + K_{h_0} \tag{46}$$

and inserting the putative expansion for v shows that $a_{0\ell} = a_{1\ell} = 0$ for all ℓ, and $a_{21} = \frac{1}{3}(K_{h_0} - |\beta|^2)$ and $a_{2\ell} = 0$ for $\ell > 1$, ie $v \sim a_{21} y^2 \log y + a_{20} y^2 + \mathcal{O}(y^3 (\log y)^\ell)$ for some ℓ. Inductively we can solve for each of the coefficients $a_{j\ell}$ with $j > 2$ using that

$$\left(-\partial_y^2 + \frac{2}{y^2}\right) y^j (\log y)^\ell = y^{j-2} (\log y)^{\ell-2} \big((-j(j-1)+2)(\log y)^2 - \ell(2j-1)\log y - \ell(\ell-1)\big).$$

Note that the coefficient a_{20} is not formally determined in this process and different choices will lead to different formal expansions, and also that there are increasingly high powers of $\log y$ higher up in the expansion.

Now use Borel summation to choose a polyhomogeneous function u_0 with this expansion. This has a Nahm pole at $y = 0$ and satisfies $N(u_0) = f = \mathcal{O}(y^\ell)$ for all ℓ, as desired. □

We next turn to the construction of a similar approximate solution to all orders in the presence of knot singularities. To carry this out, we first review a geometric construction from [22] which is at the heart of the regularity theorem quoted in Section 3.4 for the full extended Bogomolny equations and the analogous result for (42) which we describe below.

If $p \in \Sigma$, we define the blowup of $\Sigma \times \mathbb{R}^+$ at $(p, 0)$ to consist of the disjoint union $(\Sigma \times \mathbb{R}^+) \setminus \{(p, 0)\}$ and the hemisphere S^2_+, which we regard as the set of inward-pointing unit normal vectors at $(p, 0)$, and denote by $[\Sigma \times \mathbb{R}^+; \{(p, 0)\}]$, or more simply, just $(\Sigma \times \mathbb{R}^+)_p$ There is a blowdown map which is the identity away from $(p,0)$ and maps the entire hemisphere to this point. This set is endowed with the unique minimal topology and differential structure such that the lifts of smooth functions on $\Sigma\times\mathbb{R}^+$ and polar coordinates around $(p,0)$ are smooth. We use spherical coordinates (R,ψ,θ) around this point, so $R=0$ is the hemisphere and $\psi=0$ defines the original boundary $y=0$ away from $R=0$. This is a smooth manifold with corners of codimension two.

Now fix a nonzero element $\alpha \in H^0(L^{-2}K)$ and denote its divisor by $\sum_{j=1}^N n_j p_j$. For each j, choose a small ball $\widehat{B}_j$ and a local holomorphic coordinate z so that $p_j=\{z=0\}$, and write $|\alpha|^2=\sigma_j^2 r^{2n_j}$ there, with $r=|z|$ and $\sigma_j>0$. Extend r from the union of these balls to a smooth positive function on $\Sigma\setminus\{p_1,\dots,p_N\}$. By the existence of isothermal coordinates, we write $g_0=e^{2\phi}\overline{g}_0$, where $\overline{g}_0$ is flat on each $\widehat{B}_j$, and set $g=g_0+dy^2$ and $\overline{g}=\overline{g}_0+dy^2$. Then $\Delta_{g_0}=e^{-2\phi}\Delta_{\overline{g}_0}$ in these balls, and by dilating $\overline{g}_0$, we can assume that $e^{-2\phi}=1$ at each p_j. We denote by $(\Sigma\times\mathbb{R}^+)_{p_1,\dots,p_N}$ the blowup of $\Sigma\times\mathbb{R}^+$ at the collection of points $\{p_1,\dots,p_N\}$.

Proposition 4.5 *With all notation as above, there exists a function u_0 which is polyhomogeneous on $(\Sigma\times\mathbb{R}^+)_{p_1,\dots,p_N}$ and which satisfies $N(u_0)=f$ with f smooth and vanishing to all orders as $y\to 0$ (ie at all boundary components of the blowup).*

Proof In a manner analogous to the previous proposition, we construct a polyhomogeneous series expansion for u_0 term-by-term, but now at each of the boundary faces of $(\Sigma\times\mathbb{R}^+)_{p_1,\dots,p_N}$.

The initial term of this expansion involves the model solutions U_n. Choose nonintersecting balls $\widehat{B}_j$ with $B_j\Subset\widehat{B}_j$ and an open set $\widehat{B}_0\subset\Sigma\setminus\bigcup_{j=1}^N\overline{B}_j$ so that $\bigcup_{j=0}^N\widehat{B}_j=\Sigma$. Let $\{\chi_j\}$ be a partition of unity subordinate to the cover $\{\widehat{B}_j\}$ with $\chi_j=1$ on B_j for $j\geq 1$. We lift each of these functions from Σ to the blowup of $\Sigma\times\mathbb{R}^+$. Finally, set $G_j:=U_{n_j}-\log\sigma_j$, where $G_0:=U_0-\log|\alpha|=-\log y-\log|\alpha|$. Now define

$$\hat{u}_0:=\sum_{j=0}^N\chi_j G_j. \tag{47}$$

We compute that $N(\hat{u}_0)=f_0$, where f_0 is polyhomogeneous and is bounded at the original boundary $\psi=0$ and has leading term of order R^{-1} at each of the "front" faces where $R=0$.

Our goal is to iteratively solve away all of the terms in the polyhomogeneous expansion of f_0. This must be done separately at the two types of boundary faces. It turns out to be necessary to first solve away the series at $R=0$ and after that the series at $\psi=0$. The reason for doing things in this order is that, as we now explain, the iterative problem that must be solved at the $R=0$ front faces is global on each hemisphere, and the solutions "spread" to the boundary of this hemisphere, ie where $\psi=0$. By contrast, the iterative problem at the original boundary is completely local in the y directions and may be done uniformly up to the corner where $R=\psi=0$, so its solutions do not spread back to the front faces.

For simplicity, we assume that there is only one front face, and we begin by considering the model case $(\mathbb{C}\times\mathbb{R}^+)_0$, on which the linearization of (42) at U_n can be written as

$$(48)\quad L_n=-\partial_R^2-\frac{2}{R}\partial_R-\frac{1}{R^2}\Delta_{S^2_+}+2r^{2n}e^{2U_n}=-\partial_R^2-\frac{2}{R}\partial_R+\frac{1}{R^2}(-\Delta_{S^2_+}+T(\psi)),$$

where the potential equals

$$T(\psi)=\frac{(n+1)^2}{\sin^2\psi S_n(\psi)^2}.$$

In general terms, L_n is a relatively simple example of an "incomplete iterated edge operator", as explained in more detail in [22], based on the earlier development of this class in [2; 3]. We need relatively little of this theory here and quote from [22] as needed. In the present situation, we can regard L_n as a conic operator over the cross-section S^2_+. (It is the fact that this link of the cone itself has a boundary which makes L_n an "iterated" edge operator.)

The crucial fact is that the operator

$$J=-\Delta_{S^2_+}+T(\psi)$$

induced on this conic link has discrete spectrum. The proof of this is based on the observation that $T(\psi)\sim 1/\psi^2$ as $\psi\to 0$. It can then be shown using standard arguments — see [2; 3] — that the domain of J as an unbounded operator on $L^2(S^2_+)$ is compactly contained in L^2. This implies the discreteness of the spectrum. Another proof which provides more accurate information uses that J is itself an incomplete uniformly degenerate operator, as analyzed thoroughly in [20]. The main theorem in that paper produces a particular degenerate pseudodifferential operator G which inverts J on L^2. It is also shown there that $G\colon L^2(S^2_+)\to\psi^2H^2_0(S^2_+)$ (where H^2_0 is the scale-invariant Sobolev space associated to the vector fields $\psi\partial_\psi$ and $\psi\partial_\theta$). The compactness of $\psi^2H^2_0(S^2_+)\hookrightarrow L^2(S^2_+)$ follows from the L^2 Arzelà–Ascoli theorem. There is an accompanying regularity theorem: if $(J-\lambda)w=f$ where (for simplicity) f is smooth and vanishes to all orders at $\psi=0$ and $\lambda\in\mathbb{R}$ (or more generally can be any bounded polyhomogeneous function), then w is polyhomogeneous with an expansion of the form

$$w \sim \sum w_{j\ell}(\theta)\psi^{\gamma_j}(\log\psi)^{\ell} \quad \text{with } w_{j\ell} \in \mathcal{C}^{\infty}(S^1).$$

As usual, there are only finitely many log terms for each exponent γ_j. These exponents are the indicial roots of the operator J, and a short calculation shows that these satisfy $2 = \gamma_0 < \gamma_1 < \cdots$. Note that the lowest indicial root equals 2, so solutions all vanish to at least order 2 at $\psi = 0$, which is in accord with our knowledge about the behavior of solutions to the linearization of (42) at the model Nahm pole solution $-\log y$.

Denote the eigenfunctions and eigenvalues of J by $\mu_i(\psi,\theta)$ and λ_i. Since $T(\psi) > 0$, $\lambda_i > 0$ for each i. The restriction of L_n to the i^{th} eigenspace is now an ODE $L_{n,i} = -\partial_R^2 - 2R^{-2}\partial_R + R^{-2}\lambda_i$. Seeking solutions of the form $R^{\delta}\mu_i(\psi,\theta)$ leads to the corresponding indicial roots

$$\delta_i^{\pm} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4\lambda_i},$$

which are the only possible formal rates of growth or decay of solutions to $L_n u = 0$ as $R \to 0$. To satisfy the generalized Nahm pole condition, we only consider exponents greater than -1, ie the sequence $0 < \delta_1^+ < \delta_2^+ < \cdots$. We now conclude the following:

Lemma 4.6 *Suppose that $f \sim \sum f_{j\ell}(\psi,\theta)R^{\gamma_j}(\log R)^{\ell}$ is polyhomogeneous at the face $R = 0$ on $(\mathbb{C}\times\mathbb{R}^+)_0$, where all $f_{j\ell}$ are polyhomogeneous with nonnegative coefficients at $\psi = 0$ on S^2_+. Then there exists a polyhomogeneous function u such that $L_n u = f + h$, where h is polyhomogeneous at $\psi = 0$ and vanishes to all orders as $R \to 0$. At $R \to 0$, $u \sim \sum u_{j\ell}R^{\gamma'_j}(\log R)^{\ell}$; the exponents γ'_j are all of the form $\gamma_i + 2$, where γ_j appears in the list of exponents in the expansion for f, or else $\delta_i^+ + \ell$ with $\ell \in \mathbb{N}$. Each coefficient function $u_{j\ell}$, as well as the entire solution u itself and the error term h, vanish like ψ^2 at the boundary $\psi = 0$.*

Using the same result, we may clearly generate a formal solution to our semilinear elliptic equation in exactly the same way. Therefore, using this lemma, we may now choose a function $\hat{u}_1$ which is polyhomogeneous on $(\Sigma\times\mathbb{R}^+)_{p_1\cdots p_N}$ and such that $N(\hat{u}_0 + \hat{u}_1) = f_1$, where f_1 vanishes to all orders at $R = 0$ and is polyhomogeneous and vanishes like ψ^2 at $\psi = 0$. The lowest exponent in the expansion for $\hat{u}_1$ equals $\min\{1, \delta_0^+ > 0\}$.

The final step in our construction of an approximate solution is to carry out an analogous procedure at the original boundary $y = 0$ away from the front faces. This can be done almost exactly as above. In this case, (46) can be thought of as an ODE in y with "coefficients" which are operators acting in the z variables, so we are effectively just solving a family of ODEs parametrized by z. This may be done uniformly up to the corner $R = \psi = 0$. We omit details since they are the same as before. We obtain after this step a final correction term $\hat{u}_2$ which is polyhomogeneous and vanishes to all orders at $R = 0$, and which satisfies

$$N(\hat{u}_0 + \hat{u}_1 + \hat{u}_2) = f,$$

where f vanishes to all orders at all boundaries of $(\Sigma \times \mathbb{R}^+)_{p_1 \cdots p_N}$. □

The calculations above are useful not just for calculating formal solutions to the problem, but also for understanding the regularity of actual solutions to the nonlinear equation $N(u) = 0$ which satisfy the generalized Nahm pole boundary conditions with knots. The new ingredient that must be added is a parametrix G for the linearization of N at the approximate solution u_0. This operator G is a degenerate pseudodifferential operator for which there is very precise information known concerning the pointwise behavior of the Schwartz kernel. This is explained carefully in [21] for the simple Nahm pole case and in [22] for the corresponding problem with knot singularities. We shall appeal to that discussion and the arguments there and simply state:

Proposition 4.7 *Let u be a solution to (42) which is of the form $u = u_0 + v$, where v is bounded as $y \to 0$ (in particular, as $\psi \to 0$ and $R \to 0$). Then u is polyhomogeneous at the two boundaries $\psi = 0$ and $R = 0$ of the blowup $(\Sigma \times \mathbb{R}^+)_{p_1,\dots,p_N}$, and its expansion is fully captured by that of u_0.*

4.5 Existence of solutions

We now come to the construction of solutions to (42) on the entire space $\Sigma \times \mathbb{R}^+$ which satisfy the asymptotic $\mathrm{SL}(2,\mathbb{R})$ conditions as $y \to \infty$ and which also satisfy the generalized Nahm pole boundary conditions with knot singularities at $y = 0$. We employ the barrier method. The main ingredients in the construction of the barrier functions are the approximate solutions u_0 and u_∞ obtained above.

We first consider this problem in the simpler case.

Proposition 4.8 *If $E = K^{1/2} \oplus K^{-1/2}$ and $\varphi = \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}$, ie there are no knot singularities, then there exists a solution u to (42) which is smooth for $y > 0$, asymptotic to u_∞ as $y \to \infty$ (and which satisfies the Nahm pole boundary condition at $y = 0$).*

Proof Choose a smooth nonnegative cutoff function $\tau(y)$ which equals 1 for $y \le 2$ and which vanishes for $\tau \ge 3$, and define $\hat{u} = \tau(y)u_0 + (1-\tau(y))u_\infty$. We consider the operator

$$\hat{N}(v) = N(\hat{u}+v) = -(\partial_y^2 + \Delta_{g_0})v + e^{2\hat{u}}(e^{2v}-1) + |\beta|^2 e^{-2\hat{u}}(1-e^{-2v}) + f,$$

where $f = N(\hat{u})$ is smooth on $\Sigma \times \overline{\mathbb{R}}^+$, vanishes to infinite order at $y=0$ and vanishes identically for $y \ge 3$.

We now find barrier functions for this equation. Indeed, we compute that if $0 < \epsilon < 1$, then

$$\hat{N}(Ay^\epsilon) = A\epsilon(1-\epsilon)y^{\epsilon-2} + e^{2\hat{u}}(e^{2Ay^\epsilon}-1) + |\beta|^2 e^{-2\hat{u}}(1-e^{-2Ay^\epsilon}) + f.$$

The second and third terms on the right are nonnegative because $Ay^\epsilon > 0$, and we can certainly choose A sufficiently large that the entire right-hand side is positive for all $y > 0$.

We can improve this supersolution for y large. Indeed,

$$\hat{N}(A'e^{-\epsilon y}) \ge -A'\epsilon^2 e^{-\epsilon y} + e^{2\hat{u}}(2A'e^{-\epsilon y}) + |\beta|^2 e^{-2\hat{u}}(1-e^{-2A'e^{-\epsilon y}}) + f,$$

and if ϵ is sufficiently small and A' is sufficiently large, then the entire right-hand side is positive, at least for $y \ge 1$, say.

We now define $v^+ = \min\{Ay^\epsilon, A'e^{-\epsilon y}\}$. The calculations above show that v^+ is a supersolution to the equation. Essentially the same equations show that $v^- = \max\{-Ay^\epsilon, -A'e^{-\epsilon y}\}$ is a subsolution.

We now invoke Proposition 4.1 to conclude that there exists a solution v to $\hat{N}(v) = 0$, or equivalently a solution $u = \hat{u} + v$ to $N(u) = 0$, which satisfies $|u + \log y| \le Ay^\epsilon$ as $y \to 0$ and $|u - u_\infty| \le A'e^{-\epsilon y}$ as $y \to \infty$. The regularity theorem for (42) shows that this solution is polyhomogeneous at $y = 0$, and hence must have an expansion of the same type as $\hat{u}$, and a similar but more standard argument can be used to produce a better exponential rate of decay as $y \to \infty$. □

Proposition 4.9 *Let $E = L \oplus L^{-1}$ and $\varphi = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right)$ be a stable Higgs bundle, and let (p_j, n_j) be the "knot data" determined by α. Then there exists a solution u to (42) of the form $u = \hat{u} + v$ where $v \to 0$ as $y \to 0$ and as $y \to \infty$.*

Proof We proceed exactly as before, writing

$$\hat{N}(v) = N(\hat{u}+v) = -(\partial_y^2 + \Delta_{g_0})v + |\alpha|^2 e^{2\hat{u}}(e^{2v}-1) + |\beta|^2 e^{-2\hat{u}}(1-e^{-2v}) + f,$$

with $f = N(\hat{u})$ vanishing to all orders as $y \to 0$ and identically for $y \geq 3$. The same barrier functions obviously work in the region $y \geq 3$, and also in the region near $y = 0$ away from the knot singularities.

To construct barriers near a knot $(p,0)$ of weight n, recall the explicit structure of $\hat{u}$ near this point and expand the nonlinear term $e^{2v} - 1$ one step further to write, in some small neighborhood of the front face created by blowing up this point,

$$\hat{N}(v) = \left(-\partial_R^2 - \frac{2}{R}\partial_R + \frac{1}{R^2}(-\Delta_{S^2_+} + \tilde{T})\right)v + ke^{2U_n}(e^{2v} - 1 - 2v) \\ + |\beta|^2 e^{-2U_n}(1 - e^{-2v}) + f.$$

Here k is a strictly positive function which contains all the higher-order terms in the expansion for $\hat{u}$, and $\tilde{T}$ is a slight perturbation of the term T appearing in the linearization L_n. Let μ_0 denote the ground state eigenfunction for this operator on S^2_+. The corresponding eigenvalue λ_0' is a small perturbation of λ_0, which we showed earlier was strictly greater than 0. Now compute

$$\hat{N}(AR^\epsilon \mu_0(\psi,\theta)) = (\lambda_0' - \epsilon(\epsilon+1))AR^{\epsilon-2}\mu_0 + f + E,$$

where E is the sum of the two terms involving $e^{\pm 2U_n}$. As before, since $v \geq 0$ implies $e^{2v} - 1 - 2v \geq 0$ and $1 - e^{-2v}$, we have that $E \geq 0$, and if ϵ is sufficiently small, then this first term on the right has positive coefficient, and dominates f. We have thus produced a local supersolution near $(p,0)$. The full supersolution is

$$v^+ = \min\{AR^\epsilon \mu_0, A' y^{\epsilon/2}, A'' e^{-\epsilon y}\}.$$

We have chosen to use the exponents ϵ and $\frac{\epsilon}{2}$ in the first two terms here in order to ensure that the first term is smaller than the second in the interior of the front face $R = 0$; indeed, $AR^\epsilon \mu_0 < A'(R\sin\psi)^{\epsilon/2}$ when $R < (A'/A)^{2/\epsilon}(\sin\psi)^{\epsilon/2}$. This means that v^+ agrees with $A' y^{\epsilon/2}$ near the original boundary and with $AR^\epsilon \mu_0$ near the other boundaries, and, as before, with the exponentially decreasing term when y is large.

A very similar calculation with the same functions produces a subsolution v^-. Altogether, we deduce, by Proposition 4.1 again, the existence of a solution $u = \hat{u} + v$ to $N(u) = 0$ with the correct asymptotics. □

5 Uniqueness

In this section, we prove a uniqueness theorem for solutions of the extended Bogomolny equations satisfying the (generalized) Nahm pole boundary condition. This will be phrased in terms of the associated Hermitian metrics. The key to this is the subharmonicity of the Donaldson metric, which we recall in the first subsection.

5.1 The distance on Hermitian metrics

Suppose that H is a Hermitian metric on a bundle E, with compatible data (A,ϕ,ϕ_1), which satisfies the extended Bogomolny equations. As we have discussed, it is possible to choose a holomorphic gauge which is parallel in the y direction such that $\mathcal{D}_1=\partial_{\bar z}$, $\mathcal{D}_2=\operatorname{ad}\varphi$ and $\mathcal{D}_3=\partial_y$. In this gauge, the Hermitian metric H determines the gauge fields by

(49) $\partial^A=\partial+H^{-1}\partial H,\quad \varphi^\star=H^{-1}\varphi^\dagger H,\quad \partial_y^{\mathcal{A}}=\partial^{A_y}+i\phi_1=\partial_y+H^{-1}\partial_y H,$

where of course ∂ is the complex differential on Σ and, in this trivialization, $\varphi^\dagger=\varphi^\dagger=\overline{\varphi}^\top$. We can then write the extended Bogomolny equations as

$$\partial_{\bar z}(H^{-1}\partial H)+[\varphi^{\star H},\varphi]+h_0^2\partial_y(H^{-1}\partial_y H)=0,$$

where $h_0^2|dz|^2$ is the Riemannian metric on Σ.

Following [10], we define the distance between Hermitian metrics,

(50) $$\sigma(H_1,H_2)=\operatorname{Tr}(H_1^{-1}H_2)+\operatorname{Tr}(H_2^{-1}H_1)-4,$$

and recall from that paper two important properties:

(i) $\sigma(H_1,H_2)\geq 0$, with equality if and only if $H_1=H_2$.

(ii) A sequence of Hermitian metric H_i converges to H in the usual $\mathcal{C}^0$ norm if and only if $\sup_\Sigma\sigma(H_i,H)\to 0$.

Lemma 5.1 *Suppose that H_1 and H_2 are both harmonic metrics. Then the complex gauge transform $h:=H_1^{-1}H_2$ satisfies*

(51) $$\partial_{\bar z}(h^{-1}\partial^{A_1}h)+\partial_y(h^{-1}\partial_y^{\mathcal{A}_1}h)+[h^{-1}[\varphi^\star,h],\varphi]=0.$$

Proof In holomorphic gauge,

$$A_2=H_2^{-1}\partial H_2=h^{-1}H_1^{-1}\partial H_1h+h^{-1}\partial h=H_1^{-1}\partial H_1+h^{-1}\partial^{A_1}h,$$

hence $\partial_{\bar z}(H_2^{-1}\partial H_2)-\partial_{\bar z}(H_1^{-1}\partial H_1)=\partial_{\bar z}(h^{-1}\partial^{A_1}h)$.

Similarly,

$$H_2^{-1}\partial_y H_2=H_1^{-1}\partial_y H_1+h^{-1}(\partial_y h+[H_1^{-1}\partial_y H_1,h])=H_1^{-1}\partial_y H_1+h^{-1}\partial_y^{\mathcal{A}_y}h.$$

Hence $\partial_y(H_2^{-1}\partial_y H_2)-\partial_y(H_1^{-1}\partial_y H_1)=\partial_y(h^{-1}\partial_y^{\mathcal{A}_y}h)$.

Finally,

$$[\varphi^{\star H_2},\varphi]-[\varphi^{\star H_1},\varphi]=[h^{-1}[\varphi^{\star H_1},h],\varphi].$$

Altogether, we deduce the stated equation from the harmonic metric equations

$$\partial_{\bar z}(H_j^{-1}\partial H_j)+[\varphi^{\star H_j},\varphi]+h_0^2\partial_y(H_j^{-1}\partial_y H_j)=0 \quad \text{for } j=1,2. \qquad \square$$

We next show that σ is subharmonic.

Proposition 5.2 *Define $h=H_1^{-1}H_2$ as above, where H_1 and H_2 satisfy the extended Bogonomy equation. Then $(\Delta+\partial_y^2)\sigma\geq 0$ on $\Sigma\times(0,+\infty)$.*

Proof We first compute

$$\begin{aligned}\partial_{\bar z}\partial_z\operatorname{Tr}(h)&=\operatorname{Tr}(\partial_{\bar z}\partial^{A_1}h)=\operatorname{Tr}(\partial_{\bar z}(hh^{-1}\partial^{A_1}h))\\&=\operatorname{Tr}(\partial_{\bar z}(h)h^{-1}\partial^{A_1}h)+\operatorname{Tr}(h\partial_{\bar z}(h^{-1}\partial^{A_1}h))\\&\geq\operatorname{Tr}(h\partial_{\bar z}(h^{-1}\partial^{A_1}h)),\end{aligned}\tag{52}$$

since $\operatorname{Tr}(BhB^\star)\geq 0$ for any matrix B.

Continuing on,

$$\begin{aligned}\partial_y^2\operatorname{Tr}(h)&=\operatorname{Tr}(\partial_y\partial_y^{\mathcal{A}_1}h)=\operatorname{Tr}((\partial_y h)h^{-1}\partial_y^{\mathcal{A}_1}h)+\operatorname{Tr}\big(h(\partial_y(h^{-1}\partial_y^{\mathcal{A}_1}h))\big)\\&\geq\operatorname{Tr}\big(h(\partial_y(h^{-1}\partial_y^{\mathcal{A}_1}h))\big),\end{aligned}\tag{53}$$

where we use $\partial_y=(\partial_y^{\mathcal{A}_1})^\star$ and that $\star$ is the conjugate transpose with respect to H_1.

Finally,

$$0=\operatorname{Tr}([[\varphi^\star,h],\varphi])=\operatorname{Tr}([h,\varphi]h^{-1}[\varphi^\star,h])+\operatorname{Tr}(h[h^{-1}[\varphi^\star,h],\varphi]).\tag{54}$$

Since $\operatorname{Tr}([h,\varphi]h^{-1}[\varphi^\star,h])\geq 0$, we obtain $\operatorname{Tr}(h[h^{-1}[\varphi^\star,h],\varphi])\leq 0$.

Putting these together gives

$$\begin{aligned}&(\partial_{\bar z}\partial_z+h_0^2\partial_y^2)\operatorname{Tr}(h)\\&\qquad\geq\operatorname{Tr}\big(h\partial_{\bar z}(h^{-1}\partial^{A_1}h)+h_0^2h(\partial_y(h^{-1}\partial_y^{\mathcal{A}_1}h))\big)\\&\qquad\geq\operatorname{Tr}\big(h\partial_{\bar z}(h^{-1}\partial^{A_1}h)+h_0^2h(\partial_y(h^{-1}\partial_y^{\mathcal{A}_1}h+h[h^{-1}[\varphi^\star,h],\varphi]))\big)\\&\qquad\geq 0,\end{aligned}\tag{55}$$

and dividing by h_0^2 proves the claim. $\square$

5.2 Asymptotics of the Hermitian metric

In order to apply the subharmonicity of $\sigma(H_1, H_2)$ from the last subsection, we need to understand the asymptotics of this function near $y=0$. This, in turn, relies on a detailed examination of the asymptotics of the Hermitian metric.

Proposition 5.3 *Fix a Higgs bundle $\left(E \cong L^{-1} \oplus L,\ \varphi = \begin{pmatrix} t & \alpha \\ \beta & -t \end{pmatrix}\right)$. For any $p \in \Sigma$, choose an open set U_p around $(p,0)$ in $\Sigma \times \mathbb{R}^+$. Let H be a solution to the Hermitian extended Bogomolny equations (11); as explained earlier, H is polyhomogeneous on $(\Sigma \times \mathbb{R}^+)_{p_1 \cdots p_N}$ (where the p_j are the zeros of α).*

(i) *Suppose in some local trivialization in U_p that $\varphi|_{U_x} = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$, where q is holomorphic. Suppose also that*

$$H = \begin{pmatrix} \mathcal{O}(y^{-1}) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}. \tag{56}$$

Here $\mathcal{O}(y^s)$ indicates a polyhomogeneous expansion with lowest-order term a smooth multiple of y^s. Suppose also that H satisfies the Nahm pole boundary condition in unitary gauge. Then

$$H \sim \begin{pmatrix} y^{-1} g_0 + \mathcal{O}(1) & o(1) \\ o(1) & y g_0^{-1} + \mathcal{O}(1) \end{pmatrix}, \tag{57}$$

where $o(1)$ indicates a polyhomogeneous expansion with positive leading exponent.

(ii) *Suppose that in a local trivialization, $\varphi|_{U_p} = \begin{pmatrix} t & z^n \\ q & -t \end{pmatrix}$, where $z=0$ is the point p and q is holomorphic. If, in spherical coordinates,*

$$H = \begin{pmatrix} \mathcal{O}(y^{-1} R^{-n}) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, \tag{58}$$

then

$$H = \begin{pmatrix} \mathcal{O}(y^{-1} R^{-n}) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(yR^n) \end{pmatrix}. \tag{59}$$

Proof We first address (i). Write $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ and consider a gauge transformation g for which $H = g^2$. Then we have $g^\dagger = g$ and $g = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}$, where a and d are real functions and $ad - b\bar{b} = 1$. We then compute

$$\phi_z = g\varphi g^{-1} = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ -\bar{b} & a \end{pmatrix} = \begin{pmatrix} bdq - a\bar{b} & -b^2 q + a^2 \\ d^2 q - \bar{b}^2 & -bdq + a\bar{b} \end{pmatrix}. \tag{60}$$

By Proposition 3.8, the Nahm pole boundary condition requires that

$$bdq - a\overline{b} \sim o(1), \quad d^2q - \overline{b}^2 \sim o(1), \quad -b^2q + a^2 \sim \frac{g_0}{y} + \mathcal{O}(1). \tag{61}$$

By definition,

$$H = g^2 = \begin{pmatrix} a^2 + b\overline{b} & ab + bd \\ \overline{b}a + \overline{b}d & d^2 + b\overline{b} \end{pmatrix}.$$

The leading term of $d^2 + b\overline{b}$ is positive, hence b and d are bounded. Combining this with (61) and the relation $ad - b\overline{b} = 1$, we obtain

$$a \sim y^{-1/2} g_0^{1/2}, \quad d \sim y^{1/2} g_0^{-1/2}, \quad b = o(y^{1/2}) \tag{62}$$

and thus

$$H = \begin{pmatrix} a^2 + b\overline{b} & ab + bd \\ \overline{b}a + \overline{b}d & d^2 + b\overline{b} \end{pmatrix} = \begin{pmatrix} y^{-1}g_0 + o(y^{-1}) & o(1) \\ o(1) & yg_0^{-1} + o(y) \end{pmatrix}. \tag{63}$$

As for (ii), we compute

$$\begin{aligned} \phi_z = g\varphi g^{-1} &= \begin{pmatrix} a & b \\ \overline{b} & d \end{pmatrix} \begin{pmatrix} t & z^n \\ q & -t \end{pmatrix} \begin{pmatrix} d & -b \\ -\overline{b} & a \end{pmatrix} \\ &= \begin{pmatrix} bdq - a\overline{b}z^n + atd + |b|^2 t & -b^2q + a^2z^n - 2bat \\ d^2q - z^n\overline{b}^2 + 2td\overline{b} & -bdq + a\overline{b}z^n - |b|^2t - adt \end{pmatrix}. \end{aligned} \tag{64}$$

By Proposition 3.9, the knot singularity implies that

$$\begin{aligned} bdq - a\overline{b}z^n + atd + |b|^2 t &\sim \mathcal{O}(1), \\ -b^2q + a^2z^n - 2ba &\sim z^n e^{U_n} + \cdots, \\ -bdq + a\overline{b}z^n - |b|^2t - adt &\sim \mathcal{O}(1). \end{aligned} \tag{65}$$

As before,

$$H = g^2 = \begin{pmatrix} a^2 + b\overline{b} & ab + bd \\ \overline{b}a + \overline{b}d & d^2 + b\overline{b} \end{pmatrix},$$

where $d^2 + b\overline{b} \sim \mathcal{O}(1)$, so by the same positivity, d and b are both $\mathcal{O}(1)$. Next, $e^{U_n} = f(\psi)/yR^n$, where f is regular. From $-b^2q + a^2z^n - 2ba \sim z^n e^{U_n}$ we get $a \sim y^{-1/2}R^{-n/2}$. In addition, since $ab + bd = \mathcal{O}(1)$ and $ad - b\overline{b} = 1$, we see that $b \sim y^{1/2}R^{n/2}$, so $d \sim y^{1/2}R^{n/2}$. Altogether, H has the form (59). □

Proposition 5.4 *Suppose*

$$H_j = \begin{pmatrix} p_j & q_j \\ q_j^\dagger & s_j \end{pmatrix} \quad \text{for } j = 1, 2$$

are two solutions which both satisfy the Nahm pole boundary condition at $y=0$ and have the same limit as $y\to\infty$. Then $H_1=H_2$.

Proof By Propositions 3.11 and 5.3, we see that as $y\to 0$, $p_j\sim y^{-1}g_0+\cdots$, $s_j\sim yg_0^{-1}+\cdots$ and $q_j\sim o(1)$. We claim that this implies that $\sigma(H_1,H_2)\to 0$ as $y\to 0$. First,

$$H_1^{-1}H_2=\begin{pmatrix} s_1p_2-q_1q_2^\dagger & \star \\ \star & -q_1^\dagger q_2+p_1s_2\end{pmatrix},$$

so

$$\operatorname{Tr}(H_1^{-1}H_2)=s_1p_2-q_1q_2^\dagger-q_1^\dagger q_2+p_1s_2=2+o(1). \tag{66}$$

The same holds for $\operatorname{Tr}(H_2^{-1}H_1)$. This proves the claim.

We have now see that $\sigma(H_1,H_2)$ is nonnegative and subharmonic, and approaches 0 as $y\to 0$ and also as $y\to\infty$, hence $\sigma(H_1,H_2)\equiv 0$, ie $H_1=H_2$. □

Proposition 5.5 *Let H_1 and H_2 be two Hermitian metrics which are both solutions with a knot singularity of degree n at $(p,0)$. Then there exists a constant C such that $\sigma(H_1,H_2)\le C$ in a neighborhood U of $(p,0)$.*

Proof Write

$$H_j=\begin{pmatrix} a_j & b_j \\ b_j^\dagger & d_j\end{pmatrix} \quad \text{for } j=1,2.$$

By Propositions 3.11 and 5.3,

$$a_j\sim y^{-1}R^{-n},\quad d_j\sim yR^n,\quad b_j=o(1),\quad b_j^\dagger=o(1).$$

Thus $\operatorname{Tr}(H_1H_2^{-1})=a_1d_2-b_1b_2^\dagger-b_1^\dagger b_2+d_1a_2=\mathcal{O}(1)$, and, similarly, $\operatorname{Tr}(H_2^{-1}H_1)=\mathcal{O}(1)$. The result follows immediately. □

We next recall the Poisson kernel of $\Delta_g=\Delta_{g_0}+\partial_y^2$. For any $p\in\Sigma$, $P_p(z,y)$ is the unique function on $\Sigma\times\mathbb{R}^+$ which satisfies $\Delta_g P_q(z,y)=0$, $P|_{y=0}=\delta_q$ and $P(z,y)\to 1/\operatorname{Area}(\Sigma)$ as $y\to\infty$.

Theorem 5.6 *Suppose that there exist two Hermitian metrics H_1 and H_2 which are solutions and satisfy the Nahm pole boundary condition with knot singularities at p_j of degree n_j, as determined by the component α in the Higgs field $\varphi=\begin{pmatrix} t & \alpha \\ \beta & -t\end{pmatrix}$. Suppose also that H_1 and H_2 have the same limit as $y\to\infty$. Then $H_1=H_2$.*

Proof By Proposition 5.3, $\sigma(H_1, H_2) \to 0$ as $y \to 0$ and $z \notin \{p_1, \dots, p_N\}$. Near each p_j there is a neighborhood U_j where $\sigma(H_1, H_2)|_{U_j} \leq C$.

Now define $Q(z, y)$ to equal the sum of Poisson kernels $\sum_{j=1}^N P_{p_j}(z, y)$. Then, for any $\epsilon > 0$, $(\Delta_{g_0} + \partial_y^2)(\sigma(H_1, H_2) - \epsilon Q) \geq 0$, and $\sigma(H_1, H_2) - \epsilon Q \leq 0$ as $y \to 0$ and as $y \to \infty$. This means that $\sigma(H_1, H_2) \leq \epsilon Q$. Since this is true for every $\epsilon > 0$, we conclude that $\sigma(H_1, H_2) \leq 0$, ie $H_1 = H_2$. □

6 Solutions with knot singularities on $\mathbb{C} \times \mathbb{R}^+$

We now consider the extended Bogomolny equations on $\mathbb{C} \times \mathbb{R}^+$ with generalized Nahm pole boundary conditions and a finite number of knot singularities.

6.1 Degenerate limit

Consider a trivial bundle E over $\mathbb{C} \times \mathbb{R}^+$, as in [27; 11], the limiting behavior of the classical Jones polynomial indicates that one expects that for solutions of the extended Bogomolny equations on $\mathbb{C} \times \mathbb{R}^+$, $\phi \to 0$ and $\phi_1 \to 0$ as $y \to \infty$. The equation $\mathcal{D}_3 \varphi = 0$ also implies that the conjugacy class of φ is independent of y, and as argued in these papers, this implies that if Q is any invariant polynomial, then $\partial_y Q(\varphi) = 0$, hence that φ is necessarily nilpotent.

Based on these heuristic considerations, we consider a trivial rank 2 holomorphic bundle over $\mathbb{C}$ and assume $\varphi = \begin{pmatrix} 0 & p(z) \\ 0 & 0 \end{pmatrix}$. We can assume $p(z)$ is a polynomial as, up to a complex gauge transform, the equivalent class of the Higgs bundle only depends on the zeros of the upper triangular part of φ. In general, the vanishing section determined by the line bundle has the form $s = \begin{pmatrix} R(z) \\ S(z) \end{pmatrix}$. Consider the section $K(z) := (s \wedge \varphi(s))(z) = p(z) S(z)^2$ of the determinant bundle, which we can naturally identify with a holomorphic function on $\mathbb{C}$. Its zero set defines a positive divisor D.

If the singular monopoles all have order 1, as $K(z) := (s \wedge \varphi(s))(z) = p(z) S(z)^2$, we obtain that $S(z)$ will not have zeros. Up to a complex gauge transform $g = \begin{pmatrix} 1 & -R/S \\ 0 & 1 \end{pmatrix}$, we can assume, in the same trivialization, $\varphi = \begin{pmatrix} 0 & p(z) \\ 0 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In general, we can only assume $\varphi = \begin{pmatrix} t & p \\ q & -t \end{pmatrix}$ and the vanishing line bundle correspond to $s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with the nilpotent condition that $t^2 + pq = 0$.

Although we expect to be able to solve extended Bogomolny equations with knot singularities corresponding to any divisor, the equation will generally not reduce to a scalar one, except in the special case where $\varphi = \begin{pmatrix} 0 & p(z) \\ 0 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and it gives an $\mathrm{SL}(2,\mathbb{R})$ structure. Now the extended Bogomolny equations reduce to

$$-(\Delta + \partial_y^2)v + |p(z)|^2 e^{2v} = 0, \tag{67}$$

and we shall search for a solution for which $v \to -C \log y$ as $y \to \infty$.

Remark It is not enough to simply require that $v \to -\infty$ as $y \to \infty$. Indeed, if $p(z) \equiv 1$, then z–independent solutions solve the ODE $-u'' + e^{2u} = 0$. One solution is $-\log y$, but there is an additional family $\log(C/\sinh(Cy))$ for any $C > 0$. These are the only global solutions to this ODE. The solutions in this second family grow like $-Cy$ as $y \to \infty$, and

$$\phi_1 \to C \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}.$$

These solutions appear in [19] and are described by Gaiotto and Witten [11] as a real symmetry breaking phenomenon at $y \to \infty$.

6.2 Existence

In this section, we will prove:

Proposition 6.1 *Let $p(z)$ be any polynomial on $\mathbb{C}$ of degree $N_0 > 1$. Then there exists a solution u to (67) satisfying the generalized Nahm pole conditions with knot determined by the divisor $D = \sum n_j p_j$ of the polynomial p, and which is asymptotic to $-(N_0 + 1)\log R - \log\sin\psi + \mathcal{O}(1)$ as $R \to \infty$, uniformly in $(\psi,\theta) \in S^2_+$.*

Proof As before, first construct a function $\hat{u}$ which is an approximate solution to this equation with boundary conditions to all orders in all asymptotic regimes, and then use the method of barriers to find a correction term which gives the exact solution.

We first pass to the blowup of $\mathbb{C} \times \mathbb{R}^+$ around the points $(p_j, 0)$, and in an additional step, also take the radial compactification as $R \to \infty$. This gives a compact manifold with corners, which we call $\widehat{X}$ for simplicity; there are boundary faces $F_1, \dots, F_N$, each hemispheres corresponding to the blowups at the zeros of p, another boundary face F_∞, also a hemisphere, corresponding to the radial compactification at infinity, and the original boundary B, which is a disk with N smaller disks removed.

The first step in the construction of $\hat{u}$ is to use the approximate solutions near each of these faces. Around F_j for $j = 1, \dots, N$, we use U_{n_j}; near F_∞ we use U_{N_0}, but now of course with $R \to \infty$ rather than near 0, and finally near B we use $-\log y$. Pasting these together gives a polyhomogeneous function $\widehat{u}_0$ on $\widehat{X}$ for which $N(\widehat{u}_0) = f_0$ blows up like $1/R_j$ near each F_j, decays like R^{-3} near F_∞ and blows up like $-\log y$ near $y = 0$. Here we are denoting the nonlinear operator by N as before.

The second step is to correct the expansions, or, equivalently, to solve away the terms in the expansions of f_0, at each of these boundary faces. Near each F_j this is done exactly as in the last section. Near F_∞ it is done in a completely analogous manner, solving away the terms of order R^{-3-j} using correction terms of order R^{-1-j}. Near F_j we are using the solvability of the operator J_{n_j}, while near F_∞ we use the operator J_{N_0}. Finally, exactly as before, we solve away the terms in the expansion of the remainder as $y \to 0$ along B. This may be done uniformly up to the boundaries of B. Taking Borel sums of each of these expansions, there exists a polyhomogeneous function $\widehat{u}_1$ on $\widehat{X}$ which satisfies $N(\widehat{u}_0 + \widehat{u}_1) = f_1$, where f_1 vanishes to all orders at every boundary component of $\widehat{X}$. The approximate solution is $\widehat{u} = \widehat{u}_0 + \widehat{u}_1$.

Now write $\widehat{N}(v) = N(\widehat{u} + v)$. We expand this as

$$\widehat{N}(v) = -\Delta_{\bar{g}} v + e^{2\widehat{u}} |p(z)|^2 (e^{2v} - 1) + f_1.$$

We construct a supersolution using the following three constituent functions: first, $R_\infty^{-\epsilon} \mu_0^{N_0}$ near F_∞ (where $\mu_0^{N_0}$ is the ground state eigenfunction for J_{N_0}); next, $R_j^{\epsilon} \mu_0^{n_j}$ near F_j; and finally, $y^{\epsilon/2}$ near B. We then take

$$v^+ = \min\{R_\infty^{-\epsilon} \mu_0^{N_0}, R_1^{\epsilon} \mu_0^{n_1}, \dots, R_N^{\epsilon} \mu_0^{n_N}, y^{\epsilon/2}\}.$$

It is straightforward to check that $\widehat{N}(v^+) \geq 0$. With the obvious changes, we also obtain a function v^- for which $\widehat{N}(v^-) \leq 0$.

Proposition 4.1 now implies that there exists a solution v to this equation. By construction, $u = \widehat{u} + v$ satisfies all the required boundary conditions. □

As in Section 4, this existence theorem is accompanied by some sharp estimates for the solution u.

Proposition 6.2 *The solution u obtained in the previous proposition is polyhomogeneous on $\widehat{X}$. In particular, it has a full asymptotic expansion as $R \to \infty$, where the leading term is the model solution U_{N_0}.*

This, in turn, leads to a uniqueness theorem for the scalar equations:

Theorem 6.3 *Let $p(z)$ be a polynomial on $\mathbb{C}$ of degree $N_0 > 1$. Suppose that u_1 and u_2 are two solutions to (67) satisfying the generalized Nahm pole conditions with knot determined by the zeros of polynomial p at $y = 0$. Assume also that as $R \to \infty$, $u_i \sim U_{N_0} + R^{-\epsilon}$ for $i = 1, 2$. Then $u_1 = u_2$.*

Proof By (67),

$$-(\Delta+\partial_y^2)(u_1-u_2)+|p(z)|^2(e^{2u_1}-e^{2u_2}) = \bigl(-(\Delta+\partial_y^2)+|p(z)|^2F(u_1,u_2)\bigr)w = 0.$$

Here $w = u_1 - u_2$ and $F(u_1, u_2) = (e^{2u_1} - e^{2u_2})/(u_1 - u_2)$. By the assumption that both u_1 and u_2 satisfy the same boundary conditions, and using the regularity theory for solutions, we obtain that $\lim_{y\to 0} w = 0$, while by the hypothesis on decay at infinity, $\lim_{R\to\infty} w = 0$ as well. Noting that $F(u_1, u_2) \geq 0$, no matter whether $u_1 < u_2$ or $u_1 \geq u_2$, the maximum principle implies that $w \equiv 0$, ie $u_1 \equiv u_2$. □

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