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Inradius collapsed manifolds

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We study collapsed manifolds with boundary, where we assume a lower sectional curvature bound, two side bounds on the second fundamental forms of boundaries and upper diameter bound. Our main concern is the case when inradii of manifolds converge to zero. This is a typical case of collapsing manifolds with boundary. We determine the limit spaces of inradius collapsed manifolds as Alexandrov spaces with curvature uniformly bounded below. When the limit space has codimension one, we completely determine the topology of inradius collapsed manifold in terms of singular I–bundles. General inradius collapse to almost regular spaces are also characterized. In the general case of unbounded diameters, we prove that the number of boundary components of inradius collapsed manifolds is at most two, where the disconnected boundary happens if and only if the manifold has a topological product structure.

53C20, 53C21, 53C23

1 Introduction

We are concerned with collapsing phenomena of Riemannian manifolds with boundary under a lower sectional curvature bound. The study of collapse of closed manifolds has a long history. In the case of two side bounds on sectional curvatures, a deep general theory was established by Cheeger, Fukaya and Gromov [7]. Then for the case

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of lower sectional curvature bound, in Yamaguchi [32], Fukaya and Yamaguchi [9] and Kapovitch, Petrunin and Tuschmann [14], the structure of the first Betti numbers and the fundamental groups with their topological rigidity were determined through a fibration theorem. Later on, those results were partly extended to the case of a lower Ricci curvature bound by Cheeger and Colding [5; 6], Colding and Naber [8] and Kapovitch and Wilking [15]. In particular, the general manifold structure results of lower-dimensional collapsed manifolds under a lower sectional curvature bound were established by Shioya and Yamaguchi [27; 28] and Yamaguchi [34].

In those results, it is crucial to study Alexandrov spaces with curvature bounded below which appear as the Gromov–Hausdorff limit spaces. In particular, Perelman's topological stability theorem has played significant roles. In connection with the study of Alexandrov spaces, the collapsing phenomena of three-dimensional closed Alexandrov spaces with curvature bounded below has been classified in recent work of Mitsuishi and Yamaguchi [19].

For collapsing Riemannian manifolds with boundary, there is pioneering work by Wong [30; 31] on this subject after the investigation in the noncollapsing and bounded curvature case due to Kodani [16] and Anderson, Katsuda, Kurylev, Lassas and Taylor [2]. In the study of convergence and collapsing Riemannian manifolds with boundary, it is obvious that the main problem is to control the boundary behavior in a geometric way. It is in [30] that a nice extension procedure over the boundary was first carried out to study collapsed manifolds with boundary under a lower sectional curvature bound. The study of collapse of three-dimensional Alexandrov spaces with boundary is now undergoing in the work [18] of Mitsuishi and Yamaguchi, where all the details of collapses will be made clear.

In the present paper, partly motivated by [18], we develop and extend results in [31] to a great extent. Let $\mathcal{M}(n,\kappa,\lambda,d)$ denote the set of all isometry classes of n–dimensional compact Riemannian manifolds M with boundary whose sectional curvature, second fundamental form and diameter satisfy

$$K_M \ge \kappa, \quad |\Pi_{\partial M}| \le \lambda, \quad \operatorname{diam}(M) \le d.$$

Every Riemannian manifold in $\mathcal{M}(n,\kappa,\lambda,d)$ can be glued with a warped cylinder along their boundaries in such a way that the resulting space becomes an Alexandrov space with curvature bounded below having C^0–Riemannian structure and that its boundary is totally geodesic [30]. Investigating such a cylindrical extension, Wong proved that $\mathcal{M}(n,\kappa,\lambda,d)$ is precompact with respect to the Gromov–Hausdorff distance. He also proved that if $\mathcal{M}(n,\kappa,\lambda,d,v)$ denotes the set of all elements $M \in \mathcal{M}(n,\kappa,\lambda,d)$ having volume $\mathrm{vol}(M) \ge v > 0$, then it contains only finitely many homeomorphism types.

Under the situation above, the main problem we are concerned with in this paper is as follows:

Problem 1.1 Let M_i be a sequence in $\mathcal{M}(n,\kappa,\lambda,d)$ converging to a length space N with respect to the Gromov–Hausdorff distance.

(1) Characterize the structure of N.

(2) Find geometric and topological relations between M_i and N for large enough i.

The *inradius* of M is defined as the largest radius of metric ball contained in the interior of M,

$$\mathrm{inrad}(M) := \sup_{x\in M} d(x,\partial M).$$

In the present paper, we first consider the case of $\mathrm{inrad}(M_i)$ converging to zero. We prove in Corollary 3.13 that if $\mathrm{inrad}(M_i)$ converges to zero, then M_i actually dimension collapses in the sense that any limit space N has dimension

$$\dim N \le n-1.$$

Therefore, in this case, we say that M_i *inradius collapses*. The inradius collapse is a typical case of collapsing of manifolds with boundary. Actually in the forthcoming paper [35], we show that if a sequence M_i in $\mathcal{M}(n,\kappa,\lambda,d)$ converges to a topological closed manifold or a closed Alexandrov space, then M_i inradius collapses.

The main results in this paper are stated as follows. The first one is about the limit spaces of inradius collapse.

Theorem 1.2 *Let $M_i \in \mathcal{M}(n,\kappa,\lambda,d)$ inradius collapse to a length space N with respect to the Gromov–Hausdorff distance. Then N is an Alexandrov space with curvature $\ge c(\kappa,\lambda)$, where $c(\kappa,\lambda)$ is a constant depending only on κ and λ.*

It should be noted that M_i are not Alexandrov spaces unless $\mathrm{II}_{\partial M_i} \ge 0$, and that the constant $c(\kappa,\lambda)$ really depends on both κ and λ. Moreover, if one assumes only $\mathrm{II}_{\partial M_i} \ge -\lambda^2$ or $\mathrm{II}_{\partial M_i} \le \lambda^2$ instead of $|\mathrm{II}_{\partial M_i}| \le \lambda^2$, there are counterexamples to Theorem 1.2 (see Examples 3.16, 3.17, 3.18 and 3.19).

Let $\mathcal{M}(n,\kappa,\lambda)$ denote the set of all isometry classes of n–dimensional complete Riemannian manifolds M satisfying

$$K_M \geq \kappa, \quad |\mathrm{II}_{\partial M}| \leq \lambda.$$

This family is also precompact with respect to the pointed Gromov–Hausdorff convergence. Theorem 1.2 actually holds true for the limit of manifolds in $\mathcal{M}(n,\kappa,\lambda)$ with respect to the pointed Gromov–Hausdorff convergence (see Theorem 6.3).

Next we discuss the topological structure of inradius collapsed manifolds. First consider the case of inradius collapse of codimension one. We can give a complete characterization of codimension-one inradius collapsed manifolds as follows:

Theorem 1.3 *Let $M_i \in \mathcal{M}(n,\kappa,\lambda,d)$ inradius collapse to an $(n-1)$–dimensional Alexandrov space N. Then there is a singular I–fiber bundle*

$$I \to M_i \xrightarrow{\pi_i} N$$

whose singular locus coincides with ∂N.

Remark 1.4 Let $D^2_+(i)$ be the upper half-disk on xy–plane of radius ϵ_i with $\epsilon_i \to 0$ as $i \to \infty$, and $J_i := D^2_+(i) \cap \{y=0\}$. It follows from the proof of Theorem 1.3 that M_i becomes a gluing of the I–bundle $N \tilde{\times} I_i$ over N and $D^2_+(i)$–bundle $\partial N \tilde{\times} D^2_+(i)$ over ∂N,

$$M_i = N \tilde{\times} I_i \cup \partial N \tilde{\times} D^2_+(i),$$

where $I_i = [-\epsilon_i, \epsilon_i]$ and the gluing is done via $\partial N \tilde{\times} I_i = \partial N \tilde{\times} J_i$, and $\tilde{\times}$ denotes either the product or a twisted product. Thus M_i collapses to N as the result of shrinking of $D^2_+(i)$ and I_i to points. In particular, M_i has the same homotopy type as N.

Next, we consider inradius collapse to almost regular spaces. An Alexandrov space N is called *ϵ–almost regular* if any point of N has the space of directions whose volume is greater than $\mathrm{vol}\,\mathbb{S}^{\dim N-1} - \epsilon$, where $\mathbb{S}^m$ denotes the unit m–sphere. We say that N is *almost regular* if N is ϵ–almost regular for an $\epsilon > 0$ small enough compared with $\dim N$.

Theorem 1.5 *Let a sequence M_i in $\mathcal{M}(n,\kappa,\lambda,d)$ inradius collapse to an Alexandrov space N, and suppose that the limit of ∂M_i is almost regular and*

$$\mathrm{vol}(\Sigma_x(N)) > \tfrac{1}{2}\,\mathrm{vol}\,\mathbb{S}^{\dim N-1}$$

for all $x \in N$. Then the topology of M_i can be classified into the following two types:

(a) *There exists a locally trivial fiber bundle*

$$F_i \times I \to M_i \to N,$$

where F_i *is a closed almost nonnegatively curved manifold in a generalized sense as in* [32].

(b) *There exists a locally trivial fiber bundle*

$$\mathrm{Cap}_i \to M_i \to N,$$

where Cap_i (*resp.* $\partial\,\mathrm{Cap}_i$) *is an almost nonnegatively curved manifold with boundary* (*resp. a closed almost nonnegatively curved connected manifold*) *in a generalized sense as in* [32].

In general, ϵ–almost regularity of N implies that of the limit of ∂M_i (Proposition 4.30). However the converse is not true (see Example 3.21).

It should also be pointed out that several fibration theorem were obtained in [31] in some cases, where the nonnegativity of the second fundamental form $\mathrm{II}_{\partial M_i} \ge 0$, or the upper bound $K_{M_i} \le \kappa^2$ and the lower bound for the injectivity radius $\mathrm{inj}(M_i) \ge i_0 > 0$ were assumed.

Next we discuss the number of boundary components of inradius collapsed manifolds, where we do not assume the diameter bound.

Theorem 1.6 *There exists a positive number* $\epsilon = \epsilon_n(\kappa, \lambda)$ *such that if* M *in* $\mathcal{M}(n, \kappa, \lambda)$ *satisfies* $\mathrm{inrad}(M) < \epsilon$, *then*

(1) *the number* k *of components of* ∂M *is at most two*;

(2) *if* $k = 2$, *then* M *is diffeomorphic to* $W \times [0, 1]$, *where* W *is a component of* ∂M.

Theorem 1.6(1) was stated in [31, Theorem 5]. However it seems to the authors that the argument there is unclear (see Remark 6.1). Theorem 1.6 may be considered as a generalization of a result of Gromov [10] and Alexander and Bishop [1], where an I–bundle structure was found for an inradius collapsed manifold under the two-sides bound on sectional curvature. It should be pointed out that the constants $\epsilon(\kappa, \lambda)$ in [10; 1] are explicit and independent of n while our constant $\epsilon_n(\kappa, \lambda)$ is neither. This is because our argument is by contradiction.

The organization and the outline of the proofs are as follows.

In Section 2, we first recall basic notions and facts on the Gromov–Hausdorff convergence and Alexandrov spaces with curvature bounded below. Then we focus on Wong's extension procedure of a Riemannian manifold with boundary by gluing a warped cylinder along their boundaries. By Kosovskiĭ [17], the result of the gluing is a $C^{1,\alpha}$–manifold with C^0–Riemannian metric, and becomes an Alexandrov space with curvature bounded below. This construction is quite effective and used in an essential way in the present paper.

In Section 3, we describe limit spaces of glued Riemannian manifolds with boundary. The limit spaces also have gluing structure. In this section we focus on the estimate of multiplicities of gluing, the intrinsic metric structure of the limit space and a general description of the limit spaces of extensions.

In Section 4, we determine the metric structure of limit spaces. First we study the spaces of directions of the limit space at gluing points, and prove that the gluing map preserves the length of curves. This implies that the gluing in the limit space is done metrically in a natural manner, and yields significant structure results (see Theorem 4.32) on the limits including Theorem 1.2.

Those structure results are applied in Section 5 to obtain the fiber structures of inradius collapsed manifolds. Theorems 1.3 and 1.5 are proved there. To prove Theorem 1.3, we need to analyze the singularities of the singular I–fiber bundle in details. To prove Theorem 1.5, we apply an equivariant fibration-capping theorem from [34].

To prove Theorem 1.6, we consider the case of unbounded diameters in Section 6. Applying the results in Section 4, we obtain basically three types on local connectedness of the boundary of an inradius collapsed complete manifold, according to the types of the local limit spaces. After such local observation, Theorem 1.6 follows from a monodromy argument.

Our approach can be applied to the general case of noninradius collapse of Riemannian manifolds with boundary. As a continuation of the present paper, in [35], we obtain the structure of limit spaces, stabilities of topological types and volumes, in the general framework of noninradius collapse/convergence, and get an obstruction to the general collapse.

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2 Preliminaries

In order to make the paper more accessible, we fix some basic definitions, notation and conventions:

- $\tau(\delta)$ is a function such that $\lim_{\delta\to 0}\tau(\delta)=0$.
- For topological spaces X and Y, $X\approx Y$ means X is homeomorphic to Y.
- The distance between two points x and y in a metric space is denoted by $d(x,y)$, $|x,y|$ or $|xy|$.
- For a point x and a subset A of a metric space X, $B(x,r)=B^X(x,r)$ and $B(A,r)=B^X(A,r)$ denote open r–balls in X around x and A, respectively.
- For a metric space (X,d) and $r>0$, the rescaled metric space (X,rd) is denoted by rX.
- The Euclidean cone $K(\Sigma)$ over a metric space (Σ,ρ) is $\Sigma\times[0,\infty)$ equipped with the metric d defined as
$$d((x_1,t_1),(x_2,t_2))=\bigl(t_1^2+t_2^2-2t_1t_2\cos(\min\{\rho(x_1,x_2),\pi\})\bigr)^{1/2}$$
for any two points $(x_1,t_1),(x_2,t_2)\in\Sigma\times[0,\infty)$.
- For a subspace M of a metric space $(\widetilde{M},d_{\widetilde{M}})$, M^{ext} denotes $(M,d_{\widetilde{M}})$, which is called the exterior metric of M.
- The metric d of a connected metric space (X,d) induces a length metric d_{int} of X defined as the infimum of the length of all curves joining two given points. We denote by X^{int} the new metric space (X,d_{int}).
- The length of a curve γ is denoted by $L(\gamma)$.

2.1 The Gromov–Hausdorff convergence

A (not necessarily continuous) map $f\colon X\to Y$ between two metric spaces X and Y is called an *ε–approximation* if it satisfies

(1) $|d(x,y)-d(f(x),f(y))|<\varepsilon$ for all $x,y\in Y$,

(2) $f(X)$ is ε–dense in Y, ie $B(f(X),\varepsilon)=Y$.

The Gromov–Hausdorff distance $d_{\mathrm{GH}}(X,Y)$ is defined as the infimum of those ε such that there are ε–approximations $f\colon X\to Y$ and $g\colon Y\to X$.

A map $f\colon (X,x)\to (Y,y)$ between two pointed metric spaces is called a *pointed ε–approximation* if it satisfies

(1) $f(x)=y$,

(2) $|d(x,y)-d(f(x),f(y))|<\varepsilon$ for all $x,y\in B^X(x,1/\varepsilon)$,

(3) $f(B^X(x,1/\varepsilon))$ is ε–dense in $B^Y(y,1/\varepsilon)$.

The pointed Gromov–Hausdorff distance $d_{\mathrm{pGH}}((X,x),(Y,y))$ is defined as the infimum of those ε such that there are pointed ε–approximations $f\colon (X,x)\to (Y,y)$ and $g\colon (Y,y)\to (X,x)$.

Consider a pair (X,Λ) of a metric space X and a group Λ of isometries of X. For such pairs (X,Λ) and (Y,Γ), a triple (f,φ,ψ) of maps $f\colon X\to Y$, $\varphi\colon \Lambda\to\Gamma$ and $\psi\colon \Gamma\to\Lambda$ is called an *equivariant ε–approximation* from (X,Λ) to (Y,Γ) if

(1) f is an ε–approximation;

(2) if $\lambda\in\Lambda$ and $x\in X$, then $d(f(\lambda x),(\varphi\lambda)(fx))<\varepsilon$;

(3) if $\gamma\in\Gamma$ and $x\in X$, then $d(f(\psi(\gamma)x),\gamma(fx))<\varepsilon$.

The *equivariant Gromov–Hausdorff distance* $d_{\mathrm{eGH}}((X,\Lambda),(Y,\Gamma))$ is defined as the infimum of those ε such that there are ε–approximations from (X,Λ) to (Y,Γ) and from (Y,Γ) to (X,Λ).

2.2 Alexandrov spaces

Let X be a geodesic metric space, where every two points of X can be joined by a shortest geodesic. For a fixed real number κ and a geodesic triangle Δpqr in X with vertices p, q and r, denote by $\tilde\Delta pqr$ a *comparison triangle* in the complete simply connected model surface M^2_κ with constant curvature κ. This means that $\tilde\Delta pqr$ has the same side lengths as the corresponding ones in Δpqr. Here we suppose that the perimeter of Δpqr is less than $2\pi/\sqrt{\kappa}$ if $\kappa>0$. The metric space X is called an *Alexandrov space with curvature $\geq\kappa$*, or sometimes *Alexandrov space* for short if we do not emphasize the lower curvature bound, if each point of X has a neighborhood U satisfying the following: for any geodesic triangle in U with vertices p, q and r and for any point x on the segment qr, we have $|px|\geq|\tilde p\tilde x|$, where $\tilde x$ is the point on the segment $\tilde{q}\tilde{r}$ satisfying $|\tilde{q}\tilde{x}| = |qx|$. From now on we assume that an Alexandrov space is always finite-dimensional.

For an Alexandrov space X with curvature bounded below by κ, let $\alpha\colon [0, s_0] \to X$ and $\beta\colon [0, t_0] \to X$ be two geodesics parametrized by unit speed starting from a point x. The *angle* between α and β is defined by $\angle(\alpha, \beta) = \lim_{s,t\to 0} \tilde{\angle}\,\alpha(s)x\beta(t)$, where $\tilde{\angle}\,\alpha(s)x\beta(t)$ denotes the angle of a comparison triangle $\tilde{\Delta}\alpha(s)x\beta(t)$ at the point $\tilde{x}$. Two geodesics α and β from $x \in X$ are called *equivalent* if $\angle(\alpha, \beta) = 0$. We denote by $\Sigma'_x(X)$ the set of equivalent classes of geodesics emanating from x. The *space of directions* at x, denoted by $\Sigma_x = \Sigma_x(X)$, is the completion of $\Sigma'_x(X)$ with the angle metric. A direction of minimal geodesic from p to x is also denoted by $\uparrow_p^x$. Let X be n–dimensional. Then Σ_x is an $(n-1)$–dimensional compact Alexandrov space with curvature ≥ 1.

A point $x \in X$ is called *regular* if Σ_x is isometric to $\mathbb{S}^{n-1}$. Otherwise we call x a singular point. We denote by X^{reg} (resp. X^{sing}) the set of all regular points (resp. singular points) of X.

The *tangent cone* at $x \in X$, denoted by $T_x(X)$, is the Euclidean cone $K(\Sigma_x)$ over Σ_x. It is known that $T_x(M) = \lim_{r\to 0}\big(\frac{1}{r}M, x\big)$.

For a closed subset A of X and $p \in A$, *the space of directions* $\Sigma_p(A)$ of A at p is defined as the set of all $\xi \in \Sigma_p(X)$ which can be written as the limit of directions in $\Sigma_p(X)$ from p to points $p_i \in A$ with $|p, p_i| \to 0$:

$$\xi = \lim_{i\to\infty} \uparrow_p^{p_i} .$$

For $x, y \in X \setminus A$, consider a comparison triangle on M_κ^2 having the side-length $(|A, x|, |x, y|, |y, A|)$ whenever it exists. Then $\tilde{\angle}\,Axy$ denotes the angle of this comparison triangle at the vertex corresponding to x.

For $x, y, z \in X$, we denote by $\angle xyz$ (resp. $\tilde{\angle}\,xyz$) the angle between the geodesics yx and yz at x (resp. the geodesics $\tilde{y}\tilde{x}$ and $\tilde{y}\tilde{z}$ at $\tilde{x}$ in the comparison triangle $\tilde{\Delta}xyz = \Delta\tilde{x}\tilde{y}\tilde{z}$).

Let X be an n–dimensional Alexandrov space with curvature bounded below by κ. For $\delta > 0$, a system of n pairs of points $\{a_i, b_i\}_{i=1}^n$ is called an (n, δ)*-strainer* at $x \in X$ if it satisfies

$$\tilde{\angle}_\kappa\, a_i x b_i > \pi - \delta, \quad \tilde{\angle}_\kappa\, a_i x a_j > \tfrac{1}{2}\pi - \delta, \quad \tilde{\angle}_\kappa\, b_i x b_j > \tfrac{1}{2}\pi - \delta, \quad \tilde{\angle}_\kappa\, a_i x b_j > \tfrac{1}{2}\pi - \delta$$

for every $1 \le i \ne j \le n$. If $x \in X$ has an (n,δ)–strainer, then we say x is (n,δ)–strained. In this case, we call x *δ–regular*. We call X *almost regular* if every point of X is δ_n–regular for some $\delta_n < 1/(100n)$. It is known that a small neighborhood of any almost regular point is almost isometric to an open subset in $\mathbb{R}^n$.

Inductively on the dimension, the boundary ∂X is defined as the set of points $x \in X$ such that Σ_x has nonempty boundary $\partial\Sigma_x$. We denote by $D(X)$ the double of X, which is also an Alexandrov space with curvature $\ge \kappa$ (see [21]). By definition, $D(X) = X \amalg_{\partial X} X$, where two copies of X are glued along their boundaries.

A boundary point $x \in \partial X$ is called *δ–regular* if x is δ–regular in $D(X)$. We say that X is *almost regular with almost regular boundary* if every point of X is δ–regular in $D(X)$ for $\delta < 1/(100n)$.

In Section 5.1, we need the following result on the dimension of the interior singular point sets. We set $\operatorname{int} X := X \setminus \partial X$.

Theorem 2.1 ([4]; see also [20]) *We have*

$$\dim_H(X^{\text{sing}} \cap \operatorname{int} X) \le n-2, \quad \dim_H(\partial X)^{\text{sing}} \le n-2,$$

where $(\partial X)^{\text{sing}} = D(X)^{\text{sing}} \cap \partial X$.

Theorem 2.2 ([21; 22]; see also [13]) *If a sequence X_i of n–dimensional compact Alexandrov spaces with curvature $\ge \kappa$ Gromov–Hausdorff converges to an n–dimensional compact Alexandrov space X, then X_i is homeomorphic to X for large enough i.*

A subset E of an Alexandrov space X is called *extremal* [23] (see also [26]) if every distance function $f = \operatorname{dist}_q$ with $q \in M \setminus E$ has the property that if $f|_E$ has a local minimum at $p \in E$, then $df_p(\xi) \le 0$ for every $\xi \in \Sigma_p(X)$. Extremal subsets possess quite important properties.

Theorem 2.3 [23] *Let E be an extremal subset of X.*

(1) *For every $p \in E$, $\Sigma_p(E)$ is an extremal subset of $\Sigma_p(X)$;*

(2) *E is totally quasigeodesic in the sense that any nearby two points of E can be joined by a quasigeodesic* (see [24]).

(3) *E has a topological stratification.*

Theorem 2.3(1)–(2) implies the following:

Corollary 2.4 *For an extremal subset E of X and $p \in E$,* $\dim \Sigma_p(E) \le \dim E - 1$.

Suppose that a compact group G acts on X as isometries. Then the quotient space X/G is an Alexandrov space [4]. Let F denote the set of G–fixed points.

Proposition 2.5 [23] *$\pi(F)$ is an extremal subset of X/G, where $\pi\colon X \to X/G$ is the projection.*

Boundaries of Alexandrov spaces are typical examples of extremal subsets.

Proposition 2.6 [34, Proposition 5.10] *Let X be an Alexandrov space with curvature bounded below having nonempty boundary ∂X which is not necessarily compact. Then ∂X has a collar neighborhood.*

An n–dimensional Alexandrov space is called *smoothable* if it is a Gromov–Hausdorff limit of n–dimensional closed Riemannian manifolds with a uniform lower sectional curvature bound.

Theorem 2.7 [12] *Let X be a smoothable Alexandrov space. Then for any $p \in X$, every iterated space of directions*

$$\Sigma_{\xi_k}(\Sigma_{k-1}(\cdots(\Sigma_{\xi_1}(\Sigma_p(X))\cdots))$$

is homeomorphic to a sphere, where

$$\xi_1 \in \Sigma_p(X), \quad \xi_2 \in \Sigma_{\xi_1}(X), \quad \dots, \quad \xi_k \in \Sigma_{\xi_{k-1}}(\cdots(\Sigma_{\xi_1}(\Sigma_p(X))\cdots)).$$

2.3 Manifolds with boundary and gluing

In this section, we consider a Riemannian manifold M with boundary in $\mathcal{M}(n,\kappa,\lambda,d)$. First, we recall some fundamental properties of ∂M, which were derived by Wong [30]. We also recall Wong's cylindrical extension procedure based on Kosovskiĭ's gluing theorem [17].

Let M be a Riemannian manifold with boundary, and ∂M^α denote a boundary component of ∂M. Then $(\partial M^\alpha)^{\mathrm{int}}$ means ∂M^α with intrinsic length metric.

The following is an immediate consequence of the Gauss equation.

Proposition 2.8 *For every $M \in \mathcal{M}(n,\kappa,\lambda)$, ∂M has a uniform lower sectional curvature bound $K_{\partial M} \ge K$, where $K = K(\kappa,\lambda)$.*

Proposition 2.9 [30] *Let $M \in \mathcal{M}(n,\kappa,\lambda,d)$.*

(1) *There exists a constant $D = D(n,\kappa,\lambda,d)$ such that any boundary component ∂M^{α} has intrinsic diameter bound*

$$\mathrm{diam}((\partial M^{\alpha})^{\mathrm{int}}) \le D.$$

(2) *∂M has at most J components, where $J = J(n,\kappa,\lambda,d)$.*

It follows from Proposition 2.9 that every boundary component of $M \in \mathcal{M}(n,\kappa,\lambda,d)$ is an Alexandrov space with curvature $\ge K$ and diameter $\le D$, where $K = K(\kappa,\lambda)$ and $D = D(n,\kappa,\lambda,d)$.

In general, a Riemannian manifold with boundary is not necessarily an Alexandrov space. Wong [30] carried out a gluing of warped cylinders and M along their boundaries in such a way that the resulting manifold becomes an Alexandrov space having totally geodesic boundary.

This is based on Kosovskiĭ's gluing theorem:

Theorem 2.10 [17] *Let M_0 and M_1 be Riemannian manifolds with boundaries Γ_0 and Γ_1, respectively, with sectional curvature $K_{M_i} \ge \kappa$ for $i = 0, 1$. Assume that there exists an isometry $\phi\colon \Gamma_0 \to \Gamma_1$, and let M denote the space with length metric obtained by gluing M_0 and M_1 along their boundaries via ϕ. Let L_i for $i = 0, 1$ be the second fundamental form of $\Gamma := \Gamma_0 \cong_{\phi} \Gamma_1 \subset M$ with respect to the normal inward to M_i. Then M is an Alexandrov space with curvature $\ge \kappa$ if and only if the sum $L := L_1 + L_2$ is positive semidefinite.*

Remark 2.11 Actually, for every $\delta > 0$, a smooth Riemannian metric g_δ on M is constructed in [17] in such a way that the sectional curvature of g_δ is greater than $\kappa(\delta)$ with $\lim_{\delta\to 0}\kappa(\delta) = \kappa$ and that (M, g_δ) Gromov–Hausdorff converges to M as $\delta \to 0$.

Now let us recall the extension construction in [30].

Let M be an n–dimensional complete Riemannian manifold with boundary satisfying

$$K_M \ge \kappa, \quad \lambda^- \le \mathrm{II}_{\partial M} \le \lambda^+,$$

where $\mathrm{II}_{\partial M}$ denotes the second fundamental form of ∂M with respect to the inward unit normal to M. Let $\bar{\lambda} := \min\{0, \lambda^-\}$. Then for arbitrary $t_0 > 0$ and $0 < \varepsilon_0 < 1$

there exists a monotone nonincreasing function $\phi\colon [0,t_0]\to\mathbb{R}^+$ satisfying

$$\phi''(t)+K\phi(t)\le 0,\quad \phi(0)=1,\quad \phi(t_0)=\varepsilon_0,\quad -\infty<\phi'(0)\le\bar{\lambda},\quad \phi'(t_0)=0$$

for some constant $K=K(\lambda,\varepsilon_0,t_0)$. Now consider the warped product metric on $\partial M\times[0,t_0]$ defined by

$$g(x,t)=dt^2+\phi^2(t)g_{\partial M}(x),$$

where $g_{\partial M}$ is the Riemannian metric of ∂M induced from that of M. We denote by $\partial M\times_\phi[0,t_0]$ the warped product. It follows from the construction that

(2-1)
- $\mathrm{II}_{\partial M\times\{0\}}\ge|\min\{0,\lambda^-\}|$,
- $\mathrm{II}_{\partial M\times\{t_0\}}\equiv 0$,
- the sectional curvature of $\partial M\times_\phi[0,t_0]$ is greater than a constant $c=c(\kappa,\lambda^\pm,\varepsilon_0,t_0)$,
- the second fundamental form of $\partial M\times\{t\}$ is given by

$$\mathrm{II}_{\partial M\times\{t\}}(V,W)=\frac{\phi'(t)}{\phi(t)}g(V,W)$$

  for vector fields V,W on $\partial M\times\{t\}$.

Clearly, $\partial M\times\{0\}$ in $\partial M\times_\phi[0,t_0]$ is canonically isometric to ∂M. Thus we can glue M and $\partial M\times_\phi[0,t_0]$ along ∂M and $\partial M\times\{0\}$. The resulting space

$$\widetilde{M}:=M\amalg_{\partial M}(\partial M\times_\phi[0,t_0])$$

carries the structure of a differentiable manifold of class $C^{1,\alpha}$ with C^0–Riemannian metric [17]. Obviously M is diffeomorphic to $\widetilde{M}$.

Proposition 2.12 [30] *For $M\in\mathcal{M}(n,\kappa,\lambda)$, we have*

(1) *$\widetilde{M}$ is an Alexandrov space with curvature $\ge\tilde{\kappa}$, where $\tilde{\kappa}=\tilde{\kappa}(\kappa,\lambda)$;*

(2) *the exterior metric M^{ext} is L–bi-Lipschitz homeomorphic to M for the uniform constant $L=1/\varepsilon_0$;*

(3) $\operatorname{diam}(\widetilde{M})\le\operatorname{diam}(M)+2t_0$.

The notion of warped product also works for metric spaces.

Let X and Y be metric spaces, and $\phi\colon Y \to \mathbb{R}_+$ a positive continuous function. Then the warped product $X \times_\phi Y$ is defined as follows (see [29]). For a curve $\gamma = (\sigma, \nu)\colon [a,b] \to X \times Y$, the length of γ is defined as

$$L_\phi(\gamma) = \sup_{|\Delta|\to 0} \sum_{i=1}^{k} \sqrt{\phi^2(\nu(s_i))|\sigma(t_{i-1}), \sigma(t_i)|^2 + |\nu(t_{i-1}), \nu(t_i)|^2},$$

where $\Delta\colon a = t_0 < t_1 < \cdots < t_k = b$ and s_i is any element of $[t_{i-1}, t_i]$. The warped product $X \times_\phi Y$ is defined as the topological space $X \times Y$ equipped with the length metric induced from L_ϕ.

Proposition 2.13 [29, Proposition B.2.6] *Let X_i be a convergent sequence of length spaces. If Y is a compact length space, we have*

$$\lim_{\mathrm{GH}}(X_i \times_\phi Y) = (\lim_{\mathrm{GH}} X_i) \times_\phi Y.$$

3 Descriptions of limit spaces and examples

Under the notation in Section 2.3, throughout this section unless otherwise stated, we assume $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ Gromov–Hausdorff converges to a compact length space N, where $\mathrm{inrad}(M_i) \to 0$. Let $\widetilde{M}_i$ converge to a compact Alexandrov space Y, and M_i^{ext} converge to a closed subset X of Y under the convergence $\widetilde{M}_i \to Y$.

Here we fix some notation used later on:

- C_{M_i} denotes $\partial M_i \times_\phi [0, t_0]$.
- $C_{M_i,t}$ denotes the subspace $\partial M_i \times_\phi \{t\}$ in C_{M_i}.
- For $C_{M_i} \subset \widetilde{M}_i$, $C_{M_i}^{\mathrm{ext}}$ denotes $(C_{M_i}, d_{\widetilde{M}_i})$.

In this section, we first investigate the relation between the limit C (resp. C_0) of C_{M_i} (resp. of ∂M_i) and Y (resp. X), and discuss the intrinsic structure of X and prove that X^{int} is isometric to N (Proposition 3.9). Then we describe the metric structure of Y (Proposition 3.11)

3.1 Descriptions of X and Y

Under the notation presented in the beginning of this section, in view of Proposition 2.9 and (2-1), passing to a subsequence, we may assume that C_{M_i} converges to some compact Alexandrov space C with curvature $\geq K = K(\kappa, \lambda)$. Here C_{M_i} is not necessarily connected, and therefore the convergence $C_{M_i} \to C$ should be understood componentwisely. It follows from Proposition 2.13 that

$$C = C_0 \times_\phi [0, t_0], \quad C_0 = \lim_{i\to\infty} (\partial M_i)^{\mathrm{int}},$$

where $(\partial M_i)^{\mathrm{int}}$ denotes ∂M_i endowed with length metric induced by its original metric. For simplicity we write

$$C_0 := C_0 \times \{0\}, \quad C_t := C_0 \times \{t\} \subset C.$$

Since the identity map $\iota_i\colon C_{M_i} \to C_{M_i}^{\mathrm{ext}}$ is 1–Lipschitz, we can define a surjective 1–Lipschitz map $\eta\colon C \to Y$ in the limit. More precisely, define $\eta\colon C \to Y$ by

$$\eta = \lim_{i\to\infty} g_i \circ \iota_i \circ f_i,$$

where $f_i\colon C \to C_{M_i}$ and $g_i\colon \widetilde{M}_i \to Y$ are componentwise ε_i–approximations with $\lim \varepsilon_i = 0$.

From now on, we consider

$$\eta_0 := \eta|_{C_0\times\{0\}}\colon C_0 \to X,$$

which is also a surjective 1–Lipschitz map with respect to the exterior metrics of C_0 and X, and hence with respect to the interior metrics, too.

The following two lemmas are obvious.

Lemma 3.1 *The map $\eta\colon C \setminus C_0 \to Y \setminus X$ is a bijective local isometry.*

Lemma 3.2 *For $(p,t) \in C \setminus C_0$, we have $|\eta(p,t), X| = t$.*

We now study the multiplicities of the gluing map η_0.

Lemma 3.3 *For every $x \in X$, we have the following:*

(1) $\#\eta_0^{-1}(x) \le 2$.

(2) *Suppose $\#\eta_0^{-1}(x) = 2$ for some $x \in X$, and take $p_k \in C_0$ for $k = 1, 2$ with $\eta_0(p_k) = x$. Then $\Sigma_x(Y)$ is isometric to a spherical suspension with the two vertices $\{\xi_1, \xi_2\}$, where*

$$\xi_k := \uparrow_x^{\eta(p_k,t_0)}.$$

Proof Suppose that $\#\eta_0^{-1}(x) \geq 3$ and take $p_i \in \eta_0^{-1}(x)$ and let $y_i := \eta(p_i, t)$ for some $t > 0$ and for $i = 1, 2, 3$. We show that $|y_i, y_j| = 2t$, or, equivalently,

$$\tilde{\angle}\, y_i x y_j = \pi, \tag{3-1}$$

if $t < \frac{1}{2}\phi(t_0)|p_i p_j|_{C_0^{\mathrm{int}}}$, where $i < j$ and $i, j \in \{1, 2, 3\}$. It turns out that the geodesics $y_1 y_2$ and $y_1 y_3$ branch at x, which is impossible since Y is an Alexandrov space with curvature bounded below. The conclusions (1) and (2) follow immediately.

Let $\gamma\colon [0, \ell] \to Y$ be a minimal geodesic in Y joining y_i and y_j. If γ meets X, we certainly have $|y_i, y_j| = 2t$. Suppose that γ does not meet X. Then $\tilde{\gamma} = \eta^{-1}(\gamma)$ is well defined by Lemma 3.1 and is a minimal geodesic joining (p_i, t) and (p_j, t). Write $\tilde{\gamma}$ as $\tilde{\gamma}(s) = (\sigma(s), \nu(s)) \in C_0 \times_\phi [0, t_0]$. Then we have

$$\begin{aligned} L(\gamma) = L(\tilde{\gamma}) &= \int_0^\ell \sqrt{\phi^2(\nu(s))|\dot{\sigma}(s)|^2 + |\dot{\nu}(s)|^2}\, ds \\ &\geq \int_0^\ell \phi(t_0)|\dot{\sigma}(s)|\, dt \geq \phi(t_0)|p_i, p_j|_{C_0^{\mathrm{int}}}. \end{aligned}$$

Thus we have $|y_i, y_j| = L(\gamma) \geq \phi(t_0)|p_i, p_j|_{C_0^{\mathrm{int}}}$. On the other hand, the triangle inequality shows that $|y_i, y_j| \leq 2t < \phi(t_0)|p_i, p_j|_{C_0^{\mathrm{int}}}$. This is a contradiction, and therefore γ meets X and $|y_i, y_j| = 2t$. □

Next we construct a good approximation map $\widetilde{M}_i \to Y$, which helps us to grasp a whole picture on several convergences.

Let $\psi_i\colon \partial M_i = C_{M_i,0} \to C_0$ be an ϵ_i–approximation with $\lim_{i\to\infty} \epsilon_i = 0$.

Lemma 3.4 [29] *The map $\Psi_i\colon C_{M_i} \to C$ defined by*

$$\Psi_i(p, t) = (\psi_i(p), t)$$

is an ϵ_i'–approximation with $\lim_{i\to\infty} \epsilon_i' = 0$. Actually, for any approximation map $\Psi_i'\colon C_{M_i} \to C$ there is a $\psi_i\colon \partial M_i = C_{M_i,0} \to C_0$ such that $|\Psi_i(p,t), \Psi_i'(p,t)| < \epsilon_i'$ for $\Psi_i = (\psi_i, \mathrm{id})$.

Proof This follows from Proposition 2.13. □

Recall that $\eta\colon C \setminus C_0 \to Y \setminus X$ is a locally isometric bijection. In particular, for every $y = (p, t_0) \in C_{t_0} \subset Y$, there is a unique minimal geodesic $\gamma_y\colon [0, t_0] \to Y$ between X

and y such that $\gamma_y(0) \in X$ and $\gamma(t_0) = y$. Actually γ_y is defined as $\gamma_y(t) = \eta(p,t)$. Define $g_i^*\colon C_{M_i}^{\text{ext}} \to Y$ by

$$g_i^*(p,t) = \eta \circ \Psi_i \circ \iota_i^{-1}(p,t) = \eta(\psi_i(p),t). \tag{3-2}$$

Proposition 3.5 *The map $g_i^*\colon C_{M_i}^{\text{ext}} \to Y$ defined above provides an ϵ_i''–approximation, where $\lim \epsilon_i'' = 0$.*

Let $g_i\colon C_{M_i}^{\text{ext}} \to Y$ be any ϵ_i–approximation such that $g_i = g_i^*$ on C_{M_i,t_0}, namely $g_i(p,t_0) = g_i^*(p,t_0)$.

For the proof of Proposition 3.5, it suffices to show the following:

Lemma 3.6 $|g_i(p,t), g_i^*(p,t)| < \epsilon_i''$ *for all* $(p,t) \in C_{M_i}^{\text{ext}}$.

Proof We have to show that

$$\lim_{i\to\infty} \sup_{(p,t)\in C_{M_i}} |g_i(p,t), g_i^*(p,t)| = 0.$$

Suppose the contrary. Then there are subsequence $\{j\} \subset \{i\}$ and $(p_j,t_j) \in C_{M_j}$ such that

$$|g_j(p_j,t_j), g_j^*(p_j,t_j)| \ge c > 0 \tag{3-3}$$

for some constant c independent of j. Passing to a subsequence, we may assume that $(\psi_j(p_j),t_j)$ converges to $(p_\infty,t_\infty) \in C$. Let $\gamma_j(t) = (p_j,t)$ for $0 \le t \le t_0$, which is a minimal geodesic in $C_{M_j}^{\text{ext}}$ between ∂M_j and C_{M_i,t_0}. Now $g_j^* \circ \gamma_j(t) = \eta(\psi_j(p_j),t)$ converges to a minimal geodesic $\gamma_\infty(t) = \eta(p_\infty,t)$ realizing the distance between X and $(p_\infty,t_0) \in C_{t_0} \subset Y$. Since g_j is an ϵ_j–approximation, any limit of $g_j \circ \gamma_j$, say $\hat\gamma$, must also be a minimal geodesic between X and (p_∞,t_0). From the uniqueness of such geodesics, we have $\gamma_\infty(t) = \hat\gamma_\infty(t)$, which contradicts (3-3). □

Remark 3.7 Proposition 3.5 will be effectively used in Section 6.

Next, we determine the intrinsic structure of X, and prove Proposition 3.9 below, which will be crucial in our start for the description of Y in terms of N (see Proposition 3.11)

Recall that $X \subset Y$ is the limit of M_i^{ext} under the convergence $\widetilde{M}_i \to Y$. By Proposition 2.12, the identity $\iota_i\colon M_i^{\text{ext}} \to M_i$ is an L–bi-Lipschitz homeomorphism. Therefore we have that:

Lemma 3.8 *For a subsequence, $\iota_i\colon M_i^{\text{ext}} \to M_i$ converges to an L–bi-Lipschitz homeomorphism $\iota_\infty\colon X \to N$.*

Proposition 3.9 *X^{int} is isometric to N.*

Sublemma 3.10 *X is connected.*

Proof Take an ϵ_i–approximation $\varphi_i\colon \widetilde{M}_i \to Y$ such that $\varphi_i(M_i) \subset X$ and $\lim \epsilon_i = 0$. For every $x, y \in X$, choose $p_i, q_i \in M_i$ such that $\varphi_i(p_i) \to x$ and $\varphi_i(q_i) \to y$. Let $\gamma_i\colon [0,1] \to M_i$ be a minimal geodesic in M_i joining p_i to q_i. Then the Lipschitz curve $\iota_i^{-1} \circ \gamma_i\colon [0,1] \to M_i^{\text{ext}}$ converges to a Lipschitz curve in X joining x to y under the convergence $\widetilde{M}_i \to Y$. □

Proof of Proposition 3.9 Passing to a subsequence if necessary, we may assume that the L–Lipschitz map $\iota_i\colon M_i^{\text{ext}} \to M_i$, where $L = 1/\epsilon_0$, converges to a surjective map $h\colon X \to N$ satisfying

$$|x, y|_Y \le |h(x), h(y)|_N \le L|x, y|_Y$$

for every $x, y \in X$. Let $\sigma\colon [0,d] \to N$ be a minimal geodesic joining $h(x)$ and $h(y)$. Then we have

$$|h(x), h(y)|_N = L(\sigma) \ge L(h^{-1}(\sigma)) \ge |x, y|_{X^{\text{int}}}.$$

Next we show the reverse inequality. Let $\gamma\colon [0,\ell] \to X$ be a minimal geodesic in X^{int} joining x to y. For any $\varepsilon > 0$, take a subdivision Δ of γ: $x = x_0 < x_1 < \cdots < x_\alpha < \cdots x_k = y$ such that, denoting by γ_Δ the broken geodesic consisting of minimal geodesics joining $x_{\alpha-1}$ and x_α in Y for $1 \le \alpha \le k$, we have

(1) $|L(\gamma_\Delta) - |x, y|_{X^{\text{int}}}| < \varepsilon$;

(2) $\max_t |\gamma_\Delta(t), X| < \varepsilon$.

Take $p_\alpha^i \in M_i$ converging to x_α under the convergence $\widetilde{M}_i \to Y$, and denote by γ_Δ^i a broken geodesic consisting of minimal geodesics joining $p_{\alpha-1}^i$ and p_α^i in $\widetilde{M}_i$ for $1 \le \alpha \le k$. Note that, for large enough i,

(1) $|L(\gamma_\Delta) - L(\gamma_\Delta^i)| < \varepsilon$;

(2) $\max_t |\gamma_\Delta^i(t), M_i| < \varepsilon$.

Let $\sigma_i := \pi_i \circ \gamma_\Delta^i$, where $\pi_i \colon \widetilde{M}_i \to M_i$ is the canonical projection defined by $\pi_i(p,t) = p$. From the warped product metric construction, we have $L(\gamma_\Delta^i) \ge \phi(\varepsilon) L(\sigma_i)$ for large i. It follows that

$$|x,y|_{X^{\mathrm{int}}} \ge L(\gamma_\Delta) - \varepsilon > L(\gamma_\Delta^i) - 2\varepsilon \ge \phi(\varepsilon) L(\sigma_i) - 2\varepsilon \ge \phi(\varepsilon)|p_i, q_i|_{M_i} - 2\varepsilon,$$

where $p_i \to x$ and $q_i \to y$ under $\widetilde{M}_i \to Y$. Letting $|\Delta| \to 0$ and $i \to \infty$, we conclude that $|x,y|_{X^{\mathrm{int}}} \ge |h(x), h(y)|_N$. This completes the proof. □

Let $X^{\mathrm{int}} \cup_{\eta_0} C_0 \times_\phi [0,t_0]$ denote the length space obtained by the result of gluing of the two length spaces X^{int} and $C_0 \times_\phi [0,t_0]$ by the map $\eta_0 \colon C_0 \times 0 \to X^{\mathrm{int}}$.

Proposition 3.11 *Y is isometric to the length space*

$$X^{\mathrm{int}} \cup_{\eta_0} C_0 \times_\phi [0,t_0].$$

Proof Let $Z := X^{\mathrm{int}} \cup_{\eta_0} C_0 \times_\phi [0,t_0]$, and $\Phi \colon Y \to Z$ be the canonical map. Note that Φ is bijective. For every $y_0, y_1 \in Y$, let $\gamma \colon [0,\ell] \to Y$ be a minimal geodesic joining y_0 and y_1. Decompose γ into the two parts

$$\gamma = \gamma_{Y \setminus X} \cup \gamma_X,$$

where $\gamma_{Y\setminus X} = \gamma \cap (Y \setminus X)$ and $\gamma_X = \gamma \cap X$. Let $\gamma_{Y\setminus X} = \cup_\alpha \gamma_\alpha$ be the at-most countable union consisting of open arc components of $\gamma_{Y\setminus X}$. For any $\epsilon > 0$, take γ_α of length $\le \epsilon$ such that the endpoints z_α and w_α of γ_α are contained in X if such a γ_α exists. Take $p_i, q_i \in M_i$ such that $p_i \to z_\alpha$ and $q_i \to w_\alpha$ under the convergence $\widetilde{M}_i \to Y$. For a minimal geodesic γ_i joining p_i and q_i in $\widetilde{M}_i$, let $\sigma_i := \pi_i(\gamma_i)$, where $\pi_i \colon \widetilde{M}_i \to M_i$ is the projection. Note that $\max |\gamma_i(t), M_i| < \epsilon$. Using the warped metric structure, we have $L(\gamma_i) \ge \phi(\epsilon) L(\sigma_i)$, which implies

$$|z_\alpha, w_\alpha|_Y \ge |p_i, q_i|_{\widetilde{M}_i} - o_i \ge \phi(\epsilon)|p_i, q_i|_{M_i} - o_i,$$

where $\lim o_i = 0$. Letting $i \to \infty$, we have $|z_\alpha, w_\alpha|_Y \ge \phi(\epsilon)|z_\alpha, w_\alpha|_{X^{\mathrm{int}}}$. Now we replace γ_α by a minimal geodesic joining z_α and w_α in X^{int}. Repeating this procedure at most countably many times if necessary, we construct a Lipschitz curve $\hat{\gamma}$ joining y_0 to y_1 such that in the decomposition

$$\hat{\gamma} = \hat{\gamma}_{Y\setminus X} \cup \hat{\gamma}_X,$$

$\hat{\gamma}_{Y\setminus X}$ (resp. $\hat{\gamma}_X$) consists of finitely many Y–minimal geodesics each of length $\geq \epsilon$ (resp. finitely many X–minimal geodesic) and that

$$|y_0, y_1|_Y = L(\gamma) \geq \phi(\epsilon)L(\hat{\gamma}) \geq \phi(\epsilon)|\Phi(y_0), \Phi(y_1)|_Z.$$

Letting $\epsilon \to 0$, we conclude that $|y_0, y_1|_Y \geq |\Phi(y_0), \Phi(y_1)|_Z$.

Next, taking a Z–minimal geodesic joining $\Phi(y_0)$ and $\Phi(y_1)$ and replacing it by a Lipschitz curve in a similar way, we obtain the reverse inequality $|y_0, y_1|_Y \leq |\Phi(y_0), \Phi(y_1)|_Z$. This completes the proof. □

Remark 3.12 Both Propositions 3.9 and 3.11 hold true for pointed Gromov–Hausdorff limits of inradius collapsed manifolds (see Section 6). Moreover, in the above proofs, we do not need the assumption of inradius collapse. Therefore Propositions 3.9 and 3.11 also hold for Gromov–Hausdorff limits of noninradius collapsed manifolds.

The reason why we use the naming of inradius collapse partly comes from the following corollary:

Corollary 3.13 *If $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ inradius collapses to N, then we have*

(1) $\dim M_i > \dim N$;

(2) $\lim \mathrm{vol}(M_i) = 0$.

Proof (1) From Lemma 3.8 and Proposition 3.11, we have

$$\dim M_i = \dim \widetilde{M}_i \geq \dim Y \geq \dim X + 1 = \dim N + 1.$$

(2) We proceed by contradiction. Suppose $\mathrm{vol}(M_i) > v_0 > 0$ for some constant v_0 independent of i. By Proposition 2.12, there is a uniform bound V with $\mathrm{vol}(\partial M_i) \leq V$. Choose any $\epsilon_0 \in (0,1)$ and $t_0 \in (0, v_0/(2V))$, and perform the extension procedure with warping function as in Section 2.3. Then C_{M_i} has volume

$$\mathrm{vol}(C_{M_i}) < Vt_0 < \tfrac{1}{2}v_0.$$

Passing to a subsequence, we may assume that $\widetilde{M}_i$ converges to Y. Since $\mathrm{vol}(\widetilde{M}_i) \geq v_0$, we have $\dim Y = n$. It follows from the volume convergence that

$$\mathrm{vol}(Y) = \lim \mathrm{vol}(\widetilde{M}_i) \geq v_0.$$

However,

$$\mathrm{vol}(Y) = \mathrm{vol}(Y \setminus X) + \mathrm{vol}(X) = \mathrm{vol}(C_0 \times_\phi [0, t_0]) < V_0 t_0 \leq \tfrac{1}{2}v_0,$$

which is a contradiction. □

Remark 3.14 Wong proved $\dim M_i > \dim N$ in [31, Lemma 1] under the condition that N is an absolute Poincaré duality space. In [35], we shall show that if N is a closed topological manifold or a closed Alexandrov space, then M_i inradius collapses. Hence Corollary 3.13 gives another version of Wong's result. It should also be noted that the conclusion of Corollary 3.13 holds for limit spaces of inradius collapsed manifolds with respect to the pointed Gromov–Hausdorff topology (see Corollary 6.2).

Definition 3.15 In view of Lemma 3.3 and Proposition 3.11, we make an identification $N = X^{\mathrm{int}}$ and set, for $k = 1, 2$,

$$N_k = X_k := \{x \in X \,|\, \#\eta_0^{-1}(x) = k\}, \quad C_0^k := \{p \in C_0 \mid \eta_0(p) \in X_k\}.$$

3.2 Examples

We exhibit some examples of collapse of manifolds with boundary. All the examples except Example 3.23 are inradius collapses.

Example 3.16 Let $\mathbb{S}^n(r) := \bigl\{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} (x_i)^2 = r^2\bigr\}$. For $0 \le a < r$ and small $\epsilon > 0$, define $M_\epsilon := \{x \in \mathbb{S}^n(r) \mid a \le x_{n+1} \le a + \epsilon\}$. Then $K_{M_\epsilon} = 1/r^2$ and

$$-\frac{a+\epsilon}{r\sqrt{r^2 - (a+\epsilon)^2}} \le \Pi_{\partial M_\epsilon} \le \frac{a}{r\sqrt{r^2 - a^2}}.$$

Now M_ϵ inradius collapses to $N := \mathbb{S}^{n-1}(\sqrt{r^2 - a^2}\,)$, where the limit space is an Alexandrov space with curvature $\ge 1/(r^2 - a^2)$. Note that $N_2 = N$, and that the limit Y of $\widetilde{M}_\epsilon$ is isometric to the form

$$Y = (\mathbb{S}^{n-1}(\sqrt{r^2 - a^2}\,) \amalg \mathbb{S}^{n-1}(\sqrt{r^2 - a^2}\,)) \times_\phi [0, t_0]/(f(x), 0) \sim (x, 0),$$

where f is the canonical involution on $\mathbb{S}^{n-1}(\sqrt{r^2 - a^2}\,) \amalg \mathbb{S}^{n-1}(\sqrt{r^2 - a^2}\,)$. Equivalently, Y is isometric to the warped product

$$\mathbb{S}^{n-1}(\sqrt{r^2 - a^2}\,) \times_{\tilde{\phi}} [-t_0, t_0],$$

where $\tilde{\phi}(t) = \phi(|t|)$.

Example 3.17 Let $T^2 \subset \mathbb{R}^3$ be a torus smoothly imbedded in $\mathbb{R}^3$, and let M_ϵ be a closed ϵ–neighborhood of T^2 in $\mathbb{R}^3$ for small $\epsilon > 0$. Then, as $\epsilon \to 0$, M_ϵ inradius collapses to T^2, where the limit space T^2 has negative curvature somewhere, while M_ϵ is flat.

Examples 3.16 and 3.17 show that the lower Alexandrov curvature bound of the limit in Theorem 1.2 really depends on the lower sectional curvature bound $K_M \geq \kappa$ and $\lambda \geq |\Pi_{\partial M}|$.

Example 3.18 [31] Let $N \subset \mathbb{R}^2 \times 0 \subset \mathbb{R}^3$ be a nonconvex domain with smooth boundary, and let M'_ϵ denote the closure of ϵ–neighborhood of N in $\mathbb{R}^3$. After a slight smoothing of M'_ϵ, we obtain a flat Riemannian manifold M_ϵ with boundary such that $\Pi_{\partial M_\epsilon} \geq -\lambda$ for some $\lambda > 0$ independent of ϵ. Note that M_ϵ inradius collapses to N, where N has no lower Alexandrov curvature bound.

This example shows that Theorem 1.2 does not hold if one drops the upper bound $\lambda \geq \Pi_{\partial M}$.

Example 3.19 Let $N \subset \mathbb{R}^2$ be the union of the unit circle $\{(x,y) \mid x^2+y^2=1\}$ and the segment $\{(x,y) \mid x=0,\ -1 \leq y \leq 1\}$. Let M_ϵ be the intersection of the closed ϵ–neighborhood of N in $\mathbb{R}^2$ and the unit disk $\{(x,y) \mid x^2+y^2 \leq 1\}$. After slight smoothing of M_ϵ, it is a compact surface with $K_{M_\epsilon} \equiv 0$ and $\mathrm{II}_{\partial M_\epsilon} \leq \lambda^2$ for some λ. However $\inf \mathrm{II}_{\partial M_\epsilon} \to -\infty$ as $\epsilon \to 0$, and M_ϵ inradius collapses to N, which is not an Alexandrov space with curvature bounded below.

This example shows that Theorem 1.2 does not hold if one drops the lower bound $-\lambda^2 \leq \mathrm{II}_{\partial M}$.

Example 3.20 Let $\pi\colon P \to N$ be a Riemannian double covering between closed Riemannian manifolds with the deck transformation $\varphi\colon P \to P$. Define $\Phi\colon P \times [-\epsilon,\epsilon] \to P \times [-\epsilon,\epsilon]$ by

$$\Phi(x,t) = (\varphi(x), -t),$$

and consider $M_\epsilon := P \times [-\epsilon,\epsilon]/\Phi$, which is a twisted I–bundle over N. Note that $M_\epsilon \in \mathcal{M}(n,\kappa,0,d)$ for some κ and d, and that M_ϵ inradius collapses to N as $\epsilon \to 0$. In this case, we have $N_2 = N$. Note that the limit Y of $\widetilde{M}_\epsilon$ is isometric to the form

$$Y = P \times_\phi [0,t_0]/(\varphi(x),0) \sim (x,0),$$

or, equivalently, Y is doubly covered by the warped product

$$P \times_{\tilde{\phi}} [-t_0, t_0].$$

Example 3.21 Let N be a convex domain in $\mathbb{R}^{n-1} \times 0 \subset \mathbb{R}^{n+1}$ with smooth boundary. Let M'_ϵ denote the intersection of the boundary of the ϵ–neighborhood of N in $\mathbb{R}^{n+1}$

with the upper half space $H_+ = \{(x_1, \dots, x_{n+1}) \mid x_{n+1} \ge 0\}$. After a slight smoothing of M'_ϵ, we obtain a nonnegatively curved Riemannian manifold M_ϵ with totally geodesic boundary. Note that M_ϵ inradius collapses to N as $\epsilon \to 0$. Note also that $(\partial M_\epsilon)^{\mathrm{int}}$, a smooth approximation of the boundary of ϵ–neighborhood of N in $\mathbb{R}^n$, converges to the double $D(N)$ of N. It follows that $N_1 = \partial N$ and $N_2 = N \setminus \partial N$, and that the limit Y of $\widetilde{M}_\epsilon$ is isometric to the form

$$Y = D(N) \times_\phi [0, t_0]/(r(x), 0) \sim (x, 0),$$

where $r\colon D(N) \to D(N)$ denotes the canonical reflection of $D(N)$.

Next let us consider more general examples. The following ones come from Example 1.2 in [32], where general examples of collapse of closed manifolds were given.

Example 3.22 Let $\hat{\pi}\colon M \to N$ be a fiber bundle over a closed manifold N with fiber F having nonempty boundary and with the structure group G such that

(1) G is a compact Lie group;

(2) F has a G–invariant metric g_F of nonnegative curvature which smoothly extends to the double $D(F)$.

Fix a bi-invariant metric b on G and a metric h on N. Let $\pi\colon P \to N$ be the principal G–bundle associated with $\hat{\pi}\colon M \to N$. Define a G–invariant metric g_ϵ on P by

$$g_\epsilon(u, v) = h(d\pi(u), d\pi(v)) + \epsilon^2 b(\omega(u), \omega(v)),$$

where ω is a G–connection on P. Define a metric $\tilde{g}_\epsilon$ on $P \times D(F)$ as

$$\tilde{g}_\epsilon = g_\epsilon + \epsilon^2 g_F.$$

For the G–action on $P \times D(F)$ defined by $(p, f) \cdot g = (pg, g^{-1} f)$, $\tilde{g}_\epsilon$ is G–invariant and invariant under the action of reflection of $D(F)$. Therefore it induces a metric $g_{D(M),\epsilon}$ on $D(M) = P \times D(F)/G$. Since $g_{D(M),\epsilon}$ is invariant under the action of reflection of $D(M)$, it induces a metric $g_{M,\epsilon}$ on M with totally geodesic boundary such that $(M, g_{M,\epsilon})$ inradius collapse to (N, h) under a lower sectional curvature bound.

Example 3.23 Let M be a compact manifold with boundary, and suppose that a compact Lie group of positive dimension effectively acts on M which extends to the action on $D(M)$. Suppose that $D(M)$ has a G–invariant and reflection-invariant smooth metric g. As in Example 1.2 of [32], one can construct a metric $g_{D(M),\epsilon}$ on $D(M)$ which collapses to $(D(M), g_{D(M),\epsilon})/G$ under a lower curvature bound. It follows that the metric $(M, g_{M,\epsilon})$ induced by $g_{D(M),\epsilon}$ also collapses to $(M, g_{M,\epsilon})/G$ under a lower curvature bound. Note that $(M, g_{M,\epsilon})$ has totally geodesic boundary.

4 Metric structure of limit spaces

Let $X \subset Y$ and N be as in Section 3. The main purpose of this section is to show that Y and N are actually isometric to C/η_0 and C_0/η_0, respectively. To study how this gluing is made, we first analyze the tangent cones of C, C_0, Y and X at gluing points, and their relations via the differential $d\eta_0$ of the gluing map η_0. It turns out that the identification map η_0 preserves length of curves. Finally, we see that N is isometric to a quotient of C_0^{int} by an isometric $\mathbb{Z}_2$–action. (see Proposition 4.30), which implies Theorems 1.2 and 4.32.

4.1 Preliminary argument

In this subsection, we study geodesic behavior in C and the property of a rescaling limit of the map $\eta\colon C \to Y$. These will be useful in the next subsection to investigate geodesic behavior in Y.

Let $\tilde{\pi}\colon C \to C_0$ and $\pi\colon Y \to X$ be the projections. To be precise, let $\pi(y) := \eta_0 \circ \tilde{\pi}(\eta^{-1}(y))$, which is a surjective Lipschitz map. For every $p \in C_0$, let $\tilde{\gamma}_+(t) = (p, t)$ and $\gamma_+(t) = \eta(\tilde{\gamma}(t))$ for $t \in [0, t_0]$. We call $\tilde{\gamma}_+$ (resp. γ_+) a *perpendicular to* C_0 (resp. to X) at p (resp. at $\eta_0(p)$). The map $\tilde{\pi}$ and π are the projections along perpendiculars. Note that $\eta\colon C \setminus C_0 \to Y \setminus X$ is a locally isometric bijective map. Therefore $C \setminus C_0$ and $Y \setminus X$ are isometric to each other with respect to their *length* metrics.

For simplicity, we use the notation

$$C_t := \{x \in C \mid d(C_0, x) = t\}, \quad C_t^Y := \{y \in Y \mid d(X, y) = t\}$$

for every $t \in (0, t_0]$. We also denote by

$$\tilde{\pi}_t\colon C \to C_t, \quad \pi_t\colon Y \setminus X \to C_t,$$

the canonical projections along perpendiculars. Recall that

$$X_1 = \{x \in X \mid \#\eta_0^{-1}(x) = 1\}, \quad X_2 = \{x \in X \mid \#\eta_0^{-1}(x) = 2\},$$
$$C_0^k = \{p \in C_0 \mid \eta_0(p) \in X_k\} \quad \text{for } k = 1, 2.$$

First we investigate the behavior of geodesics in C. To do this we make use of the Gromov–Hausdorff convergence $C_{M_i} \to C$.

Recall that for every $t \in [0, t_0]$, we set

$$C_{M_i,t} = \{x \in C_{M_i} \mid d(x, \partial M_i) = t\}.$$

We also use t to denote the distance functions on C and C_{M_i} from C_0 and ∂M_i, respectively.

Let $\gamma\colon [0,\ell] \to C$ be a unit-speed geodesic, and $\xi = \frac{\partial}{\partial t}$ the unit vector field on C. Take a geodesic γ_i in C_{M_i} such $\gamma_i \to \gamma$. We denote by Π^i_t the second fundamental form of $C_{M_i,t}$,

$$\Pi^i_t(V,W) = -\langle \nabla_V \xi_i, W\rangle \quad \text{for } V, W \in T(C_{M_i,t}),$$

where $\xi_i = \frac{\partial}{\partial t}$ is the unit vector field on C_{M_i}. Consider the function $\rho_i(s) = t(\gamma_i(s)) = |\gamma_i(s), \partial M_i|$. We have

$$\begin{aligned}\rho_i'(s) &= \langle \xi_i(\gamma_i(s)), \dot\gamma_i(s)\rangle,\\ \rho_i''(s) &= \langle \nabla_{\dot\gamma_i^T}\xi_i, \dot\gamma_i^T\rangle = -\Pi(\dot\gamma_i^T, \dot\gamma_i^T) = \frac{\phi'(\rho_i(s))}{\phi(\rho_i(s))}|\dot\gamma_i^T(s)|^2,\end{aligned}$$

where $\dot\gamma_i^T(s)$ is the component of $\dot\gamma_i(s)$ tangent to $C_{M_i,\rho_i(s)}$. Note that $0 \ge \rho_i''(s) \ge -c$ for some uniform constant $c > 0$. In particular, we have:

Lemma 4.1 *ρ_i and ρ are concave functions.*

Lemma 4.2 *For every $t \in [0, t_0]$ and $p_1, p_2 \in C_t$, we have*

$$\left| \frac{|p_1,p_2|_{C_t^{\text{int}}}}{|p_1,p_2|_C} - 1\right| < O(|p_1,p_2|_C^2).$$

Proof Let $\gamma\colon [0,\ell] \to C$ be a unit-speed minimal geodesic joining p_1 to p_2. Take a unit-speed minimal geodesic $\gamma_i\colon [0,\ell_i] \to C_{M_i}$ such that $\gamma_i \to \gamma$ under the Gromov–Hausdorff convergence $C_{M_i} \to C$. We may assume that $\rho_i(\gamma_i(0)) = \rho_i(\gamma_i(\ell_i)) = t$. Putting

$$\rho_i(s) = \rho_i(\gamma_i(s)) = |\gamma_i(s), \partial M_i|,$$

$\rho_i(s)$ takes a maximum $t_i = \rho_i(u_i) > t$ at some $u_i \in (0,\ell)$. By the mean value theorem, we obtain

$$\frac{\rho_i(u_i) - t}{u_i} = \rho_i'(v_i), \quad \frac{t - \rho_i(u_i)}{\ell_i - u_i} = \rho_i'(v_i'), \quad \frac{\rho_i'(v_i') - \rho_i'(v_i)}{v_i' - v_i} = \rho''(w_i)$$

for some $0 < v_i < u_i < v_i' < \ell$ and $v_i < w_i < v_i'$. Adding the first two equalities, we get

$$\rho_i(u_i) - t \le \frac{(\ell_i - u_i)u_i}{\ell_i}(v_i' - v_i)(-\rho_i''(w_i)) \le c|\gamma_i(0), \gamma_i(\ell_i)|^2. \tag{4-1}$$

Setting $t^* := \max_{[0,\ell]} \rho$ and letting $i \to \infty$, we have

$$t^* - t = \max_{[0,\ell]} \rho - t \le c|p_1, p_2|^2, \tag{4-2}$$

and hence

$$\left| \frac{\phi(t)}{\phi(t^*)} - 1 \right| \le c'|p_1, p_2|^2. \tag{4-3}$$

Let $\pi_t \colon C \to C_t$ be the canonical projection. Since π_t has Lipschitz constant $\dfrac{\phi(t)}{\phi(t^*)}$ on the domain bounded by C_t and C_{t^*}, it follows from (4-3) that

$$|p_1, p_2|_{C_t^{\mathrm{int}}} \le L(\pi_t \circ \gamma) \le \frac{\phi(t)}{\phi(t^*)}|p_1, p_2| < (1 + O(|p_1, p_2|^2))|p_1, p_2|. \tag{4-4}$$

This completes the proof. □

Lemma 4.3 *For every $p_1, p_2 \in C_t$ and unit-speed minimal geodesic $\gamma \colon [0, \ell] \to C$ joining p_1 to p_2, we have*

$$\rho'(0) \le C|p_1, p_2|,$$

where $\rho(s) = |\gamma(s), C_t|$.

Proof Let $\rho(s)$ take the maximum at $s = s_0$. Using the mean value theorem, we obtain $-\rho'(0)/s_0 \ge \inf \rho'' \ge -c$, from which the conclusion is immediate. □

Next we discuss a rescaling limit of the map $\eta \colon C \to Y$. Fix $p \in C_0$ and $x = \eta_0(p) \in X$, and let t_i be an arbitrary sequence of positive numbers with $\lim t_i = 0$. Passing to a subsequence, we may assume that

$$\eta_i = \eta \colon \left(\frac{1}{t_i} C, p\right) \to \left(\frac{1}{t_i} Y, x\right)$$

converges to a 1–Lipschitz map

$$\eta_\infty \colon (T_p(C), o_p) \to (T_x(Y), o_x)$$

between the tangent cones of the Alexandrov spaces. We may also assume that $\left(\frac{1}{t_i} X, x\right)$ converges to a closed subset $(T_x^*(X), o_x)$ of $(T_x(Y), o_x)$ under the convergence $\left(\frac{1}{t_i} Y, x\right) \to (T_x Y, o_x)$.

Sublemma 4.4 *$\eta_\infty\colon T_p(C)\setminus T_p(C_0)\to T_xY\setminus T_x^*(X)$ is a bijective local isometry.*

Proof Let $\tilde\rho=|\cdot,C_0|$, $\rho=|\cdot,X|$. Under the $1/t_i$–rescaling, we may assume that $\tilde\rho$ and ρ converge to the maps

$$\tilde\rho_\infty=|\cdot,T_p(C_0)|,\quad \rho_\infty=|\cdot,T_x^*(X)|,$$

respectively, satisfying $\tilde\rho_\infty=\rho_\infty\circ\eta_\infty$. For any $\tilde w\in T_p(C)\setminus T_p(C_0)$, let $\epsilon=\tilde\rho_\infty(\tilde w)$ and $w=\eta_\infty(\tilde w)$. Since $\rho_\infty(w)=\epsilon$, it is easily checked that $\eta_\infty\colon B\big(\tilde w,\frac{\epsilon}{2}\big)\to B\big(w,\frac{\epsilon}{2}\big)$ is an isometry.

Next let us show that $\eta_\infty\colon T_p(C)\setminus T_p(C_0)\to T_xY\setminus T_x^*(X)$ is bijective. Suppose that $w:=\eta_\infty(\tilde w_1)=\eta_\infty(\tilde w_2)$ for $\tilde w_j\in T_p(C)\setminus T_P(C_0)$. Take $q_1^i,q_2^i\in C$ such that q_j^i converges to $\tilde w_j$ under the $1/t_i$–rescaling. Let $\eta(q_j^i)=y_j^i$. Since y_j^i converges to the same point w, any minimal geodesic joining y_1^i and y_2^i does not meet X. This implies that $|q_1^i,q_2^i|=|y_1^i,y_2^i|$. However this must imply that $\tilde w_1=\tilde w_2$. Hence η_∞ is injective on $T_p(C)\setminus T_p(C_0)$. It is easy to see that $\eta_\infty\colon T_p(C)\setminus T_p(C_0)\to T_xY\setminus T_x^*(X)$ is surjective, and hence the proof is omitted. □

4.2 Spaces of directions and differential of η_0

In this subsection, we study the spaces of directions of C, C_0, Y and X at the points where the gluing is done, and the relation between them. We also study the differential of the gluing map η_0 at those points.

Lemma 4.5 *For every $p\in C_0$, let $\tilde\gamma_+(t)=(p,t)$ and $\gamma_+(t)=\eta(\tilde\gamma(t))$. Then*

(1) *$\Sigma_p(C)$ is isometric to the half-spherical suspension $\{\tilde\gamma_+'(0)\}*\Sigma_p(C_0)$;*

(2) *for every $s\in(0,t_0)$, $\Sigma_{(p,s)}(C)$ and $\Sigma_{\eta(p,s)}(\eta(C))$ are isometric to the spherical suspensions $\{\pm\tilde\gamma_+'(s)\}*\Sigma_p(C_0)$ and $\{\pm\gamma_+'(s)\}*\Sigma_p(C_0)$, respectively.*

Proof From the suspension structure $C=C_0\times_\phi[0,t_0]$, obviously we have $T_p(C)=T_p(C_0)\times[0,\infty)$, which implies the conclusion (1). Since both $\tilde\gamma_+$ and γ_+ are geodesic, the splitting theorem shows (2). □

Lemma 4.6 *For every $x\in X$ and $\xi\in T_x(Y)\setminus K(\Sigma_x(X))$ which is not a perpendicular direction, assume that there is a geodesic $\gamma\colon[0,\ell]\to Y$ with $\gamma'(0)=\xi$, and let*

$$\tilde\gamma=\eta^{-1}(\gamma),\quad \tilde\sigma=\tilde\pi\circ\tilde\gamma,\quad \sigma=\pi\circ\gamma,\quad p:=\tilde\gamma(0).$$

Let $\tilde{\gamma}_+$ *be the perpendicular to* C_0 *at* p, *and set* $\gamma_+ = \eta(\tilde{\gamma}_+)$. *Put*

$$\tilde{\xi} = \tilde{\gamma}'(0), \quad \tilde{\xi}_+ = \tilde{\gamma}'_+(0), \quad \tilde{v} = \tilde{\sigma}'(0), \quad \xi_+ = \gamma'_+(0).$$

Then

(1) σ *defines a unique vector* $v = \sigma'(0) \in K(\Sigma_x(X))$ *and we have*

$$\angle(\xi_+, \xi) = \angle(\tilde{\xi}_+, \tilde{\xi}), \quad \angle(\xi, v) = \angle(\tilde{\xi}, \tilde{v}),$$
$$\angle(\xi_+, \xi) + \angle(\xi, v) = \angle(\xi_+, v) = \tfrac{1}{2}\pi; \tag{4-5}$$

(2) *there is a unique limit* $\eta_\infty\colon T_p(C) \to T_x(Y)$ *of* $\eta_t = \eta\colon \big(\frac{1}{t}C, p\big) \to \big(\frac{1}{t}Y, x\big)$ *as* $t \to 0$, *and we have*

$$\eta_\infty(\tilde{v}) = v, \quad |\tilde{v}| = |v|.$$

Proof Let $\zeta \in \Sigma_x(X)$ be a direction defined by the curve σ. By definition, this means that $\zeta = \lim_{i\to\infty} \uparrow_x^{\sigma(t_i)}$ for a sequence $t_i \to 0$. Since γ is minimal, so is $\tilde{\gamma}$. Note that $\tilde{v}$ is uniquely determined since $\tilde{\sigma}$ is a shortest curve. From Lemma 4.5, we have

$$\angle(\tilde{\xi}_+, \tilde{\xi}) + \angle(\tilde{\xi}, \tilde{v}) = \angle(\tilde{\xi}_+, \tilde{v}) = \tfrac{1}{2}\pi. \tag{4-6}$$

Now we show (4-5). Consider the $1/t_i$–rescaling limits,

$$(T_x(Y), o_x) = \lim_{i\to\infty}\Big(\frac{1}{t_i}Y, x\Big), \quad (T_p(C), o_p) = \lim_{i\to\infty}\Big(\frac{1}{t_i}C, p\Big).$$

Let γ_{t_i} (resp. $\tilde{\gamma}_{t_i}$) be the perpendicular to X at $\sigma(t_i)$ (resp. to C_0 at $\tilde{\sigma}(t_i)$). Passing to a subsequence, we may assume that the quadruplet $(\gamma_+, \gamma, \sigma, \gamma_{t_i})$ converges to $(\gamma_{+\infty}, \gamma_\infty, \sigma_\infty, \gamma_{\infty 1})$ under the convergence $\big(\frac{1}{t_i}Y, x\big) \to (T_x(Y), o_x)$. For instance, this explicitly means that the Lipschitz curve $\frac{1}{t_i}\sigma(t_i t)$ converges to a Lipschitz curve $\sigma_\infty(t)$ in $T_x(Y)$. Thus $\gamma_{+\infty}$ and $\gamma_{\infty 1}$ are perpendicular to $T_x^*(X)$ at o_x and $\sigma_\infty(1)$ and γ_∞ is the geodesic from o_x with $\gamma_\infty(1) = \xi$. Here we assume that $\big(\frac{1}{t_i}X, x\big)$ converges to a closed subset $(T_x^*(X), o_x)$ of $T_x(Y), o_x)$.

Similarly passing to a subsequence, we may assume that the quadruplet $(\tilde{\gamma}_+, \tilde{\gamma}, \tilde{\sigma}, \tilde{\gamma}_{t_i})$ converges to $(\tilde{\gamma}_{+\infty}, \tilde{\gamma}_\infty, \tilde{\sigma}_\infty, \tilde{\gamma}_{\infty 1})$ under the convergence $\big(\frac{1}{t_i}C, p\big) \to (T_p(C), o_p)$. Thus $\tilde{\gamma}_{+\infty}$ and $\tilde{\gamma}_{\infty 1}$ are perpendicular to $T_p(C_0)$ at o_p and $\tilde{\sigma}_\infty(1)$ and $\tilde{\gamma}_\infty$ is the geodesic from o_p with $\tilde{\gamma}_\infty(1) = \tilde{\xi}$.

We set

$$\rho(t) = |C_0, \tilde{\gamma}(t)| = |X, \gamma(t)|.$$

Notice that:

(1) $|\xi, \sigma_\infty(1)| = |\tilde{\xi}, \tilde{\sigma}_\infty(1)| = \rho'(0)$.

(2) $\rho'(0) = |\tilde{\xi}| \sin \angle(\tilde{\xi}, \tilde{v})$.

Let $\tilde{\lambda}$ be a minimal geodesic joining $\tilde{\xi} = \tilde{\gamma}_\infty(1)$ to $\tilde{\gamma}_{+\infty}$. Let $\eta_\infty\colon T_p(C) \to T_x(Y)$ be any limit of $\eta_{t_i} = \eta\colon \big(\frac{1}{t_i}C, p\big) \to \big(\frac{1}{t_i}Y, x\big)$. Since η_∞ is 1–Lipschitz, we have

$$|\tilde{\xi}| \sin \angle(\tilde{\xi}_+, \tilde{\xi}) = L(\tilde{\lambda}) = L(\eta_\infty \circ \tilde{\lambda}) \geq |\xi| \sin \angle(\xi_+, \xi),$$

and hence

$$\angle(\xi_+, \xi) \leq \angle(\tilde{\xi}_+, \tilde{\xi}). \tag{4-7}$$

Next we show that

$$\angle(\xi, \zeta) = \angle(\tilde{\xi}, \tilde{v}). \tag{4-8}$$

Put for simplicity

$$\begin{aligned} \tilde{\theta} &:= \angle(\tilde{\xi}, \tilde{v}), \quad \tilde{\theta}_i := \angle(\tilde{\gamma}'(t_i), T_{\tilde{\gamma}(t_i)} C_{\rho(t_i)}), \\ \theta &:= \angle(\xi, \zeta), \quad \theta_i := \angle(\gamma'(t_i), T_{\gamma(t_i)} C^Y_{\rho(t_i)}). \end{aligned}$$

From the warping product structure of C, we easily have

$$\lim_{i\to\infty} \tilde{\theta}_i = \tilde{\theta}.$$

On the other hand, under the convergence $\big(\frac{1}{t_i}C, p\big) \to (T_p(C), o_p)$ (resp. under the convergence $\big(\frac{1}{t_i}Y, x\big) \to (T_xY, o_x)$), we may assume that C_{st_i} converges to some space, denoted by $C_{s\infty}$. (resp. $C^Y_{st_i}$ converges to some space $C^Y_{s\infty}$). Then we have

(1) $\tilde{\theta} = \angle(\tilde{\gamma}'_\infty(0), \tilde{\sigma}'_\infty(0))$;

(2) $\tilde{\theta} = \lim \tilde{\theta}_i = \angle(\tilde{\gamma}'_\infty(1), T_{\tilde{\gamma}_\infty(1)}(C_{\rho'(0)\infty}))$.

Since $\angle\big(\tilde{\gamma}'_\infty(1), \tilde{\gamma}'_{\infty 1}(\rho'(0))\big) = \frac{1}{2}\pi - \tilde{\theta}$, we have

$$\angle o_p \tilde{\gamma}_\infty(1) \tilde{\sigma}_\infty(1)) = \tfrac{1}{2}\pi - \tilde{\theta}. \tag{4-9}$$

On the other hand, since $\eta\colon C \setminus C_0 \to C^Y \setminus X$ is a local isometry, we have

$$\tilde{\theta}_i = \theta_i.$$

From the lower semicontinuity of angles, we have

$$\lim \theta_i = \angle(\gamma'_\infty(1), T_{\gamma_\infty(1)}(C_{\rho'(0)\infty})).$$

It follows from the spherical suspension structure of $\Sigma_{\gamma_\infty(1)} T_p(C)$ that

$$\angle\big(\gamma'_\infty(1), \gamma'_{\infty 1}(\rho'(0))\big) = \tfrac{1}{2}\pi - \angle(\gamma'_\infty(1), T_{\gamma_\infty(1)}(C_{\rho'(0)\infty})) = \tfrac{1}{2}\pi - \tilde{\theta},$$

and hence

$$\angle o_x \xi \sigma_\infty(1) = \tfrac{1}{2}\pi - \tilde{\theta}. \tag{4-10}$$

By (4-9) and (4-10), the two Euclidean triangles $\triangle o_x \xi \sigma_\infty(1)$ and $\triangle o_p \tilde{\xi} \tilde{\sigma}_\infty(1)$ are congruent to each other, and we conclude that $\angle(\xi, \zeta) = \angle(\tilde{\xi}, \tilde{v})$, as required.

The first variation formula immediately implies $\angle(\xi_+, \zeta) \ge \frac{1}{2}\pi$. It follows from (4-7) and (4-8) that

$$\tfrac{1}{2}\pi \le \angle(\xi_+, \zeta) \le \angle(\xi_+, \xi) + \angle(\xi, \zeta) \le \angle(\tilde{\xi}_+, \tilde{\xi}) + \angle(\tilde{\xi}, \tilde{v}) = \tfrac{1}{2}\pi. \tag{4-11}$$

Thus we conclude that

$$\angle(\xi_+, \xi) + \angle(\xi, \zeta) = \angle(\xi_+, \zeta) = \tfrac{1}{2}\pi,$$

which shows the uniqueness of ζ. Namely, σ determines a unique direction at x. Note that

$$\begin{aligned} v := \sigma'(0) = \lim_{i\to\infty} \frac{|x, \sigma(t_i)|}{t_i} \zeta = |o_x, \sigma_\infty(1)|\zeta, \\ |v| = |o_x, \sigma_\infty(1)| = |o_p, \tilde{\sigma}_\infty(1)| = |\tilde{v}|. \end{aligned} \tag{4-12}$$

Since $\eta_\infty(\tilde{v}) = v$, this shows that η_∞ does not depend on the choice of $t_i \to 0$. □

Remark 4.7 The argument in the proof of Lemma 4.6 also shows that

$$|\tilde{\xi}| \cos\tilde{\theta} = |o_p, \tilde{\sigma}_\infty(1)| = |o_x, \sigma_\infty(1)| \le L(\sigma_\infty|_{[0,1]}) \le L(\tilde{\sigma}_\infty|_{[0,1]}) = \cos\tilde{\theta},$$

which implies that

(4-13) σ_∞ is minimizing in the direction $\sigma'(0)$.

Corollary 4.8 *For every $x \in X$ and $\xi \in \Sigma_x(Y) \setminus \Sigma_x(X)$ which is not a perpendicular direction, there is a unique perpendicular direction $\xi_+ \in \Sigma_x(Y)$ to X at x and a unique $v \in \Sigma_x(X)$ such that*

$$\angle(\xi_+, \xi) + \angle(\xi, v) = \angle(\xi_+, v) = \tfrac{1}{2}\pi. \tag{4-14}$$

Proof This immediately follows from Lemma 4.6 and a limit argument. □

By Lemma 4.6, for every geodesic γ in Y starting from $x \in X$ such that $\gamma'(0) \in \Sigma_x(Y) \setminus \Sigma_x(X)$, the Lipschitz curve $\sigma = \pi(\gamma)$ determines a unique direction $[\sigma] \in \Sigma_x(X)$. In general, we call such a direction $[\sigma]$ an *intrinsic direction* if σ is a Lipschitz curve in X starting from x and having a unique direction $[\sigma] = \sigma'(0)$ in the sense that for any sequence $t_i \to 0$, $\uparrow_x^{\sigma(t_i)}$ converges to $[\sigma]$.

The next lemma shows that every direction in $\Sigma_x(X)$ can be approximated by intrinsic directions.

Lemma 4.9 *For every $v \in \Sigma_x(X)$, we have the following*:

(1) *For any perpendicular direction $\xi_+ \in \Sigma_x(Y)$, we have*

$$\angle(\xi_+, v) = \tfrac{1}{2}\pi.$$

(2) *There are intrinsic directions $[\sigma_i] \in \Sigma_x(X)$ satisfying*

$$\lim[\sigma_i] = v.$$

Proof For every $v \in \Sigma_x(X)$ take a sequence $y_i \in X$ with $y_i \to x$ and $v_i := \uparrow_x^{y_i} \to v$. Let $\mu_i\colon [0, s_i] \to Y$ be a minimal geodesic from x to y_i. Let γ_+ be a perpendicular to X at x with $\gamma_+'(0) = \xi_+$. Let λ_i be a minimal geodesic joining $\gamma_+(t_0)$ to y_i. Considering perpendiculars to X through the points of λ_i and taking the limit, we obtain a perpendicular γ_{y_i} to X at y_i. Let $\gamma_i\colon [0, t_i] \to Y$ be a minimal geodesic from x to $\gamma_{y_i}(s_i)$, and set

$$\sigma_i(t) := \pi(\gamma_i(t)), \quad \tilde{\gamma}_i = \eta^{-1}(\gamma_i), \quad \tilde{\sigma}_i = \tilde{\pi}(\tilde{\gamma}_i).$$

By Lemma 4.6, σ_i defines a unique direction $\hat{v}_i \in \Sigma_x(X)$ such that

$$\angle(\xi_+, \xi_i) + \angle(\xi_i, \hat{v}_i) = \angle(\xi_+, \hat{v}_i) = \tfrac{1}{2}\pi, \tag{4-15}$$

where $\xi_i = \gamma_i'(0)$. Note that $y_i = \sigma_i(t_i)$.

We now use an argument similar to that of Lemma 4.6. Consider the convergence

$$\left(\frac{1}{t_i}Y, x\right) \to (T_x(Y), o_x), \quad \left(\frac{1}{t_i}C, p\right) \to (T_p(C), o_p).$$

Passing to a subsequence, we may assume that ξ_i converges to some $\xi \in \Sigma_x(Y) \subset T_x(Y)$. We may also assume that

(a) $\gamma_i(t_i s)$ and $\sigma_i(t_i s)$ converge to a geodesic $\gamma_\infty(s)$ and a Lipschitz curve $\sigma_\infty(s)$, respectively;

(b) $\tilde\gamma_i(t_i s)$ and $\tilde\sigma_i(t_i s)$ converge to geodesics $\tilde\gamma_\infty(s)$ and $\tilde\sigma_\infty(s)$ in $T_p(C)$, respectively.

Let $\eta_\infty\colon (T_p(C), o_p) \to (T_xY, o_x)$ be the 1–Lipschitz map defined in Lemma 4.6(2) as the limit of

$$\eta_i = \eta\colon \left(\frac{1}{t_i}C, p\right) \to \left(\frac{1}{t_i}Y, x\right).$$

Note that $\eta_\infty(\tilde\sigma_\infty(s)) = \sigma_\infty(s)$. Consider the geodesic triangles

$$\Delta_{o_x} := \Delta o_x\gamma_\infty(1)\sigma_\infty(1) \subset T_x(Y),$$
$$\Delta_{o_p} := \Delta o_p\tilde\gamma_\infty(1)\tilde\sigma_\infty(1) \subset T_p(C).$$

An argument similar to that in Lemma 4.6 implies that

$$\angle o_x\gamma_\infty(1)\sigma_\infty(1) = \angle o_p\tilde\gamma_\infty(1)\tilde\sigma_\infty(1). \tag{4-16}$$

It should be remarked that in the case of Lemma 4.6, the geodesic $\gamma\colon [0,\ell] \to Y$ in the direction ξ was given in the beginning, and we considered the points $\gamma(t_i)$ with $t_i \to 0$. On the other hand, in the present case, we have only the geodesic $\gamma_i\colon [0,t_i] \to Y$. Therefore we take a point $z_i \in Y$ instead, in such a way that

$$\tilde\angle x\gamma_i(t_i)z_i > \pi - o_i, \quad |\gamma_i(t_i), z_i| = t_i,$$

where $\lim o_i = 0$. Then, with almost parallel argument, we obtain (4-16) and that Δ_{o_x} and Δ_{o_p} are congruent to each other as Euclidean flat triangles. In particular, we conclude that

$$|o_x, \sigma_\infty(1)| = |o_p, \tilde\sigma_\infty(1)|. \tag{4-17}$$

Since $L(\sigma_\infty) \le L(\tilde\sigma_\infty)$, this implies that σ_∞ is a minimal geodesic in the direction v. As in Lemma 4.6, (4-16) also implies that

$$\angle(\xi_+, \xi) + \angle(\xi, v) = \angle(\xi_+, v) = \tfrac{1}{2}\pi, \tag{4-18}$$

where $\xi = \lim \xi_i$. This proves (1). It follows from (4-15) and (4-18) that

$$v = \lim \hat v_i, \tag{4-19}$$

which shows (2).

Furthermore, it follows from

$$\frac{L(\tilde\sigma_i)}{t_i} \ge \frac{L(\sigma_i)}{t_i} \ge \frac{L(\mu_i)}{t_i}, \quad \lim \frac{L(\tilde\sigma_i)}{t_i} = |o_p, \tilde\sigma_\infty(1)|, \quad \lim \frac{L(\mu_i)}{t_i} = |o_x, \sigma_\infty(1)|$$

that

$$\lim \frac{L(\sigma_i)}{t_i} = L(\sigma_\infty). \tag{4-20}$$ □

Let $\Sigma_x^0(X)$ denote the set of intrinsic directions $[\sigma] \in \Sigma_x(Y)$ of Lipschitz curves $\sigma\colon [0,\epsilon) \to X$ starting from x such that the direction $[\sigma]$ is uniquely determined. From Lemma 4.9, we immediately have the following:

Proposition 4.10 *$\Sigma_x(X)$ coincides with the closure of $\Sigma_x^0(X)$ in $\Sigma_x(Y)$.*

The space of direction $\Sigma_x(X)$ was originally defined in an extrinsic way; see Section 2.2. Proposition 4.10 shows that it coincides with the one defined in an intrinsic way.

For $x \in X_1$ (resp. $x \in X_2$), let $\xi_+ \in \Sigma_x(Y)$ (resp. $\xi_\pm \in \Sigma_x(Y)$) be the unique (resp. the two) direction (resp. directions) of the perpendicular (resp. perpendiculars) to X at x.

Corollary 4.11 *For every $x \in X$, we have the following*:

(1) *If $x \in X_1$, then*

$$\Sigma_x(X) = \left\{v \in \Sigma_x(Y) \mid \angle(\xi_+, v) = \tfrac{1}{2}\pi\right\}.$$

(2) *If $x \in X_2$, then $\Sigma_x(Y)$ is isometric to the spherical suspension $\{\xi_\pm\} * \Sigma_x(X)$.*

In either case, $\Sigma_x(X)$ is an Alexandrov space with curvature ≥ 1 of dimension equal to $\dim Y - 2$.

Proof (1) is a direct consequence of Corollary 4.8 and Lemma 4.9. (2) is a direct consequence of Lemma 3.3, Corollary 4.8 and Lemma 4.9. In an Alexandrov space Σ with curvature ≥ 1, for any $\xi \in \Sigma$, the set $\left\{v \in \Sigma \mid |v,\xi| \geq \tfrac{1}{2}\pi\right\}$ is convex, which implies the last conclusion. □

Our next purpose is to show the following:

Proposition 4.12 *Under the convergence $\lim_{\delta\to 0}\left(\tfrac{1}{\delta}Y, x\right) = (T_x(Y), o_x)$, $\left(\tfrac{1}{\delta}X, x\right)$ converges to the Euclidean cone $\left(K(\Sigma_x(X)), o_x\right)$ as $\delta \to 0$.*

We set

$$T_x(X) = K(\Sigma_x(X))$$

and call it the *tangent cone of X at x*.

For the proof of Proposition 4.12, we need three lemmas.

Lemma 4.13 *For every minimal geodesic $\gamma\colon [0,\ell]\to Y$ joining any $x\in X$ and $y\in Y$, the curve $\sigma(t):=\pi(\gamma(t))$ has a unique direction at $t=0$, and hence defines an intrinsic direction $[\sigma]\in\Sigma_x(X)$.*

Proof By Lemma 4.6, it suffices to consider only the case $\xi:=\gamma'(0)\in\Sigma_x(X)$. Suppose that for sequences $s_j\to 0$ and $t_j\to 0$ we have limits

$$\eta:=\lim_{j\to\infty}\uparrow_x^{\sigma(s_j)},\quad \zeta:=\lim_{j\to\infty}\uparrow_x^{\sigma(t_j)}.$$

Take $\xi_i\in\Sigma_x(Y)\setminus\Sigma_x(X)$ such that $\xi_i\to\xi$ and the geodesic γ_i in the direction ξ_i is defined. Set $\sigma_i=\pi\circ\gamma_i$. By Lemma 4.6, σ_i defines an intrinsic direction $[\sigma_i]\in\Sigma_x(X)$, and passing to a subsequence we may assume that

$$\angle(\xi_+,\xi_i)+\angle(\xi_i,[\sigma_i])=\angle(\xi,[\sigma_i])=\tfrac{1}{2}\pi$$

for some perpendicular direction ξ_+ at x. It follows that $\angle(\xi_i,[\sigma_i])\to 0$ and $[\sigma_i]\to\xi$. From $\xi_i\to\xi$, we have

$$|\gamma_i(t),\gamma(t)|<o_i t,$$

where $\lim o_i=0$. Since π is 1–Lipschitz, it follows that

$$|\sigma_i(t),\sigma(t)|<o_i t, \tag{4-21}$$

which implies that

$$|\sigma_i(s_j),\sigma(s_j)|<o_i s_j,\quad |\sigma_i(t_j),\sigma(t_j)|<o_i t_j. \tag{4-22}$$

Passing to a subsequence, we may assume that there are limits

$$\alpha=\lim_{j\to\infty}\frac{|x,\sigma(s_j)|}{s_j},\quad \alpha_i=\lim_{j\to\infty}\frac{|x,\sigma_i(s_j)|}{s_j},$$
$$\beta=\lim_{j\to\infty}\frac{|x,\sigma(t_j)|}{t_j},\quad \beta_i=\lim_{j\to\infty}\frac{|x,\sigma_i(t_j)|}{t_j}.$$

Inequality (4-21) implies $\alpha_i\to\alpha$ and $\beta_i\to\beta$. On the other hand, since $\xi\in\Sigma_x(X)$, (4-12) shows $\alpha_j\to 1$ and $\beta_j\to 1$, Thus we have $\alpha=\beta=1$. Since (4-22) implies

$$|\alpha_i[\sigma_i],\alpha\eta|\le o_i,\quad |\beta_i[\sigma_i],\beta\zeta|\le o_i, \tag{4-23}$$

we conclude that

$$|[\sigma_i],\eta|\le o_i,\quad |[\sigma_i],\zeta|\le o_i,$$

and hence the uniqueness $\eta=\zeta=\xi$. □

Lemma 4.14 *For every $x, y \in X$ and every minimal geodesic $\mu\colon [0,\ell] \to Y$ joining them, let $\sigma = \pi \circ \mu$ and set $\rho(t) = |\mu(t), X|$. Then we have*

(1) $\max \rho \le O(|x,y|^2)$;

(2) $\angle(\mu'(0), [\sigma]) \le O(|x,y|)$;

(3) $|L(\sigma)/L(\mu) - 1| < O(|x,y|^2)$.

Proof (1) Let $\rho(s^*) = \max\rho$ and take $0 \le a < b \le \ell$ such that $s^* \in (a,b)$, $\rho > 0$ on (a,b) and $\rho(a) = \rho(b) = 0$. Then $\tilde\mu = \eta^{-1}(\mu|_{[a,b]})$ and $\tilde\sigma = \tilde\pi(\tilde\mu)$ are well defined. By (4-1), we have

$$\rho(s^*) = |\tilde\mu(s^*), C_0| \le O(|\tilde\mu(a), \tilde\mu(b)|^2) \le O(|\mu(a), \mu(b)|^2) \le O(|x,y|^2).$$

(2) We may assume $\mu'(0) \ne [\sigma]$. Take the smallest $s_1 \in (0,\ell]$ satisfying $\mu(s_1) \in X$. Note that $\tilde\mu = \eta^{-1}(\mu|_{0,s_1]})$ is well defined and $|x, \mu(s_1)| = |\tilde\mu(0), \tilde\mu(s_1)|$. Since $\rho(s) = |\mu(s), X| = |\tilde\mu(s), C_0|$ for $0 \le s \le s_1$, Lemmas 4.6 and 4.3 imply

$$\rho'(0) = \sin\angle(\mu'(0), [\sigma]) \le C|\tilde\mu(0), \tilde\mu(s_1)| = C|x, \mu(s_1)| \le C|x,y|.$$

(3) Take at most countable disjoint open intervals (a_i, b_i) of $[0,\ell]$ such that

- $\mu(a_i), \mu(b_i) \in X$ and $\mu((a_i,b_i)) \subset Y \setminus X$;
- $\mu(s) \in X$ for all $s \in J := [0,\ell] \setminus \cup(a_i,b_i)$.

Set

$$\mu_i = \mu|_{[a_i,b_i]}, \quad \sigma_i = \sigma|_{[a_i,b_i]}, \quad \tilde\mu_i = \eta^{-1}(\mu_i), \quad \tilde\sigma_i = \eta^{-1}(\sigma_i).$$

Let $L(J)$ denote the measure of J. Since $L(\tilde\mu_i) = L(\mu_i)$, $L(\tilde\sigma_i) \ge L(\sigma_i)$ and $\mu(J) = \sigma(J)$, Lemma 4.2 implies that

$$\begin{aligned} L(\sigma) &= \sum L(\sigma_i) + L(J) \\ &\le \sum L(\tilde\sigma_i) + L(J) \\ &\le \sum (1 + O(|\tilde\sigma_i(a_i), \tilde\sigma_i(b_i)|^2) L(\tilde\mu_i) + L(J) \\ &= \sum (1 + O(|\mu(a_i), \mu(b_i)|^2) L(\mu_i) + L(J) \\ &\le (1 + O(|x,y|^2) L(\mu). \end{aligned}$$

□

Lemma 4.15 *For any $v = [\sigma] \in \Sigma^0_x(X)$, if we consider the arc-length parameter of σ, $\sigma(\delta t)$ converges to the geodesic ray $\sigma_\infty(t)$ in $T_x(Y)$ from the origin o_x in the direction v as $\delta \to 0$ under the convergence $\left(\frac{1}{\delta}Y, x\right) \to T_x(Y), o_x)$.*

Proof Since σ determines the unique direction v, we have, for any $0 < R_1 < R_2$,

$$\lim_{\delta\to 0} \uparrow_x^{\sigma(\delta R_1)} = \lim_{\delta\to 0} \uparrow_x^{\sigma(\delta R_2)} .$$

This implies that the image $\sigma_\infty([0,\infty))$ coincides with the ray in the direction v. □

We denote by $\tau(R\,|\,\epsilon)$ a function satisfying $\lim_{\epsilon\to 0}\tau(R\,|\,\epsilon) = 0$ for any fixed R.

Proof of Proposition 4.12 We have to show that the Gromov–Hausdorff distance between $\big(\frac{1}{\delta}B(x,\delta R; X), x\big)$ and $(B(o_x, R; K(\Sigma_x(X)), o_x)$ converges to zero as $\delta\to 0$ for any fixed $R>0$. For every small $\epsilon>0$ take an ϵ–dense subset $\{[\sigma_i]\}_{i=1}^I$ of $\Sigma_x^0(X)$, and put $K := [R/\epsilon]+1$. Taking small enough δ, we may assume that σ_i are defined on $[0,\delta R]$ and that σ_i can be written as $\sigma_i = \pi(\gamma_i)$, where γ_i is a minimal geodesic in Y joining x to $\sigma_i(\delta R)$. Let us consider the sets

$$\begin{aligned} N_\infty^\epsilon &:= \Big\{\frac{kR}{K}[\sigma] \;\Big|\; 1\le i\le I,\ 0\le k\le K\Big\} \subset B\big(o_x, R; K(\Sigma_x(X))\big),\\ N^\epsilon &:= \Big\{\sigma_i\Big(\frac{\delta kR}{K}\Big) \;\Big|\; 1\le i\le I,\ 0\le k\le K\Big\} \subset \frac{1}{\delta}B(x,\delta R; X). \end{aligned}$$

First we show that both N_∞^ϵ and N^ϵ are $(\tau(R|\epsilon)+\tau(R|\delta))$–dense. For simplicity, set

$$v_{i,k} = \frac{kR}{K}[\sigma], \quad x_{i,k} = \sigma_i\Big(\frac{\delta kR}{K}\Big).$$

For every $y\in B(x,\delta R; X)$, let $\gamma\colon [0,\ell]\to Y$ be a minimal geodesic joining x to y, and let $\sigma := \pi(\gamma)$. Choose i and k with $\angle([\sigma],[\sigma_i]) < \epsilon$ and $|\ell - kR/K| < R/K < \epsilon$. Since Lemma 4.14 implies that

$$\angle(\gamma'(0),\gamma_i'(0)) \le \angle(\gamma'(0),[\sigma]) + \angle([\sigma],\sigma_i]) + \angle([\sigma_i],\gamma_i'(0)) \le \epsilon + 2\tau(\delta),$$

we obtain

$$|\gamma(\ell),\gamma_i(\ell)| \le C\ell(\epsilon+2\tau(\delta)).$$

It follows from Lemma 4.14 that

$$\begin{aligned} |y, x_{i,k}| &\le |\gamma(\ell),\gamma_i(\ell)| + |\gamma_i(\ell),\sigma_i(\ell)| + |\sigma_i(\ell), x_{i,k}|\\ &\le c\ell(\epsilon+2\tau(\delta)) + (\delta R)^2 + \frac{C\delta R}{K}\\ &\le \delta(\tau(R\,|\,\epsilon)+\tau(R\,|\,\delta)). \end{aligned}$$

Thus N^ϵ is $(\tau(R|\epsilon)+\tau(R|\delta))$–dense in $\frac{1}{\delta}B(x,\delta R)$.

For every $v \in B\big(o_x, R; K(\Sigma_x(X))\big)$, take i and k satisfying $\angle(v, [\sigma_i]) < \epsilon$ and $||v| - kR/K| < R/K < \epsilon$. Then we have

$$|v, v_{i,k}| \le \frac{R}{K} + \epsilon R = \tau(R \,|\, \epsilon).$$

Hence N^{ϵ}_{∞} is $\tau(R|\epsilon)$–dense in $B\big(o_x, R; K(\Sigma_x(X))\big)$.

Finally, define $f\colon N^{\epsilon}_{\infty} \to N^{\epsilon}$ by $f(v_{i,k}) = x_{i,k}$. For simplicity put

$$w_{i,k} = \frac{kR}{K}\gamma_i'(0), \quad y_{i,k} = \gamma_i\Big(\frac{\delta k R}{K}\Big).$$

By Lemma 4.14, we then have

$$\begin{aligned} \big||x_{i,k}, x_{j,\ell}| - |y_{i,k}, y_{j,\ell}|\big| &< 2C(\delta R)^2, \\ \big||y_{i,k}, y_{j,\ell}| - |w_{i,k}, w_{j,\ell}|\big| &< \tau(R \,|\, \delta), \\ \big||w_{i,k}, w_{j,\ell}| - |v_{i,k}, v_{j,\ell}|\big| &< \tau(R \,|\, \delta), \end{aligned}$$

which implies that f is $\tau(R|\delta)$–approximation. In this way, we conclude that

$$d_{\mathrm{GH}}\Big(\Big(\frac{1}{\delta}B(x, \delta R), x\Big), \Big(B(o_x, R; T_x(X))\Big)\Big) < \tau(R \,|\, \epsilon) + \tau(R \,|\, \delta). \qquad \square$$

Lemma 4.16 *Fix any $x \in X$ and take $p \in C_0$ with $\eta_0(p) = x$. Then, for every $y \in X$, there is a point $q \in \eta_0^{-1}(y)$ such that*

$$\left|\frac{|x, y|_Y}{|p, q|_C} - 1\right| < \tau_x(|x, y|_Y),$$

where $\tau_x(t)$ is a function depending on x with $\lim_{t\to 0} \tau_x(t) = 0$.

Proof Suppose the lemma does not hold. Then, since η_0 is 1–Lipschitz, we have a sequence $y_i \in X$ with $\lim y_i = x$ such that for every $q_i \in \eta_0^{-1}(y_i)$,

$$\frac{|x, y_i|_Y}{|p, q_i|_C} < 1 - \epsilon \tag{4-24}$$

for some $\epsilon > 0$ independent of i.

We proceed as in the proof of Lemma 4.9. Let $\mu_i\colon [0, s_i] \to Y$ be a minimal geodesic from x to y_i, and take a perpendicular γ_{y_i} to X at y_i. Let $\gamma_i\colon [0, t_i] \to Y$ be a minimal geodesic from x to $\gamma_{y_i}(s_i)$, and set

$$\sigma_i(t) := \pi(\gamma_i(t)), \quad \tilde\gamma_i = \eta^{-1}(\gamma_i), \quad \tilde\sigma_i = \tilde\pi(\tilde\gamma_i).$$

Let $q_i := \tilde{\sigma}(t_i)$. By passing to a subsequence if necessary, under the convergences

$$\Big(\frac{1}{t_i}Y, x\Big) \to (T_x(Y), o_x), \quad \Big(\frac{1}{t_i}C, p\Big) \to (T_p(C), o_p),$$

we may assume that the triplet $(\mu_i(t_i s), \gamma_i(t_i s), \sigma_i(t_i s))$ (resp. the pair $(\tilde{\gamma}_i(t_i s), \tilde{\sigma}_i(t_i s))$) converges to a triplet $(\mu_\infty(s), \gamma_\infty(s), \sigma_\infty(s))$ (resp. a double $(\tilde{\gamma}_\infty(s), \tilde{\sigma}_\infty(s))$. From (4-17), we see that

$$\lim \frac{|x, y_i|_Y}{t_i} = |o_x, \sigma_\infty(1)| = |o_p, \tilde{\sigma}_\infty(1)| = \lim \frac{|p, q_i|_C}{t_i},$$

which yields a contradiction to the hypothesis (4-24),

$$\lim_{i\to\infty} \frac{|x, y_i|_Y}{|p, q_i|_C} = 1. \qquad \square$$

For any $p \in C_0$, by Lemma 4.6, as $t \to 0$, $\eta_0\colon \big(\frac{1}{t}C_0, p\big) \to \big(\frac{1}{t}X, x\big)$ converges to a 1–Lipschitz map $(d\eta_0)_p\colon T_p(C_0) \to T_x(X)$, which is called the *differential* of η_0 at p. Note that the surjectivity of $\eta_0\colon C_0 \to X$ implies that of $(d\eta_0)_p\colon T_p(C_0) \to T_x(X)$.

Lemma 4.6 immediately implies the following:

Proposition 4.17 *For every $p \in C_0$, the differential $d\eta_0\colon T_p(C_0) \to T_x(X)$ satisfies*

$$|d\eta_0(\tilde{v})| = |\tilde{v}|$$

for every $\tilde{v} \in T_p(C_0)$. In particular, $\eta_0\colon C_0 \to X$ preserves the length of Lipschitz curves in C_0.

By Proposition 4.17, $d\eta_0$ provides a surjective 1–Lipschitz map $d\eta_0\colon \Sigma_p(C_0) \to \Sigma_x(X)$.

Remark 4.18 By Corollary 4.11, $x \in X_2$ is a regular point of Y if and only if the tangent cone $T_x(X)$ is isometric to $\mathbb{R}^{m-1}$, where $m = \dim Y$. For this reason, in that case we call x a *regular point* of X, and set $X^{\mathrm{reg}} := X \cap Y^{\mathrm{reg}}$. Later we show that every $x \in X_1$ is a *singular point* of X unless $X = X_1$ (see Corollary 4.35).

Proposition 4.19 *For every $p \in C_0^2$, we have:*

(1) *the differential $d\eta_p$ provides an isometry $d\eta_p\colon T_p(C) \to T_x^+(Y)$ which preserves the half suspension structures of both $\Sigma_p(C) = \{\tilde{\xi}_+\} * \Sigma_p(C_0)$ and $\Sigma_x^+(Y) := \{\xi_+\} * \Sigma_x(X)$, where $T_x^+(Y) = T_x(X) \times \mathbb{R}_+$.*

(2) *$p \in C_0^{\mathrm{reg}}$ if and only if $x \in X^{\mathrm{reg}}$. In this case, $(d\eta_0)_p\colon T_p(C_0) \to T_x(X)$ is a linear isometry.*

Proof (1) For every $\widetilde{v}_1, \widetilde{v}_2 \in \Sigma_p(C_0)$, put $v_i := d\eta_0(\widetilde{v}_i)$. We show that $\angle(\widetilde{v}_1, \widetilde{v}_2) = \angle(v_1, v_2)$. Let $\widetilde{\xi}_i$ (resp. ξ_i) be the midpoint of the geodesic joining $\widetilde{\xi}_+$ to $\widetilde{v}_i$ (resp. ξ_+ to v_i). Note that $d\eta(\widetilde{\xi}_i) = \xi_i$. We may assume that there are geodesics $\widetilde{\gamma}_i(t)$ with $\widetilde{\gamma}_i'(0) = \widetilde{\xi}_i$, and set $\gamma_i(t) := \eta(\widetilde{\gamma}_i(t))$. Since $T_x(Y) = T_x(X) \times \mathbb{R}$, any minimal geodesic joining $\gamma_1(t)$ and $\gamma_2(t)$ does not meet X for any small $t > 0$. It follows from the fact that $\eta\colon C \setminus C_0 \to Y \setminus X$ is locally isometric that

$$|\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t)| = |\gamma_1(t), \gamma_2(t)|,$$

which implies that $\angle(\widetilde{\xi}_1, \widetilde{\xi}_2) = \angle(\xi_1, \xi_2)$. From the suspension structures, we conclude that $\angle(\widetilde{v}_1, \widetilde{v}_2) = \angle(v_1, v_2)$.

(2) is an immediate consequence of (1). □

4.3 Gluing maps

Using the results of the last subsection, we study the metric properties of the gluing map.

From Lemma 3.3, we can define a map $f\colon C_0 \to C_0$ as follows: for an arbitrary point $p \in C_0$, let $f(p) := q$ if $\{p, q\} = \eta_0^{-1}(\eta_0(p))$, where q may be equal to p if $\eta_0(p) \in X_1$. Note that f is an involutive map, ie $f^2 = \mathrm{id}$. Moreover:

Lemma 4.20 *$f\colon C_0 \to C_0$ is a homeomorphism.*

Proof Since f is involutive, it suffices to prove that f is continuous. For a sequence p_i converging to a point p in C_0, we show that $f(p_i) \to f(p)$. Set $x = \eta_0(p)$ and $x_i = \eta_0(p_i)$.

Case 1 ($x \in X_1$) In this case, $f(p) = p$. If $x_i \in X_1$, then $f(p_i) = p_i$, and we have nothing to do. Suppose $x_i \in X_2$. Let $\gamma^{\pm}_{x_i}$ be the two perpendiculars to X at x_i. Letting $s_i = |x, x_i|$, consider minimal geodesics $\gamma_i^{\pm}$ joining x to $\gamma^{\pm}_{x_i}(s_i)$. If we set $\widetilde{\sigma}_i^{\pm} := \widetilde{\pi} \circ \widetilde{\gamma}_i^{\pm}$, where $\widetilde{\gamma}_i^{\pm} = \eta^{-1}(\gamma_i^{\pm})$, then $\widetilde{\sigma}_i^{\pm}$ are minimal geodesics joining p to p_i and $f(p)$ to $f(p_i)$, respectively. Lemma 4.16 then shows that

$$\left| \frac{|x, x_i|}{|p, p_i|} - 1 \right| < \tau_x(|x, x_i|), \quad \left| \frac{|x, x_i|}{|f(p), f(p_i)|} - 1 \right| < \tau_x(|x, x_i|), \tag{4-25}$$

which implies $f(p_i) \to f(p)$.

Case 2 $(x \in X_2)$ In this case, $f(p) \neq p$. Let $\gamma_x^{\pm}$ be the two perpendiculars to X at x. By Corollary 4.11, we have

$$\angle \gamma_x^+(s_0) x_i \gamma_x^-(s_0) \geq \tilde{\angle} \gamma_x^+(s_0) x_i \gamma_x^-(s_0) > \pi - \tau(s_0) \tag{4-26}$$

for small enough $s_0 > 0$. If $x_i \in X_1$, then both $\angle(\xi_i^+, \uparrow_{x_i}^{\gamma_x^+(s_0)})$ and $\angle(\xi_i^+, \uparrow_{x_i}^{\gamma_x^-(s_0)})$ become small, yielding a contradiction to (4-26). Thus we have $x_i \in X_2$. Then, in a way similar to Case 1, we have the formula (4-25), which implies $f(p_i) \to f(p)$. □

Corollary 4.21 *$\eta_0|_{C_0^2} \colon C_0^2 \to X_2$ is a double covering space and X_2 is open in X.*

Proof For $x \in X_2$, set $\eta_0^{-1}(x) = \{p_1, p_2\}$, and take an open neighborhood D_1 of p_1 in C_0 such that $D_1 \cap f(D_1)$ is empty. We set $D_2 = f(D_1)$. We show that $E := \eta_0(D_i)$ is open in X. Suppose that E is not open, and take $y \in E$ for which there are $y_i \in X \setminus E$ converging to y. Let $\{q_1, q_2\} := \eta_0^{-1}(y)$ with $q_k \in D_k$ for $k = 1, 2$. Applyinjg Lemma 4.16 to $y_i \to y$ and $q_k \in \eta_0^{-1}(y)$, we have $q_{k,i} \in \eta_0^{-1}(y_i)$ such that

$$\left| \frac{|q_k, q_{k,i}|}{|y, y_i|} - 1 \right| < \tau_y(|y, y_i|) \quad \text{for } k = 1, 2.$$

This implies that $q_{k,i} \in D_k$ and $y_i \in E$ for large i. Since this is a contradiction, E is open. Similarly one can show that each restriction $\eta_0|_{D_k} \colon D_k \to E$ is an open map, and hence is a homeomorphism. □

Corollary 4.22 *Y and X are homeomorphic to the quotient spaces $C_0 \times_\phi [0, t_0]/f$ and C_0/f, respectively, where $(x, 0)$ and $(f(x), 0)$ are identified for every $x \in C_0$.*

Corollary 4.23 *If the inradius of $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ converges to zero, then the number of components of ∂M_i is at most two for large enough i.*

Proof Since f is an involutive homeomorphism, f gives a transposition of two components of C_0. The conclusion is immediate from the connectedness of X. □

Remark 4.24 In Theorem 1.6, we remove the diameter bound to get the diameter-free result.

Lemma 4.25 *$\eta_0|_{C_0^2} \colon (C_0^2)^{\mathrm{int}} \to X_2^{\mathrm{int}}$ is a local isometry.*

Proof For every $p \in C_0^2$, by Corollary 4.21 it is possible to take relatively compact open subsets $D \ni p$ and $E \ni \eta_0(p)$ of C_0^2 and X_2, respectively, such that $\eta_0\colon D \to E$ is a homeomorphism. We may assume that D and E are small enough so as to satisfy that for every $x, y \in E$, there is a minimal geodesic $\gamma\colon [0,1] \to X_2$ joining x to y. We must show that $\eta_0\colon D \to E$ is an isometry with respect to the interior distances of C_0 and X, respectively. Since η_0 is 1–Lipschitz, it suffices to show that $g := \eta_0^{-1}\colon E \to D$ is 1–Lipschitz. We do not know if $g \circ \gamma$ is a Lipschitz curve yet. However, by Proposition 4.17, $g \circ \gamma$ has the speed $v_{g\circ\gamma}(t)$ (see [3]),

$$v_{g\circ\gamma}(t) = \lim_{\epsilon\to 0} \frac{|g\circ\gamma(t), g\circ\gamma(t+\epsilon)|}{|\epsilon|},$$

which is equal to the speed $v_\gamma(t)$ of γ, and therefore

$$|x,y| = L(\gamma) = \int_0^1 v_{g\circ\gamma}(t)\,dt = L(g\circ\gamma) \geq |g(x), g(y)|. \qquad \square$$

Lemma 4.26 *If X_1 has nonempty interior in X, then $X = X_1$ and $\eta_0\colon (C_0)^{\mathrm{int}} \to X^{\mathrm{int}}$ is an isometry.*

Proof If the interior U of X_1 is nonempty, then $V := \eta_0^{-1}(U) \subset C_0^1$ is open in C_0. From the nonbranching property of geodesics in Alexandrov spaces, we have $V = C_0$ and $X = X_1$. An argument similar to the proof of Lemma 4.25 shows that $\eta_0\colon (C_0)^{\mathrm{int}} \to X^{\mathrm{int}}$ is an isometry. $\square$

Proposition 4.27 *$f\colon (C_0)^{\mathrm{int}} \to (C_0)^{\mathrm{int}}$ is an isometry.*

Proof If C_0 is disconnected, Lemma 4.25 immediately implies the conclusion. Therefore we may assume that C_0 is connected. For $x \in X_2$ with $\eta_0^{-1}(x) = \{q_1, q_2\}$, by Lemma 4.25, we can take disjoint open sets $D_i \ni q_i$ for $i = 1, 2$ and $E \ni x$ such that $\eta_0^i := \eta_0|_{D_i}\colon D_i \to E$ are isometries. Thus $f|_{D_1} = (\eta_0^2)^{-1} \circ \eta_0^1\colon D_1 \to D_2$ is an isometry with respect to the interior distances. Note that f is the identity on C_0^1, and by Lemma 4.25, $f\colon (C_0^2)^{\mathrm{int}} \to (C_0^2)^{\mathrm{int}}$ is a local isometry. For every $p_1, p_2 \in C_0$ we show that $|f(p_1), f(p_2)| = |p_1, p_2|$. This is obvious if $p_1, p_2 \in C_0^1$. Let $\gamma\colon [0,1] \to C_0$ be a minimal geodesic joining p_1 to p_2. If $p_1, p_2 \in C_0^2$, applying Lemma 4.25, we may assume that γ meets C_0^1. Let $t_0 \in (0,1)$ be the smallest parameter with $\gamma(t_0) \in C_0^1$. By Lemma 4.25, we have $|f(p_1), f(\gamma(t_0))| = |p_1, \gamma(t_0)|$. Therefore the nonbranching property of geodesics in Alexandrov space implies that $\gamma \cap C_0^1$ consists of only the single point $\gamma(t_0)$, and therefore we also have $|f(p_2), f(\gamma(t_0))| = |p_2, \gamma(t_0)|$. It follows that

$$\begin{aligned}|f(p_1), f(p_2)| &\le |f(p_1), f(\gamma(t_0))| + |f(\gamma(t_0)).f(p_2)| \\ &\le |p_1, \gamma(t_0)| + |\gamma(t_0), p_2| = |p_1, p_2|.\end{aligned}$$

Repeating this, we also have $|p_1, p_2| \le |f(p_1), f(p_2)|$ and $|f(p_1), f(p_2)| = |p_1, p_2|$. The cases of $p_1 \in C_0^1$ and $p_2 \in C_0^2$ are similar, and hence are omitted. □

4.4 Structure theorems

In this subsection, making use of the results on gluing maps in the last subsection, we obtain structure results for limit spaces.

We begin with:

Lemma 4.28 *X_2 is convex in X.*

Proof Suppose this is not the case. Then we have a minimal geodesic $\gamma\colon [0,1] \to X$ joining points $x, y \in X_2$ such that γ is not entirely contained in X_2. Let t_1 be the first parameter with $\gamma(t_1) \in X_1$. Set $z := \gamma(t_1)$. By Lemma 4.25, for any $p \in \eta_0^{-1}(x)$, there exists a unique geodesic $\tilde{\gamma}\colon [0, t_1] \to C_0$ such that $\tilde{\gamma}(0) = p$ and $\eta_0 \circ \tilde{\gamma}(t) = \gamma(t)$ for every $t \in [0, t_1]$. Put $\tilde{z} := \tilde{\gamma}(t_1) \in C_0^1$, and take $\tilde{v} \in \Sigma_{\tilde{z}}(C_0)$ such that $(d\eta_0)_{\tilde{z}}(\tilde{v}) = \frac{d}{dt}\gamma(t_0) \in \Sigma_z(X)$. Let $\tilde{\gamma}_1\colon [0, t_1] \to C_0$ and $\gamma_1\colon [0, t_1] \to X$ be the reverse geodesic to $\tilde{\gamma}$ and $\gamma_{[0,t_1]}$, namely $\tilde{\gamma}_1(t) = \tilde{\gamma}(t_1 - t)$, $\gamma_1(t) = \gamma(t_1 - t)$, and set $\tilde{\gamma}_2(t) := f(\tilde{\gamma}_1(t))$. Since $(d\eta_0)_{\tilde{z}}$ preserves norm and is 1–Lipschitz, we have

$$\angle(\tilde{v}, \tilde{\gamma}_i'(0)) \ge \angle\Big(\frac{d}{dt}\gamma(t_1), \frac{d}{dt}\gamma_1(0)\Big) = \pi$$

for $i = 1, 2$. Since $\tilde{\gamma}_1'(0) \ne \tilde{\gamma}_2'(0)$, this is impossible in the Alexandrov space C_0. □

Lemma 4.29 *For every $x, y \in X$, let $\gamma\colon [0,1] \to X$ be a minimal geodesic joining x to y, and let $p \in C_0$ be such that $\eta_0(p) = x$. Then there exists a minimal geodesic $\tilde{\gamma}\colon [0,1] \to C_0$ starting from p such that $\eta_0 \circ \tilde{\gamma} = \gamma$. If $x \in X_2$, then $\tilde{\gamma}$ is uniquely determined.*

In particular, if X_1 is not empty, then C_0 is connected.

Proof Suppose $x \in X_2$. It follows from Lemmas 4.25 and 4.28 and Proposition 4.27 together with the nonbranching property of geodesics in Alexandrov spaces that $\gamma([0,1)) \subset X_2$.

Suppose $x \in X_1$. If $\gamma \subset X_1$, the conclusion is immediate. Otherwise, by an argument similar to the above, we have $\gamma((0,1)) \subset X_2$. Thus we have a lift $\hat{\gamma}\colon (0,1) \to C_0^2$ of $\gamma|_{(0,1)}$, which extends to a required lift $\tilde{\gamma}\colon [0,1] \to C_0$ of γ. □

Proposition 4.30 *$N = X^{\mathrm{int}}$ is isometric to C_0^{int}/f.*

Proof In the case of $X = X_1$ or $X = X_2$, the conclusion follows from Lemma 4.26 or Lemma 4.25, respectively. Next assume that both X_1 and X_2 are nonempty. We set $Z := C_0^{\mathrm{int}}/f$, which is an Alexandrov space. Letting $\zeta\colon C_0^{\mathrm{int}} \to Z$ be the projection, we decompose Z as

$$Z = Z_1 \cup Z_2 \quad \text{with } Z_i := \zeta(C_0^i) \text{ for } i = 1, 2.$$

For every $\zeta(p) \in Z_1$, $\Sigma_{\zeta(p)}(Z)$ is isometric to $\Sigma_p(C_0)/f_*$, where $f_*\colon \Sigma_p(C_0) \to \Sigma_p(C_0)$ is the isometry induced by f. Since X_1 is a proper subset of X, f_* defines a nontrivial isometric $\mathbb{Z}_2$–action on $\Sigma_p(C_0)$. Thus $\zeta(p)$ is a singular point of Z, ie $\zeta(p) \in Z^{\mathrm{sing}}$, and therefore $Z_1 \subset Z^{\mathrm{sing}}$. Thus $Z^{\mathrm{reg}} \subset Z_2$. Now, by Proposition 4.19, there exists an isometry $F_0\colon Z_2 \to X_2^{\mathrm{int}}$. Since Z^{reg} is convex in Z (see [25]), F_0 defines a 1–Lipschitz map $F_1\colon (Z^{\mathrm{reg}})^{\mathrm{ext}} \to X$ which extends to a 1–Lipschitz map $F\colon Z \to X$, where $(Z^{\mathrm{reg}})^{\mathrm{ext}}$ denotes the exterior metric of Z^{reg}.

Conversely, since X_2 is convex in X by Lemma 4.28, F_0^{-1} defines a 1–Lipschitz map $G_1\colon (X_2)^{\mathrm{int}} \to Z_2$ which extends to a 1–Lipschitz map $G\colon X^{\mathrm{int}} \to Z$ satisfying $G \circ F = 1_Z$. Therefore X^{int} must be isometric to Z. □

Proof of Theorem 1.2 By Proposition 4.27, $f\colon C_0^{\mathrm{int}} \to C_0^{\mathrm{int}}$ is an involutive isometry. By Propositions 3.9 and 4.30, N is isometric to C_0^{int}/f. Since C_0^{int} is an Alexandrov space with curvature $\geq c(\kappa, \lambda)$, so is N. □

In view of Proposition 2.5, Proposition 4.30 immediately implies:

Corollary 4.31 *X_1 is an extremal subset of X^{int}.*

Theorem 4.32 *Let $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ inradius collapse to a compact length space N. Let $\widetilde{M}_i$ Gromov–Hausdorff converge to Y, and M_i^{ext} converge to $X \subset Y$ under the convergence $\widetilde{M}_i \to Y$. Then*

(1) *X^{int} is isometric to N;*

(2) *Y is isometric to $C_0^{\mathrm{int}} \times_\phi [0, t_0]/(f(x), 0) \sim (x, 0)$, or equivalently, isometric to the quotient by an isometric involution $\tilde{f} = (f, -\mathrm{id})$,*

$$C_0^{\mathrm{int}} \times_{\tilde{\phi}} [-t_0, t_0]/\tilde{f},$$

where $\tilde{\phi}(t) = \phi(|t|)$.

In particular, Y is a singular I–bundle over N, where singular fibers occur exactly on X_1 unless $X = X_1$.

Compare Examples 3.16, 3.20 and 3.21.

Proof of Theorem 4.32 (1) is just Proposition 3.9. (2) follows immediately from Propositions 3.11 and 4.30. □

Remark 4.33 Theorem 4.32 can be generalized to the unbounded diameter case (see Section 6).

Proposition 4.34 *If $x \in X_1$ then $\Sigma_x(X)$ is isometric to the quotient space $\Sigma_p(C_0)/f_*$ and $\Sigma_x(Y)$ is isometric to the quotient space $\Sigma_p(C)/f_*$, where $f_*\colon \Sigma_p(C_0) \to \Sigma_p(C_0)$ is the isometry induced by f.*

Proof Take an f–invariant neighborhood U_p of p in C_0, where $\eta_0(p) = x$. It is easy to check that $V_x := \eta_0(U_p)$ is a neighborhood of x isometric to U_p/f. The conclusion follows immediately. □

Corollary 4.35 *Let $\dim N = m$. Suppose that both X_1 and X_2 are nonempty. Then every element $x \in X_1$ satisfies that*

$$\mathrm{vol}\,\Sigma_x(X) \le \tfrac{1}{2}\,\mathrm{vol}\,\mathbb{S}^{m-1}.$$

In particular, $\dim(X_1 \cap \partial X) \le m-1$ and $\dim(X_1 \cap \mathrm{int}\, X) \le m-2$.

Proof For $x \in X_1$, take $p \in C_0$ with $\eta_0(p) = x$. Note that C_0 is connected by Lemma 4.29. If $f_*\colon \Sigma_p(C_0) \to \Sigma_p(C_0)$ is the identity, then the nonbranching property of geodesics in Alexandrov spaces implies that f is the identity on C_0. Therefore, f_* must be nontrivial on $\Sigma_p(C_0)$. The conclusion follows since

$$\mathrm{vol}\,\Sigma_x(X) = \tfrac{1}{2}\,\mathrm{vol}\,\Sigma_p(C_0) \le \tfrac{1}{2}\,\mathrm{vol}\,\mathbb{S}^{m-1}.$$ □

By Corollary 4.35, if every $x \in X$ satisfies that

$$\mathrm{vol}\, \Sigma_x(X) > \tfrac{1}{2}\, \mathrm{vol}\, \mathbb{S}^{m-1},$$

then $X = X_1$ or $X = X_2$.

Next let us consider such a case. If $X = X_1$, then by Lemma 4.26, η_0 is an isometry. If $X = X_2$, then by Lemma 4.25, η_0 is a locally isometric double covering. Therefore it is straightforward to see the following:

Corollary 4.36 *If $X = X_1$ or X_2, then Y can be classified by N as follows:*

(1) *If $X = X_1$, then Y is isometric to $N \times_\phi [0, t_0]$.*

(2) *If $X = X_2$, then either Y is isometric to the gluing*

$$N \times_{\tilde{\phi}} [-t_0, t_0],$$

with length metric, or else Y is a nontrivial I–bundle over N, and is doubly covered by

$$C_0^{\mathrm{int}} \times_{\tilde{\phi}} [-t_0, t_0],$$

where $\tilde{\phi}(t) = \phi(|t|)$.

Compare Examples 3.16 and 3.20.

From now, we write for simplicity as $C_0 := C_0^{\mathrm{int}}$.

5 Inradius collapsed manifolds with bounded diameters

In this section, we investigate the structure of inradius collapsed manifolds M_i, applying the structure results for limit spaces in Section 4. First we study the case of inradius collapse of codimension one to determine the manifold structure. To carry out this, some additional considerations on the limit spaces are needed to determine the singularities of singular I–fibered spaces. In the second part of this section, we study inradius collapse to almost regular spaces.

5.1 Inradius collapse of codimension one

We consider $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ that inradius collapse to an $(n-1)$–dimensional Alexandrov space N. Then, by Theorem 2.2, M_i is homeomorphic to Y, and by Theorem 4.32, we have

$$Y = C_0 \times_{\tilde{\phi}} [-t_0, t_0]/\tilde{f}, \quad N = C_0/f,$$

where $\tilde{f} = (f, -\mathrm{id})$ is an isometric involution. and the singular locus of the singular I–bundle structure on Y defined by the above form coincides with C_0^1 unless $X \neq X_1$. Later, in Lemma 5.5, we show that $\eta_0(C_0^1) = \partial N$.

Assuming that N has nonempty boundary, we begin with construction of singularity models of singular I–fibered spaces around each boundary component of the limit space N.

By Proposition 2.6, each component $\partial_\alpha N$ of ∂N has a collar neighborhood V_α. Let $\varphi\colon V_\alpha \to \partial_\alpha N \times [0,1)$ be a homeomorphism. Let $\pi\colon Y \to N$ be the projection. Then the I–fiber structure on $\pi^{-1}\varphi^{-1}(\{p\} \times [0,1)$ is isomorphic to the form

$$R_{t_0} := [0,1) \times [-t_0, t_0]/(0,y) \sim (0,-y),$$

with the projection $\pi\colon R_{t_0} \to [0,1)$ induced by $(x,y) \to x$. Therefore $\pi^{-1}(V_\alpha)$ is an R_{t_0}–bundle over $\partial_\alpha N$.

Now we define two singularity models for the singular I–bundle $\pi^{-1}(V_\alpha)$: one is the case when $\pi^{-1}(V_\alpha)$ is a trivial R_{t_0}–bundle over $\partial_\alpha N$, and the other one is the case of a nontrivial R_{t_0}–bundle.

Definition 5.1 (1) First, set

$$\mathcal{U}_1(\partial_\alpha N) := \partial_\alpha N \times R_{t_0},$$

and define $\chi\colon \mathcal{U}_1(\partial_\alpha N) \to \partial_\alpha N \times [0,1)$ by $\chi(p,x,y) = (p,x)$ for $(p,x,y) \in \partial_\alpha N \times R_{t_0}$. This gives $\mathcal{U}_1(\partial_\alpha N)$ the structure of a singular I–bundle over $\partial_\alpha N \times [0,1)$ whose singular locus is $\partial_\alpha N \times 0$. We call this *the product singular I–bundle model* around $\partial_\alpha N$.

(2) For the second model, suppose that $\partial_\alpha N$ admits a double covering space $\rho\colon P_\alpha \to \partial_\alpha N$ with the deck transformation φ. Let

$$\mathcal{U}_2(\partial_\alpha N) := (P_\alpha \times R_{t_0})/\Phi,$$

where Φ is the isometric involution on $P_\alpha \times R_{t_0}$ defined by $\Phi = (\varphi, g)$, where $g\colon R_{t_0} \to R_{t_0}$ is the reflection induced from $(x,y) \to (x,-y)$. Define $\chi\colon \mathcal{U}_2(\partial_\alpha N) \to \partial_\alpha N \times [0,1)$ by $\chi\big([(p,[x,y])]\big) = (\rho(p), x)$ for $(p,x,y) \in P_\alpha \times R_\epsilon$. This gives $\mathcal{U}_2(\partial_\alpha N)$ the structure of a singular I–bundle over $\partial_\alpha N \times [0,1)$ whose singular locus is $\partial_\alpha N \times 0$. The second model is a twisted one, and is doubly covered by the first model $\mathcal{U}_1(P_\alpha) = P_\alpha \times R_\epsilon$. We call this the *twisted singular I–bundle model* around $\partial_\alpha N$.

Example 5.2 Let us consider the codimension one inradius collapse in Example 3.21. Recall that the limit space Y of $\widetilde{M}_\epsilon$ is isometric to the form

$$Y = D(E) \times_{\tilde{\phi}} [-t_0, t_0]/(x,t) \sim (r(x), -t),$$

where $r\colon D(E) \to D(E)$ denotes the canonical reflection of $D(E)$. If $\pi\colon Y \to E$ denotes the projection, then $\pi^{-1}(V)$ is isomorphic to the product singular I–bundle model around ∂E, where V is any collar neighborhood of ∂E.

Example 5.3 Let Q_ϵ denote the space obtained from the disjoint union of two copies of the completion $\bar{R}_\epsilon$ of R_ϵ glued along each segment $1 \times [-\epsilon, \epsilon]$ of the boundaries,

$$Q_\epsilon = \bar{R}_\epsilon \amalg_{1\times[-\epsilon,\epsilon]} \bar{R}_\epsilon.$$

Let $r\colon Q_\epsilon \to Q_\epsilon$ be the reflection induced from $(x,y) \to (x,-y)$. Let

$$M_\epsilon = (\mathbb{S}^1(1) \times Q_\epsilon)/(z,p) \sim (-z, r(p)).$$

As $\epsilon \to 0$, M_ϵ inradius collapses to $\mathbb{S}^1\left(\frac{1}{2}\right) \times [0,2]$. Let $\pi_\epsilon\colon M_\epsilon \to \mathbb{S}^1\left(\frac{1}{2}\right) \times [0,2]$ be the projection induced by $[z,(x,y)] \to (z,x)$. Then both $\pi_\epsilon^{-1}\big(\mathbb{S}^1\left(\frac{1}{2}\right) \times [0,1)\big)$ and $\pi_\epsilon^{-1}\big(\mathbb{S}^1\left(\frac{1}{2}\right) \times (1,2]\big)$ are solid Klein bottles and their I–fiber structures are isomorphic to the twisted singular I–bundle model around respective boundary of $\mathbb{S}^1\left(\frac{1}{2}\right) \times [0,2]$.

The following is a detailed version of Theorem 1.3:

Theorem 5.4 *Let $M_i \in \mathcal{M}(n,\kappa,\lambda,d)$ inradius collapse to an $(n-1)$–dimensional Alexandrov space N. Then there is a singular I–fiber bundle*

$$I \to M_i \xrightarrow{\pi_i} N$$

whose singular locus coincides with ∂N.

More precisely:

(1) *If N has no boundary, then M_i is homeomorphic to a product $N \times I$ or a twisted product $N \tilde{\times} I$.*

(2) *If N has nonempty boundary, each component $\partial_\alpha N$ of ∂N has a neighborhood V such that $\pi_i^{-1}(V)$ is isomorphic to either the product or the twisted singular I–fiber bundle around $\partial_\alpha N$.*

(3) *If $\pi_i^{-1}(V)$ is isomorphic to the product singular I–fiber bundle for some component $\partial_\alpha N$, then M_i is homeomorphic to $D(N) \times [-1,1]/(x,t) \sim (r(x),-t)$, where r is the canonical reflection of the double $D(N)$.*

Recall that

$$Y = C_0 \times_{\tilde{\phi}} [-t_0, t_0]/\tilde{f},$$

where $\tilde{f} = (f, -\mathrm{id})$, and C_0 and Y are the noncollapsing limits of $(\partial M_i)^{\mathrm{int}}$ and $\widetilde{M}_i$, respectively. Therefore both C_0 and $Y \setminus C_{t_0}$ are smoothable spaces in the sense of [12]. See also Remark 2.11.

Let $F \subset C_0$ denote the fixed-point set of the isometry $f\colon C_0 \to C_0$. By Proposition 2.5 and Theorem 2.3, $\eta_0(F)$ is an extremal subset of N and it has a topological stratification.

Lemma 5.5 *$\eta_0(F)$ coincides with ∂N if f is not the identity.*

We postpone the proof of Lemma 5.5 for a moment.

Proof of Theorem 5.4 (1) By Lemma 5.5, if N has no boundary, F is empty, and therefore either $N = N_1$ or $N = N_2$. If $N = N_1$, then $C_0 = N$ and Y is homeomorphic to $N \times I$. If $N = N_2$, then $N = C_0/f$ has no boundary, and Y is homeomorphic to either $N \times I$ or $C_0 \times [-1, 1]/(x, t) \sim (f(x), -t)$, which is a twisted I–bundle over N.

(2) Suppose N has nonempty boundary. Note that

$$N_1 = \eta_0(F).$$

By Proposition 2.6, each component $\partial_\alpha N$ of ∂N has a collar neighborhood V_α. Let $\varphi\colon V_\alpha \to \partial_\alpha N \times [0, 1)$ be a homeomorphism. Since M_i is homeomorphic to Y for large i, the projection from Y to N induces a map $\pi_i\colon M_i \to N$.

By the I–fiber structure of Y, $\pi_i^{-1}(\varphi^{-1}(x \times [0, 1))$ is canonically homeomorphic to R_{t_0}. In particular, $\pi_i^{-1}(V_\alpha)$ is an R_{t_0}–bundle over $\partial_\alpha N$. If this bundle is trivial, $\pi_i^{-1}(V_\alpha)$ is isomorphic to the product singular I–bundle structure $\mathcal{U}_1(\partial_\alpha N) = \partial_\alpha N \times R_{t_0}$.

Suppose this bundle is nontrivial and let P_α be the boundary of $\pi_i^{-1}\big(\varphi^{-1}\big(\partial_\alpha N \times \{\tfrac{1}{2}\}\big)\big)$, which is a double covering of ∂N_α. Let $\Phi = (\varphi, g)$, and let $\rho\colon P_\alpha \to \partial_\alpha N$ be the projection.

Lemma 5.6 *$\pi_i^{-1}(V_\alpha)$ is isomorphic to the twisted singular I–bundle structure*

$$\mathcal{U}_2(\partial_\alpha N) = (P_\alpha \times R_{t_0})/\Phi.$$

Proof Note that

$$\mathcal{U}_2(\partial_\alpha N) := (P_\alpha \times R_{t_0})/(p,x,y) \sim (\varphi(p), x, -y),$$
$$\pi_i^{-1}(V_\alpha) = \pi_i^{-1}\varphi^{-1}(\partial_\alpha N \times [0,1).$$

We define a map $\Psi\colon \mathcal{U}_2(\partial_\alpha N) \to \pi_i^{-1}(V_\alpha)$ as follows: Note that for each $(p,x) \in P_\alpha \times [0,1)$, $\{p, \varphi(p)\}$ can be identified with the boundary of the I–fiber $I_{\rho(p),x} := \pi_i^{-1}\varphi^{-1}(\rho(p)\times\{x\})$. Define $\Psi(p,x,y)$ for $-t_0 \le y \le t_0$ to be the arc on the fiber $I_{\rho(p),x}$ from p to $\varphi(p)$. It is easy to see that $\Psi\colon \mathcal{U}_2(\partial_\alpha N) \to \pi_i^{-1}(V_\alpha)$ gives an isomorphism between I fibered spaces. □

(3) Put $\operatorname{int} N := N \setminus \partial N$ for simplicity.

Assertion 5.7 *There is an isometric imbedding $g\colon N \to C_0$ such that $\eta_0 \circ g = 1_N$.*

Proof Set $F_\alpha := \eta_0^{-1}(\partial_\alpha N)$. Since the bundle $\pi_i^{-1}(V_\alpha)$ is nontrivial, we may assume that F_α is two-sided in the sense that the complement of F_α in some connected neighborhood of it is disconnected. Thus there is a connected neighborhood V_α of $\partial_\alpha N$ in $\operatorname{int} N$ for which there is an isometric imbedding $g_\alpha\colon V_\alpha \to C_0 \setminus F$ such that $\eta_0 \circ g_\alpha = 1_{V_\alpha}$.

Let W be the maximal connected open subset of $\operatorname{int} N$ for which there is an isometric imbedding $g_0\colon W \to C_0 \setminus F$ such that $\eta_0 \circ g_0 = 1_W$ and $g_0(W) \supset g_\alpha(V_\alpha)$. We only have to show that $W = \operatorname{int} N$. Otherwise, there is a point $x \in \partial W \cap \operatorname{int} N$. Take a connected neighborhood W_x of x in $\operatorname{int} N$ such that $\eta_0^{-1}(W_x)$ is a disjoint union of open sets U_1 and U_2 such that $\eta_0\colon U_i \to W_x$ is an isometry for $i = 1,2$. Obviously one U_i, say U_1, meets $g_0(W)$ and the other does not. We extend g_0 to $g_1\colon W \cup W_x \to C_0 \setminus F$ by requiring $g_1|_{W_x} = \eta_0^{-1}\colon W_x \to U_1$. Since g_1 is an isometric imbedding, this is a contradiction to the maximality of W.

Thus we have an isometric imbedding $g_0\colon \operatorname{int} N \to C_0 \setminus F$. Since $\operatorname{int} N$ is convex and η_0 is 1–Lipschitz, g_0 preserves the distance. It follows that g_0 extends to an isometric imbedding $g\colon N \to C_0$ which preserves distance. □

Assertion 5.7 shows that every component of F is two-sided. It follows that $C_0 = D(N)$ and that f is the reflection of the double $D(N)$. This completes the proof of Theorem 5.4. □

Proof of Lemma 5.5 Obviously $\partial N \subset \eta_0(F)$. Suppose that $\eta_0(F) \cap (\operatorname{int} N)$ is not empty.

Sublemma 5.8 *$\dim(\eta_0(F)\cap \operatorname{int} N)\le m-2$, where $m:=\dim N$.*

Proof If $\dim(\eta_0(F)\cap \operatorname{int} N)=m-1$, then the top-dimensional strata S of the intersection $\eta_0(F)\cap \operatorname{int} N$ is a topological $(m-1)$–manifold, and therefore it meets the m–dimensional strata of N because $N^{\mathrm{sing}}\cap \operatorname{int} N$ has codimension ≥ 2 (Theorem 2.1). Take $p\in\eta_0^{-1}(S)$. It is now easy to see that f is the reflection with respect to $\eta_0^{-1}(S)$ in a small neighborhood of p. It follows that S is a subset of ∂N, which contradicts the hypothesis. □

Take a point $x=\eta_0(p)\in\eta_0(F)\cap \operatorname{int} N$, and consider the directional derivative $f_*\colon \Sigma_p(C_0)\to\Sigma_p(C_0)$ of f at p, which is again an isometric involution with fixed-point set

$$F_*:=\Sigma_p(F).$$

By Corollary 2.4 and Sublemma 5.8, $\dim F_*\le m-3$ while $\dim \Sigma_p(C_0)=m-1$. Repeating this, we have a finite sequence of directional derivatives of f, $f_*\dots$, each of which is an isometric involution

$$f_{*k}\colon \Sigma_{*k}(C_0)\to\Sigma_{*k}(C_0),$$

where $\Sigma_{*k}(C_0)$ denotes a k–iterated space of directions,

$$\Sigma_{*k}(C_0)=\Sigma_{\xi_{k-1}}\bigl(\cdots\bigl(\Sigma_{\xi_1}(\Sigma_p(C_0))\bigr)\cdots\bigr),$$

and ξ_i is taken from the fixed-point set of the iterated directional derivatives,

$$\xi_1\in\Sigma_p(F),\quad \xi_2\in\Sigma_{\xi_1}(F_*),\quad \dots,\quad \xi_k\in\Sigma_{\xi_{i-1}}(F_{*(k-1)}),$$

and F_{*i} denotes the fixed-point set of $f_{*i}\colon \Sigma_{*i}(C_0)\to\Sigma_{*i}(C_0)$, which coincides with $\Sigma_{\xi_{i-1}}(F_{*(i-1)})$.

Note that the iterated space of directions $\Sigma_{*k}(C_0)$ has dimension $m-k$, and the iterated fixed-point set $F_{*k}\subset\Sigma_{*k}(C_0)$ has dimension $\le m-k-2$. It follows that for some $k\le m-2$, F_{*k} becomes a finite set. It follows that for any $\xi_{k+1}\in F_{*k}$,

$$f_{*(k+1)}\colon \Sigma_{\xi_{k+1}}(\Sigma_{*k}(C_0))\to\Sigma_{\xi_{k+1}}(\Sigma_{*k}(C_0))$$

has no fixed points. Put

$$D:=C_0\times_{\tilde\varphi}[-t_0,t_0],$$

and let $\tilde f$ be an isometric involution on D defined by $\tilde f=(f,-\mathrm{id})$. From Theorem 4.32,

$$Y=D/\tilde f.$$

Let $x = \eta_0(p)$, $p = (p, 0)$ and $\xi_i \in \Sigma_{\xi_{i-1}}(F_{*(i-1)})$ for $1 \le i \le k+1$ be as above. Note that we have isometric identifications

$$\Sigma_x(Y) = \Sigma_p(D)/\tilde{f}_*, \quad \Sigma_x(X) = \Sigma_p(C_0)/f_*.$$

Let $\zeta_1 \in \Sigma_x(\eta_0(F)) \subset \Sigma_x(X) \subset \Sigma_x(Y)$ be the element corresponding to $\xi_1 \in \Sigma_p(F) \subset \Sigma_p(C_0) \subset \Sigma_p(D)$. Note that

$$\Sigma_p(D) = \{\xi_\pm\} * \Sigma_p(C_0)$$

and $\tilde{f}_* = (f_*, -\mathrm{id})$ interchanges ξ_+ and ξ_- and preserves $\Sigma_p(C_0)$. Next consider

$$\Sigma_{\zeta_1}(\Sigma_x(Y)) = \Sigma_{\xi_1}(\Sigma_p(D))/\tilde{f}_{**},$$

where $\tilde{f}_{**}$ denotes the directional derivative of f_* at ζ_1. Note that $\Sigma_{\xi_1}(\Sigma_p(D))$ is still isometric to $\{\xi_\pm\} * \Sigma_{\xi_1}(\Sigma_p(C_0))$ and $\tilde{f}_{**} = (f_{**}, -\mathrm{id})$ interchanges ξ_+ and ξ_- and preserves $\Sigma_{\xi_1}(\Sigma_p(C_0))$. Similarly and finally we consider

$$\Sigma_{\zeta_{k+1}}(\Sigma_{*k}(Y)) = \Sigma_{\xi_{k+1}}(\Sigma_{*k}(D))/\tilde{f}_{*k+1}, \tag{5-1}$$

where $\zeta_{k+1} \in \Sigma_{*k}(Y)$ is the element corresponding to $\xi_{k+1} \in \Sigma_{*k}(D)$, and $\tilde{f}_{*k+1} = (f_{*k+1}, -\mathrm{id})$ freely acts on $\Sigma_{\xi_{k+1}}(\Sigma_{*k}(D))$. Recall that

$$\ell := \dim \Sigma_{\xi_{k+1}}(\Sigma_{*k}(D)) = m - k \ge 2.$$

Note that the iterated spaces of directions of the smoothable spaces $Y \setminus C_{t_0}$ must be all homeomorphic to spheres (Theorem 2.7). However, (5-1) shows that $\Sigma_{\zeta_{k+1}}(\Sigma_{*k}(Y))$ is homeomorphic to a quotient $\mathbb{S}^\ell/\mathbb{Z}_2$ for $\ell \ge 2$ by a free $\mathbb{Z}_2$–action, which is a contradiction. This completes the proof of Lemma 5.5. □

5.2 Inradius collapse to almost regular spaces

Next we consider the case where M_i inradius collapses to an almost regular Alexandrov space N. The idea of using an equivariant fibration-capping theorem in [34] was inspired by the recent work [18].

First we recall this theorem. Let X be a k–dimensional complete Alexandrov space with curvature $\ge \kappa$ possibly having nonempty boundary. We denote by $D(X)$ the double of X, which is also an Alexandrov space with curvature $\ge \kappa$. (see [21]). By definition, $D(X) = X \cup X^*$ glued along their boundaries, where X^* is another copy of X.

A (k,δ)–strainer $\{(a_i,b_i)\}$ of $D(X)$ at $p\in X$ is called *admissible* if $a_i\in X$ and $b_j\in X$ for every $1\le i\le k$ and $1\le j\le k-1$ (clearly, $b_k\in X^*$ if $p\in\partial X$ for instance). Let $R^D_\delta(X)$ denote the set of points of X at which there are admissible (k,δ)–strainers. It has the structure of a Lipschitz k–manifold with boundary. Note that every point of $R^D_\delta(X)\cap\partial X$ has a small neighborhood in X almost isometric to an open subset of the half space $\mathbb{R}^k_+$ for small δ.

If Y is a closed domain of $R^D_\delta(X)$, then the δ_D*–strain radius* of Y is defined as the infimum of positive numbers ℓ such that there exists an admissible (k,δ)–strainer of length $\ge\ell$ at every point in Y, denoted by $\delta_D\text{–str rad}(Y)$.

For a small $\nu>0$, we put

$$Y_\nu := \{x\in Y \mid d(\partial X,x)\ge\nu\}.$$

We use the special notation

$$\partial_0 Y_\nu := Y_\nu\cap\{d_{\partial X}=\nu\},\quad \operatorname{int}_0 Y_\nu := Y_\nu-\partial_0 Y_\nu.$$

Let M^n be another n–dimensional complete Alexandrov space with curvature $\ge\kappa$ having no boundary. Let $R_\delta(M)$ denote the set of all (n,δ)–strained points of M.

A surjective map $f\colon M\to X$ is called an ϵ*–almost Lipschitz submersion* if

(1) it is an ϵ–approximation;

(2) for every $p,q\in M$,

$$\left|\frac{d(f(p),f(q))}{d(p,q)}-\sin\theta_{p,q}\right|<\epsilon,$$

where $\theta_{p,q}$ denotes the infimum of $\angle qpx$ when x runs over $f^{-1}(f(p))$.

In the definition above, if $\dim X=n$, $\sin\theta_{p,q}$ is replaced by 1.

Now let a Lie group G act on M^n and X as isometries. Let

$$d_{\mathrm{eGH}}((M,G),(X,G))$$

denote the equivariant Gromov–Hausdorff distance as defined in Section 2.1. We need to assume the following on the existence of slice for G–orbits:

Assumption 5.9 For each $p\in X$, there is a *slice* L_p at p. Namely $U_p:=GL_p$ provides a G–invariant tubular neighborhood of Gp which is G–isomorphic to $G\times_{G_p}L_p$.

Obviously, Assumption 5.9 is automatically satisfied if G is discrete. By [11], it also holds true if G is compact.

Theorem 5.10 (equivariant fibration-capping theorem [34, Theorem 18.9]) *Let X and G be as above such that X/G is compact. Given k and $\mu > 0$ there exist positive numbers $\delta = \delta_k$, $\epsilon_{X,G}(\mu)$ and $\nu = \nu_{X,G}(\mu)$ satisfying the following: Suppose $X = R_\delta^D(X)$ and δ_D–str rad$(X) > \mu$. Suppose $M = R_{\delta_n}(M)$ and $d_{\mathrm{eGH}}((M,G),(X,G)) < \epsilon$ for some $\epsilon \le \epsilon_{X,G}(\mu)$. Then there exists a G–invariant decomposition*

$$M = M_{\mathrm{int}} \cup M_{\mathrm{cap}}$$

of M into two closed domains glued along their boundaries, and a G–equivariant Lipschitz map $f\colon M \to X_\nu$ such that

(1) *M_{int} is the closure of $f^{-1}(\mathrm{int}_0 X_\nu)$, and $M_{\mathrm{cap}} = f^{-1}(\partial_0 X_\nu)$;*

(2) *the restrictions $f|_{M_{\mathrm{int}}}\colon M_{\mathrm{int}} \to X_\nu$ and $f|_{M_{\mathrm{cap}}}\colon M_{\mathrm{cap}} \to \partial_0 X_\nu$ are*
   - (a) *locally trivial fiber bundles;*
   - (b) *$\tau(\delta,\nu,\epsilon/\nu)$–Lipschitz submersions.*

Here, $\tau(\epsilon_1,\dots,\epsilon_k)$ denotes a function depending on an a priori constant and ϵ_i satisfying

$$\lim_{\epsilon_i \to 0} \tau(\epsilon_1,\dots,\epsilon_k) = 0.$$

Remark 5.11 If X has no boundary, then X_ν is replaced by X, $M_{\mathrm{cap}} = \varnothing$ and $M = N$ in the statement above.

We go back to the situation of Theorem 1.5. Assume that M_i inradius collapses to an almost regular Alexandrov space N. Let us consider the double and the partial double of $\widetilde{M}_i$ and Y, respectively,

$$D(\widetilde{M}_i) := \widetilde{M}_i \amalg_{\partial \widetilde{M}_i} \widetilde{M}_i, \quad W := Y \amalg_{C_{t_0}} Y,$$

where two copies of Y are glued along C_{t_0}. From Perelman's result [21], both $D(\widetilde{M}_i)$ and W are Alexandrov spaces. Note that both $D(M_i)$ and W admit canonical isometric $\mathbb{Z}_2$–actions by the reflections.

The proof of the following lemma is standard, and hence omitted.

Lemma 5.12 *$(D(\widetilde{M}_i),\mathbb{Z}_2)$ converges to $(W,\mathbb{Z}_2)$ with respect to the equivariant Gromov–Hausdorff convergence.*

Proof of Theorem 1.5 By Lemma 5.12, for any $\varepsilon > 0$, if i is large,

$$d_{\mathrm{eGH}}((D(\widetilde{M}_i), \mathbb{Z}_2), (W, \mathbb{Z}_2)) < \varepsilon.$$

By Theorem 4.32 together with our assumption on N, Y is almost regular with almost regular boundary. Hence, $W = R^D_\delta(W)$ and δ_D–str rad$(W) > \mu$ for some $\mu > 0$. Thus, by Theorem 5.10 and Remark 5.11, there exists a $\mathbb{Z}_2$–equivariant capping fibration

$$\tilde{f}_i\colon D(\widetilde{M}_i) \to W_\nu,$$

where

$$W_\nu = \{x \in W \mid d(x, \partial W) \ge \nu\}.$$

Notice that W_ν is homeomorphic to W because of the form of Y. Obviously, $\tilde{f}_i$ induces a map $f_i\colon \widetilde{M}_i \to Y$. From Theorem 4.32, Y is isometric to $C_0^{\mathrm{int}} \times_{\tilde{\phi}} [-t_0, t_0]/\tilde{f}$ with $\tilde{f} = (f, -\mathrm{id})$. Moreover, by the remark after Corollary 4.35, $\eta_0\colon C_0 \to X$ is either an isometry or a locally isometric double covering.

Case (a) (the action of f on C_0 is nontrivial) In this case, Y is a nontrivial I–bundle over N, and hence $\partial Y = C_{t_0}$. Hence W has no boundary. Thus in this case, $f_i\colon \widetilde{M}_i \to Y$ is a fiber bundle with fiber F_i which are closed, almost nonnegatively curved manifolds. This implies that $\widetilde{M}_i$ and hence M_i is an $F_i \times I$–bundle over N.

Case (b) (the action of f on C_0 is trivial) In this case, Y is isometric to $N \times_\phi [0, t_0]$, and therefore ∂Y consists of $\eta(C_0) = X$ and $\eta(C_{t_0})$. Thus ∂W consists of two copies of $\eta_0(C_0)$. Therefore, by Theorem 5.10, there exists a $\mathbb{Z}_2$–invariant decomposition

$$D(\widetilde{M}_i) = (D(\widetilde{M}_i))_{\mathrm{int}} \cup (D(\widetilde{M}_i))_{\mathrm{cap}} \tag{5-2}$$

of $D(\widetilde{M}_i)$ into two closed domains glued along their boundaries such that

(1) $(D(\widetilde{M}_i))_{\mathrm{int}}$ is the closure of $\tilde{f}_i^{-1}(\mathrm{int}_0\, W_\nu)$ and $(D(\widetilde{M}_i))_{\mathrm{cap}} = \tilde{f}_i^{-1}(\partial_0 W_\nu)$,

(2) $\tilde{f}_i|_{(D(\widetilde{M}_i))_{\mathrm{int}}}\colon (D(\widetilde{M}_i))_{\mathrm{int}} \to W_\nu$ and $\tilde{f}_i|_{(D(\widetilde{M}_i))_{\mathrm{cap}}}\colon (D(\widetilde{M}_i))_{\mathrm{cap}} \to \partial_0 W_\nu$ are locally trivial fiber bundles,

where

$$\partial_0 W_\nu := \{x \in W \mid d(x, \partial W) = \nu\}, \quad \mathrm{int}_0\, W_\nu := W_\nu \setminus \partial_0 W_\nu.$$

Since (5-2) is $\mathbb{Z}_2$–invariant, it induces a decomposition

$$\widetilde{M}_i = (\widetilde{M}_i)_{\mathrm{int}} \cup (\widetilde{M}_i)_{\mathrm{cap}}.$$

Since $\tilde{f}_i$ is $\mathbb{Z}_2$–equivariant, these fibrations induce fibrations

$$F_i \to (\widetilde{M}_i)_{\mathrm{int}} \to Y_\nu, \quad \mathrm{Cap}_i \to (\widetilde{M}_i)_{\mathrm{cap}} \to \partial_0 Y_\nu,$$

where $Y_\nu = \{y \in Y \mid d(y, X) \ge \nu\}$ and $\partial_0 Y_\nu := \{y \in Y \mid d(x, X) = \nu\}$. By construction, $\partial\mathrm{Cap}_i$ is homeomorphic to F_i. Note that every cylindrical geodesic in the warped cylinder $C_i \subset \widetilde{M}_i$ is almost perpendicular to the fibers [32, Lemma 3.1; 33, Lemma 4.1]. This implies that $(\widetilde{M}_i)_{\mathrm{int}}$ is homeomorphic to $\partial_1(\widetilde{M}_i)_{\mathrm{int}} \times [0,1]$, where $\partial_1(\widetilde{M}_i)_{\mathrm{int}}$ is a component of $\partial(\widetilde{M}_i)_{\mathrm{int}}$. Therefore $\widetilde{M}_i$ and hence M_i is homeomorphic to $(\widetilde{M}_i)_{\mathrm{cap}}$. Noting $\partial_0 Y_\nu$ is homeomorphic to N, we obtain a fiber bundle

$$\mathrm{Cap}_i \to M_i \to N. \qquad \square$$

6 The case of unbounded diameters

In this section we provide the proof of Theorem 1.6. In view of Corollary 4.23, for the proof of Theorem 1.6(1), it suffices to consider inradius collapsed manifolds with unbounded diameters.

Remark 6.1 Theorem 1.6(1) was stated in [31, Theorem 5], where the following argument was employed: if $k \ge 3$ and if $p \in M$ is the furthest point from ∂M, then $B(p, r)$, where $r = \mathrm{inrad}(M)$, touches ∂M at least three points. However it seems to the authors that this is unclear.

6.1 Description of pointed inradius collapse

In the case of unbounded diameter, we do not know the uniform boundedness of the numbers of boundary components of inradius collapsed manifolds yet. This forces us to reconsider the descriptions of limit spaces in Section 3.

Let $\mathcal{M}(n,\kappa,\lambda)$ (resp. $\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$) denote the set of all isometry classes of n–dimensional complete Riemannian manifolds M (resp. pointed complete Riemannian manifolds (M, p) with $p \in \partial M$) satisfying

$$K_M \ge \kappa, \quad |\Pi_{\partial M}| \le \lambda.$$

We carry out the extension procedure for M to obtain $\widetilde{M}$. Let

$$\widetilde{\mathcal{M}}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$$

denote the set of all $(\widetilde{M}, M, p)$ with $M \in \mathcal{M}(n,\kappa,\lambda)$ and $p \in \partial M$. We let

$$\partial_0 \widetilde{\mathcal{M}}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$$

be the set of all pointed Gromov–Hausdorff limit spaces (Y, X, x) of sequences $(\widetilde{M}_i, M_i, p_i)$ in $\widetilde{\mathcal{M}}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$ with

$$\operatorname{inrad}(M_i) \to 0.$$

From now on, $(\widetilde{M}_i, M_i, p_i)$ and $(\widetilde{M}, M, p)$ are always assumed to be elements in $\widetilde{\mathcal{M}}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$.

Now suppose that a sequence $(M_i, p_i) \in \mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$ converges to a complete length space (N, q) with $\operatorname{inrad}(M_i) \to 0$, while $(\widetilde{M}_i, M_i, p_i)$ converges to (Y, X, x). In a way similar to Proposition 3.9, we see that X^{int} is isometric to N.

Next we describe Y as

$$Y = X \cup_{\eta_0} C_0 \times_\phi [0, t_0],$$

as in the bounded diameter case. In the bounded diameter case, the number of components of ∂M_i is uniformly bounded, and C_0 is the componentwise Gromov–Hausdorff limit of ∂M_i. In the case of unbounded diameter, we do not know the boundedness of the number of components of ∂M_i yet. This is why we need a bit careful consideration to define C_0, which will be carried out in the following.

Let

$$C^Y_{t_0} := \{y \in Y \mid |X, y| = t_0\}.$$

We begin with the decomposition of $C^Y_{t_0}$ into the connected components,

$$C^Y_{t_0} = \coprod_{\alpha \in A} C^\alpha_{t_0}.$$

Set

$$C^\alpha_0 := \frac{1}{\phi(t_0)} C^\alpha_{t_0}, \quad C^\alpha = C^\alpha_0 \times_\phi [0, t_0],$$

and

$$C := \coprod_{\alpha \in A} C^\alpha, \quad C_0 = \coprod_{\alpha \in A} C^\alpha_0 \times \{0\}.$$

Note that each component of C and C_0^{int} is an Alexandrov space with curvature $\geq c(\kappa, \lambda)$.

Each $p \in C_0$ can be identified with the element of $C^Y_{t_0}$, which we write as $\eta(p,t_0)$, and there is a unique perpendicular $\gamma^{\eta(p,t_0)}(t)$ for $0 \le t \le t_0$ to X satisfying $\gamma^{\eta(p,t_0)}(t_0) = \eta(p,t_0)$. We then define the surjective 1–Lipschitz map $\eta\colon C \to Y$ as

$$\eta(p,t) = \gamma^{\eta(p,t_0)}(t).$$

Obviously $\eta\colon C \setminus C_0 \to Y \setminus X$ is a bijective locally isometric map.

Let $\varphi_i\colon B^{\widetilde{M}_i}(p_i, 1/\delta_i) \to B^Y(x, 1/\delta_i)$ be a δ_i–approximation, with $\lim \delta_i = 0$. Note that for each component $C^\alpha_{t_0}$ of $C^Y_{t_0}$ and any fixed point $y_\alpha \in C^\alpha_{t_0}$, there is a component, say $\partial^\alpha \widetilde{M}_i$, of $\partial \widetilde{M}_i$ for which we have a point $q^\alpha_i \in \partial^\alpha \widetilde{M}_i$ satisfying $|\varphi_i(q^\alpha_i), y_\alpha| < \delta_i$. For a distinct component $C^\beta_{t_0}$ and $y_\beta \in C^\beta_{t_0}$, we also have a component $\partial^\beta \widetilde{M}_i$ of $\partial \widetilde{M}_i$ for which there is a point $q^\beta_i \in \partial^\beta \widetilde{M}_i$ satisfying $|\varphi_i(q^\beta_i), y_\beta| < \delta_i$. Since $|C^\alpha_{t_0}, C^\beta_{t_0}| \ge 2t_0$, it is easily checked that $\partial^\alpha \widetilde{M}_i$ and $\partial^\beta \widetilde{M}_i$ are distinct components, and hence $|\partial^\alpha \widetilde{M}_i, \partial^\beta \widetilde{M}_i| \ge 2t_0$. Thus we have that

$$\lim_{i\to\infty} (\partial^\alpha \widetilde{M}_i, q^\alpha_i) = (C^\alpha_{t_0}, y_\alpha), \quad \lim_{i\to\infty} (\partial^\beta \widetilde{M}_i, q^\beta_i) = (C^\beta_{t_0}, y_\beta), \tag{6-1}$$

under the convergence $(\widetilde{M}_i, p_i) \to (Y, x)$. In particular, the component $\partial^\alpha \widetilde{M}_i$ is uniquely determined by $C^\alpha_{t_0}$.

Now, for the map

$$\eta_0 := \eta|_{C_0}\colon C_0 \to X,$$

from an argument similar to the bounded diameter case, we see that

$$\#\eta_0^{-1}(x) \le 2$$

for every $x \in X$. Thus we can define the involution $f\colon C_0 \to C_0$ as in Section 4. Note that all the results in Sections 4.1, 4.2 and 4.3 still hold true for the present situation of noncompact C_0, because the arguments there are local. In particular, we see that

(6-2) the number of components of C_0 is at most two.

Thus we see that all the results in Section 4 holds true except Corollary 4.23, and therefore we conclude that

$$Y = C_0^{\mathrm{int}} \times_{\tilde\phi} [-t_0, t_0]/\tilde f,$$

where $\tilde\phi(t) = \phi(|t|)$, and $N = X^{\mathrm{int}}$ is isometric to C_0^{int}/f.

Now we immediately have the following:

Corollary 6.2 *If $(M_i, p_i) \in \mathcal{M}(n, \kappa, \lambda)$ inradius collapse to (N, q) with respect to the pointed Gromov–Hausdorff convergence, then we have* $\dim M_i > \dim N$.

Theorem 6.3 *Let a sequence of pointed complete Riemannian manifolds (M_i, p_i) in $\mathcal{M}(n, \kappa, \lambda)$ inradius collapse to a pointed length space (N, q) with respect to the pointed Gromov–Hausdorff convergence. Then N is an Alexandrov space with curvature $\geq c(\kappa, \lambda)$, where $c(\kappa, \lambda)$ is a constant depending only on κ and λ.*

To have Corollary 4.23 in the case when Y is noncompact is a main purpose of the rest of this section.

Remark 6.4 Property (6-2) only shows that there are at most two components of ∂M_i meeting a bounded region from the reference point p_i. Thus (6-2) does not immediately imply that the number of components of ∂M_i is at most two. This is because the convergence $(\widetilde{M}_i, p_i) \to (Y, p)$ is only under the *pointed* Gromov–Hausdorff topology. Namely, there is still a possibility that some component of ∂M_i disappears to infinity under that convergence.

To overcome the difficulty stated in Remark 6.4, we investigate the local connectedness of ∂M_i in more detail. To carry out this, it is helpful to consider a special pointed Gromov–Hausdorff approximations similar to (3-2), which is verified below.

Let $\iota_{\partial M} \colon ((\partial M)^{\mathrm{int}}, d_{\partial M^{\mathrm{int}}}) \to ((\partial M)^{\mathrm{ext}}, d_{\partial M^{\mathrm{ext}}})$ be the canonical map, where $(\partial M)^{\mathrm{ext}}$ is equipped with the exterior metric in M. For each $p \in M$ take a point $q \in \partial M$ satisfying $|p, q| = |p, \partial M|$. Setting $\omega(p) = q$, we define a nearest-point map $\omega_M \colon M \to \partial M$. It should be noted that although ω_M is not continuous in general, it will not affect the argument below (compare Proposition 3.5).

For $(\widetilde{M}, M, p)$ and $(Y, X, x) \in \partial_0 \widetilde{\mathcal{M}}\mathcal{M}(n, \kappa, \lambda)_{\mathrm{pt}}$ with

$$Y = X \cup_{\eta_0} C_0 \times_\phi [0, t_0],$$

as described above, set

$$\partial M^{\mathrm{int}}(p, 1/\delta) := (\partial M \cap B^{\widetilde{M}}(p, 1/\delta), d_{\partial M^{\mathrm{int}}}),$$
$$C_0^{\mathrm{int}}(p_0, 1/\delta) := (C_0 \cap B^C(p_0, 1/\delta, d_{C_0^{\mathrm{int}}}),$$

where $p_0 \in \eta_0^{-1}(x)$. Note that if $q, q' \in \partial M^{\mathrm{int}}(p, 1/\delta)$ belong to distinct components of ∂M, then the distance between them in $\partial M^{\mathrm{int}}(p, 1/\delta)$ is infinity: $d_{\partial M^{\mathrm{int}}}(q, q') = \infty$.

Similarly, we also consider

$$\partial \widetilde{M}^{\mathrm{int}}(p, 1/\delta) := (\partial \widetilde{M} \cap B^{\widetilde{M}}(p, 1/\delta), d_{\partial \widetilde{M}}),$$
$$C_{t_0}(p_0, 1/\delta) := (C_{t_0} \cap B^{C}(p_0, 1/\delta), d_{C_{t_0}}).$$

Sublemma 6.5 *The number of components of ∂M_i which intersect $\partial M_i^{\mathrm{int}}(p_i, 1/\delta_i)$ is at most two.*

Proof Suppose that there are three points $q_i^\alpha \in \partial M_i^{\mathrm{int}}(p_i, 1/\delta_i)$ which belong to three distinct components $\partial^\alpha M_i$ for $1 \le \alpha \le 3$. Let $\hat{q}_i^\alpha \in \partial \widetilde{M}(p_i, 1/\delta)$ be the image of q_i^α under the projection to $\partial \widetilde{M}_i$ along perpendiculars, which belongs to the component $\partial^\alpha \widetilde{M}_i$ corresponding to $\partial^\alpha M_i$. Since C_0 as well as C_{t_0} has at most two components, we may assume that $\varphi_i(q_i^1)$ and $\varphi_i(q_i^2)$ are in the same component of C_0. It turns out that $\varphi_i(\hat{q}_i^1)$ and $\varphi_i(\hat{q}_i^2)$ are in the same component of C_{t_0}, which contradicts (6-1). □

Definition 6.6 For $(\widetilde{M}, M, p)$ and $(Y, X, x) \in \partial_0 \widetilde{\mathcal{M}}\mathcal{M}(n, \kappa, \lambda)_{\mathrm{pt}}$ with

$$Y = X \cup_{\eta_0} C_0 \times_\phi [0, t_0],$$

we define the pointed Gromov–Hausdorff starred distance

$$d^*_{\mathrm{pGH}}((\widetilde{M}, M, p), (Y, X, x)) \tag{6-3}$$

as the infimum of those $\delta > 0$ such that

(1) there exists a componentwise δ–approximation

$$\psi\colon (\partial M)^{\mathrm{int}}(p, 1/\delta) \to C_0^{\mathrm{int}}(x, 1/\delta);$$

(2) the map $\varphi\colon B^{M^{\mathrm{ext}}}(p, 1/\delta) \to B^{X^{\mathrm{ext}}}(x, 1/\delta)$ defined by

$$\varphi = \eta_0 \circ \psi \circ \iota_{\partial M}^{-1} \circ \omega_M$$

is a δ–approximation:

$$\begin{array}{ccccc} M^{\mathrm{ext}} & \xrightarrow{\omega_M} & \partial M^{\mathrm{ext}} & \xrightarrow{\iota_M^{-1}} & \partial M^{\mathrm{int}} \\ \varphi \big\downarrow & & & & \big\downarrow \psi \\ X^{\mathrm{ext}} & & \xleftarrow{\quad\eta_0\quad} & & C_0^{\mathrm{int}} \end{array}$$

(3) the map $\Phi\colon B^{\widetilde{M}}(p,1/\delta)\to B^Y(x,1/\delta)$ defined by

$$\Phi(q)=\begin{cases}\varphi(q) & \text{if } q\in M\cap B^{\widetilde{M}}(p,1/\delta),\\ (\varphi(q_0),t) & \text{if } q=(q_0,t)\in\partial M\times_\phi[0,t_0]\cap B^{\widetilde{M}}(p,1/\delta),\end{cases}$$

is a δ–approximation.

This definition is justified by the following lemma:

Lemma 6.7 *Let the sequence $(\widetilde{M}_i,M_i,p_i)\in\widetilde{\mathcal{M}}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$ converge to (Y,X,x) in $\partial_0\widetilde{\mathcal{M}}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$ for the usual pointed Gromov–Hausdorff topology. Then there exists a componentwise δ_i–approximation*

$$\psi_i\colon(\partial M_i)^{\mathrm{int}}(p_i,1/\delta_i)\to C_0^{\mathrm{int}}(x,1/\delta_i)$$

with $\lim\delta_i=0$ *such that the maps*

$$\varphi_i\colon B^{M_i^{\mathrm{ext}}}(p_i,1/\delta_i)\to B^{X^{\mathrm{ext}}}(x,1/\delta_i),\quad \Phi_i\colon B^{\widetilde{M}_i}(p_i,1/\delta_i)\to B^Y(x,1/\delta_i)$$

defined as in Definition 6.6 via ψ_i are δ_i'–approximations with $\lim\delta_i'=0$.

Proof Let $\lambda_i\colon B^{\widetilde{M}_i}(p_i,1/\epsilon_i)\to B^Y(x,1/\epsilon_i)$ be an ϵ_i–approximation with $\lim\epsilon_i=0$. We may assume that when it is restricted to the boundary, it provides a componentwise ϵ_i–approximation $\lambda_i^{t_0}\colon B^{\widetilde{M}_i}(p_i,1/\epsilon_i)\cap\partial\widetilde{M}_i\to B^Y(x,1/\epsilon_i)\cap C_{t_0}^Y$. Since $\partial\widetilde{M}_i$ and C_{t_0} are convex and $1/\phi(t_0)$–homothetic to $(\partial M_i)^{\mathrm{int}}$ and C_0, respectively, $\lambda_i^{t_0}$ gives a componentwise $\epsilon_i/\phi(t_0)$–approximation

$$\psi_i\colon(\partial M_i)^{\mathrm{int}}\big(p_i,1/(\phi(t_0)\epsilon_i)\big)\to C_0^{\mathrm{int}}\big(p_0,1/(\phi(t_0)\epsilon_i)\big).$$

Let $\delta_i:=\phi(t_0)\epsilon_i$, and define φ_i and Φ_i as in Definition 6.6. In a way similar to Proposition 3.5, one can easily show that the restriction

$$\Phi_i\colon B^{\widetilde{M}_i}(p_i,1/\delta_i)\setminus M_i\to B^Y(x,1/\delta_i)\setminus X$$

is a δ_i'–approximation with $\lim\delta_i'=0$. In particular, this implies that

$$\varphi_i\colon B^{M_i^{\mathrm{ext}}}(p_i,1/\delta_i)\cap\partial M_i\to B^{X^{\mathrm{ext}}}(x,1/\delta_i)$$

is also a δ_i'–approximation. Let $\nu_i:=\mathrm{inrad}(M_i)$. Since $B^{M_i^{\mathrm{ext}}}(p_i,1/\delta_i)\cap\partial M_i$ is ν_i–dense in $B^{M_i^{\mathrm{ext}}}(p_i,1/\delta_i)\cap\partial M_i$, φ_i is certainly a δ_i''–approximation with $\lim\delta_i''=0$. Since Φ_i is a natural extension of φ_i, Φ_i is also a δ_i''–approximation. □

Lemma 6.8 *For each $\delta > 0$ there exists a positive number $\epsilon = \epsilon(\delta)$ such that if (M, p) in $\mathcal{M}(n, \kappa, \lambda)_{\mathrm{pt}}$ satisfies $\operatorname{inrad}(M) < \epsilon$, then*

$$d^{*}_{\mathrm{pGH}}((\widetilde{M}, M, p), (Y, X, x)) < \delta$$

for some (Y, X, x) contained in $\partial_0 \widetilde{\mathcal{M}}\mathcal{M}(n, \kappa, \lambda)_{\mathrm{pt}}$.

Proof Lemma 6.8 follows from Lemma 6.7 and the precompactness of $\widetilde{\mathcal{M}}\mathcal{M}(n, \kappa, \lambda)_{\mathrm{pt}}$ combined with a contradiction argument. □

If $(Y, X, x) \in \partial_0 \widetilde{\mathcal{M}}\mathcal{M}(n, \kappa, \lambda)_{\mathrm{pt}}$ satisfies the conclusion of Lemma 6.8 for $(M, p) \in \mathcal{M}(n, \kappa, \lambda)_{\mathrm{pt}}$, we call it a *$\delta$–limit* of $(\widetilde{M}, M, p)$, which is also denoted by $\mathcal{Y}(M, p)$ for simplicity:

$$\mathcal{Y}(M, p) = (Y, X, x).$$

6.2 Local/global connectedness of boundary

In this subsection, using Lemma 6.8, we first investigate the local behavior of connectedness of boundary of a inradius collapsed manifold, and then provide the proof of Theorem 1.6.

Definition 6.9 Let $(Y, X, x) \in \partial_0 \widetilde{\mathcal{M}}\mathcal{M}(n, \kappa, \lambda)$ and $y \in X$. We call y a *single point* (resp. *double point*) if $\#\eta_0^{-1}(y) = 1$ (resp. $\#\eta_0^{-1}(y) = 2$). We say that (Y, X, x) is single (resp. double) if every element of X is single (resp. double). If (Y, X, x) is neither single nor double, it is called *mixed*. We also say that (Y, X, x) is single (resp. double) in scale R if every element of $X \cap B^Y(x, R)$ is single (resp. double). If (Y, X, x) is neither single nor double in scale R, it is called mixed in scale R.

From now on, to prove Theorem 1.6, we analyze the local structure of ∂M about the connectedness when $\operatorname{inrad}(M) < \epsilon$. By Lemma 6.8, for any $p \in M$, there exists a δ–limit $\mathcal{Y}(M, p) = (Y, X, x)$ together with

(1) a δ–approximation $\psi\colon (\partial M)^{\mathrm{int}}(p, R) \to C_0^{\mathrm{int}}(p, R)$;

(2) a δ–approximation $\varphi := \eta_0 \circ \psi \circ \iota_{\partial M}^{-1} \circ \omega_M\colon B^{M^{\mathrm{ext}}}(p, R) \to B^{X^{\mathrm{ext}}}(x, R)$.

Note that for every $p_1, p_2 \in B^{M^{\mathrm{ext}}}(p, R)$,

$$\begin{aligned} |\varphi(p_1), \varphi(p_2)|_{X^{\mathrm{int}}} &\le L|\varphi(p_1), \varphi(p_2)|_{X^{\mathrm{ext}}} \\ &\le L(|p_1, p_2|_{M^{\mathrm{ext}}} + \delta) \le L(|p_1, p_2|_{M^{\mathrm{int}}} + \delta). \end{aligned} \tag{6-4}$$

Those approximation maps are effectively used in the proofs of the following lemma:

Lemma 6.10 *For any $R > 0$ there exists $\delta_0 > 0$ satisfying the following: For every $0 < \delta \le \delta_0$, let $\epsilon = \epsilon(\delta) > 0$ be as in Lemma 6.8. For each M in $\mathcal{M}(n, \kappa, \lambda)$ with* $\operatorname{inrad}(M) < \epsilon$ *and for each $p \in \partial M$, we have the following: Let $\mathcal{Y}(M, p)$ any δ–limit of (M, p).*

(1) *If $\mathcal{Y}(M, p)$ is single in scale R, then every $p_1, p_2 \in \partial M \cap B^{\widetilde{M}}(p, R)$ can be joined by a curve in ∂M of length $\le L|p_1, p_2|_M + (L+1)\delta$.*

(2) *If $\mathcal{Y}(M, p)$ is double in scale R, then there exists a point $p' \in \partial M$ with $|p, p'|_M < \delta$ such that every $q \in \partial M \cap B^{\widetilde{M}}(p, R)$ can be joined to p or p' by a curve in ∂M of length $\le L|p, q|_M + (L+2)\delta$.*

(3) *If $\mathcal{Y}(M, p)$ is mixed in scale R, then there exists a point $p_0 \in \partial M \cap B^{\widetilde{M}}(p, R)$ such that every point q in $\partial M \cap B^{\widetilde{M}}(p, R)$ can be joined to p_0 by a curve in ∂M of length $L|p_0, q|_M + (L+2)\delta$.*

Proof Let $(Y, X, x) := \mathcal{Y}(M, p)$, and ψ, φ be approximation maps as above.

(1) Put $x_i := \varphi(p_i) \in X$ for $i = 1, 2$. Take $\tilde{x}_i \in C_0$ such that $\eta_0(\tilde{x}_i) = x_i$. Lemma 4.29 shows $|\tilde{x}_1, \tilde{x}_2| = |x_1, x_2|$. Since ψ is a δ–approximation and $\psi(p_i) = \tilde{x}_i$, it follows from (6-4) that

$$|p_1, p_2|_{\partial M^{\mathrm{int}}} < |\tilde{x}_1, \tilde{x}_2|_{C_0^{\mathrm{int}}} + \delta = |x_1, x_2|_{X^{\mathrm{int}}} + \delta < L|p_1, p_2|_M + (L+1)\delta.$$

(2) Set $x := \varphi(p)$, $y := \varphi(q)$. Since (Y, X, x) is double in scale R, we can put $\{\tilde{x}_1, \tilde{x}_2\} := \eta_0^{-1}(x)$ and $\{\tilde{y}_1, \tilde{y}_2\} := \eta_0^{-1}(y)$. Let $\gamma\colon [0,1] \to X$ be a minimal geodesic joining x to y. From Lemma 4.29, there are lifts $\tilde{\gamma}_i\colon [0,1] \to C_0$ of γ starting from $\tilde{x}_i$, where we may assume $\tilde{\gamma}_i(1) = \tilde{y}_i$ and $\tilde{x}_1 = \psi(p)$. If $\psi(q) = \tilde{y}_1$, then

$$|p, q|_{\partial M^{\mathrm{int}}} < |\tilde{x}_1, \tilde{y}_1|_{C_0^{\mathrm{int}}} + \delta = |x, y|_{X^{\mathrm{int}}} + \delta < L|p, q|_M + (L+1)\delta.$$

If $\psi(q) = \tilde{y}_2$, then take a point p' with $|\psi(p'), \tilde{x}_2| < \delta$. Then, similarly, we have $|p', q|_{\partial M^{\mathrm{int}}} < L|p, q|_M + (L+2)\delta$.

(3) Let $x_0 \in X$ be a single point with $|x, x_0| \le R$, and take $\tilde{x}_0 \in C_0$ and $p_0 \in \partial M$ such that $\eta_0(\tilde{x}_0) = x_0$ and $|\psi(p_0), \tilde{x}_0| < \delta$ Let $\gamma\colon [0,1] \to X$ be a minimal geodesic from x_0 to $\varphi(q)$. Since $\tilde{x}_0 \in C_0^1$, there is a unique minimal geodesic $\tilde{\gamma}\colon [0,1] \to C_0$ from $\tilde{x}_0$ to $\psi(q)$ with $\eta_0 \circ \tilde{\gamma} = \gamma$. We then have

$$\begin{aligned}
|p_0, q|_{\partial M^{\mathrm{int}}} &< |\tilde{x}_0, \psi(q)|_{C_0^{\mathrm{int}}} + \delta = |x_0, \varphi(q)|_{X^{\mathrm{int}}} + \delta \\
&\le |\varphi(q), \varphi(p_0)|_{X^{\mathrm{int}}} + |\varphi(p_0), x_0|_{X^{\mathrm{int}}} + \delta \\
&\le L|p_0, q|_M + (L+2)\delta.
\end{aligned}$$

□

From now on, for a fixed $R>0$, let $\delta=\delta_0(n,\kappa,\lambda,R)>0$ and $\epsilon=\epsilon(\delta)>0$ be as determined in Lemma 6.10.

Lemma 6.11 *If $M\in\mathcal{M}(n,\kappa,\lambda)$ has inradius $\operatorname{inrad}(M)<\epsilon$ and disconnected boundary, then every δ–limit $\mathcal{Y}(M,p)$ is double in scale R for every $p\in\partial M$.*

Proof Suppose that some δ–limit $\mathcal{Y}(M,p)=(Y,X,x)$ is single or mixed in scale R. First note that by Lemma 6.10(1), (3), any points q_1,q_2 in $\partial M\cap B^{\widetilde{M}}(p,R)$ can be joined by a curve in ∂M. Take a point $p_\alpha\in\partial M$ contained in a component different from the component containing p. Let $c\colon[0,\ell]\to M$ be a unit-speed minimal geodesic in M from p to p_α. For each k with $1\le k\le[2\ell/R]$, take $p_k\in\partial M$ with $\bigl|p_k,c\bigl(\frac12 kR\bigr)\bigr|_M<\epsilon$. Note that

$$B^{\widetilde{M}}(p_k,R)\cap B^{\widetilde{M}}(p_{k+1},R)\cap\partial M\neq\varnothing$$

for each $1\le k\le[2\ell/R]-1$. By applying Lemmas 6.10 to p_k together with a standard monodromy argument, we see that p_α can be joined to p by a curve in ∂M, which is a contradiction. □

Lemma 6.12 *Suppose that $M\in\mathcal{M}(n,\kappa,\lambda)$ has inradius $\operatorname{inrad}(M)<\epsilon$ and disconnected boundary. For any $p\in\partial M$, let (Y,X,x) be any δ–limit for (M,p). For any $y\in X$, take distinct points $y_1\neq y_2\in C^Y_{t_0}$ such that $|y_i,y|=t_0$. Then $|y_1,y_2|=2t_0$.*

Remark 6.13 In Lemma 6.12, we need the assumption on the disconnectedness of ∂M. Namely, for some (Y,X,x) which is double, the conclusion of Lemma 6.12 does not hold. For instance, take the Möbius band

$$Y=S^1_\ell\,\tilde{\times}_{\tilde{\phi}}\,[-t_0,t_0].$$

If the length ℓ of $X=S^1_\ell$ is smaller than t_0, then $|y_1,y_2|<2t_0$ for every $y\in X$ and $y_i\in C^Y_{t_0}$ with $|y_i,X|=t_0$ for $i=1,2$.

Proof of Lemma 6.12 First note that (Y,X,x) is double in scale R. Suppose $|y_1,y_2|<2t_0$, and take a minimal geodesic γ joining them in Y. Then γ does not meet X, and therefore we can project γ to $C^Y_{t_0}$. The obtained curve $\pi_{t_0}(\gamma)$ joins y_1 and y_2 in $C^Y_{t_0}$. Thus the two elements $\tilde{y}_1$ and $\tilde{y}_2$ of $\eta_0^{-1}(y)$ can be joined in C_0. Take $q_1,q_2\in\partial M$ such that $|\psi(q_k.),\tilde{y}_k|<\delta$ for $k=1,2$. Lemma 6.10(2) shows that every $p'\in\partial M\cap B^{\widetilde{M}}(p,R)$ can be joined to q_1 or q_2 by a curve in ∂M. By a monodromy argument as in Lemma 6.11, we can conclude that every $q\in\partial M$ can be joined to q_1 or q_2 by a curve in ∂M, which is a contradiction. □

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6 We assume that $\operatorname{inrad}(M) < \epsilon$.

(1) Suppose that ∂M is disconnected. By Lemma 6.11, every δ–limit $\mathcal{Y}(M,p)$ is double in scale R for every $p \in M$. Take p_α and p_β from distinct components of ∂M. For every $p \in \partial M$, let $c\colon [0,\ell] \to M$ be a unit-speed curve in M from p_α to p_β through p. For each k with $1 \le k \le [2\ell/R]$, take $p_k \in \partial M$ with $\bigl|p_k, c\bigl(\frac{1}{2}kR\bigr)\bigr|_M < \epsilon$. By applying Lemma 6.10(2) to each p_k together with a standard monodromy argument as in Lemma 6.11, we see that p can be joined to p_α or p_β by a curve in ∂M. Therefore we conclude that the number of boundary components of M is at most two.

(2) Suppose that ∂M has two components. By Lemma 6.11, any δ–limit $\mathcal{Y}(M,p) = (Y,X,x)$ is double in scale R for every $p \in \partial M$. Therefore, for any $y \in X$, there are distinct $y_1 \ne y_2 \in C^Y_{t_0}$ with $|y_k, y| = t_0$ for $k = 1,2$. Lemma 6.12 shows that

$$|y_1, y_2| = 2t_0.$$

Let W be a component of $\partial\widetilde{M}$, and consider the distance function d_W from W. The above observation shows that for any $p \in M$, there exists a point $q \in \partial\widetilde{M}$ such that

$$\tilde{\angle}\, Wpq > \pi - \tau(\delta).$$

That is, d_W is $\frac{1}{2}\pi - \tau(\delta)$–regular on a neighborhood of M in $\widetilde{M}$. This makes it possible to define locally defined gradient-like vector fields for d_W on neighborhoods of the points of M. Then, by gluing those local gradient-like vector fields, we get a globally defined gradient-like vector field V on $\widetilde{M}$ whose support is contained in a neighborhood of M. It is now straightforward to obtain a diffeomorphism between $\widetilde{M}$ and $W \times [0,1]$ by means of integral curves of V. □

Theorem 6.14 (Gromov [10]; Alexander and Bishop [1]) *There exists a positive number $\epsilon = \epsilon(n,\kappa,\lambda)$ such that if $M \in \mathcal{M}(n,\kappa,\lambda)$ has the two side bounds on sectional curvature $|K_M| \le \kappa^2$ in addition and if the inradius satisfies $\operatorname{inrad}(M) < \epsilon$, then either M or its double cover is diffeomorphic to a product $W \times [0,1]$, where W is a closed manifold.*

The following example shows that Theorem 6.14 does not hold in the connected boundary case if one drops the upper sectional curvature bound $K_M \le \kappa^2$. Namely there are some $M \in \mathcal{M}(n,\kappa,\lambda,d)$ with connected boundary and with small inradius that are not finitely covered by any topological product of the form $W \times [0,1]$, where W is a closed manifold.

Example 6.15 Let N be a compact surface of genus one with connected boundary, and consider a Riemannian metric on N such that ∂N has a cylindrical neighborhood U_ϵ. Namely there is an isometric imbedding $f\colon S^1_\ell \times [0,\epsilon) \to U_\epsilon$ such that $f(S^1_\ell \times 0) = \partial N$, where $\ell = L(\partial N)$. Consider a segment $I = \{(x,0,0) \mid 0 \le x \le 2\epsilon\}$ in the xyz–space $\mathbb{R}^3$, and let D_ϵ denote the intersection of the boundary of the ϵ–neighborhood of I with $\{x \le \epsilon, z \le 0\}$. Let

$$J_\epsilon := D_\epsilon \cap \{x = \epsilon\}, \qquad K_\epsilon := D_\epsilon \cap \{x \le 0, z = 0\},$$
$$L_\epsilon := D_\epsilon \cap \{z = -\epsilon\}, \quad E_\epsilon := D_\epsilon \cap \{0 \le x \le \epsilon\}.$$

Note that J_ϵ and K_ϵ (resp. L_ϵ) are segments of length $\pi\epsilon$ (resp. length ϵ). Since there is an isometry $\varphi\colon \overline{U}_\epsilon \times \left[-\frac{1}{2}\pi\epsilon, \frac{1}{2}\pi\epsilon\right] \to S^1_\ell \times E_\epsilon$, we have an obvious gluing to obtain a three-dimensional Riemannian manifold M_ϵ with totally geodesic boundary,

$$M_\epsilon = N \times \left[-\tfrac{1}{2}\pi\epsilon, \tfrac{1}{2}\pi\epsilon\right] \amalg_\varphi S^1_\ell \times D_\epsilon.$$

Note that after slight smoothing of M_ϵ, we may assume that $M_\epsilon \in \mathcal{M}(3,\kappa,0,d)$ for some κ and d, and it inradius collapses to N as $\epsilon \to 0$. Note that M_ϵ is homeomorphic to $N \times I \cup \partial N \times D^2$ as in Theorem 5.4.

Note that any finite cover $\widehat{M}_\epsilon$ of M_ϵ is not homeomorphic to $W \times [0,1]$ for any closed surface. Otherwise a finite cover $\widehat{M}_\epsilon$ of M_ϵ is homeomorphic to $W \times [0,1]$ as above. Since M_ϵ has the same homotopy type as N, $\pi_1(M_\epsilon)$ is a free group generated by two elements. It turns out that $\pi_1(\widehat{M}_\epsilon) = \pi_1(W)$ is a free group, which is a contradiction.

7 Remark on locally convex manifolds

In the argument so far, the assumption $|\mathrm{II}_{\partial M}| \le \lambda^2$ was used to have lower sectional curvature bound $K_{\partial M} \ge c(\kappa,\lambda)$. It is a challenging problem to study the case where only the lower bound $\mathrm{II}_{\partial M} \ge -\kappa$ is assumed.

In the case of locally convex boundary in the sense that $\mathrm{II}_{\partial M} \ge 0$, the Gauss equation implies $K_{\partial M} \ge \kappa$ as long as $K_M \ge \kappa$. Therefore, taking $\phi(t) = 1$ as the warping function, we can extend M to $\widetilde{M} = M \cup \partial M \times [0,t_0]$, and proceed by the same argument as in the previous sections, to obtain the results corresponding to Theorems 1.3, 1.5 and 1.6.

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