

The classification of Lagrangians nearby the Whitney immersion

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The Whitney immersion is a Lagrangian sphere inside the four-dimensional symplectic vector space which has a single transverse double point of Whitney self-intersection number $+1$. This Lagrangian also arises as the Weinstein skeleton of the complement of a binodal cubic curve inside the projective plane, and the latter Weinstein manifold is thus the “standard” neighbourhood of Lagrangian immersions of this type. We classify the Lagrangians inside such a neighbourhood which are homologically essential, and which are either embedded or immersed with a single double point; they are shown to be Hamiltonian isotopic to either product tori, Chekanov tori, or rescalings of the Whitney immersion.

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1 Introduction

In the following $(\mathbb{C}P^2, \omega_{\text{FS}})$ is taken to denote the complex projective plane endowed with the Fubini–Study symplectic form, where the latter has been normalised so that a line is of symplectic area equal to $\int_{\ell} \omega_{\text{FS}} = \pi$. Our main result concerns classification up to Hamiltonian isotopy of embedded Lagrangian tori and immersed Lagrangian spheres inside the open symplectic manifold

$$V := \mathbb{C}P^2 \setminus (\ell_{\infty} \cup C) \subset (\mathbb{C}P^2, \omega_{\text{FS}}),$$

where $\ell_{\infty} \subset \mathbb{C}P^2$ denotes the line at infinity and where $C := \overline{\{z_1 z_2 = 1\}} \subset \mathbb{C}P^2$ is a smooth conic. In other words, V is the complement of the binodal cubic curve $\ell_{\infty} \cup C$. The fact that (V, ω_{FS}) is a Liouville manifold for a family of inequivalent Liouville forms λ_r , $d\lambda_r = \omega_{\text{FS}}$, parametrised by $r \in (0, \frac{3}{2})$ will play an important role in our proof; see [Section 6.1](#) for their construction. (In fact, it is well known that V even admits the structure of a Weinstein manifold, but this will not be needed.)

In [Section 1.3](#) we give an explicit description of a one-parameter family

$$\Pi_s: V \rightarrow (-1, 1) \times (0, +\infty) \quad \text{for } s \in \left(0, \frac{\pi}{2}\right)$$

of Lagrangian fibrations, the fibres of which project to simple closed curves in \mathbb{C} that encircle the value $1 \in \mathbb{C}$ under the standard Lefschetz fibration $(z_1, z_2) \mapsto z_1 z_2$. All fibres of Π_s are embedded Lagrangian tori except $\Pi_s^{-1}(0, 1)$, which is singular; it consists of a Lagrangian sphere having one transverse double point of Whitney self-intersection number equal to $+1$, and for which the symplectic action class

$$\int_{[\cdot]} \omega_{\text{FS}}: H_2(\mathbb{C}P^2 \setminus \ell_{\infty}, \Pi_s^{-1}(0, 1)) \rightarrow \mathbb{R}$$

assumes precisely the values ns for $n \in \mathbb{Z}$. This singular fibre is a Lagrangian incarnation of the so-called Whitney immersion, which becomes exact inside V for the aforementioned Liouville form $\lambda_{3s/\pi}$; see [Lemma 6.2\(1\)](#).

All the Lagrangian fibres of Π_s are well-studied objects, going back to work by Y V Chekanov [\[6\]](#) and by Y Eliashberg and L Polterovich [\[15\]](#); for that reason we call them *standard*. The embedded Lagrangian tori are of two different types: product tori, including the monotone Clifford torus, and monotone Chekanov tori. (Monotonicity here refers to the tori when considered inside $(\mathbb{C}P^2 \setminus \ell_{\infty}, \omega_{\text{FS}})$.) Our main result can be roughly stated as follows: any Lagrangian inside (V, ω_{FS}) with the same classical properties as those of a fibre $\Pi_s^{-1}(u_1, u_2)$ is actually Hamiltonian isotopic to a fibre.

The Lagrangian fibrations Π_s can be well understood by using the theory of almost toric systems as developed by M Symington [34]; the unique singular fibre corresponds to the unique node of the base diagram (corresponding to a singularity of “focus-focus” type), while varying the parameter s is equivalent to performing a so-called “nodal slide”. The almost toric systems whose singularity consists of one single such node constitute the simplest examples of nontrivial Lagrangian torus fibrations, and they have therefore been an important example when studying the SYZ conjecture in mirror symmetry; see eg D Auroux [4].

The Hamiltonian classification results for Lagrangian submanifolds are scarce (except, of course, when the symplectic manifold is of dimension two and the Lagrangian thus is a curve). The only known results for closed Lagrangians exist in the present setting of four-dimensional symplectic manifolds, where strong results have been obtained for embedded discs by Eliashberg and Polterovich [14], spheres by R Hind [20], and tori by the author together with E Goodman and A Ivrii [12]. These three works all utilise the technique of positivity of intersection in different ways; recall that positivity of intersection for pseudoholomorphic curves is a purely four-dimensional phenomenon.

In higher dimensions the only currently known Hamiltonian isotopy classifications hold for Lagrangians on the *flexible side* of symplectic topology, by the work of Eliashberg and E Murphy [13]. These results apply to Lagrangians with conical singularities over *loose* Legendrians. Without going into the details concerning these flexible Lagrangians, we would just like to point out that their singularities are more complicated than a transverse double point, which is the only type of singularity that we consider here.

One of the results proven in [12] was the nearby Lagrangian conjecture for $T^*\mathbb{T}^2$; this is a Weinstein manifold with skeleton that is an embedded Lagrangian torus. Since (V, ω_{FS}) can be endowed with a Weinstein structure for which the Lagrangian Whitney immersion is the skeleton, our result can be interpreted as a result in line with the nearby Lagrangian conjecture for a generically immersed Lagrangian sphere. Namely, our result in particular provides the following Hamiltonian classification of, for example, the Lagrangian tori which are homologically essential in some small neighbourhood of such an immersed Lagrangian.

1.1 Preliminaries

We begin by swiftly covering the notions needed to formulate our results. The experienced reader can safely skip this subsection.

For convenience, we will often utilise the standard symplectic identification

$$\varphi: (\mathbb{C}P^2 \setminus \ell_\infty, \omega_{\text{FS}}) \xrightarrow{\cong} (B^4, \omega_0), \quad [z_1 : z_2 : 1] \mapsto \frac{1}{\sqrt{1 + \|z_1\|^2 + \|z_2\|^2}}(z_1, z_2),$$

with inverse

$$(\tilde{z}_1, \tilde{z}_2) \mapsto \frac{1}{\sqrt{1 - \|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2}}(\tilde{z}_1, \tilde{z}_2),$$

in order to realise V as an embedding

$$V \cong \tilde{V} := B^4 \setminus \varphi(C) \subset (B^4, \omega_0)$$

into the standard symplectic unit ball. The linear symplectic form

$$\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$$

is exact with primitive $\lambda_{\text{std}} = x_1 dy_1 + x_2 dy_2$, which thus is a Liouville form for the symplectic form on V as well. (This Liouville form does not, however, make V into a Liouville domain.) We will often switch between these two realisations of (V, ω_{FS}) in order to work with the description which is most suitable for our different needs.

Recall that a two-dimensional immersion $\iota: L \hookrightarrow V$ is *Lagrangian* if $\iota^* \omega_{\text{FS}} \equiv 0$. A Lagrangian immersion is *weakly exact* if $\int_\alpha \omega_{\text{FS}} = 0$ for all $\alpha \in \pi_2(V, L)$. Given a choice of Liouville form λ for $\omega_{\text{FS}} = d\lambda$, we say that the Lagrangian is *exact* in the case when $\iota^* \lambda$ is an exact one-form, and *strongly exact* if the primitive moreover can be chosen to be constant when restricted to each preimage set $\iota^{-1}(\text{pt})$ for $\text{pt} \in \iota(L)$. Note that exact Lagrangian embeddings, as well as *strongly exact* Lagrangian immersions, necessarily also are weakly exact. More generally, the *symplectic action class* is given by $[\lambda|_{TL}] \in H^1(L, \mathbb{R})$; this class also depends on the choice of Liouville form.

Another important class associated to a Lagrangian submanifold is the *Maslov class*

$$\mu_L: H_2(V, L) \rightarrow \mathbb{Z}$$

which takes values in the even integers for an oriented Lagrangian; see eg McDuff and Salamon [29] for more details. For general closed curves on L there is also a notion of Maslov class induced by the trivialisation of $\mathbb{C}^2 \supset B^4 \supset \tilde{V}$; this Maslov class will be denoted by $\mu_L^{\mathbb{C}^2}: H^1(L) \rightarrow \mathbb{Z}$. Note that the equality $\mu_L^{\mathbb{C}^2} \circ \partial = \mu_L$ holds, where $\partial: H_2(V, L) \rightarrow H_1(L)$ is the connecting homomorphism.

The classification that we are pursuing is that of Lagrangians up to a *Hamiltonian isotopy* $\phi_{H_t}^t$, ie a smooth isotopy whose infinitesimal generator satisfies $\iota_{X_t} \omega_{\text{FS}} = -dH_t$

for a smooth family of functions $H_t: V \rightarrow \mathbb{R}$; this function is called the generating *Hamiltonian*. A standard result shows that a smooth path of Lagrangian embeddings $L_t \subset (V, \omega_{FS} = d\lambda)$, also called a *Lagrangian isotopy*, is generated by a global Hamiltonian isotopy if and only if the symplectic action is *constant* for the path relative an arbitrary choice of Liouville form λ . The difference

$$\text{Flux}(\{L_s\}_{s \in [a,b]}) := [\lambda|_{TL_b}] - [\lambda|_{TL_a}] \in H^1(\mathbb{T}^2)$$

of symplectic action classes (suitably identified) is called the *symplectic flux* of the Lagrangian isotopy $\{L_t\}_{t \in [a,b]}$. A Lagrangian isotopy L_t for $t \in [0, 1]$ is thus generated by a Hamiltonian if and only if the corresponding *symplectic flux-path*, defined as

$$t \mapsto \text{Flux}(\{L_s\}_{s \in [0,t]}) \in H^1(\mathbb{T}^2),$$

vanishes for all $t \in [0, 1]$.

1.2 Result

The result that we show here is a classification of the Lagrangians inside (V, ω_{FS}) up to Hamiltonian isotopy under the assumption that they satisfy properties similar to those of the fibres of Π_s . Our main result is as follows.

Theorem A *Let $L \subset (V, \omega_{FS})$ be either an embedded Lagrangian torus or an immersed Lagrangian sphere with a single transverse double point. Assume that **at least** one of the following two conditions is satisfied:*

- (1) *The class $[L] \in H_2(V) \cong \mathbb{Z}$ is nonzero in homology.*
- (2) (a) *If L is an embedded torus, then for any homotopy class $\alpha \in \pi_2(V, L)$ the implication*

$$\mu_L(\alpha) = 2 \implies \int_{\alpha} \omega_0 \leq 0$$

holds.

- (b) *If L is an immersed sphere, then property (a) holds for any Lagrangian torus resulting from a Lagrangian surgery applied to its double point.*

Then L is Hamiltonian isotopic inside (V, ω_{FS}) to a standard Lagrangian. In other words, there exists a nonempty subset of values $s \in (0, \pi/2)$ (possibly the entire interval) such that L is Hamiltonian isotopic to a unique fibre of the Lagrangian fibration $\Pi_s: V \rightarrow (-1, 1) \times (0, +\infty)$.

It is not difficult to see that both of the conditions in [Theorem A](#) are satisfied for standard Lagrangians, ie the fibres of the fibrations Π_s ; we refer to [Section 1.3](#) for more details.

Remark 1.1 We point out the following, with more details given in [Proposition 1.4](#):

(1) Condition (2) of [Theorem A](#) is automatically satisfied in the case when the Lagrangian (embedded or immersed) is weakly exact, or if the Lagrangian is embedded and has vanishing Maslov class. Further, note that all fibres $\Pi_s^{-1}(u_1, u_2)$ have vanishing Maslov class when considered inside (V, ω_{FS}) , while they are weakly exact if and only if $u_1 = 0$.

(2) The Hamiltonian isotopy classes of the strongly exact immersed spheres $\Pi_s^{-1}(0, 1)$ are all different, with each uniquely determined by the parameter $s \in (0, \pi/2)$. Contrary to this, every fixed torus fibre $\Pi_{s_0}^{-1}(u_1, u_2)$ that is not weakly exact (ie $u_1 \neq 0$) is Hamiltonian isotopic to a unique fibre $\Pi_s^{-1}(u_1, u'_2)$ for any other choice of $s \in (0, \pi/2)$ as well. For the weakly exact tori the situation is more complicated. A given weakly exact torus fibre of Clifford type is Hamiltonian isotopic to a fibre of the fibrations Π_s only for a certain strict subinterval of parameters $s \in (0, s_0) \subsetneq (0, \pi/2)$, while for a torus of Chekanov type the corresponding subinterval is of the form $(s_0, \pi/2)$.

[Theorem A](#) gives conditions for when a Lagrangian torus inside $(\mathbb{C}P^2, \omega_{FS})$ is Hamiltonian isotopic to a torus of either Clifford or Chekanov type in terms of the linking properties with a binodal cubic. In particular, [Theorem A](#) shows there are precisely two different monotone Lagrangian tori which are exact in the complement of the binodal cubic up to Hamiltonian isotopy. It has been shown by R Vianna [\[36\]](#) that there are *infinitely* many different Hamiltonian isotopy classes of monotone Lagrangian tori inside $(\mathbb{C}P^2, \omega_{FS})$ which, moreover, can be realised as exact Lagrangians inside the complement of the *smooth* cubic curve.

The central technique used in the proof of [Theorem A](#) is to consider the limit of pseudoholomorphic foliations by conics when stretching the neck around the Lagrangian torus. Note that there is a natural holomorphic conic fibration on V , which is the restriction of the Lefschetz fibration $z_1 z_2$ on $\mathbb{C}P^2 \setminus \ell_\infty$ to the complement of the smooth fibre C above $1 \in \mathbb{C}$. Here we study the foliations given by such conics that satisfy an additional tangency to C at ℓ_∞ for arbitrary almost complex structures (which are still required to be standard near ℓ_∞).

The idea to use pseudoholomorphic foliations and neck stretching to classify Lagrangian tori goes back to H Hofer and K M Luttinger. This program was carried out in the recent work of the author together with Goodman and Ivrii [12], where foliations by degree-one spheres were considered; see [12, Section 1.2] for more details concerning the history of the problem. One notable result obtained using these techniques was the positive answer to the so-called nearby Lagrangian conjecture for \mathbb{T}^2 ; see [12, Theorem B]. In the course of proving [Theorem A](#) we also need to provide a sharpening of this result:

Theorem B *Suppose that $L \subset (T^*\mathbb{T}^2, d\lambda_{\mathbb{T}^2})$ is a Lagrangian embedding which is either weakly exact, homologically essential, or a Lagrangian torus of vanishing Maslov class. In all these cases, L is Hamiltonian isotopic to the graph of a closed one-form in $\Omega^1(\mathbb{T}^2)$. Moreover, for any convex subset $U \subset \mathbb{R}^2$:*

- (1) *If $L \subset \mathbb{T}^2 \times U \subset T^*\mathbb{T}^2$, then the Hamiltonian isotopy can be taken to be supported inside $\mathbb{T}^2 \times U$.*
- (2) *For any $\theta \in S^1$ consider the properly embedded Lagrangian disc*

$$\begin{aligned} \dot{D}_{\mathbf{p}^0}(\theta) &:= (S^1 \times \{\theta\}) \times (\{p_1^0\} \times (-\infty, p_2^0]) \subset \mathbb{T}^2 \times \mathbb{R}^2 = T^*\mathbb{T}^2, \\ \mathbf{p}^0 &:= (p_1^0, p_2^0), \end{aligned}$$

with one interior point removed. If

$$L \cap \dot{D}_{\mathbf{p}^0}(e^{is}) = \partial \dot{D}_{\mathbf{p}^0}(e^{is}) = S^1 \times \{e^{is}\} \times \{\mathbf{p}^0\}$$

holds for all $|s| < \epsilon$, then the Hamiltonian isotopy can be assumed to be supported outside of the subset

$$\bigcup_{|s| < \delta} \dot{D}_{\mathbf{p}^0}(e^{is}) = S^1 \times e^{i[-\delta, \delta]} \times \{p_1^0\} \times (-\infty, p_2^0]$$

for some $0 < \delta < \epsilon$ sufficiently small (note that for symplectic action reasons, we may not be able to Hamiltonian isotope the Lagrangian to the constant section $\mathbb{T}^2 \times \{\mathbf{p}^0\}$).

- (3) *If $L \subset \mathbb{T}^2 \times U$ holds in addition to the assumptions of (2), then the Hamiltonian isotopy produced there can moreover be taken to have support contained inside $\mathbb{T}^2 \times U$.*

We prove this result in [Section 9](#). Note that (1) is a fairly straightforward consequence of [12, Theorem 7.1], while (2) requires a more careful study of its proof. Part (3) finally follows without too much additional work by simply combining (1) and (2).

In [Section 3](#) we show that (V, ω_{FS}) is a Liouville domain with completion $(\widehat{W}, d\lambda)$, which is a Liouville manifold whose skeleton is the Whitney sphere itself. A version of the Weinstein neighbourhood theorem shows that \widehat{W} serves as a standard neighbourhood for any immersed Lagrangian sphere having a single self-intersection point of positive sign. The classification given by [Theorem A](#) can thus be interpreted as a result in line with the nearby Lagrangian conjecture for such an *immersed* Lagrangian sphere. In addition, note that (V, ω_{FS}) is a Weinstein manifold that, in some sense, is not too distant from the cotangent bundle of a torus, since it can be obtained from $(DT^*\mathbb{T}^2, d\lambda_{\mathbb{T}^2})$ by attaching a single Weinstein two-handle along the conormal lift of a simple closed geodesic.

The complete Liouville manifold $(\widehat{W}, d\lambda)$ admits a surjective Lagrangian *almost toric* fibration $\widehat{\pi}: \widehat{W} \rightarrow \mathbb{R}^2$ with a single nodal fibre in the sense of Symington [\[34\]](#). The nodal fibre is an immersed Lagrangian sphere, while all other fibres are embedded tori. The process of a *nodal slide* introduced in the aforementioned work can be applied to the node, thereby producing a one-parameter family $\widehat{\pi}_s$ of almost toric fibrations of the same type. The Hamiltonian isotopy class of the nodal fibres for different values of $s \in \mathbb{R}$ live in distinct Hamiltonian isotopy classes. We fix our convention so that $\widehat{\pi}_0^{-1}(0)$ is a strongly exact immersed Lagrangian sphere for the Liouville form λ . Recall that since the fibration is assumed to be almost toric, the map $\widehat{\pi}_0$ is determined up to an affine transformation by the corresponding Lagrangian foliation.

Corollary 1.2 *A Lagrangian $L \subset (\widehat{W}, d\lambda)$ that satisfies either of the assumptions in [Theorem A](#) is Hamiltonian isotopic to a fibre of $\widehat{\pi}_s$ for some $s \in \mathbb{R}$. If L moreover is strongly exact with respect to the Liouville form λ , and if $\widehat{\pi}_0(L)$ is contained inside a subset $O \subset \mathbb{R}^2$ which is star-shaped with respect to the origin (in the above affine coordinates), then the Hamiltonian isotopy may be taken to have support inside the preimage $\widehat{\pi}_0^{-1}(O)$.*

Remark 1.3 We do not know whether it is possible to confine the above Hamiltonian isotopy to the preimage of the star-shaped subset $O \subset \mathbb{R}^2$ without the assumption of L being strongly exact. A direct consequence of [Theorem A](#) tells us that this stronger result is true at least when the preimage $\pi_0^{-1}(O)$ is symplectomorphic to $(V, c\omega_{\text{FS}})$ for some $c > 0$.

Proof Consider the fibre of $\widehat{\pi}_s$ that has the same classical invariants as those of L , and observe that

- when L is a sphere, this is the nodal fibre for a uniquely determined value of $s \in \mathbb{R}$,
- when L is an embedded torus which is not weakly exact, there is a unique such representative up to Hamiltonian isotopy, which moreover appears as fibres in the fibrations $\hat{\pi}_s$ for any choice of $s \in \mathbb{R}$, and
- when L is a weakly exact embedded torus, there are precisely two such Hamiltonian isotopy classes, and both can be assumed to be obtained by appropriate action-preserving Lagrangian surgeries on the immersed sphere $\hat{\pi}_s^{-1}(0)$ for a uniquely determined value of $s \in \mathbb{R}$.

Using Proposition 3.8 one constructs a symplectic embedding $\iota_s: (V, c\omega_{\text{FS}}) \hookrightarrow (\hat{W}, d\lambda)$ for some $c \gg 0$ satisfying the properties that both L and the Lagrangian fibre of $\hat{\pi}_s$ pinpointed above are contained inside the image of ι_s . Theorem A finally implies that L is Hamiltonian isotopic inside V to a fibre of $\Pi_{s'}$ for some $s' \in (0, \pi/2)$. Since the same is true also for the aforementioned fibre of $\hat{\pi}_s$, we have thereby managed to produce our Hamiltonian isotopy contained entirely inside $\iota_s(V)$.

For the last point, it is sufficient to apply the negative Liouville flow of λ to the Hamiltonian isotopy, to make sure that it stays inside the required subset. To that end, note that the Liouville form preserves exact Lagrangian submanifolds, and that it retracts the subset $\hat{\pi}_0^{-1}(O)$ onto the immersed Lagrangian sphere $\hat{\pi}_0^{-1}(0)$. See Section 3.2 for more details. □

1.3 A family of Lagrangian fibrations

Since the work of Chekanov [6] it has been known that (B^4, ω_0) admits two types of monotone Lagrangian tori that live in different Hamiltonian isotopy classes but whose classical invariants agree; these are the so-called Clifford and Chekanov tori. These two types of Lagrangian tori can also be realised as weakly exact Lagrangian tori inside (V, ω_{FS}) ; see Eliashberg and Polterovich [15] and Auroux [4]. Furthermore, as shown in the latter article, these tori arise as the leaves of Lagrangian torus fibrations on $\mathbb{C}P^2$, which also were studied by J Pascaleff [32] and by Vianna [35]. In particular, [35] makes use of the convenient language of almost toric fibrations and their deformations as introduced by Symington [34].

In this subsection we recall an explicit description of a one-parameter family of such Lagrangian fibrations on (V, ω_{FS}) . Alternatively, this fibration can be constructed by using the language of almost-toric base diagrams. More precisely, one can deform

the standard moment polytope of B^4 (ie the standard fibration by product tori) by a so-called nodal trade, followed by a nodal slide; the parameter of the slide induces the parameter in our family of fibrations. The reason for instead choosing the more explicit approach is as follows. First, we want the Lagrangian fibrations to be *compatible* with the standard Lefschetz fibration

$$f: (\mathbb{C}P^2 \setminus \ell_\infty, \omega_{FS}) \rightarrow \mathbb{C}, \quad (z_1, z_2) \mapsto z_1 \cdot z_2,$$

by holomorphic conics in the following sense: the restriction of f to a smooth fibre L of the Lagrangian fibration is a smooth S^1 -bundle over the simple closed curve $f(L) \subset \mathbb{C}$. Second, we want the fibration to restrict to a fibration on $V = \mathbb{C}P^2 \setminus (\ell_\infty \cup C)$; ie we want the smooth Lagrangian fibres to be disjoint from the smooth conic C .

Before starting, we say a few more words about the Lefschetz fibration f . Note that it has a unique singular fibre

$$C_{\text{nodal}} := f^{-1}(0) = \ell_1 \cup \ell_2$$

consisting of a union

$$\ell_1 := \overline{\mathbb{C} \times \{0\}} \quad \text{and} \quad \ell_2 := \overline{\{0\} \times \mathbb{C}}$$

of two lines. All conic fibres of f pass through the two points $q_i := \ell_i \cap \ell_\infty$ while being tangent to $v_i := T_{q_i} \ell_i$. We will call f a *symplectic Lefschetz fibration*, even if we make no claims concerning the symplectic triviality outside of a compact subset.

Now we are ready to commence with the construction of the Lagrangian fibrations Π_s . For convenience we will work with the identification $\tilde{V} \subset (B^4, \omega_0)$ of (V, ω_{FS}) , where the fibration takes the form

$$\tilde{f} := f \circ \varphi^{-1}: B^4 \rightarrow \mathbb{C}, \quad (\tilde{z}_1, \tilde{z}_2) \mapsto \frac{\tilde{z}_1 \tilde{z}_2}{1 - \|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2},$$

where $(\tilde{z}_1, \tilde{z}_2)$ are the standard complex coordinates on $B^4 \subset \mathbb{C}^2$. (This notation is useful for distinguishing the two different complex coordinates $\tilde{z}_i, z_i: V \rightarrow \mathbb{C}$.)

Begin by constructing a smooth one-parameter family of diffeomorphisms

$$\Psi_s: \mathbb{C} \rightarrow \mathbb{C} \quad \text{for } s \in \left(0, \frac{\pi}{2}\right)$$

that satisfy

$$\Psi_s(t) = t \quad \text{for } t \in [0, 1] \quad \text{and} \quad \overline{\Psi_s(z)} = \Psi_s(\bar{z}),$$

and which all are equal to the identity outside of some compact subset (that necessarily depends on the parameter s). We moreover demand that they satisfy the following.

Let $\eta_c := \{\|z - 1\| = c\} \subset \mathbb{C}$ be the foliation of $\mathbb{C} \setminus \{1\}$ by concentric circles centred at $1 \in \mathbb{C}$, where η_1 is the leaf passing through the origin. Then:

(1) The curve

$$(\Psi_s \circ \tilde{f})^{-1}(\eta_1) \cap \{\tilde{z}_1 = \tilde{z}_2\} \subset \tilde{V} \subset B^4$$

is an immersion of the form (α_s, α_s) for an immersed figure-8 curve $\alpha_s \subset D^2_{1/\sqrt{2}}$ satisfying $-\alpha_s = \alpha_s$. We require that the symplectic area of either of the two bounded components of $\alpha_s \subset (D^2_{1/\sqrt{2}}, dx \wedge dy)$ be equal to $s/2 \in (0, \pi/4)$ (ie the symplectic area inside B^4 bounded by (α_s, α_s) is equal to $s \in (0, \pi/2)$).

(2) There is a smooth extension of the family to include also diffeomorphisms

$$\Psi_0: \mathbb{C} \setminus [0, 1] \xrightarrow{\cong} \mathbb{C} \setminus D^2(1) \quad \text{and} \quad \Psi_{\pi/2}: \mathbb{C} \setminus (-\infty, 0] \xrightarrow{\cong} B^2(1) \subset \mathbb{C},$$

where Ψ_0 is the identity outside of some compact subset, and $\Psi_{\pi/2}$ is the identity near $1 \in \mathbb{C}$.

The dependence of the curves $\gamma_c := \Psi_s^{-1}(\eta_c)$ on the parameter $s \in (0, \pi/2)$ is exhibited in Figure 1.

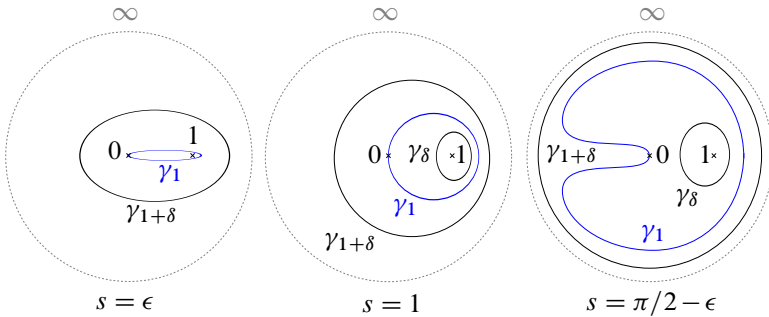


Figure 1: The image $\gamma_c := \Psi_s^{-1}(\eta_c)$ of the leaves in the foliation of \mathbb{C} by the concentric circles $\eta_c = \{\|z - 1\| = c\}$ for different values of the parameter $s \in (0, \pi/2)$.

We can finally define the smooth one-parameter family of Lagrangian fibrations to be

$$\tilde{\Pi}_s: \tilde{V} \rightarrow (-1, 1) \times (0, +\infty), \quad (\tilde{z}_1, \tilde{z}_2) \mapsto (\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2, \|\Psi_s(\tilde{f}(\tilde{z}_1, \tilde{z}_2)) - 1\|^2).$$

We also write $\Pi_s := \tilde{\Pi}_s \circ \varphi$ for the corresponding fibration on V .

It is immediate by the construction that these fibrations are compatible with the standard symplectic Lefschetz fibration f ; see Figures 2, 3 and 4, for a schematic depiction of this. The fibres $\Pi_s^{-1}(u_1, u_2)$ with $u_2 > 1$ and $u_2 < 1$ will be called tori of Clifford

type and Chekanov type, respectively. Note that a torus of Clifford type is fibred over a curve whose winding number around $0 \in \mathbb{C}$ is equal to one, while a torus of Chekanov type is fibred over such a curve with zero winding.

Later we will also make use of the limit case when $s = 0$. In this case, by property (2) above, we obtain an induced Lagrangian fibration

$$\Pi_0: V \setminus f^{-1}[0, 1] \rightarrow (-1, 1) \times (1, +\infty),$$

whose fibres all will turn out to be embedded Lagrangian tori.

We summarise the important properties of the Lagrangian fibres of Π_s in the following proposition, the proof of which we postpone to [Section 2.2](#).

Proposition 1.4 *All fibres of Π_s are compact Lagrangian immersions contained inside (V, ω_{FS}) . Furthermore:*

- (1) *The fibres $L_{Wh}(s) := \Pi_s^{-1}(0, 1)$ for $s \in (0, \pi/2)$ constitute a one-parameter family of weakly exact Lagrangian immersions of the sphere, each having a single transverse double point and Whitney self-intersection number equal to $+1$. It is the case that $[L_{Wh}(s)] \in H_2(V) \cong \mathbb{Z}$ is a generator. Moreover, the primitive of the pullback $\lambda_{std}|_{TL_{Wh}(s)}$ has potential difference equal to $\Delta(L_{Wh}(s)) = s$ at the double point; two different such spheres are hence not Hamiltonian isotopic.*
- (2) *A fibre $\Pi_s^{-1}(u_1, u_2)$ for $(u_1, u_2) \neq (0, 1)$ is an embedded Lagrangian torus which is homologous to the generator $[L_{Wh}(s)] \in H_2(V)$, and its Maslov class evaluates to zero on any element in $H_2(V, L)$. Such a fibre is weakly exact (and hence also monotone) if and only if $u_1 = 0$. Moreover:*
 - (a) *A weakly exact torus of Clifford type (resp. Chekanov type) is Hamiltonian isotopic inside $B^4 \supset \tilde{V}$ to a Clifford torus (resp. Chekanov torus).*
 - (b) *Any fibre $\Pi_s^{-1}(u_1, u_2)$ with $u_1 \neq 0$ (that is, any fibre that is not weakly exact) is Hamiltonian isotopic inside V to a unique fibre of the form $(\Pi_1)^{-1}(u_1, u'_2)$ for some appropriate $u'_2 \in (0, +\infty)$.*

The nonzero homology groups for this space are $H_i(V) = \mathbb{Z}$ for $i = 0, 1, 2$. By [Proposition 1.4\(2\)\(a\)](#) combined with the classical result of Chekanov [\[6\]](#) we deduce the following: weakly exact (or monotone) tori of Clifford and Chekanov types are never Hamiltonian isotopic, while such tori that are not weakly exact (or are nonmonotone) are all Hamiltonian isotopic to product tori.

Finally, we make the following comment regarding the apparent nonsymmetry that

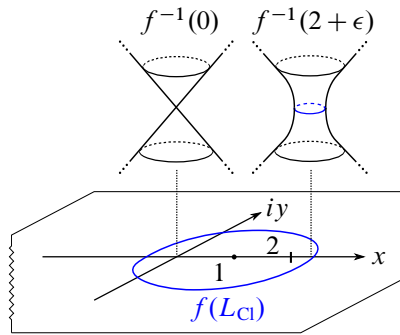


Figure 2: A Clifford torus $L_{Cl} := \Pi_s^{-1}(0, u_2)$ with $u_2 > 1$ is fibred over a curve in the base of the Lefschetz fibration which encircles the critical value.

holds between the tori of Clifford and Chekanov types when considered in conjunction with the Lefschetz fibration f ; this nonsymmetry disappears when one considers the Liouville completion of V as constructed in Section 3.2.

Proposition 1.5 (Propositions 3.8 and 3.5) *The Liouville completion $(\widehat{W}, d\lambda)$ of (V, ω_{FS}) admits a global symplectomorphism $I: (\widehat{W}, d\lambda) \rightarrow (\widehat{W}, d\lambda)$ which is of order two and fixes the Whitney immersion setwise, while interchanging the two sheets at the double point. Moreover, the involution I interchanges the Hamiltonian isotopy classes of the Clifford and Chekanov tori.*

As shown by Chekanov [6], we cannot find a similar symplectomorphism defined on all of $(\mathbb{C}P^2, \omega_{FS})$ or (B^4, ω_0) ; the Clifford and Chekanov tori cannot be interchanged by a globally defined symplectomorphism there.

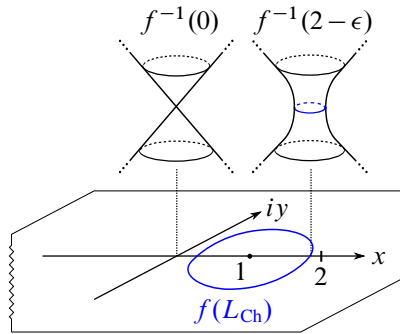


Figure 3: A Chekanov torus $L_{Ch} := \Pi_s^{-1}(0, u_2)$ with $u_2 < 1$ is fibred over a curve in the base of the Lefschetz fibration which does not encircle the critical value.

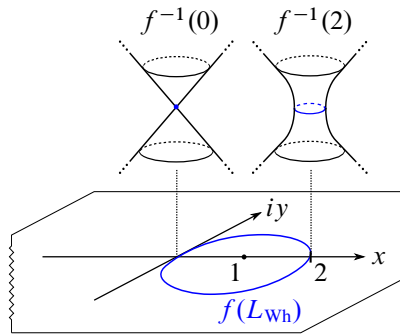


Figure 4: A Whitney immersion $L_{Wh} := \Pi_s^{-1}(0, 1)$ is fibred over a curve in the base of the Lefschetz fibration which passes through the critical value.

Acknowledgements I am grateful for having a very stimulating environment at the University of Cambridge, where the main ideas leading to this paper were born; thanks to Ivan Smith, Jack Smith, Dmitry Tonkonog, and Renato Vianna for useful discussions and a shared interest in Lagrangian tori. I would also like to thank my former PhD advisor Tobias Ekholm for introducing these questions to me during my thesis, and for stressing the importance of the technique of neck stretching in symplectic topology; it is obvious that the latter tool has played a very crucial role here. The author is supported by the grant KAW 2016.0198 from the Knut and Alice Wallenberg Foundation.

2 Properties of the Lagrangian fibration

Here we investigate crucial properties of the family of fibrations

$$\Pi_s: \tilde{V} \rightarrow (-1, 1) \times (0, +\infty) \quad \text{for } s \in \left(0, \frac{\pi}{2}\right).$$

We will see that the unique singular fibre is the immersed Lagrangian sphere $L_{Wh}(s) := \Pi_s^{-1}(0, 1)$, while all other fibres are smoothly embedded Lagrangian tori.

We start by treating the Lagrangian condition of the fibres, which will turn out to be a direct consequence of the following statement.

Lemma 2.1 *For any embedded path $\gamma \subset \mathbb{C}$ and $u_1 \in (-1, 1)$, the intersection*

$$\tilde{f}^{-1}(\gamma) \cap \{\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2 = u_1\} \subset (B^4, \omega_0)$$

is a Lagrangian submanifold outside of the origin. (The origin is in general a singular point.)

Proof The characteristic distribution of the hypersurface $\{\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2 = u_1\}$, ie the line field that is the kernel of the restriction of ω_0 , is generated by the vector field $(d/dt)(e^{it}\tilde{z}_1, e^{-it}\tilde{z}_2)$. (This can be checked, for example, by computing the Euclidean gradient of the function $\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2$, which is equal to $-2i(d/dt)(e^{it}\tilde{z}_1, e^{-it}\tilde{z}_2)$.) Since $\tilde{f}^{-1}(\gamma) \cap \{\|z_1\|^2 - \|z_2\|^2 = u_1\}$ is foliated by closed integral curves of the aforementioned characteristic distribution away from the origin, the claim now follows. \square

Below we will see that the fibres are compact subsets of $\mathbb{C}P^2 \setminus \ell_\infty$. The topology of the fibres can then be investigated by hand without too much difficulty; $L_{Wh}(s)$ is an immersed sphere having a transverse self-intersection whose Whitney self-intersection number is equal to $+1$, while the smooth fibres all are tori. (Recall that any smooth Lagrangian foliation by closed surfaces must consist of torus leaves by a version of the Arnold–Liouville theorem.)

Lemma 2.2 • *The fibre $\Pi_s^{-1}(u_1, u_2)$ with $(u_1, u_2) \neq (0, 1)$ is a smooth and compact Lagrangian torus living inside $\tilde{V} \subset (B^4, \omega_0)$. In fact these tori with fixed $u_1 = u_1^0 \neq 0$ provide a smooth foliation of the noncompact hypersurface $\Sigma_{u_1^0} \cap \tilde{V}$, where*

$$\Sigma_{u_1^0} := \{\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2 = u_1^0\} \cap B^4$$

is an open solid torus.

- *The fibre $L_{Wh}(s) = \Pi_s^{-1}(0, 1)$ is a Lagrangian sphere with a single transverse double point with Whitney self-intersection number equal to $+1$.*

Proof The Lagrangian condition was proven in [Lemma 2.1](#).

The fibres are compact subsets of B^4 , since the Lefschetz fibration $\tilde{f}(\tilde{z}_1, \tilde{z}_2) = \tilde{z}_1\tilde{z}_2/(1 - \|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2)$ is proper when restricted to any subset $\Sigma_{u_1^0}$. Indeed, any sequence $\mathbf{z}(k) := (\tilde{z}_1(k), \tilde{z}_2(k)) \in B^4$ inside $\Sigma_{u_1^0}$ which converges to a point in ∂D^4 must satisfy the positive lower bounds

$$\epsilon < \|\tilde{z}_1(k)\|^2 \quad \text{and} \quad \|\tilde{z}_2(k)\|^2 < 1 \quad \text{for } k \gg 0$$

for some $\epsilon > 0$ (here we use $u_1^0 \in (-1, 1)$). However, such a sequence is now seen to map to an unbounded sequence $\tilde{f}(\mathbf{z}(k)) \in \mathbb{C}$, since the numerators of $\tilde{f}(\mathbf{z}(k))$ are positive and bounded from below while the denominators tend to 0. \square

2.1 Action properties

Recall that $\lambda_{\text{std}} = x_1 dy_1 + x_2 dy_2$ is the standard Liouville form defined on

$$(B^4, \omega_0 = d\lambda_{\text{std}}) \supset \tilde{V}.$$

We start with the following action computation for the immersed sphere.

Lemma 2.3 *The Lagrangian sphere $L_{\text{Wh}}(s) = \Pi_s^{-1}(0, 1)$ has an action difference equal to $\Delta = s \in (0, \pi/2)$ at the two preimages of its double point, where the action is computed by taking a primitive of the pullback of the Liouville form λ_{std} .*

We proceed to investigate the symplectic action of the torus fibres. But first, we need to fix the choice of a basis of each such fibre.

Lemma 2.4 (1) *For any Lagrangian torus fibre $L = \Pi_s^{-1}(u_1, u_2)$ there is a canonical choice of generator $e_0 \in H_1(L)$ of $\ker(H_1(L) \rightarrow H_1(V)) \cong \mathbb{Z}$ induced by the inclusion of a fibre, which is determined uniquely by the requirement that*

$$\int_{e_0} \lambda_{\text{std}} = \pi \cdot u_1.$$

Moreover, the Maslov class evaluates to $\mu_L^{\mathbb{C}^2}(e_0) = 0$ on this element.

(2) *When $L \subset V \setminus C_{\text{nodal}}$ moreover is satisfied, ie when $u_2 \neq 1$, any relative cycle $(D, \partial D) \rightarrow (V, L)$ with $[\partial D] = e_0$ has the property that $D \bullet \ell_i = (-1)^i$ for each of the two lines $f^{-1}(0) = \ell_1 \cup \ell_2$ in the nodal conic.*

Proof These statements are easily checked. We simply note that $\int_{e_0} \lambda_{\text{std}} = \pi \cdot u_1$ holds if we represent the class e_0 by a closed curve of the form $\theta \mapsto (e^{i\theta}a, e^{-i\theta}b) \in B^4$ for suitable $a, b \in \mathbb{C}$ satisfying $(a, b) \in L$, and thus $\|a\|^2 - \|b\|^2 = u_1$. □

The choice of the homology class e_0 can be seen to vary continuously with the fibres. In the following manner, we then proceed to extend e_0 to a (not globally defined) basis $\langle e_0, e_1 \rangle = H_1(L)$ for the fibres, where $\mu_L^{\mathbb{C}^2}(e_1) = 2$. Recall that, as shown in [34, Section 4.3], the bundle of standard tori is nontrivial due to the presence of the fibre with a nodal singularity. For that reason, no global and continuous choice of basis exists; also see Remark 2.6.

Lemma 2.5 *We can uniquely specify classes $e_1 \in H_1(L)$ of the torus fibres such that $\langle e_0, e_1 \rangle$ is a basis by requiring, for any relative cycle $(D, \partial D) \rightarrow (B^4, L)$ with $[\partial D] = e_1$, that:*

- (1) $D \bullet C = 1$.
- (2) (a) **In the case $u_2 < 1$** $D \bullet \ell_i = 0$ for each of the two lines $\ell_i \subset f^{-1}(0)$, $i = 1, 2$, in the nodal conic.
- (b) **In the case $u_2 > 1$ and $u_1 \geq 0$** $D \bullet \ell_1 = 1$ and $D \bullet \ell_2 = 0$.
- (c) **In the case $u_2 > 1$ and $u_1 < 0$** $D \bullet \ell_2 = 1$ and $D \bullet \ell_1 = 0$.

The Maslov class satisfies $\mu_L^{\mathbb{C}^2}(e_1) = 2$ in all cases above.

Proof The proof, which consists of an explicit verification, is left to the reader. □

An immediate consequence of the above two lemmas is that a Lagrangian torus fibre $\Pi_s^{-1}(u_1, u_2)$ is monotone if and only if $u_1 = 0$.

Remark 2.6 In the complement of the ray $\Pi_s^{-1}(0, u_2)$ for $u_2 > 1$ of torus fibres, the bases determined by the above lemmas can be seen to coincide and vary continuously. However, the basis vector e_1 is not unambiguously defined over the monotone tori $\Pi_s^{-1}(0, u_2)$ for $u_2 > 1$ of Clifford type; for these tori there are the two choices induced by the continuous extension for the tori with $u_1 > 0$, and another one which is the continuous extension of the basis for the tori with $u_1 < 0$. Nevertheless, since these are monotone tori, and since the two different choices of basis differ by a cycle of Maslov index zero, this ambiguity is irrelevant as far as computations of the symplectic action are concerned.

Using the basis constructed above we are now ready to describe the global behaviour of the symplectic actions of the different fibres.

Lemma 2.7 *The symplectic action of a torus $\Pi_s^{-1}(u_1, u_2)$ satisfies*

$$\int_{e_0} \lambda_{\text{std}} = \pi \cdot u_1 \quad \text{and} \quad \int_{e_1} \lambda_{\text{std}} = A_s(u_1, u_2)$$

for a function

$$A_s: (-1, 1) \times (0, +\infty) \rightarrow \left(0, \frac{\pi}{2}\right)$$

which is smooth in all variables (s, u_1, u_2) and which satisfies:

- (1) For each fixed s and u_1 , we have

$$A_s(\{u_1\} \times (0, +\infty)) = \left(0, \frac{1}{2}\pi(1 - |u_1|)\right) \subset \mathbb{R}.$$

- (2) $\partial_{u_2} A_s > 0$ away from $(u_1, u_2) = (0, 1)$.

- (3) A_s has a continuous extension to the compactification $[-1, 1] \times [0, +\infty]$ with the properties
 - (a) $A_s(u_1, +\infty) = \frac{1}{2}\pi(1 - |u_1|)$,
 - (b) $A_s(u_1, 0) = 0$,
 - (c) $A_s(\pm 1, u_2) = 0$.
- (4) $\lim_{s \rightarrow 0} A_s(u_1, 1) = 0$ while $\lim_{s \rightarrow \pi/2} A_s(u_1, 1) = \frac{1}{2}\pi(1 - |u_1|)$.

See Figure 5.

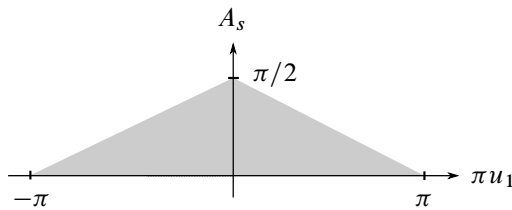


Figure 5: The image of the values $(\pi u_1, A_s)$ of the symplectic action class evaluated on the pair (e_0, e_1) of basis vectors of $H_1(\Pi_s^{-1}(u_1, u_2))$.

Proof By continuity, it suffices to establish the claims whenever $u_1 \neq 0$. In fact, the case $u_1 = 0$ is easy to check by hand since the fibration is sufficiently explicit for those parameters. Furthermore, one can argue by symmetry and restrict to the case $u_1 > 0$. We continue to exhibit these torus fibres with additional care.

Recall that the one-parameter family of tori for fixed $s \in (0, \pi/2)$ and $u_1 \neq 0$ provides a foliation of the hypersurface

$$\Sigma_{u_1} := \{\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2 = u_1\} \cap \tilde{V}$$

by Lemma 2.2, where Σ_{u_1} is a solid torus. Further, we exhibit an explicit foliation of this solid torus by symplectic discs

$$D_\theta(u_1) := \{(e^{i\theta} \sqrt{u_1 + \|z\|^2}, z) : 2\|z\|^2 + u_1 < 1\} \subset B^4$$

of total symplectic area equal to $(\pi/2)(1 - u_1)$. (Recall that we here are considering the case $u_1 > 0$.) Also, note that the smooth conic $\tilde{C} := \tilde{f}^{-1}(1)$ intersects each symplectic disc $D_\theta(u_1)$ transversely in a single point $(e^{i\theta} \sqrt{u_1 + t^2}, e^{-i\theta} t)$ for a unique value of $t \in (0, \sqrt{(1 - u_1)}/2)$.

The one-parameter family of Lagrangian torus fibres $\Pi_s^{-1}(u_1, \cdot)$ for fixed s and u_1 can be seen to foliate the solid torus, and also to intersect each symplectic disc $D_\theta(u_1)$ in a

foliation by simple closed curves encircling the unique intersection point $D_\theta(u_1) \cap \tilde{C} = \{p_\theta(u_1)\} \subset D_\theta(u_1)$. (A Lagrangian obviously cannot have a full tangency to a symplectic leaf, and hence it is automatically transverse to each such symplectic disc when considered inside the solid torus.) Since $A_s(u_1, u_2)$ simply is the area inside the symplectic disc bounded by the curve $D_\theta(u_1) \cap \Pi_s^{-1}(u_1, u_2)$, properties (1), (2) and (3) can now be seen to follow.

Property (4) is a consequence of the satisfaction of property (2) by the family Ψ_s of diffeomorphisms in the construction of Π_s given in Section 1.3. We argue as above by considering the symplectic area inside the discs foliating the solid torus Σ_{u_1} . To that end we note that, for any $\epsilon > 0$, there exists a sufficiently small neighbourhood $U \subset \mathbb{C}$ of either of the subsets $[0, 1]$ (the case $s = 0$) or $(-\infty, 0]$ (the case $s = \pi/2$) such that the discs $D_\theta(u_1)$ intersected with $f^{-1}U$ may be assumed to be of symplectic area bounded from above by $\epsilon > 0$ independently of $u_1 \in (-1, 1)$. □

2.2 Proof of Proposition 1.4

The Lagrangian property for the fibres was shown in Lemma 2.2. The claims concerning the action in Proposition 1.4(1) are a consequence of Lemma 2.3. The topological considerations can be investigated by hand.

For property (2)(a) we refer to the work of A Gadbled [17] concerning the different guises of the Chekanov torus.

Property (2)(b) is finally shown with the below lemma, which finishes the proof. □

Lemma 2.8 *Consider any Lagrangian torus fibre $\Pi_{s'}^{-1}(u_1, u_2)$ which is **not** weakly exact, ie with $u_1 \neq 0$. For any $s' \in (0, \pi/2)$ this Lagrangian torus is Hamiltonian isotopic inside (V, ω_{FS}) to a unique torus fibre of the form $\Pi_{s'}^{-1}(u_1, u'_2)$.*

Proof The Hamiltonian isotopy is constructed by hand, by considering a suitable family of Lagrangian tori contained inside the solid torus,

$$\Sigma_{u_1} = \{\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2 = u_1\} \cap \tilde{V} \quad \text{with } u_1 \neq 0,$$

all of which can be taken to be standard fibres for different values of $s \in (0, \pi/2)$.

For the uniqueness, it is sufficient to note that any Hamiltonian isotopy must preserve the basis element e_0 as constructed in Lemma 2.4; recall that this is a preferred generator of $\ker(H_1(\mathbb{T}^2) \rightarrow H_1(V)) \cong \mathbb{Z}$ induced by the inclusion of the torus. The action computation in Lemma 2.7 thus implies that any Hamiltonian isotopy connecting two

fibres $\Pi_s^{-1}(u_1, u_2)$ and $\Pi_{s'}^{-1}(u'_1, u'_2)$ must satisfy $u_1 = u'_1$. The uniqueness of the parameter u'_2 is also a consequence of Lemma 2.7. □

3 Standard neighbourhood of a sphere with self-intersection number +1

In this section we give a careful description of the symplectic geometry of the standard neighbourhood of a Lagrangian sphere with a single transverse double point. In dimension $2k$ there are two different such Lagrangian spheres up to diffeomorphism; the two cases are determined by their Whitney self-intersection number I , which can be either $+1$ or -1 . Note that the situation is different in odd dimensions, where the local model for a Lagrangian sphere with a single transverse self-intersection is unique. Here we are only interested in the case when $k = 1$ and the self-intersection number is equal to $+1$. This is the only sphere which appears as an isolated singular fibre in a Lagrangian torus fibration.

We present the neighbourhood of the immersed sphere as a self-plumbing of the cotangent bundle of an embedded sphere. We moreover show that it naturally carries the structure of a Liouville domain, which thus can be completed to a Liouville manifold. We also construct a Lagrangian torus fibration on this space, together with an induced symplectic embedding of (V, ω_{FS}) that preserves the torus fibres.

3.1 The self-plumbing of the cotangent bundle of a sphere

We commence with construction of the symplectic manifold of interest. First, consider the standard symplectic unit cotangent bundle

$$DT^*D^2 := \{\|q\| \leq 1, \|p\| \leq 1\} \subset (T^*\mathbb{R}^2, d\lambda_{\mathbb{R}^2})$$

of the unit disc. This is a smooth manifold with boundary with corners, where

$$\begin{aligned} \partial DT^*D^2 &= \partial_V DT^*D^2 \cup \partial_H DT^*D^2, \\ \partial_V DT^*D^2 &:= DT^*_{\partial D^2} D^2 = \{\|q\| = 1, \|p\| \leq 1\}, \\ \partial_H DT^*D^2 &:= ST^*D^2 = \{\|q\| \leq 1, \|p\| = 1\}. \end{aligned}$$

It is also convenient to use the canonical identification

$$DT^*D^2 \hookrightarrow \mathbb{C}^2, \quad (q, p) \mapsto iq + p,$$

with the corresponding subset of the complex plane.

We also need the unit cotangent bundle

$$DT^*(S^1 \times [-1, 1]) := \{\|p_{\theta,q}\| \leq 1\} \subset (T^*(S^1 \times [-1, 1]), d\lambda_{S^1 \times [-1,1]})$$

of the cylinder, with the coordinates $p_{\theta,q} = (p_\theta, p_q)$ on the cotangent fibres induced by the standard coordinates $(\theta, q) \in S^1 \times [-1, 1]$. Note that we here have made implicit use of the flat product metric on $S^1 \times [-1, 1]$ when speaking about the unit codisc bundle.

For some small $\epsilon > 0$ we consider the open neighbourhoods

$$\begin{aligned} \{q = (q_1, q_2) = r(\cos \theta, \sin \theta) : r \in [1 - \epsilon, 1 + \epsilon]\} &\subset DT^*D^2_{1+\epsilon}, \\ \{p = (p_1, p_2) = r(\cos \theta, \sin \theta) : r \in [1 - \epsilon, 1 + \epsilon]\} &\subset D_{1+\epsilon}T^*D^2 \end{aligned}$$

of the pieces $\partial_V DT^*D^2$ and $\partial_H DT^*D^2$ of the boundary, respectively. There are symplectic inclusions

$$\phi_V: \{q = r(\cos \theta, \sin \theta) : r \in [1 - \epsilon, 1 + \epsilon]\} \hookrightarrow D_2T^*(S^1 \times [-1 - \epsilon, -1 + \epsilon])$$

defined by

$$(q, p) \mapsto ((\theta, -1 + (r - 1)), (p_\theta = r(-\sin \theta, \cos \theta) \bullet p, p_q = (\cos \theta, \sin \theta) \bullet p)),$$

and

$$\phi_H: \{p = r(\cos \theta, \sin \theta) : r \in [1 - \epsilon, 1 + \epsilon]\} \hookrightarrow D_2T^*(S^1 \times [1 - \epsilon, 1 + \epsilon])$$

defined by

$$(q, p) \mapsto ((\theta + \frac{1}{2}\pi, 1 - (r - 1)), (p_\theta = -r(-\sin \theta, \cos \theta) \bullet q, p_q = (\cos \theta, \sin \theta) \bullet q)).$$

Note that the restrictions

$$\begin{aligned} \phi_V|_{\partial_V DT^*D^2} &\xrightarrow{\cong} DT^*_{\{q=-1\}}(S^1 \times [-1, 1]), \\ \phi_H|_{\partial_H DT^*D^2} &\xrightarrow{\cong} DT^*_{\{q=1\}}(S^1 \times [-1, 1]) \end{aligned}$$

are diffeomorphisms.

Using the above symplectomorphisms it is possible to perform the gluing

$$W = \frac{DT^*D^2 \sqcup D_\epsilon T^*(S^1 \times [-1, 1])}{x \sim \phi_V(x), x \sim \phi_H(x)} \quad \text{for } 0 < \epsilon \ll 1,$$

producing a symplectic manifold (W, ω) . This symplectic manifold is naturally identified with the self-plumbing of a two-sphere. In [Section 3.2](#) we will exhibit a natural primitive of the symplectic form, giving it the structure of a Liouville domain.

Observe that

$$L_0 := 0_{D^2} \cup DT^*_{(0,0)}D^2 \cup 0_{S^1 \times [-1,1]} \subset (W, \omega)$$

is the Lagrangian immersion of a sphere with one transverse double point

$$\{q = 0 = p\} = 0_{D^2} \cap DT^*_{(0,0)}D^2.$$

The Whitney self-intersection number of this sphere can be computed to be +1.

There is a symplectomorphism

$$\begin{aligned} I_1: (DT^*D^2, d\lambda_{\mathbb{R}^2}) &\xrightarrow{\cong} (DT^*D^2, d\lambda_{\mathbb{R}^2}), \\ I_1((q_1, q_2), (p_1, p_2)) &= ((-p_2, p_1), (q_2, -q_1)), \end{aligned}$$

that satisfies $I_1^2 = \text{Id}_{DT^*D^2}$. Likewise, the symplectomorphism

$$\begin{aligned} I_2: (DT^*(S^1 \times [-1, 1]), d\lambda_{S^1 \times [-1,1]}) &\xrightarrow{\cong} (DT^*(S^1 \times [-1, 1]), d\lambda_{S^1 \times [-1,1]}), \\ I_2((\theta, q), (p_\theta, p_q)) &= ((\theta, -q), (p_\theta, -p_q)), \end{aligned}$$

is of order two, ie $I_2^2 = \text{Id}_{DT^*(S^1 \times [-1,1])}$. Since one can check that

$$I_2 \circ \phi_V = \phi_H \circ I_1 \quad \text{and} \quad I_2 \circ \phi_H = \phi_V \circ I_1$$

are satisfied, we obtain an induced symplectomorphism I of (W, ω) whose unique fixpoint is the origin $(0, 0) \in DT^*D^2$. To summarise:

Proposition 3.1 *The induced symplectomorphism*

$$I: (W, \omega) \rightarrow (W, \omega)$$

is of order two and fixes the immersed Lagrangian sphere L_0 setwise (ie $I(L_0) = L_0$), while reversing its orientation and interchanging the two sheets at its unique double point.

Weinstein’s symplectic neighbourhood theorem [37] (also see [29]) shows that any Lagrangian submanifold L has a neighbourhood symplectomorphic to a neighbourhood of the zero section of $(T^*L, d\lambda_L)$ of the cotangent bundle equipped with the standard symplectic form, while moreover identifying the Lagrangian with the zero section. It is well known that the result generalises to establish a standard symplectic neighbourhood also of a Lagrangian immersion with transverse double points; see eg [34, Proposition 4.8]. In particular, it is the case that:

Proposition 3.2 For any two-dimensional Lagrangian immersion $L \subset (W', \omega')$ of a sphere with a single transverse double point whose Whitney self-intersection number equals $+1$, there is a symplectic embedding

$$\phi: (U, \omega') \hookrightarrow (W, \omega)$$

of a neighbourhood $U \subset W'$ of L which moreover satisfies $\phi(L) = L_0$.

Later in Proposition 3.8 this result is extended to also preserve locally defined Lagrangian torus fibrations.

3.2 Extending the neighbourhood to a complete Liouville manifold

Here we construct a suitable Liouville form defined on all of (W, ω) by interpolating between natural Liouville forms on the pieces DT^*D^2 and $DT^*(S^1 \times [-1, 1])$. First, consider the Liouville form

$$\lambda_0 := \frac{1}{2} \sum_{i=1}^2 (p_i dq_i - q_i dp_i) = \lambda_{\mathbb{R}^2} - \frac{1}{2} d \left(\sum_{i=1}^2 q_i p_i \right)$$

defined on $T^*\mathbb{R}^2$ and which clearly satisfies $I_1^* \lambda_0 = \lambda_0$.

Fix a smooth function $h: [-1, 1] \rightarrow \mathbb{R}$ that satisfies

- $h(-q) = -h(q)$,
- $h(q) = 2 + q$ near $q = -1$ and $h(q) = -2 + q$ near $q = 1$,
- $h'(q) < 2$ for $q \in [-1, 1]$, $h'(q) \geq 0$ for $q \leq -\frac{2}{3}$, $h'(q) = 0$ for $q \in [-\frac{2}{3}, -\frac{1}{3}]$, and $h'(q) \leq 0$ for $q \in [-\frac{1}{3}, 0]$.

In particular, we note that $h(q)$ is necessarily nonvanishing outside the subset $[-\frac{1}{3}, \frac{1}{3}]$, while it is constant on $\{q : \frac{1}{3} \leq |q| \leq \frac{2}{3}\}$.

Then, using the function h , we construct the Liouville form

$$\lambda := \lambda_{S^1 \times [-1, 1]} - \frac{1}{2} d(h(q) p_q) = p_q dq + p_\theta d\theta - \frac{1}{2} d(h(q) p_q)$$

defined on $T^*(S^1 \times [-1, 1])$, which also clearly satisfies $I_2^* \lambda_{S^1 \times [-1, 1]} = \lambda_{S^1 \times [-1, 1]}$. The Liouville vector field induced by λ can be seen to be given by

$$(3-1) \quad \zeta_\lambda := p_\theta \partial_{p_\theta} + p_q \left(1 - \frac{1}{2} h'(q) \right) \partial_{p_q} + \frac{1}{2} h(q) \partial_q,$$

and we use ϕ_λ^t to denote the corresponding Liouville flow. (This flow is well defined at least for negative times $t \leq 0$.)

Lemma 3.3 We have

$$\phi_V^* \lambda = \lambda_0 \quad \text{and} \quad \phi_H^* \lambda = \lambda_0,$$

and these Liouville forms thus combine to a smooth Liouville form λ on all of $(W, \omega = d\lambda)$ invariant under the symplectomorphism I .

Proof Observe that

$$dq = dr = \frac{q_1 dq_1 + q_2 dq_2}{\sqrt{q_1^2 + q_2^2}} = \cos \theta dq_1 + \sin \theta dq_2,$$

$$d\theta = \frac{-q_2 dq_1 + q_1 dq_2}{q_1^2 + q_2^2} = r^{-1}(-\sin \theta dq_1 + \cos \theta dq_2),$$

from which we deduce that

$$\phi_V^* \lambda_{S^1 \times [-1, 1]} = \sum_{i=1}^2 p_i dq_i = \lambda_{\mathbb{R}^2}.$$

Combining this with the relation $h(q) p_q \circ \phi_V = \sum_{i=1}^2 q_i p_i$ we deduce that

$$\phi_V^* \lambda = \lambda_{\mathbb{R}^2} - \frac{1}{2} d \left(\sum_{i=1}^2 q_i p_i \right) = \lambda_0.$$

Since both λ_0 and λ are invariant under I_1 and I_2 , respectively, and since $I_2 \circ \phi_H = \phi_V \circ I_1$, it is also a direct consequence that

$$\phi_H^* \lambda = \lambda_0. \quad \square$$

Next we construct the smooth function

$$\rho: W \rightarrow \mathbb{R}_{\geq 0}$$

specified by the following:

- Inside DT^*D^2 , the function ρ is given by

$$\rho(\mathbf{q}, \mathbf{p}) = (p_1 q_1 + p_2 q_2)^2 + (q_1 p_2 - q_2 p_1)^2.$$

- Inside $DT^*(S^1 \times [-1, 1])$, it is given by

$$\rho((\theta, q), (p_\theta, p_q)) = g(q)^2 p_q^2 + p_\theta^2$$

for a smooth function $g(|q|) > 0$ defined as follows: $g(q) = |h(q)|$ outside of $[-\frac{1}{3}, \frac{1}{3}]$, while $g(q) \equiv h(-\frac{2}{3})$ is constant for $q \in [-\frac{2}{3}, \frac{2}{3}]$. Observe that $g(q)$ thus in particular is nonvanishing.

The function ρ is smooth on W . Furthermore, it is the case that:

Lemma 3.4 *The Liouville vector field ζ_λ on $(W, \omega = d\lambda)$ corresponding to the primitive λ satisfies the following properties in a neighbourhood of L_0 :*

(1) *The Liouville form λ vanishes on TL_0 , and its backward flow satisfies*

$$\bigcap_{n \in \mathbb{Z}_{\geq 0}} \phi_\lambda^{-n}(W) = L_0.$$

(2) *The form λ , and hence ζ_λ , as well as the function ρ , are all preserved by the symplectomorphism I .*

(3) *$\rho^{-1}(0) = L_0$, and the vector field ζ_λ satisfies $d\rho(\zeta_\lambda) = f \cdot \rho$ for some function $f: W \rightarrow \mathbb{R}_{>0}$ which is identically 2 outside the set*

$$\{(\theta, q, p_\theta, p_q) : q \in [-\frac{1}{3}, \frac{1}{3}]\} \subset DT^*(S^1 \times [-1, 1]).$$

In particular, for all $a > 0$ sufficiently small, the level set $Y_a := \rho^{-1}(a) \subset (W, d\lambda)$ is a smooth contact-type hypersurface with induced contact form $\alpha_a := \lambda|_{TY_a}$, and the restriction

$$I|_{Y_a}: (Y_a, \alpha_a) \xrightarrow{\cong} (Y_a, \alpha_a)$$

is a strict contactomorphism.

Proof (1) This follows from property (3) (to be proven below).

(2) We leave this to the reader to check.

(3) First we establish the claim $\rho^{-1}(0) = L_0$. Inside $DT^*(S^1 \times [-1, 1])$, it is clear that ρ vanishes precisely along the zero section. Further, by the definition of ρ , inside DT^*D^2 we have $\rho = 0$ if and only if the two vectors $(q_1, q_2), (p_1, p_2) \in \mathbb{R}^2$ are simultaneously orthogonal and collinear. In other words $\rho = 0$ if and only if either $(q_1, p_2) = 0$ or $(p_1, q_2) = 0$ in that subset. This shows the claim.

The computation of $d\rho(\zeta_\lambda)$ is left to the reader. Inside the subset DT^*D^2 we use the expression

$$\zeta_\lambda = \frac{1}{2} \sum_{i=1}^2 (q_i \partial_{q_i} + p_i \partial_{p_i}),$$

while in $DT^*(S^1 \times [-1, 1])$ we use (3-1). □

Finally, we use the Liouville flow ϕ_λ^t generated by ζ_λ in order to produce the following completion of the symplectic manifold $(W, d\lambda)$. Consider the sublevel set $W_a := \rho^{-1}[0, a]$, which is a Liouville manifold with a contact boundary by Lemma 3.4. Attaching half of the corresponding symplectisation, ie

$$([0, +\infty) \times Y_a, d(e^t \alpha_a)),$$

along its boundary, there is a smooth extension of the Liouville form λ on W_a by $e^t \alpha_a$ on this cylindrical end. This produces a smooth Liouville form with a complete Liouville flow, and we denote by

$$(\widehat{W}, d\lambda) = (W_a, \lambda) \cup ([0, +\infty) \times Y_a, d(e^t \alpha_a))$$

the resulting complete Liouville manifold which contains (W_a, ω) as a subdomain. Recollecting the previously established results, we can conclude that:

Proposition 3.5 (1) *The Liouville form λ vanishes along TL_0 of the immersed sphere $L_0 \subset (\widehat{W}, d\lambda)$, and thus is exact, and its backward flow satisfies*

$$\bigcap_{n \in \mathbb{Z}_{\leq 0}} \phi_\lambda^{-n}(\widehat{W} \cap \{t \leq N\}) = L_0$$

for any $N \geq 0$.

- (2) *There is a smooth function $\widehat{\rho}: \widehat{W} \setminus L_0 \rightarrow \mathbb{R}_{>0}$ uniquely defined by the property that $\widehat{\rho}^{-1}(a) = Y_a$ together with $\widehat{\rho} \circ \phi_\lambda^t = e^t \cdot \widehat{\rho}$ for all $t \in \mathbb{R}$ (in particular, the level sets $(\widehat{\rho}^{-1}(a), \lambda|_{T(\widehat{\rho}^{-1}(a))})$ are hypersurfaces of contact type for λ).*
- (3) *The symplectomorphism $I: (W_a, d\lambda) \rightarrow (W_a, d\lambda)$ extends to an exact symplectomorphism of $(\widehat{W}, d\lambda)$ of order two which fixes L_0 setwise, preserves the Liouville form λ , and preserves each level set $\widehat{\rho}^{-1}(s)$, $s \in [a, +\infty)$ (where it consequently acts by contactomorphism preserving the contact form $\lambda|_{T(\widehat{\rho}^{-1}(s))}$).*

Proof The properties follow more or less directly from Lemma 3.4, and by construction. For property (3) we have to use that the Liouville flow of λ is invariant under I on $W_a \subset \widehat{W}$ by Lemma 3.4(2), and that $\widehat{\rho}^{-1}(a) = \rho^{-1}(a) = \partial W_a$ is fixed by I . Hence we can smoothly extend I to all of $(\widehat{W}, d\lambda)$ by requiring that I commute with the Liouville flow of λ , ie that

$$I \circ \phi_\lambda^t = \phi_\lambda^t \circ I$$

be satisfied. □

3.3 A singular Lagrangian torus fibration

Following Symington’s construction in [34, Section 4.2] we consider the map

$$\pi = (\pi_1, \pi_2): W_a \rightarrow \mathbb{R}^2$$

which is defined by

$$\begin{aligned} \pi(\mathbf{q}, \mathbf{p}) &= (q_1 p_2 - q_2 p_1, q_1 p_1 + q_2 p_2), \\ \pi((\theta, q), \mathbf{p}_{(\theta, q)}) &= (p_\theta, g(|q|) p_q) \end{aligned}$$

for $(\mathbf{q}, \mathbf{p}) \in DT^*D^2$ and $((\theta, q), \mathbf{p}_{(\theta, q)}) \in DT^*(S^1 \times [-1, 1])$, respectively. Here we have used the previously defined smooth function $g(q) > 0$ that satisfies $g(q) = |\mp 2 + q|$ near $q = \pm 1$, and we further assume that $a > 0$ is sufficiently small.

Lemma 3.6 *The map π is smooth, inducing a singular Lagrangian torus fibration, the unique singular fibre of which is given by our previously constructed Lagrangian immersion*

$$\pi^{-1}(0, 0) = L_0$$

of a sphere. Moreover, denoting the reflection of the second coordinate in \mathbb{R}^2 by $R(u_1, u_2) = R(u_1, -u_2)$, we have

$$\pi \circ I = R \circ \pi$$

and in particular I preserves the fibres of π setwise.

Proof We show that the fibres of $(q_1 p_2 - q_2 p_1, q_1 p_1 + q_2 p_2) \in \mathbb{R}^2$ are Lagrangian inside $T^*\mathbb{R}^2$. The remaining claims are straightforward to check.

The Lagrangian condition is most easily seen by using the fact that any complex curve inside \mathbb{C}^2 becomes Lagrangian for the symplectic form

$$\Re(dz_1 \wedge dz_2) = dx_1 \wedge dx_2 - dy_1 \wedge dy_2.$$

Indeed, we can set $z_1 = q_1 - iq_2$ and $z_2 = p_1 + ip_2$, thus turning

$$\pi(\mathbf{q}, \mathbf{p}) = z_1 z_2$$

into a holomorphic Lefschetz fibration. □

Proposition 3.7 *There exists a Lagrangian torus fibration $\hat{\pi}: \hat{W} \rightarrow \mathbb{R}^2$, where $\hat{\pi}$ is onto \mathbb{R}^2 and submersive outside of the origin, and for which $\hat{\pi}^{-1}(0) = L_0$ is the unique singular fibre. Furthermore, the fibration can be taken to satisfy:*

(1) $\hat{\pi}|_{Y_a} = \pi$, and for any $\mathbf{v} \in \pi(Y_a) = \{u_1^2 + u_2^2 = a\}$ we have

$$\hat{\pi}(\phi_\lambda^t(\hat{\pi}^{-1}(\mathbf{v}))) = e^{t/2} \cdot \mathbf{v} \quad \text{for } t \geq 0,$$

and in particular $\hat{\rho} = \|\hat{\pi}\|^2$ holds inside $\hat{\rho}^{-1}(a, +\infty) = \hat{W} \setminus W_a$.

(2) $\hat{\pi} = \pi$ in some neighbourhood of L_0 .

(3) The Liouville flow ϕ_λ^t applied to any fibre of $\hat{\pi}$ is again Hamiltonian isotopic to a fibre of $\hat{\pi}$.

Proof Recall that $a \equiv \rho|_{Y_a} \equiv \|\pi\|^2|_{Y_a}$ is satisfied by construction. In view of Proposition 3.5(2), the Liouville flow applied to the family of tori $\pi^{-1}(\mathbf{v}) \subset Y_a$ for $\mathbf{v} \in \{\|(u_1, u_2)\|^2 \equiv a\}$ produces a smooth torus fibration $f: \hat{W} \setminus L_0 \rightarrow \mathbb{R}^2 \setminus \{0\}$ which coincides with π when restricted to the hypersurface Y_a . This torus fibration (defined only in the complement of L_0) can thus be made to satisfy property (1) by construction.

What suffices is then to perform a suitable interpolation between these two fibrations. To that end, we argue as follows. First, using the fact that the two fibrations coincide along Y_a by construction, we can use the classical Arnold–Liouville theorem [34, Theorem 2.3] in order to find a symplectomorphism ϕ_0 defined inside $W_{a+\epsilon} \setminus W_{a-\epsilon}$ such that

- ϕ_0 is the identity along Y_a , and
- $\pi \circ \phi_0 = f$.

The construction of ϕ_0 is standard; see the proof of Proposition 3.8 for more details.

The fact that ϕ_0 is a symplectomorphism that restricts to the identity along Y_a implies that the differential must satisfy $D\phi_0 = \text{Id}_{T\hat{W}}$ along $T_{Y_a}\hat{W}$. Hence, after shrinking $\epsilon > 0$, a standard argument shows that $\phi_0 = \phi_{H_t}^1$ holds for a Hamiltonian isotopy that again can be taken to satisfy $\phi_{H_t}^t|_{Y_a} = \text{Id}$.

After an appropriate cutoff of this Hamiltonian isotopy, we construct a symplectomorphism ϕ_1 that satisfies

- $\phi_1 = \text{Id}$ inside $W_{a+\epsilon} \setminus W_a$, and
- $\phi_1 = \phi_0$ inside $W_{a-\epsilon/2} \setminus W_{a-\epsilon}$.

This provides us with the sought interpolation of the two torus fibrations.

To show (3), note that one can readily find a path of Lagrangian fibres of $\hat{\pi}$ realising the same symplectic flux-paths as that induced by the Liouville flow applied to the

given fibre. (In fact, above the subset $\{u_1^2 + u_2^2 \geq a\}$, the positive Liouville flow maps fibres to fibres by construction.) The result is then a consequence of Lemma 6.11. \square

Recall that Lagrangian fibrations with a unique singular fibre were constructed for the symplectic manifold (V, ω_{FS}) in Section 1.3. The fact that the fibration constructed here has similar properties gives us a convenient way to construct a symplectic embedding of $(V, \omega_{FS}) \cong (\mathbb{C}P^2 \setminus (\ell_\infty \cup C), \omega_{FS})$ by utilising the Arnold–Liouville theorem.

Proposition 3.8 *Given any $s \in (0, \pi/2)$ and $c > 0$, there is a symplectic embedding*

$$\iota_s: (V, c\omega_0) \hookrightarrow (\widehat{W}, d\lambda), \quad \iota_s(\Pi_s^{-1}(0, 1)) = L_0,$$

for which the following is satisfied:

- (1) *Given an arbitrarily small neighbourhood $U \subset (-1, 1) \times (0, +\infty)$ of $(0, 1) \in (-1, 1) \times (0, +\infty)$ (ie the unique critical value of Π_s), we may moreover assume that ι_s maps fibres $\Pi_s^{-1}(p)$ for $p \notin U \setminus \{(0, 1)\}$ to fibres of $\widehat{\pi}$.*
- (2) *For $c \gg 0$ sufficiently large, the image $\iota_s(V)$ projects to a star-shaped subset $\widehat{\pi}(\iota(V_s)) \subset \mathbb{R}^2$.*

Proof The Arnold–Liouville theorem [34, Theorem 2.3], together with its generalisation [34, Proposition 4.8] to Lagrangian fibrations with nodal singular fibres, shows the existence of the embeddings.

More precisely, the generalised version of the Arnold–Liouville theorem provides us with a symplectomorphism $\iota_s: (O, c\omega_0) \hookrightarrow (\widehat{W}, d\lambda)$ from a neighbourhood $O \subset V$ of the singular Lagrangian fibre $\Pi_s^{-1}(0, 1) \subset V$ to a neighbourhood $\iota(O) \ni \pi^{-1}(0, 0)$, which moreover maps

- the singular fibre $\Pi_s^{-1}(0, 1)$ to the singular fibre $\pi^{-1}(0, 0)$, and
- all fibres of Π_s to fibres of π outside of some neighbourhood $\Pi_s^{-1}(U \setminus \{(0, 1)\})$, as required.

Here we use the fact that the unique singular fibres $\Pi_s^{-1}(0, 0)$ and $\pi^{-1}(0, 0)$ of the two fibrations are “nodes” as defined in [34, (4.3)]. (Observe that it is not possible to find a symplectomorphism that preserves also the fibres near the singular fibre in general; see [34, Remark 3.9].)

Next we must extend the map ι_s from $O \subset V$ to all of V . This is a simple matter of applying the classical Arnold–Liouville theorem. Namely, for sufficiently small and

simply connected $U_1 \subset (-1, 1) \times (+\infty) \setminus \{0, 1\}$ and $U_2 \subset \mathbb{R}^2 \setminus \{(0, 0)\}$, it provides us with symplectic identifications of neighbourhoods $\Pi_s^{-1}(U_1)$ and $\hat{\pi}^{-1}(U_2)$ with neighbourhoods of the form $\mathbb{T}^2 \times A_i \subset T^*\mathbb{T}^2$. The extension is then created by patching together these identifications. Recall that the identification supplied by the Arnold–Liouville theorem is canonical up to fibrewise translations in $T^*\mathbb{T}^2$ and the discrete action of $\text{Gl}(2, \mathbb{Z})$ by pullbacks of the corresponding linear diffeomorphisms of \mathbb{T}^2 . (The discreteness is crucial for this argument.)

We are left with showing property (2). We show that if $v \in \mathbb{R}^2$ is in the image of $\hat{\pi} \circ \iota_s$, then necessarily $e^{-t}v$ for $t \geq 0$ is in the image as well, from which the sought star-shaped property follows.

For $c \gg 0$ the torus fibres above the complement of some given compact neighbourhood of $(0, 1) \in (-1, 1) \times (0, +\infty)$ may be assumed to map to torus fibres of $\hat{\pi}$ contained inside $\hat{W} \setminus W_a$. By Proposition 3.7(1), the forward Liouville flow ϕ_λ^t preserves the fibres in the same subset. Furthermore, whenever $L := \hat{\pi}^{-1}(v) \subset \hat{W} \setminus W_a$, the image $\hat{\pi}(\phi_\lambda^t(L))$ is simply the radial rescaling $e^{t/2} \cdot v$ by the same result.

For this reason it now suffices to show that the Liouville flow preserves also the torus fibres of Π_s , at least up to Hamiltonian isotopy. Indeed, the convexity properties satisfied by the values of the symplectic action on these fibres, as shown in Figure 5, combined with Lemma 6.11, implies that all tori $\phi_{i_s^* \lambda}^{-t}(\Pi_s^{-1}(u_1, u_2))$ for small $t \geq 0$ again are Hamiltonian isotopic to fibres of Π_s . From this it then follows that the image of $\hat{\pi}(\iota_s(V))$ is invariant under multiplication by e^{-t} outside of the subset $\hat{\pi}(W_a) = \{\|u_1\|^2 + \|u_2\|^2 \leq a\} \subset \mathbb{R}^2$, as sought. □

4 Pencils of pseudoholomorphic conics

In this section we assume that we are given a tame almost complex structure on $(\mathbb{C}P^2, \omega_{\text{FS}})$ which coincides with the standard integrable complex structure i near the divisor $\ell_\infty \subset \mathbb{C}P^2$ at infinity. A pseudoholomorphic line inside $\mathbb{C}P^2$ is a pseudoholomorphic curve of degree one. One of the first examples of the power of the technique of pseudoholomorphic curves in symplectic topology was Gromov’s result from [19] which shows that $\mathbb{C}P^2$ is foliated by pseudoholomorphic lines for any choice of tame almost complex structure.

Theorem 4.1 [19] *For any tame almost complex structure, the pseudoholomorphic lines that pass through a given point $\text{pt} \in \mathbb{C}P^2$ are embedded symplectic spheres which form the leaves of a smooth foliation of the complement $\mathbb{C}P^2 \setminus \{\text{pt}\}$ of that point.*

A pseudoholomorphic conic inside $\mathbb{C}P^2$ is a pseudoholomorphic curve of degree two. The adjunction formula, together with positivity of intersection [28], allows us to conclude that:

Lemma 4.2 *A pseudoholomorphic conic is either a smoothly embedded sphere, a nodal sphere consisting of the union of two different pseudoholomorphic lines, or a two-fold branched cover of a pseudoholomorphic line.*

Proof We show that the only singularities are nodes and branch points; the rest follows from elementary applications of the adjunction formula and positivity of intersection.

Consider a line passing through a singular point as well as a smooth point on the conic (its existence follows from Gromov’s result, Theorem 4.1, concerning the classification of pseudoholomorphic lines). Unless the line is contained inside the conic, the singular point contributes with at least +2 to the algebraic intersection number (see [28]), while the intersection at the smooth point contributes with +1. Since a line and a conic intersect with algebraic intersection number +2, positivity of intersection implies that the line must be contained inside the conic. \square

Now we fix the two points $q_1 = [1 : 0 : 0]$ and $q_2 = [0 : 1 : 0] \in \ell_\infty$ at the line at infinity, together with the complex tangent vectors $v_i \subset T_{q_i}\mathbb{C}P^2$ for $i = 1, 2$ to the two lines

$$\ell_1 := \{[Z_1 : 0 : Z_3] \in \mathbb{C}P^2\} \quad \text{and} \quad \ell_2 := \{[0 : Z_2 : Z_3] \in \mathbb{C}P^2\}$$

at the two respective points. Note that there is a Lefschetz fibration

$$f: \mathbb{C}P^2 \setminus \ell_\infty \rightarrow \mathbb{C}, \quad f(z_1, z_2) = z_1 z_2,$$

whose fibres are precisely those conics satisfying the specified tangencies $v_i \subset T_{q_i}\mathbb{C}P^2$ for $i = 1, 2$. This Lefschetz fibration has a unique singular fibre $f^{-1}(0)$; this is the standard nodal conic $C_{\text{nodal}} \subset \mathbb{C}P^2$ given as the union of the coordinate lines.

Here we show how Gromov’s strategy for establishing a foliation by pseudoholomorphic lines extends to give an analogous result also for conics. Namely, for an arbitrary tame almost complex structure J which is standard at infinity, there exists a fibration f_J by J -holomorphic conics having properties similar to the standard fibration f .

We first need to introduce a couple of notions. Denote by \mathcal{M}_J the moduli space of J -holomorphic conics and by $\mathcal{M}_J(v_1, v_2) \subset \mathcal{M}_J$ the subspace of conics satisfying the two tangencies v_i for $i = 1, 2$. For a smooth family J_s for $s \in I^k$ of tame

almost complex structures on $(\mathbb{C}P^2, \omega_{FS})$, all of which are assumed to be standard near ℓ_∞ , we denote by $C_{\text{nodal}}^s \subset \mathbb{C}P^2$ the union of the two unique J_s -holomorphic lines satisfying the tangencies $v_i \subset T_{q_i} \mathbb{C}P^2$. (Recall Gromov’s result, [Theorem 4.1](#).) The lines $C_{\text{nodal}}^s \in \mathcal{M}_{J_s}(v_1, v_2)$ can of course be considered to be a nodal conic. Further, the holomorphicity of ℓ_∞ implies that C_{nodal}^s intersects the line at infinity transversely precisely in the two points q_i . The node $x_{\text{nodal}}^s \in \mathbb{C}P^2$ of C_{nodal}^s , which must be different from the two points q_i , is thus contained inside $\mathbb{C}P^2 \setminus \ell_\infty$.

We are now ready to formulate the existence result for conic foliations.

Theorem 4.3 *The conics in $\mathcal{M}_{J_s}(v_1, v_2)$ form a smooth foliation of*

$$\mathbb{C}P^2 \setminus (\ell_\infty \cup \{x_{\text{nodal}}^s\})$$

with symplectic leaves, whose unique singular fibre is given by the nodal conic C_{nodal}^s with node x_{nodal}^s . There is an induced family of symplectic fibrations

$$f_{J_s}: \mathbb{C}P^2 \setminus \ell_\infty \rightarrow \mathbb{C},$$

which are submersions outside of the singularity of the nodal conic, and which depend smoothly on the parameter $s \in I^k$.

Under the further assumption that $J = i$ holds inside some given subset of the form $f^{-1}(U)$ with $U \subset \mathbb{C}$, one can ensure that $f_{\tilde{J}}|_{f^{-1}(U)} = f$ is satisfied there. In particular, this can be assumed to hold above some subset $U \subset \mathbb{C}$ whose complement is compact.

Proof The proof of the existence of the foliation relies on the well-known fact that tame almost complex structures form a contractible space [19]. As a consequence, the tame almost complex structures which are standard at infinity also form a contractible space. We may thus extend the family J_s for $s \in I^k$ to a smooth family $J_{(s,s)}$ parametrised by $(s, s) \in I \times I^k$, where $J_{(0,s)} \equiv i$ and $J_{(1,s)} = J_s$.

The transversality of the space of conics for all almost complex structures $J_{(s,s)}$ and the foliation property are then both consequences of the below automatic transversality result, [Lemma 4.4](#), together with a cobordism argument applied to the moduli space $\bigcup_{(s,s) \in I \times I^k} \mathcal{M}_{J_{s,s}}(v_1, v_2)$ of conics satisfying the given tangencies. Note that [Lemma 4.4](#) implies that the evaluation map from the moduli space is a *diffeomorphism* defined locally near any given conic (except at the node). The global foliation property is then a consequence of the facts that

- for any fixed s , we have $\mathcal{M}_{J_{0,s}}(v_1, v_2) \setminus \{2\ell_\infty\} \cong \mathbb{C}$, where we use $2\ell_\infty$ to denote the two-fold cover of the line at infinity branched at $\{q_1, q_2\}$,
- the conics $\mathcal{M}_{J_{0,s}}(v_1, v_2) \setminus \{2\ell_\infty\}$ foliate $\mathbb{C}^2 \setminus \{0\}$, and
- the almost complex structures $J_{(s,s)}$ are all standard near ℓ_∞ , and hence there exists a neighbourhood $2\ell_\infty \in U \subset \mathcal{M}_{J_{s,s}}(v_1, v_2)$ of solutions which persist for all $(s, s) \in I \times I^k$.

The third point above combined with positivity of intersection also implies that no line can satisfy both tangency conditions v_i for $i = 1, 2$ simultaneously. Hence, the two lines satisfying these tangency conditions join to form a nodal conic $C_{\text{nodal}}^{(s,s)}$, as sought.

To produce the symplectic fibrations $f_{J_{(s,s)}}: \mathbb{C}P^2 \setminus \ell_\infty \rightarrow \mathbb{C}$ whose fibres are the leaves in our conic foliation, we use the fact that the evaluation map from the moduli space is a diffeomorphism away from the node.

First, by positivity of intersection together with $[2\ell_\infty] \bullet [2\ell_\infty] = 4$, two conics in this family must intersect precisely in the two points $\{q_1, q_2\} \subset \ell_\infty$.

Then, we fix standard affine holomorphic coordinates $[1 : z_1 : z_2] \in \mathbb{C}P^2$ around $q_1 = [1 : 0 : 0]$ in which ℓ_∞ is given by $\{z_2 = 0\}$. Since the almost complex structures considered are standard near q_1 , each conic u has a uniquely determined power series expansion of the form

$$z \mapsto (z_1, z_2) = (0, z) + \sum_{k \geq 2} (a_k^{(s,s)}(u)z^k, 0), \quad a_k \in \mathbb{C},$$

after a suitable reparametrisation of the domain (depending smoothly on the conic C). The map $f_{J_{(s,s)}}: \mathbb{C}^2 \rightarrow \mathbb{C}$ which along each $J_{(s,s)}$ -holomorphic conic $u \subset \mathbb{C}^2$ with $u \in \mathcal{M}_{J_{(s,s)}}(v_1, v_2)$ takes as value the corresponding coefficient $a_2^{(s,s)}(u)$ is a smooth function by the foliation property. We end by arguing that this is a fibration of the sought form.

First we show that

$$a_2^{(s,s)}: \mathcal{M}_{J_{(s,s)}}(v_1, v_2) \rightarrow \mathbb{C}$$

is injective. To that end, note that two different conics would intersect with local intersection index at least 3 at q_1 if they would have the same coefficient $a_2^{(s,s)}$ in the above expansion. Together with positivity of intersection, this is then in contradiction with $[2\ell_\infty] \bullet [2\ell_\infty] = 4$, taking into account that the local intersection index at the other point $q_2 \in \ell_\infty$ is at least 2.

Then we claim that, when $s = 0$, this construction gives back the standard fibration $f = f_i$. A topological argument now shows that $f_{J(s,s)}$ is surjective for all (s, s) .

It remains to show that $f_{J(s,s)}$ is submersive away from the node of the singular conic. This is a consequence of the foliation property, together with the last statement of Lemma 4.4, by which $a_2^{(s,s)}$ is submersive. □

The following automatic transversality result was crucial in the above proof.

Lemma 4.4 *Any smooth conic (ie a conic which is neither nodal nor a branched cover) inside $\mathcal{M}_J(v_1, v_2)$ is a regular solution to this moduli problem for an arbitrary tame J . Consequently, $\mathcal{M}_J(v_1, v_2)$ is a smooth two-dimensional manifold. Furthermore, the section normal to some conic $u \in \mathcal{M}(v_1, v_2)$ corresponding to a nonzero vector in $T_u\mathcal{M}_J(v_1, v_2)$ vanishes precisely at the two points q_i for $i = 1, 2$, where it moreover has zeros of precisely order two.*

Proof The statement is a fairly straightforward consequence of the automatic transversality result in [22]; we proceed to give the argument.

Consider the space \mathcal{M}_J of embedded J -holomorphic conics $u: \mathbb{C}P^1 \rightarrow \mathbb{C}P^2$, together with a fixed solution $u_0 \in \mathcal{M}_J(v_1, v_2)$ satisfying the tangency conditions $Du_0(T_0\mathbb{C}P^1) = v_1$ and $Du_0(T_\infty\mathbb{C}P^1) = v_2$. Recall that these conics are embedded by Lemma 4.2. The kernel of the linearised $\bar{\partial}$ -problem *disregarding reparametrizations* is thus a complex five-dimensional space by the aforementioned automatic transversality; indeed, the cokernel vanishes and the Fredholm index is equal to $(n - 3)\chi(\mathbb{C}P^1) + 2c_1(\mathbb{C}P^2)[u_0] = -1(2) + 12 = 10$.

We need to show that the infinitesimal evaluation map

$$(u, \text{pt}_1, \text{pt}_2) \mapsto ((u(\text{pt}_1), Du(T_{\text{pt}_1}\mathbb{C}P^1)), (u(\text{pt}_2), Du(T_{\text{pt}_2}\mathbb{C}P^1)))$$

is transverse to the pair $((q_1, v_1), (q_2, v_2))$. Since we consider an embedded conic, we can identify the solutions near $u_0 \in \mathcal{M}_J$ with certain sections $\sigma_u \in \Gamma(\nu_{u_0})$ in the normal bundle of u_0 . We make the choice of appropriate holomorphic coordinates near the two points q_i (recall that u_0 is holomorphic near these two points), and can then assume that the normal bundle is holomorphic there, and that $\bar{\partial}_J$ for the sections σ_u actually is the standard Cauchy–Riemann operator near the two points $\{0, \infty\} = u_0^{-1}\{q_1, q_2\} \subset \mathbb{C}P^1$.

From this point of view, the problem boils down to showing that the map

$$\mathcal{M}_J \ni u \mapsto ((\sigma_u(0), \sigma'_u(0)), (\sigma_u(\infty), \sigma'_u(\infty))) \in ((v_{u_0})_0)^2 \times ((v_{u_0})_\infty)^2 \cong (\mathbb{C}^2)^2$$

is submersive to the origin for u close to u_0 . The differential of this map is a linear map

$$\Phi: \mathbb{C}^5 \cong T_{u_0}\mathcal{M}_J \ni \xi \mapsto ((\xi(0), \xi'(0)), (\xi(\infty), \xi'(\infty))) \in \mathbb{C}^4,$$

where $\xi \in T_{u_0}\mathcal{M}_J$ again can be seen as a section in the normal bundle of u_0 . Moreover,

- ξ can be identified with a holomorphic map to \mathbb{C} near the two points $\{0, \infty\} = u_0^{-1}\{q_1, q_2\} \subset \mathbb{C}P^1$ in the above coordinates;
- every geometric intersection of ξ with the zero section contributes positively to the algebraic intersection number.

The second statement is the main technical result of [22].

In conclusion, when Φ is not surjective, one can readily find a section $\xi \in \Gamma(v_{u_0})$ with a sufficiently high vanishing at the points $\{0, 1\} = u_0^{-1}\{q_1, q_2\}$ that the algebraic intersection index there is equal to at least $+5$ there. Using the aforementioned result concerning positivity of intersection, this is in contradiction with the fact that the Euler number of v_{u_0} is equal to 4. In order to see the claimed vanishing, we argue as follows. When Φ is not surjective, then the linear subspace $\ker \Phi \subset \mathbb{C}^2$ is of real dimension ≥ 3 . In this situation one thus finds a one-dimensional subspace which satisfies the vanishing $\xi''(0) = 0$ as well.

The claim concerning the vanishing of the section corresponding to the infinitesimal variation in $T_u\mathcal{M}_J(v_1, v_2)$ is shown similarly, using the positivity of intersection from [22] together with the fact $[u] \bullet [u] = 4$. To that end, note that any section in the normal bundle coming from a nonzero variation in $T_u\mathcal{M}_J(v_1, v_2)$ automatically vanishes to order *at least* two at both points q_i for $i = 1, 2$, by the tangency condition. \square

4.1 Normalising the fibration

For us it is necessary to perform a further normalisation of the conic fibration supplied by Theorem 4.3. In particular, we want to make the fibration standard outside of a compact subset, and to make the nodal conic standard near its node.

Remark 4.5 The normalised fibration will still not define a symplectic Lefschetz fibration in the complement of the line ℓ_∞ at infinity in the normal sense; see eg [31] for the definition. The reason is that the requirement for the fibration to be symplectically trivial outside of a compact subset is not satisfied even for the standard fibration f .

Theorem 4.6 Assume we are given a conic fibration f_J as produced by [Theorem 4.3](#), where f_J is standard inside a subset of the form $f^{-1}U$ with $U \subset \mathbb{C}$, where $\mathbb{C} \setminus U$ is compact. It is possible to find a one-parameter family f_{J_t} for $t \in [0, 1]$ of such conic fibrations, where $J_0 = i$ and $f_{J_0} = f$ both are standard, for which

- the fibres of f_{J_1} coincide with the fibres of f_J outside some small neighbourhood of the form

$$O_\epsilon := (B_\epsilon^4(q_1) \cup B_\epsilon^4(q_2)) \setminus (\ell_\infty \cup f^{-1}F) \quad \text{for } 1 \gg \epsilon > 0,$$

where $F \subset U$ is an arbitrary closed subset. Moreover, the almost complex structure J_1 can be taken to coincide with J outside of the neighbourhood $O_{\epsilon'}$ for some $0 < \epsilon' \ll \epsilon$;

- the nodal conics $C_{\text{nodal}}^{J_t}$ each coincide with the standard nodal conic C_{nodal} near ℓ_∞ and are moreover each given as the preimage $C_{\text{nodal}}^{J_t} = f_{J_t}^{-1}(0)$; and
- $f_{J_t} = f$ holds outside of some compact subset of \mathbb{C}^2 .

Proof There exists a path J_t from $J_0 = i$ to $J_1 = J$ with corresponding fibrations f_{J_t} by J_t -holomorphic conics. See eg the proof of [Theorem 4.3](#).

We start to normalise the foliations near the two points $\{q_1, q_2\}$, making them coincide with the standard foliation there. Note that the smooth foliation property outside of these two points then allows us to find a deformation J'_t of the path of almost complex structures, where still $J'_0 = i$, for which the deformed foliations are J'_t -holomorphic. (Here may assume that $J' = i$ still holds in a possibly smaller neighbourhood of ℓ_∞ .)

The symplectic foliation is deformed by carefully replacing the coefficients in the power series expansions near the points $\{q_1, q_2\}$ that were described in the proof of [Theorem 4.3](#). For simplicity we will here consider the case of the fixed fibration f_J ; the general one-parameter case is proven without any additional difficulty.

First we recall our choice of power series expansions near q_i for the leaves. Take the standard affine holomorphic coordinates $[1 : z_1 : z_2] \in \mathbb{C}P^2$ around $q_1 = [1 : 0 : 0]$ in which ℓ_∞ is given by $\{z_2 = 0\}$. Since the almost complex structures considered are standard near q_1 , each conic $u \in \mathcal{M}_J(v_1, v_2)$ has a uniquely determined power series expansion of the form

$$z \mapsto (z_1, z_2) = (0, z) + \sum_{k \geq 2} (a_k(u)z^k, 0), \quad a_k \in \mathbb{C},$$

after a suitable reparametrisation of the domain. In analogous affine coordinates near q_2 , the leaves can be written as

$$z \mapsto (z_1, z_2) = (z, 0) + \sum_{k \geq 2} (0, b_k(u)z^k, 0), \quad b_k \in \mathbb{C}.$$

The coefficients b_k and a_k of nonminimal degree in the above power series can be replaced by functions of the form

$$\beta_r(\|z\|) \cdot a_k(u) \quad \text{and} \quad a_r(\|z\|) \cdot b_k(u) \quad \text{for } k \geq 3,$$

where β_r is a bump function satisfying $\beta'_r(t) \geq 0$ and $\beta_r(t) = 1$ for $t \geq r$, $\beta_r(t) = 0$ near $t = 0$, and in addition $|\beta'_r| \leq 2/r$. Note that such a deformation does not deform those leaves which already are standard near the points q_i . We claim that, when $t_0 = r > 0$ is taken to be sufficiently small, this deformation is still a symplectic foliation, since only the higher-order terms are deformed. Here we use the facts that the inequality

$$\|d(\beta_r(\|z\|) \cdot z^k)\| \leq 2r^{k-1} + kr^{k-1} = (k+2)r^{k-1} \quad \text{for } k \geq 3$$

holds in the region D_r^2 containing the support of β'_r . (Here $r > 0$ is sufficiently small.) In this manner, we can make the leaves of the foliation coincide with leaves of the standard foliation near the two points q_i . This finishes the claim in the first bullet point.

For the second bullet point, we need to satisfy

$$a_2(C_{\text{nodal}}^J) = b_2(C_{\text{nodal}}^J) = 0.$$

We proceed as follows. Recall that all conic fibres except $2\ell_\infty$ are graphical over the second and first affine coordinate around q_1 and q_2 , respectively, as described above. The foliation can then readily be deformed by replacing the coefficients $a_2(u)$ and $b_2(u)$ by coefficients of the form

$$\phi_{\beta_r(2\|z\|)}(a_2(u)) \quad \text{and} \quad \psi_{\beta_r(2\|z\|)}(b_2(u))$$

for suitable smooth and compactly supported isotopies $\phi_t, \psi_t: \mathbb{C} \rightarrow \mathbb{C}$ which both satisfy $\phi_t = \text{Id}_{\mathbb{C}}$ for all $t \geq \frac{1}{2}$. For the symplectic condition of the deformed foliation, note that

$$\begin{aligned} \|d(\phi_{\beta_r(2\|z\|)}(a_1(u)) \cdot z^2)\| &\leq c_1(u) \cdot r, \\ \|d(\psi_{\beta_r(2\|z\|)}(b_2(u)) \cdot z^2)\| &\leq c_1(u) \cdot r \end{aligned}$$

are satisfied inside $\{\|z\| \leq r\}$ for $r > 0$ sufficiently small, where

$$c_1: \mathcal{M}(v_1, v_2) \setminus \{2\ell_\infty\} \rightarrow \mathbb{R}_{\geq 0}$$

is a continuous function depending on the choice of isotopy which is constant outside of some compact subset. Since the isotopies are compactly supported, it thus suffices to take $r > 0$ sufficiently small in order to guarantee the symplectic condition.

What remains is the last bullet point. We need to make each fibre of f_J coincide with *one and the same* fibre of f near both points q_i for $i = 1, 2$. This can be done by the same method as in the proof of the second bullet point, using suitable isotopies ϕ_t and ψ_t of \mathbb{C} . Once this has been done, it is a simple matter of reparametrising the target \mathbb{C} of the fibration in order to achieve the sought property. \square

In addition it will be useful to consider the following normalisation of the fibration over a path in the base starting at the image of the nodal conic. Let

$$\gamma: [0, 1] \hookrightarrow \mathbb{C}$$

be an embedded path which

- coincides with the canonical inclusion $\gamma_0: [0, 1] \hookrightarrow \mathbb{C}$ near its boundary, and
- is homotopic to the canonical inclusion γ_0 through embeddings γ_t with $\gamma_1 = \gamma$, all coinciding in some neighbourhood of the boundary.

Theorem 4.7 *Assume that $f_{J_t}: (\mathbb{C}P^2 \setminus \ell_\infty, \omega_{FS}) \rightarrow \mathbb{C}$ is a smooth path of symplectic conic fibrations with $J_0 = i$ and $J_1 = J$, where f_{J_t} all are fixed outside of a compact subset of $\mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})$. Under the above assumptions on γ , there then exists a compactly supported Hamiltonian isotopy*

$$\phi_{H_t}^t: (\mathbb{C}P^2 \setminus (\ell_\infty \cup C_{\text{nodal}}), \omega_{FS}) \rightarrow (\mathbb{C}P^2 \setminus (\ell_\infty \cup C_{\text{nodal}}), \omega_{FS})$$

which maps $f_J^{-1}(\gamma(x))$ to $f^{-1}(x)$ for each $x \in [0, 1]$.

Proof Recall that the symplectic fibrations f_{J_t} give rise to a parallel transport along the extended curves γ_t by integrating a suitably normalised characteristic vector field inside $f_{J_t}^{-1}\gamma_t$. Using this parallel transport, starting from the conic fibre $f_{J_t}^{-1}(\epsilon) = f^{-1}(\epsilon)$ for some small $\epsilon > 0$, we obtain a compactly supported isotopy

$$\varphi_t: f_{J_0}^{-1}(\epsilon) \times [\epsilon, 1 + \epsilon] \hookrightarrow \mathbb{C}P^2 \setminus (\ell_\infty \cup C_{\text{nodal}})$$

of hypersurfaces where

- $\varphi_t^* \omega_{FS}$ is constant,
- $\varphi_t(f_{J_0}^{-1}(\epsilon) \times \{x\}) = f_{J_t}^{-1}(\gamma_t(x))$, and
- $\varphi_t = \varphi_0$ holds outside of a compact subset, as well as near C_{nodal} .

The independence of t asserted in the last two points follows from the assumption that the fibrations all are standard outside of a compact subset of $\mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})$.

The standard symplectic neighbourhood theorem [29] then provides us with an extension of the family φ_t to a family of open symplectic embeddings

$$\begin{aligned} \Phi_t: (f_{J_0}^{-1}(\epsilon) \times [\epsilon, 1 + \epsilon] \times [-\epsilon', \epsilon'], \omega) &\hookrightarrow (\mathbb{C}P^2 \setminus \ell_\infty, \omega_{\text{FS}}), \\ \Phi_t|_{f_{J_0}^{-1}(\epsilon) \times [\epsilon, 1 + \epsilon] \times \{0\}} &= \varphi_t, \end{aligned}$$

fixed outside of a compact subset of each $f_{J_0}^{-1}(\epsilon) \times \{(x, y)\}$. For the last property, we again rely on the assumption that the fibrations are standard outside of a compact subset.

Finally, since the support of the above family of symplectomorphisms is of the aforementioned form, a standard fact now shows that it can be generated by a compactly supported Hamiltonian, ie $\Phi_t = \phi_{G_t}^t \circ \Phi_0$, where G_t moreover can be taken to vanish in some neighbourhood of $f^{-1}[0, \epsilon]$. A suitable cutoff of this Hamiltonian then generates the sought global Hamiltonian isotopy of $(\mathbb{C}P^2 \setminus \ell_\infty, \omega_{\text{FS}})$. □

4.2 Hamiltonian isotopies of symplectic surfaces with smooth self-intersection

Here we recall and establish facts concerning Hamiltonian isotopy of nodal symplectic surfaces, as well as symplectic surfaces with more complicated discrete self-intersections. First recall that a smooth family of embedded symplectic surfaces can be generated by a Hamiltonian isotopy by the following basic result.

Proposition 4.8 [33, Proposition 0.3] *Let $\Sigma_t \subset (X^4, \omega)$ be a smooth isotopy of symplectic surfaces, where the isotopy moreover is fixed inside some (possibly empty) subset $F \subset X$. Then there exists a Hamiltonian $H_t: X \rightarrow \mathbb{R}$ for which $\Sigma_t = \phi_{H_t}^t(\Sigma_0)$. The Hamiltonian H_t can moreover be taken to vanish on any given open subset $U \subset F$.*

There is no analogous results for *nodal* symplectic surfaces; for example, the tangent planes at the node are not generically symplectically orthogonal. We now argue that we still can find a Hamiltonian isotopy that generates a path of eg nodal symplectic surfaces, albeit after an initial deformation near its nodes. First, a discrete self-intersection locus may be assumed to remain fixed in the family after a Hamiltonian isotopy. [Proposition 4.9](#) then makes it possible to deform the surfaces near the self-intersection locus in order to make the symplectomorphism class of the singularities constant. Finally, we can apply [Proposition 4.8](#) to this deformed family.

Let $\Sigma_t \subset (X^4, \omega)$ be a smooth family of symplectic immersions of a finite union of closed surfaces whose self-intersection loci each consist of precisely the k number of points $p^1, \dots, p^k \in \Sigma_t$ fixed in the family. (These points are all isolated singularities by assumption, but they are not necessarily transverse double points.)

Proposition 4.9 *After a deformation of Σ_t through smooth paths of symplectic immersions of the same kind, where the deformation can be assumed to be supported in an arbitrarily small neighbourhood of p^1, \dots, p^k , we obtain a path $\tilde{\Sigma}_t$ of symplectic immersions that satisfies $\tilde{\Sigma}_0 = \Sigma_0$ and which is fixed in a **neighbourhood** of its self-intersections.*

- (1) *If, in addition, the path of immersion Σ_t is fixed when restricted to a number of components $A \subset \Sigma_t$, then we may assume $A \subset \tilde{\Sigma}_t$ to be fixed as well.*
- (2) *If, in addition, $p^i \in \Sigma_t$ is a transverse double point for all t , and if the corresponding sheets of Σ_0 and Σ_1 coincide near p^i , then $\tilde{\Sigma}_1 = \Sigma_1$ can be assumed to hold near p^i .*

Proof The deformation follows as in the proof of Corollary 3.7 in [12]. Roughly speaking, we consider the links of the singularities of the immersions (ie the points p^1, \dots, p^k). The singularities consist of intersections of a number of smooth sheets. The links are thus transverse links inside the standard contact three-sphere (S^3, ξ_{std}) , where each component of the link is a standard transverse unknot. The deformation is obtained by a suitable smooth interpolation, by attaching symplectic cylinders given as the trace of an isotopy through transverse links.

For (2), we need to use the fact that the space consisting of linear symplectic two-planes that are transverse to another given linear symplectic two-plane is contractible. \square

5 Properties derived from broken conic fibrations

In this section $L \subset V \subset (\mathbb{C}P^2, \omega_{\text{FS}})$ will always be used to denote an embedded Lagrangian *torus* satisfying at least one of the conditions of [Theorem A](#). Our goal here is to establish [Theorem 5.12](#), ie to construct a pseudoholomorphic conic fibration in the sense of [Section 4](#) that is compatible with L . Recall that compatibility of the fibration with the torus implies that the latter is fibred over an embedded closed curve in the base. In the entire section we rely heavily on the technique of stretching the neck, which is performed on the unit normal bundle of the Lagrangian torus.

5.1 A neck-stretching sequence

We follow [12, Section 2] in the construction of a sequence of almost complex structures which *stretch the neck* around an embedding of the unit cotangent bundle of the Lagrangian torus $L \subset (\mathbb{C}P^2 \setminus \ell_\infty, \omega_{FS})$. This is a sequence J_τ with $\tau \geq 0$ of compatible almost complex structures on $(\mathbb{C}P^2, \omega_{FS})$ satisfying the following properties:

- $J_\tau = i$ in a neighbourhood of ℓ_∞ as well as in a neighbourhood of the smooth conic $C \subset \mathbb{C}P^2$.
- In a fixed Weinstein neighbourhood

$$\phi: (D_{3\epsilon}T^*\mathbb{T}^2, d\lambda_{\mathbb{T}^2}) \hookrightarrow (V, \omega_{FS}), \quad \phi(0_{\mathbb{T}^2}) = L,$$

the almost complex structure takes the form

$$J_\tau \partial_{\theta_i} = -\rho_\tau(\|p\|)\partial_{p_i}$$

for a function $\rho_\tau: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$ satisfying $\rho_\tau(t) = \epsilon$ for $t \leq \epsilon$ and $\rho_\tau(t) = t$ for $t \geq 2\epsilon$, while $\int_\epsilon^{2\epsilon} \rho(t) dt \geq \tau$.

- J_τ is fixed outside of the above Weinstein neighbourhood $\phi(D_{3\epsilon}T^*\mathbb{T}^2) \subset \mathbb{C}P^2$.

In Section 8 a variation of the above construction will be used, where we stretch the neck around two disjoint Lagrangian tori simultaneously. In that case, the sequence is constructed in the analogous manner utilising disjoint Weinstein neighbourhoods of the two tori.

The above choices also specify the following important compatible almost complex structures:

- The compatible almost complex structure J_{std} on $T^*\mathbb{T}^2$ which is given by $J_{std} \partial_{\theta_i} = -\rho_1(\|p\|)\partial_{p_i}$.
- The compatible almost complex structure J_{cyl} defined on

$$(T^*\mathbb{T}^2 \setminus 0_{\mathbb{T}^2}, d\lambda_{\mathbb{T}^2}) = (\mathbb{R} \times ST^*\mathbb{T}^2, d(e^t \lambda_{\mathbb{T}^2}|_{ST^*\mathbb{T}^2}))$$

by $J_{cyl} \partial_{\theta_i} = -\|p\| \partial_{p_i}$. It is important to note that this almost complex structure is cylindrical with respect to the contact form $\alpha := \lambda_{\mathbb{T}^2}|_{ST^*\mathbb{T}^2}$ induced by the flat Riemannian metric on \mathbb{T}^2 .

- The compatible almost complex structure J_∞ defined on $\mathbb{C}P^2 \setminus L$ which coincides with J_{cyl} inside the above Weinstein neighbourhood $\phi(D_{3\epsilon}T^*\mathbb{T}^2)$ and with J_τ , where $\tau \geq 0$ is arbitrary, in the complement $\mathbb{C}P^2 \setminus \phi(D_{3\epsilon}T^*\mathbb{T}^2)$.

Recall that the periodic Reeb orbits of α correspond to lifts of closed geodesics on \mathbb{T}^2 induced by the flat metric. A basic but very crucial fact is that these geodesics all live in Bott manifolds $\Gamma_\alpha \cong S^1$, which are in bijection with the nonzero homology classes $\alpha \in H_1(L) \setminus \{0\}$ of the corresponding geodesics. In particular, there are no closed contractible geodesics for the flat metric.

When speaking about finite-energy pseudoholomorphic spheres we mean pseudoholomorphic spheres inside either $(T^*\mathbb{T}^2, J_{\text{std}})$, $(\mathbb{R} \times ST^*\mathbb{T}^2, J_{\text{cyl}})$, or $(\mathbb{C}P^2 \setminus L, J_\infty)$ with a finite number of punctures asymptotic to periodic Reeb orbits on $(S^*\mathbb{T}^2, \alpha)$. By a classical result this condition is equivalent to that of having finite so-called Hofer energy; see [24] and [23]. In the following all punctured pseudoholomorphic spheres will tacitly be assumed to be holomorphic for one of the almost complex structures as described above, and to be of finite Hofer energy.

A one-punctured pseudoholomorphic sphere is called a *plane* while a two-punctured pseudoholomorphic sphere is called a *cylinder*.

By a *broken pseudoholomorphic conic* we mean a pseudoholomorphic building of at least two levels whose components satisfy the following topological condition: gluing all the domains of the components at their nodes we obtain a sphere without punctures, on which the maps compactify to give a continuous cycle $S^2 \rightarrow \mathbb{C}P^2$ of degree two. Of course there also exist closed J_∞ -holomorphic curves inside $\mathbb{C}P^2 \setminus L$ without punctures; eg the conic $C \subset \mathbb{C}P^2$ is J_∞ -holomorphic by the assumptions made on J_∞ . Such curves will be called *unbroken*. Similarly we also consider *(un)broken pseudoholomorphic lines* inside $\mathbb{C}P^2 \setminus L$.

It is immediate from the SFT compactness theorem that the limit of a sequence of J_τ -holomorphic conics (resp. lines) as $\tau \rightarrow +\infty$ is a broken or unbroken pseudoholomorphic conic (resp. line); see [5] or [8].

Remark 5.1 It is not clear a priori that the converse holds, ie that all broken pseudoholomorphic conics or lines arise as such limits; this would require a rather strong form of pseudoholomorphic gluing in the Bott setting under consideration. Since we do not rely on such a result, we must instead use somewhat roundabout arguments based upon (asymptotic) positivity of intersection of pseudoholomorphic buildings, combined with automatic transversality for the components in the building, in order to rule out certain unwanted broken configurations.

When considering limits of pseudoholomorphic conics or lines in the four-dimensional setting, the following crucial property is a consequence of positivity of intersection [28].

Lemma 5.2 *The limit of pseudoholomorphic lines or conics under a neck-stretching sequence consists of components all of which are (possibly trivial) branched covers of embedded punctured pseudoholomorphic spheres. In the case of a limit of lines, it is moreover the case that two different components have disjoint interiors.*

Another important technical feature of the dimension where we are working is that the Fredholm index of a punctured sphere can be forced to be nonnegative under certain mild assumptions; this facilitates transversality arguments and analysis significantly. More precisely, we have:

Lemma 5.3 [12, Lemma 3.3] *If all simple J_∞ -holomorphic punctured spheres inside $\mathbb{C}P^2 \setminus L$ are of nonnegative Fredholm index, then the same is true for all J_∞ -holomorphic punctured spheres. Since a plane has odd Fredholm index, its index is thus at least one in this case. Moreover, if the index of a such a punctured plane inside $\mathbb{C}P^2 \setminus (\ell_\infty \cup L)$ is equal to one, then it is simply covered with a simply covered asymptotic. Finally, for a generic almost complex structure J_∞ as in Section 5.1, all simply covered curves can indeed be assumed to be transversely cut out and hence of nonnegative index.*

Lemma 5.4 [12, Proposition 3.5] *Assume that there exists no punctured pseudoholomorphic spheres inside $\mathbb{C}P^2 \setminus L$ of negative index. After perturbing J_∞ inside some compact subset of $U \setminus \ell_\infty$, where $U \subset \mathbb{C}P^2$ is an arbitrarily small neighbourhood of ℓ_∞ , we may assume that any curve which is either*

- *a broken line satisfying a fixed-point constraint at $q \in \ell_\infty$, or*
- *a broken conic satisfying two fixed tangency conditions, one at each of two points $q_1, q_2 \in \ell_\infty$,*

*has a top level consisting of precisely two planes of index **one**, and possibly several cylinders of index **zero**. (Here we take appropriate constraint(s) into account when considering the index of a component.) Moreover, the components passing through ℓ_∞ are transversely cut out when considered with the appropriate point and tangency condition, respectively.*

Remark 5.5 In the above lemma we do not assume that the broken curve is an actual limit under a neck-stretching sequence.

Proof First we note that the Fredholm index for the corresponding nonbroken solutions

is equal to two in both of the cases, where the moduli spaces are considered with the corresponding point or tangency constraint. Therefore, the proof of [12, Proposition 3.5] carries over immediately to the current situation, once the following transversality result has been established: for generic almost complex structures of the type considered, the constrained moduli spaces are transversely cut out and assume their expected dimensions.

Since we are in a very particular situation the sought transversality is not difficult to establish. In the unbroken case, this is a consequence of automatic transversality; see Lemma 4.4. In the case of a broken curve, positivity of intersection implies that the component satisfying the additional constraint(s) necessarily is simply covered and traverse to ℓ_∞ . The transversality properties can now be achieved by finding explicit deformations by hand, while using standard transversality techniques [30, Section 3]. \square

Most work in the remaining part of this section is to sharpen the above result, by showing that a broken conic consists of precisely two components in its top level, where each component moreover intersects ℓ_∞ , as is depicted in Figure 6. In order for this to hold, it is crucial to use one of the two conditions in the assumptions of Theorem A.

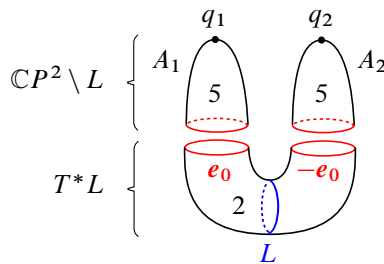


Figure 6: The broken conic in the generic case. The two planes A_i each intersect ℓ_∞ transversely in the point q_i , where they moreover are tangent to the smooth conic C . The two planes join to form a continuous embedding of a sphere.

5.2 Solid tori foliated by pseudoholomorphic planes and consequences

When a broken line or conic consists of a plane that is disjoint from ℓ_∞ , we can under certain assumptions use the techniques from [12] in order to construct an embedded solid torus with boundary equal to L , which is foliated by discs, one of which is close to the aforementioned plane.

Proposition 5.6 [12, Section 5] *Let $L \subset (\mathbb{C}P^2 \setminus \ell_\infty, \omega_{FS})$ be an embedded Lagrangian torus and consider any compatible almost complex structure $J_\infty \subset \mathbb{C}P^2 \setminus L$ that satisfies the properties of Lemma 5.4. Assume that there exists a pseudoholomorphic plane $A \subset \mathbb{C}P^2 \setminus (L \cup \ell_\infty)$ asymptotic to L which arises as a component of a generic broken line satisfying a fixed-point constraint at one of the points $q_i \in \ell_\infty$. Then there exists a smooth embedding of a solid torus*

$$(S^1 \times D^2, S^1 \times S^1) \hookrightarrow (\mathbb{C}P^2 \setminus \ell_\infty, L)$$

foliated by embedded J -holomorphic discs in the same homotopy class as A , where $J = J_\infty$ can be taken to be satisfied outside of an arbitrarily small neighbourhood of L .

Using the above we show that a Lagrangian torus that satisfies either of the conditions in the assumptions of Theorem A can be disjointed from the standard nodal conic by a Hamiltonian isotopy.

Lemma 5.7 *Assume that either condition (1) or (2) of Theorem A is satisfied. For an almost complex structure J_∞ that satisfies the properties of Lemma 5.4, the image of the nodal conic is not broken. It follows that the embedded Lagrangian torus $L \subset (V, \omega_{FS})$ can be disjointed from the standard nodal conic $C_{\text{nodal}} = \ell_1 \cup \ell_2$ by a Hamiltonian isotopy supported inside V .*

Proof Assume that the nodal conic is broken. Lemma 5.4 implies that the broken line in the nodal conic has a component that is a plane of index one which is disjoint from ℓ_∞ . Using either condition (2), or condition (1) in conjunction with the existence of the solid torus produced by Proposition 5.6 that bounds L (and contains a perturbation of the pseudoholomorphic plane as a compressing disc), we conclude that the plane necessarily intersects C . Since such a line already is tangent to C at one of the points $q_i \in \ell_\infty$ this finally leads to a contradiction with the total intersection number, $[\ell_\infty] \bullet C = 2$, of a line and a conic. For the last part we need positivity of intersection [28].

What remains is to construct the Hamiltonian isotopy. Start by considering a compatible almost complex structure J for which the aforementioned broken conic $\ell_1^J \cup \ell_2^J$ as well as C and ℓ_∞ all are J -holomorphic simultaneously. Then interpolate between J and J_0 through compatible almost complex structures, all for which $C \cup \ell_\infty$ remains pseudoholomorphic. Gromov’s result, Theorem 4.1, produces a smoothly varying family Σ_t of symplectic immersions where $\Sigma_0 = \ell_1 \cup \ell_2 \cup C \cup \ell_\infty$ and $\Sigma_1 = \ell_1^J \cup \ell_2^J \cup C \cup \ell_\infty$. Deforming this family at times $t > 0$ using a Hamiltonian

isotopy constructed by hand, we may assume that the position of the node remains fixed in the family. In particular, the node now coincides with $\ell_1 \cap \ell_2$ for all $t \geq 0$.

Proposition 4.9(1) applies to this family, yielding a family $\tilde{\Sigma}_t$ to which **Proposition 4.8** can be applied. Here $\tilde{\Sigma}_0 = \Sigma_0$, while we also have $C, \ell_\infty \subset \tilde{\Sigma}_t$, and $\tilde{\Sigma}_1 \subset V \setminus L$ is a small perturbation of Σ_1 . In other words, the global Hamiltonian isotopy produced by **Proposition 4.8** that generates the isotopy $\tilde{\Sigma}_{1-t}$ of symplectic surfaces is the one we seek. □

Lemma 5.8 *Assume that either condition (1) or (2) of **Theorem A** is satisfied for the Lagrangian torus $L \subset (V, \omega_{FS})$. There exists a basis $\langle e_0, e_1 \rangle = H_1(L)$ with the property that e_0 generates the kernel of the canonical map $H_1(L) \rightarrow H_1(V)$ induced by the inclusion $L \subset V$, while any disc $(D, \partial D) \rightarrow (\mathbb{C}P^2 \setminus \ell_\infty, L)$ with boundary in the homology class e_1 must satisfy $D \bullet C = a_1 > 0$.*

Proof Take any basis $\langle e_0, e_1 \rangle = H_1(L) = \mathbb{Z}^2$ for which e_0 is in the kernel of $H_1(L) \rightarrow H_1(V) = \mathbb{Z}$. It suffices to show that e_1 satisfies the property that any disc $(D, \partial D) \rightarrow (\mathbb{C}P^2 \setminus \ell_\infty, L)$ with boundary in the homology class e_1 must satisfy $D \bullet C \neq 0$.

Using **Proposition 5.6** applied to a suitable broken line, we produce a solid torus inside $\mathbb{C}P^2 \setminus \ell_\infty$ with boundary equal to L , which is foliated by J -holomorphic discs $D(\theta)$ of Maslov index two. The compatible almost complex structure J on $\mathbb{C}P^2$ can moreover be chosen so that $C \cup \ell_\infty$ is J -holomorphic.

Under either of the assumptions of **Theorem A**, positivity of intersection [28] implies that the discs foliating the above solid torus must intersect C positively. Since $[\partial D(\theta)] = ke_0 + le_1$ the statement follows. □

Lemma 5.9 *Assume that either condition (1) or (2) of **Theorem A** is satisfied for the Lagrangian torus $L \subset (V, \omega_{FS})$, and consider a basis $\langle e_0, e_1 \rangle = H_1(L)$ as in **Lemma 5.8**. Any broken J_∞ -holomorphic conic satisfying the given tangency conditions at q_i consists of components all whose asymptotics are geodesics in homology classes of the form $ke_0 \in H_1(L)$ with $k \in \mathbb{Z} \setminus \{0\}$.*

Remark 5.10 In the above lemma we make no assumptions on the genericity of J_∞ .

Corollary 5.11 *The same is true also for an almost complex structure J_∞ on $\mathbb{C}P^2 \setminus (L \cup L')$ obtained by stretching the neck also around an additional Lagrangian torus $L' \subset V \setminus L$, with the only caveat that any component then also may have additional punctures asymptotic to L' .*

Proof The statements follow by positivity of intersection with C together with topological considerations in conjunction with [Lemma 5.8](#). Recall that, as prescribed by the tangency conditions satisfied by the broken conic, it intersects C with algebraic intersection index $+4$ at the two points $q_i \in C$ for $i = 1, 2$. In other words, there are precisely two intersections of this building and the unbroken conic C , both which occur at the two points $q_i \in \ell_\infty$.

Consider a puncture p appearing as an asymptotic of a top-level component A in the building, where the asymptotic at p corresponds to a geodesic in the homology class $ke_0 + le_1 \in H_1(L)$. Compactify A to a continuous chain with boundary on L in the homology class $-(ke_0 + le_1) \in H_1(L)$ (note the sign!). Choose an arbitrary null-homology B of $ke_0 + le_1 \in H_1(L)$ inside $\mathbb{C}P^2 \setminus \ell_\infty$, and create a cycle $A \cup B$.

In this manner we have produced a cycle in $H_2(\mathbb{C}P^2)$ of degree $d = A \bullet \ell_\infty$, where d equals either 0, 1 or 2. The intersection number of the cycle and C is precisely $2d + la_1 = 2d$. The first term arises from the intersections of A and ℓ_∞ (these occur at $\{q_1, q_2\} \subset \ell_\infty$ by positivity of intersection) while the second term arises from the intersections of B and C . Since $a_1 > 0$ by [Lemma 5.8](#) we must have $l = 0$, as sought. □

5.3 Constructing a compatible fibration of conics

We are now ready to state the result concerning the existence of a pseudoholomorphic conic foliation compatible with a Lagrangian torus $L \subset (V, \omega_{FS})$ that satisfies one of the assumptions of [Theorem A](#). We postpone its proof to [Section 5.4](#).

Theorem 5.12 *After a Hamiltonian isotopy of $L \subset V \setminus (\ell_\infty \cup C \cup C_{nodal})$, there exists a tame almost complex structure J on $(\mathbb{C}P^2, \omega_{FS})$, where $J = i$ is standard near the divisor $\ell_\infty \cup C \cup C_{nodal}$, for which an application of [Theorem 4.3](#) gives rise to a fibration $f_J: (V, \omega_{FS}) \rightarrow \mathbb{C}$ by J -holomorphic conics which is compatible with the Lagrangian torus in the following sense:*

- (1) *The restriction $f_J|_L$ makes the torus a smooth S^1 -fibre bundle over the embedded closed curve*

$$\sigma := f_J(L) \subset \mathbb{C} \setminus \{0 = f_J(C_{nodal}), 1 = f_J(C)\},$$

which has winding number one around $1 \in \mathbb{C}$. The fibres of $f_J|_L$ are, moreover, closed curves inside L which are contractible inside V and of Maslov index zero.

- (2) The fibre of $f_J^{-1}(s)$ for any $s \in \sigma$ coincides with the standard fibre $f^{-1}(s)$ outside of a compact subset.
- (3) The curve σ is either disjoint from the curve $[0, 1] \subset \mathbb{C}$ or is of the form $s_0 + i[-\delta, \delta]$ in some neighbourhood of $[0, 1]$, where $s_0 \in (0, 1)$. Furthermore, we may assume that $f_J = f$ holds inside the same neighbourhood.

Remark 5.13 In the case when $L \subset \mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})$ already is satisfied, it is not necessary to apply a Hamiltonian isotopy to L in order to achieve parts (1) and (2) of [Theorem 5.12](#).

In view of [Theorem 5.12\(3\)](#) there are two possibilities for a Lagrangian torus: the closed curve $\sigma \subset \mathbb{C} \setminus \{0, 1\}$ over which the torus is fibred either has winding number around $0 \in \mathbb{C}$ equal to $w = 1$ (in the case when it is disjoint from $[0, 1] \subset \mathbb{C}$) or $w = 0$ (in the case when it intersects $[0, 1] \subset \mathbb{C}$ in a single point). When $w = 1$, we say that the torus is in *Clifford position* (see [Figure 8](#)), and when $w = 0$, *Chekanov position* (see [Figure 7](#)). These Lagrangian tori will later be shown to be Hamiltonian isotopic to standard tori of the corresponding types.

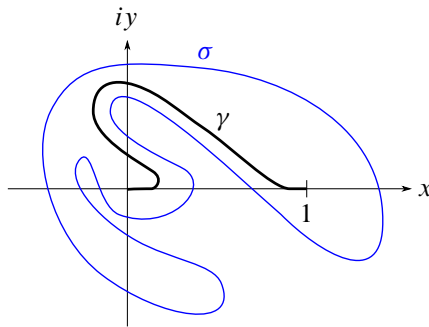


Figure 7: The image of a torus in Clifford in position under a compatible fibration, before carrying out the deformation to make the fibration standard above $[0, 1] \subset \mathbb{C}$.

We now state some consequences of [Theorem 5.12](#).

Corollary 5.14 Under the assumption of either condition (1) or (2) of [Theorem A](#), the Lagrangian torus $L \subset \mathbb{C}P^2 \setminus (\ell_\infty \cup C)$ is smoothly isotopic inside the same subset to a fibre of the Lagrangian fibration $\Pi_s: V \rightarrow (-1, 1) \times (0, +\infty)$ through totally real tori. In particular, conditions (1) and (2) of [Theorem A](#) are equivalent for embedded Lagrangian tori.

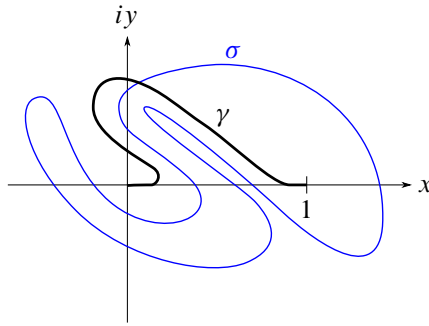


Figure 8: The image of a torus in Chekanov position under a compatible fibration, before carrying out the deformation in order to make the fibration standard above $[0, 1] \subset \mathbb{C}$.

Proof The smooth isotopy is readily constructed by first moving the torus within the S^1 -family of conic fibres σ to a Lagrangian torus that lives near, say, the point q_1 . The latter Lagrangian torus can then be smoothly isotoped through Lagrangian tori compatible with the standard Lefschetz fibration $f: \mathbb{C}P^2 \setminus (\ell_\infty \cup C) \rightarrow \mathbb{C}$ to a standard representative. The first part of the isotopy is not necessarily through Lagrangian tori, but at least it is through *totally real* tori. The reason is that, since these tori all live inside a three-dimensional hypersurface σ foliated by pseudoholomorphic curves, while they by construction have no full tangency to any of the curves, they cannot have any complex tangencies.

We end by noting that conditions (1) and (2) of **Theorem A** are equivalent merely by the existence of this formal Lagrangian isotopy; the standard tori are obviously in the class of the generator of $H_2(V)$, and all continuous discs of nonzero Maslov index on them must intersect C with a nonzero algebraic intersection number. \square

Later we will establish that the tori actually are Lagrangian isotopic, and not just formally Lagrangian isotopic. Recall that this already is known to be the case inside the larger space $(\mathbb{C}P^2 \setminus \ell_\infty, \omega_{FS}) \supset V$ by the main result of [12]; constructing a Lagrangian isotopy in the complement of the divisor C will turn out to require some additional work.

The property of having vanishing Maslov class for a torus satisfying either assumption of **Theorem A** has the following important and nontrivial consequence.

Proposition 5.15 *Any Lagrangian torus inside (V, ω_{FS}) which satisfies either of the assumptions of **Theorem A** is not Hamiltonian displaceable inside V .*

Proof By [Corollary 5.14](#) the Maslov class of the torus L vanishes when considered inside V .

Since the expected dimension of a pseudoholomorphic disc of Maslov index zero is equal to -1 , there are no such somewhere injective discs when the almost complex structure is generic. This follows by a standard argument; see eg [\[30\]](#). Then, by Lazzarini's result [\[27\]](#), it is the case that the space of nonconstant pseudoholomorphic discs with boundary on L is *empty* for such a generic almost complex structure. Indeed, otherwise we would be able to extract a somewhere injective pseudoholomorphic disc by appealing to Lazzarini's result.

The Hamiltonian nondisplaceability is then a direct consequence of the main result of [\[7\]](#). In order to see that this result can be applied, we note that the complement of a union of positive pseudoholomorphic divisors in a closed symplectic manifold is a noncompact but tame symplectic manifold. \square

5.4 The proof of [Theorem 5.12](#)

We start by performing a Hamiltonian isotopy in order to disjoin L from the standard nodal conic; this is possible by [Lemma 5.7](#).

Now we assume that J_∞ is chosen generically as in [Lemma 5.4](#), so that there are no pseudoholomorphic planes inside $\mathbb{C}P^2 \setminus L$ of negative index. We may assume that the nodal conic $\ell_1 \cup \ell_2$ tangent to C at $\{q_1, q_2\} \subset \ell_\infty$ is J_∞ -holomorphic.

Proposition 5.16 *Consider the basis $\langle e_0, e_1 \rangle = H_1(L)$ as in [Lemma 5.8](#). Any broken conic tangent to C at $q_i \in \ell_\infty$ for $i = 1, 2$ which appears as the limit of pseudoholomorphic conics when stretching the neck satisfies the following: its top level consists of precisely two planes $A_i \subset \mathbb{C}P^2 \setminus L$ for $i = 1, 2$, where*

- $A_i \cap \ell_\infty = \{q_i\}$, where it satisfies the tangency $v_i \subset T_{q_i} \mathbb{C}P^2$, and
- A_i is asymptotic to a geodesic on L in the class $(-1)^{i+1} e_0$ (after a suitable choice of sign for the generator e_0),

while the remaining components consists of a single cylinder contained inside T^*L . Moreover, for the above choice of sign for e_0 , the disc $(D, \partial D) \rightarrow (V, L)$ with $[\partial D] = e_0$ is of Maslov index zero and satisfies $D \bullet \ell_i = (-1)^i$. See [Figure 6](#).

Proof We begin by noting that a broken conic intersects the smooth conic C precisely at the two points $q_i \in \ell_\infty$ for $i = 1, 2$; this follows from positivity of intersection [\[28\]](#).

First we argue that a broken conic does not contain a plane that is disjoint from ℓ_∞ . Under the assumption of condition (2) of [Theorem A](#) this is immediate from [Lemma 5.4](#).

Under the assumption of condition (1) we must work a bit harder: By [Lemma 5.9](#) we can conclude that the plane appearing in the broken conic must be asymptotic to a geodesic in the class $k\mathbf{e}_0$ with $k \neq 0$ and that it is disjoint from C . In fact, since the plane is of index one by [Lemma 5.4](#), we must have $k = \pm 1$. Positivity of intersection [\[28\]](#), together with [Lemma 5.8](#), now implies that the moduli space containing this plane is compact, and hence homeomorphic to S^1 ; if the family of planes break in the sense of SFT compactness [\[5\]](#), then we can readily extract a different plane in the compactification which is asymptotic to an orbit in the homology class $k'\mathbf{e}_0 + l'\mathbf{e}_1$ with $l' \neq 0$. (The nonzero intersection number with C is then a contradiction.) We then argue as in [\[11\]](#): by appealing to automatic transversality [\[38\]](#) we can then use this moduli space to construct a continuous chain inside $\mathbb{C}P^2 \setminus \ell_\infty$ with boundary equal to $m[L]$ for some $m \neq 0$, ie to yield a null-homology of $m[L]$. By positivity of intersection, this chain is moreover disjoint from C , which is in contradiction with condition (1).

By the above, together with [Lemmas 5.4](#) and [5.9](#), we thus conclude that the broken conic consists of precisely the two planes A_i for $i = 1, 2$ in its top level, where $A_i \cap \ell_\infty = q_i$.

We then show that any disc $(D, \partial D) \rightarrow (B^4, L)$ with boundary in the homology class $[\partial D] = \mathbf{e}_0 \in H_1(L)$ must satisfy $D \bullet \ell_i = (-1)^i$, after an appropriate choice of sign for the generator. Indeed, we can complete the plane A_i intersecting q_i with a single puncture asymptotic to $k\mathbf{e}_0$ to a cycle in the class of a line by adding a $-k$ number of such discs, and then use the fact that

$$A_i \bullet \ell_j = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where again we have appealed to positivity of intersection. From this we also conclude that the plane intersecting q_i must be primitive and, moreover, asymptotic to a geodesic in the class $(-1)^{i+1}\mathbf{e}_0$.

It now follows by a topological argument that the remaining top components, ie those which are disjoint from ℓ_∞ , must have total symplectic area equal to

$$\int_{\mathbf{e}_0} \lambda_{\text{std}} - \int_{\mathbf{e}_0} \lambda_{\text{std}} = 0.$$

Hence, there can be no top-level components disjoint from ℓ_∞ and, again by positivity of intersection, the top-level component consists of precisely the two planes A_i for $i = 1, 2$.

A final index calculation shows that e_0 is of Maslov index zero. \square

We now proceed with the proof of [Theorem 5.12](#).

(1) We run a neck stretching with $J_\infty = i$ near $\ell_\infty \cup C_{\text{nodal}} \cup C$. The SFT compactness theorem [[5](#); [8](#)], together with [Theorem 4.3](#), allows us to extract *broken* pseudoholomorphic conics as limits from the moduli spaces $\mathcal{M}_{J_\tau}(v_1, v_2)$. In view of [Proposition 5.16](#) this conic consists of precisely two planes A_i for $i = 1, 2$ in the top level, where the plane A_i satisfies the tangency v_i at $q_i \in \ell_\infty$. Here we must use a generic almost complex structure J_∞ .

The rest of the argument follows the same ideas as [[12](#), Section 5.4]; we need to vary the components in the top level of the obtained broken conic in order to produce a whole one-parameter family of broken conics.

By the nonexistence of planes of Maslov index two inside $V \setminus L$ that are asymptotic to L , as follows from [Proposition 5.16](#) (also see [Lemma 5.8](#)), we conclude that the planes A_i are contained inside compact components of its moduli spaces of planes satisfying a tangency to C at $\{q_1, q_2\} \subset \ell_\infty$. Namely, a broken configuration arising as the SFT limit of such planes would consist of at least one plane contained in the top level $\mathbb{C}P^2 \setminus L$, which is disjoint from ℓ_∞ and C by positivity of intersection. However, since the only geodesic on L which is contractible inside V has vanishing Maslov class by [Proposition 5.16](#), this contradicts the genericity of the almost complex structure chosen.

Since automatic transversality is satisfied by the main result of [[38](#)], the component of the moduli space of planes tangent to C at q_i that contains the plane A_i is diffeomorphic to S^1 for each $i = 1, 2$. In order to achieve automatic transversality with the tangency condition, we must argue as in the proof of [Lemma 4.4](#) (but for one tangency condition instead of two). More precisely, we need to infer that the infinitesimal variations of solutions that vanish to order two at the point q_i must vanish to order precisely two there, while vanishing nowhere else (including asymptotically at the puncture). This is a consequence of positivity of intersection together with the computation of the asymptotic intersection number made in the subsequent paragraph.

The argument from [[12](#), Lemma 5.13], based upon the computations of the asymptotic winding of the eigenvectors from [[21](#)], shows that the asymptotic evaluation map is

a diffeomorphism onto the space of orbits. The computation is the same in this case, despite the fact that the Fredholm index of the plane A_i is equal to five (instead of one). To that end, we use the fact that the index of this plane again is equal to one, when we consider it together with the tangency constraint at $q_i \in \ell_\infty$.

Now we recall how to exclude two planes having the same asymptotic orbit by the aforementioned calculation: Two different planes asymptotic to the same orbit have a new intersection point after a small holomorphic perturbation (the holomorphic perturbation exists in view of the aforementioned automatic transversality result). In this way, two planes with the same asymptotic and tangency at q_i must coincide; otherwise we could readily construct two cycles in degree two that intersect with algebraic intersection number at least five by assembling suitable pseudoholomorphic planes asymptotic to the torus which are both tangent to C at q_i for $i = 1, 2$.

Finally, we can use the smoothing procedure from [12, Section 5.3] in order to assemble the planes in these components of the moduli space to form a S^1 -family of closed embedded pseudoholomorphic conics, each intersecting L in a closed curve, and all being tangent to v_i at q_i . The claim that the Maslov index of this curve is zero when considered on L is an immediate consequence of Proposition 5.16.

The existence of the global symplectic conic foliation is finally a consequence of Theorem 4.3.

The claim about the winding number of σ around the projection of the fibre C is an immediate consequence of either of the conditions on L made in the assumptions of Theorem A.

(2) This part is a straightforward consequence of the normalisation carried out by Theorem 4.6.

- (3) One can construct an embedded curve $\gamma \subset \mathbb{C}$ which connects 0 to 1 and which
- coincides with $[0, 1] \subset \mathbb{C}$ near its boundary,
 - is isotopic to the latter standard embedding through embeddings of the same type, and
 - either is disjoint from σ (when the torus is in Clifford position) or intersects it transversely in a single point (when the torus is in Chekanov position).

An application of Theorem 4.7 finally normalises the conics above the path γ .

In order to make the fibration standard inside a whole neighbourhood of $f^{-1}([0, 1])$ we perform a reiteration of the entire neck-stretching argument, while using almost complex structures which are standard in the aforementioned neighbourhood. (Note that the neighbourhood has been made disjoint from L by the previous application of [Theorem 4.7.](#)) □

6 Liouville forms and inflation

Here we construct a family of Liouville forms on the complement of the holomorphic divisor $D_0 \cup D_1 \cup D_2 \cup D_3 \subset \mathbb{C}P^2$, where $D_0 = \ell_\infty$, $D_1 = \ell_1$, $D_2 = \ell_2$, and $D_3 = C$. The main use of these different Liouville forms is to deform Lagrangian isotopies to Hamiltonian isotopies, by applications of the corresponding Liouville flows. This process is called inflation, and is described in [Section 6.2.](#) As a part of the construction, we also obtain the natural one-parameter family λ_r of Liouville forms on $(V = \mathbb{C}P^2 \setminus (D_0 \cup D_3), \omega_{FS})$ that we study in [Section 6.1.](#)

The Fubini–Study Kähler form normalised so that $\int_{\ell_\infty} \omega_{FS} = \pi$ can be expressed as

$$\begin{aligned} \omega_{FS} &:= \frac{1}{4} dd^c \log \left(\frac{1}{1 + \|z_1\|^2 + \|z_2\|^2} \right) \\ &= -\frac{1}{2} i \partial \bar{\partial} \log \left(\frac{1}{1 + \|z_1\|^2 + \|z_2\|^2} \right) = \frac{1}{2} i \partial \bar{\partial} \log (1 + \|z_1\|^2 + \|z_2\|^2) \\ &= \frac{i}{2(1 + \|z_1\|^2 + \|z_2\|^2)} \sum_{i=1}^2 dz_i \wedge d\bar{z}_i + \frac{-i}{2(1 + \|z_1\|^2 + \|z_2\|^2)^2} \sum_{i,j} \bar{z}_i z_j dz_i \wedge d\bar{z}_j \end{aligned}$$

in the affine chart $\mathbb{C}^2 = \mathbb{C}P^2 \setminus \ell_\infty$. Here we have used the notation $d^c f(\cdot) := df(i \cdot)$.

Consider the antitautological line bundle $E := \text{Tot}(\mathcal{O}(1)) \rightarrow \mathbb{C}P^2$ with sheaf of holomorphic sections $\mathcal{O}(1)$. In other words, the line bundle $E^{\otimes 3} = \det(T\mathbb{C}P^2)$ is the anticanonical line bundle with sheaf of holomorphic sections $\mathcal{O}(3)$ on $\mathbb{C}P^2$. We endow E with the Hermitian metric determined by the following condition: for the standard trivialisation over the affine chart $\mathbb{C}^2 = \mathbb{C}P^2 \setminus \ell_\infty$ this metric takes the value

$$\|s\|_E^2 = \frac{\|s\|^2}{1 + \|z_1\|^2 + \|z_2\|^2}$$

on the locally defined section $s: \mathbb{C}^2 \rightarrow \mathbb{C}$. We also consider the induced metrics

$$\|s\|_{E^{\otimes n}}^2 = \frac{\|s\|^2}{(1 + \|z_1\|^2 + \|z_2\|^2)^n}$$

on the n^{th} tensor power $E^{\otimes n}$ of the line bundle. Take holomorphic sections s_{ℓ_∞} and s_C of E and $E^{\otimes 2}$ respectively, which in the trivialisation over $\mathbb{C}^2 = \mathbb{C}P^2 \setminus \ell_\infty$ are given by

$$s_{\ell_\infty}(z_1, z_2) = 1 \quad \text{and} \quad s_C(z_1, z_2) = z_1 z_2 - 1.$$

Similarly, there are sections s_{ℓ_i} of E which in the trivialisation over the affine coordinate chart $\mathbb{C}^2 = \mathbb{C}P^2 \setminus \ell_\infty$ take the form $s_{\ell_1} = z_2$ and $s_{\ell_2} = z_1$. Note that $(s_{\ell_i}) = \ell_i$.

Using the above sections and Hermitian metrics we are now ready to define a family of Liouville forms on the complement $\mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})$ of a particular singular divisor. Let $\mathbf{r} = (r_0, \dots, r_3) \in \mathbb{R}^4$ be numbers satisfying $r_0 + \dots + r_3 = 1$. Using the formula for the curvature we compute

$$\omega_{\text{FS}} = \frac{1}{4} d d^c (r_0 \log \|s_{\ell_\infty}\|_E^2 + r_1 \log \|s_{\ell_1}\|_E^2 + r_2 \log \|s_{\ell_2}\|_E^2 + \frac{1}{2} r_3 \log \|s_C\|_{E^{\otimes 2}}^2),$$

and hence

$$\lambda_{\mathbf{r}} := \frac{1}{4} d^c (r_0 \log \|s_{\ell_\infty}\|_E^2 + r_1 \log \|s_{\ell_1}\|_E^2 + r_2 \log \|s_{\ell_2}\|_E^2 + \frac{1}{2} r_3 \log \|s_C\|_{E^{\otimes 2}}^2)$$

is a family of Liouville forms defined in the complement of the divisor $\bigcup_{\{i:r_i \neq 0\}} D_i$.

6.1 A family of Liouville forms on $V = \mathbb{C}P^2 \setminus (\ell_\infty \cup C)$

In this subsection we endow (V, ω_{FS}) with a family of Liouville forms. Also compare to the construction in Section 3.2, where the Liouville form is more carefully adapted to the Lagrangian fibrations considered. Setting $r_1 = r_2 = 0$, $r_0 = 1 - r$, and $r_3 = r$ in the construction above, we obtain a family

$$\lambda_r := \frac{1}{4} (1 - r) d^c \log \|s_{\ell_\infty}\|_E^2 + \frac{1}{8} r d^c \log \|s_C\|_{E^{\otimes 2}}^2 \quad \text{for } r \in [0, 1]$$

of Liouville forms for ω_{FS} defined on all of V . Denote by

$$\phi_{\lambda_r}^t : (V, \omega_{\text{FS}}) \rightarrow (V, e^{-t} \omega_{\text{FS}})$$

the corresponding Liouville flow.

Remark 6.1 It makes sense to also consider the limit cases λ_r with $r = 0$ and $r = 1$.

(1) The Liouville form λ_0 is identified with the standard radial Liouville form, which is the pullback of

$$\lambda_{\text{std}} = \frac{1}{2} \|\tilde{z}_1\|^2 d\tilde{\theta}_1 + \frac{1}{2} \|\tilde{z}_2\|^2 d\tilde{\theta}_2 = \frac{1}{4} i (\tilde{z}_1 d\bar{\tilde{z}}_1 - \bar{\tilde{z}}_1 d\tilde{z}_1) + \frac{1}{4} i (\tilde{z}_2 d\bar{\tilde{z}}_2 - \bar{\tilde{z}}_2 d\tilde{z}_2)$$

on $(B^4, \omega_0 = d\lambda_{\text{std}})$ under the symplectomorphism φ . By Gromov’s theorem [19] there are no exact closed Lagrangian embeddings with respect to this Liouville form.

(2) The Liouville λ_1 is defined on $(\mathbb{C}P^2 \setminus C, \omega_{\text{FS}})$ and vanishes along the Lagrangian embedding

$$\overline{\{z_2 = -\bar{z}_1\}} \subset (\mathbb{C}P^2, \omega_{\text{FS}})$$

of $\mathbb{R}P^2$, which thus in particular is exact. Note that $\mathbb{C}P^2 \setminus C$ is symplectomorphic to the disc cotangent bundle of $\mathbb{R}P^2$.

Different Lagrangian fibres of Π_s become exact for different Liouville forms in the family λ_r . Here we give a description, using the conventions from Section 2.1 for the computation of the symplectic action.

Lemma 6.2 (1) *The Lagrangian fibre $\Pi_s^{-1}(0, u_2)$ is exact and, in the case of the immersed sphere $u_2 = 1$, even strongly exact, precisely for the Liouville form λ_r with parameter $r = (2/\pi)A_s(0, u_2)$ in the above family (note that $(2/\pi)A_s(0, u_2) \in (0, 1)$).*

(2) *The backward Liouville flow $\phi_{\lambda_r}^{-t}$ for $t \geq 0$ is complete on V for all $r \in (0, 1)$.*

(3) *For any $A \in (0, \pi/2)$, the backward Liouville flow $\phi_{\lambda_{(2/\pi)A}}^{-t}$ applied to the torus fibre $\Pi_s^{-1}(u_1, u_2)$ rescales both the quantities*

$$\int_{e_0} \lambda_{\text{std}} = \pi \cdot u_1 \quad \text{and} \quad \int_{e_1} \lambda_{\text{std}} - A = A_s(u_1, u_2) - A$$

by multiplication with e^{-t} , where the function A_s is as given in Lemma 2.7. (Here we have used the flow to identify the basis in homology.)

Proof (1) This follows from an explicit computation of the symplectic action of $e_1 \in H_1(L)$. The crucial identities are

$$\int_{e_1} \frac{1}{4} d^c \log \|s_{\ell_\infty}\|_E^2 = \int_{e_1} \lambda_{\text{std}} = A_s(u_1, u_2),$$

$$\int_{e_1} \frac{1}{8} d^c \log \|s_C\|_E^2 = A_s(u_1, u_2) - \frac{1}{2}\pi.$$

(2) The Liouville vector field is equal to the gradient

$$\nabla\left(-\frac{1}{4}(1-r) \log \|s_{\ell_\infty}\|_E^2 - \frac{1}{8}r \log \|s_C\|_E^2\right).$$

Since this function has no critical points outside of a compact subset of $\mathbb{C}P^2 \setminus (\ell_\infty \cup C)$, and since it moreover blows up along the divisor, the completeness is now immediate.

(3) The symplectic action of e_0 coincides with the symplectic area of a suitable disc living inside V ; its rescaling properties are thus immediate from the basic property that the Liouville flow rescales the symplectic form by e^t .

For the rescaling properties of the symplectic action of e_1 , one can, for example, use the fact that the fibre $\Pi_s^{-1}(0, u)$ with $A_s(0, u) = A$ is exact for the Liouville form $\lambda_{(3/2)A}$. Hence, the corresponding symplectic actions for this torus, as well as its images under the Liouville flow, vanish.

For an arbitrary torus L it is now a simple matter of finding a two-dimensional chain with boundary on $L \cup \Pi_s^{-1}(0, u)$, where the boundary on L moreover lives in the class e_1 (and whose symplectic area hence coincides with the symplectic action of e_1 on L). \square

In the proof of Proposition 7.2 we will need to use the fact that the backward Liouville flow corresponding to λ_r contracts V into any neighbourhood of the subset

$$\{\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2 = 0\} \cap V.$$

This is a consequence of the following.

Lemma 6.3 *The Liouville vector field corresponding to λ_r for any $r \in [0, 1]$ evaluates positively (resp. negatively) on the exterior derivative*

$$d(\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2) = d\left(\frac{\|z_1\|^2 - \|z_2\|^2}{1 + \|z_1\|^2 + \|z_2\|^2}\right)$$

inside the subset $\{\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2 > 0\}$ (resp. $\{\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2 < 0\}$) of $\mathbb{C}P^2 \setminus (\ell_\infty \cup C)$. It follows that the same is true also for the Liouville flows corresponding to λ_r with $r = (r - \eta, \eta, \eta, 1 - r - \eta) \in \mathbb{R}_{>0}^4$.

Proof We prove the fact for the parameters $r = 0$ and $r = 1$; since the Liouville forms for the remaining parameters are convex interpolations of λ_0 and λ_1 , the statement then follows in general.

For the Liouville form λ_0 the statement follows by Remark 6.1(1).

For the Liouville form λ_1 we argue as follows. By the proof of Lemma 6.2 the corresponding Liouville vector field is given as the gradient

$$-\nabla \frac{1}{8} \log \|s_C\|_E^2$$

for the Fubini–Study metric. It thus suffices to show that the gradient

$$V := \nabla \frac{\|z_1\|^2 - \|z_2\|^2}{1 + \|z_1\|^2 + \|z_2\|^2} = 2(z_1 \partial_{z_1} + \bar{z}_1 \partial_{\bar{z}_1} - z_2 \partial_{z_2} - \bar{z}_2 \partial_{\bar{z}_2})$$

evaluates positively (resp. negatively) on

$$d(-\|s_C\|_{E^{\otimes 2}}^2) = -d \frac{\|z_1 z_2 - 1\|^2}{(1 + \|z_1\|^2 + \|z_2\|^2)^2}$$

in the subset $\{\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2 > 0\}$ (resp. $\{\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2 < 0\}$), since

$$\begin{aligned} d(-\|s_C\|_{E^{\otimes 2}}^2)(V) &= \frac{4\|z_1 z_2 - 1\|^2(\|z_1\|^2 - \|z_2\|^2)}{(1 + \|z_1\|^2 + \|z_2\|^2)^3} \\ &= \frac{4\|z_1 z_2 - 1\|^2}{1 + \|z_1\|^2 + \|z_2\|^2} (\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2). \quad \square \end{aligned}$$

6.2 Inflation via the Liouville flow

Recall that a Lagrangian isotopy is generated by a Hamiltonian if and only if the corresponding symplectic flux-path vanishes identically. Since we are considering the case of an exact symplectic manifold $(X, \omega = d\lambda)$, the latter condition can be formulated as having a Lagrangian isotopy under which the pullback of λ to L is a path of closed one-forms on L that are *constant in cohomology*. In cases when we can find suitable symplectic divisors in the complement of the Lagrangian isotopy, the technique of “inflation” along the divisors can in favourable situations be used to deform a Lagrangian isotopy into one with vanishing flux-path.

Remark 6.4 In the proof of the nearby Lagrangian conjecture for the torus [12, Theorem 7.1] the Lagrangian isotopy was turned into a Hamiltonian isotopy by applying the fibrewise addition of suitable Lagrangian sections in $T^*\mathbb{T}^2$ (ie deformation by nonexact symplectomorphisms). That procedure can be reinterpreted as an inflation as well; first one compactifies $DT^*\mathbb{T}^2$ to $S^2 \times S^2$, and then one inflates along appropriate lines inside the compactifying divisor.

In our case the inflation will be performed along the divisors $D_0 = \ell_\infty$, $D_1 = \ell_1$, $D_2 = \ell_2$, and $D_3 = C$ living inside $\mathbb{C}P^2$, with different parameters $r_0, r_1, r_2, r_3 \geq 0$, one parameter for each of the divisors. In practice we found it most efficient to perform the inflation by constructing a family of Liouville forms on $\mathbb{C}P^2 \setminus (D_0 \cup \dots \cup D_3)$ parametrised by $\mathbf{r} = (r_0, r_1, r_2, r_3) \in \mathbb{R}^4$. Again we emphasise that it also would be possible to use a more hands-on approach as in eg [12, Section 6].

We start by investigating where these different Liouville flows can be integrated:

Lemma 6.5 Assume that $r_i \geq 0$ holds for all $i = 0, 1, 2, 3$. Then the backward Liouville flow of $\lambda_{\mathbf{r}}$ exists for all times when restricted to $\mathbb{C}P^2 \setminus \bigcup_{\{i:r_i \neq 0\}} D_i$.

The Liouville vector field that corresponds to $\lambda_{\mathbf{r}}$ equals the negative gradient $-\nabla f_{\mathbf{r}}/4$ (taken with respect to the Fubini–Study metric) of the function

$$f_{\mathbf{r}} := r_0 \log \|s_{\ell_\infty}\|_E^2 + r_1 \log \|s_{\ell_1}\|_E^2 + r_2 \log \|s_{\ell_2}\|_{E \otimes 3}^2 + \frac{1}{2} r_3 \log \|s_C\|_{E \otimes 3}^2.$$

Hence Lemma 6.5 is a direct consequence of the following behaviour of $f_{\mathbf{r}}$ that we proceed to establish.

Lemma 6.6 Whenever all $r_i \geq 0$ are nonnegative, $f_{\mathbf{r}} \leq C_{\mathbf{r}}$ is bounded uniformly from above by a constant depending continuously on \mathbf{r} . If, moreover, $r_i > 0$ holds for some $i \in \{0, 1, 2, 3\}$, then for any $c < 0$, there exists a neighbourhood $U_i \supset D_i$ inside B^4 on which $f_{\mathbf{r}}|_{U_i \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})} \leq c$.

Proof The uniform bound is straightforward to show for any of the four terms in the above expression of $f_{\mathbf{r}}$, using the property that all sections s_{ℓ_∞} , s_{ℓ_i} , and s_C are holomorphic.

Concerning the behaviour near the divisor D_i in the case when $r_i > 0$, the claim is a standard consequence of the fact that the holomorphic section corresponding to the i^{th} term vanishes along that divisor, together with the above uniform bound. □

We will now see that the effect of the backward Liouville flow $\phi_{\lambda_{\mathbf{r}}}^{-t}$ corresponding to $\lambda_{\mathbf{r}}$ on a Lagrangian $L \subset \mathbb{C}P^2 \setminus (C \cup \ell_\infty \cup C_{\text{nodal}})$ in the complement of this singular divisor corresponds to the Lagrangian isotopy induced by performing an appropriate “inflation” along the same divisors. In other words, we need to investigate which effect the Liouville flow has on the symplectic action of the Lagrangian.

It is clear that the change of symplectic action under the flow only depends on the homotopy class of the Lagrangian. Here we consider two cases, characterised in terms of the existence of a basis satisfying certain properties. For a Lagrangian torus $L \subset \mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})$, any basis

$$H_1(L) = \langle f_1, f_2 \rangle$$

is induced by a basis

$$F_i \in H_2(B^4, L) \quad \text{with} \quad [\partial F_i] = f_i \quad \text{for } i = 1, 2$$

of the corresponding relative homology group. In the following we are interested in two types of bases that are characterised as follows:

- **A basis of Clifford type** $F_i \bullet C = 1$ for $i = 1, 2$, while $F_i \bullet \ell_j = \delta_i^j$.
- **A basis of Chekanov type** $F_0 \bullet \ell_i = (-1)^i$ for $i = 1, 2$, while $F_0 \bullet C = 0$, and $F_1 \bullet C = 1$, while $F_1 \bullet \ell_i = 0$ for $i = 1, 2$.

The fibres of Π_S of Clifford and Chekanov types also admit bases of the respective types; in the case of the Clifford torus both basis elements have Maslov index two, while in the Chekanov case F_0 and F_1 have Maslov index zero and two, respectively.

Remark 6.7 (1) The property of admitting a basis as above only depends on the homotopy class of the torus inside $\mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})$. Furthermore, the two cases are mutually exclusive for a given torus.

(2) The basis of Chekanov type coincides with the basis considered in Lemmas 2.3 and 2.4 for the same tori, ie $f_i = e_i$.

The Liouville flow has the following effect on the symplectic action of a Lagrangian torus in the two aforementioned cases:

Proposition 6.8 *Take some*

$$r \in \{(r_0, \dots, r_3) : r_i \geq 0, r_0 + \dots + r_3 = 1\}$$

and consider the symplectic action

$$A_i(t) := \int_{f_i} \lambda_{\text{std}} \quad \text{for } i = 1, 2$$

of the image L_t of a Lagrangian torus $L \subset \mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})$ under the time- $(-t)$ Liouville flow of λ_r , where L_t moreover is assumed to be contained inside the same subset for $t \in [0, t_0]$. Then:

- If $\langle f_1, f_2 \rangle$ is a basis of Clifford type, then

$$A_1(t) = \pi(r_1 + \frac{1}{2}r_3) + e^{-t}(A_1(0) - \pi(r_1 + \frac{1}{2}r_3)),$$

$$A_2(t) = \pi(r_2 + \frac{1}{2}r_3) + e^{-t}(A_2(0) - \pi(r_2 + \frac{1}{2}r_3)),$$

for all $t \in [0, t_0]$.

- If $\langle f_1, f_2 \rangle$ is a basis of Chekanov type, then

$$A_1(t) = \pi(r_1 - r_2) + e^{-t}(A_1(0) - \pi(r_1 - r_2)),$$

$$A_2(t) = \frac{1}{2}\pi r_3 + e^{-t}(A_2(0) - \frac{1}{2}\pi r_3),$$

for all $t \in [0, t_0]$.

Here we have used the Lagrangian isotopy to make the identification $H_1(L_t) \cong H_1(L)$.

Remark 6.9 In view of Lemma 6.6, assuming that $r_0, \dots, r_3 > 0$ is sufficient to guarantee that $L_t \subset \mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})$ for the entire backward Liouville flow.

Proof The statement is a consequence of the following more general properties. Consider a surface $\Sigma \subset \mathbb{C}P^2$ which intersects D_i transversely with algebraic intersection number equal to $n_i \in \mathbb{Z}$, while having boundary contained in the complement of $D_0 \cup \dots \cup D_3$. We assume that all intersections $\Sigma \pitchfork D_i$ are transverse. The Liouville flow ϕ_r^{-t} with $r_i > 0$ acts on $\Sigma \setminus (D_0 \cup \dots \cup D_3)$ by an isotopy, and this punctured surface can be suitably completed near $D_0 \cup \dots \cup D_3$ to extend this isotopy to a smooth isotopy $\Sigma_t \subset \mathbb{C}P^2$ of surfaces, where $\Sigma_0 = \Sigma$, and $\Sigma_t \cdot D_i = n_i$ for all $t \geq 0$. The general claim that shows the statement is that

$$\int_{\Sigma_t} \omega_{\text{FS}} = e^{-t} \int_{\Sigma} \omega_{\text{FS}} + (1 - e^{-t})(n_0 r_0 + n_1 r_1 + n_2 r_2 + \frac{1}{2} n_3 r_3)$$

whenever $r_i > 0$. This holds, since the area of any such surface can be computed to be

$$\int_{\Sigma_t} \omega_{\text{FS}} = \int_{\partial \Sigma_t} \lambda_r + n_0 r_0 + n_1 r_1 + n_2 r_2 + \frac{1}{2} n_3 r_3,$$

while $(\phi_r^{-t})^* \lambda_r = e^{-t} \lambda_r$ is satisfied. (The contributions for each transverse intersection with a divisor, ie the latter terms, are preferably computed for a model problem consisting of a small complex disc contained in a fibre normal to the divisor.) \square

Corollary 6.10 For $t \in [0, 1]^k$ smoothly parametrising a family

$$\tilde{L}_t := \Pi_{s(t)}^{-1}(u_1(t), u_2(t)) \subset \mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}}) \quad \text{with } u_2(t) \neq 1$$

of Lagrangian standard torus fibres, there exists smooth functions

$$r(t) \in \{(r_0(t), \dots, r_3(t)) : r_i(t) > 0, r_0(t) + \dots + r_3(t) = 1\}$$

and $\alpha(t) \in \mathbb{R}_{\geq 0}$ for which the path

$$L_t := \phi_{r(t)}^{-\alpha(t)}(\Pi_{s(0)}^{-1}(u_1(0), u_2(0))) \subset \mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})$$

of Lagrangian tori realises the same symplectic flux-path as \tilde{L}_t . We may moreover assume that $\alpha(t) = 0$ holds whenever $(u_1(t), u_2(t)) = (u_1(0), u_2(0))$.

Proof We prove the statement in the case of tori of Chekanov type, ie in the case $u_2(t) \in (0, 1)$. The case of tori of Clifford type is similar. Without loss of generality we can consider the case when $s(t) \equiv \pi/2 - \epsilon$ for $\epsilon > 0$ sufficiently small. Indeed, we may realise the path \tilde{L}_t of tori by tori in the same homotopy class, and of the same action, while the path $s(t)$ is constant.

Denote by $(A_1(\mathbf{t}), A_2(\mathbf{t}))$ the symplectic actions of the tori $\tilde{L}_{\mathbf{t}}$ with respect to the basis of Chekanov type, and write

$$(a_1(\mathbf{t}), a_2(\mathbf{t})) := (A_1(\mathbf{t}), A_2(\mathbf{t})) - (A_1(0), A_2(0)).$$

For a small number $\delta > 0$ we choose a smooth bump function $\beta: \mathbb{R}_{\geq 0} \rightarrow [-\delta^{-1}, \delta]$ which satisfies $\beta(t) = t$ near $t = 0$, while $\beta(t) = -\delta^{-1}$ for all $t \geq \delta$. Then we construct the smooth and nonnegative function

$$\alpha(\mathbf{t}) := -\log \left(\frac{e^{\beta\|(a_1(\mathbf{t}), a_2(\mathbf{t}))\|^2}}{1 + e^{\beta\|(a_1(\mathbf{t}), a_2(\mathbf{t}))\|^2}} \right).$$

In view of Proposition 6.8 it then suffices to take a suitable path $\mathbf{r}(\mathbf{t})$ that satisfies

$$\begin{aligned} & (\pi(r_1(\mathbf{t}) - r_2(\mathbf{t})), \frac{1}{2}\pi r_3(\mathbf{t})) \\ &= (A_1(0) + (1 + e^{\beta\|(a_1(\mathbf{t}), a_2(\mathbf{t}))\|^2})a_1(\mathbf{t}), A_2(0) + (1 + e^{\beta\|(a_1(\mathbf{t}), a_2(\mathbf{t}))\|^2})a_2(\mathbf{t})). \end{aligned}$$

That this is possible when $\delta > 0$ in the construction of the above bump function is chosen to be sufficiently small can be readily seen using Lemma 2.7. Namely, by this lemma the values of $(A_1(\mathbf{t}), A_2(\mathbf{t}))$ and hence also

$$(A_1(0), A_2(0)) + (1 + e^{\beta(\|(a_1(\mathbf{t}), a_2(\mathbf{t}))\|^2)})(a_1(\mathbf{t}), a_2(\mathbf{t}))$$

are restricted to the interior of the polytope in Figure 5; also see Remark 6.7(2). □

6.3 Flux-paths that determine the Hamiltonian isotopy class

A Lagrangian isotopy with a nonzero symplectic flux-path cannot be generated by a Hamiltonian isotopy. Furthermore, it is known that two Lagrangian isotopies starting at the same Lagrangian torus (ie the Clifford torus) and realising the *same* symplectic flux-paths still need not end at Hamiltonian isotopic tori (we may, for example, end at either the Clifford or the Chekanov torus). In contrast to this, we here show that when the symplectic flux-path can be generated by a complete negative Liouville flow, then the Hamiltonian isotopy class is indeed determined by the flux-path.

Lemma 6.11 *Consider a symplectic manifold $(X^{2n}, \omega = d\lambda)$, with a **complete** negative Liouville flow ϕ^{-t} induced by λ , and a smooth Lagrangian submanifold $L \subset X$. For any Lagrangian isotopy L_t with $L_0 = L$ that satisfies the property that the symplectic actions of $\phi^{-t}(L)$ and L_t agree for all $t \in [0, T]$, it is the case that $\phi^{-T}(L)$ and L_T are Hamiltonian isotopic inside X^{2n} .*

Proof By Weinstein’s Lagrangian neighbourhood theorem one readily proves the following: the set of numbers $S \in [0, T]$ satisfying the property that $\phi^{-t}(L)$ and L_t are Hamiltonian isotopic for all $t \in [0, S]$ forms an interval which is open inside $[0, T]$ and contains zero. It suffices to show that this interval also is closed, since it then is equal to $[0, T]$.

Thus, we take $S_0 > 0$ such that the property is true for all $S < S_0$. By the same reason why the previously established openness property holds, there exists some $\epsilon > 0$ such that L_{S_0-t} is Hamiltonian isotopic to $\phi^t(L_{S_0})$ whenever $0 \leq t \leq \epsilon$.

By our assumptions, $\phi^{-(S_0-\epsilon)}(L)$ is Hamiltonian isotopic to $L_{S_0-\epsilon}$, and by the previous paragraph, the Lagrangian $L_{S_0-\epsilon}$ is Hamiltonian isotopic to $\phi^\epsilon(L_{S_0})$. Since Hamiltonian isotopies are preserved by the conformal symplectomorphism $\phi^{-\epsilon}$, it finally follows that, as sought, $\phi^{-S_0}(L) = \phi^{-\epsilon} \circ \phi^{-(S_0-\epsilon)}(L)$ is Hamiltonian isotopic to $\phi^{-\epsilon} \circ \phi^\epsilon(L_{S_0}) = L_{S_0}$. (The assumption of completeness is needed to guarantee that $\phi^{-\epsilon}$ is well defined on the whole Hamiltonian isotopy from $\phi^{-(S_0-\epsilon)}(L)$ to $\phi^\epsilon(L_{S_0})$.) \square

7 The Hamiltonian isotopy to a standard torus

In this section we prove [Theorem A](#) in the embedded case. In other words, given a Lagrangian torus $L \subset (V, \omega_{\text{FS}})$ satisfying the assumptions made in that theorem, we produce a Hamiltonian isotopy to a standard torus. The first step of the proof consists of applying [Theorem 5.12](#). From now on we hence assume that L has been placed in either Clifford or Chekanov position by the Hamiltonian isotopy produced by the aforementioned result and thus, in particular, that we have $L \subset \mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})$. The proof of [Theorem A](#) for L in Clifford position is given in [Section 7.1](#), and in [Section 7.2](#) when L is in Chekanov position.

7.1 Classification up to Hamiltonian isotopy for tori in Clifford position

The argument presented in [Sections 7.2](#) and [7.3](#), which concerns the case of a torus in Chekanov position, applies also to the present case of tori in Clifford position after minor modifications. However, we here choose a different path with a somewhat simpler argument. This argument also turns out to be useful later when we classify the immersed Lagrangian spheres.

After the application of [Theorem 5.12](#) the torus can be assumed to be contained inside $V \setminus f^{-1}[0, 1]$. Since $f^{-1}[0, 1]$ deformation retracts onto $f^{-1}(0)$, it follows from [Corollary 5.14](#) that the torus is homologically essential in the same subset. The existence

of the Hamiltonian isotopy is now a direct consequence of **Theorem B** combined with the following lemma:

Lemma 7.1 *There exists a symplectomorphism*

$$\Phi: (V \setminus f^{-1}[0, 1], \omega_{FS}) \xrightarrow{\cong} \mathbb{T}^2 \times U \subset (T^*\mathbb{T}^2, d\lambda_{\mathbb{T}^2})$$

where

$$U := \{(p_1, p_2) : |p_1| < \pi, \max\{0, p_1\} < p_2 < \frac{1}{2}(p_1 + \pi)\} \subset \mathbb{R}^2$$

is the bounded convex subset shown in **Figure 9**. This symplectomorphism can moreover be taken to satisfy:

- (1) For any fixed $(u, v) \in (-1, 1) \times (1, +\infty]$, each torus fibre $\Pi_s^{-1}(u, v)$ is mapped to the standard torus $\Phi(\Pi_s^{-1}(u, v)) = \mathbb{T}^2 \times \{\mathbf{p}_{(u,v)}\}$ whenever $s > 0$ is sufficiently small.
- (2) The punctured Lagrangian disc

$$\{(z, -\bar{z}) \in \mathbb{C}^2\} \cap f^{-1}(-\infty, 0) \subset V \setminus f^{-1}[0, 1]$$

(ie the standard Lefschetz thimble with a point removed) is mapped into the Lagrangian annulus

$$(S^1 \times \{0\}) \times (\{0\} \times \mathbb{R}) \subset T^*\mathbb{T}^2.$$

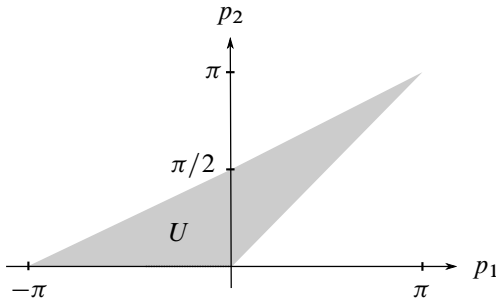


Figure 9: The image $U \subset \mathbb{R}^2$ of the values (p_1, p_2) of the symplectic action class evaluated on the pair (e_0, \tilde{e}_1) of basis vectors of $H_1(\Pi_s^{-1}(u_1, u_2))$ with $u_2 > 1$.

Proof The symplectomorphism is constructed by considering the Lagrangian torus fibration Π_0 on $V \setminus f^{-1}[0, 1]$ from **Section 1.3** obtained as the “limit” of the fibrations Π_s as $s \rightarrow 0$. Using the classical Arnold–Liouville theorem [34, Theorem 2.3] we

can find locally defined fibre-preserving symplectomorphisms from this torus fibration into the standard fibration $\mathbb{T}^2 \times \{\mathbf{p}\}$. Since the space of fibres is simply connected (even contractible), the maps can be patched together to give a well-defined global symplectomorphism. This symplectic map is an embedding, as follows from the action considerations in Lemma 2.7 (the Lagrangian tori in the fibration are determined uniquely by their symplectic actions).

To see that the image is $\mathbb{T}^2 \times U \subset T^*\mathbb{T}^2$ as claimed, we choose action–angle coordinates (p_1, p_2) that correspond to the following global continuous choice of basis of $H_1(L)$ for the tori of Clifford type in the fibration. First make the global choice of basis vector \mathbf{e}_0 as specified in Lemma 2.5. Then choose the second basis vector $\tilde{\mathbf{e}}_1$ by following the recipe in the case $u_1 < 0$ in Lemma 2.5(2)(c). The shape of the image is now a consequence of the action computations made in Lemma 2.7; also see Figures 5 and 9. (Observe that, while the obtained basis $\langle \mathbf{e}_0, \tilde{\mathbf{e}}_1 \rangle$ of course satisfies $\tilde{\mathbf{e}}_1 = \mathbf{e}_1$ in the region $u_1 < 0$, it satisfies $\tilde{\mathbf{e}}_1 = \mathbf{e}_1 + \mathbf{e}_0$ in the region $u_1 \geq 0$.)

- (1) This property follows from the construction of the fibration Π_0 in Section 1.3.
- (2) The Lagrangian punctured disc inside V under consideration can alternatively be described as

$$\Pi_s^{-1}(\{0\} \times (1, +\infty)) \cap f^{-1}(-\infty, 0)$$

for any $s \geq 0$. The action properties of the tori in the family $\Pi_s^{-1}(\{0\} \times (1, +\infty))$ imply that, after an appropriate choice of coordinates on $T^*\mathbb{T}^2$, the image of the punctured disc lives inside the subset

$$(S^1 \times S^1) \times (\{0\} \times \mathbb{R}) \subset T^*\mathbb{T}^2,$$

while it intersects the Lagrangian tori $S^1 \times S^1 \times \{(0, t)\}$ cleanly in embedded closed curves. Hence, using the Lagrangian condition for the punctured disc, its image can be seen to be of the form $(S^1 \times \{\theta_0\}) \times (\{0\} \times \mathbb{R})$ for some fixed $\theta_0 \in S^1$. □

7.2 Classification up to Lagrangian isotopy for tori in Chekanov position

In the case of a torus in Chekanov position we must start by constructing a mere Lagrangian isotopy inside the subset $\mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})$ that deforms it to a standard torus.

Proposition 7.2 *Any Lagrangian torus $L \subset (\mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}}), \omega_{\text{FS}})$ in Chekanov position is Lagrangian isotopic inside the same subset to a standard torus.*

Proof Consider a sufficiently small neighbourhood

$$U \subset V = \mathbb{C}P^2 \setminus C$$

of the subset $\{\|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2 = 0\} \cap V$. Using standard techniques, such as the symplectic neighbourhood theorem, one readily constructs a symplectic embedding $\psi: (U, \omega_{FS}) \hookrightarrow (W, d\lambda)$ into the neighbourhood of the self-plumbing of the sphere as constructed in Section 3. Recall that there exists a symplectic involution I of $(W, d\lambda)$ which interchanges tori of Chekanov and Clifford types; see Proposition 1.5. We may furthermore require that

- the nodal conic C_{nodal} is identified with the symplectic nodal surface

$$\{p_1 = \pm q_2, p_2 = \mp q_1\},$$

which is fixed setwise by I (while its two sheets are interchanged), and

- the image $\psi(U) \subset W$ is invariant under the symplectic involution I .

Using Lemma 6.3 we can see that the backward Liouville flow of $\lambda_{(1/2-\eta, \eta, \eta, 1/2-\eta)}$ for some small $\eta \in (0, 1)$ is a Lagrangian isotopy of L confined to $\mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})$ that produces a Lagrangian isotopy from L to a torus

$$\phi_{\lambda_{(1/2-\eta, \eta, \eta, 1/2-\eta)}}^{-T}(L) \subset U \setminus C_{\text{nodal}}.$$

An application of Theorem 5.12 to the torus

$$L' := \psi^{-1} \circ I \circ \psi \circ \phi^{-T}(L) \subset V \setminus C_{\text{nodal}}$$

can readily be seen to make it compatible with a conic fibration while being in Clifford position; see Proposition 1.5 together with the homotopical consideration in Remark 6.7(1). In Section 7.1 we constructed a Hamiltonian isotopy from L' to a standard torus of Clifford type. There is a Lagrangian isotopy that is contained inside $V \setminus C_{\text{nodal}}$ from that torus to a standard torus of Clifford type that moreover is contained inside $U \setminus C_{\text{nodal}}$. Denote by L'_t this entire Lagrangian isotopy.

The sought Lagrangian isotopy is then obtained by an application of the Liouville flow of $\lambda_{(1/2-\eta, \eta, \eta, 1/2-\eta)}$ to the Lagrangian isotopy L'_t , with the purpose of making it confined to the subset $U \setminus C_{\text{nodal}}$, followed by an application of the involution $\psi^{-1} \circ I \circ \psi$, which is defined in the same subset. Indeed, note that this involution interchanges standard tori of Clifford and Chekanov types (at least up to Hamiltonian isotopy) by Proposition 1.5. □

7.3 From Lagrangian to Hamiltonian isotopy for tori in Chekanov position

Proposition 7.2 gives a Lagrangian isotopy $L_t \subset (\mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}}), \omega_{\text{FS}})$ from $L_0 = L$ in Chekanov position to a standard torus. In order to correct the symplectic flux, it is sufficient to find a standard torus of Chekanov type of the same action as L , as the following lemma shows.

Lemma 7.3 *Let $L_t \subset (\mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}}), \omega_{\text{FS}})$ be a Lagrangian isotopy of tori in Chekanov position from $L_0 = L$ to $L_1 = \Pi_s^{-1}(u_1, u_2)$, where $s \in (0, \pi/2)$ and $(u_1, u_2) \in (-1, 1) \times (0, 1)$. Assume that there exists a Lagrangian isotopy*

$$L_t^{\text{std}} := \Pi_{s(t)}^{-1}(u_1(t), u_2(t)) \subset \mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}}) \quad \text{with } u_2(t) \in (0, 1)$$

of standard tori satisfying $s(1) = s$ and $(u_1(1), u_2(1)) = (u_1, u_2)$ such that the symplectic actions of L_t^{std} and L_t coincide for all t (when using the identifications induced by the isotopies). Then L is Hamiltonian isotopic inside $\mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}})$ to the standard torus $\Pi_{s(0)}^{-1}(u_1(0), u_2(0))$.

Proof By our assumptions, after concatenating the Lagrangian isotopy L_t with the Lagrangian isotopy L_{1-t}^{std} , we may produce a Lagrangian isotopy L_t of the same type but that satisfies $L_0 = L$ and $L_1 = \Pi_{s(0)}^{-1}(u_1(0), u_2(0))$. In particular, the symplectic actions of $L_0 = L$ and the standard torus L_1 now coincide.

In the remainder of this proof we construct a deformation L_t of the path relative endpoints $t = 0, 1$ to a Lagrangian isotopy with vanishing symplectic flux-path by applying suitable negative Liouville flows; the main ingredient is [Corollary 6.10](#).

Let $L_t^{\text{std}} := \Pi_{s(t)}^{-1}(u_1(t), u_2(t))$ be the path of standard tori being of the same action as (the new version of) the tori L_t . Note that, since (the new version of) the Lagrangian isotopy L_t starts and ends at a Lagrangian of the same symplectic action, L_t^{std} is actually a *loop* of standard tori, ie $L_0^{\text{std}} = L_1^{\text{std}}$. Since $u_2(t) \in (0, 1)$, this loop is moreover contractible; ie there exists a homotopy $L_{t,\tau}^{\text{std}} := \Pi_{s(t,\tau)}^{-1}(u_1(t, \tau), u_2(t, \tau))$ of loops of standard tori satisfying

$$\begin{aligned} \Pi_{s(t,0)}^{-1}(u_1(t, 0), u_2(t, 0)) &= \Pi_{s(t)}^{-1}(u_1(t), u_2(t)), \\ \Pi_{s(t,1)}^{-1}(u_1(t, 1), u_2(t, 1)) &\equiv \Pi_{s(0)}^{-1}(u_1(0), u_2(0)), \\ \Pi_{s(i,\tau)}^{-1}(u_1(i, \tau), u_2(i, \tau)) &\equiv \Pi_{s(0)}^{-1}(u_1(0), u_2(0)) \quad \text{for } i = 0, 1, \end{aligned}$$

where $u_2(t, \tau) \in (0, 1)$ moreover is satisfied. Using [Corollary 6.10](#) we then find smooth paths

$$r(t, \tau) \in \{\mathbb{R}^4 : r_i > 0, r_0 + \dots + r_3 = 1\},$$

$$\alpha(t, \tau) \geq 0 \quad \text{with } \alpha(0, \tau) \equiv \alpha(1, \tau) \equiv 0,$$

for which $\phi_{r(t, \tau)}^{-\alpha(t, \tau)}(\Pi_{s(t)}^{-1}(u_1(t), u_2(t)))$, and hence $\phi_{r(t, \tau)}^{-\alpha(t, \tau)}(L_t)$, realises the same symplectic flux-path as $L_{t, \tau}^{\text{std}}$. In particular, $\phi_{r(t, 1)}^{-\alpha(t, 1)}(L_t)$ is the sought Hamiltonian isotopy from the torus $\phi_{r(0, 1)}^{-\alpha(0, 1)}(L_0) = L_0 = L$ to the standard torus

$$\phi_{r(1, 1)}^{-\alpha(1, 1)}(\Pi_{s(0)}^{-1}(u_1(0), u_2(0))) = \Pi_{s(0)}^{-1}(u_1(0), u_2(0)). \quad \square$$

The nondisplaceability result in [Proposition 5.15](#) can now be used to show that the assumptions of the previous lemma indeed are satisfied.

Lemma 7.4 *Let $L_t \subset (\mathbb{C}P^2 \setminus (\ell_\infty \cup C \cup C_{\text{nodal}}))$, ω_{FS} be a Lagrangian isotopy of tori in Chekanov position from $L_t = L$ to $L_1 = \Pi_s^{-1}(u_1, u_2)$, where $s \in (0, \pi/2)$ and $(u_1, u_2) \in (-1, 1) \times (0, 1)$. Then the assumptions of [Lemma 7.3](#) can be assumed to hold. In particular L is Hamiltonian isotopic to a standard torus.*

Proof It suffices to show that there exists a standard torus of Chekanov type that is of the same symplectic action as L_t for all $t \in [0, 1]$. This is the case for the image of L_t under the negative Liouville flow $\phi_{1/2-\eta, \eta, \eta, 1/2-\eta}^{-T}$ for some small $\eta > 0$ and sufficiently large $T \gg 0$, as can be seen by referring to the symplectic action computations in [Proposition 6.8](#).

Since $\phi_{1/2-\eta, \eta, \eta, 1/2-\eta}^{-T}(\Pi_s^{-1}(u_1, u_2))$ is Hamiltonian isotopic to a standard torus by [Lemma 6.11](#), we can in fact conclude the following by appealing to [Lemma 7.3](#): the image $\phi_{1/2-\eta, \eta, \eta, 1/2-\eta}^{-T}(L)$ of our torus is Hamiltonian isotopic to a standard torus $\Pi_{s'}^{-1}(u'_1, u'_2)$ of Chekanov type.

From this we will now deduce that there exists a standard torus of the same symplectic action as L_t . Consider the symplectic embedding

$$\iota_{s'}: (V, \omega_{\text{FS}}) \hookrightarrow (\widehat{W}, d\lambda)$$

into its Liouville completion as provided by [Proposition 3.8](#), which takes torus fibres to torus fibres. Since $\iota_{s'} \circ \phi_{1/2-\eta, \eta, \eta, 1/2-\eta}^{-T}(L)$ is Hamiltonian isotopic to a standard torus fibre $\widehat{\pi}: \widehat{W} \rightarrow \mathbb{R}^2$, we can use [Lemma 6.11](#) to conclude that

$$\widetilde{L} := \phi_\lambda^T \circ \iota_{s'} \circ \phi_{1/2-\eta, \eta, \eta, 1/2-\eta}^{-T}(L)$$

is Hamiltonian isotopic to a fibre $\widehat{\pi}^{-1}(a, b)$ also. Since $\phi_\lambda^t \circ \iota_{s'} \circ \phi_{1/2-\eta, \eta, \eta, 1/2-\eta}^{-t}(L)$

is a Hamiltonian isotopy from $\iota_{s'}(L)$ to \tilde{L} , we conclude that the latter fibre $\hat{\pi}^{-1}(a, b)$ must be contained in the image of $\iota_{s'}$, as sought. If not, then we have produced a Hamiltonian displacement of $\iota_{s'}(L)$, which is in contradiction with Proposition 5.15. \square

8 The Hamiltonian isotopy to a standard immersed sphere

In this section we let $\tilde{L} \subset (B^4, \omega_0)$ be an immersed Lagrangian sphere that satisfies one of the conditions in the assumption of Theorem A. The goal is to prove the theorem by finding a Hamiltonian isotopy to a standard sphere. We begin with a preliminary lemma which holds for arbitrary such spheres.

Lemma 8.1 *For any immersed Lagrangian sphere $\tilde{L} \subset (B^4, \omega_0 = d\lambda_{\text{std}})$ with a single transverse double point, a primitive $f: \tilde{L} \rightarrow \mathbb{R}$ of $\lambda_{\text{std}}|_{T\tilde{L}}$ satisfies the property that the difference of its values at the two preimages $\{p, q\}$ of the double point is bounded by $|f(p) - f(q)| \leq \pi/2$.*

Proof This is a consequence of [9, Corollary 1.3], by which the minimal positive symplectic action of a Lagrangian torus satisfies the same upper bound $\pi/2$. Indeed, performing a “zero-area” Lagrangian surgery on \tilde{L} (this can always be done in an arbitrarily small neighbourhood of the double point) we obtain a monotone Lagrangian torus that is contained inside B^4 and whose minimal value of the modulus of the nonzero symplectic action is equal to precisely $|f(p) - f(q)|$. (For such a surgery, the symplectic area of a disc with a corner is *the same* as the corresponding area of the disc with boundary on the resolved Lagrangian.) \square

In view of the above lemma, combined with Proposition 1.4(1), when taking $s = |f(p) - f(q)|$, we conclude that \tilde{L} and $L_{\text{Wh}}(s)$ both are strongly exact for the Liouville form $\lambda_{3s/\pi}$ on (V, ω_{FS}) ; see Lemma 6.2.

Consider the two nonmonotone tori $T_{\pm}^0 := \Pi_s^{-1}(\pm\epsilon, 1)$ fibred over the closed curve $f(T_{\pm}^0) = \Psi_s^{-1}(1 + e^{i\theta}) \subset \mathbb{C}$ in the standard conic Lefschetz fibration. The subset of the fibres bounded by these two Lagrangian tori is a one-parameter family

$$A^0(\theta) := f^{-1}(\Psi_s^{-1}(1 + e^{i\theta})) \cap \{ \|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2 \leq \epsilon \} \quad \text{for } \theta \in S^1$$

of holomorphic annuli, whose boundaries thus provide foliations of the two tori. All annuli are embedded and smooth except for the “nodal” annulus $A^0(\pi)$, which consists of two embedded discs intersecting transversely in a single point. In addition, the annuli $\{A^0(\theta)\}_{\{\theta \neq \pi\}}$ intersect the smooth part (ie the complement of the double point)

of the Whitney immersion $L_{\text{Wh}}(s)$ in a foliation by closed curves, while the double point of $L_{\text{Wh}}(s)$ intersects $A^0(\pi)$ precisely in its node.

A basis $\langle e_0^\pm, e_1^\pm \rangle = H_1(T_\pm^0)$ with the properties in Lemma 5.8 can be fixed in the following manner. We let e_0^\pm be the boundary inside T_\pm^0 of either holomorphic disc in the nodal annulus $A^0(\pi)$; their orientations are moreover chosen so that the symplectic action of e_0^\pm satisfies $\pm \int_{e_0^\pm} \lambda_{\text{std}} > 0$ on $H_1(T_\pm^0)$. We then let e_1^\pm be the unique class of Maslov index two inside \mathbb{C}^2 for which $\int_{e_1^\pm} \lambda_{\text{std}} = s$.

Consider a Weinstein neighbourhood of \tilde{L} identified with a neighbourhood of the standard sphere $L_{\text{Wh}}(s) = \Pi_s^{-1}(0, 1)$ by a symplectomorphism Φ satisfying $\Phi(L_{\text{Wh}}) = \tilde{L}$ (here we have used Proposition 3.2). The map Φ is constructed so that it induces the identity map in homology; to that end recall that $H_i(V) \cong \mathbb{Z}$ for $i = 0, 1$, and that V deformation retracts onto the Whitney sphere by Propositions 3.5 and 3.8.

Since \tilde{L} satisfies the assumption of Theorem A, the same holds also for the induced Lagrangian tori $T_\pm := \Phi(T_\pm^0)$, which can be assumed to be arbitrarily close to \tilde{L} . Moreover, using this symplectomorphism we can produce a family $A(\theta) := \Phi(A^0(\theta))$ of symplectic annuli containing our immersed Lagrangian sphere $\tilde{L} \subset \bigcup_\theta A(\theta)$. Using the same identification Φ , we also obtain an induced basis $\langle e_0^\pm, e_1^\pm \rangle = H_1(T_\pm)$ from the choice of basis for $H_1(T_\pm^0)$ made above. We formulate its properties in a lemma.

Lemma 8.2 *The ordered basis*

$$\langle e_0^\pm, e_1^\pm \rangle = H_1(T_\pm)$$

satisfies the properties of Lemma 5.8. Furthermore, the induced orientation on $[T_\pm] \in H_2(V)$ is equivalent to the orientation of a standard torus $[\Pi_s^{-1}(u_1, u_2)] \in H_2(V)$ that is induced by the action–angle coordinates and the standard orientation of the base $\{(u_1, u_2)\} \subset \mathbb{R}^2$ (more precisely, orienting the base gives an orientation of the cotangent bundle of the torus, which is canonically isomorphic to the tangent bundle by the Lagrangian condition).

Lemma 8.3 *For a generic neck-stretching sequence around either of the tori T_\pm , the broken conic fibres produced by Proposition 5.16 satisfy the property that the unique plane in the building which passes through q_1 (resp. q_2) is asymptotic to a closed geodesic in the class e_0^\pm (resp. $-e_0^\pm$).*

Proof Because the assumptions of Theorem A are satisfied for the tori T_\pm , they are Hamiltonian isotopic to standard tori $\Pi_s^{-1}(u_1^\pm, u_2^\pm)$. By Lemma 8.2, we must

have $\int_{e_0^\pm} (\lambda_{\text{std}}) = \pi \cdot u_1^\pm$ (the sign of the right-hand side is the nontrivial part of this equivalence, for which we must use the statement concerning the orientation induced by the basis). The claim now follows from the topological fact that the area of a plane that has intersection number $+1$ with ℓ_∞ is of symplectic area equal to $\pi - \int_{\mathbf{g}} \lambda_{\text{std}} = \pi - \pi u_1^\pm$, where $\mathbf{g} \in H_1(T^\pm)$ is the class of the geodesic to which the plane is asymptotic (note the sign!). \square

We proceed to perform a neck-stretching *simultaneously* around the two disjoint Lagrangian tori $T_+ \cup T_-$. It can be explicitly seen that it is possible to choose the neck-stretching sequence J_τ so that the above family $A(\theta)$ of symplectic annuli stay pseudoholomorphic for all $\tau \geq 0$. To that end, it is important to use the fact that they all are nicely embedded near their boundaries, and that they intersect the tori T_\pm cleanly in a foliation by embedded closed curves. Moreover, we can achieve the following for the limit almost complex structure J_∞ defined on $\mathbb{C}P^2 \setminus (T_+ \cup T_-)$:

Lemma 8.4 *The almost complex structure J_∞ on $\mathbb{C}P^2 \setminus (T_+ \cup T_-)$ may be taken to satisfy:*

- (1) $C(\theta) := A(\theta) \setminus (T_+ \cup T_-)$ for $\theta \neq \pi$ are embedded J_∞ -holomorphic cylinders (ie two-punctured spheres) inside $\mathbb{C}P^2 \setminus (T_- \cup T_+)$, one of the punctures of which is asymptotic to a closed geodesic in the class $-e_0^+$ on T_+ , while the other puncture is asymptotic to a closed geodesic in the class e_0^- on T_- .
- (2) The curve $C(\pi)$ is a nodal J_∞ -holomorphic cylinder consisting of two transversely intersecting pseudoholomorphic planes P_\pm , where the plane P_\pm is asymptotic to a closed geodesic on T_\pm in the class $\mp e_0^\pm$.
- (3) Away from the node $P_+ \cap P_-$, the punctured spheres $C(\theta)$ provide a smooth foliation of an embedded three-manifold $\bigcup_\theta C(\theta) \setminus \{P_+ \cap P_-\}$, and the asymptotic evaluation maps $\{C(\theta)\} \rightarrow \Gamma_{\pm e_0} \cong S^1$ from the cylinders to its asymptotic orbit on either torus T_\pm are diffeomorphisms.
- (4) The Fredholm index of each cylinder $C(\theta)$ with $\theta \neq \pi$ is $\text{index}(C(\theta)) = 0$, while each of the two punctured planes involved in the nodal cylinder $C(\pi)$ is of index -1 .

Proof (1)–(3) These properties can be explicitly checked by hand.

(4) This follows easily from the formulation of the Fredholm index in terms of the Maslov index; see [12, Section 3.1]. \square

The planes involved in $C(\pi)$ are of negative Fredholm index, and are hence clearly not transversely cut out. Neither of the cylinders $C(\theta)$ has the correct dimension, since the cylinders come in a one-dimensional family while their index is equal to zero. In other words, an almost complex structure J_∞ as above is never regular. For that reason, additional care must be taken in order to control the structure of the broken conics. First we need to exclude the existence of other pseudoholomorphic planes of negative index. By the following lemma, this is possible by perturbing J_∞ away from \tilde{L} .

Lemma 8.5 *There exists an arbitrarily small closed neighbourhood $\Phi(U) \subset \mathbb{C}P^2$ of $\bigcup_\theta A(\theta) \supset T_\pm$, where $U \subset \mathbb{C}P^2$ is a sufficiently small neighbourhood of $\bigcup_\theta A^0(\theta) \supset T_\pm^0$, and an almost complex structure J_U on $\Phi(U) \setminus (T_+ \cup T_-)$ for which*

- (1) J_U can be extended to an almost complex structure J_∞ on $\mathbb{C}P^2 \setminus (T_+ \cup T_-)$ satisfying the properties of Lemma 8.4;
- (2) J_U has the property that, given an arbitrary such extension, any J_∞ -holomorphic plane inside $\mathbb{C}P^2 \setminus (T_+ \cup T_-)$ which is not equal to a branched cover of P_\pm must leave the neighbourhood $\Phi(U)$.

In particular, after choosing an extension J_∞ to be generic outside of $\Phi(U)$, we may assume that the planes of negative index are precisely the planes P_\pm together with their branched covers.

Proof There is a bijection between pseudoholomorphic planes $P \subset \Phi(U) \setminus (T_+ \cup T_-)$ contained inside the neighbourhood of interest, and pseudoholomorphic planes

$$\Phi^{-1}(P) \subset U \setminus (T_+^0 \cup T_-^0) \subset \mathbb{C}P^2 \setminus (T_+^0 \cup T_-^0),$$

given that the almost complex structures are chosen so that the symplectomorphism Φ is a biholomorphism. The complex structure satisfying the sought properties will be constructed on the “standard model” $U \setminus (T_+^0 \cup T_-^0)$, which then will be pushed forward to $\Phi(U)$ under the locally defined symplectomorphism Φ . In other words, we need to find a suitable neighbourhood U of $\bigcup_\theta A^0(\theta)$, together with a suitable almost complex structure J_∞ on $\mathbb{C}P^2 \setminus (T_-^0 \cup T_+^0)$, for which the branched covers of the two planes

$$P_\pm^0 \subset C^0(\pi) \subset \mathbb{C}P^2 \setminus (T_+^0 \cup T_-^0)$$

asymptotic to T_\pm^0 comprise all the J_∞ -holomorphic planes contained inside U .

We begin by constructing an almost complex structure J_∞ on $(\mathbb{C}P^2 \setminus (T_+^0 \cup T_-^0), \omega_{FS})$ by deforming the standard complex structure i near $T_+^0 \cup T_-^0$ in order to obtain a

concave cylindrical end there. With some additional care, this construction can be performed so that

- J_∞ is made to satisfy the analogous properties of Lemma 8.4 (for T_\pm^0 instead of T_\pm), and
- all i -holomorphic conic fibres $f^{-1}(z)$ that pass through T_\pm^0 give rise to three J_∞ -holomorphic punctured spheres $f^{-1}(z) \setminus (T_+^0 \cup T_-^0)$. The latter three components consist of two planes $A_i^\infty(\theta)$ for $i = 1, 2$ which are tangent to C at $A_i^\infty(\theta) \cap \ell_\infty = \{q_i\}$, together with a cylinder $C^0(\theta)$.

Using the second property, one obtains a broken pseudoholomorphic conic by adjoining two cylinders in the bottom level to these three punctured spheres in the top level, as shown in Figures 10 and 11.

We continue by arguing that the simply covered J_∞ -holomorphic planes inside $V \setminus (T_+^0 \cup T_-^0)$ are precisely the two planes P_\pm^0 . We argue by contradiction, assuming the existence of a J_∞ -holomorphic plane $P \subset U \setminus T_+^0$ asymptotic to T_+^0 ; the argument is analogous in the case of T_-^0 . The only asymptotic of P is a closed geodesic in the class $-ke_0^+$ on T_+^0 for some $k > 0$. (Recall that $k < 0$ is impossible by positivity of symplectic area for pseudoholomorphic curves.)

A topological consideration shows that

$$P \bullet (P_-^0 \cup A_2^\infty(\pi)) = k[P_+^0] \bullet (P_-^0 \cup A_2^\infty(\pi)) = k.$$

For the first equality, we use that T_+ is disjoint from the line $P_-^0 \cup A_2^\infty(\pi) \subset \mathbb{C}P^2$ and that the connecting homomorphism $H_2(\mathbb{C}^2, T_+) \xrightarrow{\cong} H_1(T_+)$ is an isomorphism. Positivity of intersection then shows that, unless $P = kP_+^0$, the plane P also must intersect at least one of the components $C^0(\theta)$ or $A_2^\infty(\theta)$ for all θ sufficiently close to π . (This follows by positivity of intersection of P and $C_\pi^0 \cup A_2^\infty(\pi)$.)

Now choose a θ as above, with the additional requirement that all asymptotics arising in the broken conic with top components $A_1^\infty(\theta) \cup C^0(\theta) \cup A_2^\infty(\theta) \subset \mathbb{C}P^2 \setminus T_-^0$ are different from the asymptotic of P . In other words,

$$P \cup k \cdot A_1^\infty(\theta) \quad \text{and} \quad A_1^\infty(\theta) \cup C^0(\theta) \cup A_2^\infty(\theta) = f^{-1}(z)$$

inside $\mathbb{C}P^2 \setminus T_-^0$ compactify to two cycles in $\mathbb{C}P^2$ that intersect in a subset where they both are J_∞ -holomorphic. Positivity of intersection leads to the final contradiction, since one computes the intersection number

$$(P \cup k \cdot A_1^\infty(\theta_0)) \bullet f^{-1}(z) > k[\ell_\infty] \bullet [f^{-1}(z)] = 2k$$

between the cycles given as the completions of the respective buildings. Recall that the intersection at $q_1 \in \ell_\infty$ of the two buildings is a tangency that precisely gives the contribution $2k$. □

Lemma 8.6 *Consider a neck-stretching sequence around T_+ (resp. T_-) for which $P_\pm \subset \mathbb{C}P^2 \setminus T_\pm$ is the unique simply covered J_∞ -holomorphic plane of index -1 . Then the limit of lines passing through $q_1 \in \ell_\infty$ (resp. $q_2 \in \ell_\infty$) while being tangent to C is a broken line containing a branched cover of P_+ (resp. P_-) in its limit.*

Proof We prove the statement in the case of the line through q_1 , and while stretching the neck around the Lagrangian torus T_+ ; the proof is analogous in the other case.

If the limit is a broken line, then the statement is a direct consequence of [Corollary 5.14](#) together with the fact that there are no contractible geodesics on the flat T_+ ; by the assumptions there simply are no other pseudoholomorphic planes inside $V \setminus T_+$ to which the broken line can limit except the branched covers of P_+ . Note that, by elementary topological reasons, it is clear that the building indeed must contain a plane.

It thus suffices to show that the line necessarily converges to a *broken* line. We argue by contradiction and assume that the limit line passing through q_1 , denoted by $\ell_1^{J_\infty}$, is unbroken.

We can perturb the almost complex structure J_∞ through a path $J_\infty(t)$ for $t \in [0, \epsilon]$ of almost complex structures with $J_\infty(0) = J_\infty$ in order to obtain the regular almost complex structure $J_\infty(\epsilon)$. The lines passing through q_1 which are tangent to C may be assumed to remain unbroken during this isotopy, giving rise to a smooth family $\ell_1^{J_\infty(t)}$ of such lines.

Then we examine the limit of the conic fibration when stretching the neck around T_+ in the case of the generic almost complex structure $J_\infty(\epsilon)$. By [Proposition 5.16](#) together with [Lemma 8.3](#), we see that the cycle P_+ (which need not be pseudoholomorphic for the perturbed almost complex structure) necessarily intersects $\ell_1^{J_\infty(1)}$, and hence also $\ell_1^{J_\infty}$, with *negative* intersection number. Indeed, the plane $A_1 \subset \mathbb{C}P^2 \setminus T_+$ in the broken conic with asymptotic Reeb orbit covering the same geodesic as the corresponding asymptotic of P_+ (but necessarily with the opposite orientation of the geodesic) can be completed to a cycle $A_1 \cup P_+$ in the class of a line; then we use the fact that $A_1 \bullet \ell_1^{J_\infty} = 2$. (There is a similar argument in the proof of [Proposition 5.16](#).)

Finally, since P_+ and $\ell_1^{J_\infty}$ both are J_∞ -holomorphic, the negative intersection number $P_+ \bullet \ell_1^{J_\infty} < 0$ gives the sought contradiction. □

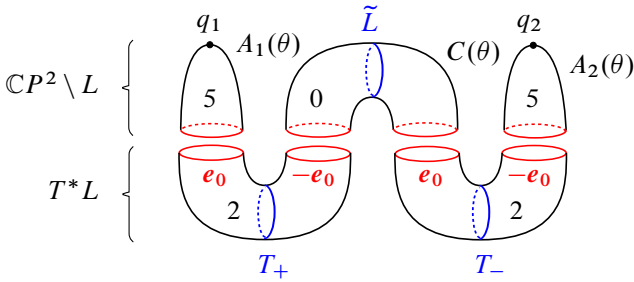


Figure 10: The broken conic in the generic case. The two planes $A_i(\theta)$ each intersect ℓ_∞ transversely in the point q_i , where they moreover are tangent to the smooth conic C . The two planes join to form a continuous embedding of a sphere.

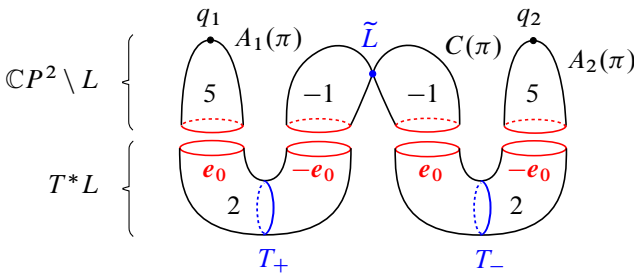


Figure 11: The broken conic in the generic case. The two planes $A_i(\pi)$ each intersect ℓ_∞ transversely in the point q_i , where they moreover are tangent to the smooth conic C . The two planes join to form a continuous embedding of a sphere.

Proposition 8.7 For a generic almost complex structure satisfying the conclusions of Lemma 8.5, there exists a family of embedded broken conics parametrised by $\theta \in S^1$, all of which are tangent to C at the two points $q_1, q_2 \in \ell_\infty$, and consisting of the following three components in the top level $\mathbb{C}P^2 \setminus (T_+ \cup T_-)$, varying smoothly with θ ,

- two planes $A_i(\theta) \subset \mathbb{C}P^2 \setminus (T_+ \cup T_-)$ for $i = 1, 2$, where $A_1(\theta)$ is asymptotic to T_+ and tangent to C at $q_1 \in \ell_\infty$, and where $A_2(\theta)$ is tangent to C at $q_2 \in \ell_\infty$ and asymptotic to T_- , together with
- the cylinder $C(\theta) = \Phi(C^0(\theta))$ constructed above,

while the bottom level consists of two cylinders inside T^*T_\pm . Moreover, all curves $A_i(\theta) \setminus \ell_\infty$ and $C(\theta) \setminus \ell_\infty$ are mutually disjoint and foliate the three-dimensional variety that is given by their union. See Figures 10 and 11 for a schematic depiction of this.

Proof Using Lemma 8.6 we conclude that the nodal conics converge to a broken conic consisting of (possibly trivial) branched covers of both planes P_{\pm} under a neck-stretching limit. Arguing by using positivity of intersection, we conclude that these planes occur only once in each such broken line, and that they both are simply covered. The lines are thus seen to have components in the top level consisting of P_{\pm} together with planes $A_i \subset \mathbb{C}P^2 \setminus (T_+ \cup T_-)$ passing through q_i for $i = 1, 2$.

We then argue that the planes A_i necessarily live in compact components of its moduli spaces consisting of planes tangent to C at $q_i \in \ell_{\infty}$. Namely, any broken building arising as an SFT limit of such planes would involve an additional plane P_{\pm} . Since every such a plane intersects P_{\mp} transversely, this would contradict positivity of intersection when combined with the fact that the original planes A_i were disjoint from $P_+ \cup P_-$; recall that the SFT compactness theorem implies that we can extract a convergent sequence which converges uniformly in the C^{∞} -topology on compact subsets. Finally we can argue as in the proof of Theorem 5.12 (ie using automatic transversality and the asymptotic intersection number) in order to show the existence of the required S^1 -families $A_i(\theta)$ of planes. □

8.1 The proof of Theorem A in the case of a sphere

Proposition 8.7 produces a family of broken conics asymptotic to $T_+ \cup T_-$ that contains the immersed sphere \tilde{L} . We then use the techniques from [12, Section 5.3] in order to smooth the broken conics near $T_+ \cup T_-$. Recall that this smoothing is performed so that the produced conics in the S^1 -family

$$\overline{A_1(\theta) \cup C(\theta) \cup A_2(\theta)} \subset (\mathbb{C}P^2, \omega_{FS}) \quad \text{for } \theta \in S^1$$

each intersect the union $T_+ \cup T_-$ of tori in two embedded and disjoint closed curves that are homotopically nontrivial. In addition, we may assume that the symplectic annuli $C(\theta)$ are left undeformed by this smoothing procedure, since they already compactify to a smooth surface with boundary on T_{\pm} by their construction.

An application of Theorem 4.3 combined with Theorem 4.6 now produces a globally defined smooth conic fibration $f_J: \mathbb{C}P^2 \setminus \ell_{\infty} \rightarrow \mathbb{C}$ satisfying the following crucial properties:

- $f_J^{-1}(0)$ is the nodal conic,
- $f_J^{-1}(1) = C$ is our standard smooth conic, and
- $f_J = f$ near ℓ_{∞} .

By construction, the immersed sphere $\tilde{L} \subset \bigcup_{\theta} C(\theta)$ satisfies the property that $f_J(\tilde{L}) = \sigma \subset \mathbb{C}$ is a closed curve encircling $f_J(C) = 1 \in \mathbb{C}$ with winding number one and passing through $0 \in \mathbb{C}$; see [Figure 12](#) for an example. Note that both tori T_{\pm} also live over the same curve σ by construction. In the four steps below we then perform additional normalisations of the fibration f_J .

Step I: Normalising the nodal conic By construction, the nodal conic $f_J^{-1}(0)$ can be assumed to have a node modelled on the standard conic C_{nodal} ; recall that this nodal conic coincides with the nodal annulus $C(\pi)$ which, in turn, is symplectomorphic to the standard nodal annulus $C^0(\pi) \subset C_{\text{nodal}}$ by construction. After an application of a suitable Hamiltonian isotopy we may also assume that $\ell_i^J \subset f_J^{-1}(0) = \ell_1^J \cup \ell_2^J$ coincides with ℓ_i near the node for $i = 1, 2$.

Arguing as in the proof of [Lemma 5.7](#), we can readily produce a Hamiltonian isotopy that deforms $f_J^{-1}(0)$ to C_{nodal} : First we consider a family of symplectic nodal conics that connects $f_J^{-1}(0)$ to C_{nodal} produced by Gromov’s result, [Theorem 4.1](#), for an interpolation of tame almost complex structures from J to i . Then we invoke [Proposition 4.9\(2\)](#) to normalise the node as well as the intersections with ℓ_{∞} . Finally, we can apply [Proposition 4.8](#).

Step II: Normalising a neighbourhood of the nodal conic It is now the case that the double point of \tilde{L} intersects C_{nodal} precisely in its node. After a Hamiltonian isotopy supported in an arbitrarily small neighbourhood $f_J^{-1}B_{2\epsilon}^2$ of this node, one readily makes sure that \tilde{L} moreover coincides with $L_{\text{Wh}}(s)$ inside $f^{-1}B_{\epsilon}^2$ for any fixed choice of $s \in (0, \pi/2)$. Reiterating the previous construction in this setting, we can now assume that the fibration f_J produced satisfies the additional properties

- $f_J = f$, and
- $\tilde{L} = L_{\text{Wh}}(s)$

in some neighbourhood of C_{nodal} . The image of the immersed sphere after this modification is shown in [Figure 13](#).

For a suitable such modification, we may in addition assume that there exists an embedded path $\gamma \subset \mathbb{C}$ from 0 to 1, also shown in [Figure 13](#), satisfying the properties that

- (1) $\gamma \cap \sigma = \{0\}$,
- (2) γ coincides with $[0, 1]$ near its boundary point $\{0, 1\} \subset \mathbb{C}$, and
- (3) γ is isotopic to $[0, 1]$ through paths γ_t coinciding with $[0, 1]$ near its boundary, where $\gamma_0 = \gamma$ and $\gamma_1 = [0, 1]$.

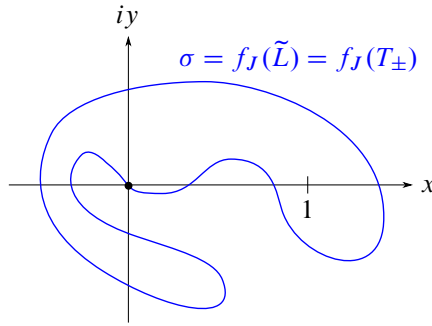


Figure 12: The image of \tilde{L} and T_{\pm} under f_J coincide by construction.

For the last point, it might be necessary to first deform the sphere \tilde{L} near its double point by an explicit rotation via an application of the Reeb flow on the standard contact sphere. We proceed to give the details.

Consider a Hamiltonian of the form $h(\|\tilde{z}\|^2)$ defined on

$$(B^4 \setminus \{0\}, \omega_0) \cong (CP^2 \setminus \ell_{\infty}, \omega_{FS})$$

with support in a small neighbourhood of the origin. We require $h(\|\tilde{z}\|^2) = \frac{1}{4}k\pi\|\tilde{z}\|^2$ near the origin for some suitable $k \in \mathbb{Z}$. Note that the corresponding Hamiltonian diffeomorphisms given as the time-1 maps are of the form $\tilde{z} \mapsto e^{i\theta(\|\tilde{z}\|^2)}\tilde{z}$, where $\theta(\|\tilde{z}\|^2) \equiv k\pi/2$ near the origin. In particular, this Hamiltonian diffeomorphism fixes the two coordinate lines $\{\tilde{z}_i = 0\}$ for $i = 1, 2$.

Inspecting the standard fibration $\tilde{f}(\tilde{z}_1, \tilde{z}_2) = \tilde{z}_1\tilde{z}_2/(1 - \|\tilde{z}_1\|^2 - \|\tilde{z}_2\|^2)$ in these coordinates, we see that the deformation of \tilde{L} by such a Hamiltonian diffeomorphism still

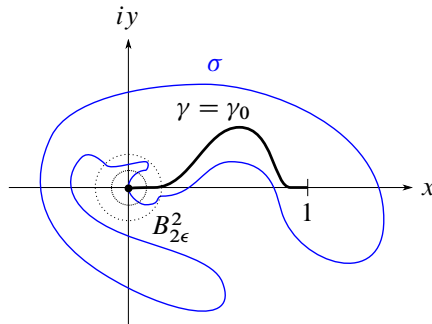


Figure 13: Here we have deformed the sphere \tilde{L} inside $f^{-1}(B_{2\epsilon}^2)$ in order to make it standard inside $f^{-1}(B_{\epsilon}^2)$.

has an image under f_J which is a curve that passes through the origin, but which now spins around it with a number k of additional half-turns. Here we note that the double point of the sphere is $0 \in B^4$, and that its two sheets are tangent to the two Lagrangian discs $\{(z, \pm\bar{z})\} \subset \mathbb{C}^2$ there.

Step III: Normalise along a path from 0 to 1 As in the proof of [Theorem 5.12](#), the isotopy from γ to $[0, 1]$ induces a Hamiltonian isotopy that disjoins \tilde{L} from $f^{-1}(0, 1]$. A second reiteration of the whole argument in this subsection now allows us to infer that, in addition to the above, it is the case that $f_J = f$ is satisfied in some neighbourhood of $f^{-1}([-\epsilon, 1])$, and that \tilde{L} moreover coincides with a standard sphere in the same neighbourhood. See [Figure 14](#) for a schematic depiction of this situation.

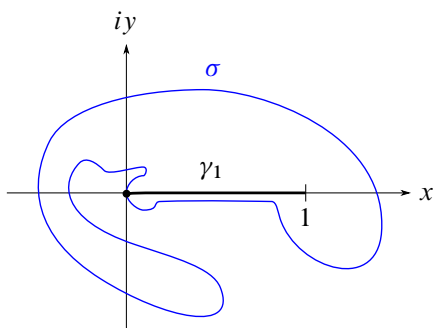


Figure 14: Here we have deformed the sphere \tilde{L} to make it standard inside a neighbourhood of $f^{-1}[0, 1]$.

Step IV: Resolving the double point Finally, we perform a Lagrangian surgery in a neighbourhood of C_{nodal} , replacing \tilde{L} there (where it coincides with a standard immersed sphere) with a piece of a standard torus of Clifford type; this is shown in [Figure 15](#). Denote by L the obtained Lagrangian torus, which is contained in an arbitrarily small neighbourhood of \tilde{L} . Note that the standard Lagrangian disc $\{(z, -\bar{z}) : \|z\| \leq \sqrt{\epsilon}\} \subset V$ may be assumed to intersect L cleanly precisely along its boundary.

The proof of [Theorem A](#) is finalised by a Hamiltonian isotopy from L to a standard torus, where this Hamiltonian isotopy is supported away from the Lagrangian disc $\{(z, -\bar{z}) : \|z\| \leq \sqrt{\epsilon}\} \subset V$. To construct this Hamiltonian isotopy we apply parts (2) and (3) of [Theorem B](#), after first using the symplectic identification of $(\mathbb{C}P^2 \setminus (\ell_\infty \cup f^{-1}[0, 1]), \omega_{\text{FS}})$ and a suitable subset $\mathbb{R} \times U \subset T^*\mathbb{T}^2$ as produced by [Lemma 7.1](#).

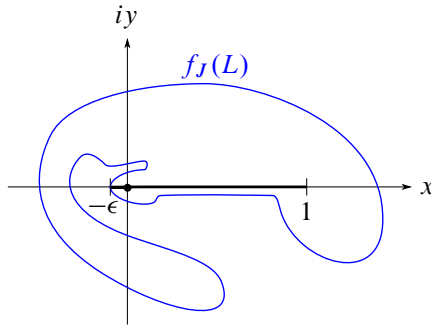


Figure 15: The resolution of the double point of \tilde{L} creates a Lagrangian torus L of Clifford type. The image under f_J of the Lagrangian disc $\{(z, -\bar{z}) : \|z\| \leq \sqrt{\epsilon}\}$ is equal to the path $[-\epsilon, 0] \subset \mathbb{C}$.

The final Hamiltonian isotopy from \tilde{L} to a standard immersed sphere is then constructed by means of a simple modification by hand of the above Hamiltonian isotopy from L to a standard Clifford torus. □

9 The proof of Theorem B

In this section we prove the refined version of the nearby Lagrangian conjecture for \mathbb{T}^2 formulated in Theorem B. Denote by $\iota_L: L \hookrightarrow T^*\mathbb{T}^2$ the Lagrangian embedding under consideration, and let $\pi: T^*\mathbb{T}^2 \rightarrow \mathbb{T}^2$ be the canonical bundle projection.

We begin by deducing the existence of a Hamiltonian isotopy to a graphical Lagrangian under any of the different assumptions made on the embedding, ie that ι_L is

- (1) weakly exact,
- (2) homological essential, or
- (3) a Lagrangian embedding of a torus with vanishing Maslov class.

In view of [12, Theorem 7.1], such a Hamiltonian isotopy exists whenever ι_L is a homotopy equivalence; we thus need the following result.

Lemma 9.1 *The three assumptions (1), (2) and (3) on ι_L are equivalent, and each condition is moreover equivalent to ι_L being a homotopy equivalence.*

Proof First note that any closed Lagrangian $L \subset T^*\mathbb{T}^2$ which is homologically essential with \mathbb{Z} -coefficients is orientable, and is hence a torus by the Lagrangian

adjunction formula. By [3, Theorem 4.1.1], a Lagrangian torus $L \subset T^*\mathbb{T}^2$ has vanishing Maslov class if and only if it is homologically essential. As shown by Arnold [2], and independently by E Giroux [18], the latter is in turn equivalent to ι_L being a homotopy equivalence. In conclusion, we have shown that (2) and (3) both are equivalent to ι_L being a homotopy equivalence. It is clear from elementary topology that these conditions also imply weak exactness.

What remains is showing that (1) implies that ι_L is a homotopy equivalence. If L weakly exact, Lemma 9.2 shows that L is exact after a translation by a section corresponding to the graph of a suitable closed one-form. By [1] and [25], the exactness now implies that L indeed is a torus whose inclusion is homotopy equivalent to the zero section. \square

Given a closed one-form $\alpha \in \Omega^1 M$ we use $\tau_\alpha: (T^*M, d(\lambda_M + \alpha)) \rightarrow (T^*M, d(\lambda_M))$ to denote the (not necessarily exact) symplectomorphism given by fibrewise addition with the section α . The identification $T^*\mathbb{T}^2 = \mathbb{T}^2 \times \mathbb{R}^2$ provides us with the “constant” one-forms $p_1 d\theta_1 + p_2 d\theta_2$ and we write $\tau_{(p_1, p_2)}$ for the corresponding symplectomorphism.

Lemma 9.2 *If $L \subset (T^*M, d\lambda_M)$ is a closed connected Lagrangian submanifold which is weakly exact, and if M is connected with $\pi_1(M)$ abelian, then the fibrewise translation $\tau_\alpha(L)$ by a suitable closed one-form $\alpha \in \Omega^1 M$ is exact.*

Proof Since $\pi_1(M)$ is abelian, it follows that $H_1(T^*M) = \pi_1(T^*M)$, and hence any element

$$\gamma \in \ker(H_1(L) \xrightarrow{[\iota_L]} H_1(T^*M))$$

can be lifted to $\tilde{\gamma} \in \pi_1(L)$ that is contained in the image of the connecting homomorphism $\pi_2(T^*M, L) \rightarrow \pi_1(L)$.

By the previous paragraph, together with the assumption of weak exactness of L , it now follows that $[\iota_L^* \lambda_M]$ vanishes on $\ker[\iota_L] \subset H_1(L; \mathbb{R})$. The symplectic action class can thus be written as $[\iota_L^* \lambda_M] = -[(\pi \circ \iota_L)^* \alpha]$ for some closed one-form $\alpha \in \Omega^1 M$.

For any section $\alpha \in \Omega^1 M$ of a closed one-form, the pullback of λ_M under ι_L and its fibrewise translation $\tau_\alpha \circ \iota_L$ by α satisfies

$$[(\tau_\alpha \circ \iota_L)^* \lambda_M] = [(\pi \circ \iota_L)^* \alpha] + [\iota_L^* \lambda_M] = 0$$

in $H^1(L, \mathbb{R})$, which shows the statement. \square

We are now ready to prove parts (1)–(3) of [Theorem B](#).

Proof of (1) As already established, there is a Hamiltonian isotopy L_t from the Lagrangian torus $L = L_0$ to a section L_1 of a closed one-form, which may be assumed to be a constant section $L_1 = \mathbb{T}^2 \times \{\mathbf{p}_0\}$. Unless $\mathbf{p}_0 \in U \subset \mathbb{R}^2$, it follows that L is displaceable from itself by a Hamiltonian isotopy, which thus contradicts, for example, Floer’s original work [16] or the result in [26] by F Laudenbach and J-C Sikorav.

In order to see that the entire Hamiltonian isotopy can be assumed to be contained inside U , we note the following. First, the fibrewise rescalings

$$\sigma_s : (T^*\mathbb{T}^2, d\lambda_{\mathbb{T}^2}) \rightarrow (T^*\mathbb{T}^2, e^{-s} d\lambda_{\mathbb{T}^2}), \quad (\boldsymbol{\theta}, \mathbf{p}) \mapsto (\boldsymbol{\theta}, e^s \mathbf{p}),$$

preserve exact Lagrangian submanifolds. Hence, the induced isotopy by such a rescaling of an exact Lagrangian submanifold can thus be generated by a Hamiltonian isotopy. Since $\tau_{\mathbf{p}_0}^{-1}(L)$ is exact, we now see that the isotopy $\tau_{\mathbf{p}_0} \circ \sigma_s \circ \tau_{\mathbf{p}_0}^{-1}$ acts on our Lagrangian L by Hamiltonian isotopy.

Since the conformal symplectic isotopy

$$\tau_{\mathbf{p}_0} \circ \sigma_s \circ \tau_{\mathbf{p}_0}^{-1} : (T^*\mathbb{T}^2, d\lambda_{\mathbb{T}^2}) \rightarrow (T^*\mathbb{T}^2, e^{-t} d\lambda_{\mathbb{T}^2})$$

contracts any compact subset into an arbitrarily small neighbourhood of $\mathbb{T}^2 \times \{\mathbf{p}_0\}$ as $s \rightarrow -\infty$, and since it preserves the subset $\mathbb{T}^2 \times U \supset \mathbb{T}^2 \times \{\mathbf{p}_0\}$ by the convexity assumptions, it now follows that we can deform the Hamiltonian isotopy L_t to the Hamiltonian isotopy $\tau_{\mathbf{p}_0} \circ \sigma_s \circ \tau_{\mathbf{p}_0}^{-1}(L_t)$ for $s \gg 0$, which is contained entirely inside $\mathbb{T}^2 \times U$. Finally, we also need to preconcatenate with the Hamiltonian isotopy $\tau_{\mathbf{p}_0} \circ \sigma_s \circ \tau_{\mathbf{p}_0}^{-1}(L)$, which also is contained entirely inside $\mathbb{T}^2 \times U$ for the very same reasons. □

Proof of (2) The geometric assumptions made on the Lagrangian enable us to construct a Hamiltonian isotopy which “frees up” space around L above $S^1 \times e^{i[-\epsilon, \epsilon]} \subset \mathbb{T}^2$.

Lemma 9.3 *There exists a Hamiltonian isotopy $\phi_{H_t}^t$ with support in the complement of $\bigcup_{|s| \leq \epsilon} \dot{D}_{\mathbf{p}^0}(e^{is})$, where*

$$\phi_{H_t}^1(L) \cap (S^1 \times e^{i[-\epsilon/2, \epsilon/2]} \times \mathbb{R} \times (-\infty, p_2^0]) = S^1 \times e^{i[-\epsilon/2, \epsilon/2]} \times \{\mathbf{p}^0\}$$

is satisfied for the time-1 map.

See [Figure 16](#) for an example of the effect of the above symplectic isotopy in the case $\mathbf{p}^0 = (0, 0)$.

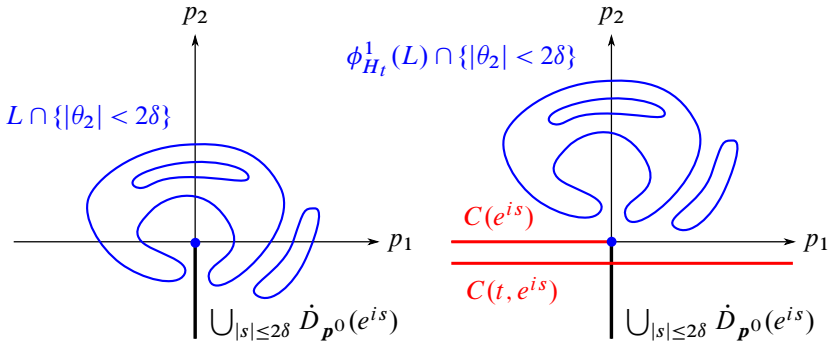


Figure 16: The canonical projection of $L \cap (S^1 \times (-\epsilon, \epsilon) \times \mathbb{R}^2)$ to \mathbb{R}^2 before and after the deformation provided by Lemma 9.3. Here $p^0 = (0, 0)$ and $2\delta := \epsilon/2$. In this subset the deformation is simply a translation of $L \setminus S^1 \times (-\epsilon, \epsilon) \times \{p^0\}$ in the positive p_2 -direction.

Proof We act on the subset of L contained inside

$$S^1 \times (-\epsilon, \epsilon) \times O \subset T^*\mathbb{T}^2 \quad \text{with } O := \mathbb{R}^2 \setminus (\{p_1^0\} \times (-\infty, p_2^0])$$

by the symplectic isotopy given by the fibrewise translation $\tau_{th(\theta_2)} d\theta_2$. Here $h(\theta_2) \geq 0$ is a smooth bump function with nonempty support contained inside $(-\epsilon/2, \epsilon/2)$. Here it is crucial to use the assumptions on the intersection properties of L and the Lagrangian disc in order to infer that this produces a Lagrangian isotopy.

Finally we argue that the above Lagrangian isotopy has a vanishing symplectic flux-path and, hence, that it is a Hamiltonian isotopy, as sought. It suffices to consider the change of symplectic action on a closed loop $\gamma \subset L$ which is homotopic to $\{pt\} \times S^1 \times \{0\}$ inside $T^*\mathbb{T}^2$ (recall that L is homotopic to the zero section). Since the degree of the projection $T^*\mathbb{T}^2 \rightarrow S^1$ to the second factor of $\mathbb{T}^2 = S^1 \times S^1$ restricted to γ is one, and since the part of γ contained inside $S^1(-\epsilon, \epsilon) \times \{0\} \subset L$ is fixed by the deformation, one readily computes that the flux is unchanged (by degree reasons the total integral over the remaining sheets of γ where the isotopy is supported is equal to zero). □

We now replace L with the Lagrangian produced by Lemma 9.3 and will write

$$\delta := \frac{\epsilon}{8}.$$

Under these assumptions it is possible to establish a refined Lagrangian isotopy result — this is the only result in this section whose proof requires hard techniques.

Proposition 9.4 For L as above, there exists a **Lagrangian** isotopy L_t from $L_0 = L$ to a graphical Lagrangian torus L_1 , where the isotopy satisfies the additional property that

$$L_t \cap (S^1 \times e^{i(-2\delta, 2\delta)} \times \mathbb{R} \times (-\infty, p_2^0]) = S^1 \times \{\theta_2\} \times \{p_1^t, p_2^0\},$$

where p_1^t depends smoothly on $t \in [0, 1]$.

We prove this proposition in Section 9.1 by following the steps of the proof of [12, Theorem 7.1] while taking additional care.

Given Proposition 9.4, Theorem B(2) is now a direct consequence of the straightforward result Lemma 9.5. □

Lemma 9.5 Consider the Lagrangian isotopy L_t produced by Proposition 9.4. There exists a smooth path of one-forms

$$\alpha_t = (p_1 - p_1^t) d\theta_1 + g_t(\theta_2) d\theta_2$$

supported inside $S^1 \times e^{i(-2\delta, 2\delta)}$, and where $g_t = 0$ moreover holds in $e^{i(-\delta, \delta)}$, for which $\tau_{\alpha_t}(L_t)$ is a Hamiltonian isotopy.

In other words, $\tau_{\alpha_t}(L_t)$ is a Hamiltonian isotopy which starts at $L_0 = L$, satisfies

$$\tau_{\alpha_t}(L_t) \cap (S^1 \times e^{i(-\delta, \delta)} \times \mathbb{R} \times (-\infty, p_2^0]) = S^1 \times e^{i(-\delta, \delta)} \times \{\mathbf{p}^0\}$$

for all $t \in [0, 1]$, and ends at the graphical Lagrangian torus $\tau_{\alpha_1}(L_1)$.

Proof The functions $g_t(\theta_2)$ are constructed to satisfy

$$\int_{S^1} g_t(\theta_2) d\theta_2 = \int_{\{e^{i0}\} \times S^1} \lambda_{\mathbb{T}^2}|_{TL_0} - \int_{\{e^{i0}\} \times S^1} \lambda_{\mathbb{T}^2}|_{TL_t}$$

for all $t \in [0, 1]$. Using the fact that $L \subset T^*\mathbb{T}^2$ is a homotopy equivalence, the resulting deformed isotopy is now seen to have a vanishing symplectic flux-path, as sought. □

Proof of (3) Consider the Hamiltonian isotopy L_t constructed in (2), where L_1 is a Lagrangian that is the graph of the closed one-form α on \mathbb{T}^2 . Since $L = L_0 \subset \mathbb{T}^2 \times U$ we can readily find a different closed one-form β , coinciding with α near $\{\theta_2 = 0\}$, whose graph is contained inside $\mathbb{T}^2 \times U$. Here we have used the fact that there is a Hamiltonian isotopy from L_0 to a constant section inside $\mathbb{T}^2 \times U$ by (1). We then construct a Hamiltonian isotopy \tilde{L}_t from $\tilde{L}_0 = L$ to the graph of β which satisfies

the requirements of (2) as follows: concatenate the original path L_t with a path of graphical Lagrangians from the fibrewise convex interpolation $t\beta + (1-t)\alpha$.

What remains is to deform the Hamiltonian isotopy constructed above relative its endpoints in order to make it confined to the subset $\mathbb{T}^2 \times U \supset L$. This we do by arguing as in (1), ie by considering the deformed Hamiltonian isotopy

$$\tau_\beta \circ \sigma_s \circ \tau_\beta^{-1}(\tilde{L}_t) \quad \text{for } s \gg 0.$$

Here it is important to use the property that

$$U \setminus \{p_2 < p_2^0\} \subset \mathbb{R}^2$$

is a *star-shaped* set with respect to the point p^0 , in order to ensure that the requirements in (2) are satisfied also for this new Hamiltonian isotopy. □

9.1 Proof of Proposition 9.4

Here we show the existence of the Lagrangian isotopy L_t . After a suitable translation by the constant section $-(p_1^0 d\theta_1 + p_2^0 d\theta_2)$ it suffices to consider the case when $p_1^0 = p_2^0 = 0$.

The starting point of the proof is the observation that

$$C(e^{is}) := S^1 \times \{e^{is}\} \times (-\infty, p_1^0) \times \{p_2^0\} \subset T^*\mathbb{T}^2 \setminus L$$

is a smooth one-dimensional family of embedded two-punctured symplectic spheres, ie cylinders, which moreover are pseudoholomorphic for the standard cylindrical almost complex structure J_{cyl} on $T^*\mathbb{T}^2 \setminus 0_L$ (see Section 5.1). Here $C(e^{is})$ is parametrised by (θ_1, p_1) with $p_1 < p_1^0 = 0$, while the family is parametrised by $s \in (-4\delta, 4\delta)$. Note that the cylinder is asymptotic to Reeb chords of $T^*\mathbb{T}^2 \setminus L$ at its convex (ie near $+\infty$) as well as concave (ie near L) ends. In fact this is a “trivial” cylinder inside the symplectisation $T^*\mathbb{T}^2 \setminus 0_{T^2} \xrightarrow{\cong} \mathbb{R} \times ST^*\mathbb{T}^2$, in the sense that each $C(e^{is})$ is invariant under translation of the symplectisation coordinate.

In addition, for all $s \in (-4\delta, 4\delta)$ and $p_2 < p_2^0$, there are J_{cyl} -holomorphic cylinders

$$C(p_2, e^{is}) := S^1 \times \{e^{is}\} \times \mathbb{R} \times \{p_2\} \subset T^*\mathbb{T}^2 \setminus L$$

having both punctures asymptotic to Reeb orbits in the convex end (ie near $+\infty$).

Of course, since L only partially coincides with $0_{\mathbb{T}^2}$, the almost complex structure J_{cyl} is typically not defined on all of $T^*\mathbb{T}^2 \setminus L$. However, we can find a well-defined

compatible almost complex structure J on $(T^*\mathbb{T}^2 \setminus L, d\lambda_{\mathbb{T}^2})$ which is cylindrical outside of a compact subset, and which agrees with J_{cyl} in a neighbourhood of the union of cylinders

$$C(e^{is}) \cup C(p_2, e^{is}) \quad \text{with } |s| \leq 4\delta \text{ and } p_2 < p_2^0.$$

Here it is crucial that these cylinders all are disjoint from L ; see Figure 16. (This was achieved by the previous application of Lemma 9.3.)

We now follow the argument given in [12, Section 7]. First, we compactify $D_R T^*\mathbb{T}^2$ to $S^2 \times S^2$ for some $R \gg 0$ as described in [12, Section 7]. The above cylinders $C(e^{is})$ now become one-punctured pseudoholomorphic spheres inside the latter space, ie pseudoholomorphic planes, given that J is carefully chosen. In fact, we may require J to be equal to the standard product complex structure near the divisor

$$C_\infty := (S^2 \times \{0, \infty\}) \sqcup (\{0, \infty\} \times S^2) \subset S^2 \times S^2,$$

and thus, in particular, this divisor J -holomorphic.

Second, arguing as in [12] by using automatic transversality, positivity of intersection, and the SFT compactness theorem, we obtain the following result, which is the core of the argument here:

Proposition 9.6 [12, Section 6] *The above family $\{C(e^{is})\}_{|s|<4\delta}$ of J -holomorphic planes inside $S^2 \times S^2 \setminus L$ lives inside a regularly cut out and compact component of its moduli space $\mathcal{M} = \{C(\theta)\}_{\theta \in S^1} \cong S^1$. Furthermore, the planes $C(\theta)$ for $\theta \in S^1$ foliate the hypersurface*

$$\bigcup_{\theta \in S^1} C(\theta) \subset T^*\mathbb{T}^2 \setminus L$$

*diffeomorphic to $S^1 \times C(e^{i0})$, and the asymptotic evaluation map $\mathcal{M} \rightarrow \Gamma$ is a diffeomorphism, where we have used $\Gamma \cong S^1$ to denote the corresponding S^1 -family of Reeb orbits on ST^*L .*

By construction, the discs $C(\theta)$ with $\theta \in e^{i(-4\delta, 4\delta)}$ may be assumed to coincide with planes of the form

$$B^2 \times \{e^{i\theta}\} \subset S^2 \times S^2,$$

where $B^2 \subset S^2$ denotes the lower open hemisphere.

Lemma 9.7 [12, Section 7.1] *After a deformation of $C(\theta)$ with $\theta \in S^1$ supported near the divisor $C_\infty \subset S^2 \times S^2$ we may further assume that each $C(\theta)$ with $\theta \in S^1$ coincides with the J_{cyl} -holomorphic cylinder*

$$B^2 \times \{e^{i\theta}\} \subset S^2 \times S^2$$

in that neighbourhood.

Third, after deforming the planes $\{C(\theta) : \theta \notin e^{i(-3\delta, 3\delta)}\}$ as in [12, Section 5.3], we may assume that the compactifications $D(\theta) \subset (S^2 \times S^2, L)$ for all $\theta \in S^1$ yield a smoothly embedded solid torus

$$(\tilde{\mathcal{T}} := D^2 \times S^1, S^1 \times S^1) \hookrightarrow (S^2 \times S^2, L).$$

Using the initial identification

$$S^2 \times S^2 \setminus C_\infty \xrightarrow{\cong} D_R T^* \mathbb{T}^2$$

we extend the image of $\tilde{\mathcal{T}}$ to a proper embedding

$$((D^2 \setminus \{0\}) \times S^1, S^1 \times S^1) \xrightarrow{\cong} (\dot{\mathcal{T}}, \partial \dot{\mathcal{T}}) \subset (T^* \mathbb{T}^2, L)$$

foliated by symplectic punctured discs $\dot{D}(\theta)$. Using the existence of our initial punctured pseudoholomorphic spheres $C(e^{i^s})$ and $C(p_2, e^{i^s})$, we can make sure that this “punctured solid torus” is of a standard form.

Lemma 9.8 *In addition to the above properties, we may assume that:*

- (1) *The punctured disc leaves $\dot{D}(\theta)$ for $\theta \in e^{i(-3\delta, 3\delta)}$ coincide with*

$$S^1 \times \{\theta\} \times \mathbb{R}_{\leq 0} \times \{0\},$$

while $\dot{D}(\theta) \setminus D_R T^ \mathbb{T}^2$ all are contained inside the latter subset.*

- (2) *The intersection*

$$\dot{\mathcal{T}} \cap (S^1 \times e^{i(-3\delta, 3\delta)} \times \mathbb{R} \times \mathbb{R}_{< 0})$$

is empty.

Proof (1) This property can be achieved by [Lemma 9.7](#).

(2) This is a consequence of positivity of intersection with the cylinders $C(p_2, s^{i^s})$ with $p_2 < 0$, which can be assumed to remain pseudoholomorphic during all of the steps taken above. □

Lastly, we need to correct the monodromy map induced by the characteristic distribution of the solid torus. Recall that this monodromy is a symplectomorphism

$$\phi: (S^1 \times \{e^{i0}\} \times \mathbb{R}_{\leq 0} \times \{0\}, d\lambda_{\mathbb{T}^2}) \rightarrow (S^1 \times \{e^{i0}\} \times \mathbb{R}_{\leq 0} \times \{0\}, d\lambda_{\mathbb{T}^2})$$

obtained by integrating the line field

$$\ker(d\lambda_{\mathbb{T}^2}|_{T\dot{\gamma}}) \subset T\dot{\gamma},$$

which clearly is transverse to all symplectic leaves $\dot{D}(\theta)$. The goal of this deformation is to make this symplectomorphism preserve the foliation

$$\gamma_p := S^1 \times \{e^{i0}\} \times \{p\} \times \{0\} \subset S^1 \times \{e^{i0}\} \times \mathbb{R}_{\leq 0} \times \{0\} = \dot{D}(e^{i0})$$

by simple closed curves. Observe that this is already the case for the curve γ_0 as well as γ_{-P} for all $P \gg 0$, as follows from the Lagrangian condition of L and Lemma 9.8(1), respectively.

The deformation of the family of annuli is performed by a so-called symplectic suspension of a suitable Hamiltonian isotopy $\phi_{H_t}^t$ of the symplectic punctured disc

$$\dot{D}(e^{i0}) = S^1 \times \{e^{i0}\} \times \mathbb{R}_{\leq 0} \times \{0\} \subset (T^*\mathbb{T}^2, d\lambda_{\mathbb{T}^2}).$$

The argument for the existence of Hamiltonian isotopies of the disc in [12, Lemma 7.4] carries over to show the following.

Lemma 9.9 *There exists a Hamiltonian isotopy*

$$\phi_{H_t}^t: (\dot{D}(e^{i0}) \cong D^2 \setminus \{0\}, \omega) \xrightarrow{\cong} (\dot{D}(e^{i0}) \cong D^2 \setminus \{0\}, \omega) \quad \text{for } t \in [0, \delta],$$

where $H_t = 0$ for all $t \notin [\delta/2, \delta)$, which

- is generated by a nonnegative time-dependent Hamiltonian $H_t \geq 0$ that has compact support, and satisfies $H_t|_{\partial D^2} \equiv 0$ for all t , and
- satisfies the property that $\phi_{H_t}^\delta \circ \phi$ preserves the leaves of the foliation $\{\gamma_p\}$ by embedded closed curves.

Proof We begin by recalling the standard fact that any symplectomorphism of an annulus that preserves the boundary setwise is Hamiltonian isotopic, while fixing the boundary setwise, to one which preserves the standard foliation by meridian curves. Whenever a boundary component is fixed pointwise by the symplectomorphism, this Hamiltonian isotopy may moreover be assumed to fix the same component pointwise.

To see this claim, first we point out that a Hamiltonian isotopy can be explicitly constructed in order to deform the symplectomorphism to one which is the identity on both boundary components. We can then use [10, Proposition A.4], together with the fact that the standard Dehn twist and hence its powers are compactly supported symplectomorphisms that preserve the standard foliation by meridians.

After adding a time-dependent constant we may assume that the Hamiltonian constructed above satisfies $H_t|_{\partial D^2} \equiv 0$. In order to make it nonnegative everywhere, we can simply add a suitable autonomous Hamiltonian which is constant along any meridian in the foliation. (The generated isotopy now performs an additional rotation along the meridian curves.) Note that the same technique also can be used to make H_t vanish near the origin $0 \in D^2$, as opposed to being merely constant there. \square

The symplectic suspension that we use is the locally defined symplectomorphism

$$(z, (e^{is}, p_2)) \\ \mapsto (\phi_{H_t}^{s+3\delta}(z), e^{is}, p_2 - H_{s+3\delta}(\phi_{H_t}^{s+3\delta}(z))) \quad \text{for } z \in D(e^{i0}) \text{ and } s \in [-3\delta, -2\delta].$$

When applying this symplectomorphism to the family $S^1 \times e^{i(-3\delta, -2\delta)} \times \mathbb{R}_{\leq 0} \times \{0\}$ of punctured discs, the properties in Lemma 9.8 satisfied by $\dot{\mathcal{T}}$ imply the following: the deformation of $\dot{\mathcal{T}}$ obtained by excising the above family and replacing it by its suspension $\dot{\mathcal{T}}'$ is embedded, and has a symplectic monodromy that preserves all leaves $\{\gamma_p\}$ for $p \leq 0$ of the foliation of $\dot{D}(e^{i0})$.

Given the above, the Lagrangian isotopy is finally constructed to be contained completely inside the hypersurface $\dot{\mathcal{T}}'$ foliated by punctured symplectic discs. More precisely, we consider the Lagrangian tori L_t in $\dot{\mathcal{T}}'$ uniquely determined by the property that they intersect the disc $\dot{D}(0)$ precisely in the curve γ_{-Ct} for some $C \gg 0$ sufficiently large. Indeed, by the assumption that all annuli are standard outside of a compact subset of $T^*\mathbb{T}^2$, it necessarily follows that these tori all are graphical for $t = 1$. \square

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