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# Toric geometry of $G_2$–manifolds

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We consider $G_2$–manifolds with an effective torus action that is multi-Hamiltonian for one or more of the defining forms. The case of $T^3$–actions is found to be distinguished. For such actions multi-Hamiltonian with respect to both the three- and four-form, we derive a Gibbons–Hawking type ansatz giving the geometry on an open dense set in terms a symmetric $3 \times 3$ matrix of functions. This leads to particularly simple examples of explicit metrics with holonomy equal to $G_2$. We prove that the multimoment maps exhibit the full orbit space topologically as a smooth four-manifold containing a trivalent graph as the image of the set of special orbits and describe these graphs in some complete examples.

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## 1 Introduction

The Gibbons–Hawking ansatz [27] furnishes a way of constructing hyperkähler four-manifolds with circle symmetry. More generally, the classifications of complete hypertoric manifolds (see eg Bielawski [8] and Dancer and Swann [17]) show that moment-map techniques, similar to the Delzant construction of symplectic geometry, can be useful when exploring Ricci-flat metrics.

Metrics of holonomy $G_2$ are known to be Ricci-flat. What is perhaps less familiar is that also, in this setting, one has a notion of (multi)symplectic geometry; see Madsen and Swann [36; 37]. It is therefore natural to ask what should be the analogue of toric or hypertoric geometry in this context.

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The first question to consider is which tori can act in a multi-Hamiltonian way on a torsion-free $G_2$–manifold. We find in Section 2 that the torus must have rank between $2$ and $4$. A dimension count then reveals that the case that best mimics hypertoric geometry is when a three-torus is multi-Hamiltonian for both the defining three-form and its Hodge dual four-form: this is the only case where the dimension of the orbit space matches the dimension of the target space for the multimoment map. This "toric" case with an effective $T^3$–action enjoys several immediate properties in common with the standard toric and hypertoric situation. In particular, we see that all stabilisers are again connected subtori, in this case of dimension at most $2$, and that the multimoment maps provide local coordinates on the manifold of principal orbits, so an open dense set of $M$ becomes a $T^3$–bundle over a four-manifold.

In Section 3, we derive the analogue of the Gibbons–Hawking ansatz for toric $G_2$–manifolds $M$. The crucial local datum is now a smooth positive-definite section $V \in \Gamma(U, S^2(\mathbb{R}^3))$ on an open set in $U \subset \mathbb{R}^4$. This determines the curvature of the $T^3$–bundle and must satisfy a pair of PDEs: one is a divergence-free condition on $V$ and the other system is a quasilinear elliptic second-order PDE. These differential operators are natural for the action of $\mathrm{GL}(3,\mathbb{R})$ resulting from change of basis for the Lie algebra $\mathfrak{t}^3$ of $T^3$, and are nearly uniquely specified by this property. The divergence-free equation is essentially one used in continuum mechanics.

The above description, in terms of $V$, applies at points that have trivial $T^3$–stabiliser. In Section 4, we obtain a good understanding of the differential topology near singular orbits. As in the hypertoric case, one finds that $M/T^3$ is homeomorphic to a smooth manifold. This is unlike the situation for toric symplectic manifolds, where the orbit space is a manifold with corners; see Karshon and Lerman [32]. Our main result is that such a homeomorphism is realised via the multimoment maps. Furthermore, the image of the singular orbits in the four-manifold $M/T^3$ is a trivalent graph, whose edges are straight lines in multimoment map coordinates. These results are obtained by first studying flat models, including $S^1 \times \mathbb{C}^3$, where the graph has a single vertex where three edges meet, and $T^2 \times \mathbb{R} \times \mathbb{C}^2$, where the graph has one edge and no vertex.

Our distinguished case of $G_2$–manifolds that are multi-Hamiltonian for $T^3$ has the good feature that there are nontrivial complete examples with full holonomy $G_2$. Indeed, the Bryant–Salamon $G_2$–structure on the spin bundle of $S^3$ [13] is such an example, as are the generalisations of Brandhuber, Gomis, Gubser and Gukov [11] and Bazaikin and Bogoyavlenskaya [5; 10]. We study the Bryant–Salamon example in some detail, showing how it fits into the general framework. In particular, the associated trivalent graph is connected with two vertices and the multimoment map provides a global homeomorphism $M/T^3 \to \mathbb{R}^4$.

If one is willing to compromise on completeness, our approach produces particularly simple Riemannian metrics with (restricted) holonomy equal to $\mathrm{G}_2$; see Examples 5.2 and 5.5.

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# 2 $\mathrm{G}_2$–manifolds with multi-Hamiltonian torus actions

Let $M$ be a connected 7–manifold. A $\mathrm{G}_2$–structure on $M$ is determined by a 3–form $\varphi$ that is pointwise linearly equivalent to the form

$$\varphi_0 = e^{123} - e^1(e^{45} + e^{67}) - e^2(e^{46} + e^{75}) - e^3(e^{47} + e^{56}),$$

where $E_1, \dots, E_7$ is a basis of $V \cong \mathbb{R}^7$, $e^1, \dots, e^7$ is its dual basis of $V^*$, wedge signs are suppressed and $e^{123} = e^1 \wedge e^2 \wedge e^3$, etc. We shall sometimes refer to $E_1, \dots E_7$ (and its dual) as an *adapted basis*.

The $\mathrm{GL}(V)$–stabiliser of $\varphi_0$ is the compact 14–dimensional Lie group $\mathrm{G}_2 \subset \mathrm{SO}(V)$. In fact, $\varphi_0$ uniquely determines both the inner product $g_0 = \sum_{j=1}^7 (e^j)^2$ and volume element $\mathrm{vol}_0 = e^{1234567}$ via the relation

$$6g_0(X,Y)\,\mathrm{vol}_0 = (X \lrcorner \varphi_0) \wedge (Y \lrcorner \varphi_0) \wedge \varphi_0,$$

for all $X, Y \in V$ (see [12]). Correspondingly, $\varphi$ determines a metric $g$ and a volume form vol on $M$. From this, it also follows that we have an additional dual 4–form, $*\varphi$, pointwise equivalent to

$$*\varphi_0 = e^{4567} - e^{23}(e^{45} + e^{67}) - e^{31}(e^{46} + e^{75}) - e^{12}(e^{47} + e^{56}).$$

We also get a cross-product operation via $g(X\times Y, Z)=\varphi(X,Y,Z)$. Three-dimensional subspaces of $T_pM$ closed under the cross-product are *associative*; their orthogonal complements are *coassociative*.

Following standard terminology, we say that $(M,\varphi)$ is a $G_2$*–manifold* if the $G_2$–structure is torsion-free, hence the (restricted) holonomy group $\mathrm{Hol}_0(g)$ is contained in $G_2\subset SO(7)$. This implies $g$ is Ricci-flat. It is well known [23] that being torsion-free, in this context, is equivalent to the condition that $\varphi$ be closed and coclosed.

We are interested in $G_2$–manifolds that come with an effective action of a torus $T^k$ on $M$ that preserves $\varphi$, hence also $*\varphi$ and the metric $g$. Such an action gives us a map

$$\xi\colon \mathbb{R}^k \cong \mathfrak{t}^k \to \mathfrak{X}(M), \tag{2-1}$$

which is a Lie algebra antihomomorphism. Subsequently, we shall often write $\xi_p$ for the image of $\xi$ at $p\in M$. This is a subspace of $T_pM$ of dimension at most $k$.

**Definition 2.1** [37, Definition 3.5] Let $N$ be a manifold equipped with a closed $(k+1)$–form $\alpha$, and $G$ an abelian Lie group acting on $N$ preserving $\alpha$. A *multimoment map* for this action is an invariant map $\nu\colon N\to\Lambda^k\mathfrak{g}^*$ such that

$$d\langle\nu, W\rangle = \xi(W)\lrcorner\alpha$$

for all $W\in\Lambda^k\mathfrak{g}$; here $\xi(W)\in\Gamma(\Lambda^kTM)$ is the unique multivector determined by $W$ via $\xi$.

We say that such a torus symmetry on a $G_2$–manifold is *multi-Hamiltonian* if there are multimoment maps associated with $(\varphi, T^k)$ and/or $(*\varphi, T^k)$. This requires that $k\geqslant 2$ for nontriviality. A discussion of circle-invariant $G_2$–metrics can be found in [2], and such metrics were also at the heart of the constructions in [25].

Given an effective torus action by $T^k$ on $(M,\varphi)$, it is obvious that $k\leqslant 7$ as we have the following well-known observation:

**Lemma 2.2** *Let $N$ be an $n$–manifold with an effective action of a torus $T^k$. Then $k\leqslant n$ and the principal stabiliser is trivial.*

**Proof** It suffices to prove the final statement. As $T^k$ is abelian, conjugation is trivial. Therefore different isotropy subgroups $H_p$ belong to different isotropy types. It follows that the principal stabiliser can be obtained as the intersection of all stabilisers, $\bigcap_{p\in N}H_p$, and so is the trivial group by effectiveness of the action. □

If $N$ is a compact Ricci-flat manifold, then each Killing vector field is parallel [9]. It follows by [7, Corollary 6.67] that $(N, h)$ has a finite cover in the form of a Riemannian product $T^l \times N_1^{n-l}$, some $k \leqslant l \leqslant n$, of a flat torus and compact simply connected Ricci-flat manifold $N_1$. In particular, for a compact $G_2$–manifold with an effective $T^k$–action, $\mathrm{Hol}_0(g)$ is a proper subgroup of $G_2$. From Berger's classification [6], it follows that the restricted holonomy is trivial, $SU(2)$ or $SU(3)$. Correspondingly, we must have $l = 7$, $l = 3$ or $l = 1$, respectively.

As our main interest is the case of full holonomy, we will often concentrate on the case when $M$ is noncompact.

Focusing on multi-Hamiltonian actions, we have already established that our torus must have rank between $2$ and $7$. It turns out there are further restrictions.

**Proposition 2.3** *If $T^k$ acts effectively on a $G_2$–manifold and is multi-Hamiltonian, then $2 \leqslant k \leqslant 4$.*

The proof of Proposition 2.3 is an immediate consequence of Lemmas 2.4 and 2.5 below.

**Lemma 2.4** *Suppose $W$ is a 5–dimensional subspace of $(V, \varphi_0)$. Then $W$ contains both associative and coassociative subspaces.*

**Proof** Choose an orthonormal basis $E_1$, $E_2$ for $W^\perp$. Then $E_3 = E_1 \times E_2$ lies in $W$. Thus $W$ contains the coassociative subspace $\langle E_1, E_2, E_3\rangle^\perp$. Furthermore, $E_1$, $E_2$, $E_3$ can be extended to a $G_2$ adapted basis for $V$. For this basis, $E_4 \times E_7 = E_3$, so $\langle E_3, E_4, E_7\rangle$ is an associative subspace of $W$. □

The following observation states that a necessary condition for an action to be multi-Hamiltonian is that the orbits be "isotropic":

**Lemma 2.5** *If a torus action of $T^k$ on $N$ is multi-Hamiltonian for a closed differential form $\alpha$ of degree $r \leqslant k$, then $\alpha|_{\Lambda^r \xi} \equiv 0$.*

*If $b_1(N) = 0$, this condition is also sufficient for the $T^k$–action to be multi-Hamiltonian.*

**Proof** Consider the fundamental vector fields $\xi(V_1), \dots, \xi(V_{r-1})$ associated with vectors $V_1, \dots, V_{r-1} \in \mathfrak{t}^k \cong \mathbb{R}^k$, and let $\nu'$ be a component of the multimoment map $\nu\colon N \to \Lambda^{r-1}\mathfrak{t}^k$ that satisfies

$$d\nu' = \alpha(\xi(V_1), \dots, \xi(V_{r-1}), \cdot).$$

By invariance of the multimoment map, we have for any $V_r \in \mathfrak{t}^k$ that

$$0 = \mathcal{L}_{\xi(V_r)}\nu' = \xi(V_r) \lrcorner d\nu' = \alpha(\xi(V_1), \dots, \xi(V_r)).$$

It follows that $\alpha$ vanishes on $\Lambda^r \xi$, as required.

As $T^k$ preserves $\alpha$, the 1–form $\alpha(\xi(V_1), \dots, \xi(V_{r-1}), \cdot)$ is closed and therefore exact, say equal to $d\nu'$, when $b_1(N) = 0$. The condition $\alpha|_{\Lambda^r \xi} \equiv 0$ implies invariance of $\nu'$, since $T^k$ is connected. □

The upshot of Proposition 2.3 is that there are potentially 7 possible cases that can occur: $T^2$ multi-Hamiltonian for $\varphi$, $T^3$ multi-Hamiltonian for either $\varphi$ or $*\varphi$, $T^3$ multi-Hamiltonian for both $\varphi$ and $*\varphi$, $T^4$ acts multi-Hamiltonian for $\varphi$ or $*\varphi$, and $T^4$ acts multi-Hamiltonian for both $\varphi$ and $*\varphi$. In reality, the last situation cannot occur, as we shall explain below.

Let $M_0 \subset M$ denote the subset of points $p$ such that the map $\xi$ of (2-1) is injective. It follows by Lemma 2.2 that $M_0$ is open and dense, since it contains the set of principal orbits $M_0'$. Note that $M_0'$ is the total space of a principle $T^k$–bundle.

## 2.1 Two-torus actions

This case was studied in [36], so we shall only give a brief summary.

Given a multi-Hamiltonian action for $\varphi$, the multimoment map $\nu$ is an invariant scalar function $M \to \Lambda^2(\mathfrak{t}^2)^* \cong \mathbb{R}$. For $t \in \nu(M)$, if the action of $T^2$ is free on the level set $\nu^{-1}(t)$, then the reduction $N = \nu^{-1}(t)/T^2$ is a 4–manifold carrying three symplectic forms of the same orientation, induced by

$$U_1 \lrcorner \varphi, \quad U_2 \lrcorner \varphi \quad \text{and} \quad U_1 \wedge U_2 \lrcorner *\varphi,$$

where $U_i$ generate the $T^2$–action. In interesting cases this triple is not hyperkähler, but does fit in to the framework of [24].

Conversely, the $G_2$–manifold $(M, \varphi)$ can be recovered from the 4–manifold $N$ by building a two-torus bundle over it. One then equips the total space of this bundle with a suitable $SU(3)$–structure and reconstructs the original $G_2$–holonomy manifold via an adapted "Hitchin flow".

Known complete $G_2$–manifolds with a multi-Hamiltonian $T^2$–action include the Bryant–Salamon metrics on the space of antiselfdual 2–forms over a complete selfdual positive Einstein manifold [13].

## 2.2 Three-torus actions

The main interest here will be for actions that are multi-Hamiltonian for both $\varphi$ and $*\varphi$, so that we have multimoment maps $(\nu,\mu)\colon M\to\mathbb{R}^3\times\mathbb{R}$. This is the only case in which the dimension of $M/T^k$ matches that of the target space for the multimoment maps. Being multi-Hamiltonian for $\varphi$, it follows by Lemma 2.5 that $\varphi|_{\Lambda^3\xi}\equiv 0$. This condition was studied in [29, Section IV], where it is shown that $\mathrm{G}_2$ acts transitively on the set of such three-planes. Indeed, for $p\in M_0$, for any orthonormal $X_2, X_3\in\xi_p$, there is an adapted basis where these correspond to $E_6$ and $E_7$. The $\mathrm{G}_2$–stabiliser of $\{E_6,E_7\}$ is an $\mathrm{SU}(2)$ acting on $\langle E_2,E_3,E_4,E_5\rangle\cong\mathbb{C}^2$. Using this action, we see that we can extend to a basis $X_1$, $X_2$, $X_3$ of $\xi_p$ and have $X_1$ identified with $E_5$. Now $\widehat{\theta}_i=e^{i+4}$, for $i=1,2,3$, are dual to $X_1,X_2,X_3$: $\widehat{\theta}_i(X_j)=\delta_{ij}$ and $\widehat{\theta}_i(X)=0$ for $X\perp\langle X_1,X_2,X_3\rangle$. Putting

$$\alpha_i=X_j\wedge X_k\lrcorner\varphi=-e^i,\quad \beta=X_1\wedge X_2\wedge X_3\lrcorner*\varphi=-e^4,$$

where $(i\ j\ k)=(1\,2\,3)$, corresponding to the differentials of the multimoment maps at $p$, the $\mathrm{G}_2$–structure at $p\in M_0$ takes the form

$$\begin{aligned}\varphi&=-\alpha_{123}-\alpha_1(\beta\widehat{\theta}_1-\widehat{\theta}_{23})-\alpha_2(\beta\widehat{\theta}_2-\widehat{\theta}_{31})-\alpha_3(\beta\widehat{\theta}_3-\widehat{\theta}_{12}),\\ *\varphi&=\widehat{\theta}_{123}\beta+\alpha_{23}(\beta\widehat{\theta}_1-\widehat{\theta}_{23})+\alpha_{31}(\beta\widehat{\theta}_2-\widehat{\theta}_{31})+\alpha_{12}(\beta\widehat{\theta}_3-\widehat{\theta}_{12}).\end{aligned}\tag{2-2}$$

We shall return to this expression later on, in Section 3, refining it to give a $\mathrm{G}_2$–analogue of the Gibbons–Hawking ansatz.

As in the hypertoric case, there are no points with discrete stabiliser. In particular, $M_0$ is the total space of a principal $T^3$–bundle over the corresponding orbit space.

**Lemma 2.6** *Suppose* $T^3$ *acts effectively on a manifold* $M$ *with* $\mathrm{G}_2$*–structure* $\varphi$ *such that the orbits are isotropic,* $\varphi|_{\Lambda^3\xi_p}=0$. *Then each isotropy group is connected and of dimension at most two, hence trivial, a circle or* $T^2$.

**Proof** Let $p\in M$ have isotropy group $H\leqslant T^3$. Then $H$ is an abelian group acting on $V=T^\perp$, where $T=T_p(T^3\cdot p)$ is the tangent space to the orbit. As $T^3\cdot p$ has an neighbourhood that can be identified with the normal bundle $T^3\times_H V$ and this neighbourhood necessarily intersects principal orbits, the action on $V$ is faithful. Adding the trivial $H$–module $T$ to $V$, we have that the $H$–action on $T_pM=T\oplus V$ preserves the $\mathrm{G}_2$–structure. As $\mathrm{G}_2$ has rank 2, we get $\dim H\leqslant 2$.

If $\dim H = 0$, then at $p$, any generators $U_1$, $U_2$, $U_3$ of the $T^3$ have the property that their cross-products span $T_pM$. As the $T^3$–action preserves the $\mathrm{G}_2$–structure, this implies that $H$ fixes every element of $T_pM$. Thus $H$ is trivial.

For $\dim H = 1$, the space $T$ is spanned by two linearly independent vectors $U_1$ and $U_2$. It follows that $H$ preserves the nonzero vector $U_1 \times U_2$ in $V$ and must act as a subgroup of $\mathrm{SU}(2)$ on the orthogonal complement. Thus $H$ is a one-dimensional abelian subgroup of $\mathrm{SU}(2)$. This forces the identity component $H_0$ to be a maximal torus of $\mathrm{SU}(2)$, so conjugate to $T^1 = \{\mathrm{diag}(\exp(i\theta), \exp(-i\theta)) \mid \theta \in \mathbb{R}\}$. But any matrix in $\mathrm{SU}(2)$ commuting with $T^1$ is diagonal, so belongs to $T^1$. Thus $H \cong T^1$, which is connected.

If $\dim H = 2$, then $H$ is a subgroup of $\mathrm{SU}(3)$, so its identity component is a maximal torus. Again this is conjugate to a group of diagonal matrices

$$\mathrm{diag}\big(\exp(i\theta), \exp(i\varphi), \exp\left(-i(\theta+\varphi)\right)\big)$$

and any other matrix commuting with this group is of this form. Thus $H \cong T^2$ and is connected. □

The classical example of a complete $\mathrm{G}_2$–holonomy manifold with a multi-Hamiltonian $T^3$–action is the spin bundle of $S^3$ equipped with its Bryant–Salamon structure [13]; see Section 5.1.2. Additional complete examples can be found in [11; 5; 10].

## 2.3 Four-torus actions

If a torus $T^4$ is multi-Hamiltonian for $\varphi$, then the multimoment map has 6 components as its image is in $\Lambda^2(\mathfrak{t}^4)^* \cong \mathbb{R}^6$.

**Lemma 2.7** *Suppose $(M, \varphi)$ admits an effective $T^4$–action that is multi-Hamiltonian for $\varphi$. If $p \in M_0$, then $\xi_p \subset T_pM$ is coassociative.*

**Proof** Take a pair $E_1$, $E_2$ of orthonormal vectors in $\xi_p$. As $\varphi|_{\Lambda^3\xi} \equiv 0$, we have that $E_3 = E_1 \times E_2$ lies in $\xi_p^\perp$. We may extend $E_1$, $E_2$, $E_3$ to an adapted basis $E_1, \dots, E_7$. Using the stabiliser $\mathrm{SU}(2)$ of $E_1$, $E_2$ in $\mathrm{G}_2$, we may ensure that $E_4 \in \xi_p$. Then the relations $E_1 \times E_4 = E_5$ and $E_2 \times E_4 = E_6$ give $\xi_p^\perp = \langle E_3, E_5, E_6 \rangle$, and so $\xi_p = \langle E_1, E_2, E_4, E_7 \rangle$. In particular, $\xi_p^\perp$ is associative and $\xi_p$ is coassociative. □

A local description of G$_2$–manifolds with $T^4$–symmetry whose orbits are coassociative is given in [3], and also discussed in [20]. Essentially these correspond to positive minimal immersions into $\mathbb{R}^{3,3} \cong H^2(T^4)$, and this in turn is the image of the multimoment map.

If $T^4$ is multi-Hamiltonian for $*\varphi$, we get a multimoment map with 4 components as it has values in $\Lambda^3 \mathfrak{t}^4 \cong \mathbb{R}^4$.

**Lemma 2.8** *Suppose $T^4$ acts effectively on $(M, \varphi)$ and is multi-Hamiltonian for $*\varphi$. If $p \in M_0$, then the 4–dimensional subspace $\xi_p \leqslant T_p M$ contains an associative subspace. In particular, the action can not be multi-Hamiltonian for $\varphi$.*

**Proof** Choose a pair of orthonormal vectors $E_1, E_2 \in \xi_p$ and extend these to an adapted basis for $T_p M$. As before, we may now use the stabiliser $\mathrm{SU}(2) \leqslant \mathrm{G}_2$ of $E_1$, $E_2$ to ensure that $E_4 \in \xi_p$. Now $*\varphi|_{\Lambda^4 \xi} \equiv 0$ implies that $E_7 = E_1 \times E_2 \times E_4$ lies in $\xi_p^{\perp}$. Therefore, $\xi_p = \langle E_1, E_2, E_4, v\rangle$ with $v$ a unit vector in $\langle E_3, E_5, E_6\rangle$.

As $\langle E_3, E_4, E_7\rangle$ is associative, there is a circle subgroup of G$_2$ that acts via multiplication by $e^{it}$ on $\mathbb{C}^2 \cong \langle E_1 + iE_2, E_5 + iE_6\rangle$. Using this, we may ensure that $v \in \langle E_3, E_5\rangle$. Writing $v = xE_3 + yE_5$, we find that $E_1 \times v = -xE_2 + yE_4$, so that $\xi_p$ contains the associative subspace $\langle E_1, xE_2 - yE_4, xE_3 + yE_5\rangle$. □

All currently known examples of complete G$_2$–manifolds with a multi-Hamiltonian action of $T^4$ have reduced holonomy.

## 3 Toric G$_2$–manifolds: local characterisation

Motivated by the discussion in Section 2, we introduce the following terminology:

**Definition 3.1** A *toric* G$_2$*–manifold* is a torsion-free G$_2$–manifold $(M, \varphi)$ with an effective action of $T^3$ multi-Hamiltonian for both $\varphi$ and $*\varphi$.

The purpose of this section is to derive an analogue of the Gibbons–Hawking ansatz [27; 28] for toric G$_2$–manifolds, more specifically obtaining a local form for a toric G$_2$–structure and describing the torsion-free condition in these terms. An independent derivation of such equations with an extension to SU(2)–actions was obtained by [14] after our announcement [43].

So assume $(M,\varphi)$ is a toric $\mathrm{G}_2$–manifold, with $T^3$ acting effectively. Let $U_1$, $U_2$, $U_3$ be infinitesimal generators for the $T^3$–action; then these give a basis for $\xi_p \leqslant T_pM$ for each $p \in M_0$. Denote by $\theta = (\theta_1,\theta_2,\theta_3)^t$ the dual basis of $\xi_p^* \leqslant T_p^*M$:

$$\theta_i(U_j) = \delta_{ij} \quad \text{and} \quad \theta(X) = 0 \quad \text{for all } X \perp U_1, U_2, U_3.$$

For brevity we write $\theta_{ab}$ for $\theta_a \wedge \theta_b$, etc.

Let $\nu = (\nu_1,\nu_2,\nu_3)^t$ and $\mu$ be the associated multimoment maps; these satisfy

$$d\nu_i = U_j \wedge U_k \lrcorner \varphi = (U_j \times U_k)^\flat, \quad (i\ j\ k) = (1\,2\,3), \qquad d\mu = U_1 \wedge U_2 \wedge U_3 \lrcorner *\varphi.$$

It follows from Section 2.2 that $(d\nu, d\mu)$ has full rank on $M_0$ and induces a local diffeomorphism $M_0/T^3 \to \mathbb{R}^4$. We define a $3\times 3$ matrix $B$ of inner products given by

$$B_{ij} = g(U_i,U_j),$$

and on $M_0$ we put $V = B^{-1} = \det(B)^{-1}\operatorname{adj}(B)$.

In these terms, we have the following local expression for the $\mathrm{G}_2$–structure:

**Proposition 3.2** *On $M_0$, the 3–form $\varphi$ and 4–form $*\varphi$ are*

$$\varphi = -\det(V)d\nu_{123} + d\mu \wedge d\nu^t \operatorname{adj}(V)\theta + \mathfrak{S}_{ijk}\, \theta_{ij} \wedge d\nu_k,$$

$$*\varphi = \theta_{123}d\mu + \frac{1}{2\det(V)}(d\nu^t \operatorname{adj}(V)\theta)^2 + \det(V)d\mu \wedge \mathfrak{S}_{ijk}\, \theta_i \wedge d\nu_{jk}.$$

*The associated $\mathrm{G}_2$–metric is given by*

$$g = \frac{1}{\det V}\theta^t \operatorname{adj}(V)\theta + d\nu^t \operatorname{adj}(V)d\nu + \det(V)d\mu^2. \tag{3-1}$$

We note that $M_0$ comes with a coassociative foliation with $T^3$–symmetry whose leaves are specified by setting $\nu$ equal to a constant. The corresponding distribution is given by the kernel of $d\nu_{123}$. In particular, the restriction of $*\varphi$ to the each leaf is $\theta_{123}d\mu$.

**Proof** We start by choosing an auxiliary symmetric matrix $A > 0$ such that $A^2 = B^{-1}$ which is possible as $B$ is positive-definite. Then we set $X_i = \sum_{j=1}^3 A_{ij}U_j$ and observe that

$$g(X_i,X_j) = (ABA)_{ij} = (A^2B)_{ij} = \delta_{ij},$$

showing that the triplet $(X_1,X_2,X_3)$ is orthonormal. It follows that we can apply the formulas (2-2) for $\varphi$ and $*\varphi$.

We make the identification $\mathbb{R}^3 \cong \Lambda^2\mathbb{R}^3$ via contraction with the standard volume form. Then if we let $\Lambda^2 A$ denote the induced action of $A$ on $\Lambda^2\mathbb{R}^3$, we can get

$$\Lambda^2 A = \det(A)A^{-1}.$$

In these terms, we have that

$$\alpha = (\Lambda^2 A)d\nu, \quad \beta = \det(A)d\mu \quad \text{and} \quad \widehat{\theta} = A^{-1}\theta = \frac{1}{\det(A)}(\Lambda^2 A)\theta.$$

Turning to the expressions for the $G_2$ three-form, we start by noting that

$$\alpha_{123} = \det(\Lambda^2 A)d\nu_{123}$$

and that $\alpha_q(\beta\widehat{\theta}_q - \widehat{\theta}_{rs})$ equals

$$\sum_{i=1}^{3}(\Lambda^2 A)_{qi}\,d\nu_i\biggl(\sum_{j=1}^{3}(\Lambda^2 A)_{qj}\,d\mu\theta_j - \det(B)\sum_{k,l=1}^{3}(\Lambda^2 A)_{rk}(\Lambda^2 A)_{sl}\theta_{kl}\biggr),$$

where $(qrs) = (123)$. Summing these terms gives

$$\begin{aligned}\varphi = -\det(\Lambda^2 A)d\nu_{123} + d\mu\sum_{i,j=1}^{3} d\nu_i\,(\Lambda^2 A)^2_{ij}\theta_j \\ + \det(B)\sum_{i,k,l=1}^{3}(\Lambda^2 A)_{1i}(\Lambda^2 A)_{2k}(\Lambda^2 A)_{3l}(d\nu_i\theta_{kl} + d\nu_k\theta_{li} + d\nu_l\theta_{ik}),\end{aligned}$$

which is simplified by observing that the expression in the second line above reduces to give $d\nu_1\theta_{23} + d\nu_2\theta_{31} + d\nu_2\theta_{12}$, as required by the multimoment map relations. The asserted expression for $\varphi$ thus follows by noting that $(\Lambda^2 A)^2 = B/\det(B) = \operatorname{adj}(V)$.

To rephrase the 4–form expression, we observe that

$$\widehat{\theta}_{123}\beta = \theta_{123}d\mu,$$

consistent with the multimoment map condition, and that $\alpha_{rs}(\beta\widehat{\theta}_q - \widehat{\theta}_{rs})$ equals

$$\sum_{i,j=1}^{3}(\Lambda^2 A)_{ri}(\Lambda^2 A)_{sj}\,d\nu_{ij}\biggl(\sum_{k=1}^{3}(\Lambda^2 A)_{qk}\,d\mu\theta_k - \frac{1}{\det(A)^2}\sum_{k,l=1}^{3}(\Lambda^2 A)_{rk}(\Lambda^2 A)_{sl}\theta_{kl}\biggr)$$

for $(q\,r\,s) = (1\,2\,3)$. Upon summation, this quickly gives the stated expression for $*\varphi$.

Finally, for the metric we have

$$\begin{aligned} g = \widehat{\theta}^t\widehat{\theta} + \alpha^t\alpha + \beta^2 &= (A^{-1}\theta)^t A^{-1}\theta + (\Lambda^2 A\,d\nu)^t \Lambda^2 A\,d\nu + \det(A)^2 d\mu^2 \\ &= \theta^t\left(\frac{1}{\det(V)}\operatorname{adj}(V)\right)\theta + d\nu^t \operatorname{adj}(V)\,d\nu + \det(V)\,d\mu^2, \end{aligned}$$

as claimed. □

**Remark 3.3** The expression for $*\varphi$ may also be written as

$$*\varphi = \theta_{123}d\mu - \mathop{\mathfrak{S}}_{ijk}\mathop{\mathfrak{S}}_{pqr} V_{ip}d\nu_{jk}\theta_{qr} + \det(V)d\mu \wedge \mathop{\mathfrak{S}}_{ijk}\theta_i \wedge d\nu_{jk}. \tag{3-2}$$

**Remark 3.4** In the above, we have a natural action of $\mathrm{GL}(3,\mathbb{R})$, corresponding to changing the basis of $\mathfrak{t}^3$. This action can sometimes be used to simplify arguments as it allows us to assume $V$ is diagonal or the identity matrix at a given point provided only the $\mathbb{R}^3 = \widetilde{T^3}$–action is of relevance.

## 3.1 The torsion-free condition

Whilst it is true that any toric $\mathrm{G}_2$–manifold can be expressed as in Proposition 3.2, the $\mathrm{G}_2$–structure captured by these formulas is not automatically torsion-free.

Computing $d\varphi$ and $d{*}\varphi$ involves the exterior derivatives of $\theta$. By our observations in Section 2.2, we may think of $\theta$ as a connection 1–form and its exterior derivative

$$d\theta = \omega = (\omega_1, \omega_2, \omega_3)^t$$

is therefore a curvature 2–form (and as such represents an integral cohomology class). In terms of our parametrisation for the base space, via multimoment maps, we can write the curvature components of $\omega$ in the form

$$\omega_l = \mathop{\mathfrak{S}}_{ijk}(z_l^i\,d\nu_i\,d\mu + w_l^i\,d\nu_{jk}).$$

For convenience, we collect these curvature coefficients in two $3\times 3$ matrices $Z = (z_j^i)$ and $W = (w_j^i)$.

Closedness of $\varphi$ now becomes

$$\begin{aligned} 0 = -d\det(V)\wedge d\nu_{123} + d\mu(d\nu)^t \operatorname{adj}(V)\omega + d\mu(d\nu)^t d(\operatorname{adj}(V))\wedge\theta \\ + \mathop{\mathfrak{S}}_{ijk}(\omega_i\,d\nu_j - \omega_j\,d\nu_i)\theta_k. \end{aligned} \tag{3-3}$$

More explicitly, by wedging with $d\nu_i$, these equations completely determine the 9 curvature functions $z_i^j$:

$$z_i^l = \frac{\partial \operatorname{adj}(V)_{kl}}{\partial \nu_j} - \frac{\partial \operatorname{adj}(V)_{jl}}{\partial \nu_k}, \tag{3-4}$$

where $(i\ j\ k) = (1\,2\,3)$. Note, in particular, that the above expressions imply that $Z$ is traceless: $\operatorname{tr}(Z) = 0$.

In addition, upon wedging with $d\mu$, we see that (3-3) forces $W$ to be symmetric: $w_j^i = w_i^j$. Finally, it follows by wedging (3-3) with $\theta_{123}$ that

$$\left\langle \operatorname{adj}(V), \frac{\partial V}{\partial \mu} - W \right\rangle = 0, \tag{3-5}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $M_3(\mathbb{R}) \cong \mathbb{R}^9$.

Addressing coclosedness of $\varphi$, we use (3-2) to get

$$\begin{aligned} 0 &= d*\varphi \\ &= \underset{ijk}{\mathfrak{S}}\, \omega_i \theta_{jk} d\mu - \underset{ijk}{\mathfrak{S}}\, \underset{pqr}{\mathfrak{S}}\, dV_{ip} \wedge d\nu_{jk} \theta_{qr} - \underset{ijk}{\mathfrak{S}}\, \underset{pqr}{\mathfrak{S}}\, V_{ip} d\nu_{jk} (\omega_q \theta_r - \theta_q \omega_r) \\ &\qquad + d(\det(V)) \wedge d\mu\, \underset{ijk}{\mathfrak{S}}\, \theta_i d\nu_{jk}. \end{aligned} \tag{3-6}$$

The curvature functions $w_j^i$ are computed from the wedge product of (3-6) with $d\nu_i \theta_j$ to be

$$w_i^j = \frac{\partial V_{ij}}{\partial \mu} \tag{3-7}$$

and it follows that (3-5) automatically holds. If instead we wedge (3-6) with $d\mu\theta_i$ we find that

$$\sum_{i=1}^{3} \frac{\partial V_{ij}}{\partial \nu_i} = 0 \quad \text{for } j = 1, 2, 3. \tag{3-8}$$

We shall occasionally refer to this first-order underdetermined elliptic PDE system as the "divergence-free" condition. Coincidentally, (3-8) appears in the study of (linear) elasticity in continuum mechanics, expressing that the stress tensor is divergence-free (see eg [21; 22]). This equation together with the expression for $\operatorname{adj} V$ allows us to rewrite the coefficients $z_j^i$ as

$$z_l^i = \sum_{a=1}^{3} \frac{\partial V_{jl}}{\partial \nu_a} V_{ka} - \frac{\partial V_{kl}}{\partial \nu_a} V_{ja} \quad \text{for } (i\ j\ k) = (1\,2\,3). \tag{3-9}$$

One may now check that there are no further relations from (3-3) or (3-6).

There are only 6 additional equations, arising from the condition $d\omega = 0$. Using (3-7), (3-8) and (3-9), these equations can be expressed in the form of a second-order nonlinear elliptic PDE without zeroth-order terms:

$$L(V) + Q(dV) = 0. \tag{3-10}$$

Here the operator $L$ is given by

$$L = \frac{\partial^2}{\partial\mu^2} + \sum_{i,j} V_{ij} \frac{\partial^2}{\partial\nu_i \partial\nu_j},$$

and so has the same principal symbol as the Laplacian for the metric $d\mu^2 + d\nu^t B d\nu$, which, up to a conformal factor of $\det(V)$, is the same as the restriction of the $G_2$–metric (3-1) to the horizontal space. The operator $Q$ is the quadratic form in $dV$ given explicitly by

$$Q(dV)_{ij} = -\sum_{a,b=1}^{3} \frac{\partial V_{ia}}{\partial\nu_b} \frac{\partial V_{jb}}{\partial\nu_a}.$$

In summary, we have that the torsion-free condition determines $Z$ and $W$ together with three first-order equations and six second-order equations. We therefore have the following local description of toric $G_2$–manifolds:

**Theorem 3.5** *Any toric $G_2$–manifold can be expressed in the form of Proposition 3.2 on the open dense subset of principal orbits for the $T^3$–action.*

*Conversely, given a principal $T^3$–bundle over an open subset $\mathcal{U} \subset \mathbb{R}^4$, parametrised by $(\nu, \mu)$, together with $V \in \Gamma(\mathcal{U}, S^2(\mathbb{R}^3))$ that is positive-definite at each point. Then the total space comes equipped with a $G_2$–structure of the form given in Proposition 3.2. This structure is torsion-free, and hence toric, if and only if the curvature matrices $Z$ and $W$ are determined by $V$ via (3-4) and (3-7), respectively, and $V$ satisfies the divergence-free condition (3-8) together with the nonlinear second-order elliptic system (3-10).* □

Using this characterisation, it is not difficult to construct many explicit incomplete examples of toric $G_2$–manifolds (see Section 5.2).

As one would expect, solutions with $V$ constant are trivial in the following sense:

**Corollary 3.6** *A toric $\mathrm{G}_2$–manifold with $V$ constant is flat and hence locally isometric to $\mathbb{R}^7$.*

**Proof** If $V$ is constant, we may assume $V \equiv 1$. Now $\det(V) = 1$ everywhere and therefore $M_0 = M$. Consequently, by Proposition 3.2, we have a global orthonormal coframe $e^1, \dots, e^7$ satisfying $de^i = 0$ for all $1 \leqslant i \leqslant 7$. □

Let us conclude this section by remarking that (3-8) can be integrated to obtain what in a sense may be seen as an analogue of the local potential for hypertoric manifolds (cf [8]). The following observation is also known from continuum mechanics:

**Proposition 3.7** *Assume that $V \in \Gamma(\mathcal{U}, S^2(\mathbb{R}^3))$ satisfies (3-8), with $\mathcal{U} \subset \mathbb{R}^3$ simply connected. Then there exists $A \in \Gamma(\mathcal{U}, S^2(\mathbb{R}^3))$ such that*

$$V_{ii} = \frac{\partial^2 A_{jj}}{\partial \nu_k^2} + \frac{\partial^2 A_{kk}}{\partial \nu_j^2} - 2\frac{\partial^2 A_{jk}}{\partial \nu_j \partial \nu_k}, \quad V_{ij} = \frac{\partial^2 A_{ik}}{\partial \nu_j \partial \nu_k} + \frac{\partial^2 A_{jk}}{\partial \nu_k \partial \nu_i} - \frac{\partial^2 A_{ij}}{\partial \nu_k^2} - \frac{\partial^2 A_{kk}}{\partial \nu_i \partial \nu_j}, \tag{3-11}$$

*where $(i\,j\,k) = (1\,2\,3)$.*

**Proof** We begin by noting that (3-8) can be written more concisely as $d{*_3}(V d\nu) = 0$, where $\nu = (\nu_1, \nu_2, \nu_3)^t$ and $*_3$ is the flat Hodge star operator with respect to $\nu$. It follows that $*_3 V d\nu$ is exact, ie $V d\nu = {*_3} d(W d\nu)$ for some $W \in \Gamma(\mathcal{U}, M_3(\mathbb{R}))$. The symmetry of $V$ is then

$$\frac{\partial W_{iq}}{\partial \nu_p} - \frac{\partial W_{ip}}{\partial \nu_q} = \frac{\partial W_{js}}{\partial \nu_r} - \frac{\partial W_{jr}}{\partial \nu_s} \quad \text{for } (j\,p\,q) = (1\,2\,3) = (i\,r\,s).$$

For $i = j$ this relation is trivial. For $i \neq j$, order $i$ and $j$ and take $k$ such that $(i\,j\,k) = (1\,2\,3)$. Then $p = k$, $q = i$, $r = j$, $s = k$, so the symmetry is

$$-\frac{\partial (W_{ii} + W_{jj})}{\partial \nu_k} + \frac{\partial W_{ik}}{\partial \nu_i} + \frac{\partial W_{jk}}{\partial \nu_j} = 0.$$

This is the same as

$$d{*_3}(\widetilde{W} d\nu) = 0,$$

where $\widetilde{W} = W^T - (\operatorname{tr} W)1_3$, which is a divergence-free condition. Thus $\widetilde{W} d\nu = {*_3} d(A d\nu)$ for some $A \in \Gamma(\mathcal{U}, M_3(\mathbb{R}))$. It follows that $A$ determines the symmetric matrix $V$. In detail, we have $\widetilde{W}_{ij} = \partial A_{iq}/\partial \nu_p - \partial A_{ip}/\partial \nu_q$ for $(j\,p\,q) = (1\,2\,3)$, so, using $W = \widetilde{W}^T - \frac{1}{2}(\operatorname{tr}\widetilde{W})1_3$, we get

$$V_{ij} = \frac{\partial W_{iq}}{\partial v_p} - \frac{\partial W_{ip}}{\partial v_q} = \frac{\partial}{\partial v_p}\biggl(\frac{\partial A_{qs}}{\partial v_r} - \frac{\partial A_{qr}}{\partial v_s} - \tfrac{1}{2}\delta_{iq}\sum_{t=1}^{3}\biggl(\frac{\partial A_{tv}}{\partial v_u} - \frac{\partial A_{tu}}{\partial v_v}\biggr)\biggr) \\ - \frac{\partial}{\partial v_q}\biggl(\frac{\partial A_{ps}}{\partial v_r} - \frac{\partial A_{pr}}{\partial v_s} - \tfrac{1}{2}\delta_{ip}\sum_{t=1}^{3}\biggl(\frac{\partial A_{tv}}{\partial v_u} - \frac{\partial A_{tu}}{\partial v_v}\biggr)\biggr)$$

for $(j\,p\,q) = (1\,2\,3) = (i\,r\,s) = (t\,u\,v)$. To simplify this, consider separately the cases where $i = j$ and where $i \neq j$. First, for $i = j$, we get $p = r$ and $q = s$ distinct from $i$, so

$$V_{ii} = \frac{\partial^2 A_{rr}}{\partial v_s^2} + \frac{\partial^2 A_{ss}}{\partial v_r^2} - \frac{\partial^2 (A_{sr} + A_{rs})}{\partial v_r \partial v_s}.$$

For $i \neq j$, again rearrange and introduce $k$ so that $(i\,j\,k) = (1\,2\,3)$. Then $p = k$, $q = i$, $r = j$, $s = k$ and

$$V_{ij} = \frac{\partial}{\partial v_k}\biggl(\frac{\partial A_{ik}}{\partial v_j} - \frac{\partial A_{ij}}{\partial v_k} - \tfrac{1}{2}\delta_{ii}\sum_{t=1}^{3}\biggl(\frac{\partial A_{tv}}{\partial v_u} - \frac{\partial A_{tu}}{\partial v_v}\biggr)\biggr) \\ - \frac{\partial}{\partial v_i}\biggl(\frac{\partial A_{kk}}{\partial v_j} - \frac{\partial A_{kj}}{\partial v_k} - \tfrac{1}{2}\delta_{ik}\sum_{t=1}^{3}\biggl(\frac{\partial A_{tv}}{\partial v_u} - \frac{\partial A_{tu}}{\partial v_v}\biggr)\biggr),$$

which reduces to an expression that only depends on the symmetric part of $A$, so we may take $A$ to be symmetric. □

Note that the right-hand side of (3-11) is not elliptic, so a rewriting of Theorem 3.5 looses ellipticity of that system. The papers [22; 21] contain a description of the kernel of $A \mapsto V(A)$.

## 3.2 Digression: natural PDEs for toric $G_2$–manifolds

As we have already seen, toric $G_2$–manifolds come with an associated action of $GL(3,\mathbb{R})$. Thus a way of approaching (3-10), is to understand how $L$ and $Q$ transform with respect to this action.

The general linear group $GL(3,\mathbb{R})$ acts by changing the basis of $\mathfrak{t}^3$ and so of $\xi_p \cong \mathbb{R}^3$ at $p \in M_0$. It is useful to write $GL(3,\mathbb{R}) \cong \mathbb{R}^\times \times SL(3,\mathbb{R})$ and accordingly express irreducible representations in the form $\ell^p\Gamma_{a,b}$, where $\Gamma_{a,b}$ is an irreducible representation of $SL(3,\mathbb{R})$ (see eg [4]) and $\ell$ is the standard one-dimensional representation of

$\mathbb{R}^\times \to \mathbb{R}\setminus\{0\}$ given by $t \mapsto t$. As an example, this means that we have for $p \in M_0$ that $\xi_p = \ell^1\Gamma_{0,1}$.

So let $U = (\mathbb{R}^3)^* = \ell^{-1}\Gamma_{1,0}$, viewed as a representation of $\mathrm{GL}(3,\mathbb{R})$. Then $V \in S^2(U) = \ell^{-2}\Gamma_{2,0}$. The collection of first-order partial derivatives $V^{(1)} = (V_{ij,k}) = (\partial V_{ij}/\partial v_k)$ is then an element of $S^2(U)\otimes \ell^{-3}U^* = \ell^{-4}\Gamma_{2,0}\otimes\Gamma_{0,1}$. As a $\mathrm{GL}(3,\mathbb{R})$ representation this decomposes as

$$S^2(U)\otimes \ell^{-3}U^* = \ell^{-4}\Gamma_{1,0}\oplus \ell^{-4}\Gamma_{2,1},$$

with the projection to $\Gamma_{1,0}$ being just the contraction $S^2(\Gamma_{1,0})\otimes\Gamma_{0,1}\to\Gamma_{1,0}$, and $\Gamma_{2,1}$ denoting the kernel of this map. The divergence-free equation (3-8) just says this contraction is zero, so $V^{(1)} \in \ell^{-4}\Gamma_{2,1}$.

The operator $Q$ is a symmetric quadratic operator on $V^{(1)}$ with values in $S^2(U)$. Thus we may think of $Q(dV)$ as an element of the space $\ell^6 S^2(\Gamma_{2,1})^*\otimes S^2(\Gamma_{1,0})$. This space contains exactly one submodule isomorphic to $\ell^6$ as $S^2(\Gamma_{1,0})^*$ is a submodule of $S^2(\Gamma_{2,1})^*$. Direct computations show that $Q(dV)$ belongs to $\ell^6$.

Similarly, we may discuss the second-order terms in (3-10). We have $V^{(2)} = (V_{ij,kl}) \in R = (S^2(U)\otimes S^2(\ell^{-3}U^*))\cap(\ell^{-6}\Gamma_{2,1}\otimes\Gamma_{0,1})$. Now, ignoring the $\partial^2 V/\partial\mu^2$ term, $L(V)$ is built from a product of $V$ with $V^{(2)}$ and takes values in $S^2(U)$. So $L(V) \in S^2(U)^*\otimes R^*\otimes S^2(U)$. In this case, there are two submodules isomorphic to $\ell^6$, but only one appears in $L(V)$, corresponding to the contractions

$$S^2(U^*)\otimes(S^2(U^*)\otimes S^2(\ell^3 U))\otimes S^2(U)\to\ell^6.$$

Contracting in this way is arguably the most natural choice.

Finally, addressing the terms of $L$ involving $\partial^2 V/\partial\mu^2$, we have that $\partial/\partial\mu$ is an element of $\ell^{-3}$, and therefore $\partial^2 V/\partial\mu^2$ belongs to $\ell^6 S^2(U)^*\otimes S^2(U)$. In fact, it is easy to see that $\partial^2 V/\partial\mu^2$ belongs to the one-dimensional summand isomorphic to $\ell^6$ as we are tracing.

In conclusion, we have that $L$ and $Q$ are preserved up to scale by $\mathrm{GL}(3,\mathbb{R})$ change of basis, and this specifies $Q$ uniquely.

**Proposition 3.8** *Under the action of* $\mathrm{GL}(3,\mathbb{R})$, $L(V)$ *and* $Q(dV)$ *transform as elements of* $\ell^6$. *Moreover, up to scaling,* $Q$ *is the unique* $S^2(U)$*–valued quadratic form in* $dV$ *with this property.* □

# 4 Behaviour near singular orbits

In our description of toric $\mathrm{G}_2$–manifolds, we have so far been focusing on the regular part $M_0 \subset M$. We now turn to address what happens near a singular orbit for the $T^3$–action.

## 4.1 Flat models

For a complete hyperkähler manifold with a tri-Hamiltonian action of $T^n$ it is known that the hyperkähler moment map induces a homeomorphism $M/T^n \to \mathbb{R}^n$ (see [17; 42]). In this section, we establish the analogous result for toric $\mathrm{G}_2$–manifolds for flat models with a singular orbit; later we will prove this in general. There are two cases to consider as the singular orbit can be either $S^1$ or $T^2$, corresponding to a stabiliser of dimension 2 or 1.

**4.1.1 Two-dimensional stabiliser** Consider the flat model $M = S^1 \times \mathbb{C}^3$ equipped with the 3–form

$$\varphi = \tfrac{i}{2} dx \wedge (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3) + \mathrm{Re}(dz_1 \wedge dz_2 \wedge dz_3),$$

with dual 4–form

$$*\varphi = \mathrm{Im}(dz_1 \wedge dz_2 \wedge dz_3) \wedge dx - \tfrac{1}{8}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3)^2,$$

where $z_j = x_j + i y_j$ for $j = 1, 2, 3$ are standard complex coordinates on $\mathbb{C}^3$.

There is a natural effective $T^3$–action on $M$: writing $T^3 = S^1 \times T^2$, the $T^2$ acts as a maximal torus of $\mathrm{SU}(3)$ on $\mathbb{C}^3$ and the remaining circle acts naturally on the $S^1$ factor. Correspondingly, we have generating vector fields given by

$$U_1 = \frac{\partial}{\partial x},\ U_2 = 2\,\mathrm{Re}\left(i\left(z_1 \frac{\partial}{\partial z_1} - z_3 \frac{\partial}{\partial z_3}\right)\right),\ U_3 = 2\,\mathrm{Re}\left(i\left(z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3}\right)\right).$$

It follows that the matrix $B$ is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & |z_1|^2 + |z_3|^2 & |z_3|^2 \\ 0 & |z_3|^2 & |z_2|^2 + |z_3|^2 \end{pmatrix}$$

and so $V$ takes the form

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (|z_2|^2 + |z_3|^2)/A & -|z_3|^2/A \\ 0 & -|z_3|^2/A & (|z_1|^2 + |z_3|^2)/A \end{pmatrix},$$

where $A = |z_1 z_2|^2 + |z_3 z_1|^2 + |z_2 z_3|^2$. We have that $M_0$ is the complement of the following sets: $M^{T^2} = S^1 \times \{0\}$, where the singular stabiliser is $T^2 = \{1\} \times T^2 \leqslant S^1 \times T^2 = T^3$; $M^{S^1_i} = S^1 \times \{z_j = z_k = 0,\ z_i \neq 0\}$ for $(i\ j\ k) = (1\,2\,3)$, which all have singular stabiliser circles $S^1_i \leqslant T^2 \leqslant T^3$.

For the multimoment maps, we first compute

$$d\mu = U_1 \wedge U_2 \wedge U_3 \lrcorner *\varphi = d\,\mathrm{Im}(z_1 z_2 z_3),$$

giving that, up to addition of a constant, $\mu = \mathrm{Im}(z_1 z_2 z_3)$. Similarly, we find $\nu_1 = -\mathrm{Re}(z_1 z_2 z_3)$, from $U_2 \wedge U_3 \lrcorner \varphi$, and

$$d\nu_2 = U_3 \wedge U_1 \lrcorner \varphi = \tfrac{1}{2} d(|z_2|^2 - |z_3|^2).$$

So, again up to addition of a constant, $\nu_2 = \frac{1}{2}(|z_2|^2 - |z_3|^2)$. Finally, we have that $\nu_3 = -\frac{1}{2}(|z_1|^2 - |z_3|^2)$. Summarising, the multimoment maps are

$$\nu_1 + i\mu = -\overline{z_1 z_2 z_3}, \quad \nu_2 = \tfrac{1}{2}(|z_2|^2 - |z_3|^2), \quad \nu_3 = -\tfrac{1}{2}(|z_1|^2 - |z_3|^2).$$

**Proposition 4.1** *The multimoment map* $(\nu, \mu)$: $S^1 \times \mathbb{C}^3 \to \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$ *induces a homeomorphism* $(S^1 \times \mathbb{C}^3)/T^3 = \mathbb{C}^3/T^2 \to \mathbb{R}^4$.

As the referee points out, this map $\mathbb{C}^3/T^2 \to \mathbb{R}^4$ has also been considered in [1].

**Proof** Let us introduce some new variables. Putting $t = |z_3|^2$, we have $|z_1|^2 = t - a$ and $|z_2|^2 = t - b$, where $a = 2\nu_3$ and $b = -2\nu_2$. For $c = |\mu|^2 + |\nu_1|^2 = |z_1|^2 |z_2|^2 |z_3|^2$, we have the relation

$$f(t) := t(t-a)(t-b) = c.$$

Note that $f$ has zeros at $0$, $a$ and $b$. The constraints $|z_i|^2 \geqslant 0$ imply $t \geqslant x := \max\{0, a, b\}$. Now $f(t) \to \infty$ as $t \to \infty$, so $f([x, \infty)) = [0, \infty)$ and $f$ is strictly monotone increasing on $[x, \infty)$. Thus $f(t) = c$ has a unique solution $t = t(a, b, c) \geqslant x$ for each $a, b \in \mathbb{R}$ and each $c \geqslant 0$.

Write $\rho$: $\mathbb{C}^3/T^2 \to \mathbb{R}^4$ for the map induced by $(\nu, \mu)$. Given $(p, q) \in \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$, let $t = t(2p_3, -2p_2, q^2 + p_1^2)$, where $t(a, b, c)$ is as defined above. Now $\rho(z_1, z_2, z_3) = (p, q)$ if and only if $(|z_1|^2, |z_2|^2, |z_3|^2) = (t - 2p_3, t + 2p_2, t)$ and $z_1 z_2 z_3 = (iq - p_1)$. One sees that these equations are consistent, $\rho$ is surjective and solutions are unique up to the action of $T^2 \leqslant \mathrm{SU}(3)$. Thus $\rho$ is a continuous bijection $\mathbb{C}^3/T^2 \to \mathbb{R}^4$.

But $\mathbb{C}^3/T^2$ is homeomorphic to $\mathbb{R}^4$. Indeed, it follows from the results of [30] that $S^5/T^2$ is homeomorphic to $S^3$, so the claimed result follows by considering the cones on these spaces.

To be explicit, we note that

$$\begin{aligned} S^5 &= \{(z_1, z_2, z_3) \mid |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\} \\ &= \{(t_1^{1/2}e^{iu}, t_2^{1/2}e^{iv}, t_3^{1/2}e^{iw}) \mid t_i \geqslant 0,\ t_1 + t_2 + t_3 = 1\} \end{aligned}$$

with $T^2$–action induced by $(e^{i\theta}, e^{i\phi})\cdot(e^{iu}, e^{iv}, e^{iw}) = (e^{i(\theta+u)}, e^{i(\phi+v)}, e^{i(w-\theta-\phi)})$. Each $T^2$–orbit contains a representative with $u = v = w$. Furthermore, this representative is unique modulo $\frac{2\pi}{3}$ unless some $t_i$ is zero, since $\theta + u = \phi + v = w - \theta - \phi \pmod{2\pi}$ implies the common value $a$ satisfies $3a = u + v + w \pmod{2\pi}$ and each such $a$ gives a unique solution for $\theta$ and $\phi$ mod $2\pi$.

Topologically the two-simplex $\{(t_1, t_2, t_3) \mid t_i \geqslant 0,\ t_1 + t_2 + t_3 = 1\}$ is a unit disc $\{w \in \mathbb{C} \mid |w|^2 \leqslant 1\}$. The quotient $S^5/T^2$ has circle fibres over the interior of the disc that collapse to points on the boundary. Thus $S^5/T^2$ is topologically

$$\{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} = S^3.$$

Now $\rho$ is a continuous bijection $\mathbb{R}^4 = \mathbb{C}^3/T^2 \to \mathbb{R}^4$. By Brouwer's invariance of domain (see [35, Theorem 7.12]), it follows that $\rho$ is a homeomorphism. □

**4.1.2 One-dimensional stabiliser** The previous model contains points with stabiliser $S^1$, but we can also provide a simple standard model in this case. Let $M = (T^2 \times \mathbb{R}) \times \mathbb{C}^2$ with the 3–torus split as $T^3 = T^2 \times S^1$, the first $T^2$–factor acting on the corresponding torus in the first factor of $M$, and the $S^1$–factor acting as the maximal torus of SU(2) on $\mathbb{C}^2$. Introduce standard (local) coordinates $x$, $y$, $u$ for $T^2 \times \mathbb{R}$ and $(z, w)$ for $\mathbb{C}^2$.

The $\mathrm{G}_2$ 3–form may be written as

$$\varphi = du \wedge dx \wedge dy - du \wedge \tfrac{i}{2}(dz \wedge d\overline{z} + dw \wedge d\overline{w}) - \mathrm{Re}((dx - i\,dy) \wedge dz \wedge dw),$$

with dual 4–form

$$\begin{aligned} *\varphi = \tfrac{1}{8}(dz \wedge d\overline{z} + dw \wedge d\overline{w})^2 + dx \wedge dy \wedge \tfrac{i}{2}(dz \wedge d\overline{z} + dw \wedge d\overline{w}) \\ + du \wedge \mathrm{Im}((dx - i\,dy) \wedge dz \wedge dw). \end{aligned}$$

The generating vector fields are then

$$U_1 = \frac{\partial}{\partial x}, \quad U_2 = \frac{\partial}{\partial y}, \quad U_3 = -2\,\mathrm{Re}\biggl(i\biggl(z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}\biggr)\biggr).$$

The matrix $V$ is now

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/(|z|^2+|w|^2) \end{pmatrix}.$$

We compute the multimoment maps:

$$\begin{aligned} d\mu &= U_1 \wedge U_2 \wedge U_3 \lrcorner *\varphi = d\big(\tfrac{1}{2}(|z|^2-|w|^2)\big), \\ d\nu_1 &= U_2 \wedge U_3 \lrcorner \varphi = d\,\mathrm{Re}(zw), \\ d\nu_2 &= U_3 \wedge U_1 \lrcorner \varphi = d\,\mathrm{Im}(zw), \\ d\nu_3 &= U_1 \wedge U_2 \lrcorner \varphi = du. \end{aligned}$$

Thus, we may take

$$\mu = \tfrac{1}{2}(|z|^2-|w|^2), \quad \nu_1 + i\nu_2 = zw, \quad \nu_3 = u.$$

Note that, as expected, $(\mu, \nu_1, \nu_2)$ are just the standard hyperkähler moment maps for the action of $S^1$ on $\mathbb{H} = \mathbb{C}^2$. We know that this is essentially the Hopf fibration $S^3 \to S^2$ on distance spheres in $\mathbb{H} = \mathbb{R}^4$ and $\mathbb{R}^3$. Indeed,

$$\mu^2 + \nu_1^2 + \nu_2^2 = \tfrac{1}{4}(|z|^4 - 2|z|^2|w|^2 + |w|^4) + |z|^2|w|^2 = \tfrac{1}{4}(|z|^2+|w|^2)^2,$$

so 3–spheres of radius $r$ are mapped to 2–spheres of radius $\frac{1}{2}r^2$. Again we get:

**Proposition 4.2** *The multimoment map* $(\nu, \mu)$: $(T^2 \times \mathbb{R}) \times \mathbb{C}^2 \to \mathbb{R}^4$ *induces a homeomorphism* $((T^2 \times \mathbb{R}) \times \mathbb{C}^2)/T^3 = \mathbb{R} \times \mathbb{H}/S^1 \to \mathbb{R}^4$. □

## 4.2 Comparing with the flat models

We now turn to general toric G$_2$–manifolds $(M, \varphi)$. One way of obtaining a first feel for the behaviour of the multimoment maps near singular stabilisers is by comparing with the flat models. In order to do so, it turns out useful to recall some basic facts about Killing fields.

**4.2.1 Killing vector fields** If a vector field $X$ on $(M, g)$ is Killing, then this implies that $\nabla X$ is skew-adjoint, normalises the holonomy algebra and

$$\nabla^2_{A,B} X = -R_{X,A} B.$$

For the last result — cf [33] (see also [7]) — we use that $X$ preserves the Levi-Civita connection,

$$[X, \nabla_A B] = \nabla_{[X,A]} B + \nabla_A [X, B] = \nabla_{[X,A]} B + \nabla_A \nabla_X B - \nabla_A \nabla_B X, \tag{4-1}$$

to get

$$\begin{aligned} R_{X,A}B &= \nabla_X \nabla_A B - \nabla_A \nabla_X B - \nabla_{[X,A]} B = \nabla_X \nabla_A B - [X, \nabla_A B] - \nabla_A \nabla_B X \\ &= \nabla_{\nabla_A B} X - \nabla_A \nabla_B X = -\nabla^2_{A,B} X. \end{aligned}$$

It follows that at a zero $p$ of $X$, we have $(\nabla^2 X)_p = 0$ and

$$(\nabla^3_{A,B,C} X)_p = (-(\nabla_A(R_X))_B C)_p = (-(\nabla_A R)_{X,B} C - R_{\nabla_A X, B} C)_p = -(R_{\nabla_A X,B} C)_p.$$

Note also that at such a $p$, the endomorphism $(\nabla X)_p$ on $T_pM$ gives the infinitesimal action of the one-parameter group generated by $X$.

If $X$ and $Y$ are two commuting Killing vector fields with $X_p = 0$, then we claim that the endomorphisms $\nabla X$ and $\nabla Y$ commute at $p$. To see this, let $A$ be an arbitrary vector field. Then, at $p$, we have $\nabla_X \cdot = 0$, so using (4-1) gives

$$\begin{aligned} [\nabla X, \nabla Y]_p(A) &= (\nabla_{\nabla_A Y} X - \nabla_{\nabla_A X} Y)_p = ([\nabla_A Y, X] - \nabla_{\nabla_A X} Y)_p \\ &= (\nabla_{[A,X]} Y + \nabla_A [Y, X] - \nabla_{\nabla_A X} Y)_p = (\nabla_{\nabla_X A} Y)_p = 0, \end{aligned}$$

as claimed.

Finally, for a vector field $X$ preserving $\varphi$, we get that $X$ is Killing and

$$0 = L_X \varphi = d(X \lrcorner \varphi) = \boldsymbol{a}\varphi(\nabla X, \cdot, \cdot) = \varphi(\nabla X, \cdot, \cdot) + \varphi(\cdot, \nabla X, \cdot) + \varphi(\cdot, \cdot, \nabla X),$$

where $\boldsymbol{a}$ is the alternation map, which shows that $\nabla X \in \mathfrak{g}_2$.

**4.2.2 Near points with two-dimensional stabiliser** Let $p \in M$ be a point with $\mathrm{Stab}_{T^3}(p) \cong T^2$. We may identify $T_pM$ linearly with $\mathbb{R} \times \mathbb{C}^3 = T_{(1,0)}(S^1 \times \mathbb{C}^3)$ in the standard model of Section 4.1.1, so that the $\mathrm{G}_2$–forms agree at this point. We have an equivariant diffeomorphism between a neighbourhood of $0 \in T_pM$ and a neighbourhood of $p \in M$ via the local tubular model $T^3 \times_{\mathrm{Stab}(p)} \mathbb{C}^3 \cong T^3/T^2 \times \mathbb{C}^3$, the map on the $\mathbb{C}^3$ part being given by the Riemannian exponential map. The elements of $\mathrm{Stab}(p)$ act on $\mathbb{R} \times \mathbb{C}^3$ linearly as a maximal torus in $\mathrm{SU}(3)$. We may choose our linear identification so that this is the standard diagonal subgroup and may choose our generators $U_2$ and $U_3$ for $\mathrm{Stab}(p)$ so that

$$(\nabla U_2)_p = \mathrm{diag}(i, 0, -i), \quad (\nabla U_3)_p = \mathrm{diag}(0, i, -i)$$

in this model.

Let us now specify a choice of $U_1$. We note that the $T^3$–orbit of $p$ is $T^3/\mathrm{Stab}(p) \times \{0\}$ in the local model. This orbit is the fixed-point set of $\mathrm{Stab}(p)$, so is totally geodesic.

For any $U$ generating $T^3/\operatorname{Stab}(p)$, we thus have $(\nabla_U U)_p \in \mathbb{R}U$. But $(\nabla U)_p$ is an element of $\mathfrak{g}_2 \subset \mathfrak{so}(7)$, so $(\nabla_U U)_p = 0$. As the splitting $\mathbb{R} \times \mathbb{C}^3$ is orthogonal, it follows that $(\nabla U)_p \in \mathfrak{su}(3)$. Now each $U_i$ vanishes at $p$, so the endomorphisms $(\nabla U_i)_p$ commute with $(\nabla U)_p$, by Section 4.2.1. As $(\nabla U_2)_p$ and $(\nabla U_3)_p$ generate a maximal torus of $\mathfrak{su}(3)$, it follows that $(\nabla U)_p = a(\nabla U_2)_p + b(\nabla U_3)_p$ for some $a, b \in \mathbb{R}$. Putting $U_1 = U - aU_2 - bU_3$, we still have that $U_1$ generates $T^3/\operatorname{Stab}(p)$ and get $(\nabla U_1)_p = 0$. If we wish, we may assume that $(U_1)_p$ is of length $1$.

Now consider the multimoment maps. For $\nu_2$, we have

$$(\nabla \nu_2)_p = (d\nu_2)_p = (U_3 \wedge U_1 \lrcorner \varphi)_p = 0,$$

since $(U_3)_p = 0$. Similarly $\nabla \nu_3 = 0 = \nabla \nu_1 = \nabla \mu$ at $p$. Furthermore,

$$(\nabla^2 \nu_2)_p = ((\nabla \varphi)(U_3, U_1, \cdot) + \varphi(\nabla U_3, U_1, \cdot) + \varphi(U_3, \nabla U_1, \cdot))_p = \varphi(\nabla U_3, U_1, \cdot)_p$$

agrees with the flat model at $p$, and similarly for $(\nabla^2 \nu_3)_p$. For $\nu_1$, we have

$$(\nabla^2 \nu_1)_p = (\varphi(\nabla U_2, U_3, \cdot) + \varphi(U_2, \nabla U_3, \cdot))_p = 0,$$

as both $U_2$ and $U_3$ vanish at $p$. Similarly, $(\nabla^2 \mu)_p = 0$.

For third-order derivatives, we have

$$(\nabla^3 \nu_2)_p = (\varphi(\nabla^2 U_3, U_1, \cdot) + 2\varphi(\nabla U_3, \nabla U_1, \cdot) + \varphi(U_3, \nabla^2 U_1, \cdot))_p = 0,$$

since $(\nabla^2 U_3)_p = 0$ by Section 4.2.1, and $(\nabla U_1)_p = 0$ by our choice of $U_1$. Similarly, $(\nabla^3 \nu_3)_p = 0$. On the other hand,

$$\begin{aligned}(\nabla^3 \nu_1)_p &= (\varphi(\nabla^2 U_2, U_3, \cdot) + 2\varphi(\nabla U_2, \nabla U_3, \cdot) + \varphi(U_2, \nabla^2 U_3, \cdot))_p \\ &= 2\varphi(\nabla U_2, \nabla U_3, \cdot)_p,\end{aligned}$$

which agrees with the flat model, as does $(\nabla^3 \mu)_p$.

Let us now compute fourth-order derivatives. Firstly,

$$\begin{aligned}(\nabla^4 \nu_2)_p &= (\varphi(\nabla^3 U_3, U_1, \cdot) + 3\varphi(\nabla^2 U_3, \nabla U_1, \cdot) + 3\varphi(\nabla U_3, \nabla^2 U_1, \cdot) \\ &\qquad\qquad + \varphi(U_3, \nabla^3 U_1, \cdot))_p \\ &= \varphi(\nabla^3 U_3, U_1, \cdot)_p + 3\varphi(\nabla U_3, \nabla^2 U_1, \cdot)_p \\ &= -\varphi(R_{\nabla U_3, \cdot}\, \cdot, U_1, \cdot)_p - 3\varphi(\nabla U_3, R_{U_1, \cdot}\, \cdot, \cdot)_p,\end{aligned}$$

with a similar expression for $(\nabla^4 \nu_3)_p$. For $\nu_1$ and $\mu$, the same type of computation gives $(\nabla^4 \nu_1)_p = 0 = (\nabla^4 \mu)_p$. In conclusion, we have shown:

**Lemma 4.3** *Let $p \in M$ be a point with stabiliser $T^2$ whose infinitesimal generators are $U_2$ and $U_3$. Then the multimoment maps $\nu_2$ and $\nu_3$ agree with the flat model to order* 3 *and $\nu_1$ and $\mu$ agree with the flat model to order* 4. □

**4.2.3 Near points with one-dimensional stabiliser** In this case, we need less detailed information. Let $p \in M$ have $\operatorname{Stab}_{T^3}(p) \cong S^1$. We take the infinitesimal generator for this stabiliser to be $U_3$. Let $U_1$ and $U_2$ be two vector fields of the $T^3$–action that generate the quotient $T^3/\operatorname{Stab}(p) \cong T^2$. We take them to be of unit length and orthogonal at $p$. Then $U_1$ and $U_2$ are invariant under $U_3$ as is their $\mathrm{G}_2$–cross-product $U_1 \times U_2 = \varphi(U_1, U_2, \cdot)^\sharp$. We have $T_pM = \mathbb{R}^3 \times \mathbb{C}^2$ linearly, with $\mathbb{R}^3 = \langle U_1, U_2, U_1 \times U_2 \rangle_p$ and $\mathbb{C}^2$ the orthogonal complement. This identification may be chosen so that $(\nabla U_3)_p$ acts as the element $\operatorname{diag}(i, -i)$ in $\mathfrak{su}(2)$ on $\mathbb{C}^2$. The local model is $T^3 \times_{\operatorname{Stab}(p)} (\mathbb{R} \times \mathbb{C}^2) \cong (T^2 \times \mathbb{R}) \times \mathbb{C}^2$, with $T^2 \times \mathbb{R} \times \{0\}$ the fixed-point set of $U_3$, so totally geodesic. Now $d\nu_3 = (U_1 \times U_2)^\flat$ is nonzero and therefore provides a transverse coordinate to a six-dimensional level set, and $d\nu_1 = 0 = d\nu_2 = d\mu$ are zero at $p$. The three second derivatives $\nabla^2\nu_1$, $\nabla^2\nu_2$ and $\nabla^2\mu$ are specified by $U_i$ for $i = 1, 2$ and $\nabla U_3$ at $p$ and so all agree with the standard flat model at $p$.

**4.2.4 Images of singular orbits** First consider a point $p$ with stabiliser $S^1$. The previous section provides an integral basis $U_1$, $U_2$, $U_3$ of $\mathfrak{t}^3$ with $(U_3)_p = 0$. Furthermore, this is true for all points of $T^2 \times \mathbb{R}$ in the local model. It follows that $\nu_1$, $\nu_2$ and $\mu$ are constant on this set, and so the image under $(\nu, \mu)$ of this family of singular orbits is a straight line parametrised by the values of $\nu_3$.

Now for points $p$ with $T^2$–stabiliser, these lie on a circle $T^3p$. The normal bundle is modelled on $\mathbb{C}^3$ and there are three families of points with stabiliser $S^1$. These families meet at $p$ and correspond to the complex coordinate axes in $\mathbb{C}^3$. There is thus an integral basis $U_1$, $U_2$, $U_3$ of $\mathfrak{t}^3$ with $U_2 = 0 = U_3$ at $p$ and such that $U_2$, $U_3$ and $-U_2 - U_3$ generate the $S^1$ stabilisers of the three families. The images of the families under $(\nu, \mu)$ all have the same constant $\mu$– and $\nu_1$–coordinates, and provide the three half-lines meeting at the image of $p$ lying in $\nu_3$, $\nu_2$ or $\nu_2 - \nu_3$ constant.

Summarising, we have:

**Lemma 4.4** *For $p \in M \setminus M_0$, we have* $\operatorname{rank} B_p \leqslant 2$. *The image in $M/T^3$ of the union $M \setminus M_0$ of singular orbits consists of trivalent graphs lying in sets $\mu =$ constant with edges that are straight lines of rational slope in the $\nu$–coordinates. At each vertex the three primitive integral slope vectors sum to zero*; *in particular, these edges lie in a plane.* □

### 4.3 Deforming to the flat model

Let $\varphi$ be a torsion-free $\mathrm{G}_2$–structure on the ball $B_2(0) \subset \mathbb{R}^7$ with centre 0 and radius 2. Choose linear coordinates $(x_1, \dots, x_7)$ on $\mathbb{R}^7$ so that $\varphi|_0 = \varphi_0|_0$, where $\varphi_0$ is the standard constant coefficient $\mathrm{G}_2$–form. Our aim is to construct a family of torsion-free $\mathrm{G}_2$–structures $\varphi_t$ for $t \in (0,1]$, with $\varphi_1 = \varphi$ and with $\varphi_t$ converging to $\varphi_0$ on $\overline{B_1(0)}$ in each $C^k$–norm.

For $t \in (0,1]$, define a linear diffeomorphism $\lambda_t \colon \mathbb{R}^7 \to \mathbb{R}^7$ by $\lambda_t(x) = tx$. Note that $\lambda_t^* \varphi_0 = t^3 \varphi_0$, so let us take $\varphi_t$ to be

$$\varphi_t = t^{-3} \lambda_t^* \varphi \quad \text{for } t \in (0,1].$$

We have $\varphi = \varphi_0 + \psi$ where $\psi \in \Omega^3(B_2(0))$ is smooth and has $\psi|_0 = 0$. It follows that

$$\psi = \sum_{|I|=3} f_I \, dx_I,$$

where $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge dx_{i_3}$ for $I = (i_1, i_2, i_3) \in \{1, \dots, 7\}^3$ and $f_I$ is smooth with $f_I(0) = 0$. We may therefore write $f_I(x) = \sum_{k=1}^7 x_k h_{I,k}(x)$ with $h_{I,k}$ smooth. We have $\lambda_t^* \psi = \sum_I (\lambda_t^* f_I) t^3 dx_I$ and $(\lambda_t^* f_I)(x) = \sum_k t x_k h_{I,k}(tx)$, so $\|\lambda_t^* f_I\|_{C^0} \leqslant t \|f_I\|_{C^0}$. Thus, putting $\psi_t = t^{-3} \lambda_t^* \psi$, so $\varphi_t = \varphi_0 + \psi_t$, we get $\|\psi_t\|_{C^0} \leqslant t \|\psi\|_{C^0}$. Thus $\varphi_t \to \varphi_0$ in $C^0(\overline{B_1(0)})$ as $t \searrow 0$.

The Riemannian metric $g_t$ defined by $\varphi_t$ satisfies

$$g_t = t^{-2} \lambda_t^* g,$$

where $g = g_1$. The same types of computations as above show that $g_t \to g_0 = \sum_{i=1}^7 dx_i^2$ in $C^0$ as $t \searrow 0$. Let $\nabla^t$ be the Levi-Civita connection of $g_t$ and write its Christoffel symbols as $(\Gamma_t)_{ij}^k$. We claim that $\nabla^t \to \nabla^0$, meaning that $(\Gamma_t)_{ij}^k \to 0$, as $t \searrow 0$.

We have

$$(g_t)_{ij}(x) = \delta_{ij} + t \sum_{k=1}^7 x_k h_{ijk}(tx)$$

for some smooth functions $h_{ijk}$. Thus

$$\frac{\partial}{\partial x_l} (g_t)_{ij}(x) = t h_{ijl}(tx) + t^2 \sum_{k=1}^7 x_k \frac{\partial h_{ijk}}{\partial x_l}(tx)$$

and

$$(g_t^{-1})_{ij}(x) = \delta_{ij} + t \sum_{k=1}^{7} x_k \widetilde{h}_{ijk}(tx)$$

for some smooth functions $\widetilde{h}_{ijk}$. This gives

$$2(\Gamma_t)_{ij}^k(x) = t(h_{ijl} + h_{jil} - h_{lij})(tx) + O(t^2)$$

and hence $(\Gamma_t)_{ij}^k \to 0$, as claimed.

Now note that $0 = \nabla^t \varphi_t = \nabla^t \varphi_0 + \nabla^t \psi_t$, so $\nabla^t \psi_t = -\nabla^t \varphi_0 \to 0$ in $C^0$ as $t \searrow 0$. It follows that $\varphi_t \to \varphi_0$ in $C^1$. Iterating, noting that each derivative adds an extra factor of $t$, we get the claimed convergence in $C^k$.

If $U$ is a linear symmetry of $\mathbb{R}^7$ that preserves $\varphi$, then it is also a symmetry of $\varphi_t$, since $U$ commutes with dilations. Furthermore, if $X = \sum_{i=1}^7 v_i \partial/\partial x_i$ is a constant-coefficient vector field preserving $\varphi$ then it is also a symmetry of $\varphi_t$. Indeed, the one-parameter group generated by $X$ is $T_s(x) = x + sv$. Now, for any $f \in C^\infty(V)$ we have $(\lambda_t)_* X = tX$. This gives

$$L_X \varphi_t = t^{-3} L_X \lambda_t^* \varphi = t^{-3}(X \lrcorner d\lambda_t^* \varphi + d(X \lrcorner \lambda_t^* \varphi)) = t^{-2} \lambda_t^* L_X \varphi = 0,$$

which is the claimed symmetry.

Note that we now also get that the multimoment maps converge to those of flat space as $t \searrow 0$.

## 4.4 Identifications of the quotients

Consider a compact group G acting linearly on a finite-dimensional vector space $V$. A main result of [40] — cf [38] — is that any smooth G–invariant function is necessarily a smooth function of any set of generators for the ring of G–invariant polynomials on $V$. Suppose $\sigma_1, \dots, \sigma_k$ is a minimal set of such polynomial generators, meaning that no subset generates. Then the statement gives that $\sigma$ induces a diffeomorphism of $V/\mathrm{G}$ with $\sigma(V) \subset \mathbb{R}^k$ with respect to the "smooth structures": a function on $V/\mathrm{G}$ is smooth if its pullback to $V$ is smooth; a function on $\sigma(V)$ is smooth if it has local extensions to smooth functions in open $\mathbb{R}^k$–neighbourhoods of each point.

In our cases we are interested in two models:

(i) $\mathrm{G} = S^1$ acting on $V = \mathbb{R}^4 = \mathbb{C}^2$ as a maximal torus in $\mathrm{SU}(2)$, and

(ii) $\mathrm{G} = T^2$ acting on $V = \mathbb{R}^6 = \mathbb{C}^3$ as a maximal torus in $\mathrm{SU}(3)$.

Let us consider each of these in turn. For (i), let $(z,w)$ be standard complex coordinates. Then $S^1$ acts as $e^{i\theta}(z,w) = (e^{i\theta}z, e^{-i\theta}w)$. The invariant polynomials are generated by $(\sigma_1,\dots,\sigma_4)$:

$$\sigma_1 + i\sigma_2 = zw, \quad \sigma_3 = \tfrac{1}{2}(|z|^2 - |w|^2), \quad \sigma_4 = \tfrac{1}{2}(|z|^2 + |w|^2).$$

Note that these satisfy the relations

$$\sigma_4 \geqslant 0, \quad \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \sigma_4^2. \tag{4-2}$$

For (ii), write $(z_1,z_2,z_3)$ for the standard coordinates in the flat model, as above. This time the ring of polynomial invariants is generated by five elements,

$$\sigma_1 + i\sigma_2 = -\overline{z_1z_2z_3}, \quad \sigma_3 = \tfrac{1}{2}(|z_2|^2 - |z_3|^2), \quad \sigma_4 = \tfrac{1}{2}(|z_3|^2 - |z_1|^2), \quad \sigma_5 = |z_3|^2,$$

satisfying the relations

$$\sigma_5 \geqslant \max\{0, -2\sigma_3, 2\sigma_4\}, \quad \sigma_1^2 + \sigma_2^2 = \sigma_5(\sigma_5 + 2\sigma_3)(\sigma_5 - 2\sigma_4). \tag{4-3}$$

We have chosen our generators in such a way that $\sigma_1,\dots,\sigma_{k-1}$ correspond to the relevant multimoment maps in the flat models. Our work in Section 4.1 on the flat models shows that in both cases the map $\sigma(V) \to \mathbb{R}^{k-1}$ given by $(\sigma_1,\dots,\sigma_{k-1},\sigma_k) \mapsto (\sigma_1,\dots,\sigma_{k-1})$ is a homeomorphism. For the nonflat cases, we have the multimoment maps giving us invariant functions that agree with $\sigma_1,\dots,\sigma_{k-1}$ to certain orders. As Schwarz gives that $V/\mathrm{G}$ is diffeomorphic to $\sigma(V)$, the aim is now to show that these still give homeomorphisms $\sigma(V) \to \mathbb{R}^{k-1}$. For the case of one-dimensional stabilisers this is what [8] does, albeit in a hyperkähler context, but the local model is the same. We discuss this briefly as preparation for the six-dimensional case.

For the four-dimensional model we may proceed as follows. Let $V$ denote the slice with its $S^1$–action. Write $\pi\colon V \to V/S^1$ for the projection. Use $W = \mathbb{R}^4 = U \times \mathbb{R}$ with $U = \mathbb{R}^3$. Let $F_0$ be the linear projection $W \to U$. Let $S = \sigma(V) \subset W$ be the semialgebraic set given by (4-2).

On the four-dimensional slice $V$, we have (restrictions of) the multimoment map functions $\nu_1$, $\nu_2$ and $\mu$. Collect these into a single function $m = (\nu_1,\nu_2,\mu)\colon V \to \mathbb{R}^3$. This is a smooth invariant function, so by Schwarz it is induced by a smooth function on $S$. Write

$$m = f \circ \sigma, \quad f\colon S \to \mathbb{R}^3.$$

Note that $f$ smooth means it extends to a smooth function in a neighbourhood of any given point; we use the same name for a choice of such a smooth extension in a neighbourhood of $0 \in W$.

By Section 4.2.3, we know that the first two covariant derivatives at the origin of $\nu_1$, $\nu_2$ and $\mu$ agree with those of $\sigma_1$, $\sigma_2$ and $\sigma_3$, respectively. So $m$ agrees with $m_0 = (\sigma_1, \sigma_2, \sigma_3)$ to order 2 near the origin and $f = F_0 + \tilde{f}$ with $\tilde{f}$ smooth. In the slice coordinates at the origin, $\tilde{f} \circ \sigma$ vanishes to order 2 and all the $\sigma_i$ have degree 2, so $\tilde{f}$ vanishes to order 1 in $\sigma$. In other words,

$$\tilde{f}(\sigma) = \sum_{i,j=1}^{4} \sigma_i \sigma_j f_{ij}(\sigma),$$

where each $f_{ij}$ is smooth. In particular, the derivative of $\tilde{f}$ has norm bounded above by $c\|\sigma\|$ on this neighbourhood and the mean value theorem gives

$$\|\tilde{f}(x) - \tilde{f}(y)\| \leqslant c(\|x\| + \|y\|)\|x - y\|. \tag{4-4}$$

Consider points $q_1$ and $q_2$ in the slice near the fixed point $p = 0$. Write $x = \sigma(q_1)$, $y = \sigma(q_2)$. Then

$$\begin{aligned} \|m(q_1) - m(q_2)\| = \|f(x) - f(y)\| &= \|F_0(x) - F_0(y) + \tilde{f}(x) - \tilde{f}(y)\| \\ &\geqslant \|F_0(x) - F_0(y)\| - c(\|x\| + \|y\|)\|x - y\|. \end{aligned} \tag{4-5}$$

But $F_0^{-1}(a) = (a, \|a\|) \in S$ and

$$\|x - y\| = \big\|(F_0(x), \|F_0(x)\|) - (F_0(y), \|F_0(y)\|)\big\| \leqslant 2\|F_0(x) - F_0(y)\|$$

gives

$$\|m(q_1) - m(q_2)\| \geqslant \tfrac{1}{2}\|x - y\| - c(\|x\| + \|y\|)\|x - y\| \geqslant \big(\tfrac{1}{2} - c(\|x\| + \|y\|)\big)\|x - y\|.$$

So, for $\|x\|, \|y\| \leqslant 1/(8c)$, we have $\|m(q_1) - m(q_2)\| \geqslant \frac{1}{4}\|x - y\|$, proving that $m$ is injective on orbits in a neighbourhood of the origin. Invoking Brouwer's invariance of domain gives that $m$ induces a homeomorphism of the quotient space in a neighbourhood of the origin.

Let us turn to the six-dimensional models. Let $V$ be the slice with its $T^2$–action and write $\pi\colon V \to V/T^2$ for the projection map. Let $W = \mathbb{R}^5 = U \times \mathbb{R}$ with $U = \mathbb{R}^4$ and write $F_0\colon W \to U$ for the linear projection. The vector space $W$ contains the semialgebraic set $S = \sigma(V)$ given by (4-3). Write $m = (\nu_1, \mu, \nu_2, \nu_3)\colon V \to \mathbb{R}^4$ for the collection of multimoment maps. By Schwarz, $m = f \circ \sigma$ for a smooth $f\colon S \to \mathbb{R}^4$.

On $V$, the first four derivatives of $\nu_1$ and $\mu$, and the first three derivatives of $\nu_2$ and $\nu_3$, agree with those of $\sigma_1$, $\sigma_2$, $\sigma_3$ and $\sigma_4$, respectively. Noting that any homogeneous polynomial in $\sigma_i$ of degree 2 is at least of degree 4 in the $z_i$ and $\overline{z}_i$, we thus have $f = F_0 + \widetilde{f}$ with

$$\widetilde{f}(\sigma) = \sum_{i,j=1}^{5} \sigma_i \sigma_j f_{ij}(\sigma)$$

for some smooth functions $f_{ij}$. This gives the estimates (4-4) and (4-5) on some neighbourhood $S_0$ of $0 \in S$.

Now consider points $x = \sigma(q)$ satisfying (4-3). To estimate $x_5$, note that

$$x_5(x_5 + 2x_3)(x_5 - 2x_4) \geqslant (x_5 - \max\{0, -2x_3, 2x_4\})^3,$$

so, as $x_5 \geqslant 0$, we have

$$|x_5| \leqslant (x_1^2 + x_2^2)^{1/3} + \max\{0, -2x_3, 2x_4\} \leqslant (x_1^2 + x_2^2)^{1/3} + 2(x_3^2 + x_4^2)^{1/2}.$$

For $\|(x_1, x_2, x_3, x_4)\| < 1$, we have

$$\begin{aligned} \|x\| = \|(F_0(x), x_5)\| &\leqslant \|F_0(x)\| + |x_5| \\ &\leqslant \|F_0(x)\| + \|F_0(x)\|^{2/3} + 2\|F_0(x)\| \\ &\leqslant \|F_0(x)\|^{2/3}(3\|F_0(x)\|^{1/3} + 1) \leqslant 4\|F_0(x)\|^{2/3}. \end{aligned}$$

So on $S_0 \cap B_1(0)$ this gives

$$\|m(q)\| = \|f(x)\| \geqslant \|F_0(x)\| - c\|x\|^2 \geqslant \bigl(\tfrac{1}{4}\|x\|\bigr)^{3/2} - c\|x\|^2 = \|x\|^{3/2}\bigl(\tfrac{1}{8} - c\|x\|^{1/2}\bigr).$$

Thus for $\|x\| \leqslant 1/(256c^2)$ we have that $\|m(q)\| > \frac{1}{16}\|x\|^{3/2}$. This implies that 0 is the only point in the neighbourhood $W_0 = \{x \in S_0 \cap B_1(0) \mid \|x\| < 1/(256c^2)\}$ that maps to 0 under $m$.

Now consider a family $\varphi_t$ of $T^3$–invariant torsion-free $G_2$–structures on $S^1 \times \sigma^{-1}(W_0)$ with $\varphi_1 = \varphi$, the structure we are interested in, and $\varphi_0$ the flat $G_2$–structure that coincides with $\varphi$ at 0. Such a family was constructed in Section 4.3 and the discussion there shows that $f_t \to f_0 = F_0$ as $t \searrow 0$. Moreover the bound $c_t$ above for $f_t$ also has $c_t \searrow 0$ and in particular $c = c_1 \geqslant c_t$ for all $t < 1$.

Let us consider the Brouwer degrees of these maps; cf [39; 18]: let $W_1 \Subset W_0$ be an open ball containing 0; for $f\colon W_0 \to \mathbb{R}^4$ of class $C^2$ the *Brouwer degree* is

$$d_B[f, W_1] = \int_{W_1} \chi(\|f(x)\|) J_f(x)\, dx,$$

where $J_f = \det Df$ is the Jacobian of $f$ and $\chi\colon [0,\infty) \to [0,\infty)$ is continuous, has the closure of its support contained in $(0, \inf_{x \in \partial W_1} \|f(x)\|)$ and satisfies $\int_{\mathbb{R}^4} \chi(\|x\|)\, dx = 1$. This definition extends to continuous functions $f$ by approximating them uniformly via smooth functions, and the degree is homotopy-invariant; it agrees with the topological degree of the map $f/\|f\|\colon \partial W_1 \to S^3$. For $z \notin f(\partial W_1)$, the Brouwer degree of $f$ at $z$ is $d_B[f, W_1, z] = d_B[f(\cdot) - z, W_1]$. At regular values $z$, the number $d_B[f, W_1, z]$ counts the points $x$ in $f^{-1}(z) \cap W_1$ with the signs of $J_f(x)$. Any homeomorphism has $d_B[f, W_1, z] = \pm 1$.

Now $F_0 = f_0$ is a homeomorphism $S \to \mathbb{R}^4$ and has degree $+1$ at all points. Furthermore, $S$ is the set of $(\sigma_1, \dots, \sigma_5) \in \mathbb{R}^5$ satisfying (4-3). Differentiating this equation, we have

$$p_5 d\sigma_5 = \sum_{i=1}^{4} p_i \, d\sigma_i$$

with

$$p_1 = 2\sigma_1, \quad p_2 = 2\sigma_2, \quad p_3 = -2\sigma_5(\sigma_5 - 2\sigma_4), \quad p_4 = 2\sigma_5(\sigma_5 + 2\sigma_3),$$
$$p_5 = (\sigma_5 + 2\sigma_3)(\sigma_5 - 2\sigma_4) + \sigma_5(\sigma_5 - 2\sigma_4) + \sigma_5(\sigma_5 + 2\sigma_3) = |z_1 z_2|^2 + |z_3 z_1|^2 + |z_2 z_3|^2,$$

where $(z_1, z_2, z_3)$ are the coordinates on $V = \mathbb{C}^3$. In particular, $\sigma_5$ is a smooth function of $(\sigma_1, \dots, \sigma_4)$ off the locus $p_5 = 0$, which is the image of the set on which two of the $z_i$ are zero, ie the image of the complex coordinate axes of $V$. But this is just the locus of points with $T^3$–stabiliser of dimension at least 1 and so is specified purely by the group action. Off this locus $dm_t$ has rank 4 and so the same is true of $df_t$. In particular, off this locus $df_t$ is a local diffeomorphism. Furthermore, on the locus but away from 0, we have $S^1$–stabilisers and from the four-dimensional models we know that $f_t$ is a local homeomorphism.

Now homotopy-invariance combined with the fact that $f_t^{-1}(0) \cap W_1 = \{0\}$ implies that each $f_t$ has degree $+1$ and at smooth points the local degrees are also $+1$. It follows that on the smooth locus inside $W_1$ the maps $f_t$ are one-to-one for all $t \in [0,1]$. However, the image $(p_5 = 0) \setminus \{0\}$ consists of three half-lines each determined the group action, in particular by which copy of $S^1 \subset T^2$ is the corresponding stabiliser. On this set $m_t$ is still a local homeomorphism and so is monotone on each half-line. As $f_t$ is a local homeomorphism it follows that the local degrees at these points are also $+1$. Thus $f_t$ is injective on $W_1$. Using Brouwer's invariance of domain, we conclude that $f$ is a homeomorphism from $W_1$ to a neighbourhood of $0 \in \mathbb{R}^4$.

Summarising the above analysis, we have shown:

**Theorem 4.5** *Let* $(M,\varphi)$ *be a toric* $G_2$*–manifold. Then* $M/T^3$ *is homeomorphic to a smooth four-manifold. Moreover, the multimoment map* $(\nu,\mu)$ *induces a local homeomorphism* $M/T^3 \to \mathbb{R}^4$*.* □

# 5 Explicit examples of toric $G_2$–manifolds

We now turn to write down some explicit examples of toric $G_2$–manifolds.

## 5.1 Some complete examples

In this section, we describe some known nonflat complete examples of toric $G_2$–manifolds.

**5.1.1 Holonomy SU(3):** $M = S^1 \times T^*S^3$ Before turning to a concrete example, it seems worthwhile explaining how it arises as a particular case of a more general construction of toric $G_2$–manifolds with holonomy in $SU(3)$. So assume we have a 6–manifold $N$ with vanishing first Betti number and equipped with a Calabi–Yau structure $(\sigma,\Psi)$. If there is an effective $T^2$–action on $N$ preserving $\sigma$ and $\Psi = \psi + i\hat{\psi}$, then we have invariant scalar functions $(\nu,\mu)\colon N \to \mathbb{R}^4$ that satisfy the relations

$$d\nu_1 = \psi(U_2,U_3,\cdot), \quad d\nu_2 = -\sigma(U_3,\cdot), \quad d\nu_3 = \sigma(U_2,\cdot), \quad d\mu = -\hat{\psi}(U_2,U_3,\cdot),$$

where $U_2$ and $U_3$ are generators for the torus action. We can now consider the torsion-free product $G_2$–structure on $M = S^1 \times N$ given by

$$\varphi = dx \wedge \sigma + \psi, \quad *\varphi = \hat{\psi} \wedge dx + \tfrac{1}{2}\sigma^2.$$

Clearly, $(M,\varphi)$ is toric with $T^3 = S^1 \times T^2$ acting in the obvious way and associated multimoment maps $(\nu,\mu)$. Theorem 4.5 now implies that $N/T^2$ is locally homeomorphic to $\mathbb{R}^4$ and Lemma 4.4 implies that the trivalent graphs lie in the surfaces $(\nu_1,\mu)$ constant.

For $(N,\sigma,\Psi)$ as above there is a special Lagrangian foliation (of an open dense subset) with $T^2$–symmetry. The leaves are given by fixing $(\nu_2,\nu_3,\mu)$ to be constant. The corresponding distribution is given by the kernel of $d\mu \wedge d\nu_{23}$, and the restriction of $\psi$ to each leaf is $\theta_{23} \wedge d\nu_1$.

As a concrete example of the above, one can take $N = T^*S^3$ with its Stenzel Calabi–Yau structure [41]. For our purposes, it is more convenient to identify $N$ with the complex sphere

$$Q = \left\{ z \in \mathbb{C}^4 \;\middle|\; \sum_{j=0}^{3} z_j^2 = 1 \right\},$$

following [31]. Specifically, one has the $\mathrm{SO}(4)$–equivariant diffeomorphism

$$T^*S^3 \to Q, \quad (p, v) \mapsto \cosh(\|v\|)p + i \sinh(\|v\|)\frac{v}{\|v\|}$$

(see [45]). In terms of $Q$, the Kähler 2–form is given by $\sigma = d\alpha$, where

$$\alpha(X)_z = \tfrac{1}{2} f'(|z|^2)\,\mathrm{Im}(X^t\overline{z}) \quad \text{for } X \in T_zQ,\ z \in Q,$$

with $f$ satisfying the differential equation

$$((f_u)^3)_u = 3k(\sinh u)^2$$

for some constant $k > 0$. The holomorphic volume form can be computed as

$$\Psi(X_1, X_2, X_3)_z = dz_{0123}(z, X_1, X_2, X_3)$$

for $X_1, X_2, X_3 \in T_zQ$ and $z \in Q$.

For the $T^2$–action, we consider $T^2 \subset \mathrm{SO}(4)$ generated by the vector fields

$$U_2(z) = (-z_1, z_0, 0, 0), \quad U_3(z) = (0, 0, -z_3, z_2).$$

In accordance with [31, Theorem 5.2] one finds that the multimoment maps are

$$\nu_1 + i\mu = \tfrac{1}{2}(\overline{z}_0^2 + \overline{z}_1^2), \quad \nu_2 = -f'(|z|^2)\,\mathrm{Im}(z_2\overline{z}_3), \quad \nu_3 = f'(|z|^2)\,\mathrm{Im}(z_0\overline{z}_1).$$

Many other examples are to be found in [1; 34] and related works.

#### 5.1.2 The cone over $S^3 \times S^3$ and its deformation

As mentioned in Section 2.2, one example of a complete toric $\mathrm{G}_2$–manifold with holonomy equal to $\mathrm{G}_2$ is the spin bundle over $S^3$ equipped with its Bryant–Salamon structure. It may be viewed as a deformation of the cone over $S^3 \times S^3$ with its nearly Kähler structure. In both cases, one can describe the $\mathrm{G}_2$–structure in terms of one-parameter families of left-invariant half-flat $\mathrm{SU}(3)$–structures on $S^3 \times S^3 \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1) \subset \mathbb{H} \times \mathbb{H}$.

To make this concrete, let us take $\{(i, 0), (j, 0), (-k, 0), (0, i), (0, j), (0, -k)\}$ as our basis of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \cong T_1(S^3 \times S^3)$. Correspondingly, the tangent space at $(p, q) \in S^3 \times S^3$ has basis

$$\begin{aligned} &E_1(p, q) = (pi, 0), \quad E_2(p, q) = (pj, 0), \quad E_3(p, q) = (-pk, 0), \\ &F_1(p, q) = (0, qi), \quad F_2(p, q) = (0, qj), \quad F_3(p, q) = (0, -qk). \end{aligned} \tag{5-1}$$

If we let $e^1, \dots, f^3$ denote the dual coframe, then $de^i = 2e^{jk}$ and $df^i = 2f^{jk}$ for $(i\,j\,k) = (1\,2\,3)$.

We have an almost-effective action of $\mathrm{Sp}(1)^3$ on $S^3 \times S^3$ given by

$$((h,k,l),(p,q)) \mapsto (hp\ell^{-1}, kq\ell^{-1})$$

that preserves the half-flat $\mathrm{SU}(3)$–structures of interest (cf [15]). By choosing a maximal torus $S^1$ in each $\mathrm{Sp}(1)$, we obtain an almost-effective action of $T^3$. Considering the quotient of $T^3$ by $\mathbb{Z}_2 = \{\pm(1,1,1)\}$, we get an effective action of a torus $T^3$. For concreteness, let us choose each maximal torus $T^1 \subset \mathrm{Sp}(1)$ to be of the form $\{e^{i\theta} \mid \theta \in \mathbb{R}\}$. In this case, we have generating vector fields given by

$$U_1(p,q) = (ip, 0), \quad U_2(p,q) = (0, iq), \quad U_3(p,q) = (-pi, -qi).$$

Following [19], we can express these vector fields in terms of (5-1) via

$$\begin{aligned}
U_1(p,q) &= \langle \bar{p}ip, i\rangle E_1(p,q) + \langle \bar{p}ip, j\rangle E_2(p,q) - \langle \bar{p}ip, k\rangle E_3(p,q),\\
U_2(p,q) &= \langle \bar{q}iq, i\rangle F_1(p,q) + \langle \bar{q}iq, j\rangle F_2(p,q) - \langle \bar{q}iq, k\rangle F_3(p,q),\\
U_3(p,q) &= -E_1(p,q) - F_1(p,q),
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$. Note that each of the maps $p \mapsto \bar{p}ip$ and $q \mapsto \bar{q}iq$ is a standard Hopf fibration $\pi_H \colon S^3 \to S^2 \subset \operatorname{Im}\mathbb{H}$. We see that the span of $U_1$, $U_2$, $U_3$ is 3–dimensional, unless $p, q \in \pi_H^{-1}(\{\pm i\}) = \{e^{i\theta}, je^{i\theta} \mid \theta \in \mathbb{R}\}$.

The nearly Kähler structure on $S^3 \times S^3$ can be expressed as

$$\begin{aligned}
\sigma &= \tfrac{2}{3\sqrt{3}}(e^1f^1 + e^2f^2 + e^3f^3),\\
\psi &= \tfrac{4}{9\sqrt{3}}(e^{23}f^1 + e^{31}f^2 + e^{12}f^3 - e^1f^{23} - e^2f^{31} - e^3f^{12}),\\
\hat{\psi} &= \tfrac{4}{27}(-2e^{123} - 2f^{123} + e^1f^{23} + e^2f^{31} + e^3f^{12} + e^{23}f^1 + e^{31}f^2 + e^{12}f^3).
\end{aligned}$$

Specifically this means that $(\sigma, \psi)$ defines an $\mathrm{SU}(3)$–structure satisfying $d\sigma = 3\psi$ and $d\hat{\psi} = -2\sigma^2$. As mentioned above, $T^3$ acts effectively, preserving the nearly Kähler structure, and we have associated multimoment maps $(\tilde{\nu}, \tilde{\mu}) \colon S^3 \times S^3 \to \mathbb{R}^4$ for the pair of closed forms $(\psi, \sigma^2)$. As $d\sigma = 3\psi$ and $d\hat{\psi} = -2\sigma^2$, it is particularly easy to compute the maps $(\tilde{\nu}, \tilde{\mu})$: by [37, Proposition 3.1] we have that $\tilde{\nu}_i = \frac{1}{3}\sigma(U_j, U_k)$ and $\tilde{\mu} = \frac{1}{2}\hat{\psi}(U_1, U_2, U_3)$.

The conical $\mathrm{G}_2$–structure on $\mathbb{R}_+ \times S^3 \times S^3$ is given by

$$\varphi_C = dr \wedge r^2\sigma + r^3\psi = d\big(\tfrac{1}{3}r^3\sigma\big), \quad *\varphi_C = r^3\hat{\psi} \wedge dr + \tfrac{1}{2}r^4\sigma^2 = d\big(-\tfrac{1}{4}r^4\hat{\psi}\big).$$

It follows that

$$U_i \wedge U_j \lrcorner \varphi = 3r^2\widetilde{\nu}_k dr + r^3 d\widetilde{\nu}_k = d(r^3\widetilde{\nu}_k),$$
$$U_1 \wedge U_2 \wedge U_3 \lrcorner *\varphi = 2r^3\widetilde{\mu} dr + \tfrac{1}{2}r^4 d\widetilde{\mu} = d\big(\tfrac{1}{2}r^4\widetilde{\mu}\big).$$

So, in terms of nearly Kähler data, the multimoment maps $(\nu^C, \mu^C)\colon \mathbb{R}_+ \times S^3 \times S^3 \to \mathbb{R}^4$ are given by $(\nu^C, \mu^C) = \big(r^3\widetilde{\nu}, \tfrac{1}{2}r^4\widetilde{\mu}\big)$. Explicitly,

$$\begin{aligned}
\nu_1^C(r,(p,q)) &= \tfrac{2}{9\sqrt{3}}r^3\langle \overline{q}iq, i\rangle,\\
\nu_2^C(r,(p,q)) &= \tfrac{2}{9\sqrt{3}}r^3\langle \overline{p}ip, i\rangle,\\
\nu_3^C(r,(p,q)) &= \tfrac{2}{9\sqrt{3}}r^3\langle \overline{p}ip, \overline{q}iq\rangle,\\
\mu^C(r,(p,q)) &= \tfrac{2}{27}r^4(\langle \overline{p}ip, j\rangle\langle \overline{q}iq, k\rangle - \langle \overline{p}ip, k\rangle\langle \overline{q}iq, j\rangle).
\end{aligned}$$

From the remarks about Hopf fibrations, it is clear that $(\nu^C, \mu^C)$ induces a map $\mathbb{R}_+ \times S^2 \times S^2 \to \mathbb{R}^4$ given by

$$(r,(v,w)) \mapsto \tfrac{2}{9\sqrt{3}}r^3\big(\langle v, i\rangle, \langle w, i\rangle, \langle v, w\rangle, \tfrac{2}{\sqrt{3}}r(\langle v, j\rangle\langle w, k\rangle - \langle v, k\rangle\langle w, j\rangle)\big).$$

Turning now to the Bryant–Salamon solution on the spin bundle of $S^3$, we begin by observing that this can be written in the form

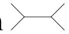
$$\begin{aligned}
\varphi_{\mathrm{BS}} &= -\tfrac{4}{3\sqrt{3}}\epsilon(e^{123} - f^{123}) + d\big(\tfrac{1}{3}(r^3-\epsilon)\sigma\big),\\
*\varphi_{\mathrm{BS}} &= \tfrac{4}{9}\epsilon\, dr \wedge (e^{123} + f^{123}) + (r^3-\epsilon)\widehat{\psi} \wedge dr + \tfrac{1}{2}r(r^3-4\epsilon)\sigma^2
\end{aligned}$$

for some $\epsilon > 0$ (see eg [11]). Then, building on the computations from the nearly Kähler case, we find that the multimoment maps for the toric Bryant–Salamon manifold are

$$\begin{aligned}
\nu_1^{\mathrm{BS}}(r,(p,q)) &= \tfrac{2}{9\sqrt{3}}(r^3-4\epsilon)\langle \overline{q}iq, i\rangle,\\
\nu_2^{\mathrm{BS}}(r,(p,q)) &= \tfrac{2}{9\sqrt{3}}(r^3-4\epsilon)\langle \overline{p}ip, i\rangle,\\
\nu_3^{\mathrm{BS}}(r,(p,q)) &= \tfrac{2}{9\sqrt{3}}(r^3-\epsilon)\langle \overline{p}ip, \overline{q}iq\rangle,\\
\mu^{\mathrm{BS}}(r,(p,q)) &= \tfrac{2}{27}r(r^3-4\epsilon)(\langle \overline{p}ip, j\rangle\langle \overline{q}iq, k\rangle - \langle \overline{p}ip, k\rangle\langle \overline{q}iq, j\rangle).
\end{aligned}$$

In this case, the matrix $V$ has inverse given by

$$V^{-1} = \begin{pmatrix}
\dfrac{4(r^3-\epsilon)}{9r} & -\dfrac{\sqrt{3}}{r}\dfrac{2\epsilon+r^3}{r^3-\epsilon}\nu_3^{\mathrm{BS}} & -\dfrac{\sqrt{3}}{r}\nu_2^{\mathrm{BS}}\\
-\dfrac{\sqrt{3}}{r}\dfrac{2\epsilon+r^3}{r^3-\epsilon}\nu_3^{\mathrm{BS}} & \dfrac{4(r^3-\epsilon)}{9r} & -\dfrac{\sqrt{3}}{r}\nu_1^{\mathrm{BS}}\\
-\dfrac{\sqrt{3}}{r}\nu_2^{\mathrm{BS}} & -\dfrac{\sqrt{3}}{r}\nu_1^{\mathrm{BS}} & \dfrac{4(r^3-4\epsilon)}{9r}
\end{pmatrix}.$$

We obtain the values of the multimoment map on the zero section of the spin bundle by continuity. Away from this zero section, the points with one-dimensional stabilisers map to the straight lines $(\varepsilon_1 t, \varepsilon_2 t, \varepsilon_1\varepsilon_2(t+k), 0)$, where $\varepsilon_i \in \{\pm 1\}$, $k = 2\epsilon/(3\sqrt{3})$ and $t > 0$. The limit $t \searrow 0$ gives points with stabiliser $T^2$ and the preimages of the interior of the line segment from $(0,0,-k,0)$ to $(0,0,k,0)$ is also a family of points with one-dimensional stabiliser. The image of the singular orbits is thus of the form .

For $r$ fixed large, $(\nu, \mu/r)$ essentially induces the map

$$(x,z,y,w) \mapsto (x, y, xy + \|z\|\|w\|\cos\theta, \|z\|\|w\|\sin\theta),$$

where $(x,z),(y,w) \in S^2 \subset \mathbb{R}\times\mathbb{C}$ and $\theta$ is the oriented angle from $z$ to $w$. On the quotient space this map is thus a homeomorphisms of topological three spheres and of global degree $1$. From the general theory, we know $(\nu,\mu)$ has local degree $+1$, so we conclude that the multimoment map is injective on the orbit space. However, varying the parameter $r$, we get a deformation retract to the ellipsoids to the line segment $\{(0,0,t,0) \mid t \in [-k,k]\}$, so the multimoment map is onto. We conclude that the multimoment map is a homeomorphism from the $T^3$–orbit space of the spin bundle onto $\mathbb{R}^4$.

**Remark 5.1** After we completed this paper, Foscolo, Haskins and Nordström [26] constructed many new examples of G$_2$–manifolds, including several examples with $T^3$–symmetry. For some of these, we find that the corresponding trivalent graphs are planar (see [44]), even though the holonomy group is the whole of G$_2$.

## 5.2 Ansätze simplifying the PDEs

From a PDE viewpoint a particular challenge is the fact that the characterisation of toric G$_2$–manifolds involves the coupled system consisting of both first-order PDEs (3-8) and a second-order system (3-10). In the following, we shall study some special cases that circumvent this complicating issue. This allows us to construct many explicit (but generally incomplete) examples of toric G$_2$–manifolds. In particular, we find that simple polynomial solutions in the variables $(\nu,\mu)$ can lead to metrics with holonomy equal to G$_2$.

**5.2.1 One variable dependence** Let us assume that $V$ depends only on the variable $\mu$, so $\partial V/\partial\nu_i = 0$ for $i = 1,2,3$. Then $Z \equiv 0$. The condition that $d\omega = 0$ now yields that $\partial^2 V_{ij}/\partial\mu^2 = 0$. So $V$ is linear in $\mu$ and thus $W$ is constant.

**Example 5.2** Taking $V = \operatorname{diag}(\mu, \mu, \mu)$ gives a solution defined for all $\mu > 0$. In this case, the associated $G_2$–metric takes the form

$$g = \frac{1}{\mu}(\theta_1^2 + \theta_2^2 + \theta_3^2) + \mu^2(d\nu_1^2 + d\nu_2^2 + d\nu_3^2) + \mu^3 d\mu^2,$$

where $d\theta_i = d\nu_j \wedge d\nu_k$ for $(i\,j\,k) = (1\,2\,3)$.

This metric has (restricted) holonomy equal to $G_2$ as can be seen eg by computing the Riemannian curvature: regarded as a 2–form $\Omega = (\Omega_{ij})$ on $T^3 \times \mathcal{U}$ with values in an associated $\mathfrak{g}_2$–bundle, the span of $\Omega_{ij}$ for $1 \leqslant i \leqslant j \leqslant 7$ has dimension $14$.

From the viewpoint of complete metrics, this situation turns out to be less interesting.

**Proposition 5.3** *Suppose $V = V(\mu)$. If $(M, \varphi)$ is complete, then it is flat and hence locally isometric to $\mathbb{R}^7$.*

**Proof** By Corollary 3.6, it suffices to show that completeness forces $V$ to be a constant matrix. So let us assume $V$ is not constant.

After adding a constant to $\mu$, if necessary, we may assume that $V(0) > 0$ and then it follows by Remark 3.4 that we can take $V(0) = 1_3$. In fact, using the action of $GL(3, \mathbb{R})$ on $S^2(\mathbb{R}^3)$, we can even assume $V$ has the form $V(\mu) = \operatorname{diag}(\lambda_1\mu+1, \lambda_2\mu+1, \lambda_3\mu+1)$, where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$.

As $V$ is not constant, there is $\lambda_i \neq 0$ such that the rank of $V$ drops (the first time) when $\mu = -1/\lambda_i$. By Lemma 4.4, we cannot be approaching a point $p \in M \setminus M_0$, ie a singular orbit, as we have $\det(B) \to \infty$. To show that this implies incompleteness of the $G_2$–metric, we use the criterion of [16, Lemma 1]: we look for a finite-length curve not contained in any compact set.

In the base space of our $T^3$–bundle, we have a curve $\gamma$, defined on $(-1/\lambda_i, 0]$, corresponding to a curve parametrised by the $\mu$–coordinate. Let $p \in M_0$ be a point projecting to $\gamma(0)$ and $\tilde{\gamma}$ the horizontal lift of $\gamma$ with $\tilde{\gamma}(0) = p$. Clearly, the curve $\tilde{\gamma}\colon (-1/\lambda_i, 0] \to M_0$ has finite length, but is not contained in any compact set. □

In the cases where $V$ depends only on one of the variables $\nu_i$, similar arguments and conclusions apply.

**5.2.2 Orthogonal Killing vectors** Let us assume $V_{ij} = 0$ for all $i \neq j$, ie the generating vector fields for the torus action are orthogonal. The $G_2$–metric now takes the form

$$g = \frac{1}{V_{11}}\theta_1^2 + \frac{1}{V_{22}}\theta_2^2 + \frac{1}{V_{33}}\theta_3^2 + V_{11}V_{22}V_{33}\Big(d\mu^2 + \frac{1}{V_{11}}d\nu_1^2 + \frac{1}{V_{22}}d\nu_2^2 + \frac{1}{V_{33}}d\nu_3^2\Big).$$

In this case, $W$ is diagonal with nonzero entries given by $w^j_j = \partial V_{jj}/\partial\mu$, and $Z$ has zeros on the diagonal and off-diagonal entries given by

$$z^j_i = -V_{kk}\frac{\partial V_{ii}}{\partial \nu_k}, \quad z^i_j = V_{kk}\frac{\partial V_{jj}}{\partial \nu_k},$$

with $(i\ j\ k) = (1\,2\,3)$.

The divergence-free condition (3-8) tells us that $\partial V_{ii}/\partial\nu_i = 0$ for $i = 1,2,3$. Then the condition $d\omega = 0$ is given by the equations

$$\frac{\partial^2 V_{ii}}{\partial\mu^2} + V_{jj}\frac{\partial^2 V_{ii}}{\partial\nu_j^2} + V_{kk}\frac{\partial^2 V_{ii}}{\partial\nu_k^2} = 0 \quad \text{for } (i\ j\ k) = (1\,2\,3), \tag{5-2}$$

together with

$$\frac{\partial V_{ii}}{\partial\nu_j}\frac{\partial V_{jj}}{\partial\nu_i} = 0 \tag{5-3}$$

for $i \neq j$.

Assume now that one has $\partial V_{ii}/\partial\nu_j \neq 0$, for some $j \neq i$. Without loss of generality, we can take $\partial V_{11}/\partial\nu_2 \neq 0$, which forces $\partial V_{22}/\partial\nu_1 = 0$. So $V_{22}$ is a function of $\nu_3$ and $\mu$ alone. By differentiating the equation (5-2) for $i = 2$, we then find that

$$\frac{\partial V_{33}}{\partial\nu_1}\frac{\partial^2 V_{22}}{\partial\nu_3^2} = 0 = \frac{\partial V_{33}}{\partial\nu_2}\frac{\partial^2 V_{22}}{\partial\nu_3^2}.$$

So either $\partial^2 V_{22}/\partial\nu_3^2$ vanishes identically, or there is an open set where $\partial V_{33}/\partial\nu_i = 0$ for $i = 1,2,3$.

In the first case, $V_{22}$, as a function of $\nu_3$, has nonvanishing derivative of order at most 1 and so is either constant or linear in that variable. Correspondingly, we have $\partial V_{22}/\partial\nu_3 = 0$ or $\partial V_{22}/\partial\nu_3 \neq 0$, respectively.

If $\partial V_{22}/\partial\nu_i = 0$ for $i = 1,2,3$, the additional information captured by (5-3) is that either $\partial V_{11}/\partial\nu_3 = 0$ or $\partial V_{33}/\partial\nu_1 = 0$ in an open neighbourhood. If $\partial V_{22}/\partial\nu_3 \neq 0$, then (5-3) moreover tells us that $\partial V_{33}/\partial\nu_2 = 0$.

Considering the case where $\partial V_{22}/\partial\nu_3 \neq 0$ and

$$\frac{\partial V_{11}}{\partial\nu_3} = 0 = \frac{\partial V_{22}}{\partial\nu_1} = \frac{\partial V_{33}}{\partial\nu_2},$$

(5-2) reduces to the equations

$$\frac{\partial^2 V_{11}}{\partial\mu^2} + V_{22}\frac{\partial^2 V_{11}}{\partial\nu_2^2} = 0, \quad \frac{\partial^2 V_{33}}{\partial\mu^2} + V_{11}\frac{\partial^2 V_{33}}{\partial\nu_1^2} = 0.$$

Differentiating the first of these expressions with respect to $\nu_3$, we find that $V_{11}$ is (at most) linear in $\nu_2$. Similarly, from differentiating the second equation above with respect to $\nu_2$, we find that $V_{33}$ is (at most) linear in $\nu_1$ as $\partial V_{11}/\partial \nu_2$ is nonzero.

After possibly relabelling indices, the above considerations imply that there are two ways to satisfy (5-2) and (5-3). The first one is to have each $V_{ii}$ (at most) a linear function in two variables as follows:

$$V_{11} = V_{11}(\mu, \nu_2), \quad V_{22} = V_{22}(\mu, \nu_3), \quad V_{33} = V_{33}(\mu, \nu_1). \tag{5-4}$$

From the viewpoint of complete metrics this is less interesting:

**Proposition 5.4** *If $(M, \varphi)$ is complete with $V$ diagonal and its entries satisfy (5-4), then $(M, \varphi)$ is flat and hence locally isometric to $(\mathbb{R}^7, \varphi_0)$.*

**Proof** This is essentially proved in the same way as Proposition 5.3. We may assume that $V(0) > 0$. Consequently, we can write $V$ in the form

$$\operatorname{diag}(\epsilon_1 \nu_2 \mu + \kappa_1 \nu_2 + \lambda_1 + 1, \epsilon_2 \nu_3 \mu + \kappa_2 \nu_3 + \lambda_2 + 1, \epsilon_3 \nu_1 \mu + \kappa_3 \nu_1 + \lambda_3 + 1).$$

By considering suitable curves (corresponding to $(0, \mu)$, $(\nu_1, 0)$ etc), we arrive at the asserted conclusion. □

The second and more interesting possibility is to have $\partial V_{33}/\partial \nu_i = 0$ for $i = 1, 2, 3$ together with

$$\frac{\partial V_{22}}{\partial \nu_1} = 0 = \frac{\partial V_{22}}{\partial \nu_2}, \quad \frac{\partial V_{11}}{\partial \nu_1} = 0.$$

In this case, (5-2) corresponds to the elliptic hierarchy

$$\frac{\partial^2 V_{11}}{\partial \mu^2} + V_{22} \frac{\partial^2 V_{11}}{\partial \nu_2^2} + V_{33} \frac{\partial^2 V_{11}}{\partial \nu_3^2} = 0, \quad \frac{\partial^2 V_{22}}{\partial \mu^2} + V_{33} \frac{\partial^2 V_{22}}{\partial \nu_3^2} = 0, \quad \frac{\partial^2 V_{33}}{\partial \mu^2} = 0. \tag{5-5}$$

So again $V_{33}$ is at most a linear function of $\mu$, and $V$ is independent of $\nu_1$. This means, in particular, that $U_2$ and $U_3$ have no zeros, ie there are no points with $T^2$–isotropy, and points with $S^1$–isotropy lie above disjoint lines parallel to the $\nu_1$–axis.

When $V_{33}$ is constant, which we can take to be $1$, the $\mathrm{G}_2$–metric is a product

$$g = \theta_3^2 + \frac{1}{V_{11}} \theta_1^2 + \frac{1}{V_{22}} \theta_2^2 + V_{11} V_{22} \Big( d\mu^2 + \frac{1}{V_{11}} d\nu_1^2 + \frac{1}{V_{22}} d\nu_2^2 + d\nu_3^2 \Big),$$

so the holonomy reduces to a subgroup of $\mathrm{SU}(3)$.

Reducing the holonomy further, one obvious solution to the elliptic system in this case is given by taking $V_{22} = 1 = V_{33}$ and $V = V_{11}(\mu, \nu_2, \nu_3)$ to be a harmonic function on $\mathbb{R}^3$. Then the associated $\mathrm{G}_2$–holonomy metric is given by

$$g = \theta_2^2 + \theta_3^2 + d\nu_1^2 + \frac{1}{V}\theta_1^2 + V(d\mu^2 + d\nu_2^2 + d\nu_3^2).$$

This has the form of a product of a flat metric on (an open set of) $T^2 \times \mathbb{R}$ and a hyperkähler metric on an $S^1$–bundle over (an open set of) $\mathbb{R}^3$.

Excluding these cases of reduced holonomy, we are thus left with analysing the equations

$$\frac{\partial^2 V_{11}}{\partial\mu^2} + V_{22}\frac{\partial^2 V_{11}}{\partial\nu_2^2} + \mu\frac{\partial^2 V_{11}}{\partial\nu_3^2} = 0, \qquad \frac{\partial^2 V_{22}}{\partial\mu^2} + \mu\frac{\partial^2 V_{22}}{\partial\nu_3^2} = 0,$$

having set $V_{33}(\mu) = \mu$.

As the following example shows, it is easy to find local (incomplete) solutions to these equations that have full holonomy.

**Example 5.5** By writing down $(\nu, \mu)$ as a power series and solving (5-5), we get solutions on trivial bundles $T^3 \times \mathcal{U}$, where $\mathcal{U} \subset \mathbb{R}^4$ is an appropriate open subset. As an example of such a solution we can take

$$V_{11}(\nu_2, \nu_3, \mu) = 2\mu^5 - 15\mu^2\nu_3^2 - 5\nu_2^2, \quad V_{22}(\nu_3, \mu) = \mu^3 - 3\nu_3^2, \quad V_{33}(\mu) = \mu.$$

As in Example 5.2, one checks by explicit computations that the associated metric has (restricted) holonomy equal to $\mathrm{G}_2$.

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